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Price of Anarchy with multiple information sources under competition

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ABSTRACT

We characterize the Price of Anarchy (PoA) in a single channel under the presence of K competing sources. As performance metric we consider the Age of Information, which measures the freshness of information in a remote system. In our main results we show that when the service times of all sources are equal the PoA is $2-\frac{1}{K}$, and that otherwise the PoA is unbounded from above. Numerical computations show that the PoA increases with the disparity of the service rates.

1. Introduction

Queueing games study the decision making of non-cooperating agents that interact in a queueing system. The *Nash Equilibrium* is one of the most studied concepts of queueing games (see [2] for a recent book on this topic) and it is defined as a set of strategies such that no agent gets benefit by changing its strategy unilaterally. Such a set of strategies might differ substantially from a global optimum strategy, leading to a performance degradation of the Nash Equilibrium. This degradation is often analyzed using the concept of Price of Anarchy [3], which is defined as the ratio of the performance at the Nash Equilibrium over the optimal performance [8].

In this article, we consider a system formed by multiple processes that need to be observed by a remote monitor. Each process has a source that generates status updates and all the sources send its status updates to the monitor through the same channel. We are interested in achieving the freshest possible information of all the sources at the destination. Timely information is a crucial factor in a wide range of information, communication, and control systems. For instance, in autonomous driving systems the state of the traffic and the location of the vehicles must be as recent as possible. The Age of Information (AoI) is a relatively new metric that measures the freshness of the knowledge we have about the status of a remote system. More specifically, the AoI of a source is defined as the time elapsed since the generation of the last successfully received packet containing information about that source. As shown in the seminal paper [4], policies that optimize performance metrics of interest in queueing theory such as throughput, delay or package-loss probability do not necessarily minimize the AoI. We refer to [11] for a recent survey on AoI.

We consider that the channel through which status updates are sent is a queue operating under the Last-Generated-First-Served discipline with preemption in service, which was proven in [1] to be an optimal policy for a system with a single source. We also assume that the sources are charged for sending status updates; namely, each source has to bear an economic expense that is proportional to its load. In this context, we formulate a non-cooperative game in which each source is a player that can choose its generation rate of status updates and aims to minimize its cost, which is the sum of its Average Age of Information (AAoI) and its payment. Our goal is to measure the inefficiency due to the competition between different sources (i.e. the Nash Equilibrium) with respect to the minimum total cost that can be achieved.

The status updates are generated according to a Poisson process and served at exponential times. We consider two different scenarios: first, the *homogeneous* case in which the service rate is the same across sources, and then the *heterogeneous* case, for which we allow the sources to be served at different rates. In the homogeneous case we give a closed-form expression for both the Nash Equilibrium and the global optimum and conclude that the PoA is $2-1/K$, where K is the number of sources. When the service

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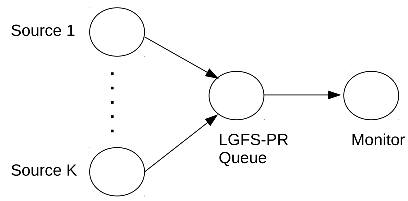

Fig. 1. The system under study.

rates are heterogeneous, even though we did not manage to characterize neither the Nash Equilibrium nor the global optimum, we provide a set of properties that lead to the unboundedness of the PoA. In other words, the performance degradation due to the selfishness of the sources can be arbitrarily large when the disparity of the service rates increases.

Some researchers have analyzed recently non-cooperative games in AoI-aware networks, but in a different setting such as considering an interference channel for which the strategy is the power level of the transmitters [6,7] or collision avoidance strategies [9].

A preliminary version of this paper was published as an extended abstract in [5].

2. Model description

Consider a system formed by K processes whose status is to be observed by a remote monitor. Each process has a source that generates status updates and sends them immediately through a transmission channel to the monitor. We assume that the transmission times from the sources to the channel and from the channel to the monitor are both negligible. Thus, the generation time of updates and the time at which they arrive to the transmission channel coincide, and similarly for the time at which updates complete service and the delivery time at the monitor. In addition, we consider that updates are served according to a Last-Generated-First-Served policy with preemption in service (LGFS-PR). Under this service discipline, when an update arrives to the queue it readily starts being served, preempting the update currently in service if any (see Fig. 1).

Updates of source n arrive to the queue according to a Poisson process of rate λ_n and are served at exponential times of rate μ_n. We denote by $\boldsymbol{\mu} = (\mu_1, \dots, \mu_K)$ the vector of service rates, by $\boldsymbol{\lambda} = (\lambda_1, \dots, \lambda_K)$ the vector of arrival rates and by $\lambda = \sum_n \lambda_n$ the total arrival rate. The AoI of source n at time t is defined as the difference between t and the generation time of the last update of this source that has been delivered to the monitor. We denote by $\Delta_n(\boldsymbol{\lambda}, \boldsymbol{\mu})$ the Average Age of Information (AAoI) of source n. In order to dissuade the sources from sending updates continuously, we consider that a system planner penalizes a source for the server usage. In fact, it can be shown that the fraction of time that updates of source n are served increases with λ_n and decreases with μ_n. Thus, in this particular model, we assume that sources are charged proportionally to the fraction λ_n/μ_n, i.e., the payment associated to source n is $c\lambda_n/\mu_n$, with $c > 0$.

