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A convergent and asymptotic Laplace method for integrals

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ABSTRACT

Watson's lemma and Laplace's method provide asymptotic expansions of Laplace integrals $F(z) := \int_0^\infty e^{-zf(t)}g(t)dt$ for large values of the parameter z. They are useful tools in the asymptotic approximation of special functions that have a Laplace integral representation. But in most of the important examples of special functions, the asymptotic expansion derived by means of Watson's lemma or Laplace's method is not convergent. A modification of Watson's lemma was introduced in [Nielsen, 1906] where, by the use of inverse factorial series, a new asymptotic as well as convergent expansion of $F(z)$, for the particular case $f(t) = t$, was derived. In this paper we go some steps further and investigate a modification of the Laplace's method for $F(z)$, with a general phase function $f(t)$, to derive asymptotic expansions of $F(z)$ that are also convergent, accompanied by error bounds. An analysis of the remainder of this new expansion shows that it is convergent under a mild condition for the functions $f(t)$ and $g(t)$, namely, these functions must be analytic in certain starlike complex regions that contain the positive axis $[0, \infty)$. In many practical situations (in many examples of special functions), the singularities of $f(t)$ and $g(t)$ are off this region and then this method provides asymptotic expansions that are also convergent. We illustrate this modification of the Laplace's method with the parabolic cylinder function $U(a, z)$, providing an asymptotic expansions of this function for large z that is also convergent.

1. Introduction

Consider Laplace integrals of the form

$$F(z) := \int_0^\infty e^{-zf(t)}g(t)dt, \qquad \Re z > x_0 > 0, \tag{1}$$

for certain $x_0 \in \mathbb{R}$, with $f(t)$ and $g(t)$ smooth enough functions, $f(t)$ real. The key point in Laplace's method is that, for large positive $|z|$ (with fixed $\arg z \in (-\pi/2, \pi/2)$), the major contribution of the integrand to the integral occurs around the absolute minimum $t = t_0$ of the phase function $f(t)$ in the integration interval $[0, \infty)$, that we assume to be unique. Then, only the local behavior of $f(t)$ and $g(t)$ at $t = t_0$ is relevant to derive an asymptotic expansion of $F(z)$ for large $|z|$ [1], [2, Chap. 2]. In particular, when $f(t) = t$ (in this case Laplace's method is indeed Watson's lemma), that minimum is obviously $t_0 = 0$. When $g(t)$ is analytic at $t = 0$, it has an asymptotic expansion at $t = 0$ of the form

$$g(t) = \sum_{k=0}^{n-1} a_k t^k + g_n(t), \qquad a_k := \frac{g^{(k)}(0)}{k!}, \tag{2}$$

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with $g_n(t) = \mathcal{O}(t^n)$ when $t \to 0^+$. This expansion converges in a certain open disk $D_r(0)$ of radius $r > 0$ centered at $t = 0$. When we replace this expansion in the above integral (with $f(t) = t$) and interchange sum and integral we obtain:

$$\int_0^\infty e^{-zt} g(t)dt \sim \sum_{k=0}^{n-1} a_k \frac{k!}{z^{k+1}} + R_n(z), \tag{3}$$

with

$$R_n(z) := \int_0^\infty e^{-zt} g_n(t)dt = \mathcal{O}(z^{-n-1}). \tag{4}$$

This last order estimate means that expansion (3) is an asymptotic expansion for large $|z|$. This is the well-known Watson's lemma [3, Chap. 2], [2, Chap. 1]. The key point is that, for large positive $|z|$, the dominant contribution to the integral (1) comes from the left end point $t = 0$ of the integration interval $[0, \infty)$. Then, only the value of $g(t)$ around that asymptotically relevant point $t = 0$, that is the sum of the first few terms of the approximation (2), is relevant for the asymptotic behavior of $F(z)$ when $|z|$ is large.

Typically, in many examples of special functions $F(z)$ (the right hand side of (1) is an integral representation of $F(z)$), the function $g(t)$ is not an entire function and then the convergence radius r of the Taylor series (2) is finite. As the integration interval $[0, \infty)$ is not contained in $D_r(0)$, the interchange of series and integral gives an expansion (3) of the function $F(z)$ that is not convergent.[1] Take for example $F(z) = e^z E_1(z)$, where $E_1(z)$ is the exponential integral [4, Sec. 6.2, eq. (6.2.2)],

$$F(z) = \int_0^\infty \frac{e^{-zt}}{t+1} dt, \qquad g(t) = \frac{1}{t+1}. \tag{5}$$

Because the singularity of $g(t)$ at $t = -1$, we have that the expansion

$$g(t) = \frac{1}{t+1} = \sum_{k=0}^\infty (-1)^k t^k,$$

is convergent in any disk $D_r(0)$ with $r < 1$. Therefore, in this particular example, formula (3) becomes

$$F(z) \sim \sum_{k=0}^\infty \frac{(-1)^k k!}{z^{k+1}}.$$

The series in the right hand side above is not convergent for any value of $z \in \mathbb{C}$.

For general phase functions $f(t)$, that is, when we consider the more general Laplace's method, the situation is similar: in most of the important examples of special functions, the functions $f(t)$ and $g(t)$ are analytic at the asymptotically relevant point t_0 (the absolute minimum of the phase function). But the convergence radius of the Taylor series of these functions at $t = t_0$ is finite and then, the convergence disk of these Taylor expansions does not contain the whole integration interval. As a consequence, the asymptotic expansion of the function $F(z)$ is not convergent; see further details in [1].

For the particular case $f(t) = t$ (Watson's lemma), it is suggested in [3, Sec. 17.3] a logarithmic change of the integration variable that transforms the unbounded integration interval $[0, \infty)$ into a bounded interval $[0, 1)$. Then, an appropriate expansion of the integrand, followed by an interchange of series and integral, results into a factorial convergent series [5]. In this paper we combine this idea with the modified Laplace's method introduced in [1] to design a new Laplace's method (for a more general phase function $f(t)$) that is also convergent.

The paper is organized as follows: In the next section we design a convergent and asymptotic method for Mellin transforms over a compact interval. This analysis is used in Section 3, where we consider the integral (1) for the special case $f(t) = t^m$, $m \in \mathbb{N}$, and we design an asymptotic and convergent generalized Watson's lemma. The main idea is a logarithmic change of the integration variable that transforms the unbounded integration interval $[0, \infty)$ considered in the Generalized Watson's lemma to the bounded integration interval $[0, 1)$ considered in Section 2. Then, in Section 4 we use the results of Section 3 combined with the modified Laplace's method introduced in [1] to design an asymptotic method for a general phase function $f(t)$ in (1) that is also convergent. The key point is to invoke the modified Laplace's method introduced in [1] that, after splitting the Laplace integral (1) at the critical point $t = t_0$, becomes the sum of two integrals of the form considered in the generalized Watson's lemma studied in Section 3. As an illustration, a convergent and asymptotic expansion of the parabolic cylinder function $U(a, z)$ for large $|z|$ is derived. Some indications for the computation of the coefficients of the convergent and asymptotic Laplace expansion derived in Section 4 are given in a separate Appendix.

In the remaining of the paper we consider that $|z| \to \infty$ along fixed rays in the half complex plane $\Re z > x_0$. We consider the principal value $(-\pi, \pi]$ for the argument of any complex variable. The symbol $\lfloor a \rfloor$ denotes the integer part of the real number a, that is, the greatest integer less than or equal to a.

[1] The convergence of (3) in not assured either when $g(t)$ is an entire function.

2. A convergent and asymptotic method for compact Mellin transforms of analytic functions

The first step in our analysis is the derivation of an asymptotic and convergent expansion of the compact Mellin integral

$$F(z) := \int_0^R (1-x^m)^{z-1} x^{s-1} f(x) dx, \quad 0 < R \le 1, \quad m \in \mathbb{N}, \quad \Re z \ge x_0 > 0, \quad \Re s > 0. \tag{6}$$

For the particular case $f(x) = 1$ and $m = 1$, the integral (6) becomes the incomplete beta function [6, Sec. 8.17],

$$B_R(s, z) := \int_0^R (1-x)^{z-1} x^{s-1} dx, \quad 0 < R \le 1, \quad \Re z, \Re s > 0.$$

As a matter of fact, this function is the basic approximant in the expansions that be derive below. If in addition we set $R = 1$, then we obtain the well-known Euler's beta function [7, Sec. 5.12].

We assume the following hypothesis on the function $f(x)$:

Hypothesis 1. Let $r > 0$ denote the radius of convergence of the Taylor series of $f(x)$ centered at the origin. Assume either that $r > R$ or that $r = R$ and $f(x) = \mathcal{O}\left((R-x)^{\sigma-1}\right)$ as $x \to R$, for some $\sigma \in (0, 1]$. If $R = 1$ it is also required that $x_0 + \sigma > 1$.

We have the following theorem.

Theorem 1. *Consider the integral (6) with the above mentioned hypothesis. Then, for $n = 1, 2, 3, \ldots$,*

$$F(z) = \frac{1}{m} \sum_{k=0}^{n-1} \frac{f^{(k)}(0)}{k!} B_{R^m}\left(\frac{k+s}{m}, z\right) + R_n(z), \tag{7}$$

The remainder $R_n(z)$ can be bounded in the form

$$|R_n(z)| \le \begin{cases} \dfrac{M}{R_0^n} B_{R^m}\left(\dfrac{n+\Re s}{m}, \Re z\right) & \text{if } r > R, \\ M(z) B(n+\Re s, \sigma) & \text{if } r = R < 1, \\ M(z) B(n+\Re s, \Re z + \sigma - 1) & \text{if } r = R = 1, \end{cases} \tag{8}$$

where M is a certain positive constant independent of n and z, and $M(z) > 0$ is independent of n. On the other hand, in the first line of (8) we have that $R_0 := r - \epsilon > R$, with $\epsilon > 0$ as small as we wish. Expansion (7) is convergent, with an exponential rate of convergence for $r > R$ and a power rate for $r = R$. More precisely, as $n \to \infty$,

$$R_n(z) = \begin{cases} \mathcal{O}\left(n^{-1}(R/R_0)^n\right) & \text{if } r > R, \ R < 1, \\ \mathcal{O}\left(n^{-\Re z} R_0^{-n}\right) & \text{if } r > R = 1, \\ \mathcal{O}\left(n^{-\sigma}\right) & \text{if } r = R < 1, \\ \mathcal{O}\left(n^{-(\Re z+\sigma-1)}\right) & \text{if } r = R = 1. \end{cases} \tag{9}$$

Expansion (7) is also an asymptotic expansion of $F(z)$ for large $|z|$: we have that, as $|z| \to \infty$, the terms of the expansion and the remainder are of the order

$$B_{R^m}\left(\frac{n+s}{m}, z\right) = \mathcal{O}\left(z^{-\frac{n+s}{m}}\right), \qquad R_n(z) = \mathcal{O}\left(z^{-\frac{n+s}{m}}\right), \qquad n = 1, 2, 3, \ldots. \tag{10}$$

Proof. Consider the Taylor series expansion of $f(x)$ at $x = 0$,

$$f(x) = \sum_{k=0}^{n-1} \frac{f^{(k)}(0)}{k!} x^k + r_n(x), \qquad |x| < r.$$

The key point of the proof is an accurate analysis of the remainder $r_n(x)$, similar to the one carried out in [8] in a different context. We consider first the case $r > R$, then the interval of integration $[0, R]$ is completely contained in the disk $D_0(r)$. Replacing the above expansion of $f(x)$ into (6), and interchanging summation and integration, we find the right hand side of (7) after a straightforward integration process, with

$$R_n(z) := \int_0^R (1-x^m)^{z-1} x^{s-1} r_n(x) dx. \tag{11}$$

The Taylor remainder $r_n(x)$ can be written in the form

$$r_n(x) = \frac{x^n}{2\pi i}\oint_C \frac{f(w)dw}{w^n(w-x)}, \qquad x \in [0,R), \tag{12}$$

where C is a simple closed circle around the origin of radius $R_0 := r - \epsilon > R$ encircling the points $w = 0$ and $w = x$ in the positive direction. The function $|f(w)|$ is bounded on C by a certain positive constant independent on x and n, and then

$$|r_n(x)| \le \frac{x^n}{2\pi R_0^n}\oint_C \left|\frac{f(w)dw}{w-x}\right| \le \bar{M}\frac{x^n}{R_0^n}, \tag{13}$$

where $\bar{M} > 0$ is independent of x and n.

