On some classes of nullnorms and h-pseudo homogeneity

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Abstract

Nullnorms are aggregation functions which generalize t-norms and t-conorms. For each nullnorm there exists an annihilator element such that, below it, the function behaves like a t-conorm, and, above it, like a t-norm. In this paper, we study some classes of nullnorms which naturally arise from well known classes of t-norms and t-conorms, such as idempotent, Archimedean, cancellative, positive and nilpotent t-norms and t-conorms. We also present the concept of h-pseudo-homogeneous nullnorm and we study when these classes of nullnorms fullfill it.

Keywords: T-norms; T-conorms; Nullnorms; Pseudo-homogeneity.

1. Introduction

Data aggregation are processes or methods that gather and summarize information. A way of formalizing the aggregation of data given in terms of real numbers (usually with values in $[0,1]$) is by means of the so-called aggregation functions. Aggregation functions are increasing functions satisfying two boundary conditions ($f(0,\dots,0)=0$ and $f(1,\dots,1)=1$). There are several classes of aggregation functions, such as, triangular norms (t-norms), triangular conorms (t-conorms), Choquet integrals, copulas, overlap functions and nullnorms, among others.

The class of nullnorms is a specific class of aggregation functions that includes t-norms and t-conorms. Since its introduction by Calvo et al [10], nullnorms have proved themselves very useful in neural networks [16, 21] and decision making [26]. Furthermore, nullnorms are also interesting from a theoretical point of view, because their structure is a combination of a t-norm and a t-conorm. Each nullnorm has an annihilator element such that, below the annihilator, the function behaves like a t-conorm, and, above the annihilator, it behaves like a t-norm. Intensive research on nullnorms has been made recently, for example, about the migrativity property in [27, 30, 39], the commuting property in [22] and the generation of orders from lattice-valued nullnorms in [18, 19].

Besides, homogeneity is an analytical property studied for many years (see, for example, [2]). Homogeneity and some generalizations of this notion have been applied in image processing and classification problems [8, 9, 11, 23]. In [37], Xie et al. introduced the idea of pseudo-homogeneity as a generalization of homogeneity and study which classes of t-norms, t-conorms and proper uninorms are pseudo-homogeneous. Since this seminal work, pseudo-homogeneity has been studied for overlap and grouping functions[1] in [29], and for t-subnorms in [25]. However, as we prove, there is no proper pseudo-homogeneous nullnorm. For overcoming this limitation, we propose a new kind of homogeneity, namely h-pseudo homogeneity, which generalizes pseudo-homogenity for t-norms and t-conorms.

In this paper, we introduce some new classes of nullnorms which are based on well-known classes of t-norms and t-conorms. We study their properties and characteristics. We also investigate h-pseudo homogeneity for these classes of nullnorms.

The structure of this paper is as follows. In Section 2, we show some preliminary concepts and results that are necessary for the remainder of the paper. In Section 3, we discuss the definition of nullnorm, some of its properties and new classes of nullnorms. In section 4, we introduce the concept of h-pseudo homogeneous and provide a characterization the nullnorms which are h-pseudo homogeneous. Finally, in Section 5, some conclusions are presented.

2. Preliminaries

In this section, we recall some concepts and results which will be used in the text.

Definition 2.1. *[20] A function $T : [0,1]^2 \to [0,1]$ is a triangular norm, t-norm for short, if it is associative, increasing, commutative and*

$$T(x,1) = x \text{ for all } x \in [0,1]. \tag{1}$$

Example 2.1. *The minimum, product, Lukasiewicz and drastic t-norms are defined for all $x, y \in [0,1]$, respectively, as $T_M(x,y) = \min(x,y)$, $T_P(x,y) = xy$, $T_L(x,y) = \max(x+y-1, 0)$,*

$$T_D(x,y) = \begin{cases} \min(x,y) & , \text{ if } \max(x,y) = 1; \\ 0 & , \text{ otherwise.} \end{cases}$$

In addition, we have the family of Schweizer-Sklar t-norms, defined for each $\beta < 0$ as

$$T_{SS}^{\beta}(x,y) = \begin{cases} (x^{\beta} + y^{\beta} - 1)^{\frac{1}{\beta}} & , \text{ if } \min(x,y) > 0; \\ 0 & , \text{ otherwise.} \end{cases}$$

Definition 2.2. *[20] A function $S : [0,1]^2 \to [0,1]$ is a triangular conorm, t-conorm for short, when is associative, increasing, commutative and*

$$S(x,0) = x \text{ for all } x \in [0,1]. \tag{2}$$

Example 2.2. *The maximum, probabilistic sum, Lukasiewicz (also called bounded sum) and drastic sum t-conorms are defined for all $x, y \in [0,1]$, respectively, as $S_M(x,y) = \max(x,y)$, $S_P(x,y) = x + y - xy$, $S_L(x,y) = \min(x+y, 1)$ and*

$$S_D(x,y) = \begin{cases} \max(x,y) & , \text{ if } \min(x,y) = 0; \\ 1 & , \text{ otherwise.} \end{cases}$$

Here we present some classes of t-norms and t-conorms which can be found in the literature. They will serve as a basis for the development of this work.

Definition 2.3. *[20] Let T be a t-norm. Then*

[1]For more details on overlaps and grouping functions see [8, 12, 13, 28].

1. T is called strict if it is continuous and, for each $x, y, z \in [0,1]$ with $0 < x$ and $y < z$, we have that $T(x,y) < T(x,z)$;
2. T is called idempotent if $T(x,x) = x$ for each $x \in [0,1]$;
3. T is called nilpotent if it is continuous and, for each $x \in (0,1)$, there is $n \in \mathbb{N}$ such that $x_T^{(n)} = 0$, where $x_T^{(0)} = 1$ and $x_T^{(n+1)} = T(x, x_T^{(n)})$;
4. T is called positive if it holds that $T(x,y) = 0$ if and only if $x = 0$ or $y = 0$;
5. T is called Archimedean if, for all $x, y \in (0,1)$, there exists $n \in \mathbb{N}$ such that $x_T^{(n)} < y$;
6. T is called cancellative if, for all $x, y, z \in [0,1]$, $T(x,y) = T(x,z) \Rightarrow x = 0$ or $y = z$.