We formulate a non-cooperative game in which each source is a player whose goal is to find its generation rate to minimize its cost function, namely

$$C_n(\boldsymbol{\lambda}, \boldsymbol{\mu}, c) = \Delta_n(\boldsymbol{\lambda}, \boldsymbol{\mu}) + c\lambda_n/\mu_n. \tag{1}$$

A solution of this game is called *Nash Equilibrium* and we denote it by $\boldsymbol{\lambda}^{ne} = (\lambda_1^{ne}, \dots, \lambda_K^{ne})$. A Nash Equilibrium is defined as a set of generation rates such that no source gets benefit from a unilateral deviation, i.e.

$$\lambda_n^{ne} \in \arg\min_{\lambda_n} C_n(\lambda_n, \boldsymbol{\lambda}_{-n}^{ne}, \boldsymbol{\mu}, c), \quad n = 1, \dots, K \tag{GAME}$$

where $\boldsymbol{\lambda}_{-n}^{ne}$ is the vector of generation rates at the Nash Equilibrium for all the sources except for source n.

In contrast with the aforementioned scenario, we can think of a setting in which the sources collaborate looking for the global benefit. In this case the mathematical problem consists in finding a vector of generation rates such that the overall cost is minimized:

$$\min_{\{\lambda_1,\dots,\lambda_K\}} \sum_{n=1}^{K} C_n(\boldsymbol{\lambda}, \boldsymbol{\mu}, c). \tag{GLOBAL-OPT}$$

A solution of this problem is called a *global optimum* and will be denoted by $\boldsymbol{\lambda}^G = (\lambda_1^G, \dots, \lambda_K^G)$.

The PoA is defined as the worst possible ratio between the cost at Nash Equilibrium and the cost at a global optimum, i.e.,

$$PoA = \sup_{\boldsymbol{\mu}, c} \frac{\sum_{n=1}^{K} C_n(\boldsymbol{\lambda}^{ne}, \boldsymbol{\mu}, c)}{\sum_{n=1}^{K} C_n(\boldsymbol{\lambda}^{G}, \boldsymbol{\mu}, c)}. \tag{POA}$$

By definition $PoA \geq 1$, and the closer it is to 1 the smaller is the inefficiency on account of the selfish strategies in the non-cooperative scenario.

3. Homogeneous service rates

We start by considering that the service rate is equal across sources, i.e., $\mu_n = \mu$ for all $n = 1, \dots, K$. The AAoI for this case [10, Thm 2a)] is

$$\Delta_n(\boldsymbol{\lambda}, \boldsymbol{\mu}) = \frac{\mu + \lambda}{\mu \lambda_n}, \qquad n = 1, \dots, K, \tag{2}$$

and thus the cost function (1) is given by

$$C_n(\boldsymbol{\lambda}, \mu, c) = \frac{1}{\mu}\left(\frac{\mu + \lambda}{\lambda_n} + c\lambda_n\right).$$

Proposition 1. *When $\mu_n = \mu$ for all $n = 1, \dots, K$, there is a unique solution of (GAME), which is a symmetric Nash Equilibrium given by*

$$\lambda_n^{ne} = \frac{(K-1) + \sqrt{(K-1)^2 + 4\mu c}}{2c}, \qquad n = 1, \dots, K. \tag{3}$$

Proof. The first order condition of the optimization problem for source n can be written as

$$c\lambda_n^2 + \lambda_n = \mu + \lambda. \tag{4}$$

By subtracting this expression for two arbitrary sources n, m it results

$$c(\lambda_n^2 - \lambda_m^2) + (\lambda_n - \lambda_m) = 0,$$

which is only possible if and only if $\lambda_n = \lambda_m$, since $c > 0$ and $\lambda_n > 0$ for all n. Using this, equation (4) can be rewritten as

$$c\lambda_n^2 - (K-1)\lambda_n - \mu = 0,$$

which is a quadratic equation whose unique positive solution is given by (3). □

Proposition 2. *When $\mu_n = \mu$ for all $n = 1, \dots, K$, there is a unique and symmetric solution of the (GLOBAL-OPT), and it is given by*

$$\lambda_n^G = \sqrt{\frac{\mu}{c}}, \qquad n = 1, \dots, K. \tag{5}$$

Proof. The objective function of (GLOBAL-OPT) with homogeneous service rates is

$$\sum_{n=1}^{K} C_n(\boldsymbol{\lambda}, \mu, c) = \sum_{n=1}^{K} \frac{1}{\mu}\left(\frac{\mu+\lambda}{\lambda_n} + c\lambda_n\right). \quad (6)$$

Setting the derivatives of (6) with respect to λ_n equal to zero and arranging terms we obtain the first order conditions

$$c + \sum_{j=1}^{K} \frac{1}{\lambda_j} = \frac{\mu+\lambda}{\lambda_n^2}, \qquad n = 1, \dots, K. \quad (7)$$

Taking the equations (7) of sources n and m and subtracting one from another we obtain

$$\frac{\mu+\lambda}{\lambda_n^2} = \frac{\mu+\lambda}{\lambda_m^2}.$$

Since the generation rates are strictly positive, this equality is possible if and only if $\lambda_n = \lambda_m$, which proves the symmetry. Using this in (7) and simplifying yields $c\lambda_n^2 - \mu = 0$, for which (5) is the only positive solution. □

Remark 1. We note that the second order condition in problems (GAME) and (GLOBAL-OPT) is $\frac{2(\mu+\lambda_{-n})}{\mu\lambda_n^3} > 0$, where $\lambda_{-n} = \lambda - \lambda_n$, ensuring that both solutions are indeed a minimum. We also note that both λ_n^{ne} and λ_n^G tend to infinity as $c \to 0$ and tend to zero as $c \to \infty$, i.e., when the payment is very high the optimal generation rates are very small, whereas when the payment is very low the generation rates are very large.