Consider now the case $r = R$; the function $f(x)$ may have an integrable singularity at $x = R$. We note that integral (12) is a constant function of ϵ, it is defined for $\epsilon = 0$ (when we set $R_0 = R$), and it is continuous as a function of ϵ, since it is the integral of an integrable function. Hence, formula (12) is also valid when $r = R$ if we take the limit $\epsilon \to 0$ in (12) and consider that C is a circle of radius $r = R$. Moreover, we perform the following translation of the integration variable: $w \mapsto v + R$, so that

$$r_n(x) = \frac{x^n}{2\pi i}\oint_{\bar{C}} \frac{f(v+R)dv}{(v+R)^n(v+R-x)}, \qquad x \in [0,R),$$

where the new integration path $\bar{C}$ is a circle of center $v = -R$ and radius R: $\bar{C} = \{v \in \mathbb{C} : |v+R| = R\}$. We make use of the fact that $(R-x)^{1-\sigma}f(x)$ is bounded as $x \to R$, that is, $v^{1-\sigma}f(v+R)$ is bounded as $v \to 0$. We find

$$|r_n(x)| \le \frac{x^n}{2\pi}\oint_{\bar{C}} \frac{\left|v^{1-\sigma}f(v+R)\right|}{|v+R|^n}\frac{|v^{\sigma-1}|}{|v+R-x|}dv \le \tilde{M}\frac{x^n}{R^n}\oint_{\bar{C}}\frac{|v^{\sigma-1}|}{|v+R-x|}dv, \quad x \in [0,R), \tag{14}$$

with $\tilde{M} > 0$ independent of n and x. After the further change of variable $v \mapsto u$ defined in the form $v = (R-x)u$, $x \neq R$, we find

$$|r_n(x)| \le \tilde{M}\frac{x^n(R-x)^{\sigma-1}}{R^n}\oint_{\bar{C}/(R-x)}\frac{|u|^{\sigma-1}}{|u+1|}du, \tag{15}$$

where the integration contour $\bar{C}/(R-x)$ is a scaled circle of center $u = -R/(R-x)$ and radius $R/(R-x)$: $\bar{C}/(R-x) = \{|u + \frac{R}{R-x}| = \frac{R}{R-x}\}$ traversed in the positive direction. In the limit $x \to R$ this scaled circle becomes the imaginary axis traversed upwards and the integral along this path in formula (15) is finite. Then the right hand side of (15) can be bounded in the form

$$|r_n(x)| \le \bar{M}\frac{x^n(R-x)^{\sigma-1}}{R^n}, \qquad x \in [0,R), \tag{16}$$

with $\bar{M} > 0$ independent of n and x.

From bounds (13) (for $r > R$) and (16) (for $r = R$) for $r_n(x)$, we find two different possibilities for the remainder $R_n(x)$ defined in (11), according to whether $r > R$ or $r = R$:

- $r > R$:

$$|R_n(z)| \le \frac{\bar{M}}{R_0^n}\int_0^R (1-x^m)^{\Re z-1}x^{n+\Re s-1}dx = \frac{\bar{M}}{R_0^n}\frac{1}{m}B_{R^m}\left(\frac{n+\Re s}{m},\Re z\right) = \frac{M}{R_0^n}B_{R^m}\left(\frac{n+\Re s}{m},\Re z\right), \tag{17}$$

  with $M := \bar{M}/m$, and the first line of (8) follows.
- $r = R$:

$$|R_n(z)| \le \frac{\bar{M}}{R^n}\int_0^R (1-x^m)^{\Re z-1}x^{n+\Re s-1}(R-x)^{\sigma-1}dx. \tag{18}$$

We distinguish two further sub-cases, depending on whether $R < 1$ or $R = 1$.

  - If $R < 1$ we use the bound $(1-x^m)^{\Re z-1} \le \tilde{M}(z) := \max\{1,(1-R^m)^{\Re z-1}\}$ independent of x, and we find

$$|R_n(z)| \le \frac{\bar{M}\,\tilde{M}(z)}{R^n}\int_0^R x^{n+\Re s-1}(R-x)^{\sigma-1}dx = \bar{M}\tilde{M}(z)R^{\Re s+\sigma-1}B(n+\Re s,\sigma) = M(z)B(n+\Re s,\sigma), \tag{19}$$

    with $M(z) := R^{\Re s+\sigma-1}\bar{M}\tilde{M}(z)$, and the second line of (8) follows.
  - If $R = 1$ we find

$$|R_n(z)| \le \bar{M}\int_0^1 x^{n+\Re s-1}(1-x)^{\Re z+\sigma-2}(1+x+\cdots+x^{m-1})^{\Re z-1}dx.$$

When $\Re z > 1$ we have the bound $(1+x+\cdots+x^{m-1})^{\Re z-1} \leq m^{\Re z-1}$. When $\Re z \leq 1$ we have $(1+x+\cdots+x^{m-1})^{\Re z-1} \leq 1$. Therefore, $(1+x+\cdots+x^{m-1})^{\Re z-1} \leq \widetilde{m}(z) := \max\{1, m^{\Re z-1}\}$ independent on n and then

$$|R_n(z)| \leq \bar{M}\widetilde{m}(z)\int_0^1 x^{n+\Re s-1}(1-x)^{\Re z+\sigma-2}dx = M(z)B(n+\Re s, \Re z+\sigma-1),$$

with $M(z) := \bar{M}\widetilde{m}(z)$, and the third line of (8) follows.

From (8) and the asymptotic behavior for large n of the complete and incomplete beta functions involved in that formula [7, eqs 5.12.1 and 5.11.12], [6, eqs. 8.17.2 and 8.18.1] we obtain the convergence rate (9) of expansion (7).

On the other hand, using again [7, eqs 5.12.1 and 5.11.12], [6, eqs. 8.17.2, 8.17.4 and 8.18.1], for the sequence $F_k(z) := B_{R^m}\left(\frac{k+s}{m}, z\right)$ we have that $\frac{F_{k+1}(z)}{F_k(z)} = \mathcal{O}\left(z^{-1/m}\right)$ as $z \to \infty$, and then the expansion (7) is formally asymptotic. Moreover, for $r > R$, the asymptotic behavior (10) follows from (17) and [7, eqs 5.12.1 and 5.11.3] or [6, eqs. 8.17.2 and 8.18.1]. For $r = R$ we split the integral in the right hand side of (18) in the form

$$\begin{aligned}\int_0^R (1-x^m)^{\Re z-1}x^{n+\Re s-1}(R-x)^{\sigma-1}dx = &\int_0^{\epsilon R}(1-x^m)^{\Re z-1}x^{n+\Re s-1}(R-x)^{\sigma-1}dx+ \\ &\int_{\epsilon R}^R (1-x^m)^{\Re z-1}x^{n+\Re s-1}(R-x)^{\sigma-1}dx,\end{aligned} \tag{20}$$

for any $0 < \epsilon < 1$. Now, in the first integral in the right hand side above we use that $(R-x)^{\sigma-1} \leq [R(1-\epsilon)]^{\sigma-1}$ $\forall x \in [0, \epsilon R]$. In the second integral we use that, for $\Re z \geq 1$, $(1-x^m)^{\Re z-1} \leq (1-(\epsilon R)^m)^{\Re z-1}$ $\forall x \in [\epsilon R, R]$. Then,

$$\begin{aligned}\int_0^R (1-x^m)^{\Re z-1}x^{n+\Re s-1}(R-x)^{\sigma-1}dx \leq &[R(1-\epsilon)]^{\sigma-1}\int_0^{\epsilon R}(1-x^m)^{\Re z-1}x^{n+\Re s-1}dx+ \\ &[1-(\epsilon R)^m]^{\Re z-1}\int_{\epsilon R}^R x^{n+\Re s-1}(R-x)^{\sigma-1}dx.\end{aligned} \tag{21}$$

The last integral above is bounded by a positive constant, say C, independent on z, and the first integral in the right hand side is an incomplete beta function. Therefore,

$$\int_0^R (1-x^m)^{\Re z-1}x^{n+\Re s-1}(R-x)^{\sigma-1}dx \leq \frac{[R(1-\epsilon)]^{\sigma-1}}{m}B_{(\epsilon R)^m}\left(\frac{n+\Re s}{m}, \Re z\right) + C[1-(\epsilon R)^m]^{\Re z-1}.$$

For large $|z|$, the second term in the right hand side is exponentially small compared to the first one, that is of the order $\mathcal{O}(z^{-(n+s)/m})$. Therefore, the asymptotic behavior (10) also follows for $r = R$. □

Example 1. Consider the integral representation of the Bessel $J_\nu(z)$ function [9, eq. (10.9.4)]: $J_\nu(z) = \frac{2(z/2)^\nu}{\pi^{1/2}\Gamma(\nu+1/2)}\int_0^1(1-t^2)^{\nu-1/2}\cos(zt)dt$. Applying the above theorem with $m = 2$ and $r > R = 1$ and after some manipulations we find the power series definition of $J_\nu(z)$ [9, eq. (10.2.2)], which is a convergent series that is also asymptotic for large $|\nu|$.

3. A convergent version of the generalized Watson's lemma

The second step of our analysis is the derivation of an asymptotic and convergent expansion of generalized Laplace transforms. For the sake of generality, when the integration interval is bounded, we let possible branch points at the end points of the integration interval. When the integration interval is unbounded, we let a possible exponential growth of the integrand at the infinity. More precisely, we consider generalized Laplace transforms of the form

$$F(z) := \int_0^b e^{-zt^m}t^{s-1}(1-t/b)^{\sigma-1}h(t)dt, \quad \Re z \geq x_0 > 0, \quad 0 < \Re\sigma \leq 1, \quad \Re s > 0, \tag{22}$$

With $m \in \mathbb{N}$ and $0 < b \leq \infty$. We assume that the factor $(1-t/b)^{\sigma-1}$ is replaced by 1 when $b = \infty$. We also need to define the following complex region.