Notice that positive t-norms are just t-norms with no zero divisors, i.e., t-norms for which there is no element $x \in (0,1]$ such that $T(x,y) = 0$ for some $y \in (0,1]$.

Definition 2.4. *[20] Let S be a t-conorm. Then*

1. S is called strict if itis continuous and, for each $x, y, z \in [0,1]$ with $x < 1$ and $y < z$, we have that $S(x,y) < S(x,z)$;
2. S is called idempotent if $S(x,x) = x$ for each $x \in [0,1]$;
3. S is called nilpotent whenever it is continuous and, for each $x \in (0,1)$, there is $n \in \mathbb{N}$ such that $x_S^{(n)} = 1$, where $x_S^{(0)} = 0$ and $x_S^{(n+1)} = S(x, x_S^{(n)})$;
4. S is called positive if it holds that $S(x,y) = 1$ if and only if $x = 1$ or $y = 1$;
5. S is called Archimedean if, for all $x, y \in (0,1)$, there exists $n \in \mathbb{N}$ such that $x_S^{(n)} > y$;
6. S is called cancellative if, for all $x, y, z \in [0,1]$, $S(x,y) = S(x,z) \Rightarrow x = 1$ or $y = z$.

Remark 2.1. *[20] The unique idempotent t-norm is T_M and the unique idempotent t-conorm is S_m.*

In addition, $k \in [0,1]$ is an annihilator element of a function $F : [0,1]^2 \to [0,1]$ if, for each $x \in [0,1]$, $F(x,k) = F(k,x) = k$. Thus, by monotonicity and from (1) and (2), we have that 0 and 1 are annihilator elements for t-norms and t-conorms, respectively.

Bijective and increasing functions $\rho : [0,1] \to [0,1]$ are called automorphisms. As it is well known, the set of automorphisms with the composition is a group. The action of an automorphism on a function $F : [0,1]^n \to [0,1]$ is the function $F^\rho : [0,1]^n \to [0,1]$ defined by

$$F^\rho(x_1, \ldots, x_n) = \rho^{-1}(F(\rho(x_1), \ldots, \rho(x_n))) \textit{ for all } x, y \in [0,1], \tag{3}$$

also called conjugate of F or isomorphic to F [31].

Definition 2.5. *[37] Let $F : [0,1]^2 \to [0,1]$ be a continuous and increasing function such that $F(x,1) = 0 \Leftrightarrow x = 0$ and let T be a t-norm. Then T is pseudo-homogeneous with respect to F whenever $T(\lambda x, \lambda y) = F(\lambda, T(x,y))$ for all $x, y, \lambda \in [0,1]$.*

Lemma 2.1. *[37, Lemma 1] Each pseudo-homogeneous t-norm is continuous.*

Theorem 2.2. *[37, Theorem 3.1] Let T be a t-norm and $F : [0,1]^2 \to [0,1]$ be a continuous and increasing function such that $F(x,1) = 0 \Leftrightarrow x = 0$. Then, T is a pseudo-homogeneous t-norm with respect to F and F is commutative if and only if $T = T_M$ and $F = T_P$.*

Theorem 2.3. *[37, Theorem 3.2] Let T be a t-norm and $F : [0,1]^2 \to [0,1]$ be a continuous and increasing function such that $F(x,1) = 0 \Leftrightarrow x = 0$. T is a pseudo-homogeneous t-norm with respect to F satisfying $T(x,x) < x$ for each $x \in [0,1]$ if and only if one of the following items holds.*

1. *$T = T_P$ and $F(x,y) = x^2 y$ for each $x, y \in [0,1]$;*

2. *for some* $\beta > 0$*,* T *and* F *are defined for each* $x, y \in [0,1]$ *as follows:*

$$T(x,y) = \begin{cases} (x^\beta + y^\beta - 1)^{\frac{1}{\beta}} & , \textit{if } x,y > 0; \\ 0 & , \textit{otherwise.} \end{cases}$$

$$F(x,y) = \begin{cases} ((xy)^\beta + x^\beta - 1)^{\frac{1}{\beta}} & , \textit{if } x,y > 0; \\ 0 & , \textit{otherwise.} \end{cases}$$

Theorem 2.4. *[37, Theorem 3.3] If* T *is a pseudo-homogeneous t-norm, then* T *can not be a non-trivial ordinal sum of continuous Archimedean t-norms.*

Definition 2.6. *[37] Let* $F : [0,1]^2 \to [0,1]$ *be an increasing function and let* S *be a t-conorm. Then* S *is pseudo-homogeneous with respect to* F *whenever* $S(\lambda x, \lambda y) = F(\lambda, S(x,y))$ *for all* $x, y, \lambda \in [0,1]$*.*

Theorem 2.5. *[37, Theorem 4.1] Let* S *be a t-norm and* $F : [0,1]^2 \to [0,1]$ *be an increasing function. Then* S *is a pseudo-homogeneous t-conorm with respect to* F *if and only if* $S = S_m$ *and* $F = T_P$*.*

3. Nullnorms

Nullnorms are generalizations of t-norms and t-conorms defined as follows.

Definition 3.1. *[10] A nullnorm is a function* $V : [0,1]^2 \to [0,1]$ *which is associative, increasing, commutative, and has an annihilator element* $k \in [0,1]$ *such that*

$$V(0,x) = x \textit{ for all } x \le k, \tag{4}$$
$$V(1,x) = x \textit{ for all } x \ge k. \tag{5}$$

Remark 3.1. *Observe for* $k = 0$*, we recover t-norms, while for* $k = 1$ *we recover t-conorms. Thus, each t-norm and each t-conorm is a nullnorm. Nullnorms with annihilator element* $k \in (0,1)$ *are called proper nullnorms.*

The following theorem shows that nullnorms are built up from a t-norm, a t-conorm and an annihilator element.