Remark 2. We can easily show that the cooperation between sources (or the presence of a centralized planner) leads to a lower generation rate, that is, $\lambda_n^{ne} > \lambda_n^G$. To see this, note that since $K \geq 2$, it follows

$$\lambda_n^{ne} = \frac{(K-1) + \sqrt{(K-1)^2 + 4\mu c}}{2c} > \frac{\sqrt{4\mu c}}{2c} = \sqrt{\frac{\mu}{c}} = \lambda_n^G.$$

From the first order conditions of (GAME), the optimal rate at Nash Equilibrium satisfies $c\left(\lambda_n^{ne}\right)^2 = (K-1)\lambda_n^{ne} + \mu$. Using also that $\lambda^{ne} = K\lambda_n^{ne}$, it results for the Nash Equilibrium that

$$\begin{aligned} C_n(\boldsymbol{\lambda}^{ne}, \mu, c) &= \frac{(\mu + \lambda^{ne}) + \mu + (K-1)\lambda_n^{ne}}{\mu\lambda_n^{ne}} \\ &= \frac{2}{\lambda_n^{ne}} + \frac{2K-1}{\mu} \\ &= \frac{4c}{(K-1) + \sqrt{(K-1)^2 + 4\mu c}} + \frac{2K-1}{\mu} \\ &= \frac{K + \sqrt{(K-1)^2 + 4\mu c}}{\mu}, \end{aligned}$$

and therefore

$$\sum_{n=1}^{K} C_n(\boldsymbol{\lambda}^{ne}, \mu, c) = \frac{K(K + \sqrt{(K-1)^2 + 4\mu c})}{\mu}. \quad (8)$$

On the other hand, from the first order condition of the (GLOBAL-OPT) we have $c(\lambda_n^G)^2 = \mu$, and thus

$$\sum_{n=1}^{K} C_n(\boldsymbol{\lambda}^G, \mu, c) = K\frac{(\mu + \lambda^G) + \mu}{\mu\lambda_n^G} = \frac{K(K + \sqrt{4\mu c})}{\mu}. \quad (9)$$

Let $\gamma = \mu c$. With (8) and (9) the expression (POA) for this model yields

$$PoA = \sup_{\gamma} \frac{\sum_{n=1}^{K} C_n(\boldsymbol{\lambda}^{ne}, \gamma)}{\sum_{n=1}^{K} C_n(\boldsymbol{\lambda}^G, \gamma)} = \sup_{\gamma} \frac{K + \sqrt{(K-1)^2 + 4\gamma}}{K + \sqrt{4\gamma}} \quad (10)$$

We note that the ratio above is a decreasing function of γ. To see that, compute its derivative with respect to γ and simplify it to obtain the expression

$$\frac{-2K\left(\sqrt{(K-1)^2 + 4\gamma} - \sqrt{4\gamma}\right) - 2(K-1)^2}{\sqrt{4\gamma}\sqrt{(K-1)^2 + 4\gamma}(K + \sqrt{4\gamma})^2} < 0.$$

Hence, the PoA is achieved when $\gamma \to 0$ and the next result follows.

Proposition 3. *When $\mu_n = \mu$ for all $n = 1, \dots, K$,*

$$PoA = 2 - \frac{1}{K}.$$

4. Heterogeneous service rates

Suppose now a system with arbitrary service rates. We first provide an explicit expression of the AAoI for heterogeneous service rates (which generalizes the result of Theorem 2a) in [10]) in the following result:

Theorem 1. *When the service rates are heterogeneous,*

$$\Delta_n(\boldsymbol{\lambda}, \boldsymbol{\mu}) = \frac{\mu_n + \lambda}{\mu_n\lambda_n}. \quad (11)$$

Proof. See Appendix A. □

Remark 3. Notice that, as a consequence of the preemption in service, the AAoI of source n does not depend on the service rate of the updates from other sources.

Unlike in the homogeneous case, we have not succeeded in deriving a closed-form expression for the optimal generation rates with an arbitrary number of sources neither in the Nash Equilibrium nor in the global optimum. However, we can derive several properties of both strategies that allow us to analyze the PoA of this case. In the following result, we provide an ordering of the generation rates at the Nash Equilibrium.

