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General overlap functions

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Abstract

As a generalization of bivariate overlap functions, which measure the degree of overlapping (intersection for non-crisp sets) of n different classes, in this paper we introduce the concept of general overlap functions. We characterize the class of general overlap functions and include some construction methods by means of different aggregation and bivariate overlap functions. Finally, we apply general overlap functions to define a new matching degree in a classification problem. We deduce that the global behavior of these functions is slightly better than some other methods in the literature.

Keywords: Overlap functions; general overlap function; aggregation functions.

1. Introduction

Fuzzy sets were introduced by Zadeh in 1965 [33], with the idea of sets with a continuum grades of membership, instead of the classical dual (yes/no) membership. This approach initiated the study of fuzzy logic [28, 32] as a class of many-valued logical systems [7, 20] that can model non-stochastic uncertainty. However, the actual concept of fuzzy logic is much wider than that of its origins [34]. Nowadays, fuzzy logic must be understood as posesing many distinct facets that have unsharp boundaries [35]. Four of the most important facets are: the logical, the set-theoretic, the relational and the epistemic facets. In this work, we are interested in the set-theoretical facet, specifically, in generating a suitable notion for counting the overlapping of n-different sets or classes whose membership degrees are represented by fuzzy sets.

As Zadeh said in [33] "More often than not, the classes of objects encountered in the real physical world do not have precisely defined criteria of membership". The problem is even more difficult if more than one class are considered simultaneously. But even if we can attribute the membership degrees of the objects to the classes, in most real problems such classes are not disjoint, but they overlap. In order to measure the overlapping between two different classes or objects, in the fuzzy community the concept of overlap function was introduced in [9]. However, this previous concept can be only applied to problems in which

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two classes are taken into account. It is not difficult to see that due to the imposed properties of overlap functions, its generalization to the case of more than 2 variables (or classes) is not direct. Some authors have already analyzed this extension and they introduced the concept of n-dimensional overlap functions [19]. In this paper, we introduce a new generalization of overlap functions that can be applied to n classes. We call this notion as general overlap function.

It is important to highlight the motivation under the novel definition of general overlap functions. The starting point is that general overlap differ from n-dimensional overlap functions in the boundary conditions they verify, which are relaxed. Specifically, in the definition of n-dimensional overlap functions, the boundary conditions, which are written as an equivalence, i.e., an "if and only if" condition, are replaced by a sufficient condition, i.e., an "if" condition. At a first glance, this replacement may not seem very significant but the real potential of general overlap functions is that they behave well for calculating the matching degree in some classification problems [3, 15, 16]. The main reason for the improvement in these problems is that when using general overlap functions for calculating the matching degree, examples with a low matching degree with the antecedent part of a fuzzy rule are not taken into account in the system.

From a theoretical point of view, we tackle the characterization and some constructions methods of general overlap functions. These constructions are based in continuous aggregation functions. We devote a last section of the paper to analyze an illustrative example of a classification problem.

This paper is organized as follows. In Section 2, we recall some preliminary notions including that of bivariate and n-dimensional overlap function as well as some construction methods. In Section 3, we present our proposal for the definition of general overlap functions as well as some remarkable facts derived from this definition and the study of its lattice structure. The characterization of general overlap functions and some construction methods are latter introduced in Section 4. We highlight the real potencial of general overlap functions in a classification problem in Section 5. We draw some conclusions in Section 6.

2. Preliminaries

In this section, we recall some concepts and properties of bivariate overlap functions, which were initially proposed in [8]. Since their introduction, these classes of functions have been largely studied, see, e.g.: [4, 11–14, 25, 29, 30]), including the extension of overlap functions to the n-dimensional case in [19]. First, we recall the well-known notion of fuzzy negation.

Definition 1. *A function $N\colon [0,1] \to [0,1]$ is said to be a fuzzy negation if the following conditions hold:*

(N1) *The boundary conditions: $N(0) = 1$ and $N(1) = 0$;*

(N2) *N is decreasing: if $x \leq y$ then $N(y) \leq N(x)$.*

The standard negation or Zadeh negation is given by $N_Z(x) = 1 - x$.

A key concept for the development of this paper is that of aggregation function [5, 6, 18].

Definition 2. *A function $A : [0,1]^n \to [0,1]$ is said to be an n-ary aggregation function if the following conditions hold:*

(A1) *A is increasing: for each $i \in \{1,\dots,n\}$, if $x_i \leq y$, then $A(x_1,\dots,x_n) \leq A(x_1,\dots,x_{i-1},y,x_{i+1},\dots,x_n)$;*

(A2) *A satisfies the boundary conditions: $A(0,\dots,0) = 0$ and $A(1,\dots,1) = 1$.*

Definition 3. *An n-ary aggregation function* $A : [0,1]^n \to [0,1]$ *is said to be conjunctive if, for any* $\vec{x} = (x_1, \ldots, x_n) \in [0,1]^n$*, it holds that*

$$A(\vec{x}) \leq \min(\vec{x}) = \min\{x_1, \ldots, x_n\}.$$

A *is said to be disjunctive if, for any* $\vec{x} = (x_1, \ldots, x_n) \in [0,1]^n$*,* A *is bounded by:*

$$A(\vec{x}) \geq \max(\vec{x}) = \max\{x_1, \ldots, x_n\}.$$

Definition 4. *[26] An aggregation function* $T : [0,1]^2 \to [0,1]$ *is said to be a t-norm if, for all* $x, y, z \in [0,1]$*, it satisfies the following properties:*

(T1) *Commutativity:* $T(x, y) = T(y, x)$*;*

(T2) *Associativity:* $T(x, T(y, z)) = T(T(x, y), z)$*;*

(T3) *Boundary condition:* $T(x, 1) = x$*.*

Example 1. *Examples of t-norms are the product* $T_P(x, y) = xy$*, the minimum* $T_{\min}(x, y) = \min\{x, y\}$*, and the Łukasiewicz t-norm* $T_L(x, y) = \max\{0, x + y - 1\}$*. Those t-norms are all continuous. Observe that any t-norm is conjunctive, that is,* $T(x, y) \leq \min\{x, y\}$*.*

Definition 5. *[26] An aggregation function* $S : [0,1]^2 \to [0,1]$ *is said to be t-conorm if, for all* $x, y, z \in [0,1]$*, it satisfies the following properties:*

(S1) *Commutativity:* $S(x, y) = S(y, x)$*;*

(S2) *Associativity:* $S(x, S(y, z)) = S(S(x, y), z)$*;*

(S3) *Boundary condition:* $S(x, 0) = x$*.*

Example 2. *Examples of t-conorms are the probabilistic sum* $S_P(x, y) = x + y - xy$*, the maximum* $T_{\max}(x, y) = \max\{x, y\}$*, and the Łukasiewicz t-conorm* $T_L(x, y) = \min\{1, x + y\}$*. Those t-conorms are all continuous. Observe that any t-conorm is disjunctive, that is,* $S(x, y) \geq \max\{x, y\}$*.*

Definitions 4 and 5 recall the notions of t-norm and t-conorm. Let us also point out that t-norms and t-conorms are particular cases of 2-ary aggregation functions typically used in the context of fuzzy sets and fuzzy logic to respectively generalize the classical conjunction and disjunction logical connectives. To this aim, sets of additional axioms or properties T1-T3 and S1-S3 are added on the definition of aggregation function to impose the required conjunctive or disjunctive behavior. Remarkably, one of these properties is associativity, which allows extending the 2-ary setting of t-norms and t-conorms to higher arities in order to perform conjunctions and disjunctions of more than 2 elements. However, associativity leaves out some interesting non-associative functions, with a clear conjunctive or disjunctive behaviour. Particularly, many conjunctive (resp. disjunctive) aggregation functions are not associative, and thus they cannot be t-norms (t-conorms), despite its potentiallity to adequately perform as conjunctive (disjunctive) connectives (see also [27] for a wider discussion on the notion of aggregation function). In this sense, the motivation behind the proposal of overlap and grouping functions is to study conjunction and disjunction without imposing associativity.