Theorem 3.1. *[10, Prop. 2] Let* $k \in [0,1]$*. A function* $V : [0,1]^2 \to [0,1]$ *is a nullnorm with annihilator element* k *if and only if there exist a t-norm* T *and a t-conorm* S *such that*

$$V(x,y) = \begin{cases} S^*(x,y) & , \textit{if } (x,y) \in [0,k)^2; \\ T^*(x,y) & , \textit{if } (x,y) \in (k,1]^2; \\ k & \textit{otherwise} , \end{cases} \tag{6}$$

where for each $x, y \in [0,k)$*,* $S^*(x,y) = \varphi^{-1}(S(\varphi(x), \varphi(y)))$ *with* $\varphi(x) = x/k$ *and for each* $x, y \in (k,1]$*,* $T^*(x,y) = \psi^{-1}(T(\psi(x), \psi(y)))$ *with* $\psi(x) = (x-k)/(1-k)$*. Furthermore, if* $k \neq 1$*, then the underlying t-norm of* V *is defined in Eq. (7); and, if* $k \neq 0$*, the underlying t-conorm of* V *is defined in Eq. (8).*

$$T_V(x,y) = \frac{V(x(1-k)+k, y(1-k)+k)-k}{1-k} \textit{ for all } x, y \in [0,1] \tag{7}$$

$$S_V(x,y) = \frac{V(kx,ky)}{k} \textit{ for all } x, y \in [0,1]. \tag{8}$$

Remark 3.2. *Observe that, for each proper nullnorm V, there exist a unique t-conorm S, a unique t-norm T and a unique $k \in (0,1)$ such that Equation (6) holds. In addition, if $k = 1$, Equation (6) holds for a unique t-conorm S and for any t-norm T. In fact, in this case $V = S$. Analogously, if $k = 0$, Equation (6) holds for a unique t-norm T and for any t-conorm S since $V = T$. A triple (S,T,k) satisfying Equation (6) for a nullnorm V will be called of generator triple of V. For the sake of clarity, we will also use the notation $V_{S,T,k}$ for meaning that $V_{S,T,k}$ is a nullnorm with (S,T,k) as generator triple.*

The following corollaries are straightforward from the characterization Theorem 3.1.

Example 3.1. *Let $k \in [0,1]$. Then the nullnorm having (S_M, T_M, k) as generator triple is:*

$$V_M^k(x,y) = \begin{cases} \max(x,y) & \text{, if } (x,y) \in [0,k]^2; \\ \min(x,y) & \text{, if } (x,y) \in [k,1]^2; \\ k & \text{, otherwise.} \end{cases} \tag{9}$$

The nullnorm having (S_P, T_P, k) as generator triple is:

$$V_P^k(x,y) = \begin{cases} x + y - \frac{xy}{k} & \text{, if } (x,y) \in [0,k]^2; \\ \frac{(x-k)(y-k)}{1-k} + k & \text{, if } (x,y) \in [k,1]^2; \\ k & \text{, otherwise.} \end{cases} \tag{10}$$

The nullnorm having (S_L, T_L, k) as generator triple is:

$$V_L^k(x,y) = \begin{cases} x + y & \text{, if } x + y < k; \\ x + y - 1 & \text{, if } x + y > 1 + k; \\ k & \text{, otherwise.} \end{cases} \tag{11}$$

The nullnorm having (S_D, T_D, k) as generator triple is:

$$V_D^k(x,y) = \begin{cases} \max(x,y) & \text{, if } \min(x,y) = 0 \text{ and } \max(x,y) < k; \\ \min(x,y) & \text{, if } \max(x,y) = 1 \text{ and } \min(x,y) > k; \\ k & \text{, otherwise.} \end{cases} \tag{12}$$

Observe that a nullnorm can be composed from any pair of a t-norm and a t-conorm, which need not to be dual. For example,

$$V_{MP}^k(x,y) = \begin{cases} \max(x,y) & \text{, if } (x,y) \in [0,k]^2; \\ \frac{(x-k)(y-k)}{1-k} + k & \text{, if } (x,y) \in [k,1]^2; \\ k & \text{, otherwise,} \end{cases} \tag{13}$$

is the nullnorm which has (S_M, T_P, k) as generator triple.

Corollary 3.1. *([10]) Let V be a nullnorm with $k \in [0,1]$ as annihilator element. Then, for each $x \in [0,1]$*

1. $V(0,x) = \min(x,k)$;
2. $V(1,x) = \max(x,k)$.

Remark 3.3. *Let V be a nullnorm with (S,T,k) as generator triple. From Remark 3.2, we observe that, if $k = 0$, then $T = T_V$; and, if $k = 1$ then $S = S_V$. In addition, if $k \in (0,1)$, then $S = S_V$ and $T = T_V$. Therefore, (S_V, T_V, k) is the generator triple of V.*

3.1. Conjugates of nullnorms

The action of automorphisms (in an algebraic sense [36]) over a class of operators generates, in general, operators of the same class. In particular, it is so for the main classes of operators in fuzzy theory, s.t. aggregation functions, t-norms, t-conorms, overlap functions, fuzzy implications, fuzzy negations, copulas, etc. In the following proposition we prove that this is also the case for nullnorms.

Proposition 3.1. *Let $V : [0,1]^2 \to [0,1]$ be a nullnorm with $k \in [0,1]$ as annihilator element. Then, for each automorphism ρ, V^ρ is also a nullnorm with annihilator element $\rho^{-1}(k)$.*

Proof: Since V is commutative and increasing and both ρ and ρ^{-1} are also increasing, trivially, V^ρ is commutative and increasing. Let $x, y, z \in [0,1]$. Then:

$$\begin{aligned} V^\rho(x, V^\rho(y,z)) &= \rho^{-1}(V(\rho(x), \rho(\rho^{-1}(V(\rho(y), \rho(z)))))) \\ &= \rho^{-1}(V(V(\rho(x), \rho(y)), \rho(z))) \\ &= V^\rho(V^\rho(x,y), z), \end{aligned}$$

and therefore, V^ρ is associative.
As $V^\rho(x, \rho^{-1}(k)) = \rho^{-1}(V(\rho(x), k)) = \rho^{-1}(k)$, then $\rho^{-1}(k)$ is an annihilator element of V^ρ.
If $x \leq \rho^{-1}(k)$, then, since ρ is increasing, $\rho(x) \leq k$ and therefore, $V^\rho(0,x) = \rho^{-1}(V(0,\rho(x))) = \rho^{-1}(\rho(x)) = x$. Analogously, if $x \geq \rho^{-1}(k)$, since ρ is increasing, $\rho(x) \geq k$, and therefore, $V^\rho(1,x) = \rho^{-1}(V(1,\rho(x))) = \rho^{-1}(\rho(x)) = x$. □

