SPARSE APPROXIMATION USING NEW GREEDY-LIKE BASES IN SUPERREFLEXIVE SPACES

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Abstract. This paper is devoted to theoretical aspects on optimality of sparse approximation. We undertake a quantitative study of new types of greedy-like bases that have recently arisen in the context of nonlinear m-term approximation in Banach spaces as a generalization of the properties that characterize almost greedy bases, i.e., quasi-greediness and democracy. As a means to compare the efficiency of these new bases with already existing ones in regards to the implementation of the Thresholding Greedy Algorithm, we place emphasis on obtaining estimates for their sequence of unconditionality parameters. Using an enhanced version of the original Dilworth-Kalton-Kutzarova method from [17] for building almost greedy bases, we manage to construct bidemocratic bases whose unconditionality parameters satisfy significantly worse estimates than almost greedy bases even in Hilbert spaces.

1. Introduction

The recent developments in the study of the efficiency of the Thresholding Greedy Algorithm (TGA for short) have given rise to new types of greedy like bases which are of interest both from the abstract point of view of functional analysis and also from the more applied nature of the problem of obtaining optimal numerical computations associated to sparse approximation by means of nonlinear algorithms.

The TGA simply takes m terms with the maximum absolute values of the coefficients from the expansion of a signal (a function) relative to a fixed representation system (a basis). Different greedy algorithms originate from different ways of choosing the coefficients of the linear combination in the m-term approximation to the signal. Another name, commonly used in the literature for m-term approximation is *sparse approximation*. Sparse approximation of functions is a powerful analytic tool which is present in many important applications to image and signal processing, numerical computation, or compressed sensing.

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It is fair to say that greedy approximation theory evolved from the study of the three main types of greedy-like bases, namely, greedy, quasi-greedy, and almost greedy bases. Greedy bases are the best for application of the TGA for sparse approximation, since for any function f in a given space $\mathbb{X}$, after m iterations it provides approximations with an error of the same order as the best m-term theoretical approximation to f. On the other hand, that a basis is quasi-greedy merely guarantees that, for any $f \in \mathbb{X}$, the TGA provides approximants that converge to f but does not guarantee the optimal rate of convergence.

Both greedy and quasi-greedy bases were introduced in the pioneering work of Konyagin and Temlyakov [23] from 1999, whereas almost greedy bases were defined shortly afterwards by Dilworth et al. [18] in what with hindsight would be, together with the work of Wojtaszczyk [27], the forerunner article on the functional analytic approach to the theory. If Konyagin and Temlyakov had characterized greedy bases as unconditional bases with the additional property of being democratic, Dilworth et al. characterized almost greedy bases as those bases that are simultaneously quasi-greedy and democratic.

In studying the optimality of the TGA, other bases have emerged which, despite being more general than quasi-greedy bases, still preserve essential properties in greedy approximation. Delving deeper into these properties is of interest both from a theoretical and a practical viewpoint. On one hand, isolating the specific features of those bases makes the theory progress; on the other hand, from a more applied approach, working with these properties leads to obtaining sharper estimates for the constants that measure the efficiency of the TGA (see [4]).

In this paper we concentrate on squeeze-symmetric bases and truncation quasi-greedy bases, in a sense that will be made explicit below, with an eye to the quantitative aspects of the theory. The central question we ask ourselves is whether these bases retain certain relevant numerical features of almost greedy and quasi-greedy bases or not. Answering this question would help us to better acknowledge their role in sparse approximation theory.

From the point of view of sparse approximation in Banach spaces with respect to the TGA, the most important numerical information of a basis $\mathcal{X} = (\boldsymbol{x}_n)_{n=1}^\infty$ is obtained through the sequence $(\boldsymbol{L}_m)_{m=1}^\infty$ of its Lebesgue parameters, which, roughly speaking, measures how far $\mathcal{X}$ is from being greedy. The growth of these parameters is linearly determined by the combination of the unconditionality parameters $(\boldsymbol{k}_m)_{m=1}^\infty$, which quantify the conditionality of $\mathcal{X}$, and the squeeze symmetric parameters, which quantify a symmetry property related to democracy (see [4, Theorem 1.5]). In the case when the basis $\mathcal{X}$ is squeeze-symmetric we have,

$$\boldsymbol{L}_m \approx \boldsymbol{k}_m, \quad m \in \mathbb{N},$$

hence for this type of bases the growth of the Lebesgue parameters is completly controlled by the growth of the unconditionality parameters.

