Variable population egalitarian ethics and the critical-level: A note*

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Abstract: This paper explores the introduction of a variable critical-level in a variable population context. We focus the attention on the "Critical-Level Egalitarian Rule", a social evaluation procedure which compares two social states as follows: (i) It reproduces the leximin criterion when applied to vectors of identical dimension and (ii) otherwise, it completes the small one with so many times a variable critical-level as to make the two vectors equal in size and applies the leximin criterion again. We prove that the use of a strict monotonic critical-level leads to the intransitivity of the social evaluation rule. This problem disappears when a weak monotonicity condition is required.

Key words: **variable population, leximin, critical-level.**

Running title: Critical-Level Egalitarian Rule

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1. Introduction

This paper deals with comparisons among vectors of real numbers of (possibly) different dimension; the context where these comparisons take place is the evaluation of social states with different population in a welfarist framework. Welfarist means that two social states are going to be compared by looking only at the utility level achieved by each member of the society in the two social states at stake. Most of the work done in the economic literature since the early utilitarians focus mainly in the case in which the number of participants -the population- is fixed. Among the rules used to make such welfarist comparisons, the utilitarian program, which chooses the social situation with the greatest utility summation, has been one of the most studied. This is the rule underlying to almost all of the theoretical and applied cost-benefit analysis.

Other social evaluation rules hark back to the egalitarian principles; examples of this line are: the welfare egalitarianism, the maximin rule or the leximin (the Pareto-efficient extension of the maximin), etc...

In many cases, such welfarist comparisons among public policies involve to compare societies with a different population. Moreover, in many relevant issues, the key questions concern to the ranking between social states with a different number of participants. The studies related to the European social security systems should invoke comparisons among societies of different size. All birth-control policies applicable to many countries of the underdeveloped world ought to be made on the basis of such comparisons among utility vectors of different size.

Again, in this setting, the utilitarian rule -which is very well defined for any size of real vectors- has been deeply analyzed in [6], for instance. In this work, the authors describe two types of economic contexts where utilitarian procedures seem to work very well. One is the timeless variable population case. The other is the notion of altruism in an intergenerational framework.

According to the later interpretation, a social policy is an action today that produces utility consequences for today and tomorrow. In this case, that social policy induces a utility vector of n components, in which the i^{th}-component represents the utility achieved by the representative individual of the i^{th} generation.

The egalitarian -and efficient- perspective, as the opposite of utilitarianism, is the leximin rule. When the population is fixed, this rule recommends the social policy which maximizes the utility of the worst-off individual -or generation, depending on the context- in a lexicographic way: if the worst-off individuals in two given situations are equally well, the rule looks at the two second worst-off individuals; or to the next worst-off individuals, up to the point in which a clear priority is reached. This rule is a "strict order": the only situations which are indifferent from the planner view to a given one, are all of its utility permutations.

Unlike the utilitarian rules, the leximin has no immediate extension to the variable dimension case. In this paper we propose a suitable extension for the leximin criterion in a variable population context. Our proposal takes as a basis the notion of a critical utility level, introduced in [8]. This idea has also been discussed in [3], [6], [9] and [10]. A critical-level is a utility amount such that the addition to a given society of a new component with precisely such a critical utility level, makes the society to be equally well-off. This notion makes sense in both contexts. In the variable population context, it forces the society to welcome new members if the welfare level of the new born is expected to be equal or greater than the minimum needed to preserve life, health and dignity of a human being. In an intergenerational context, it means that no policy should be implemented if it is expected a utility level below that critical-level for some generation in the future under consideration.

In [6], the critical-level utilitarian rules are characterized by a different set of axioms depending of the two mentioned contexts. The critical-level has also been incorporated to the maximin rule in [9].

In the former work the critical-level is supposed to be constant. In the later it is allowed this critical-level to be different for different initial societies. We find more appropriate the notion of variable critical-level. We cannot evaluate a birth-control policy in India by applying the critical-level which would be reasonable -unanimously accepted- in Canada. We cannot pretend the critical utility level to be independent of the actual utility vector of the society. At the end, any social evaluation cannot be independent of the cultural and historic background of the society itself.

This paper explores the introduction of a variable critical-level, in the same vein as Bossert, when we restrict our attention to an extended egalitarian principle. To compare two vectors of different dimensions, we complete the small one with so many times a variable critical-level as to make the two vectors equal in size, and then we apply the usual leximin criterion.