Note that if (S,T,k) is the generator triple of V, then $(S^\rho, T^\rho, \rho^{-1}(k))$ can not be the generator triple of the nullnorm V^ρ. For example, take the automorphism $\rho(x) = \sin\left(\frac{\pi}{2}x\right)$ and $k = 0.5$. Then $(V_L^k)^\rho(0.2, 0.2) = \rho^{-1}(V_L^k(0.1\pi, 0.1\pi)) = \rho^{-1}(k) = \frac{1}{3}$ and $(V_L^k)^{(\rho)}(0.2,0.2) = \rho^{-1}(k) S_L^\rho\left(\frac{0.2}{\rho^{-1}(k)}, \frac{0.2}{\rho^{-1}(k)}\right) = \frac{\rho^{-1}(S_L(\rho(0.6), \rho(0.6)))}{3} \approx 0.0067$. Therefore, $(V_L^{0.5})^\rho \neq (V_L^{0.5})^{(\rho)}$. But, by Theorem 3.1, $(S^\rho, T^\rho, \rho^{-1}(k))$ is the generator triple of some nullnorm. This leads to the following definition.

Definition 3.2. *Let V be a nullnorm having (S,T,k) as generator triple. A nullnorm with $(S^\rho, T^\rho, \rho^{-1}(k))$ as generator triple for some automorphism ρ is called indirect conjugate nullnorm of V and it is denoted by $V^{(\rho)}$.*

Remark 3.4. *For each nullnorm V and automorphism ρ, $V = (V^\rho)^{\rho^{-1}} = (V^{(\rho)})^{(\rho^{-1})}$.*

Clearly if $k = 0$ or $k = 1$, then $V^\rho = V^{(\rho)}$ for any automorphism ρ. The following proposition gives sufficient conditions for the conjugate and the indirect conjugate of a proper nullnorm to agree.

Proposition 3.2. *Let V be a nullnorm with $k \in (0,1)$ as annihilator element and let ρ be an automorphism such that $\rho(\rho^{-1}(k)x) = k\rho(x)$ for each $x \in [0,1]$. Then $V^\rho = V^{(\rho)}$ if and only if $T_{V^\rho} = (T_V)^\rho$.*

Proof: First, observe that for each $x \in [0,1]$, $\rho^{-1}(x) = \frac{\rho^{-1}(kx)}{\rho^{-1}(k)}$.
($\Rightarrow$) Direct.
($\Leftarrow$) By Eq. (8), for each $x, y \in [0,1]$, we have that

$$\begin{aligned} S_{V^\rho}(x,y) &= \frac{V^\rho(\rho^{-1}(k)x, \rho^{-1}(k)y)}{\rho^{-1}(k)} \\ &= \frac{\rho^{-1}(V(k\rho(x), k\rho(y)))}{\rho^{-1}(k)} \\ &= \rho^{-1}\left(\frac{V(k\rho(x), k\rho(y)))}{k}\right) \\ &= (S_V)^\rho(x,y). \end{aligned}$$

Since $T_{V^\rho} = (T_V)^\rho$, we have that $(S_{V^\rho}, T_{V^\rho}, \rho^{-1}(k)) = ((S_V)^\rho, (T_V)^\rho, \rho^{-1}(k))$ and hence $V^\rho = V^{(\rho)}$. □

The Proposition 3.2 provides a sufficient condition to guarantee that $V^\rho = V^{(\rho)}$ for a proper nullnorm V. Note that automorphisms of the form $\rho(x) = x^r$, for some real number $r > 0$, verify such condition. However, this condition is not necessary. In fact, for any automorphism ρ and $k \in [0,1]$, we have that $(V_M^k)^\rho = V_M^{\rho^{-1}(k)}$ and therefore $(S_M, T_M, \rho^{-1}(k))$ is the generator triple of $(V_M^k)^\rho$. This implies that $(V_M^k)^\rho = (V_M^k)^{(\rho)}$ for any automorphism ρ. Analogously, for any automorphism ρ we have that $(V_D^k)^\rho = (V_D^k)^{(\rho)}$.

Corollary 3.2. *Let S be a t-conorm, $k \in [0,1)$ and take $\rho(x) = x^r$ for some fixed real number r. Then, $(V_{(S,T_D,k)})^\rho = (V_{(S,T_D,k)})^{(\rho)}$ and $(V_{(S,T_M,k)})^\rho = (V_{(S,T_M,k)})^{(\rho)}$.*

3.2. Idempotent nullnorms

In this subsection, we recall the notion of idempotent nullnorm and some well known facts related to it.

Definition 3.3. *[7] A nullnorm V is idempotent if $V(x,x) = x$ for all $x \in [0,1]$.*

Remark 3.5. *For all $k \in [0,1]$, the unique idempotent nullnorm with k as annihilator element is V_M^k [7, 15]. Therefore, a nullnorm is idempotent if and only if its underlying t-norm and t-conorm are idempotent. It is worth to note that any conjugate of V_M^k is equal to V_M^k. The indirect conjugate of V_M^k is idempotent only for the identity automorphism.*

3.3. Nilpotent nullnorms

In this subsection we introduce the notion of nilpotent nullnorm and comment some trivial facts about it.

An element $x \in (0,1)$ is a nilpotent element of a nullnorm V with annihilator element $k \in [0,1]$ if there exists $n \in \mathbb{N}$ such that $x_V^{(n)} = k$, where $x_V^{(0)} = x$ and $x_V^{(n+1)} = V(x, x_V^{(n)})$. A nullnorm V which is continuous and every $x \in (0,1)$ is a nilpotent element of V is called of Nilpotent.

Observe that each $x \in (0,1)$ is a nilpotent element of V_D^k, trivially. But V_D^k is not a nilpotent nullnorm because it is not continuous.

Remark 3.6. *Given a nullnorm V and an automorphism ρ the following statements are equivalent:*

1. *V is nilpotent;*
2. *S_V (case $k \neq 0$) and T_V (case $k \neq 1$) are nilpotent;*
3. *V^ρ is nilpotent;*
4. *$V^{(\rho)}$ is nilpotent.*

3.4. Positive nullnorms

In this subsection, we introduce the notion of positive nullnorm, as a natural generalization of positive t-norms and positive t-conorms.