Definition 1. For any $0 < b \leq \infty$ we define the open complex region

$$S_m(b, 0) := \{t \in \mathbb{C} : \left|1-e^{-t^m}\right| < R\}, \qquad R := 1-e^{-b^m} \leq 1, \tag{23}$$

where only the branch that contains the integration interval $[0, b)$ is considered (see Fig. 1). (The "extra" argument 0 in the notation of the region $S_m(b, 0)$ will be clear later.) Observe that $R < 1$ when $0 < b < \infty$ and $R = 1$ when $b = \infty$.

We assume the following hypothesis for the function $h(t)$:

Hypothesis 2. Assume that the function $h(t)$ is analytic in the region $S_m(b, 0)$ or in a larger region $S_m(b_0, 0)$ for some $b_0 > b$ and, if $b = \infty$, we let $h(t)$ be of exponential order at the infinity: $h(t) = \mathcal{O}(e^{\alpha t^m})$ when $t \to +\infty$, with $0 < \alpha < \min\{1, x_0\}$.

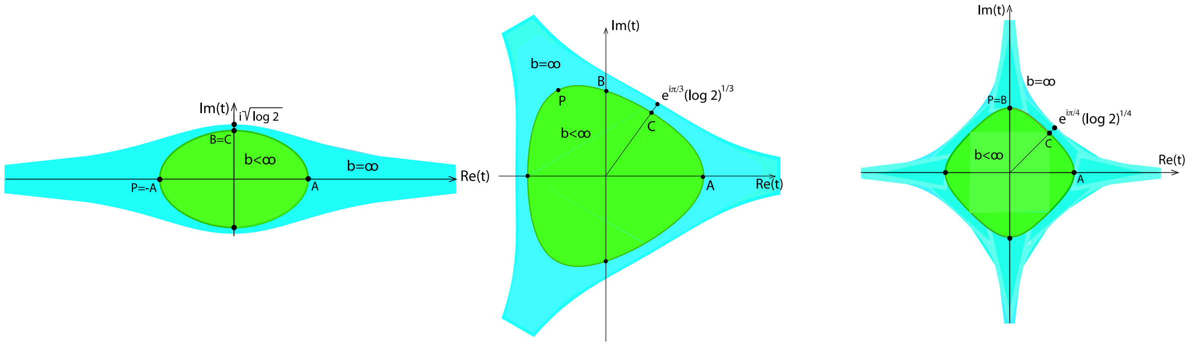

Fig. 1. Region $S_m(b,0)$ for $m=2$ (left), $m=3$ (middle) and $m=4$ (right) with $b=\infty$ (larger blue unbounded regions, including green areas) and finite b (inner green bounded figures). For any value of b, finite or infinite, the boundary of these figures is comprised by the curve $t(\theta)$ parametrized in the form $t(\theta)=[-\log(1-Re^{im\theta})]^{1/m}$, $-\pi<\theta\le\pi$, selecting the continuous branch for which $t>0$ for $\theta=0$; that is, $t(\theta)=|\log(1-Re^{im\theta})|^{1/m}e^{i\theta}$, $-\pi<\theta\le\pi$. In these figures we have highlighted four points of the boundary of $S_m(b,0)$: $A:=t(0)=[-\log(1-R)]^{1/m}>0$, $B:=t(\pi/2)=[-\log(1-Re^{im\pi/2})]^{1/m}=i|B|$, $C:=t(\pi/m)=[-\log(1+R)]^{1/m}=e^{i\pi/m}|C|$, $P:=t(2\pi/m)=e^{2i\pi/m}A$. In this figures we have depicted these points for $R<1$ (green region with $b<\infty$). When $b\to\infty$ ($R\to 1$) we have that $A\to+\infty$, $P\to e^{2i\pi/m}\infty$ and $C\to(\log 2)^{1/m}e^{i\pi/m}$ $\forall$ m. Roughly speaking, the region $S_m(b,0)$ is a circle around the origin of radius $|C|=|\log(1+R)|^{1/m}$ spiked along the m rays determined by the mth roots of the unity $e^{2\pi ik/m}$, $k=0,1,\dots,m-1$ (the point A is on the first ray, the point P is on the second one, the point C is on the middle angle between A and P). For $b=\infty$ the (blue + green) regions are unbounded, as those spikes go up to the infinity. For $b<\infty$ the (green) regions are similar, but bounded, as those spikes are bounded, and contained in $S_m(\infty,0)$: observe that $S_m(b_1,0)\subset S_m(b_2,0)$ for $b_1<b_2$.

We have the following theorem.

Theorem 2. *Consider the integral (22) with the above mentioned hypothesis. Then, for $n=1,2,3,\dots$,*

$$F(z)=\frac{1}{m}\sum_{k=0}^{n-1}A_k\,B_R\left(\frac{k+s}{m},z\right)+R_n(z), \tag{24}$$

where $R:=1-e^{-b^m}\le 1$ and A_k are the Taylor coefficients of the function

$$\tilde{h}(x):=\left(-\frac{\log(1-x^m)}{x^m}\right)^{\frac{s}{m}-1}\left(1-\frac{x}{b}\left[-\frac{\log(1-x^m)}{x^m}\right]^{1/m}\right)^{\sigma-1}h\left(x\left[-\frac{\log(1-x^m)}{x^m}\right]^{1/m}\right) \tag{25}$$

at $x=0$; assuming that, when $b=\infty$, the middle factor is replaced by 1. The Taylor coefficients A_k can be computed, either directly by using an algebraic manipulator, or by means of the following formula (see Lemma 1 in the Appendix):

$$A_k=\sum_{j=0}^{\lfloor\frac{k}{m}\rfloor}\frac{(-1)^j}{j!}\frac{B_j^{\left(\frac{k+s}{m}\right)}(1)}{(k-jm)!}\frac{d^{k-jm}}{dt^{k-jm}}\left[\left(1-\frac{t}{b}\right)^{\sigma-1}h(t)\right]_{t=0}, \tag{26}$$

where $B_n^{(\alpha)}(x)$ are the generalized Bernoulli polynomials of order α [10, Sec. 24.16], [11, Ch. VI], [12] and the factor $(1-\frac{t}{b})^{\sigma-1}$ is replaced by 1 if $b=\infty$. The remainder term is bounded in the form

$$|R_n(z)|\le\begin{cases}\dfrac{M}{R_0^n}B_R\left(\dfrac{n+\Re s}{m},\Re z\right) & \text{if } b<\infty \text{ and } \sigma=1,\\[2ex] M(z)B(n+\Re s,\Re\sigma) & \text{if } b<\infty \text{ and } \sigma\ne 1,\\[2ex] M(z)B(n+\Re s,\Re z-\alpha) & \text{if } b=+\infty,\end{cases} \tag{27}$$

for a certain $R_0>R>0$, $M>0$ independent on z and n, and $M(z)>0$ independent on n. Expansion (24) is convergent, with an exponential rate of convergence for $b<+\infty$ and $\Re\sigma=1$, and a power rate otherwise. More precisely, as $n\to\infty$,

$$R_n(z)=\begin{cases}\mathcal{O}\left(n^{-1}(R/R_0)^n\right) & \text{if } b<+\infty \text{ and } \sigma=1,\\ \mathcal{O}\left(n^{-\sigma}\right) & \text{if } b<+\infty \text{ and } \sigma\ne 1,\\ \mathcal{O}\left(n^{-(\Re z-\alpha)}\right) & \text{if } b=+\infty.\end{cases} \tag{28}$$

Expansion (24) is also an asymptotic expansion of $F(z)$ for large $|z|$: the terms of the expansion (24) and the remainder are of the order specified in (10).

Proof. Consider the change of integration variable $t \mapsto x$ defined in the form

$$t = x\left[-\frac{\log(1-x^m)}{x^m}\right]^{1/m}, \tag{29}$$

where the logarithm and the fractional power are assumed to take their principal value. The inverse is given by the formula

$$x = t\left(\frac{1-e^{-t^m}}{t^m}\right)^{1/m}, \tag{30}$$

with $x > 0$ for $t > 0$. Under this transformation, the region $S_m(b,0)$ for the variable t is transformed into the open disk $D_0(R^{1/m})$ for the variable x. Then, we find

$$F(z) = \int_0^{R^{1/m}} (1-x^m)^{z-1} x^{s-1} \tilde{h}(x)\, dx, \tag{31}$$

with $\tilde{h}(x)$ given in (25). Now, we can apply Theorem 1 to this integral with $f(x)$ replaced by $\tilde{h}(x)$ to find (24) and (27):

- When $b < +\infty$ and $\sigma = 1$, we have that $R < 1$. Moreover, the function $\tilde{h}(x)$ is analytic in a certain open disk $D_0(r)$ with $r > R_0^{1/m} > R^{1/m}$, $R_0 := 1 - e^{-b_0^m}$, and the first line of the bound (8) implies the first line of the bound (27).
- When $b < +\infty$ and $\sigma \neq 1$, we have that $R < 1$. The function $\tilde{h}(x)$ is analytic in the open disk $D_0(r)$ with $r = R^{1/m} < 1$ and the third line of the bound (8) implies the second line of the bound (27).
- When $b = +\infty$ (the middle factor in (25) is replaced by 1), we have that $R = 1$. The function $\tilde{h}(x)$ is analytic in the open disk $D_0(r)$ with $r = R = 1$ and satisfies the growth condition $\tilde{h}(x) = \mathcal{O}\left((x-1)^{-\alpha}\right)$ as $x \to 1$. Therefore, the fourth line of the bound (8) implies the third line of the bound (27) with σ replaced by $1-\alpha$. □

We illustrate Theorem 2 with the following example. This example, and also Example 3, will be used in the analysis of the Parabolic Cylinder function in Section 4.

Example 2. Consider the function

$$\bar{U_1}(a,z) := \int_0^1 (1-t)^{a-1/2} e^{-zt^2} dt, \qquad \Re a > -1/2. \tag{32}$$

It has the form of integral (22) with $b = 1$ (and then $R = 1 - e^{-1}$), $m = 2$, $s = 1$, $\sigma = a - 1/2 - \lfloor \Re a - 1/2 \rfloor$, and $h(t) = (1-t)^{a-\sigma+1/2} = (1-t)^{\lfloor \Re a - 1/2 \rfloor + 1}$. The function $h(t)$ is analytic in the region $S_2(1,0)$ and we can apply Theorem 2 to find

$$\bar{U_1}(a,z) = \frac{1}{2}\sum_{k=0}^{\infty} A_k B_{1-e^{-1}}\left(\frac{k+1}{2}, z\right). \tag{33}$$

The coefficients A_k are the Taylor coefficients at $x = 0$ of the function

$$\tilde{h}(x) = \left[-\frac{\log(1-x^2)}{x^2}\right]^{-\frac{1}{2}} \left[1 - x\left(\frac{-\log(1-x^2)}{x^2}\right)^{\frac{1}{2}}\right]^{a-1/2},$$

which can be computed using formula (26):

$$A_k = \sum_{j=0}^{\lfloor \frac{k}{2} \rfloor} \frac{(-1)^j}{j!} \frac{B_j^{\left(\frac{k+1}{2}\right)}(1)}{(k-2j)!} \frac{d^{k-2j}}{dt^{k-2j}} \left[(1-t)^{a-1/2}\right]_{t=0} = \sum_{j=0}^{\lfloor \frac{k}{2} \rfloor} \frac{(-1)^j}{j!} \frac{B_j^{\left(\frac{k+1}{2}\right)}(1)}{(k-2j)!} \left(\frac{1}{2} - a\right)_{k-2j},$$

where $(a)_k := \Gamma(a+k)/\Gamma(a)$ denotes the Pochhammer's symbol [7, Sect. 5.2.(iii)]. The first few coefficients are

$$A_0 = 1; \quad A_1 = \frac{1}{2} - a; \quad A_2 = \frac{1}{2}\left(a^2 - 2a + \frac{1}{4}\right); \quad A_3 = \frac{-1}{6}\left(a^3 - \frac{9}{2}a^2 + \frac{23}{4}a - \frac{15}{8}\right).$$

Table 1 contains a numerical experiment about approximation (33) that shows the rate of convergence of expansion (33) and its asymptotic character for large $|z|$.