We assume first the critical-level as a function of the utility vector which defines the society. Such a function satisfies the strict monotonic condition: whenever some individual becomes strictly better-off being the rest at least equally well-off, then the critical-level of the society increases. This assumption fits very well with the idea behind the variable critical-level: A richer society welcomes new members at a minimum standard of living greater than a poorer one.

Examples of that kind of variable critical-levels are arithmetic and geometric means, weighted or not.

Surprisingly enough, the use of the leximin evaluation with a strict monotonic critical-level leads to the intransitivity of the social evaluation rule. This constitutes our first impossibility result: There is no strict monotonic critical-level which produces a transitive leximin evaluation.

Next step is to weaken the strength of the monotonicity condition required for the critical-level function. We show that when we impose a weak monotonicity property, then there exists a class of variable critical-level functions for which the leximin procedure is a transitive social ordering.

2. Notation and definitions

Before introducing variable population welfare criteria, some notation is needed. The number of people in some given society will be denoted by $n \in \mathbb{Z}_{++}$ (the set of positive integers). $\mathbb{R}$ ($\mathbb{R}_{++}$) stands for the set of real (positive real) numbers and $\mathbb{R}^n$ ($\mathbb{R}^n_{++}$) is the n-fold Cartesian product of $\mathbb{R}$ ($\mathbb{R}_{++}$). A vector $u = (u_1, u_2, .., u_n) \in \mathbb{R}^n$ is interpreted as a distribution of individual utilities, where it is assumed that these utilities are fully measurable and interpersonally full comparable. Moreover, for reasons that will become clearer in the sequel, it is also assumed that for all distribution of individual utilities, $u = (u_1, u_2, .., u_n)$, $u_i \in \mathbb{R}_{++}$ $\forall i \in \{1, .., n\}$. $\mathfrak{R}$ stands for the union, for all $n \in \mathbb{Z}_{++}$, of $\mathbb{R}^n_{++}$, $\mathfrak{R} \equiv \bigcup\limits_{n} \mathbb{R}^n_{++}$. And for all $n \in \mathbb{Z}_{++}$, given $u, v \in \mathbb{R}^n_{++}$, $u \geq v, u > v, u >> v$ mean respectively, $u_i \geq v_i$ $\forall i \in \{1, .., n\}$, $u \geq v$ and $u \neq v$, $u_i > v_i$ $\forall i \in \{1, .., n\}$. Finally, for any $n \in \mathbb{Z}_{++}$ and $u \in \mathbb{R}^n_{++}$, let u^* be the permutation of u with $u^*_1 \geq u^*_2 \geq ... \geq u^*_n$.

By means of a variable population social welfare criterion, distributions of utilities which are not necessarily of the same dimension can be compared. For $\Gamma \subseteq \mathfrak{R} \times \mathfrak{R}$, a binary decision criterion on Γ, $\succ$, is a subset of Γ. If $(x, y) \in \succ$, we write $x \succ y$. We assume that $\succ$ is asymmetric with the usual interpretation that $x \succ y$ means "x is better than y". We define the indifference relation by $x \sim y \iff [\text{not } x \succ y \text{ and not } y \succ x]$, then we declare x indifferent to y when none is better than the other. We define the relation $\succeq$ by $x \succeq y \iff [x \succ y \text{ or } x \sim y]$, then $x \succeq y$ means that " x is better than or indifferent to y". Notice that the asymmetry of $\succ$ guaranties that $\succeq$ is reflexive and complete.

It is well known that many welfarist rules in the variable population context may lead to the "repugnant conclusion", introduced and discussed in [11], [12] and [13], and widely considered within the literature (see, for instance, [2], [4], [5] and [7]). In all of these works, it is assumed the existence of a utility level of neutrality. A life of an individual is worth living if its utility, as a whole, is above neutrality. We assume the existence of such a neutrality level which has been normalized to zero. All individual utilities are fully measurable and interpersonal comparable after this initial normalization. According to this interpretation, we assume that all living individuals enjoy a standard of living above neutrality; which means that the range of individual utilities is defined in the interval $(0, \infty)$. When a zero neutrality level is considered, the "repugnant conclusion" may appear. Suppose the social evaluation rule to be Classical Utilitarianism, according to which

$$u \succ v \Leftrightarrow \sum_{i=1}^{n} u_i > \sum_{j=1}^{m} v_j, \ \forall u \in \mathbb{R}^n_{++}, \forall v \in \mathbb{R}^m_{++}.$$

By adding to v any number of individuals with utilities arbitrarily close to neutrality, the social ranking between u and the new state v' (in which a number of individuals near to starvation are added to v), may change. There are, indeed, social evaluation operators which avoid this version of the "repugnant conclusion". The example suggested in [8] and [1] is a critical-level utilitarianism, in which

$$u \succ v \Leftrightarrow \sum_{i=1}^{n} (u_i - c) > \sum_{j=1}^{m} (v_j - c), \ \forall u \in \mathbb{R}^n_{++}, \forall v \in \mathbb{R}^m_{++}.$$

where c *is* a positive constant representing a constant critical-level.