Observe that, from Theorem 3.1, if V is a nullnorm with an annihilator element $k \in [0,1]$ and $\min(x,y) \leq k \leq \max(x,y)$ then $V(x,y) = k$. However, there may exist $x, y \in (k,1]$ or $x, y \in [0,k)$ such that $V(x,y) = k$. This fact leads to the following definition.

Definition 3.4. *Let V be a nullnorm with an annihilator element $k \in [0,1]$. V is positive if*

$$V(x,y) = k \Leftrightarrow \min(x,y) \leq k \leq \max(x,y).$$

Remark 3.7. *Let V be a nullnorm with an annihilator element $k \in [0,1]$ and let ρ be an automorphism. The following statements are equivalent:*

1. *V is positive;*
2. *S_V (case $k \neq 0$) and T_V (case $k \neq 1$) are positive;*
3. *V^ρ is positive;*
4. *$V^{(\rho)}$ is positive.*

3.5. Archimedean nullnorms

In abstract algebra and analysis, an ordered algebraic structure is Archimedean if it has not "infinitely large" or "infinitely small" elements. The notion of Archimedean t-norm was studied in [34], and Archimedean overlap functions were considered in [12]. In the following, we introduce the notion of Archimedean nullnorm. In particular, we consider the underlying t-norm and t-conorm as well as the conjugates of Archimedean nullnorms.

Definition 3.5. *Let V be a nullnorm with an annihilator element $k \in [0,1]$. V is Archimedean if, for each $x, y \in (0,k)$, there exists $n \in \mathbb{N}$ such that $x_V^{(n)} > y$; and, for each $x, y \in (k,1)$, there exists $n \in \mathbb{N}$ such that $x_V^{(n)} < y$.*

Remark 3.8. *Let V be a nullnorm with an annihilator element $k \in [0,1]$ and let ρ be an automorphism. The following statements are equivalent:*

1. *V is Archimedean;*
2. *S_V (case $k \neq 0$) and T_V (case $k \neq 1$) are Archimedeans;*
3. *V^ρ is Archimedean;*
4. *$V^{(\rho)}$ is Archimedean.*

3.6. Cancellative nullnorms

In Mathematics, the notion of cancellativity is a generalization of the notion of invertibility. In an algebraic structure M with a binary commutative operation $\cdot$, the cancellation law holds if, for each $x, y, z \in M$, $x \cdot y = x \cdot z$ only if $y = z$. In the case of t-norms, since 0 is an annihilator, there is no t-norm verifying the cancellative law in the above sense. Nevertheless, the cancellative law has been adapted making an exception when x is the annihilator element. With this in mind, we give the following definition of cancellative law for nullnorms:

Definition 3.6. *Let V be a nullnorm with an annihilator element $k \in [0,1]$. V satisfies the cancellative law (or it is cancellative) if $V(x,y) = V(x,z) \Rightarrow y = z$ or $x = k$ whenever $x, y, z \in [0,k]$ or $x, y, z \in [k,1]$.*

Remark 3.9. *Let V be a nullnorm with an annihilator element $k \in [0,1]$ and let ρ be an automorphism. The following statements are equivalent:*

1. *V is cancellative;*
2. *S_V (case $k \neq 0$) and T_V (case $k \neq 1$) are cancellative;*
3. *V^ρ is cancellative;*
4. *$V^{(\rho)}$ is cancellative.*

4. Pseudo-homogeneous nullnorms

Homogeneous functions are those which preserve a multiplicative scaling. That is, if all the arguments are multiplied by a fixed factor, then their output is also multiplied by the same factor (up to a power) [32]. The only homogenous t-norms and t-conorms are T_P, T_M and S_M [1]. In [37], it was proposed a flexibilization of the notion of homogeneity of functions by means of the concept of pseudo-homogeneous function for t-norms and t-conorms. It was shown that the class of pseudo-homogeneous t-norms is greater than the class of homogeneous t-norms (c.f. Theorems 2.2 and 2.3). However, the class of pseudo-homogeneous t-conorms is the same as the class of homogeneous t-conorms, i.e. the unique pseudo-homogeneous t-conorm is S_M (c.f. Theorem 2.5). This notion was also adapted and investigated for other classes of aggregation functions, such as overlap functions in [29], t-subnorms in [25] or even interval-valued uninorms, t-norms and t-conorms in [23, 24].

In this section, we consider the definition of pseudo-homogeneous nullnorms and a generalization of them.

Definition 4.1. *Let V be a nullnorm. V is called pseudo-homogeneous if it satisfies*

$$V(\lambda x, \lambda y) = G(\lambda, V(x,y)) \text{ for all } x, y, \lambda \in [0,1],$$

for a function $G : [0,1]^2 \to [0,1]$.

Since t-norms and t-conorms are nullnorms, this notion can also be used for t-norms and t-conorms. But, it must be noted that the pseudo-homogeneity proposed by [37] for t-norms, is more restrictive, because it requires the continuity of G and the condition $G(x,1) = 0 \Leftrightarrow x = 0$. Therefore, each pseudo-homogeneous t-norm and t-conorm in the sense of [37] is a pseudo-homogeneous nullnorm. However, are there proper nullnorms, i.e. nullnorms which are neither t-norms nor t-conorms, and which are pseudo-homogeneous for some function G?

The following proposition shows that the answer to the previous question is negative.

Proposition 4.1. *Let V be a proper nullnorm with an annihilator element $k \in (0,1)$. Then V is not pseudo-homogenous for any $G : [0,1]^2 \to [0,1]$.*

Proof: Suppose that V is pseudo-homogenous for some $G : [0,1]^2 \to [0,1]$. Then, on the one hand, by Definition 4.1 and Theorem 3.1, $G(k,k) = G(k, V(1,k)) = V(k, k^2) = k$; and, on the other hand, by Definition 4.1, $G(k,k) = G(k, V(0,k)) = V(0, k^2) = k^2$, which is a contradiction. □

This lack of pseudo-homogeneous proper nullnorms leads us to the introduction of a broader notion of pseudo-homogeneity for nullnorms. A first natural approach to such definition is based on considering the pseudo-homogeneity of the underlying t-norm and t-conorm.

Definition 4.2. *Let V be a nullnorm with an annihilator element $k \in [0,1]$. Then V is called underlying pseudo-homogeneous if T_V and S_V are pseudo-homogeneous for some function $F : [0,1]^2 \to [0,1]$ and $G : [0,1]^2 \to [0,1]$, respectively.*

Obviously, if V is a t-norm or a t-conorm, this notion coincides with the definitions 2.5 and 2.6. The following theorem provides a necessary condition for a nullnorm to be underlying pseudo-homogeneous.