The best one can say about the asymptotic estimates for the unconditionality parameters of (semi-normalized) quasi-greedy bases in Banach spaces is that

$$\boldsymbol{k}_m \lesssim 1 + \log m, \quad m \in \mathbb{N}, \tag{1.1}$$

(see [17, Lemma 8.2]). In turn, an asymptotic upper bound for the unconditionality parameters of truncation quasi-greedy bases in Banach spaces was estimated in [9, Theorem 5.1], where it was proved that if $\mathcal{X}$ is a (semi-normalized) truncation quasi-greedy basis of a Banach space $\mathbb{X}$ then (1.1) still holds.

Based on this, one might feel tempted to conjecture that, in spite of the fact that truncation quasi-greedy bases are a weaker form of quasi-greediness, the efficiency of the greedy algorithm for the former kind of bases is the same as the efficiency we would get for the latter. There are recent results of a more qualitative nature that substantiate this guess, such as [5, Theorem 9.14], [10, Theorem 4.3], [5, Proposition 10.17(iii)], [3, Corollary 4.5] and [2, Corollary 2.6], all of which are generalizations to truncation quasi-greedy bases of results previously obtained for quasi-greedy bases. These results improve [1, Theorem 3.1], [19, Theorem 4.2], [18, Proposition 4.4], [17, Corollary 8.6] and [18, Theorem 5.4], respectively.

It is therefore crucial to determine whether truncation quasi-greedy bases provide the same accuracy in m-term greedy approximation as quasi-greedy bases, in general Banach spaces or under certain smoothness conditions of the space. Imposing superreflexivity to the underlying Banach space is indeed a very natural restriction that leads to an improvement of the performance of the TGA. For instance, it was shown in [7, Theorem 1.1] that the unconditionality parameters of quasi-greedy bases in these spaces satisfy the sharper estimate

$$\boldsymbol{k}_m \lesssim (1 + \log m)^{1-\epsilon}, \quad m \in \mathbb{N}, \tag{1.2}$$

for some $0 < \epsilon < 1$ depending on the basis and the space.

In this paper we disprove the guess that the estimate (1.2) should pass to truncation quasi-greedy bases of super-reflexive Banach spaces (see [3, Remark 3.9]) by building squeeze-symmetric bases with "large" unconditionality parameters even inside Hilbert spaces. To the best of our knowledge, this provides the first evidence of a different behavior between the implementation of TGA for quasi-greedy bases and truncation quasi-greedy bases. In fact, the bases we construct belong to the more demanding class of bidemocratic bases! Thus, our results connect with and give more relevance to the first known examples of bidemocratic bases which are not quasi-greedy (see [2]). The method we use in our construction is of interest in the theory by itself since it permits to extend the validity of the Dilworth-Kalton-Kutzarova method (DKK method for short) to a less restritive class of bases than the ones considered in [17] and [6]. The DKK method was invented in [17] with the purpose of constructing almost greedy bases in separable Banach spaces which contain a complemented symmetric basic sequence. For the original DKK method to work, the main ingredients are a semi-normalized Schauder basis $\mathcal{X}$ of a Banach space $\mathbb{X}$ and a subsymmetric sequence space, from which we obtain a sequence space $\mathbb{Y}$ whose unit vector system is a Schauder basis fulfilling some special features. Our contribution here consists in being able to implement the DKK method with 'bases' $\mathcal{X}$ which, one one hand, are not necessarily Schauder bases and, on the other hand, need not be semi-normalized. We then study how this extension influences the properties of the resulting basis of the space $\mathbb{Y}$, with the intention to investigate its performance relative to the TGA.

2. Background and terminology

Throughout this paper we will use standard notation and terminology from Banach spaces and greedy approximation theory, as can be found, e.g., in [11]. We also refer the reader to the recent article [5] for other more especialized notation. We next single out however the most heavily used terminology.