Proposition 4. *When the service rates are heterogeneous*

$$\mu_n > \mu_m \iff \lambda_n^{ne} > \lambda_m^{ne}.$$

Proof. With formula (11), the first order condition of the non-cooperative optimization problem for source n is

$$c\lambda_n^2 + \lambda_n = \mu_n + \lambda. \quad (12)$$

By subtracting equations (12) for two arbitrary sources we obtain $(\lambda_n - \lambda_m)(1 + c(\lambda_n + \lambda_m)) = \mu_n - \mu_m$, and the result follows. □

Let us note that (GLOBAL-OPT) in the heterogeneous case is

$$\sum_{n=1}^{K} \frac{1}{\mu_n}\left(\frac{\mu_n + \lambda}{\lambda_n} + c\lambda_n\right),$$

and the first order conditions can be written as

$$\frac{1}{\mu_n}\left(\frac{\mu_n+\lambda}{\lambda_n^2}-c\right)=\sum_{m=1}^{K}\frac{1}{\mu_m\lambda_m}, \qquad n=1,\dots,K. \tag{13}$$

Let $c_{sym}=K\left(\sum_{m=1}^{K}1/\mu_m\right)$. This particular cost value allows us to characterize the solution at the global optimum. Further, since our focus is on the PoA, this choice does not reduce the applicability of our results.

Proposition 5. *When the service rates are heterogeneous,*

$$c=c_{sym} \iff \lambda_n^G=\left(\sum_{m=1}^{K}1/\mu_m\right)^{-1}, \qquad \forall n=1,\dots,K. \tag{14}$$

Proof. See Appendix B. □

Remark 4. We note that, when $c=c_{sym}$ and $\mu=K\left(\sum_{m=1}^{K}1/\mu_m\right)^{-1}$, the harmonic mean of the service rates, the optimal rate given in (5) coincides with (14):

$$\lambda_n^G=\sqrt{\frac{\mu}{c_{sym}}}=\sqrt{\frac{K\left(\sum_{m=1}^{K}1/\mu_m\right)^{-1}}{K\left(\sum_{m=1}^{K}1/\mu_m\right)}}=\left(\sum_{m=1}^{K}1/\mu_m\right)^{-1}.$$

In the remaining we will assume without loss of generality that sources are labeled such that $\mu_1>\dots>\mu_K>0$, and will denote $\tilde{\mu}=\sum_{n=1}^{K}1/\mu_n$.

Proposition 6. *When the service rates are heterogeneous,*

$$\frac{\sum_{n=1}^{K}C_n(\boldsymbol{\lambda}^{ne},\boldsymbol{\mu},c_{sym})}{\sum_{n=1}^{K}C_n(\boldsymbol{\lambda}^{G},\boldsymbol{\mu},c_{sym})}>\frac{2\lambda_K^{ne}}{3\mu_K}.$$

Proof. From conditions (13) we have that

$$c\lambda_n^G+\sum_{j=1}^{K}\frac{\mu_n\lambda_n^G}{\mu_j\lambda_j^G}=\frac{\mu_n+\lambda^G}{\lambda_n^G},$$

and the objective function of (GLOBAL-OPT) can be written as

$$\begin{aligned}\sum_{n=1}^{K}C_n(\boldsymbol{\lambda}^G,\boldsymbol{\mu},c)&=\sum_{n=1}^{K}\frac{1}{\mu_n}\left(\frac{\mu_n+\lambda^G}{\lambda_n^G}+c\lambda_n^G\right)\\&=\sum_{n=1}^{K}\frac{1}{\mu_n}\left(c\lambda_n^G+\sum_{j=1}^{K}\frac{\mu_n\lambda_n^G}{\mu_j\lambda_j^G}+c\lambda_n^G\right)\\&=\sum_{n=1}^{K}\frac{1}{\mu_n}\left(2c\lambda_n^G+\frac{\lambda^G}{\lambda_n^G}\right).\end{aligned} \tag{15}$$

Similarly, from conditions (12)

$$c\lambda_n^{ne}+1=\frac{\mu_n+\lambda^{ne}}{\lambda_n^{ne}},$$

and therefore

$$\begin{aligned}\sum_{n=1}^{K}C_n(\boldsymbol{\lambda}^{ne},\boldsymbol{\mu},c)&=\sum_{n=1}^{K}\frac{1}{\mu_n}\left(\frac{\mu_n+\lambda^{ne}}{\lambda_n^{ne}}+c\lambda_n^{ne}\right)\\&=\sum_{n=1}^{K}\frac{1}{\mu_n}\left(2c\lambda_n^{ne}+1\right).\end{aligned}$$

Setting $c=c_{sym}=K\tilde{\mu}$, from Proposition 5 we know that $\lambda_n^G=\tilde{\mu}^{-1}$ $\forall n$, and thus (15) reduces to

$$\sum_{n=1}^{K}\left(2c_{sym}\lambda_n^G+\lambda^G/\lambda_n^G\right)/\mu_n=\sum_{n=1}^{K}(2K+K)/\mu_n=3K\tilde{\mu}.$$

Moreover, from Proposition 4 and the ordering $\mu_1>\dots>\mu_K$ it follows that $\lambda_n^{ne}>\lambda_K^{ne}$, $\forall n\le K-1$, and hence

$$\begin{aligned}\frac{\sum_{n=1}^{K}C_n(\boldsymbol{\lambda}^{ne},\boldsymbol{\mu},c_{sym})}{\sum_{n=1}^{K}C_n(\boldsymbol{\lambda}^{G},\boldsymbol{\mu},c_{sym})}&=\frac{\sum_{n=1}^{K}\left(2c_{sym}\lambda_n^{ne}+1\right)/\mu_n}{3K\tilde{\mu}}\\&>\frac{\left(2c_{sym}\lambda_K^{ne}+1\right)\sum_{n=1}^{K}1/\mu_n}{3K\tilde{\mu}}\\&=\frac{2K\tilde{\mu}\lambda_K^{ne}+1}{3K}\\&>\frac{2\lambda_K^{ne}}{3\mu_K}.\quad\square\end{aligned}$$

In the next result we state that the ratio λ_K^{ne}/μ_K is unbounded from above when $c=c_{sym}$.