Theorem 4.1. *Let V be a proper nullnorm. If V is underlying pseudo-homogeneous, then $S_V = S_M$ and T_V is continuous and can not be a non-trivial ordinal sum of continuous Archimedean t-norms.*

Proof: Straightforward from Lemma 2.1 and Theorems 3.1, 2.4 and 2.5. □

Although more general than Definition 4.1, Definition 4.2 is very limited, because the underlying t-conorm of an underlying pseudo-homogeneous proper nullnorm must be S_M and only a restricted family of t-norms can be considered. In addition, Definition 4.2 is not faithfull with the original notion of homogeneity where a function f is homogeneous if $f(\lambda x, \lambda y) = \lambda f(x,y)$ and weakening of this definition which, in general, are given substituting the two product by functions satisfying some conditions. This fact justifies the introduction of a more general definition.

Definition 4.3. *A nullnorm $V_{S,T,k} : [0,1]^2 \to [0,1]$ is called h-pseudo-homogeneous for a function $G : [0,1]^2 \to [0,1]$ if for each $x, y, \lambda \in [0,1]$ it satisfies*

$$V_{S,T,k}(\lambda x, \lambda y) = G(\lambda, V_{S,T,h(k,\lambda)}(x,y)) \tag{14}$$

where $h : [0,1]^2 \to [0,1]$ is the function

$$h(k,\lambda) = \begin{cases} 1 & , \text{ if } \lambda \leq k; \\ \frac{k}{\lambda} & , \text{ otherwise.} \end{cases}$$

Observe that when $\lambda \neq 0$, then $h(k,\lambda) = \min(1, \frac{k}{\lambda})$.

Example 4.1. *Let $k \in [0,1]$.*

- V_M^k *is h-pseudo-homogeneous for* $G = T_P$*;*

- V_{MP}^k *is h-pseudo-homogeneous for*

$$G^k(a,b) = \begin{cases} ab & , \text{ if } b \leq \frac{k}{a}; \\ \frac{a^2b - ak(b+1)+k}{1-k} & , \text{ if } b > \frac{k}{a}. \end{cases} \tag{15}$$

Lemma 4.2. *Let V be a nullnorm with an annihilator element $k \in [0,1]$. If V is h-pseudo-homogeneous with respect to a function $G : [0,1]^2 \to [0,1]$, then for each $x, y \in [0,1]$,*

$$G(x,y) = \begin{cases} xy & \text{if } y \leq \frac{k}{x} \\ V(x,xy) & \text{if } \frac{k}{x} < y \end{cases}$$

Proof: If $y \leq \frac{k}{x}$ then $G(x,y) = G(x, V_{S,T,1}(0,y)) = G(x, V_{S,T,h(k,x)}(0,y)) = V_{S,T,k}(0,xy) = S^*(0,xy) = xy$. If $\frac{k}{x} < y$ then $k < \min(x,y)$ and therefore, $G(x,y) = G(x, T^*(1,y)) = G(x, V_{S,T,h(k,x)}(1,y)) = V_{S,T,k}(x,xy)$. □

Corollary 4.1. *Let V be a nullnorm with an annihilator element $k \in [0,1]$. If V is h-pseudo-homogeneous with respect to a function $G : [0,1]^2 \to [0,1]$, then for each $x \in [0,1]$, G satisfies the following properties:*

1. $G(0,x) = G(x,0) = 0$*;*

2. $G(1,x) = x$ *and* $G(x,1) = V(x,x)$*;*

3. G *is increasing in both variables;*

4. $G(k,x) = kx = G(x,k)$*;*

5. $G(x,y) = G(y,x)$ *for each* $y \in [0,1]$ *such that* $xy \leq k$*.*

Corollary 4.2. *Let (S,T,k) be the generator triple of a proper nullnorm V. If T is strict and V is h-pseudo homogeneous with respect to a function G, then, for each $x, y \in [0,1]$ such that $x \neq y$, $G(x,y) = G(y,x)$ if and only if $xy \leq k$.*

Proof: ($\Rightarrow$) If $xy > k$ and $G(x,y) = G(y,x)$ then, from Lemma 4.2, $V(x,xy) = V(y,xy)$ and since T is strict then $x = y$ which is a contradiction.

($\Leftarrow$) Direct from the last item of Corollary 4.1. □

Theorem 4.3. *Let V be a proper nullnorm with (S,T,k) as generator triple. Then V is a h-pseudo-homogeneous nullnorm if and only if $S = S_M$ and, for some function $G_T : [0,1]^2 \to [0,1]$, it holds that $T(\lambda x, \lambda y) = G_T(\lambda, T(x,y))$ for each $x, y, \lambda \in [0,1]$.*

Proof: ($\Rightarrow$) Let (S,T,k) be the generator triple of V and $x \in [0,1]$. Then $kx \leq k$ and so $V(kx,kx) \leq k$.

By Corollary 4.1, item 2, $G(1,kx) = kx$ and $G(kx,1) = V(kx,kx)$. Therefore, by item 5, $V(kx,kx) = kx$ for each $x \in [0,1]$. Since $kx \leq k$, by Equation (6), $kx = V(kx,kx) = kS\left(\frac{kx}{k}, \frac{kx}{k}\right) = kS(x,x)$, i.e. $S(x,x) = x$ for each $x \in [0,1]$. Hence, $S = S_M$. Since V is h-pseudo-homogeneous, there exists a function $G : [0,1]^2 \to [0,1]$ satisfying Eq. (14). Define the function:

$$G_T(a,b) = \frac{G\left(a(1-k)+k, b\left(1 - \frac{k}{a(1-k)+k}\right) + \frac{k}{a(1-k)+k}\right) - k}{1-k} \text{ for each } a, b \in [0,1]. \tag{16}$$