Theorem 2 requires the analyticity of the function $h(t)$ in the region $S_m(b,0)$ defined in (23) and depicted in Fig. 1 for several values of m. From a practical point of view, this region may be "too large", as the integral representations of some special functions do not satisfy this requirement (see Example 3). But, after a certain manipulation of the integral that we describe below, this condition may be relaxed, requiring the analyticity of $h(t)$ in a smaller region, enlarging in this way the range of applicability of Theorem 2.

Table 1
Absolute value of the relative error provided by the right hand side of (33) when we truncate the series after n terms, for different values of z and $a = 1.8$.

n	$z = 0.5$	$z = 2$	$z = 10$	$z = 30$	$z = 100$
0	1.26	8.18474×10^{-1}	3.0011×10^{-1}	1.54866×10^{-1}	7.93323×10^{-2}
5	8.46202×10^{-3}	3.1662×10^{-3}	5.92959×10^{-5}	1.44491×10^{-6}	2.77723×10^{-8}
10	3.15303×10^{-3}	1.06383×10^{-3}	6.64822×10^{-6}	1.88215×10^{-8}	2.5038×10^{-11}
15	1.34204×10^{-3}	4.25023×10^{-4}	1.27042×10^{-6}	4.3148×10^{-10}	7.86635×10^{-14}
20	7.06936×10^{-4}	2.15652×10^{-4}	4.16676×10^{-7}	2.36881×10^{-11}	4.40583×10^{-14}
25	4.2018×10^{-4}	1.24947×10^{-4}	1.81634×10^{-7}	2.31511×10^{-12}	4.38895×10^{-14}

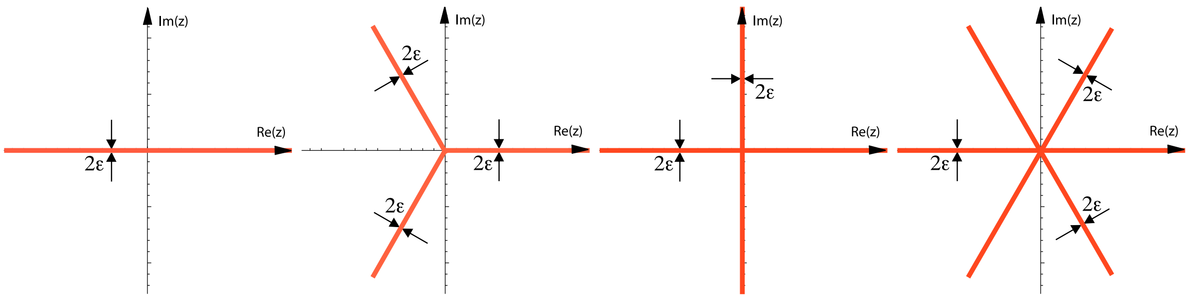

Fig. 2. Region $s_m(b,0)$ for $m=2$ (left), $m=3$ (middle left), $m=4$ (middle right) and $m=6$ (right). In every picture, the m rays in the directions $e^{2ik\pi/m}$, $k=0,1,2,\dots,m-1$, have width 2ε and length $b\varepsilon/\left|\sqrt[m]{\log 2}\right|$.

In order to understand what manipulation we need, we must first give some insight into the geometry of the transformation $t \to x$ defined in (29). The key point is that, after the change of variable $t \mapsto x$, with inverse (30), the original integration interval $t \in [0,b)$ in (22) is transformed into a new integration interval in (31): $x \in [0, R^{1/m}) \subset [0,1)$ $\forall\, b > 0$, with $R := 1 - e^{-b^m}$. Now, expansion (24) follows after a Taylor expansion of the function $\tilde{h}(x)$ given in (25) at $x = 0$ and an interchange of summation and integration in (31). The resulting expansion (24) is convergent whenever $\tilde{h}(x)$ is analytic in the disk $D_0(R^{1/m})$. And $\tilde{h}$ is analytic there whenever $h(t)$ is analytic in the image of the disk $D_0(R^{1/m})$ under the transformation (30), that is the region $S_m(b,0)$.

Then, the trick to relax the analyticity condition of Theorem 2 is a dilatation of the original integration variable $t \in [0,b)$ that squeezes the region $S_m(b,0)$ into a smaller region $s_m(b,0)$. (Again, the "extra" argument 0 in the notation will be clear in a moment.) Such a region $s_m(b,0)$ may be the following: a starlike domain with center at $t=0$ and with m arbitrarily narrow spikes of width 2ε and length $b\varepsilon/\left|\sqrt[m]{\log 2}\right|$ in the directions defined by the mth roots of the unity, with $\varepsilon > 0$ arbitrarily small (see Fig. 2).

$$s_m(b,0) := \left\{ t \in \mathbb{C} : t = (r+iy)e^{2i\pi k/m};\ 0 \le r < \frac{b\varepsilon}{\left|\sqrt[m]{\log 2}\right|};\ k = 0,1,\dots,m-1;\ -\varepsilon < y < \varepsilon \right\}. \tag{34}$$

Consider a certain *dilatation parameter* $\Lambda > 1$. After the dilatation $u \mapsto t = \Lambda u$, the region $s_m(b,0)$ for the variable u becomes the larger region $s_m(b,0) \to \Lambda s_m(b,0)$ for the new variable t. The region $\Lambda s_m(b,0)$ contains the region $S_m(b,0)$ for any $b > 0$ whenever $\varepsilon\Lambda$ is larger than the distance from the origin $t=0$ to the complementary of the region $S_m(\infty,0)$ (see Fig. 3). That distance is attained at the intersection of the boundary of $S_m(\infty,0)$ with the rays $t = re^{i(2k+1)\pi/m}$, $r > 0$, $k = 0,1,2,\dots,m-1$, which occurs at a distance $C = \left|\sqrt[m]{\log(2)}\right|$ (see Fig. 3). Therefore, the region $\Lambda s_m(b,0)$ contains the region $S_m(b,0)$ for any $b > 0$ whenever $\Lambda > \left|\sqrt[m]{\log(2)}\right|/\varepsilon$. Then,

$$F(z) := \int_0^b e^{-zu^m} u^{s-1}\left(1 - \frac{u}{b}\right)^{\sigma-1} h(u)du = \frac{1}{\Lambda^s}\int_0^{\Lambda b} e^{-(z/\Lambda^m)t^m} t^{s-1}\left(1 - \frac{t}{\Lambda b}\right)^{\sigma-1} \bar{h}(t)dt,$$

with $\bar{h}(t) := h(t/\Lambda)$. Whenever $h(u)$ is analytic in $s_m(b,0)$, $\bar{h}(t)$ is analytic in $S_m(b,0) \subset \Lambda s_m(b,0)$ and Theorem 2 may be applied to this last integral with z replaced by z/Λ^m and h by $\bar{h}$.

With this dilatation, we may avoid all the singularities of $h(t)$ off the rays $\arg t = 2k\pi/m$, $k = 0,1,2,\dots,m-1$, but not those located on these rays, as the axes of the starlike region $s_m(b,0)$ remain invariant under the dilatation $u \mapsto t = \Lambda u$ no matter how large Λ is. We still can use a second trick to avoid these singularities over the axes of $s_m(b,0)$ when $b = \infty$: an appropriate rotation of the original integration path $[0,\infty)$ whenever the function h is analytic in an appropriate sector. The effect of this rotation is a rotation of those axes that may avoid the singularities located there. Choose an angle θ satisfying $|\arg z + m\theta| < \pi/2$ (optimally $\theta = -\frac{\arg(z)}{m}$ if $\arg z \neq 0$). Now, if $e^{-\alpha t^m}h(t)$ is bounded as $t \to \infty$ in the sector $\arg t \in [0,\theta]$ (and not only for $t > 0$ as it is required in H2.2) and $h(t)$ is analytic in that sector, we can invoke Cauchy's

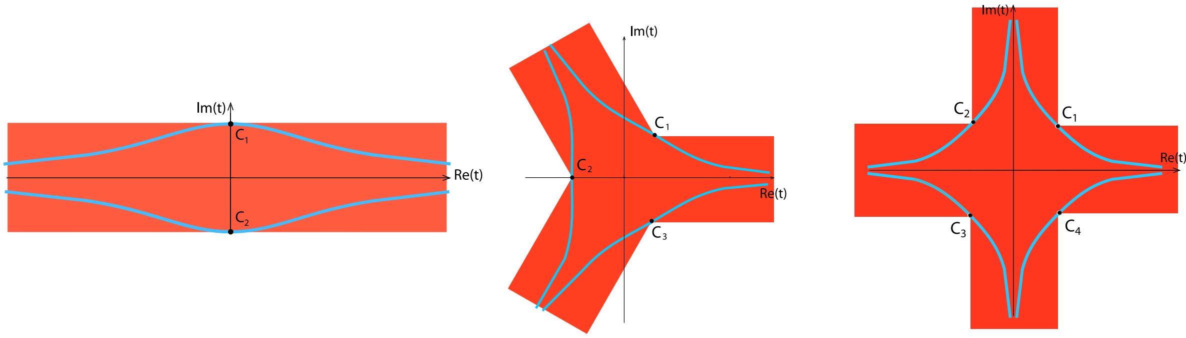

Fig. 3. The thick red star-like regions $\Lambda s_m(\infty,0)$ contain the corresponding regions $S_m(\infty,0)$ limited by the blue curves; for $m=2$ (left), $m=3$ (middle) and $m=4$ (right). In these figures $C_1, C_2, \ldots, C_m$ are the m different mth roots of $-\log(2)$, that is, $C_k=|C|e^{i(2k+1)\pi/m}$, $k=0,1,2,\ldots,m-1$, $|C|:=\left|\sqrt[m]{\log(2)}\right|$ and $\Lambda=\left|\sqrt[m]{\log(2)}\right|/\varepsilon$ for the given $\varepsilon>0$ used in the definition (34) of $s_m(b,0)$.

theorem to rotate the path of integration that angle θ. Then

$$F(z)=e^{is\theta}\int_0^\infty e^{-ze^{im\theta}t^m}t^{s-1}h(e^{i\theta}t)dt,$$

and we can apply Theorem 2 to this integral with $h(t)$ replaced by $h(e^{i\theta}t)$ and z replaced by $ze^{im\theta}$. We can combine the above two tricks to enlarge the range of applicability of Theorem 2. Then, consider again the integral (22) with the following hypotheses instead of H2:

Hypothesis 3.