The present paper introduces a variable population social welfare rule based on the use of the leximin criterion together with a *variable critical-level.*

Before presenting the notion of *critical-level,* let's add the following piece of notation. Consider two societies with utility distributions $u \in \mathbb{R}^n_{++}$, $v \in \mathbb{R}^m_{++}$, $(u, v) \in \mathbb{R}^{n+m}_{++}$ will denote the utility distribution of a society obtained by the union of the previous two.

Definition 1. *Given* $\succ \subseteq \mathfrak{R} \times \mathfrak{R}$, *a* critical-level *associated to* $\succ$ *is a function* $\varphi : \mathfrak{R} \longrightarrow \mathbb{R}_{++}$, *such that for all* $u \in \mathfrak{R}$, $(u, \varphi(u)) \sim u$. *If* $\exists u, v \in \mathfrak{R}$, $u \neq v$ *such that* $\varphi(u) \neq \varphi(v)$, *then we say that* φ *is a variable critical-level function. Otherwise we call* φ *a fixed critical-level.*

A *critical-level function* provides, for each distribution of individual utilities, the utility that an additional person should enjoy in order to keep social welfare unchanged, given that the utilities of the existing population are unaffected.

First of all, we introduce the two following "natural" properties of a *critical-level function*.

- **Critical-Level Anonymity (C-L.A.):** $\forall\, n \in \mathbb{Z}_{++}$, $\forall u \in \mathbb{R}^n_{++}$, for every permutation $\pi : \{1, .., n\} \rightarrow \{1, .., n\}$, $\varphi(u) = \varphi(\pi(u))$.

Critical-Level Anonymity says that the names of the agents do not affect the *critical-level* associated to a utility distribution.

- **Strict Monotonicity (S.Mon):** $\forall\, n \in \mathbb{Z}_{++}$, $\forall u, v \in \mathbb{R}^n_{++}$, if $u > v$ then $\varphi(u) > \varphi(v)$.

Strict Monotonicity calls for an increase of the *critical-level* whenever the utility distribution of a given society changes in such a way that no agent is worse-off and at least one agent is better-off.

Examples of *critical-level functions* satisfying these properties can be found. Consider, for instance, $\forall\, n \in \mathbb{Z}_{++}$, $\forall u \in \mathbb{R}^n_{++}$, the arithmetic and the geometric mean of $u = (u_1, u_2, .., u_n)$, denoted by $\mu(u)$ and $g(u)$, respectively. Particularly, we are interested in the analysis of the critical-level egalitarian rule, an extension of the leximin criterion to a variable population context. Before presenting this variable population welfare criterion, we introduce some additional notation. For any $n, m \in \mathbb{Z}_{++}$ with $m \geq n$, for any $u \in \mathbb{R}^n_{++}$, $v \in \mathbb{R}^m_{++}$ and for any $r \in \mathbb{R}_{++}$, $u(r)^m$ stands for the m-dimensional extension of u by means of r and $v\mid_n$ stands for the reduction to dimension n of v, that is:

$\forall u \in \mathbb{R}^n_{++}$, $u(r)^m \in \mathbb{R}^m_{++}$,

$$[u(r)^m]_i = u_i \;\forall i \in \{1, .., n\}\;,\; [u(r)^m]_k = r \;\, for\;\, k = n+1, .., m.$$

$\forall v \in \mathbb{R}^m_{++}$,

$$v\mid_n \in \mathbb{R}^n_{++},\; [v\mid_n]_i = v_i \;\forall i \in \{1, .., n\}\,.$$

Definition 2. *For $n \in \mathbb{Z}_{++}$, the leximin (lexicographic Maximin) criterion on $\mathbb{R}^n_{++}$, $>^n_L$, is defined as follows. For all $u, v \in \mathbb{R}^n_{++}$,*

$$u >^n_L v \Leftrightarrow \exists k \in \{1, .., n\} \mid u^*_i = v^*_i \; \forall i \in \{k+1, .., n\} \wedge u^*_k > v^*_k.$$