2.1. Bivariate overlap and grouping functions

Given two degrees of membership $x = \mu_A(c)$ and $y = \mu_B(c)$ of an object c into classes A and B, an overlap function is supposed to yield the degree z up to which the object c belongs to the intersection of both classes. Particularly, an overlap function is defined as a particular type of bivariate aggregation function characterized by a set of commutative, natural boundary and monotonicity properties.

Definition 6. *[8] The mapping $O : [0,1]^2 \to [0,1]$ is said to be an overlap function if the following conditions hold:*

(O1) *O is commutative;*

(O2) *$O(x, y) = 0$ if and only if $xy = 0$;*

(O3) *$O(x, y) = 1$ if and only if $xy = 1$;*

(O4) *O is increasing;*

(O5) *O is continuous.*

Overlap functions are closely related with the class of continuous t-norms, although they define different classes since the associative property is not required for the former, but it is for the latter (see for example [4, 9, 18]). In fact, as it was largely discussed in the literature (see, e.g., [4, 9, 17]), associativity implies a serious restriction, even artificial if there is not a sequence of values to be aggregated. In the following example, we can see an instance of an aggregation function that is an overlap function, but not a t-norm if $p > 1$.

Example 3. *For every $p > 0$, the bivariate aggregation function $O_p(x, y) = (\min\{x, y\})^p$ is an overlap function. Moreover, if $p \neq 1$, the bivariate function O_p is not associative, and thus it is not a continuous t-norm.*

The following result given in [9] provides a construction method to generate overlap functions by means of particular functions $f, g : [0,1]^2 \longrightarrow [0,1]$.

Theorem 1. *[9] The mapping $O : [0,1]^2 \longrightarrow [0,1]$ is an overlap function if and only if*

$$O(x, y) = \frac{f(x, y)}{f(x, y) + g(x, y)},$$

for some $f, g : [0,1]^2 \longrightarrow [0,1]$ such that

(i) *f and g are commutative;*

(ii) *f is increasing and g is decreasing;*

(iii) *$f(x, y) = 0$ if and only if $xy = 0$;*

(iv) *$g(x, y) = 0$ if and only if $xy = 1$;*

(v) *f and g are continuous functions.*

2.2. n-dimensional overlap and grouping functions

In this subsection, we present the definition of n-dimenional overlap function introduced in [19]. This notion extends the binary overlap approach into an n-dimensional case. This definition allows us to measure the overlapping degree in those situations in which an object may belong to more than two classes.

Definition 7. *[19] The mapping $\mathcal{O}: [0,1]^n \longrightarrow [0,1]$ is said to be an n-dimensional overlap function if the following properties hold:*

($\mathcal{O}$1) *$\mathcal{O}$ is commutative;*

($\mathcal{O}$2) *$\mathcal{O}(\vec{x}) = 0$ if and only if $\prod_{i=1}^n x_i = 0$;*

($\mathcal{O}$3) *$\mathcal{O}(\vec{x}) = 1$ if and only if $\prod_{i=1}^n x_i = 1$;*

($\mathcal{O}$4) *$\mathcal{O}$ is increasing;*

($\mathcal{O}$5) *$\mathcal{O}$ is continuous.*

Thus, taking into account this definition, an object c that belongs to three classes C_1, C_2 and C_3 with degrees $x_1 = 1$, $x_2 = 1$ and $x_3 = 0.3$ will not have the maximum degree of overlap since condition ($\mathcal{O}3$) of the previous definition is not satisfied. Even more, if the degrees are $x_1 = 1$, $x_2 = 1$ and $x_3 = 0$, following the second condition we will conclude that the n-dimensional degree of overlap of this object into the classification system given by the classes C_1, C_2 and C_3 will be zero. This is the reason why this first extension of the original idea of overlap proposed in [9] has been called *n-dimensional* overlap. Let us observe that this definition is closely related with the idea of intersection of n classes. Of course, and as it was pointed in [19], when we try to extend the overlap ideas from two sets into a more general scenario, different definitions or generalizations are possible.

Example 4. *It is easy to see that the following aggregation functions are n-dimensional overlap functions:*

- *The minimum powered by p element-wise: $\mathcal{M}(x_1, \ldots, x_n) = \min\limits_{1 \le i \le n} \{x_i^p\}$ with $p > 0$.*
- *The product: $\mathcal{P}(x_1, \ldots, x_n) = \prod\limits_{i=1}^{n} x_i$.*
- *The Einstein product aggregation operator: $\mathcal{EP}(x_1, \ldots, x_n) = \dfrac{\prod_{i=1}^n x_i}{1 + \prod_{i=1}^n (1 - x_i)}$*
- *The sinus induced overlap $\mathcal{S}(x_1, \ldots, x_n) = \sin \dfrac{\pi}{2} \left(\prod\limits_{i=1}^{n} x_i \right)^p$ with $p > 0$.*

3. General overlap functions

In the preceding section, we have recalled the concept of n-dimensional overlap function that extends the definition of [9, 25] for the 2-dimensional case. Nevertheless, as previously mentioned, when n classes are simultaneously considered, more (and different) generalizations can be defined. In this section, we introduce a different way of measuring the degree of overlapping. The main advantage of this new definition is that it shows a suitable behavior in classification problems. We illustrate this in the final section of this paper.

Definition 8. *A mapping $\mathcal{GO} : [0,1]^n \longrightarrow [0,1]$ is said to be a general overlap function if the following conditions hold:*

($\mathcal{GO}$1) *$\mathcal{GO}$ is commutative;*

($\mathcal{GO}$2) *If $\prod_{i=1}^{n} x_i = 0$, then $\mathcal{GO}(\vec{x}) = 0$;*

($\mathcal{GO}$3) *If $\prod_{i=1}^{n} x_i = 1$, then $\mathcal{GO}(\vec{x}) = 1$;*

($\mathcal{GO}$4) *$\mathcal{GO}$ is increasing;*

($\mathcal{GO}$5) *$\mathcal{GO}$ is continuous.*

The difference between the class of n-dimensional overlap functions and the class of general overlap functions is that the former has sufficient and necessary boundary conditions [($\mathcal{GO}$2)] and [($\mathcal{GO}$3)] while the latter has sufficient conditions. This means that the class of general overlap functions can yield 0 to some vectors $\vec{x} = (x_1, \ldots, x_n)$ such that $x_i \neq 0$ for all $i \in \{1, \ldots, n\}$. Similarly, it can yield 1 to vectors $\vec{x} = (x_1, \ldots, x_n)$ such that $x_i \neq 1$ for some $i \in \{1, \ldots, n\}$. Mathematically, this means that the class of general overlap function may have zero-divisors and one-divisors, (see e.g. [6]).