- H3.1. The function $h(t)$ is analytic in the starlike region $s_m(b,0)$ defined in (34). If $b=+\infty$, then $e^{-\alpha t^m}h(t)$ is bounded as $t\to+\infty$ for a certain $\alpha<\min\{1,x_0\}$,
  or
- H3.2. $b=+\infty$ and the function $h(t)$ is analytic in a sector $\arg t\in[0,\theta]$, with $|\arg z+m\theta|<\pi/2$, and also in the starlike region $s_m(\infty,0)$ defined in (34) rotated an angle θ:

$$s_m(\infty,\theta):=\{t\in\mathbb{C}:t=(r+iy)e^{i(\theta+2\pi k/m)};\ r\geq 0;\ k=0,1,2,\ldots,m-1;\ -\varepsilon<y<\varepsilon\},\tag{35}$$

  for a certain $\varepsilon>0$. And $e^{-\alpha t^m}h(t)$ is bounded as $t\to\infty$ in this sector for a certain $\alpha<\min\{1,x_0\}$.

We obtain the following theorem.

Theorem 3. *Consider the integral*

$$F(z):=\int_0^b e^{-zt^m}t^{s-1}(1-t/b)^{\sigma-1}h(t)dt,\qquad \Re z>x_0>0,\qquad m\in\mathbb{N},\qquad \Re s>0,\tag{36}$$

and $0<\Re\sigma\leq 1$*, with* $0<b\leq+\infty$*. If* $b=\infty$*, we assume that the factor* $(1-t/b)^{\sigma-1}$ *is replaced by* 1*. Assume that Hypotheses 3 hold. Then, for* $n=1,2,3,\ldots,$

$$F(z)=\frac{1}{m\Lambda^s}\sum_{k=0}^{n-1}A_kB_R\left(\frac{k+s}{m},\frac{z}{\Lambda^m}\right)+R_n(z),\tag{37}$$

for any $\Lambda\in\mathbb{C}$ *with* $|\Lambda|>\left|\sqrt[m]{\log(2)}\right|/\varepsilon$ *and* $\arg(\Lambda)=\theta$*, with* $\theta=0$ *if* $h(t)$ *is analytic in* $s_m(b,0)$*, and* $R:=1-e^{-(|\Lambda|b)^m}$*. The coefficients* A_k *are the Taylor coefficients of*

$$\tilde{h}(x):=\left(-\frac{\log(1-x^m)}{x^m}\right)^{\frac{s}{m}-1}\left(1-\frac{x}{b\Lambda}\left[-\frac{\log(1-x^m)}{x^m}\right]^{1/m}\right)^{\sigma-1}h\left(\frac{x}{\Lambda}\left[-\frac{\log(1-x^m)}{x^m}\right]^{1/m}\right)$$

at $x=0$*, with the middle factor replaced by* 1 *when* $b=+\infty$*. The Taylor coefficients* A_k *can be computed either, directly by using an algebraic manipulator, or by using the following formula (see Lemma 1 in the Appendix):*

$$A_k=\sum_{j=0}^{\lfloor\frac{k}{m}\rfloor}\frac{(-1)^j}{j!}\frac{B_j^{\left(\frac{k+s}{m}\right)}(1)}{(k-mj)!\,\Lambda^{k-mj}}\frac{d^{k-mj}}{dt^{k-mj}}\left[\left(1-\frac{t}{b}\right)^{\sigma-1}h(t)\right]_{t=0},\tag{38}$$

Table 2
Absolute value of the relative error provided by the right hand side of (40) when we truncate the series after n terms, for different values of z and $a = 2.6$. We have taken $\Lambda = 2e^{i\pi/6}$.

n	$z = 0.5$	$z = 2$	$z = 30$	$z = 100$	$z = 250$
0	6.07615×10^{-1}	4.77225×10^{-1}	1.84414×10^{-1}	1.08389×10^{-1}	7.08001×10^{-2}
5	1.31198×10^{-1}	3.68296×10^{-2}	1.21413×10^{-4}	3.98426×10^{-6}	2.72387×10^{-7}
10	1.18385×10^{-1}	2.50816×10^{-2}	4.67633×10^{-6}	7.70153×10^{-9}	4.13833×10^{-11}
15	1.14372×10^{-1}	2.18815×10^{-2}	1.08835×10^{-6}	2.78716×10^{-10}	2.68262×10^{-13}
20	1.10831×10^{-1}	1.92642×10^{-2}	2.4277×10^{-7}	9.90124×10^{-12}	6.95367×10^{-15}

where the factor $(1 - t/b)^{\sigma-1}$ is replaced by 1 if $b = +\infty$ and $B_k^{(\alpha)}(x)$ are the generalized Bernoulli polynomials.

The remainder $R_n(z)$ is bounded as in (27) and (28) with z replaced by z/Λ^m. Therefore expansion (37) is convergent. It is also an asymptotic expansion of $F(z)$ for large $|z|$, with the same order behavior as in Theorem 2.

Example 3. Consider the function

$$\bar{U_2}(a, z) := \int_0^\infty (t + 1)^{a-1/2} e^{-zt^2} dt, \qquad \Re a > -1/2, \qquad \Re z > 0. \tag{39}$$

It is of the form (22) or (36) with $m = 2$, $b = +\infty$ (and then $R = 1$), $s = 1$ and $h(t) = (t + 1)^{a-1/2}$. However, we cannot apply Theorem 2, as the branch point $t = -1$ is inside the region $S_2(\infty, 0)$. Nevertheless, $h(t)$ is analytic in $s_2(\infty, \theta)$ for any θ satisfying $|\arg z \pm 2\theta| \le \pi/2$ and $\varepsilon = \sin\theta$. Then, we can apply Theorem 3 with any Λ satisfying $|\arg(z) + 2\arg(\Lambda)| \le \pi/2$ or $|\arg(z) - 2\arg(\Lambda)| \le \pi/2$, $\arg \Lambda \ne 0$ and $|\Lambda| > |\sqrt{\log 2}|/\sin(\arg(\Lambda))$.

We find

$$\bar{U_2}(a, z) = \frac{1}{2\Lambda} \sum_{k=0}^{\infty} A_k B\left(\frac{k+1}{2}, \frac{z}{\Lambda^2}\right). \tag{40}$$

The coefficients A_k are the Taylor coefficients of

$$\tilde{h}(x) := \left(-\frac{\log(1 - x^2)}{x^2}\right)^{-\frac{1}{2}} \left[1 + \frac{x}{\Lambda}\left(\frac{-\log(1 - x^2)}{x^2}\right)^{\frac{1}{2}}\right]^{a-1/2}$$

at $x = 0$. They can be computed using formula (38) in the form

$$A_k = \sum_{j=0}^{\lfloor \frac{k}{2} \rfloor} \frac{(-1)^{k+j}}{j!} \frac{B_j^{\left(\frac{k+1}{2}\right)}(1)}{(k - 2j)!} \frac{\left(\frac{1}{2} - a\right)_{k-2j}}{\Lambda^{k-2j}},$$

where we have used that $\frac{d^n}{dt^n}[h(t)]_{t=0} = (-1)^n \left(\frac{1}{2} - a\right)_n$. The first few coefficients are

$$A_0 = 1, \quad A_1 = \frac{(a - 1/2)}{\Lambda}, \quad A_2 = \frac{(1/2 - a)_2}{2\Lambda^2} - \frac{1}{4}, \quad A_3 = \frac{-(1/2 - a)_3}{6\Lambda^3}.$$

We may choose for example $\arg(\Lambda) = \pi/6$ and $|\Lambda| = 2$. With this election of Λ we obtain the numerical experiment detailed in Table 2, that shows the convergent and asymptotic character of expansion (40).

4. A convergent Laplace method

Finally, we consider Laplace integrals of the form

$$F(z) := \int_0^\infty e^{-zf(t)} g(t) t^{a-1} dt, \qquad \Re z > x_0 > 0, \qquad \Re(a) > 0, \tag{41}$$

and assume the following hypotheses:

Hypothesis 4.

- H4.1. *The function $f(t)$ has only one absolute minimum in the positive real axis at a certain point $t_0 \geq 0$.*
  Then, there exists a number $m \in \mathbb{N}$ such that $f^{(m)}(t_0) > 0$ and $f^{(k)}(t_0) = 0$ for $k = 1, 2, \ldots, m-1$. If $t_0 > 0$ then m is even. Then, following the ideas of the modified Laplace method introduced in [1], we consider the Taylor polynomial of $f(t)$ of degree m at $t = t_0$:
$$p(t) := f(t_0) + \eta(t-t_0)^m, \qquad \eta := \frac{f^{(m)}(t_0)}{m!} > 0.$$
  We write $f(t) = p(t) + f_m(t)$ with
$$f_m(t) := f(t) - p(t) = \sum_{k=m+1}^{\infty} \frac{f^{(k)}(t_0)}{k!}(t-t_0)^k, \qquad |t-t_0| < r,$$
  for a certain $r > 0$. Roughly speaking, $p(t)$ is the asymptotically dominant part of the phase function $f(t)$, whereas $f_m(t)$ is subdominant (see [1] for further details on the convenience of this splitting of the phase function).
- H4.2. *The function $f(t)$ is real for real t and both, $f(t_0 - t)$ and $g(t_0 - t)$ are analytic in the region $t_0 - S_m(t_0, 0)$, with $S_m(t_0, 0)$ defined in (23).*
- H4.3. *Both functions, $f(t_0 + t)$ and $g(t_0 + t)$ are analytic in a sector $\arg(t_0 + t) \in [0, \theta]$, for a certain angle θ satisfying $|\arg(z) + m\theta| < \pi/2$, and also in the starlike region $t_0 + s_m(\infty, \theta)$ defined in (35). And there exists a number $0 < \alpha < \min\{1, x_0\}$ such that $e^{-\alpha t^m} e^{-zf_m(t_0+t)} g(t_0 + t)$ is bounded as $t \to \infty$ in the above mentioned sector.*
- H4.4. *The function $e^{-zf(t)}g(t)t^{a-1}$ is absolutely integrable on $[0, \infty)$ for $\Re z > x_0$.*

Then, from H4.1, the integral in (41) can be written in the form
$$F(z) = e^{-zf(t_0)} \int_0^{\infty} e^{-z\eta(t-t_0)^m} h(t,z) t^{a-1} dt, \qquad h(t,z) := e^{-zf_m(t)} g(t). \tag{42}$$
In this integral, the exponent of the asymptotically dominant exponential consists only of the asymptotically dominant part of the phase function, whereas the subdominant part is included in $h(t, z)$. We split up the integral at $t = t_0$ and write:
$$\begin{aligned} F(z) &= e^{-zf(t_0)} \int_0^{t_0} e^{-z\eta(t-t_0)^m} h(t,z) t^{a-1} dt + e^{-zf(t_0)} \int_{t_0}^{\infty} e^{-z\eta(t-t_0)^m} h(t,z) t^{a-1} dt \\ &= e^{-zf(t_0)} \left(F^-(z) + F^+(z)\right), \end{aligned} \tag{43}$$
where we have defined
$$F^{\pm}(z) := \int_0^{b^{\pm}} e^{-z\eta t^m} h(t_0 \pm t, z)(t_0 \pm t)^{a-1} dt, \tag{44}$$
with $b^- := t_0$ and $b^+ := \infty$. Now, we can apply Theorem 2 to $F^-(z)$ and Theorem 3 to $F^+(z)$. But the function h considered in Theorems 2 and 3 does not depend on z, and now the function h in both integrals in $F^{\pm}(z)$ does. Then, we are introducing here a new ingredient in the analysis: the function h and then the Taylor coefficients A_k in formulas (24) and (37) depend on the asymptotic variable z. This fact does not have any influence on the convergence of those expansions, and formulas (27) and (28) remain valid. But it has an effect on the asymptotic behavior of the terms of the expansions (24) and (37) and the respective remainders $R_n(z)$: formulas (10) are not longer valid. Therefore, when the function h in Theorems 2 and 3 is the function h given in (42), the asymptotic character of expansions (24) and (37) is not clear and must be proved. In the following theorem we summarize this discussion providing a convergent expansion of the integral (41) and proving its asymptotic character for large $|z|$.