In the case of egalitarian and efficient rules (the leximin), there is no immediate extension of this criterion to the domain $\mathfrak{R}$. An extension should, indeed, reproduce the usual leximin when applied to the domain $\mathbb{R}^m_{++}$ for all $m \in \mathbb{Z}_{++}$. Take, as an example, the following extension $\succ^*_L \subseteq \mathfrak{R} \times \mathfrak{R}$, such that $\forall u \in \mathbb{R}^p_{++}$, $\forall v \in \mathbb{R}^q_{++}$ with $p \leq q$

$$\begin{array}{rcl} (i) \text{ if } p & = & q, \; \succ^*_L \equiv >^p_L \;\; \forall p \in \mathbb{Z}_{++}, \\ (ii) \text{ if } p & < & q, \; u \succ^*_L v \Leftrightarrow u^* >^p_L v^* \mid_p, \text{ otherwise } v \succ^*_L u. \end{array}$$

This is a very simple extension (the leximax criterion in the ranking sets literature proceeds in a similar way -starting from the maximum-). The rule adds to the usual leximin, a new command to compare vectors of different dimension when the leximin criterion cannot reach a strict preference. In that case, the larger vector is chosen. This criterion leads straightforward to the "repugnant conclusion". Therefore, to avoid the "repugnant conclusion", we propose the following extension of the leximin criterion, which will be called the $\varphi-$critical-level egalitarian rule.

Definition 3. *For any function $\varphi : \mathfrak{R} \longrightarrow \mathbb{R}_{++}$, $\succ^\varphi_L \subseteq \mathfrak{R} \times \mathfrak{R}$ is the $\varphi-$critical-level egalitarian rule if $\forall u \in \mathbb{R}^p_{++}$, $\forall v \in \mathbb{R}^q_{++}$,*

$$u \succ^\varphi_L v \Leftrightarrow \left\{ \begin{array}{c} u >^p_L v \text{ if } p = q \\ u(\varphi(u))^q >^q_L v \text{ if } q > p \\ u >^p_L v\,(\varphi(v))^p \text{ if } p > q \end{array} \right. .$$

In spite of this seemingly reasonable extension of the leximin rule, the use of a strict monotonic and anonymous critical-level may lead to intransitivity of the strict preference. Next example shows this statement.

Example 1. *Consider the utility distribution corresponding to three different societies:*

$$u = (9, 9, 9, 9, 2) \in \mathbb{R}^5; \; v = (10, 10, 10, 2) \in \mathbb{R}^4 \text{ and } w = (20, 9.5, 2) \in \mathbb{R}^3.$$

Suppose that the critical-level function is the arithmetic mean, μ. Then, given that $\mu(v) = 8$ and $\mu(w) = 10.5$,

$u \succ_L^\mu v$ since $(9,9,9,9,2) >_L^5 (10,10,10,8,2)$,
$v \succ_L^\mu w$ since $(10,10,10,2) >_L^4 (20,10.5,9.5,2)$,
and $w \succ_L^\mu u$ since $(20,10.5,10.5,9.5,2) >_L^5 (9,9,9,9,2)$.

From the previous example, some natural questions immediately arise. How can we ensure that a *$\varphi-$critical-level egalitarian rule* is transitive? Are the above listed properties on a *critical-level function* compatible with a transitive *$\varphi-$critical-level egalitarian rule*? Next section analyzes this kind of questions.

3. The meaning of transitivity

We have assumed the social evaluation rule, $\succ$, to be asymmetric but, according to example 1 presented above, we cannot guarantee the transitivity of $\succ_L^\varphi$, our candidate to encompass the efficient egalitarianism in a variable population framework.

Next proposition characterizes the transitivity of a *$\varphi-$critical-level egalitarian rule* by means of two appealing properties: Number of Entrants Independence and Critical-Level Consistency.

Before presenting these conditions let us remark that all of the following results have been established for the weak preference, $\succsim_L^\varphi$, associated to $\succ_L^\varphi$ (see section 2). It is well known that asymmetry and negative transitivity of $\succ$ (that is, not $(x \succ y)$ and $not(y \succ z) \Rightarrow not(x \succ z)$) are equivalent to both completeness and transitivity of $\succsim$. Thus, Proposition 1 shows necessary and sufficient conditions for $\succ_L^\varphi$ to be not only transitive, but negatively transitive as well.