From the preceding comment, one easily sees that the set of general overlap functions contains the set of n-dimensional overlap functions.

Proposition 1. *If $\mathcal{O} : [0,1]^n \longrightarrow [0,1]$ is an n-dimensional overlap function, then $\mathcal{O}$ is also a general overlap function.*

It is worth mentioning that in the bivariate case, general overlap functions are a generalization of the concept of 0-overlap and 1-overlap introduced in [29], which only considers sufficient boundary conditions in 0 and 1. However, there exist instances of general overlap functions which neither are 0-overlap nor 1-overlap functions.

Example 5.

1. Every overlap function $O : [0,1]^2 \longrightarrow [0,1]$ in the sense of Definition 6 is a general overlap function, but the opposite is not true.

2. The function $\mathcal{GO}(x,y) = \max\{0, x^2 + y^2 - 1\}$ is a general overlap function but it is not a bivariate overlap function.

3. Consider $O(x,y) = (\min\{x,y\})^p$, for $p > 0$ and $T_L(x,y) = \max\{0, x + y - 1\}$. Then, the function

$$\mathcal{GO}^{T_L}(x,y) = O(x,y) * T_L(x,y)$$

is a general overlap function which it is not an overlap function.

4. Other examples are:

$$\begin{aligned} Prod\_Luk(x_1, \ldots, x_n) &= \prod_{i=1}^{n} x_i * \left(\max\left\{ \sum_{i=1}^{n} x_i - (n-1), 0 \right\} \right) \\ GM\_Luk(x_1, \ldots, x_n) &= \sqrt[n]{\prod_{i=1}^{n} x_i * \left(\max\left\{ \sum_{i=1}^{n} x_i - (n-1), 0 \right\} \right)}. \end{aligned}$$

Proposition 2. *Let $F : [0,1]^n \longrightarrow [0,1]$ be a commutative continuous aggregation function. The following statements hold:*

(i) *If F is conjunctive, then F is a general overlap function.*

(ii) *If F is disjunctive, then F is not a general overlap function.*

Proof: Consider a commutative continuous aggregation function $F : [0,1]^n \longrightarrow [0,1]$. One easily verifies that F satisfies ($\mathcal{GO}1$),($\mathcal{GO}4$) and ($\mathcal{GO}5$). Hence, it only remains to prove the boundary conditions ($\mathcal{GO}2$) and ($\mathcal{GO}3$).

Let F be a conjunctive aggregation function, i.e., the function F satisfies that $F(\vec{x}) \leq \min(\vec{x})$. Let $\vec{x} \in [0,1]$ satisfy that $\prod_{i=1}^n x_i = 0$. It holds that $F(\vec{x}) \leq \min(\vec{x}) = 0$, and, hence, $F(\vec{x}) = 0$ and ($\mathcal{GO}2$) holds. Finally, if $\vec{x} \in [0,1]$ satisfy that $\prod_{i=1}^n x_i = 1$, then it holds that $x_i = 1$ for each $i \in \{1, \ldots, n\}$ and $F(\vec{1}) = 1$. Hence, ($\mathcal{GO}3$) holds, which proves itemize (i).

Finally, let F be a disjunctive aggregation function, i.e., the function F satisfies that $F(\vec{x}) \geq \max(\vec{x})$. It holds that $F(1, 0, \ldots, 0) \geq \max(\vec{x}) = 1$, which is in contradiction with ($\mathcal{GO}2$). □

Note that the preceding proposition means that any conjunctive continuous commutative aggregation function generates an n-dimensional overlap function.

Following a similar procedure to that described in [9, 19], it is possible to characterize general overlap functions. In order to do that, let us first introduce some properties and notations.

Let us denote by $\mathfrak{O}^n$ the set of all general overlap functions. Define the order relation $\leq_{\mathfrak{O}^n} \in \mathfrak{O}^n \times \mathfrak{O}^n$, for all $\mathcal{GO}_1, \mathcal{GO}_2 \in \mathfrak{O}^n$ by:

$$\mathcal{GO}_1 \leq_{\mathfrak{O}^n} \mathcal{GO}_2 \Leftrightarrow \mathcal{GO}_1(\vec{x}) \leq \mathcal{GO}_2(\vec{x}), \text{ for all } \vec{x} \in [0,1]^n.$$

The supremum and infimum of two arbitrary general overlap functions $\mathcal{GO}_1, \mathcal{GO}_2 \in \mathfrak{O}^n$ are, respectively, the general overlap functions $\mathcal{GO}_1 \vee \mathcal{GO}_2, \mathcal{GO}_1 \wedge \mathcal{GO}_2 \in \mathfrak{O}^n$, defined, for all $\vec{x} \in [0,1]^n$ by:

$$(\mathcal{GO}_1 \vee \mathcal{GO}_2)(\vec{x}) = \max\{\mathcal{GO}_1(\vec{x}), \mathcal{GO}_2(\vec{x})\} \quad (1)$$

$$(\mathcal{GO}_1 \wedge \mathcal{GO}_2)(\vec{x}) = \min\{\mathcal{GO}_1(\vec{x}), \mathcal{GO}_2(\vec{x})\}. \quad (2)$$

The following result is immediate:

Theorem 2. *The ordered set $(\mathfrak{O}^n, \leq_{\mathfrak{O}^n})$ is a lattice.*

Remark 1. *Consider the family of general overlap functions given by:*

$$\mathcal{GO}_b(\vec{x}) = \begin{cases} 0 & \text{if } \prod_{i=1}^n x_i = 0 \\ \frac{1}{b}\prod_{i=1}^n x_i & \text{if } 0 < \prod_{i=1}^n x_i < b \\ 1 & \text{if } \prod_{i=1}^n x_i \geq b \end{cases} \quad (3)$$

One easily verifies the supremum of this family in $(0,1]$ is $\bigvee_{b \in (0,1]} \mathcal{GO}_b = \mathcal{GO}_{\sup}$ where $\mathcal{GO}_{\sup}$ is given by:

$$\mathcal{GO}_{\sup}(\vec{x}) = \begin{cases} 0 & \text{if } \prod_{i=1}^n x_i = 0 \\ 1 & \text{otherwise.} \end{cases}$$

and hence, the supremum of the lattice $(\mathfrak{O}^n, \leq_{\mathfrak{O}^n})$ is $\mathcal{GO}_{\sup}$. Similarly, it can be seen that the infimum of the lattice $(\mathfrak{O}^n, \leq_{\mathfrak{O}^n})$ is given by:

$$\mathcal{GO}_{\inf}(\vec{x}) = \begin{cases} 1 & \text{if } \prod_{i=1}^n x_i = 1 \\ 0 & \text{otherwise.} \end{cases}$$

Note that, neither $\mathcal{GO}_{\sup}$ nor $\mathcal{GO}_{\inf}$ is a general overlap function, since none of them is continuous. Hence, in the lattice $(\mathfrak{O}^n, \leq_{\mathfrak{O}^n})$ there is no bottom neither top element. This means that, similarly to the case of n-dimensional overlap functions, the lattice $(\mathfrak{O}^n, \leq_{\mathfrak{O}^n})$ is not complete.