Let $\mathbb{X}$ be an infinite-dimensional separable Banach space (or, more generally, a quasi-Banach space) over the real or complex field $\mathbb{F}$. We will denote by $\langle B\rangle$ the linear span of a subset B of $\mathbb{X}$. In turn, $[B]$ denotes the closed linear span of B. Throughout this paper by a *basis* of $\mathbb{X}$ we mean a sequence $\mathcal{X}=(\boldsymbol{x}_n)_{n=1}^\infty$ that generates the entire space, in the sense that $[\mathcal{X}]=\mathbb{X}$, and for which there is a (unique) sequence $\mathcal{X}^*=(\boldsymbol{x}_n^*)_{n=1}^\infty$ in the dual space $\mathbb{X}^*$ such that $\boldsymbol{x}_n^*(\boldsymbol{x}_k)=\delta_{k,n}$ for all k, $n\in\mathbb{N}$. We will refer to the basic sequence

$\mathcal{X}^*$ in $\mathbb{X}^*$ as to the *dual basis* of $\mathcal{X}$. If the linear span of $\mathcal{X}^*$ is w^*-dense is $\mathbb{X}^*$, i.e., if the *coefficient transform*, given by

$$\mathcal{F} = \mathcal{F}[\mathcal{X}, \mathbb{X}] \colon \mathbb{X} \to \mathbb{F}^{\mathbb{N}}, \quad f \mapsto (\boldsymbol{x}_n^*(f))_{n=1}^\infty,$$

is one-to-one, we say that the basis $\mathcal{X}$ is *total*.

The *support* of $f \in \mathbb{X}$ with respect to $\mathcal{X}$ is the set

$$\operatorname{supp}(f) = \{n \in \mathbb{N} \colon \boldsymbol{x}_n^*(f) \neq 0\}.$$

Let $\mathcal{E} = (\boldsymbol{e}_n)_{n=1}^\infty$ be the unit vector system of $\mathbb{F}^{\mathbb{N}}$, and let $(\boldsymbol{e}_n^*)_{n=1}^\infty$ be the unit functionals defined for $f = (a_k)_{k=1}^\infty \in \mathbb{F}^{\mathbb{N}}$ and $n \in \mathbb{N}$ by $\boldsymbol{e}_n^*(f) = a_n$. A *sequence space* will be a quasi-Banach space $\mathbb{Y}$ such that $c_{00} \subseteq \mathbb{Y} \subseteq \mathbb{F}^{\mathbb{N}}$ such that c_{00} is a dense subset of $\mathbb{Y}$. Note that, if $\mathbb{Y}$ is a sequence space, $\mathcal{E}$ is a total basis of $\mathbb{Y}$ whose dual basis is $(\boldsymbol{e}_n^*|_{\mathbb{Y}})_{n=1}^\infty$. Conversely, given a quasi-Banach space $\mathbb{Y}$ in which c_{00} is a dense subset, the sequence $\mathcal{E}$ is a basis if and only if $\boldsymbol{e}_n^*|_{c_{00}}$ is bounded for all $n \in \mathbb{N}$; and $\mathcal{E}$ is a total basis if and only if there is a one-to-one continuous extension $T \colon \mathbb{Y} \to \mathbb{F}^{\mathbb{N}}$ of the identity map on c_{00}. Here, we consider $\mathbb{F}^{\mathbb{N}}$ endowed with the pointwise convergence topology. If $\mathbb{Y}$ is a sequence space we will identify its dual space $\mathbb{Y}^*$ with the sequence space consisting of all $g \in \mathbb{F}^{\mathbb{N}}$ such that $\langle \cdot, g\rangle$ restricts to a functional of $\mathbb{Y}$, where $\langle \cdot, \cdot \rangle$ is the canonical dual pairing defined for $f = (a_n)_{n=1}^\infty \in c_{00}$ and $g \in (b_n)_{n=1}^\infty \in \mathbb{F}^{\mathbb{N}}$ by

$$\langle f, g\rangle = \sum_{n=1}^\infty a_n b_n.$$

Let $\mathbb{E}$ denote the set of all scalars of modulus one. Given $A \subseteq \mathbb{N}$ finite and $\varepsilon = (\varepsilon_n)_{n\in A} \in \mathbb{E}^A$ we put

$$\mathbb{1}_{\varepsilon,A}[\mathcal{X}, \mathbb{X}] = \sum_{n\in A} \varepsilon_n \, \boldsymbol{x}_n.$$

If $\varepsilon_n = 1$ for all $n \in A$, we set $\mathbb{1}_A[\mathcal{X}, \mathbb{X}] = \mathbb{1}_{\varepsilon,A}[\mathcal{X}, \mathbb{X}]$.