- **Number of Entrants Independence (N.E.I.)**: $\forall\, n \in \mathbb{Z}_{++}$, $\forall u \in \mathbb{R}^n_{++}$, if $\varphi(u) = \alpha$ then $\varphi(u, \alpha) = \alpha$.

Number of Entrants Independence requires the critical-level of a given society to be the same as the critical-level of the society obtained by adding a member to the initial one with a utility amount corresponding to its critical-level.

- **Critical-Level Consistency (C-L.C.)**: Given $\succsim \subseteq \mathfrak{R} \times \mathfrak{R}$ and $u, v \in \mathfrak{R}$, $u \succsim v \Leftrightarrow (u, \varphi(u)) \succsim (v, \varphi(v))$.

Critical-Level Consistency demands the ranking between two utility distribution to be independent of the incorporation, to each society, of a further person with the utility amount corresponding to its critical-level.

Proposition 1. *$\succsim_L^\varphi$ is transitive if and only if φ satisfies Number of Entrants Independence and $\succsim_L^\varphi$ satisfies Critical-Level Consistency.*

Proof. We will establish the proof in three steps:

- (i) If $\succsim_L^\varphi$ is transitive, then φ satisfies Number of Entrants Independence and $\succsim_L^\varphi$ satisfies Critical-Level Consistency. Firstly, we will show that if $\succsim_L^\varphi$ is transitive, then φ satisfies N.E.I. By definition of $\succsim_L^\varphi$ we have that $u \sim_L^\varphi (u, \varphi(u))$ and $(u, \varphi(u)) \sim_L^\varphi (u, \varphi(u), \varphi(u, \varphi(u)))$. Then transitivity implies $u \sim_L^\varphi (u, \varphi(u), \varphi(u, \varphi(u)))$. Therefore $(u, \varphi(u), \varphi(u)) \sim_L (u, \varphi(u), \varphi(u, \varphi(u)))$, which implies $\varphi(u, \varphi(u)) = \varphi(u)$. Secondly, we will show that if $\succsim_L^\varphi$ is transitive, then $\succsim_L^\varphi$ satisfies C-L.C. Consider $u, v \in \mathfrak{R}$ such that $u \succsim_L^\varphi v$. By definition of $\succsim_L^\varphi$, $(u, \varphi(u)) \sim_L^\varphi u$ and $(v, \varphi(v)) \sim_L^\varphi v$, therefore $(u, \varphi(u)) \sim u \succsim_L^\varphi v \sim (v, \varphi(v))$, and by transitivity $(u, \varphi(u)) \succsim_L^\varphi (v, \varphi(v))$. The converse is also true by a similar procedure.

- (ii) If φ satisfies Number of Entrants Independence and $\succsim_L^\varphi$ satisfies Critical-Level Consistency, then $\succsim_L^\varphi$ satisfies Extended Critical-Level Consistency, that is, given $u, v \in \mathfrak{R}$, $u \succsim_L^\varphi v \Leftrightarrow (u, 1^n \cdot \varphi(u)) \succsim_L^\varphi (v, 1^n \cdot \varphi(v))$. Let $u, v \in \mathfrak{R}$ such that $u \succsim_L^\varphi v$ and let $n \in \mathbb{Z}_{++}$. By applying C-L.C. we get that $u \succsim_L^\varphi v \Leftrightarrow (u, \varphi(u)) \succsim_L^\varphi (v, \varphi(v))$. Given that φ satisfies N. E. I., $\forall n \in \mathbb{Z}_{++}, \forall x \in \mathbb{R}_{++}^n, \varphi((x, \varphi(x))) = \varphi(x)$. Then it is possible applying again C-L.C. on $(u, \varphi(u))$ and $(v, \varphi(v))$, getting that $(u, \varphi(u)) \succsim_L^\varphi (v, \varphi(v)) \Leftrightarrow (u, \varphi(u), \varphi(u)) \succsim_L^\varphi (v, \varphi(v), \varphi(v))$. Therefore $u \succsim_L^\varphi v \Leftrightarrow (u, \varphi(u), \varphi(u)) \succsim_L^\varphi (v, \varphi(v), \varphi(v))$. Repeating this reasoning (n-2) times we can get $u \succsim_L^\varphi v \Leftrightarrow (u, 1^n \cdot \varphi(u)) \succsim_L^\varphi (v, 1^n \cdot \varphi(v))$.