4. Characterization and construction methods of general overlap functions

This section is devoted to analyse a characterization and some construction methods of general overlap functions, similar to the ones recalled in the preliminary section for bivariate overlap functions.

Theorem 3. *The mapping $\mathcal{GO} \in \mathfrak{O}^n$ if and only if*

$$\mathcal{GO}(\vec{x}) = \frac{f(\vec{x})}{f(\vec{x}) + h(\vec{x})}$$

for some $f, h : [0,1]^n \longrightarrow [0,1]$ satisfying the following properties

(i) *f and h are commutative;*

(ii) *f is increasing and h is decreasing;*

(iii) *If $\prod_{i=1}^n x_i = 0$, then $f(\vec{x}) = 0$;*

(iv) *If $\prod_{i=1}^n x_i = 1$, then $h(\vec{x}) = 0$;*

(v) *f and h are continuous;*

(vi) *$f(\vec{x}) + h(\vec{x}) \neq 0$ for any $\vec{x} \in [0,1]^n$.*

Proof:
($\Rightarrow$) Suppose that $\mathcal{GO}$ is a general overlap function, and take $f(\vec{x}) = \mathcal{GO}(\vec{x})$ and $h(\vec{x}) = 1 - \mathcal{GO}(\vec{x})$. It holds that $f(\vec{x}) + h(\vec{x}) = \mathcal{GO}(\vec{x}) + 1 - \mathcal{GO}(\vec{x}) = 1 \neq 0$ and the function

$$\mathcal{GO}(x) = \frac{f(x)}{f(x) + h(x)},$$

is well defined. Moreover, one easily verifies that conditions **(i)**-**(vi)** trivially hold.
($\Leftarrow$) Consider $f, h : [0,1]^n \to [0,1]$ satisfying the conditions **(i)**-**(vi)**. We show that

$$\mathcal{GO}(\vec{x}) = \frac{f(\vec{x})}{f(\vec{x}) + h(\vec{x})}$$

is a general overlap function. It is immediate that $\mathcal{GO}$ is commutative **($\mathcal{GO}$1)** and continuous **($\mathcal{GO}$5)**. Let us prove that the conditions **($\mathcal{GO}$2)**,**($\mathcal{GO}$3)** and **($\mathcal{GO}$4)** hold. Let $\vec{x} \in [0,1]^n$ such that $\prod_{i=1}^n x_i = 0$. Due to conditions **(iii)** and **(vi)**, it holds that $f(\vec{x}) = 0$ and $f(\vec{x}) + h(\vec{x}) \neq 0$, and, consequently, $\mathcal{GO}(\vec{x}) = 0$. Similarly, let $\vec{x} \in [0,1]^n$ such that $\prod_{i=1}^n x_i = 1$. Due to conditions **(iv)** and **(vi)**, it holds that $h(\vec{x}) = 0$ and $f(\vec{x}) + h(\vec{x}) \neq 0$, and, consequently, $\mathcal{GO}(\vec{x}) = \frac{f(\vec{x})}{f(\vec{x})} = 1$. Finally, let us see that **($\mathcal{GO}$4)** also holds. Consider $\vec{x}, \vec{y} \in [0,1]^n$. Without loss of generality, suppose that $\vec{x} \leq \vec{y}$. Due to condition **(ii)**, it holds that $f(\vec{x}) \leq f(\vec{y})$, and $h(\vec{y}) \leq h(\vec{x})$. Similarly, we find that $f(\vec{x})h(\vec{y}) \leq f(\vec{y})h(\vec{x})$. Therefore, $f(\vec{x})f(\vec{y}) + f(\vec{x})h(\vec{y}) \leq f(\vec{x})f(\vec{y}) + f(\vec{y})h(\vec{x})$ and thus we have

$$\mathcal{GO}(\vec{x}) = \frac{f(\vec{x})}{f(\vec{x}) + h(\vec{x})} \leq \frac{f(\vec{y})}{f(\vec{y}) + h(\vec{y})} = \mathcal{GO}(\vec{y}).$$

This completes the proof. □

Example 6. *Let us observe that Theorem 3 provides a method for building general overlap functions. For example, if we take* $f = CMin^{Tr}_{\alpha,\epsilon}$ *given by*

$$CMin^{Tr}_{\alpha,\epsilon}(\vec{x}) = \begin{cases} \min(\vec{x}) & \textit{if } \min(\vec{x}) \geq \alpha + \epsilon \\ \frac{\alpha+\epsilon}{\epsilon}(\min(\vec{x}) - \alpha) & \textit{if } \alpha < \min(\vec{x}) < \alpha + \epsilon \\ 0 & \textit{if } \min(\vec{x}) \leq \alpha \end{cases}$$

for some $0 < \alpha < \alpha + \epsilon < 1$ *and* $h(\vec{x}) = \max\{1 - x_i,\ 1 \leq i \leq n\}$*, then Theorem 3 assures that*

$$\mathcal{GO}(\vec{x}) = \frac{CMin^{Tr}_{\alpha,\epsilon}(\vec{x})}{CMin^{Tr}_{\alpha,\epsilon}(\vec{x}) + \max\{1 - x_i,\ 1 \leq i \leq n\}}$$

is a general overlap function. Moreover, one easily verifies that $\mathcal{GO}$ *is not an* n*-dimensional overlap function.*

Corollary 1. *Let* $\mathcal{GO} \in \mathfrak{O}^n$ *be built according to Theorem 3 in terms of some functions* $f, h : [0,1]^n \to [0,1]$ *satisfying the conditions* $(i) - (vi)$*. It holds that* $\mathcal{GO}$ *is idempotent if and only if, for all* $x \in [0,1]$*, it holds that:*

$$f(x, \ldots, x) = \frac{x}{1-x} h(x, \ldots, x).$$

Proof: Observe that $\mathcal{GO}$ is idempotent if and only if, for all $x \in [0,1]$,

$$\mathcal{GO}(x, \ldots, x) = \frac{f(x, \ldots, x)}{f(x, \ldots, x) + h(x, \ldots, x)} = x,$$

if and only if

$$f(x, \ldots, x) = \frac{x}{1-x} h(x, \ldots, x).$$

□

The instances of general overlap functions introduced in Example 5 may be generalized as follows.