A basis $\mathcal{X}$ of a quasi-Banach space $\mathbb{X}$ is said to be *democratic* (resp., *super-democratic*) if there is a constant $C \geq 1$ such that

$$\|\mathbb{1}_A[\mathcal{X}, \mathbb{X}]\| \leq C \,\|\mathbb{1}_B[\mathcal{X}, \mathbb{X}]\| \ \text{(resp.,} \ \ \|\mathbb{1}_{\varepsilon,A}[\mathcal{X}, \mathbb{X}]\| \leq C \,\|\mathbb{1}_{\delta,B}[\mathcal{X}, \mathbb{X}]\|)$$

for all finite subsets A and B of $\mathbb{N}$ with $|A| \leq |B|$, all $\varepsilon \in \mathbb{E}^A$, and all $\delta \in \mathbb{E}^B$. If the above inequality holds for a given C, we say that $\mathcal{X}$ is C-democratic (resp., C-super-democratic).

To measure the democracy of a basis $\mathcal{X}$ of a quasi-Banach space $\mathbb{X}$, we introduce the *upper and lower democracy functions* of the basis, defined for $m \in \mathbb{N}$ by

$$\boldsymbol{\varphi_u}[\mathcal{X}, \mathbb{X}](m) = \sup\{\|\mathbb{1}_A[\mathcal{X}, \mathbb{X}]\| : \ |A| \leq m\} \ \text{ and}$$

$$\boldsymbol{\varphi_l}[\mathcal{X},\mathbb{X}](m) = \inf\left\{\|\mathbb{1}_A[\mathcal{X},\mathbb{X}]\| \colon |A| \ge m\right\},$$

respectively. The basis $\mathcal{X}$ is democratic if and only if

$$\boldsymbol{\varphi_u}[\mathcal{X},\mathbb{X}] \le C\boldsymbol{\varphi_l}[\mathcal{X},\mathbb{X}].$$

Similarly, the basis is C-super-democratic if and only if

$$\boldsymbol{\varphi_u^s}[\mathcal{X},\mathbb{X}] \le C\boldsymbol{\varphi_l^s}[\mathcal{X},\mathbb{X}],$$

where $\boldsymbol{\varphi_u^s}[\mathcal{X},\mathbb{X}]$ and $\boldsymbol{\varphi_l^s}[\mathcal{X},\mathbb{X}]$ are, respectively, the *upper and lower super-democracy* defined for $m\in\mathbb{N}$ as

$$\boldsymbol{\varphi_u^s}[\mathcal{X},\mathbb{X}](m) = \sup\left\{\|\mathbb{1}_{\varepsilon,A}[\mathcal{X},\mathbb{X}]\| \colon |A| \le m,\, \varepsilon\in\mathbb{E}^A\right\},$$
$$\boldsymbol{\varphi_l^s}[\mathcal{X},\mathbb{X}](m) = \inf\left\{\|\mathbb{1}_{\varepsilon,A}[\mathcal{X},\mathbb{X}]\| \colon |A| \ge m,\, \varepsilon\in\mathbb{E}^A\right\}.$$

The upper super-democracy function of a basis, also called the *fundamental function* of the basis, grows as the upper democracy function. In contrast, the lower super-democracy function of the basis can grow much more slowly than the lower democracy function (see [28]).

A basis $\mathcal{X}=(\boldsymbol{x}_n)_{n=1}^\infty$ is said to be *symmetric* (resp., *subsymmetric*) if there is a constant C such that

$$\frac{1}{C}\left\|\sum_{n=1}^\infty a_n\,\boldsymbol{x}_n\right\| \le \left\|\sum_{n=1}^\infty \varepsilon_n\, a_n\,\boldsymbol{x}_{\pi(n)}\right\| \le C\left\|\sum_{n=1}^\infty a_n\,\boldsymbol{x}_n\right\|$$

for all $(a_n)_{n=1}^\infty\in c_{00}$, all $(\varepsilon_n)_{n=1}^\infty\in\mathbb{E}^\mathbb{N}$, and all bijective (resp., increasing) maps $\pi\colon\mathbb{N}\to\mathbb{N}$. If we can choose $C=1$, we say that $\mathcal{X}$ is 1-symmetric (resp., 1-subsymmetric). Any symmetric (resp., subsymmetric) basis is 1-symmetric (resp., 1-subsymmetric) under a suitable renorming of the space (see [12, 26]). Moreover, 1-symmetric bases are 1-subsymmetric. Here, we will deal with *symmetric and subsymmetric sequence spaces*, i.e., sequence spaces whose unit vector system is a 1-symmetric or 1-subsymmetric basis. The unit vector system of a subsymmetric sequence space, besides 1-unconditional, is 1-super-democratic, i.e., for each $m\in\mathbb{N}$ there is a constant $\Lambda_m=\Lambda_m[\mathbb{S}]\in(0,\infty)$ such that