- (iii) if $\succsim_L^\varphi$ satisfies Extended Critical-Level Consistency and φ satisfies Number of Entrants Independence, then $\succsim_L^\varphi$ is transitive. Let $u, v, w \in \mathfrak{R}$ such that $u \succsim_L^\varphi v, v \succsim_L^\varphi w$ and $dim(u) = r$, $dim(v) = s$ and $dim(w) = t$ with $max\{r, s, t\} = p$. By applying Extended Critical-Level Consistency, Number of Entrants Independence and taking into account the definition of $\succsim_L^\varphi$, $u \succsim_L^\varphi v \Leftrightarrow (u, 1^{p-\max\{r,s\}} \cdot \varphi(u)) \succsim_L^\varphi (v, 1^{p-\max\{r,s\}} \cdot \varphi(v)) \Leftrightarrow (u, 1^{p-r} \cdot \varphi(u)) \geq_L^p (v, 1^{p-s} \cdot \varphi(v))$. By using the same reasoning we get $v \succsim_L^\varphi w \Leftrightarrow (v, 1^{p-s} \cdot \varphi(v)) \geq_L^p (w, 1^{p-t} \cdot \varphi(w))$. Given that $\geq_L^p$is transitive, $(u, 1^{p-r} \cdot \varphi(u)) \geq_L^p (w, 1^{p-t} \cdot \varphi(w))$. Now by definition $(u, 1^{p-\max\{r,t\}} \cdot \varphi(u)) \succsim_L^\varphi (w, 1^{p-\max\{r,t\}} \cdot \varphi(w))$. Finally, by applying once more Extended Critical-Level Consistency, $u \succsim_L^\varphi w$. ■

4. An impossibility result

Once we know necessary and sufficient conditions for the $\varphi-$*critical-level egalitarian rule* to be transitive, our aim is to explore the introduction of a variable critical-level when we restrict the attention to this criterion. The unique property on a *critical-level function*, from the listed in section 1, which implies variability of the critical-level is Strict Monotonicity. This Monotonicity condition get the idea behind the variable critical-level: A richer society welcomes new members at a minimum standard of living greater than a poorer one. Unfortunately, next result shows that the use of leximin evaluation with strict monotonic critical-level leads to the intransitivity of the social evaluation rule.

Theorem 1: $\nexists \varphi : \mathfrak{R} \longrightarrow \mathbb{R}_{++}$ satisfying Strict Monotonicity and such that $\succsim_L^{\varphi}$ is transitive.

In order to proof the previous theorem we need the following lemmata.

Lemma 1: If $\succsim_L^{\varphi}$ is transitive, then φ satisfies critical-Level Anonymity.

Proof. By definition of $\succsim_L^{\varphi}$ we get that $\forall n \in \mathbb{Z}_{++}, \forall u \in \mathbb{R}_{++}^n, \forall$ *permutation* $\pi : \{1,..,n\} \rightarrow \{1,..,n\}, u \sim_L^{\varphi} \pi(u)$. Moreover we know, by proposition 1, that transitivity implies Critical-Level Consistency, therefore $(u,\varphi(u)) \sim_L^{\varphi} (\pi(u), \varphi(\pi(u)))$. Again by definition of $\succsim_L^{\varphi}$ we get $\varphi(u) = \varphi(\pi(u))$. ■

Lemma 2: $\nexists \varphi : \mathfrak{R} \longrightarrow \mathbb{R}_{++}$ satisfying

(i) Strict Monotonicity,

(ii) $\varphi(u) \geq u_2^* \ \forall n \geq 2, \forall u \in \mathbb{R}_{++}^n$, and such that

(iii) $\succsim_L^{\varphi}$ is transitive.

Proof. Suppose that $\exists \varphi : \mathfrak{R} \longrightarrow \mathbb{R}_{++}$ satisfying (i), (ii) and (iii). Let $u \in \mathbb{R}_{++}^n$. We know, by proposition 1, that transitivity of $\succsim_L^{\varphi}$ implies Number of Entrants Independence, Therefore $\varphi(u, \varphi(u), \varphi(u)) = \varphi(u)$. Consider $v = (u_1 - 1, u_2, .., u_n, \varphi(u), \varphi(u))$, by (ii) $\varphi(v) \geq v_2^*$ and by construction $v_2^* \geq \varphi(u)$, therefore $\varphi(v) \geq \varphi(u)$. Now, since $v < (u, \varphi(u), \varphi(u))$, by applying Strict Monotonicity we get $\varphi(v) < \varphi(u)$, which contradicts the previous inequality. ■