Proposition 7. *When the service rates are heterogeneous, for any given $\theta\in\mathbb{R}^+$ there exists a set of service rates $\{\mu_n\}_{n=1}^{K}$ such that if $c=c_{sym}$ then $\lambda_K^{ne}/\mu_K>\theta$.*

Proof. Consider the set of service rates

$$\mu_n=\left(M-\frac{2n}{K-1}\right)^2,\ n=1,2,\dots,K-1, \qquad \mu_K=1$$

and the cost $c=c_{sym}=K\tilde{\mu}$. For this service rates

$$\begin{aligned}\tilde{\mu}&=\frac{1}{\mu_1}+\frac{1}{\mu_2}+\dots+\frac{1}{\mu_{K-1}}+\frac{1}{\mu_K}\\&=1+K\sum_{n=1}^{K-1}\frac{1}{\left(M-\frac{2n}{K-1}\right)^2}.\end{aligned}$$

Notice that whenever $M>3$ this set of service rates preserves the ordering $\mu_1>\dots>\mu_K$ and also that $1<\tilde{\mu}<K$, and thus $K<c_{sym}<K^2$. Recall that $\lambda_{-n}=\lambda-\lambda_n$. From (12) it follows that

$$\begin{aligned}&c_{sym}(\lambda_n^{ne})^2=\mu_n+\lambda_{-n}^{ne}\iff\\&\lambda_n^{ne}=\sqrt{\frac{\mu_n+\lambda_{-n}^{ne}}{c_{sym}}}>\sqrt{\frac{\mu_n}{c_{sym}}}.\end{aligned}$$

Since $\mu_n=\left(M-\frac{2n}{K-1}\right)^2$ for $n\le K-1$ we have that

$$\lambda_n^{ne}>\frac{1}{c_{sym}^{1/2}}\left(M-\frac{2n}{K-1}\right), \qquad n=1,\dots,K-1,$$

and thus

$$\lambda_{-K}^{ne}=\sum_{n=1}^{K-1}\lambda_n^{ne}>\frac{1}{c_{sym}^{1/2}}\sum_{n=1}^{K-1}\left(M-\frac{2n}{K-1}\right)=\frac{(K-1)M-K}{c_{sym}^{1/2}}.$$

We complete the proof showing that, for any given $\theta\in\mathbb{R}^+$, by choosing M such that $M>K^3\theta^2/(K-1)$ we ensure the desired result. Indeed, from (12) with $n=K$, and keeping in mind that $\mu_K=1$ and $c_{sym}<K^2$, we obtain

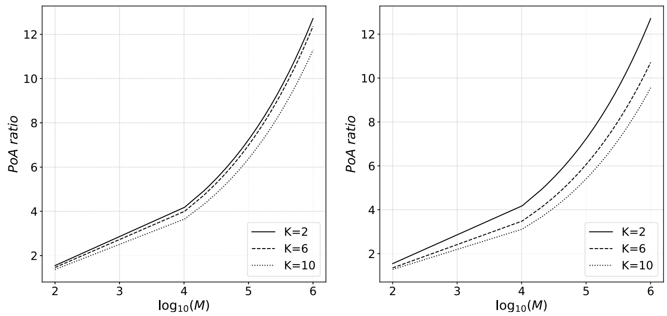

Fig. 2. Ratio $\frac{\sum_{n=1}^{K} C_n(\boldsymbol{\lambda}^{ne},\boldsymbol{\mu},c_{sym})}{\sum_{n=1}^{K} C_n(\boldsymbol{\lambda}^{G},\boldsymbol{\mu},c_{sym})}$ for Configuration 1 *(left)* and Configuration 2 *(right)*.

$$\left(\lambda_K^{ne}/\mu_K\right)^2 = \frac{\mu_K + \lambda_{-K}^{ne}}{c_{sym}} > \frac{1}{c_{sym}} + \frac{(K-1)M-K}{c_{sym}^{3/2}}$$
$$> \frac{1}{K^2} + \frac{K^3\theta^2 - K}{K^3} = \theta^2. \quad \square$$

From Propositions 6 and 7 it follows the main result of this section.

Proposition 8. *In the heterogeneous service rates model the PoA is unbounded from above.*

5. Numerical experiments

This section is intended to illustrate numerically the results stated in Propositions 7 and 8. We define two different set of parameters for which $\mu_1 > \cdots > \mu_K$: in Configuration 1 we take $\mu_K = 1$ and

$$\mu_n = M - n + 1, \ n = 1, \ldots, K-1,$$

and in Configuration 2 we let the service rates be equally spaced along the interval $[1, M]$, that is,

$$\mu_n = M - \frac{(M-1)(n-1)}{K-1}, \ n = 1, \ldots, K.$$

Note that in both configurations $M - 1 = \mu_1 - \mu_K$. The cost is $c = c_{sym}$.