Proposition 3. *Let* $\mathcal{GO} \in \mathfrak{O}^n$ *and let* $F : [0,1]^n \to [0,1]$ *be a commutative and continuous aggregation function. Then* $\mathcal{GO}F(\vec{x}) = \mathcal{GO}(\vec{x})F(\vec{x})$ *is a general overlap function.*

Proof: It is immediate that $\mathcal{GO}F$ is commutative ($\mathcal{GO}1$), increasing ($\mathcal{GO}4$) and continuous ($\mathcal{GO}5$), since $\mathcal{GO}$, F and the product operation are commutative, increasing and continuous. It remains to prove ($\mathcal{GO}2$) and ($\mathcal{GO}3$).

Let $\vec{x} \in [0,1]^n$ such that $\prod_{i=1}^n x_i = 0$. It holds that $\mathcal{GO}(\vec{x}) = 0$, and, thus

$$\mathcal{GO}F(\vec{x}) = 0 \cdot F(\vec{x}) = 0.$$

Finally, if $\vec{x} \in [0,1]^n$ such that $\prod_{i=1}^n x_i = 1$, it holds that $x_i = 1$ for each $i \in \{1, \ldots, n\}$. Since $\mathcal{GO}$ satisfies ($\mathcal{GO}3$) and F is an aggregation function, we find that

$$\mathcal{GO}F(\vec{1}) = \mathcal{GO}(\vec{1})F(\vec{1}) = 1 \cdot 1 = 1.$$

□

Corollary 2. *Let* $\mathcal{O} : [0,1]^n \to [0,1]$ *be an* n*-dimensional overlap function and let* $F : [0,1]^n \to [0,1]$ *be a commutative and continuous aggregation function. Then* $\mathcal{O}F(\vec{x}) = \mathcal{O}(\vec{x})F(\vec{x})$ *is a general overlap function.*

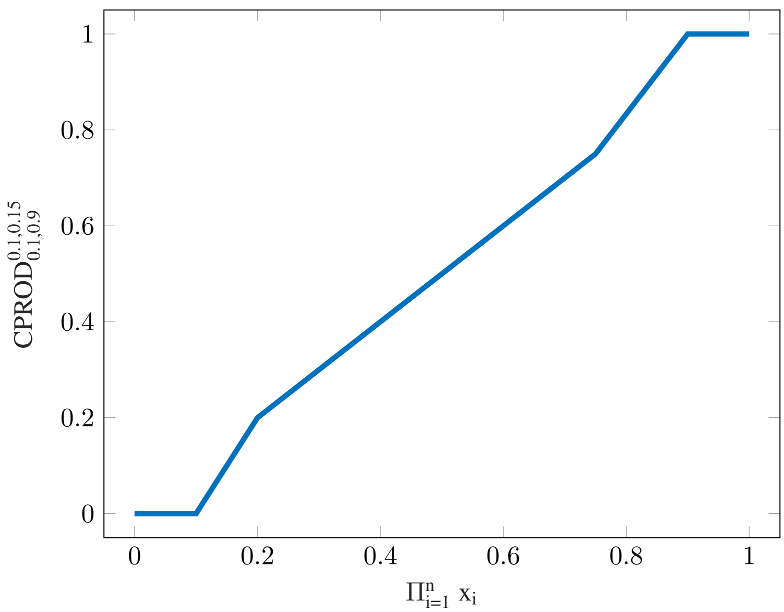

Figure 1: Graphical representation for $CPROD^{0.1,0.15}_{0.1,0.9}$ in which the $\prod_{i=1}^{n} x_i$ is represented in the x-axis.

Proof: Direct. □

Example 7. *Given $\alpha, \beta \in [0,1]$ such that $\alpha \leq \beta$, the truncated product function $PROD^{Tr}_{\alpha,\beta} : [0,1]^n \to [0,1]$ is usually defined as:*

$$PROD^{Tr}_{\alpha,\beta}(\vec{x}) = \begin{cases} 0 & \textit{if } \prod_{i=1}^{n} x_i \leq \alpha \\ \prod_{i=1}^{n} x_i & \textit{if } \alpha < \prod_{i=1}^{n} x_i < \beta \\ 1 & \textit{if } \prod_{i=1}^{n} x_i \geq \beta. \end{cases}$$

Observe that $PROD^{Tr}_{\alpha,\beta}$ is not a general overlapping function because it is not continuous. However, it is possible to build a smooth continuous version of $PROD^{Tr}_{\alpha,\beta}$. Let $\epsilon, \delta > 0$ satisfy $\alpha + \epsilon \leq \beta - \delta \leq 1$. The continuous truncated product function, $CPROD^{\epsilon,\delta}_{\alpha,\beta} : [0,1]^n \to [0,1]$, is defined as follows:

$$CPROD^{\epsilon,\delta}_{\alpha,\beta}(\vec{x}) = \begin{cases} 0 & \textit{if } \prod_{i=1}^{n} x_i \leq \alpha \\ \frac{\alpha+\epsilon}{\epsilon}\left(\prod_{i=1}^{n} x_i - \alpha\right) & \textit{if } \alpha < \prod_{i=1}^{n} x_i < \alpha + \epsilon \\ \prod_{i=1}^{n} x_i & \textit{if } \alpha + \epsilon \leq \prod_{i=1}^{n} x_i \leq \beta - \delta \\ \beta - \delta + \frac{1-(\beta-\delta)}{\delta}\left(\prod_{i=1}^{n} x_i - (\beta - \delta)\right) & \textit{if } \beta - \delta < \prod_{i=1}^{n} x_i < \beta \\ 1 & \textit{if } \prod_{i=1}^{n} x_i \geq \beta \end{cases} \quad (4)$$

A graphical representation for $CPROD^{\epsilon,\delta}_{\alpha,\beta}$ with $\alpha = 0.1$, $\epsilon = 0.1$, $\delta = 0.15$ and $\beta = 0.9$ in which the $\prod_{i=1}^{n} x_i$ is represented in the x-axis is included in 1. Note that the continuous truncated product function is a general overlap function but is not an n-dimensional overlap function.

One easily verifies that the previous example can be extended to any n-dimensional overlap function in the following way.

Definition 9. *Consider $0 < \alpha, \beta, \epsilon, \delta < 1$ such that $\alpha + \epsilon \leq \beta - \delta \leq 1$, and let $\mathcal{O}$ be an n-dimensional overlap function. The continuous truncated version of $T\mathcal{O}$-function, denoted $T\mathcal{O}^{\epsilon,\delta}_{\alpha,\beta} : [0,1]^n \longrightarrow [0,1]$, is defined, for all $\vec{x} \in [0,1]^n$, as follows:*

$$T\mathcal{O}^{\epsilon,\delta}_{\alpha,\beta}(\vec{x}) = \begin{cases} 0 & \text{if } \mathcal{O}(\vec{x}) \leq \alpha \\ \frac{\alpha+\epsilon}{\epsilon}\left(\mathcal{O}(\vec{x}) - \alpha\right) & \text{if } \alpha < \mathcal{O}(\vec{x}) < \alpha + \epsilon \\ \mathcal{O}(\vec{x}) & \text{if } \alpha + \epsilon \leq \prod_{i=1}^n x_i \leq \beta - \delta \\ \beta - \delta + \frac{1-(\beta-\delta)}{\delta}\left(\mathcal{O}(\vec{x}) - (\beta - \delta)\right) & \text{if } \beta - \delta < \prod_{i=1}^n x_i < \beta \\ 1 & \text{if } \mathcal{O}(\vec{x}) \geq \beta \end{cases} \tag{5}$$

Note that, the case $\alpha = \beta$ is the only one in which a continuous version cannot be generated, since it is impossible to find $\epsilon, \delta > 0$ satisfying $\alpha + \epsilon \leq \beta - \delta \leq 1$. Moreover, if ϵ and δ tend to zero, the behaviour of continuous and non-continuous truncated versions is similar. One easily verifies that the previous example can be extended to any n-dimensional overlap function in the following way.