Proof of Theorem 1: Suppose that $\exists \varphi : \mathfrak{R} \longrightarrow \mathbb{R}_{++}$ satisfying Strict Monotonicity and such that $\succsim_L^{\varphi}$ is transitive. By lemma 2, $\exists n \geq 2, u \in \mathbb{R}_{++}^n$ such that $\varphi(u) < u_2^*$. Consider, by lemma 1 ,u^* and construct a family of vectors $v(k,x) = (u_1^* + k, u_2^* - x, u_3^*, .., u_n^*)$ for $k \geq 0$ and $x \in [0, u_2^* - \varphi(u)) \equiv [0, b)$. There are three possibilities:

(i) $\exists x \in [0,b)$, $y \in (x,b)$ and $k>0$ such that $\varphi(v(0,x)) < \varphi(v(k,y))$. On one hand, by definition of $\succsim_L^{\varphi}, v(0,x) \succ_L^{\varphi} v(k,y)$ because $[(v(x,0)]_1 > [(v(x,0)]_2 = u_2^* - x > u_2^* - y = [(v(y,k)]_2$ and $[(v(x,0)]_j = [v((y,k)]_j \, \forall j > 2$. On the other hand and taking into account that, by hypothesis, $\varphi(v(0,x)) < \varphi(v(k,y))$ and , by Strict Monotonicity, $\varphi(v(0,x)) \leq \varphi(v(0,0)) \equiv \varphi(u) < u_2^* - y$, the introduction of the critical levels will change the ordination, that is, $(v(0,x), \varphi\left(v(0,x)\right)) \prec_L^{\varphi} (v(k,y), \varphi\left(v(k,y)\right))$. But this fact contradicts Critical-Level Consistency and, by proposition 1, transitivity.

(ii) $\exists x \in [0,b)$, $y \in (x,b)$ and $k>0$ such that $\varphi(v(0,x)) = \varphi(v(k,y))$. By taking $z = \frac{x+y}{2}$ we are in case (i) since $\varphi(v(0,x)) < \varphi(v(k,z))$.

(iii) $\forall x \in [0,b)$, $\forall y \in (x,b)$ and $\forall k > 0$, $\varphi(v(0,x)) > \varphi(v(k,y))$. In this case we define the correspondence $F_{k^*} : [0,b) \rightarrow\rightarrow \mathbb{R}_{++}$ such that $\forall \ z \in [0,b)$, and a fixed $k^* > 0$, $F_{k^*}(z) = \left[\varphi\left(v(0,z)\right), \varphi\left(v(k^*,z)\right)\right]$. Notice that, by Strict Monotonicity, F_{k^*} is always multivaluated, that is, $\forall z \in [0,b)\,, F_{k^*}(z) = [a,b]$ with $a<b$. Moreover by supposing, without loss of generality, $x>y$ and taking into account the hypothesis $\varphi(v(0,x)) > \varphi(v(k,y))$ we get that $\forall x,y \in [0,b)$, $x \neq y$, $F_{k^*}(x) \cap F_{k^*}(y) = \emptyset$. But, since the number of disjoint intervals in $\mathbb{R}_{++}$ is numerable and $[0,b)$ is not, our conclusion is absurd. ■

5. Weakening strict monotonicity

The next natural step, from the previous impossibility result, is to weaken the strict monotonicity property of the critical-level function. We propose the following:

- **Monotonicity (Mon):** $\forall \ n \in \mathbb{Z}_{++}$, $\forall u, v \in \mathbb{R}_{++}^n$, if $u >> v$ then $\varphi(u) > \varphi(v)$.

Monotonicity calls for an increase of the *critical-level* whenever the utility distribution of a given society changes in such a way that each individual is better-off.

The following result shows that when this version of monotonicity is required there exists an interesting class of variable critical-level functions compatible with the transitivity of the *φ-critical-level egalitarian rule.* This class share the spirit of what has been called “Positional Dictatorship” in social choice theory; a social evaluation rule, $\succ$, is a Positional Dictatorship if there is “a position” k such that $u \succ v \Leftrightarrow u_k^* > v_k^*$. The application of this idea to our setting may be seen in the following result and a particular case can be found in [3].