For both configurations, using a fixed-point algorithm we compute numerically the optimal generation rates and then evaluate the objective functions. As stated in Proposition 7, we observe that for any value of θ there is $M^*(\theta)$ such that for all $M \geq M^*(\theta)$, $\lambda_K^{ne}/\mu_K > \theta$, and furthermore, that $M^*(\theta)$ is non-decreasing in θ. This seems to indicate that as the range of the service rates grows, i.e. M, the ratio $\frac{\sum_{n=1}^{K} C_n(\boldsymbol{\lambda}^{ne},\boldsymbol{\mu},c_{sym})}{\sum_{n=1}^{K} C_n(\boldsymbol{\lambda}^{G},\boldsymbol{\mu},c_{sym})}$ increases.

We illustrate this in Fig. 2 where we depict the ratio $\frac{\sum_{n=1}^{K} C_n(\boldsymbol{\lambda}^{ne},\boldsymbol{\mu},c_{sym})}{\sum_{n=1}^{K} C_n(\boldsymbol{\lambda}^{G},\boldsymbol{\mu},c_{sym})}$ for different values of K when M varies from 100 to 10^6. Fig. 2 *(left)* corresponds to Configuration 1, and Fig. 2 *(right)* to Configuration 2. In both cases we observe that, regardless of K, the ratio between the total costs increases with M.

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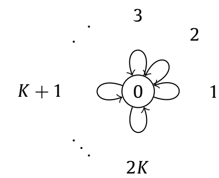

Fig. 3. The Markov chain under consideration.

Appendix A. Proof of Theorem 1

We use the Stochastic Hybrid System (SHS) technique to obtain an analytical expression of the AAoI of the M/M/1 queue with preemption in service and heterogeneous service rates. We first briefly describe this methodology, which has been introduced in [10], and we then show how to apply it in the proof of Theorem 1.

A.1. Stochastic hybrid system

A Stochastic Hybrid System is a continuous-time Markov Chain with a mixture of discrete and continuous states. The set of discrete states $\mathcal{Q}$ is finite and describes the state of the system, whereas the continuous state $\mathbf{x}(t)$ tracks the time evolution of a process of interest, the age process in our case. This process flows according to $\mathbf{x}'(t) = \mathbf{b}_k(t)$ when the system is in state k. We denote respectively by $\mathcal{L}_k$ and $\mathcal{L}'_k$ the sets of outgoing and ingoing links of state $k \in \mathcal{Q}$.

Let l be a transition from state k to state m and $r^{(l)}$ the rate at which this transition occurs. Right after the transition a transformation matrix $\mathbf{A}_l$ is applied to the continuous state $\mathbf{x}$ to indicate how the age process changes due to this transition. Remarkably, for any pair of states $k, m \in \mathcal{Q}$ (not necessarily different, so that self-transitions are included) there may exist several transitions that occur at different rates or that have a different effect on the continuous state.

Finally, we assume that the Markov Chain is ergodic and denote by π_k the stationary probability of discrete state k.

For this proof we make use of a convenient trick: each time an update is delivered to the monitor we will suppose that an exact copy of it enters service. This fake update is simply waiting to be preempted, so that it will have no effect on the age process.

Using this device, we consider a set of discrete states $\mathcal{Q} = \{0\}$, meaning that the system is always busy, either with a real or a fake update. Of course, the stationary probability is $\pi_0 = 1$. Fig. 3 depicts the Markov Chain of this model, with one single discrete state and $2K$ self-transitions, with K being the number of sources. The continuous state is $\mathbf{x}(t) = [x_0(t) \ x_1(t)]$, where the first component tracks the age of the source of interest and the second component is what the age would become if the package in service is delivered. Since both components grow linearly with time $\mathbf{b}_0 = [1 \ 1]$. For completeness we recall this result from [10] that will be needed in our proof.

Theorem 2 *([10, Thm. 4]). Let q_l be the state from which the system jumps to state q after transition $l \in \mathcal{L}'_q$. For all discrete state q, if $\mathbf{v}_q = [v_{q,0}, v_{q,1}, \ldots, v_{q,n}]$ is a non-negative solution of the system*

$$\mathbf{v}_q \sum_{l \in \mathcal{L}_q} r^{(l)} = \mathbf{b}_q \pi_q + \sum_{l' \in \mathcal{L}'_q} r^{(l')} \mathbf{v}_{q_{l'}} \mathbf{A}_{l'}, \tag{A.1}$$

then the Average Age of Information is given by $\sum_q v_{q,0}$.