Let $\lambda, x, y \in (0,1)$ and $\lambda' = \lambda(1-k)+k$. So $h(k,\lambda') = \frac{k}{\lambda'}$ and

$$\begin{aligned} G_T(\lambda, T(x,y)) &= \tfrac{G\left(\lambda', \left(1-\frac{k}{\lambda'}\right)T(x,y)+\frac{k}{\lambda'}\right)-k}{1-k} \\ &= \tfrac{G\left(\lambda', V_{S,T,\frac{k}{\lambda'}}\left(\left(1-\frac{k}{\lambda'}\right)x+\frac{k}{\lambda'}, \left(1-\frac{k}{\lambda'}\right)y+\frac{k}{\lambda'}\right)\right)-k}{1-k} \\ &= \tfrac{V((\lambda'-k)x+k, (\lambda'-k)y+k)-k}{1-k} \\ &= \tfrac{V((1-k)\lambda x+k, (1-k)\lambda y+k)-k}{1-k} \\ &= T(\lambda x, \lambda y). \end{aligned}$$

($\Leftarrow$) Let $G : [0,1]^2 \to [0,1]$ be the function:

$$G(a,b) = \begin{cases} ab & , \text{ if } ab \le k; \\ (1-k)G_T\left(\frac{a-k}{1-k}, \frac{ab-k}{a-k}\right)+k & , \text{ if } ab > k. \end{cases} \tag{17}$$

Then for each $x, y, \lambda \in [0,1]$ we have the following cases:

Case 1. If $\lambda x \le k$ and $\lambda y \le k$, then $V(\lambda x, \lambda y) = \lambda S_M(x,y) = G(\lambda, S_M(x,y)) = G(\lambda, V_{S_M,T_V,h(k,\lambda)}(x,y))$.

Case 2. If $\lambda x > k$ and $\lambda y > k$, then $h(k,\lambda) = \frac{k}{\lambda}$ and therefore, by Equation (6),

$$\begin{aligned} G(\lambda, V_{S_M,T_V,h(k,\lambda)}(x,y)) &= G(\lambda, V_{S_M,T_V,\frac{k}{\lambda}}(x,y)) \\ &= G\left(\lambda, \left(1-\tfrac{k}{\lambda}\right)T_V\left(\tfrac{\lambda x-k}{\lambda-k}, \tfrac{\lambda y-k}{\lambda-k}\right)+\tfrac{k}{\lambda}\right) \\ &= (1-k)G_T\left(\tfrac{\lambda-k}{1-k}, T_V\left(\tfrac{\lambda x-k}{\lambda-k}, \tfrac{\lambda y-k}{\lambda-k}\right)\right)+k \\ &= (1-k)T_V\left(\tfrac{\lambda x-k}{1-k}, \tfrac{\lambda y-k}{1-k}\right)+k \\ &= V(\lambda x, \lambda y). \end{aligned}$$

Case 3. If $\lambda x \le k$ and $\lambda y > k$, then $h(k,\lambda) = \frac{k}{\lambda}$ and $x \le \frac{k}{\lambda} < y$. Thereby, $xy \le \frac{k}{\lambda}$ and therefore, by Equation (6),

$$\begin{aligned} G(\lambda, V_{S_M,T_V,h(k,\lambda)}(x,y)) &= G(\lambda, V_{S_M,T_V,\frac{k}{\lambda}}(x,y)) \\ &= k \\ &= V(\lambda x, \lambda y). \end{aligned}$$

□

Therefore, there is no proper h-pseudo-homogeneous nullnorm for any G which is either nilpotent, Archimedean or cancellative.

Remark 4.1. *From Theorem 4.3, it follows that $G(x,y) = xy$ whenever $y \le k$ and every h-pseudo-homogeneous proper nullnorm is neither nilpotent, nor Archimedean, nor cancellative. In addition, observe that if V is h-pseudo-homogeneous then their underlying t-conorm is homogeneous.*

Corollary 4.3. *Let V be a proper nullnorm and $G : [0,1]^2 \to [0,1]$ be a continuous and increasing function such that $G(x,1) = 0$ if and only if $x = 0$. Then V is underlying pseudo-homogeneous if and only if V is h-pseudo homogeneous with respect to a continuous function G satisfying $G(a,b) = k \Rightarrow a = k$ and $b = 1$.*

Proof: ($\Rightarrow$) Immediate from Theorems 2.5 and 4.3.

($\Leftarrow$) From Corollary 4.1 we know that G is increasing in both variables, continuous and $G(a,b) = k \Leftrightarrow a = k$ and $b = 1$. Therefore, G_T generated from G according to Eq. (16) is continuous, increasing in both variables and satisfies $G_T(x,1) = 0 \Leftrightarrow x = 0$. From Theorem 4.3, we have that $S_V = S_M$ and $G_T(\lambda, T_V(x,y)) = T_V(\lambda x, \lambda y)$ for each $\lambda, x, y \in [0,1]$. Thereby, T_V is pseudo-homogeneous with respect to G_T and, consequently, V is underlying pseudo-homogeneous.

□

4.1. Conjugation of nullnorms and h-pseudo-homogeneity

In the following, we prove that h-pseudo-homogeneity for nullnorms is preserved for power automorphisms, i.e. automorphism defined for each $x \in [0,1]$ by $\rho(x) = x^r$ where r is a fixed positive real number. This is due to the fact that such automorphisms are the only ones for which $\rho(xy) = \rho(x)\rho(y)$ for each $x, y \in [0,1]$.

Proposition 4.2. *Let ρ be a power automorphism and let V be a nullnorm with an annihilator element $k \in [0,1]$. If V is h-pseudo-homogeneous with respect to a function $G : [0,1]^2 \to [0,1]$ and $(T_V)^\rho = T_{V^\rho}$, then V^ρ and $V^{(\rho)}$ are h-pseudo-homogeneous with respect to G^ρ.*

Proof: Let k be the annihilator element of V. By Proposition 3.2 , $\left(V_{S,T,h(k,\rho(\lambda))}\right)^\rho = V_{S^\rho,T^\rho,\rho^{-1}(h(k,\rho(\lambda)))}$ for each $\lambda \in [0,1]$.