Theorem 4. *Consider the integral (41) with the parameters m, t_0, η and the function $h(t, z)$ defined above. Assume that hypotheses H4 hold. Then, for $n = 1, 2, 3, \ldots$,*
$$F(z) = e^{-zf(t_0)} \left\{ \frac{1}{m} \sum_{k=0}^{n-1} \left[A_k^-(z) B_R\left(\frac{k+1}{m}, \eta z\right) + \frac{A_k^+(z)}{(\Lambda^+)^{\lambda^+}} B\left(\frac{k+\lambda^+}{m}, \frac{\eta z}{(\Lambda^+)^m}\right) \right] + R_n(z) \right\}, \tag{45}$$
where $R := 1 - e^{-t_0^m}$; $\lambda^+ = 1$ if $t_0 > 0$ or $\lambda^+ = a$ if $t_0 = 0$; $|\Lambda^+| > \left|\sqrt[m]{\log(2)}\right| / \varepsilon$ and $\arg(\Lambda^+) = \theta$. On the other hand, $A_k^{\pm}(z)$ are the Taylor coefficients of the function
$$\tilde{h}^{\pm}(x,z) := \left(-\frac{\log(1-x^m)}{x^m}\right)^{\frac{\lambda^{\pm}}{m}-1} \left[t_0 \pm \frac{x}{\Lambda^{\pm}}\left(-\frac{\log(1-x^m)}{x^m}\right)^{\frac{1}{m}}\right]^{\mu^{\pm}-1} h\left(t_0 \pm \frac{x}{\Lambda^{\pm}}\left[-\frac{\log(1-x^m)}{x^m}\right]^{\frac{1}{m}}, z\right) \tag{46}$$

at $x = 0$, with $\Lambda^- = \lambda^- = 1$, $\mu^- = a$ and either, $\mu^+ = a$ if $t_0 > 0$, or $\mu^+ = 1$ if $t_0 = 0$. The Taylor coefficients $A_k^{\pm}(z)$ can be computed either, directly by using an algebraic manipulator, or by using the following formula:

$$A_n^{\pm}(z) = \sum_{k=0}^{\lfloor \frac{n}{m} \rfloor} \frac{(-1)^k B_k^{(\frac{n+\lambda^{\pm}}{m})}(1)}{(\Lambda^{\pm})^{n-km}\, k!} \frac{(\pm 1)^{n-km}}{(n-km)!} \frac{d^{n-km}}{dt^{n-km}} \left[t^{\mu^{\pm}-1} h(t,z) \right]_{t=t_0}, \tag{47}$$

where $B_n^{(\alpha)}(x)$ are the generalized Bernoulli polynomials.

The remainder $R_n(z)$ is bounded by the sum of the first line of (27) with $s = 1$ and the third line of (27) with $s = \lambda^+$. Therefore, expansion (45) is convergent. It is also an asymptotic expansion of $F(z)$ for large $|z|$: the terms of the expansion between brackets inside the sum in (45) are of the order $\mathcal{O}\left(z^{\frac{k}{p}-\frac{k+1}{m}} + z^{\frac{k}{p}-\frac{k+\lambda^+}{m}}\right)$, and $R_n(z) = \mathcal{O}\left(z^{\frac{n}{p}-\frac{n+1}{m}} + z^{\frac{n}{p}-\frac{n+\lambda^+}{m}}\right)$, as $|z| \to \infty$, where $p > m$ denotes the first derivative of $f(t)$ at t_0 that does not vanish after the mth derivative (see Observation 2).

Proof. Formulas (45)–(46) and the convergence rate of expansion (45) (given by the sum of the first line of (27) with $s = 1$ and the third line of (27) with $s = \lambda^+$) follow from a direct application of Theorems 2 and 3: Using H4.2, we can apply Theorem 2 to $F^-(z)$ in the second line of (43) with $b = t_0$, $s = 1$, $\sigma = a - \lfloor \Re a \rfloor$, $h(t)$ replaced by $t_0^{a-1} h(t_0 - t, z)(1 - t/t_0)^{a-\sigma} = t_0^{a-1} h(t_0 - t, z)(1 - t/t_0)^{\lfloor \Re a \rfloor}$ and z replaced by ηz. On the other hand, using H4.3, we can apply Theorem 3 to $F^+(z)$ in the second line of (43) with $b = +\infty$, $\sigma = 1$, z replaced by ηz and either, $s = 1$ with $h(t)$ replaced by $h(t_0 + t, z)(t_0 + t)^{a-1}$ if $t_0 > 0$, or $s = a$ and $h(t)$ replaced by $h(t_0 + t, z)$ if $t_0 = 0$. Formula (47) follows directly from Lemma 1 in the Appendix.

In the remaining of this proof, we show the asymptotic character of (45). Firstly, expansion (45) is formally asymptotic: from Corollary 1 in the Appendix we have that, for $k = 0, 1, 2, \ldots$, $A_k(z) = \mathcal{O}\left(z^{\frac{k}{p}}\right)$, as $|z| \to \infty$. And then, from the asymptotic behavior of the beta and incomplete beta functions in the terms between brackets inside the sum in (45) [7, eq. (5.12.1), (5.11.12)] and [6, eq. (8.17.2), (8.17.4), (8.18.3)], we find that the terms of the expansion between brackets inside the sum in (45) are of the order $\mathcal{O}\left(z^{\frac{k}{p}-\frac{k+\lambda^+}{m}} + z^{\frac{k}{p}-\frac{k+1}{m}}\right)$ as $|z| \to \infty$.

Secondly, we study the order behavior of the remainder term $R_n(z)$ in (45). As we have mentioned above, expansion (45) follows from formula $F(z) = e^{-zf(t_0)}(F^-(z) + F^+(z))$ (see (43) and (44)), applying Theorem 2 to the integral $F^-(z)$ and Theorem 3 to the integral $F^+(z)$. Therefore,

$$R_n(z) = R_n^-(z) + R_n^+(z), \tag{48}$$

where $R_n^{\pm}(z)$ are the remainders given in Theorems 2 and 3 for $F^+(z)$ and $F^-(z)$ respectively:

$$F^{\pm}(z) = \frac{1}{m} \sum_{k=0}^{n-1} A_k^{\pm}(z) \Phi_k^{\pm}(R^{\pm}; z) + R_n^{\pm}(z), \tag{49}$$

$R^- = R = 1 - e^{-t_0^m}$, $R^+ = 1$, and

$$\Phi_k^{\pm}(c; z) := \frac{1}{(\Lambda^{\pm})^{\lambda^{\pm}}} B_c\left(\frac{k+\lambda^{\pm}}{m}, \frac{\eta z}{(\Lambda^{\pm})^m}\right), \qquad k = 0, 1, 2, \ldots. \qquad 0 \le c \le 1. \tag{50}$$

Now, the key point for the proof is an splitting of both integrals $F^{\pm}(z)$ at the point $t = |z^{-1/p}|$, for $|z| > x_0$ large enough. That is, we write $F^{\pm}(z) = F_0^{\pm}(z) + F_1^{\pm}(z)$, with

$$\begin{aligned} F_0^{\pm}(z) &= \int_0^{|z^{-1/p}|} e^{-z\eta t^m} h(t_0 \pm t, z)(t_0 \pm t)^{a-1} dt, \\ F_1^{\pm}(z) &= \int_{|z^{-1/p}|}^{b^{\pm}} e^{-z\eta t^m} h(t_0 \pm t, z)(t_0 \pm t)^{a-1} dt. \end{aligned} \tag{51}$$

(For large enough $|z|$ we have that $|z^{-1/p}| \le t_0$ for positive t_0.) The integral $F_0^-(z)$ is of the form of $F^-(z)$ with t_0 replaced by $|z^{-1/p}|$. As well as the integral $F_0^+(z)$ if we also replace $h(t_0 - t, z)(t_0 - t)^{a-1}$ by $h(t_0 + t, z)(t_0 + t)^{a-1}$. Therefore, we can apply Theorems 2 or 3 to these two integrals and we get

$$F_0^{\pm}(z) = \frac{1}{m} \sum_{k=0}^{n-1} A_k^{\pm}(z) \Phi_k^{\pm}(R_z; z) + R_{n,0}^{\pm}(z), \qquad R_z := 1 - e^{-|z^{-m/p}|}, \tag{52}$$

with $\Phi_k^{\pm}(c; z)$ given in (50) and

$$R_{n,0}^{\pm}(z) := \int_0^{(R_z)^{1/m}} (1 - x^m)^{\frac{\eta}{(\Lambda^{\pm})^m} z - 1} x^{\lambda^{\pm}-1} r_n^{\pm}(x, z) dx, \tag{53}$$

where $r_n^{\pm}(x,z)$ is the nth order Taylor remainder of $\tilde{h}^{\pm}(x,z)$ at $x=0$.

Then, on the one hand, according to the splitting described above, we have

$$F^{\pm}(z) = F_0^{\pm}(z) + F_1^{\pm}(z) = \frac{1}{m}\sum_{k=0}^{n-1} A_k^{\pm}(z)\Phi_k^{\pm}(R_z;z) + R_{n,0}^{\pm}(z) + F_1^{\pm}(z). \tag{54}$$

On the other hand we have formula (49). From (49) and (54) we find that

$$R_n^{\pm}(z) = \Psi_n^{\pm}(z) + R_{n,0}^{\pm}(z) + F_1^{\pm}(z), \tag{55}$$

with

$$\Psi_n^{\pm}(z) := \frac{1}{m}\sum_{k=0}^{n-1} A_k^{\pm}(z)[\Phi_k^{\pm}(R^{\pm};z) - \Phi_k^{\pm}(R_z;z)]. \tag{56}$$

In the remaining of the proof we study the asymptotic behavior of every one of the three terms $\Psi_n^{\pm}(z)$, $R_{n,0}^{\pm}(z)$ and $F_1^{\pm}(z)$ in the right hand side of formula (55), in order to find out the asymptotic behavior of $R_n^{\pm}(z)$, and then of $R_n(z) = R_n^{+}(z) + R_n^{-}(z)$.