A.2. Computation of the AAoI

Without loss of generality, we consider source 1 as our source of interest and compute its AAoI using the SHS method. In Ta-

Table A.1
The transitions of the Stochastic Hybrid System model.

l	$k \to m$	$r^{(l)}$	A_l	$A_l \boldsymbol{x}$	$A_l \boldsymbol{v}_k$
1	$0 \to 0$	λ_1	$\begin{bmatrix}1 & 0\\0 & 0\end{bmatrix}$	$[x_0\ 0]^T$	$[v_{00}\ 0]^T$
2	$0 \to 0$	μ_1	$\begin{bmatrix}0 & 1\\0 & 1\end{bmatrix}$	$[x_1\ x_1]^T$	$[v_{01}\ v_{01}]^T$
3	$0 \to 0$	λ_2	$\begin{bmatrix}1 & 0\\1 & 0\end{bmatrix}$	$[x_0\ x_0]^T$	$[v_{00}\ v_{00}]^T$
4	$0 \to 0$	λ_3	$\begin{bmatrix}1 & 0\\1 & 0\end{bmatrix}$	$[x_0\ x_0]^T$	$[v_{00}\ v_{00}]^T$
⋮	⋮			⋮	⋮
K+1	$0 \to 0$	λ_K	$\begin{bmatrix}1 & 0\\1 & 0\end{bmatrix}$	$[x_0\ x_0]^T$	$[v_{00}\ v_{00}]^T$
K+2	$0 \to 0$	μ_2	$\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}$	$[x_0\ x_1]^T$	$[v_{00}\ v_{01}]^T$
⋮	⋮			⋮	⋮
2K	$0 \to 0$	μ_K	$\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}$	$[x_0\ x_1]^T$	$[v_{00}\ v_{01}]^T$

ble A.1 we present the transitions of this SHS model (the values in the last column will be used to apply Theorem 2).

We now explain each of the transitions:

- $l = 1$. An incoming update of source 1 arrives and starts service. The value of $[x_0\ x_1]$ changes to $[x_0\ 0]$, i.e., x_0 is not modified since the age of the monitor does not change when an update of source 1 arrives but x_1 is reset to 0 since a fresh packet of the source of interest arrived.
- $l = 2$. An update of source 1 ends service and is delivered to the monitor. In this case, $[x_0\ x_1]$ changes to $[x_1\ x_1]$, which means that the age at the monitor is updated to the generation time of the packet that ended service. When this occurs, we put a fake update in service with the same generation time as the update delivered at the monitor that will be preempted by the next arrival.
- $l = 3$. An incoming update of source 2 arrives and starts being served. The value of $[x_0\ x_1]$ changes to $[x_0\ x_0]$, i.e., the value of x_0 is not modified since the age of the monitor does not change when an update of source 2 arrives and the value of x_1 is set to the age of the monitor to indicate that the delivery of this update will not affect the current age of source 1.

In regard to the age process transitions $l = 4, \dots, K + 1$ are identical to transition $l = 3$.

- $l = K + 2$. An update of source 2 ends service and is delivered to the monitor. In this case the value of $[x_0\ x_1]$ does not change, and a fake update starts service with the same generation time as that of the update delivered at the monitor.

Transitions $l = K + 3, \dots, 2K$ are again identical to transition $l = K + 2$.

Applying (A.1) to our case and we get

$$\begin{bmatrix} v_{00} \\ v_{01} \end{bmatrix}\left(\lambda + \mu_1 + \sum_{i=2}^{K} \mu_i\right) =$$
$$1\begin{bmatrix} 1 \\ 1 \end{bmatrix} + \lambda_1 \begin{bmatrix} v_{00} \\ 0 \end{bmatrix} + \sum_{i=2}^{K} \lambda_i \begin{bmatrix} v_{00} \\ v_{00} \end{bmatrix} + \mu_1 \begin{bmatrix} v_{01} \\ v_{01} \end{bmatrix} + \sum_{i=2}^{K} \mu_i \begin{bmatrix} v_{00} \\ v_{01} \end{bmatrix},$$

or equivalently

$$\left(\lambda + \mu_1 + \sum_{i=2}^{K} \mu_i\right) v_{00} = 1 + \left(\lambda + \sum_{i=2}^{K} \mu_i\right) v_{00} + \mu_1 v_{01}$$

$$\left(\lambda + \mu_1 + \sum_{i=2}^{K} \mu_i\right) v_{01} = 1 + \sum_{i=2}^{K} \lambda_i v_{00} + \left(\mu_1 + \sum_{i=2}^{K} \mu_i\right) v_{01}$$

Simplifying, we get that

$$\mu_1 v_{00} = 1 + \mu_1 v_{01}$$
$$\lambda v_{01} = 1 + (\lambda - \lambda_1) v_{00}.$$

From the first equation $v_{00} - v_{01} = 1/\mu_1$, and substitution into the second equation yields

$$(\lambda - \lambda_1) v_{00} - \lambda v_{01} = -1$$
$$\iff (\lambda - \lambda_1)(v_{00} - v_{01}) - \lambda_1 v_{01} = -1$$
$$\iff (\lambda - \lambda_1)/\mu_1 - \lambda_1 v_{01} = -1$$
$$\iff v_{01} = \frac{1}{\lambda_1} + \frac{\lambda - \lambda_1}{\mu_1 \lambda_1},$$

which is positive and, as a result, so is $v_{00} = v_{01} + 1/\mu_1$. Therefore, a non negative solution of (A.1) exists and from Theorem 2 we conclude that the AAoI of source 1 is equal to:

$$\sum_q v_{q,0} = v_{00} = \frac{1}{\mu_1} + v_{01} = \frac{1}{\mu_1} + \frac{1}{\lambda_1} + \frac{\lambda - \lambda_1}{\mu_1 \lambda_1} = \frac{\mu_1 + \lambda}{\mu_1 \lambda_1}.$$