Proposition 2: Let $\varphi : \mathfrak{R} \to \mathbb{R}_{++}$ defined as follows,

$$\varphi(u) = \begin{cases} u_k^* & \forall m \geq k, \ \forall u \in \mathbb{R}^m \\ u_m^* & \forall m < k, \ \forall u \in \mathbb{R}^m \end{cases},$$

Then φ satisfies Monotonicity and $\succsim_L^\varphi$ is transitive.

Proof. It is straightforward to check that φ satisfies Monotonicity. Firstly, we will show that φ satisfies Number of Entrants Independence. Consider $u \in \mathbb{R}^m$ such that $\varphi(u) = \alpha$. If $m \geq k$, $(u, \alpha)_k^* = u_k^* = \alpha$, so $\varphi((u, \alpha)) = \varphi(u)$. If $m < k$, $(u, \alpha)_{m+1}^* = u_m^* = \alpha$, therefore $\varphi((u, \alpha)) = \varphi(u)$. Secondly, we will show that $\succsim_L^\varphi$ satisfies Critical-Level Consistency. Consider $u \in \mathbb{R}_{++}^p$, $v \in \mathbb{R}_{++}^q$, and $u \sim_L^\varphi v$, and suppose, without loss of generality, that $p \leq q$. In this case, $(u, 1^{(q-p)} \cdot \varphi(u))^* = v^*$ so by Number of Entrants Independence and taking into account that $\varphi((u, 1^{(q-p)} \cdot \varphi(u))) = \varphi(v)$, we get $(u, 1^{(q-p+1)} \cdot \varphi(u))^* = (v, \varphi(v))^*$ which implies $(u, 1^{(q-p+1)} \cdot \varphi(u)) \sim_L (v, \varphi(v))$ and $(u, \varphi(u)) \sim_L^\varphi (v, \varphi(v))$. Conversely, $(u, \varphi(u)) \sim_L^\varphi (v, \varphi(v))$ implies that $(u, 1^{(q-p+1)} \cdot \varphi(u)) \sim_L (v, \varphi(v))$, then $(u, 1^{(q-p+1)} \cdot \varphi(u))^* = (v, \varphi(v))^*$, now by Number of Entrants Independence $\varphi(u) = \varphi(v)$, therefore we get $(u, 1^{(q-p)} \cdot \varphi(u))^* = v^*$, so $u \sim_L^\varphi v$. If $u \succ_L^\varphi v$, suppose that $p \leq q$, then $(u, 1^{(q-p)} \cdot \varphi(u)) >_L^q v$. Case of $\varphi(u) \geq \varphi(v)$ $(u, 1^{(q-p+1)} \cdot \varphi(u)) >_L^q (v, \varphi(v))$, then $(u, \varphi(u)) \succ_L^\varphi (v, \varphi(v))$. Case of $\varphi(u) < \varphi(v)$ $(u, 1^{(q-p)} \cdot \varphi(u))_k^* < v_k^*$ then there exist $j > k$ such that $(u, 1^{(q-p)} \cdot \varphi(u))_j^* > v_j^*$ with equality for $i > j$. Therefore, if we add $\varphi((u, 1^{(q-p)} \cdot \varphi(u))) = \varphi(u)$,by Number of Entrants Independence, and $\varphi(v)$, we get $(u, 1^{(q-p)} \cdot \varphi(u))_{j+1}^* > (v, \varphi(v))_{j+1}^*$ with equality for $i > j + 1$. Then $(u, \varphi(u)) \succ_L^\varphi (v, \varphi(v))$. Conversely, let $(u, \varphi(u)) \succ_L^\varphi (v, \varphi(v))$. Suppose that $(u, 1^{(q-p)} \cdot \varphi(u))_k^* = \varphi(u) > \varphi(v) = v_k^*$ and $(u, 1^{(q-p)} \cdot \varphi(u))_j^* = v_j^*$ for $j < k$ then $(u, 1^{(q-p)} \cdot \varphi(u)) \succ_L^\varphi (v, \varphi(v))$ which implies $u \succ_L^\varphi v$. If the first inequality between $(u, 1^{(q-p)} \cdot \varphi(u))^*$ and v^* appears before coordinate k, then obviously $u \succ_L^\varphi v$. If the first inequality between $(u, 1^{(q-p)} \cdot \varphi(u))^*$ and v^* appears after coordinate k, $\varphi(u) = \varphi(v)$ and again $u \succ_L^\varphi v$. Case of $p > q$ the reasoning is analogous. ■

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