- $\Psi_n^{\pm}(z)$. Note that the arguments of the two incomplete beta functions in the right hand side of (56) (see (50)) are the same, the incomplete beta functions only differ in their index. Then, taking into account the integral representation of the incomplete beta function [6, eq. (8.17.1)] we find that

$$\Phi_k^{\pm}(R^{\pm};z) - \Phi_k^{\pm}(R_z;z) = \frac{1}{(\Lambda^{\pm})^{\lambda^{\pm}}}\int_{R_z}^{R^{\pm}} t^{\frac{k+\lambda^{\pm}}{m}-1}(1-t)^{\frac{\eta z}{(\Lambda^{\pm})^m}-1}dt.$$

Hence

$$\left|\Phi_k^{\pm}(R^{\pm};z) - \Phi_k^{\pm}(R_z;z)\right| \le \frac{1}{|(\Lambda^{\pm})^{\lambda^{\pm}}|}\int_{1-e^{-|z^{-m/p}|}}^{1} t^{\frac{k+\Re\lambda^{\pm}}{m}-1}(1-t)^{\Re\left(\frac{\eta z}{(\Lambda^{\pm})^m}\right)-1}dt.$$

Performing the change of variables $t\mapsto u$ defined by $t = 1-e^{-|z^{-m/p}|}u$ we find

$$\begin{aligned}&\left|\Phi_k^{\pm}(R^{\pm};z) - \Phi_k^{\pm}(R_z;z)\right| \le \frac{1}{|(\Lambda^{\pm})^{\lambda^{\pm}}|}\left(e^{-|z^{-m/p}|}\right)^{\Re\left(\frac{\eta z}{(\Lambda^{\pm})^m}\right)}\times\\ &\int_0^1 (1-e^{-|z^{-m/p}|}u)^{\frac{k+\Re\lambda^{\pm}}{m}-1}u^{\Re\left(\frac{\eta z}{(\Lambda^{\pm})^m}\right)-1}du \le \frac{1}{|(\Lambda^{\pm})^{\lambda^{\pm}}|}\frac{1}{\Re\left(\frac{\eta z}{(\Lambda^{\pm})^m}\right)}e^{-\Re\left(\frac{\eta z}{(\Lambda^{\pm})^m}\right)|z^{-\frac{m}{p}}|}.\end{aligned} \tag{57}$$

It is proved in Corollary 1 in the Appendix that the coefficients $A_k^{\pm}(z)$ are polynomials in z of degree $\lfloor k/p\rfloor$. Therefore, from (56) and (57) we have that, $\forall k = 0,1,\dots,n-1$,

$$\Psi_n^{\pm}(z) = \mathcal{O}\left(z^{\lfloor n/p\rfloor -1}e^{-\Re\left(\frac{\eta z}{(\Lambda^{\pm})^m}\right)|z^{-\frac{m}{p}}|}\right), \qquad \text{as } |z|\to\infty. \tag{58}$$

- $F_1^{\pm}(z)$. We have that $f(t_0\pm t) - f(t_0) = \eta t^m + \mathcal{O}(t^p)$ as $t\to 0^+$ with $\eta>0$, and therefore $\exists\, \delta^{\pm}>0$ independent on t (and of course on z) such that

$$f(t_0\pm t) - f(t_0) - \frac{\eta}{2}t^m > 0 \qquad \text{for } 0<t<\delta^{\pm}. \tag{59}$$

On the other hand, as t_0 is an absolute minimum of $f(t)$ on $[0,\infty)$, there exist $\epsilon^{\pm}>0$ such that

$$f(t_0\pm t) - f(t_0) - \epsilon^{\pm} > 0 \qquad \text{for } t\ge\delta^{\pm}. \tag{60}$$

For large enough $|z|$ we have that $\delta^{\pm} > |z^{-1/p}|$, and we can split the integral $F_1^{\pm}(z)$ at $t=\delta^{\pm}$ and write

$$F_1^{\pm}(z) = G_1^{\pm}(z) + G_2^{\pm}(z),$$

with

$$G_1^{\pm}(z) := \int_{|z^{-1/p}|}^{\delta^{\pm}} e^{-z\eta t^m}e^{-zf_m(t_0\pm t)}g(t_0\pm t)(t_0\pm t)^{a-1}dt,$$

$$G_2^{\pm}(z) := \int_{\delta^{\pm}}^{b^{\pm}} e^{-z\eta t^m}e^{-zf_m(t_0\pm t)}g(t_0\pm t)(t_0\pm t)^{a-1}dt.$$

Using $f(t_0 \pm t) = f(t_0) + \eta t^m + f_m(t_0 \pm t)$ and (59) we find

$$|G_1^{\pm}(z)| \leq \int_{|z^{-1/p}|}^{\delta^{\pm}} |e^{-z\left[f(t_0\pm t)-f(t_0)-\eta t^m/2\right]}|\, |e^{-z\eta t^m/2}|g(t_0 \pm t)(t_0 \pm t)^{a-1}|dt$$
$$\leq \int_{|z^{-1/p}|}^{\delta^{\pm}} e^{-\eta t^m \Re z/2}|g(t_0 \pm t)(t_0 \pm t)^{a-1}|dt \leq \bar{K} \int_{|z^{-1/p}|}^{\delta^{\pm}} e^{-\eta t^m \Re z/2}$$
$$\leq \bar{K} \int_{|z^{-1/p}|}^{\infty} e^{-\eta t^m \Re z/2} = \frac{K}{(\Re z)^{1/m}} \Gamma\left(\frac{1}{m}, \frac{\eta \Re z}{2|z^{\frac{m}{p}}|}\right),$$

with K and $\bar{K}$ positive constants independent on $|z|$. From the asymptotic behavior of the incomplete gamma function [6, eq. (8.11.2)] we deduce that

$$G_1^{\pm}(z) = \mathcal{O}\left(z^{\frac{m-1}{p}-1} e^{-\frac{\eta}{2} z^{1-\frac{m}{p}}}\right), \qquad \text{as } |z| \to \infty. \tag{61}$$

On the other hand

$$G_2^{\pm}(z) := \int_{\delta^{\pm}}^{b^{\pm}} e^{-z\eta t^m} e^{-zf_m(t_0\pm t)} g(t_0 \pm t)(t_0 \pm t)^{a-1}dt = \int_{\delta^{\pm}}^{b^{\pm}} e^{-z[f(t_0\pm t)-f(t_0)]} g(t_0 \pm t)(t_0 \pm t)^{a-1}dt.$$

And then,

$$|G_2^{\pm}(z)| \leq e^{-\epsilon^{\pm}\Re z} \int_{\delta^{\pm}}^{b^{\pm}} e^{-\Re z\left[f(t_0\pm t)-f(t_0)-\epsilon^{\pm}\right]}|g(t_0 \pm t)(t_0 \pm t)^{a-1}|dt.$$

Using (60), we find that, for $\Re z > x_0$, $e^{-\Re z\left[f(t_0\pm t)-f(t_0)-\epsilon^{\pm}\right]} \leq e^{-x_0\left[f(t_0\pm t)-f(t_0)-\epsilon^{\pm}\right]}$. Taking also into account that the last integral above, with $\Re z$ replaced by x_0, is convergent by hypothesis H4.4, we conclude that

$$|G_2^{\pm}(z)| \leq K e^{-\epsilon^{\pm}\Re z},$$

with $K > 0$ independent on $|z|$. From this formula and (61) we find

$$F_1(z) = \mathcal{O}\left(e^{-\frac{\eta}{2} z^{1-\frac{m}{p}}} + e^{-z\epsilon^{\pm}}\right), \qquad \text{as } |z| \to \infty. \tag{62}$$

- $R_{n,0}^{\pm}(z)$. Recall the integral representation of $R_{n,0}^{\pm}(z)$ given in (53). The factor $r_n^{\pm}(x,z)$ is the Taylor remainder of $\tilde{h}^{\pm}(x,z)$ at $x = 0$. Then, $r_n^{\pm}(x,z)$ admits the Cauchy integral representation (12) with $f(w)$ replaced by $\tilde{h}^{\pm}(w,z)$:

$$r_n^{\pm}(x,z) = \frac{x^n}{2\pi i} \oint_{\mathcal{C}} \frac{\tilde{h}^{\pm}(w,z)}{w^n(w-x)} dw, \qquad x \in D_0(r),$$

for a certain $r > 0$ independent on $|z|$, and where we choose an integration path $\mathcal{C}$ that is a circle of center 0 and radius $2|z^{-1/p}|$ ($< r$ for large enough $|z|$), oriented in the positive sense. Since (see (52)) $R_z := 1 - e^{-|z^{-m/p}|} \simeq |z^{-m/p}|$ when $|z| \to \infty$, for sufficiently large $|z|$, both points 0 and x are contained inside the circle $\mathcal{C}$ for any $x \in [0, R_z^{1/m}]$.

Recall at this point that $\tilde{h}^{\pm}(w,z)$ is given in (46) and (42), and we can write

$$\tilde{h}^{\pm}(w) = e^{-zw^p\psi(w)}\varphi(w),$$

with

$$\psi(w) := w^{-p} f_m(t_0 \pm t(w)), \qquad t(w) := \frac{w}{\Lambda^{\pm}}\left[\frac{-\log(1-w^m)}{w^m}\right]^{\frac{1}{m}},$$

$$\varphi(w) := g(t_0 \pm t(w))\left(-\frac{\log(1-w^m)}{w^m}\right)^{\lambda^{\pm}/m-1}(t_0 \pm t(w))^{\mu^{\pm}-1}.$$

The functions $\psi(w)$ and $\varphi(w)$ are analytic in the disk $D_0(r)$ that contains the circle $\mathcal{C}$. Then, on the integration path $\mathcal{C}$ we have that

$$|\tilde{h}^{\pm}(w,z)| \leq e^{|z(2z^{-1/p})^p|\,|\psi(w)|}|\varphi(w)| = e^{2^p|\psi(w)|}|\varphi(w)| \leq \bar{K},$$

for some constant $\bar{K} > 0$ independent on $|z|$. For $w \in \mathcal{C}$ we also have $|w - x| \geq |z^{-1/p}|$ and $|w|^n = 2^n|z^{-n/p}|$. Then

$$|r_n^{\pm}(x,z)| \leq K x^n |z^{n/p}|,$$

with $K > 0$ independent on x and $|z|$. Then, from (53) we have

$$
\begin{aligned}
|R^{\pm}_{n,0}(z)| &\le \int_0^{R_z^{1/m}} (1-x^m)^{\Re\left(\frac{\eta z}{(\Lambda^{\pm})^m}\right)-1} x^{\Re\lambda^{\pm}-1} |r_n^{\pm}(x,z)| dx \\
&\le K|z^{n/p}| \int_0^{R_z^{1/m}} (1-x^m)^{\Re\left(\frac{\eta z}{(\Lambda^{\pm})^m}\right)-1} x^{n+\Re\lambda^{\pm}-1} dx \\
&= K|z^{n/p}| B_{R_z}\left(\frac{n+\Re\lambda^{\pm}}{m}, \Re\left(\frac{\eta z}{(\Lambda^{\pm})^m}\right)\right).
\end{aligned}
$$