Appendix B. Proof of Proposition 5

Proof. Note that if two sources n, m have the same service rate we can argue as in the homogeneous case to conclude that $\lambda_n^G = \lambda_m^G$ regardless of the cost c, so we will assume without loss of generality that all the service rates are different. Now, let $\tilde{\mu} = \sum_{n=1}^{K} 1/\mu_n$ and suppose that the solution of the problem is symmetric. In this case, we denote by $\lambda_n^G = \lambda_{sym}\ \forall n$. Subtracting equations (13) of sources n, m, we obtain

$$\mu_m \left(\frac{\mu_n + K\lambda_{sym}}{\lambda_{sym}^2} - c\right) - \mu_n \left(\frac{\mu_m + K\lambda_{sym}}{\lambda_{sym}^2} - c\right) = 0.$$

After simplifying we obtain that

$$\left(K/\lambda_{sym} - c\right)(\mu_m - \mu_n) = 0,$$

which implies that $c = K/\lambda_{sym}$. Inserting this expression back into equation for source n yields

$$\frac{1}{\mu_n}\left(\frac{-(\mu_n + K\lambda_{sym})}{\lambda_{sym}^2} + \frac{K}{\lambda_{sym}}\right) + \frac{1}{\lambda_{sym}} \sum_{j=1}^{K} \frac{1}{\mu_j} = 0$$

After simplifying we obtain $-\frac{1}{\lambda_{sym}^2} + \frac{\tilde{\mu}}{\lambda_{sym}} = 0$, which gives $\lambda_{sym} = 1/\tilde{\mu}$ and $c = K\tilde{\mu}$.

We now suppose that $c = K\tilde{\mu}$. Let m be the index of the source with the lowest generation rate at the global optimum, that is, $\lambda_m^G \le \lambda_n^G\ \forall n \ne m$. We know from Lemma 1 (see below) that

$$c \sum_{n=1}^{K} \frac{\lambda_n^G}{\mu_n} = \sum_{n=1}^{K} \frac{1}{\lambda_n^G}.$$

When we focus on the term $1/\lambda_m^G$ in the right hand side of the above equality, we observe that one and only one of the following can be true:

i) $\frac{1}{\lambda_m^G} = \frac{c}{K} \sum_{n=1}^{K} \frac{\lambda_n^G}{\mu_n}$, which in turn implies that $\lambda_n^G = \lambda_m^G\ \forall n \ne m$, or *ii)* $\frac{1}{\lambda_m^G} > \frac{c}{K}\sum_{n=1}^{K}\frac{\lambda_n^G}{\mu_n}$, which means that $\exists j \neq m$ such that $\frac{1}{\lambda_j^G} < \frac{c}{K}\sum_{n=1}^{K}\frac{\lambda_n^G}{\mu_n}$.

We complete the proof showing that the second option is not possible when $c = K\tilde{\mu}$. Notice that from *ii)* we can set a simple upper bound for λ_m^G:

$$1 > \lambda_m^G \frac{c}{K}\left(\sum_{n=1}^{K}\lambda_n^G/\mu_n\right) \geq \lambda_m^G\tilde{\mu}\left(\sum_{n=1}^{K}\lambda_m^G/\mu_n\right) = (\lambda_m^G\tilde{\mu})^2.$$

This implies that $\lambda_m^G < 1/\tilde{\mu}$, and this inequality means that $\mu_m/\lambda_m^G > \mu_m\tilde{\mu}$, whereas both $\lambda^G/\lambda_m^G > K$ and $\lambda_m^G/\lambda_n^G \leq 1\ \forall n \neq m$ follow from the fact that m is the source with the minimum arrival rate. Further, if we arrange the first-order equation (13) of the m-th source and write it as

$$\mu_m/\lambda_m^G + \lambda^G/\lambda_m^G = c\lambda_m^G + \mu_m\sum_{n=1}^{K}(1/\mu_n)(\lambda_m^G/\lambda_n^G),$$

we can combine these inequalities as follows

$$\begin{aligned}\mu_m\tilde{\mu} + K &< \mu_m/\lambda_m^G + \lambda^G/\lambda_m^G \\ &= c\lambda_m^G + \mu_m\sum_{n=1}^{K}(1/\mu_n)(\lambda_m^G/\lambda_n^G) \\ &\leq c\lambda_m^G + \mu_m\sum_{n=1}^{K}(1/\mu_n) \\ &= c\lambda_m^G + \mu_m\tilde{\mu},\end{aligned}$$

to finally obtain that $\lambda_m^G > 1/\tilde{\mu}$, which is a contradiction with the previous upper bound for λ_m^G. □

Lemma 1. *When the service rates are heterogeneous,*

$$c\sum_{n=1}^{K}\frac{\lambda_n^G}{\mu_n} = \sum_{n=1}^{K}\frac{1}{\lambda_n^G}. \tag{B.1}$$

Proof. We can rearrange the equations in the first order conditions (13) as

$$c\frac{\lambda_n}{\mu_n} = \frac{1}{\lambda_n} + \frac{\lambda}{\mu_n\lambda_n} - \lambda_n\sum_{j=1}^{K}\frac{1}{\mu_j\lambda_j}, \qquad n = 1, 2, \ldots, K.$$

The result follows from the summation of all equations. □

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