Proposition 4. *Let $\alpha, \beta, \epsilon, \delta$ satisfy the conditions of Definition 9 and let $\mathcal{O} : [0,1]^n \longrightarrow [0,1]$ be an n-dimensional overlap function. Then the continuous truncated $T\mathcal{O}^{\epsilon,\delta}_{\alpha,\beta}$ is a general overlap function.*

Proof: Direct. □

Remark 2. *In general, whenever $\mathcal{O} : [0,1]^n \longrightarrow [0,1]$ is an n-dimensional overlap function, then the continuous truncated $T\mathcal{O}^{\epsilon,\delta}_{\alpha,\beta}$ is a general overlap function but not an n-dimensional overlap. Only in the cases of $\alpha = 0$ and $\beta = 1$, the generated truncated is an n-dimensional and it holds that $T\mathcal{O}^{\epsilon,\delta}_{\alpha,\beta} = \mathcal{O}$. Note that truncated versions of the n-dimensional functions that only modify the boundary conditions in the bottom 0 or in the top 1 are generated when $\alpha = 0$, or $\beta = 1$, respectively.*

Note that the function $CMin^{Tr}_{\alpha,\epsilon}$ introduced in Example 6 is a general overlap function that is not an n-dimensional overlap function.

The rest of the theoretical part of the paper is devoted to study the behaviour of general overlap function in combination with some aggregation function.

First of all, we prove that $\mathfrak{O}^n$ is closed with respect to some aggregation functions, as stated by the following results, which provide a construction method for general overlap functions.

Theorem 4. *Consider $M : [0,1]^m \longrightarrow [0,1]$ be a continuous aggregation function and the tuple $\overrightarrow{\mathcal{GO}} = [\mathcal{GO}_1, \ldots, \mathcal{GO}_m] \in (\mathfrak{O}^n)^m$ of m general overlap functions. The mapping $M_{\overrightarrow{\mathcal{GO}}} : [0,1]^n \longrightarrow [0,1]$, given for all $\vec{x} \in [0,1]^n$ by*

$$M_{\overrightarrow{\mathcal{GO}}}(\vec{x}) = M(\mathcal{GO}_1(\vec{x}), \ldots, \mathcal{GO}_m(\vec{x}))$$

is a general overlap function.

Proof:

One easily verifies that $M_{\overrightarrow{\mathcal{GO}}}$ is commutative ($\mathcal{GO}1$), increasing ($\mathcal{GO}4$) and continuous ($\mathcal{GO}5$). Hence, it only remains to prove ($\mathcal{GO}2$) and ($\mathcal{GO}3$). Let $\vec{x} \in [0,1]^n$ such that $\prod_{i=1}^n x_i = 0$ then, it holds that $\mathcal{GO}_j(\vec{x}) = 0$ for all $j \in \{1, \ldots, m\}$, and, thus

$$M(\mathcal{GO}_1(\vec{x}), \ldots, \mathcal{GO}_m(\vec{x})) = M(0, \ldots, 0) = 0.$$

Finally, if $\vec{x} \in [0,1]^n$ such that $\prod_{i=1}^n x_i = 1$, it holds that $x_i = 1$ for each $i \in \{1,\dots,n\}$. Since $\mathcal{GO}_j$ satisfies ($\mathcal{GO}3$) for all $j \in \{1,\dots,m\}$, we find that

$$M(\mathcal{GO}_1(\vec{1}),\dots,\mathcal{GO}_m(\vec{1})) = M(1,\dots,1) = 1.$$

□

Corollary 3. *Consider $M : [0,1]^m \longrightarrow [0,1]$ be a continuous aggregation function and the tuple $\vec{\mathcal{O}} = [\mathcal{O}_1,\dots,\mathcal{O}_m]$ of m n-dimensional overlap functions. The mapping $M_{\vec{\mathcal{O}}} : [0,1]^n \longrightarrow [0,1]$, given for all $\vec{x} \in [0,1]^n$ by*

$$M_{\vec{\mathcal{O}}}(\vec{x}) = M(\mathcal{O}_1(\vec{x}),\dots,\mathcal{O}_m(\vec{x})),$$

is a general overlap function.

Proof: Direct. □

Corollary 4. *Let $\mathcal{GO}_1,\dots,\mathcal{GO}_m \in \mathfrak{O}^n$ be general overlap functions and consider weights $w_1,\dots,w_m \in [0,1]$ such that $\sum_{j=1}^m w_j = 1$. Then the convex sum $\mathcal{GO} = \sum_{j=1}^m w_j \mathcal{GO}_j$ is also a general overlap function.*

Proof: Since the weighted sum is a continuous commutative n-ary aggregation function, the result follows from Theorem 4. □

Example 8. *The case $m = 2$ in Theorem 4 means that the set $\mathfrak{O}^n$ is closed under any 2-dimensional overlap function and continuous t-norms (e.g.: the product, the minimum and the Łukasiewicz t-norm). Other examples of such aggregation function M are the Maximum, the Probabilistic Sum the Cross Product uninorm*[1]*:*

$$U_P(x,y) = \begin{cases} \frac{xy}{xy+(1-x)(1-y)} & \text{if } \{x,y\} \neq \{0,1\} \\ 0 & \text{otherwise.} \end{cases}$$

5. An application of general overlap functions in classification problems

In this section we present an application of general overlap functions in classification problems. These problems consist of learning a *classifier* using P training examples $x_p = (x_{p1},\dots,x_{pn})$, $p = 1,2,\dots,P$, where x_{pi} is the value of the i-th variable ($i = 1,2,\dots,n$) of the p-th training example. Each example belongs to a class $y_p \in \mathbb{C} = \{C_1, C_2, ..., C_M\}$, where M is the number of classes of the problem. The learned classifier has to be able to determine the class of new unseen testing examples.

Fuzzy Rule-Based Classification Systems (FRBCSs) [24] are widely used to solve classification problems since they provide a good balance between accuracy and interpretability [31], since the antecedents of their rules are composed of linguistic labels. In this paper we use fuzzy rules whose structure is as follows:

$$\text{Rule } R_j : \text{ If } x_1 \text{ is } A_{j1} \text{ and } \dots \text{ and } x_n \text{ is } A_{jn} \text{ then Class} = C_j \text{ with } RW_j \quad (6)$$

where R_j is the label of the j-th rule, $x = (x_1,\dots,x_n)$ is an n-dimensional example vector, A_{ji} is the fuzzy set (modelled with triangular shaped membership functions in this paper) representing the linguistic term of the j-th rule in the i-th antecedent, C_j is a class label, and $RW_j \in [0,1]$ is the rule weight [23].