Using the asymptotic behavior of the incomplete beta function [6, eq. (8.17.2), (8.17.4), (8.18.3)] we find

$$
R^{\pm}_{n,0}(z) = \mathcal{O}\left(z^{\frac{n}{p}-\frac{n+\lambda^{\pm}}{m}}\right), \qquad \text{as } |z| \to \infty. \tag{63}
$$

Finally, from (55), (58), (62) and (63) it follows that

$$
R^{\pm}_{n}(z) = \mathcal{O}\left(z^{\frac{n}{p}-\frac{n+\lambda^{\pm}}{m}}\right) + \text{exp. small terms}, \qquad \text{as } |z| \to \infty,
$$

and then, from (48),

$$
R_{n}(z) = \mathcal{O}\left(z^{\frac{n}{p}-\frac{n+\lambda^{+}}{m}} + z^{\frac{n}{p}-\frac{n+\lambda^{-}}{m}}\right) + \text{exp. small terms}, \qquad \text{as } |z| \to \infty. \quad \square
$$

Observation 1. *Observe that the first term inside the brackets in expansion (45) (and also $R_n^-(z)$ defined in the proof of Theorem 4) vanishes when $t_0 = 0$ (only the "+" terms remain).*

Observation 2. *As it has been pointed out before Theorem 4, the "extra" dependence of the function h on the asymptotic variable z introduces a new ingredient in the analysis that was not considered in Theorems 2 or 3. As it has been proved, this fact does not have any influence in the convergence of the expansions nor in the asymptotic character of expansion (45). As it is proved in Corollary 1 in the Appendix, the Taylor coefficients $A_k^{\pm}(z)$ are polynomials in z of degree $\lfloor \frac{k}{p} \rfloor$, where $p > m$ denotes the first derivative of $f(t)$ at t_0 that does not vanish after the mth derivative. The only effect of the dependence of $A_k^{\pm}(z)$ on the large variable z is that now, the asymptotic sequences $A_k^-(z)B_R((k+1)/m, \eta z)$ and $A_k^+(z)B((k+\lambda^+)/m, \eta z/(\Lambda^+)^m)$ in expansion (45) are no longer Poincaré sequences that decrease monotonically in the form $z^{-k/m}$, but sequences that decreases in the form of a sawtooth: we have, as $|z| \to \infty$, $A_k^-(z)B_R\left(\frac{k+1}{m}, \eta z\right) = \mathcal{O}\left(z^{\lfloor k/p \rfloor - \frac{k+1}{m}}\right)$ and $A_k^+(z)B\left(\frac{k+\lambda^+}{m}, \frac{\eta z}{(\Lambda^+)^m}\right) = \mathcal{O}\left(z^{\lfloor k/p \rfloor - \frac{k+\lambda^+}{m}}\right)$ (see [1] for further details).*

Example 4 (*Parabolic Cylinder Function*)**.** Consider the following integral representation of the parabolic cylinder function [13, Eq. (12).5.1],

$$
U(a,z) = \frac{e^{-z^2/4}}{\Gamma(a+1/2)} \int_0^{\infty} u^{a-1/2} e^{-\frac{u^2}{2}-zu} dt, \qquad \Re a > -\frac{1}{2}.
$$

Assume that $z < 0$ and perform the change of variable $u \mapsto t$, defined by $u = -zt$, to find the integral representation

$$
U(a,z) = \frac{(-z)^{a+1/2} e^{-z^2/4}}{\Gamma(a+1/2)} \int_0^{\infty} t^{a-1/2} e^{-z^2 f(t)} dt,
$$

with $f(t) = \frac{t^2}{2} - t = \frac{1}{2}(t-1)^2 - \frac{1}{2}$. The absolute minimum of $f(t)$ occurs at $t_0 = 1$ and $f(t)$ equals its asymptotically dominant part $p(t) = \frac{1}{2}(t-1)^2 - \frac{1}{2}$. Then, we split up the integral at $t = 1$ to find

$$
U(a,z) = \frac{(-z)^{a+1/2} e^{-z^2/4}}{\Gamma(a+1/2)} \left(U_1(a,z) + U_2(a,z)\right), \tag{64}
$$

with

$$
U_1(a,z) := e^{\frac{z^2}{2}} \int_0^1 t^{a-1/2} e^{-z^2 \frac{(t-1)^2}{2}} dt = e^{\frac{z^2}{2}} \int_0^1 (1-t)^{a-1/2} e^{-\frac{z^2}{2}t^2} dt = e^{\frac{z^2}{2}} \bar{U}_1\left(a, \frac{z^2}{2}\right), \tag{65}
$$

and

$$
U_2(a,z) := e^{\frac{z^2}{2}} \int_1^{\infty} t^{a-1/2} e^{-z^2 \frac{(t-1)^2}{2}} dt = e^{\frac{z^2}{2}} \int_0^{\infty} (t+1)^{a-1/2} e^{-\frac{z^2}{2}t^2} dt = e^{\frac{z^2}{2}} \bar{U}_2\left(a, \frac{z^2}{2}\right), \tag{66}
$$

with $\bar{U}_k\left(a, \frac{z^2}{2}\right)$, $k = 1, 2$, given in Examples 2 and 3. A convergent expansion for the parabolic cylinder function, that is also asymptotic for large $z < 0$, follows from (64), (65), (66) and the corresponding expansions for $\bar{U}_1$ and $\bar{U}_2$ given in formulas (33) and (40) respectively. ⊙

Observation 3. *Many special functions of the mathematical physics admit an integral representation of the form (41) and then, this method can applied be to obtain new series representations of those functions that have also an asymptotic property in a certain variable. This is subject of current research.*

Data availability

No data was used for the research described in the article.

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Appendix. Computation of the Taylor coefficients $A_k^{\pm}(z)$

Lemma 1. *For any $m \in \mathbb{N}$, let $\phi(t)$ be an analytic function in the region $S_m(q, 0)$ defined in (23), for some $q > 0$. Consider the function*

$$\tilde{\phi}(x) := \left(-\frac{\log(1-x^m)}{x^m}\right)^{\lambda} \phi\left(x\left[-\frac{\log(1-x^m)}{x^m}\right]^{1/m}\right), \qquad \lambda \in \mathbb{C}. \tag{67}$$

Then, $\tilde{\phi}_n$, the nth Taylor coefficient of $\tilde{\phi}$ at $x = 0$, is given by the formula

$$\tilde{\phi}_n = \sum_{k=0}^{\lfloor \frac{n}{m} \rfloor} (-1)^k \frac{B_k^{(\lambda+1+\frac{n}{m})}(1)}{k!} \frac{\phi^{(n-km)}(0)}{(n-km)!}, \tag{68}$$

where $B_k^{(\alpha)}(x)$ are the generalized Bernoulli polynomials of order α [10, Sec. 24.16], [11, Ch. VI], [12].

Proof. The function $\tilde{\phi}(x)$ is analytic in the open disk $D_0(\rho)$, for a certain $0 < \rho < 1$. From Cauchy's formula for the nth derivative of an analytic function we have

$$\tilde{\phi}^{(n)}(0) = \frac{n!}{2\pi i} \oint_C \left(-\frac{\log(1-x^m)}{x^m}\right)^{\lambda} \phi\left(x\left[-\frac{\log(1-x^m)}{x^m}\right]^{1/m}\right) \frac{dx}{x^{n+1}}, \tag{69}$$

where C is a circle of center 0 and radius $\rho < 1$ that does not contain any singularity of $\phi\left(x\left[-\frac{\log(1-x^m)}{x^m}\right]^{1/m}\right)$, and oriented in the positive sense. Consider the change of variables $x \mapsto t$ given by $t = x\left[-\frac{\log(1-x^m)}{x^m}\right]^{1/m}$ with inverse $x = t\left(\frac{1-e^{-t^m}}{t^m}\right)^{1/m}$. We find

$$\tilde{\phi}^{(n)}(0) = \frac{n!}{2\pi i} \oint_\gamma \phi(t) e^{-t^m} \left(\frac{-t^m}{e^{-t^m}-1}\right)^{\lambda+1+\frac{n}{m}} \frac{dt}{t^{n+1}}, \tag{70}$$

where γ is the image of C by this transformation, that is, the boundary of the region $S_m((-\log(1-\rho^m))^{1/m}, 0)$, which is a closed curve encircling $t = 0$ in the positive sense. Then, for small enough ρ, $S_m((-\log(1-\rho^m))^{1/m}, 0) \subset S_m(q, 0)$ and Cauchy's theorem implies that

$$\tilde{\phi}^{(n)}(0) = \frac{d^n}{dt^n}\left[\phi(t) e^{-t^m} \left(\frac{-t^m}{e^{-t^m}-1}\right)^{\lambda+1+\frac{n}{m}}\right]_{t=0}. \tag{71}$$

Furthermore, we have that

$$\frac{d^k}{dt^k}\left[e^{-t^m} \left(\frac{-t^m}{e^{-t^m}-1}\right)^{\lambda+1+\frac{n}{m}}\right]_{t=0} = \begin{cases} 0 & \text{if } k \neq 0 \pmod m, \\ (-1)^{k/m} \frac{k!}{(k/m)!} B_{\frac{k}{m}}^{(\lambda+1+\frac{n}{m})}(1) & \text{if } k = 0 \pmod m, \end{cases} \tag{72}$$

which follows from the generating function of the generalized Bernoulli polynomials [10, Sec. 24.16], and the kth derivative at $x = 0$ of a composite function of the form $g(x) = f(x^m)$. Then, (68) follows after applying Leibniz's formula to the right hand side of (71). □

A formula for the Taylor coefficients $A_n^{\pm}(z)$ of the function $\tilde{h}^{\pm}(x, z)$ considered in Theorem 4 follows from this result and Leibniz's formula for the derivative of a product. In particular, we have the following lemma.

Corollary 1. *Let $f(t)$ and $g(t)$ be analytic functions at $t = t_0$. Define $h(t, z) := e^{-zf_m(t)}g(t)$ and for any $\Lambda^{\pm} \in \mathbb{C}$ consider the function $\tilde{h}^{\pm}(x, z)$ defined in (46). Then, the nth coefficient $A_n^{\pm}(z)$ of the Taylor expansion of $\tilde{h}^{\pm}(x, z)$ at $x = 0$ is a polynomial in z of degree $\lfloor \frac{n}{p} \rfloor$, where $p > m$ is the first non-zero derivative of $f(t)$ at $t = t_0$ after the mth derivative. This means that $A_n^{\pm}(z) = \mathcal{O}\left(z^{\lfloor n/p \rfloor}\right)$ as $|z| \to \infty$.*

Proof. The Taylor coefficients of $\tilde{h}^{\pm}(x, z)$ are given by (47). There, the variable z only appears in the coefficients $h_j(z)$, that are polynomials in z of degree $\lfloor j/p \rfloor$, as shown in [1]. Taking into account the range of the index of summation we conclude that $A_n^{\pm}(z)$ is a polynomial in z of degree $\lfloor n/p \rfloor$. □

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