Since ρ preserves the t-norm T_P, ρ^{-1} also preserves T_P. Then $h(\rho^{-1}(k),\lambda) = \rho^{-1}(h(k,\rho(\lambda)))$ and therefore

$$\begin{aligned} V^\rho(\lambda x, \lambda y) &= \rho^{-1}(V(\rho(\lambda x), \rho(\lambda y))) \\ &= \rho^{-1}(G(\rho(\lambda), V_{S,T,h(k,\rho(\lambda))}(\rho(x), \rho(y)))) \\ &= G^\rho\left(\lambda, \left(V_{S,T,h(k,\rho(\lambda))}\right)^\rho (x,y)\right) \\ &= G^\rho(\lambda, V_{S^\rho,T^\rho,h(\rho^{-1}(k),\lambda)}(x,y)). \end{aligned}$$

Therefore, V^ρ is h-pseudo homogeneous with respect to G^ρ. Since, by Proposition 3.2, $V^\rho = V^{(\rho)}$ and $\left(V_{S,T,h(k,\rho(\lambda))}\right)^\rho = V_{S^\rho,T^\rho,\rho^{-1}(h(k,\rho(\lambda)))}$ for each $\lambda \in [0,1]$, then $V^{(\rho)}$ is also h-pseudo homogeneous with respect to G^ρ. □

4.2. Idempotent nullnorms and h-pseudo-homogeneity

In this subsection we will provide some results relating h-pseudo-homogeneity nullnorms and the idempotency property.

Proposition 4.3. *Let V be a h-pseudo-homogeneous nullnorm with respect to a function $G : [0,1]^2 \to [0,1]$. Then G is commutative if and only if V is idempotent.*

Proof: Let (S,T,k) be the generator triple of V. ($\Rightarrow$) Let $x \in [0,1]$. Then, by Corollary 4.1 (item 2), $V(x,x) = G(x,1) = G(1,x) = x$ and therefore V is idempotent.

($\Leftarrow$) Let $x, y \in [0,1]$. Since V is idempotent then, by Remark 3.5, we have that $V = V_M^k$ and that for each $x \in [0,1]$, $V_M^{h(k,x)}$ is also idempotent. Thus, by Eq. (14), we have that $G(x,y) = G(x, V_M^{h(k,x)}(y,y)) = V_M^k(xy,xy) = xy$ for each $x, y \in [0,1]$ which is clearly commutative. □

Corollary 4.4. *Let V be a nullnorm with an annihilator element $k \in [0,1]$ such that V is h-pseudo-homogeneous with respect to a function $G : [0,1]^2 \to [0,1]$. If G is commutative then $G = T_P$ and $V = V_M^k$.*

Proposition 4.4. *For each $k \in [0,1]$, V_M^k is h-pseudo homogeneous with respect to T_P.*

Proof: If $\lambda x, \lambda y \le k$ then $x, y \le \frac{k}{\lambda}$ and so $x, y \le h(k,\lambda)$ and $V_M^k(\lambda x, \lambda y) = k \max\left(\frac{\lambda x}{k}, \frac{\lambda y}{k}\right) = \lambda \max(x,y) = T_P(\lambda, V_M^{h(k,\lambda)}(x,y))$.

If $\lambda x, \lambda y > k$ then $x, y > \frac{k}{\lambda}$ and therefore, $k < \lambda$ which means that $h(k,\lambda) = \frac{k}{\lambda}$. So, $V_M^k(\lambda x, \lambda y) = (1-k)\min\left(\frac{\lambda x - k}{1-k}, \frac{\lambda x - k}{1-k}\right) + k = \lambda \min(x,y) = G(\lambda, T_M(x,y)) = T_P(\lambda, V_M^{\frac{k}{\lambda}}(x,y))$.

If $\lambda x < k < \lambda y$ (or $\lambda y < k < \lambda x$) we have that $x < \frac{k}{\lambda} < y$ (or $y < \frac{k}{\lambda} < x$) and $V_M^k(\lambda x, \lambda y) = k$ and $T_P(\lambda, V_M^{\frac{k}{\lambda}}(x,y)) = T_P(\lambda, \frac{k}{\lambda}) = k$. □

Corollary 4.5. *Let V be an idempotent nullnorm with an annihilator element $k \in [0,1]$. V is h-pseudo homogeneous with respect to a function G if and only if $G = T_P$ and $V = V_M^k$.*

4.3. Positive nullnorms and h-pseudo-homogeneity

Observe that V_M^k are positive nullnorms which are h-pseudo-homogeneous for T_P as proved in Proposition 4.4. But this is not the only case. In fact, V_{MP}^k is positive and is h-pseudo-homogeneous for the function G^k defined in Equation (15). Since each t-norm in the family of Schweizer-Sklar t-norms is positive, then for each $k \in (0,1)$ and $\beta < 0$, by Remark 3.7 we have that $V_{M,SS}^{k,\beta} = V_{S_M,T_{SS}^\beta,k}$ is a positive proper nullnorm.

Proposition 4.5. *For all $\beta < 0$ and $k \in (0,1)$ the positive proper nullnorm $V_{M,SS}^{k,\beta}$ is h-pseudo homogeneous for the following function:*

$$G_{M,SS}^{k,\beta}(a,b) = \begin{cases} ab & , \text{ if } ab \leq k; \\ (1-k)\left(\left(\frac{ab-k}{1-k}\right)^\beta + \left(\frac{a-k}{1-k}\right)^\beta - 1\right)^{\frac{1}{\beta}} + k & , \text{ if } ab > k. \end{cases}$$

Proof: Straightforward from Theorem 4.3 and Theorem 3.2. in [37]. □

5. Final remarks

In this paper, we have introduced new classes of nullnorms based on well-known classes of t-norms and t-conorms. We have investigated some properties of them. We have also presented a notion of pseudo-homogeneity for nullnorms. In general, the pseudo-homogeneity of a function F, as considered in [23, 24, 29, 37] is defined by $F(\lambda x, \lambda y) = G(\lambda, F(x,y))$. But here the function "F" on the right side (in our case $V_{S,T,\min(1,\frac{k}{\lambda})}$) is different from the "F" on the left side (in our case $V_{S,T,k}$). Nevertheless, $V_{S,T,\min(1,\frac{k}{\lambda})}$ and $V_{S,T,k}$ have very similar generator triples (varying just in the annihilator element). This is a novelty in the several proposed generalizations of homogeneity.
As future work, we will investigate interval-valued nullnorms in the light of the interval representation proposed in [6, 35], as it was done for other important classes of fuzzy operators like [3, 4, 5]. We will also consider admissible orders [33], as it was done in [38]. In both cases, the relation to the interval-valued pseudo-homogeneity as introduced in [23, 24] will be considered.

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