[1] See [6] for definition and properties of uninorms.

Two common metrics used to quantify the validity of fuzzy rules are the support (Equation 7) and the confidence (Equation 8), which is used as the rule weight.

$$Sup(R_j : A_j \to C_j) = \frac{\displaystyle\sum_{x_p \in ClassC_j} \mu_{A_j}(x_p)}{P} \tag{7}$$

$$RW_j = Conf(R_j : A_j \to C_j) = \frac{\displaystyle\sum_{x_p \in ClassC_j} \mu_{A_j}(x_p)}{\displaystyle\sum_{p=1}^{P} \mu_{A_j}(x_p)} \tag{8}$$

where $\mu_{A_j}(x_p)$ is the matching degree of the example x_p with the antecedent part of the fuzzy rule R_j computed as follows:

$$\mu_{A_j}(x_p) = T\left(\mu_{A_{j1}}(x_{p1}), \ldots, \mu_{A_{jn}}(x_{pn})\right) \tag{9}$$

where $\mu_{A_{ji}}(x_{pi})$ is the membership degree of the value x_{pi} to the fuzzy set A_{ji} of the rule R_j and T is a t-norm.

In this paper we consider the usage of FARC-HD [1] to accomplish the learning process because it is a state-of-the-art fuzzy classifier. The usage of both the support and the confidence plays an essential role in the generation of the fuzzy rules[2]. From Equations (7) and (8) it can be observed that they are based on the matching degree, which in turn is computed using the product t-norm. Consequently, the way the matching degree is computed is crucial in the learning process. Moreover, the matching degree is also a key operation in the fuzzy reasoning method [10], since it determines the fired rules and their firing degrees, which are the base of the subsequent components of the inference process.

In this paper, we propose to modify the computation of the matching degree by using general overlap functions instead of the product t-norm. That is, the new proposed matching degree is computed using Equation (10).

$$\mu_{A_j}(x_p) = \mathcal{GO}\left(\mu_{A_{j1}}(x_{p1}), \ldots, \mu_{A_{jn}}(x_{pn})\right) \tag{10}$$

This change, as previously stated, implies that the created fuzzy rules as well as the way in which they are used in the inference method is significantly different.

In the remainder of this section we describe the experimental framework used to study the effect of general overlap functions (Section 5.1) and we introduce the obtained results along with the corresponding statistical analysis (Section 5.2).

5.1. Experimental framework

To study the influence of the usage of general overlap functions in FARC-HD we have employed 31 datasets selected from the KEEL repository [2]. Table 1 summarizes the following characteristics of each data-set: number of examples (#Ex.), number of attributes (#Atts.) and number of classes (#Class.). We would like to point out that the *Magic, Page-blocks, Penbased, Shuttle* and *Twonorm* data-sets are stratified-sampled to 10% to improve the learning process efficiency. We have removed the missing values of *Cleveland, Crx and Wisconsin* before partitioning them.

[2]For details about the learning process see [1].

Table 1: Summary description of the considered data-sets.

Id.	Data-set	#Ex.	#Atts.	#Class.
app	Appendicitis	106	7	2
aus	Australian	690	14	2
bal	Balance	625	4	3
ban	Banana	5,300	2	2
bup	Bupa	345	6	2
cle	Cleveland	297	13	5
crx	Crx	653	15	2
eco	Ecoli	336	7	8
ger	German	1,000	20	2
gla	Glass	214	9	7
hab	Haberman	306	3	2
hea	Heart	270	13	2
ion	Ionosphere	351	33	10
iri	Iris	150	4	3
led	Led7digit	500	7	10
mag	Magic	1,902	10	2
new	New-Thyroid	215	5	3
pag	Page-blocks	548	10	5
pen	Penbased	1,992	16	10
pho	Phoneme	5,404	5	2
pim	Pima	768	8	2
rin	Ring	7,400	20	2
sat	Satimage	6,435	36	7
seg	Segment	2,310	19	7
shu	Shuttle	5,800	9	7
tae	Tae	151	5	3
tit	Titanic	2,201	3	2
two	Twonorm	740	20	2
veh	Vehicle	846	18	4
win	Wine	178	13	3
wis	Wisconsin	683	9	2

We have applied a *5-fold cross-validation model* to obtain the performance of the approaches. In each iteration a classifier is learnt using 4 folds and it is tested using the remainder one. This process is carried out 5 times using a different testing fold in each one. To measure the performance of the classifier we have used the accuracy rate. The reported final result is the average among the 5 testing results.

To give statistical support to the analysis of the results, we use the aligned Friedman ranks test [21] to detect statistical differences among a group of results and we report the obtained ranks of each method to check easily how good a method is against the remainder ones. We also consider the usage of the Holm post-hoc test [22] to find the algorithms that reject the equality hypothesis with respect to the best method according to the aligned Friedman ranks test. Furthermore, we compute the adjusted *p*-value (APV) in order to take into account the fact that multiple tests are conducted.

The configuration of FARC-HD is the one recommended by authors, which is provided by default in the KEEL software. We have used four different general overlap functions, whose identifiers and equations are introduced in Table 2. Note that, in this case, we have opted for general overlap functions in which only the boundary condition in 0 is relaxed, i.e., the considered general overlap functions satisfy that

$$\text{If} \prod_{i=1}^{n} x_i = 0 \text{ then } \mathcal{GO}(x_1, \ldots, x_n) = 0$$

and

$$\mathcal{GO}(x_1, \ldots, x_n) = 1 \text{ if and only if } \prod_{i=1}^{n} x_i = 1.$$

Table 2: General overlap functions used in the experimental study.

Identifier	Equation
Prod_Luk	$\mathcal{GO}(x_1,\ldots,x_n)=\prod_{i=1}^n x_i * (\bigvee(\sum_{i=1}^n x_i-(n-1),0))$
GM_Luk	$\mathcal{GO}(x_1,\ldots,x_n)=\sqrt[n]{\prod_{i=1}^n x_i} * (\bigvee(\sum_{i=1}^n x_i-(n-1),0))$
Prod_Nil	$\mathcal{GO}(x_1,\ldots,x_n)=\prod_{i=1}^n x_i * \left(\begin{cases}0 & \text{if } \sum_{i=1}^n x_i\leq 1\\ \bigwedge(x_1,\ldots,x_n) & \text{otherwise}\end{cases}\right)$
GM_Nil	$\mathcal{GO}(x_1,\ldots,x_n)=\sqrt[n]{\prod_{i=1}^n x_i} * \left(\begin{cases}0 & \text{if } \sum_{i=1}^n x_i\leq 1\\ \bigwedge(x_1,\ldots,x_n) & \text{otherwise}\end{cases}\right)$

The main reason for this choice is that when using these boundary conditions, examples with a low matching degree with the antecedent part of a fuzzy rule are not taken into account in the system. These examples would be in the boundaries between different fuzzy rules and consequently they could be difficult for the system. On the other hand, we want to maintain the discriminative power of the system, which is mainly driven by fuzzy rules with high matching degrees. Therefore, if we relaxed the boundary condition in the maximum value 1 we would be losing discriminative power as many different large matching degrees would be mapped into the maximum value 1.

5.2. Results obtained by general overlap functions

In Table 3 we present the obtained results in testing when using the original FARC-HD algorithm (product t-norm) as well as the ones achieved by the four general overlap functions considered in this study. The last column represents the average number of antecedents per rule (AvgAnt) obtained in each dataset[3], which is used to sort the datasets in increasing order. In this manner, we can analize the influence of the general overlap according to the number of elements to be aggregated. We have to remark that this numbers, AvgAnt, are ranged in [1, 3], since fuzzy rules are composed of at least 1 antecedent and the maximum number of antecedents allowed by FARC-HD is 3. The best result in each data-set is highlighted in **bold-face**. Furthermore, the averages for datasets whose averaged number of rules is less or equal than 2 (Mean (≤ 2)), the averages for datasets having more than 2 antecedents (Mean (> 2)) and the global mean are shown.

From these results it can be observed than the global behaviour of all the general overlap functions is slightly better than that of the product t-norm (FARC-HD). Although the global performance of all the general overlap functions is similar, the following facts can be stressed when focusing on the two scenarios determined by the average number of antecedents:

1. When the number of elements to be aggregated is less or equal than 2 the usage of the minimum nilpotent (Prod_Nil and GM_Nil) seems to be appropriate since their mean results are better than the ones of the remainder approaches. Specially good is the behaviour of Prod_Nil, since it provides the best mean result and the best performance in 7 out of the 16 datasets in this part of the study.

2. When the number of elements to be aggregated is greater than 2 the behaviour of Łukasiewicz t-norm is better. This may be due to the fact that this t-norm, when the number of elements to be aggregated is 3, only returns positive values when the average of the input values is larger than $\frac{2}{3}$. This implies

[3]The reported value is the one obtained when applying Prod_Luk. Although the values obtained with the remainder approaches are different, all of them follow a similar increasing order. Therefore, it does not affect our analysis.

Table 3: Results obtained in testing by the different approaches.

Dataset	FARC	Prod_Luk	GM_Luk	Prod_Nil	GM_Nil	AvgAnt
iri	94.00	**96.00**	**96.00**	**96.00**	**96.00**	1.00
wis	96.34	96.49	96.34	96.78	**96.92**	1.10
hab	71.22	**73.19**	72.21	72.52	70.57	1.19
new	96.28	**97.21**	**97.21**	96.74	96.74	1.30
tit	**78.87**	**78.87**	**78.87**	**78.87**	**78.87**	1.39
app	83.94	85.89	84.03	**86.84**	84.94	1.61
ban	**85.79**	85.17	85.40	84.77	84.85	1.64
win	**96.60**	94.92	91.59	94.35	94.37	1.72
pho	80.92	81.75	82.73	**83.27**	82.68	1.73
bup	67.25	66.38	67.54	**67.83**	66.67	1.75
bal	87.04	85.60	85.60	**87.52**	86.72	1.83
rin	91.08	**91.22**	90.54	88.92	90.95	1.85
two	89.19	86.22	87.97	89.59	**91.62**	1.89
shu	95.36	99.49	99.54	**99.63**	99.59	1.94
mag	**81.12**	78.76	79.44	79.60	79.76	1.94
pim	74.87	75.39	74.35	75.26	**75.78**	1.95
Mean ($\leq$ 2)	85.62	85.78	85.58	**86.16**	86.06	
pag	**94.34**	94.16	93.61	93.79	94.34	2.08
ion	88.89	88.33	88.04	**91.75**	88.05	2.19
veh	68.32	70.33	**71.16**	68.68	69.39	2.20
tae	56.28	52.97	54.95	55.66	**56.95**	2.30
gla	64.04	**69.18**	68.72	67.75	69.16	2.30
eco	80.07	79.46	79.46	78.56	**80.36**	2.44
hea	84.44	83.33	**84.81**	80.74	84.07	2.57
sat	80.71	79.93	**82.90**	74.50	72.95	2.57
seg	92.81	94.59	**94.85**	94.81	93.33	2.65
aus	**85.94**	84.93	84.93	85.36	85.22	2.65
pen	**93.18**	91.18	91.36	91.00	91.82	2.67
crx	86.53	87.60	87.91	87.91	**88.21**	2.67
ger	71.90	70.70	**73.20**	71.90	71.50	2.81
cle	55.88	**58.93**	58.24	58.23	55.90	2.90
led	69.80	**70.40**	**70.40**	**70.40**	**70.40**	2.99
Mean ($>$ 2)	78.21	78.40	**78.97**	78.07	78.11	
Global Mean	82.03	82.21	**82.38**	82.24	82.21	

that only fuzzy rules whose matching degrees are very high would be included in the inference and thus, bad rules would be discarded. In this case, the best performance is provided by GM_Luk since it obtains the best result in 6 out of the 15 datasets.

In order to support our findings, we have statistically compared the five approaches in the three scenarios (see Table 4): global behaviour and when the average number of antecedents is either less or equal than 2 or greater than 2 (shown by columns). To do so, we have applied the Aligned Friedman ranks test to conduct the multiple comparison. The obtained p-values are 2.52E-5, 0.0089 and 0.0123, respectively. The obtained ranks are shown in each cell of Table 4. Then, we applied the Holm's post-hoc test to compare, for each case, the best ranking method (the one with the lowest rank, which is highlighted in **bold-face** by columns) with the remainder approaches. The obtained APVs are the number in brackets in each cell.

From the results of the statistical test we can observe that all the approaches behave the same at global level. However, when using datasets whose averaged number of antecedents is less or equal to 2, the usage of the general overlap function Prod_Nil seems to have a positive effect on this algorithm. Finally, when the number of elements to be aggregated is greater than 2, the general overlap function GM_Luk works better than the aternative methods considered in this paper. All in all, we can conclude that general overlap functions are providing competitive results versus the ones achieved by the classical product t-norm.

Table 4: Aligned Friedman and Holm tests to compare the different general overlap functions and FARC.

	Mean (≤ 2)	Mean (> 2)	Global
FARC-HD	45.50 (0.30)	40.03 (0.74)	84.53 (1.00)
Prod_Luk	40.38 (0.61)	38.23 (0.74)	78.82 (1.00)
GM_Luk	46.78 (0.28)	**30.83**	76.10 (1.00)
Prod_Nil	**31.94**	41.87 (0.66)	**74.26**
GM_Nil	37.91 (0.61)	39.03 (0.74)	76.29 (1.00)

6. Conclusions

In this paper we have introduced a novel generalization of overlap functions which allows to simultaneously measure the overlapping of n classes. We have shown that this new notion differs in the boundary conditions, allowing the functions to have zero and one divisor. We have made a theoretical study of the lattice structure, a characterization of these functions in terms of some functions f, g satisfying suitable properties and we have studied different construction methods in terms of aggregation functions and n-dimensional overlap functions. Finally, we have applied the novel notion of general overlap function for computing the matching degree in a classification problem. We have proven that the global behavior of these functions improves the results with respect to some other methods in the literature.

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