$$\Lambda_m[\mathbb{S}] = \|\mathbb{1}_{\varepsilon,A}[\mathcal{E},\mathbb{S}]\|\ \forall\varepsilon\in\mathbb{E}^A \text{ and } \forall A\subseteq\mathbb{N} \text{ with } |A|=m.$$

We call $(\Lambda_m[\mathbb{S}])_{m=1}^\infty$ the fundamental function of $\mathbb{S}$. The sequence $(\Lambda_m[\mathbb{S}])_{m=1}^\infty$ is non-decreasing, and, in case that $\mathbb{S}$ is a Banach space, so is $(m/\Lambda_m[\mathbb{S}])_{m=1}^\infty$ [18]. In fact, the closed linear span $\mathbb{S}_0^*$ of the unit vector system of $\mathbb{S}^*$ is a subsymmetric sequence space with

$$\Lambda_m[\mathbb{S}_0^*] \approx \frac{m}{\Lambda_m[\mathbb{S}]}, \quad m\in\mathbb{N} \tag{2.1}$$

(see [24]).

In the following sections we will use subsymmetric sequence spaces whose fundamental function grows in a controlled manner, and the geometry of the underlying space plays an important role in order to ensure this steady behaviour. The next two regularity conditions formalize that pattern. We say that a sequence $(\Gamma_m)_{m=1}^\infty$ in $(0,\infty)$ has the *lower regularity property* (LRP for short) if there a positive integer b such

$$2\Gamma_m \le \Gamma_{bm}, \quad m \in \mathbb{N}.$$

We say that $(\Gamma_m)_{m=1}^\infty$ it has the *upper regularity property* (URP for short) if there a positive integer b such

$$2\Gamma_{bm} \le b\Gamma_m, \quad m \in \mathbb{N}.$$

The unfamiliar reader with the notions of Rademacher *type* and *cotype* of a Banach space can look them up in [11].

Proposition 2.1 ([18, Proposition 4.1])**.** *Let $\mathbb{S}$ be a subsymmetric sequence space.*

(i) If $\mathbb{S}$ has some nontrivial cotype, then $(\Lambda_m[\mathbb{S}])_{m=1}^\infty$ has the LRP.
(ii) If $\mathbb{S}$ has some nontrivial type, then $(\Lambda_m[\mathbb{S}])_{m=1}^\infty$ has the LRP and the URP.

From another point of view, the lattice structure induced on $\mathbb{S}$ by its unit vector system yields that $\mathbb{S}$ has some nontrivial cotype if and only if it has some nontrivial concavity, and it has some nontrivial type if and only if it is superreflexive (see [25]).

For further reference, we record a regularity result that we will need.

Lemma 2.2 (See [1])**.** *Let $(\Gamma_m)_{m=1}^\infty$ be a sequence in $(0,\infty)$ such that $(m/\Gamma_m)_{m=1}^\infty$ is non-decreasing. Then $(\Gamma_m)_{m=1}^\infty$ has the LRP if and only if $(m/\Gamma_m)_{m=1}^\infty$ has the URP. Moreover, if $(\Gamma_m)_{m=1}^\infty$ has the LRP then it satisfies the Dini condition*

$$\sum_{n=1}^m \frac{\Gamma_n}{n} \approx \Gamma_m, \quad m \in \mathbb{N}.$$

We say that a basis $\mathcal{X}$ of a quasi-Banach space $\mathbb{X}$ is *squeeze-symmetric* if there are symmetric sequence spaces $\mathbb{S}_1$ and $\mathbb{S}_2$ such that

- the *series transform*, defined by $(a_n)_{n=1}^\infty \mapsto \sum_{n=1}^\infty a_n \boldsymbol{x}_n$ is a bounded operator from $\mathbb{S}_1$ into $\mathbb{X}$,
- the coefficient transform is a bounded operator from $\mathbb{X}$ into $\mathbb{S}_2$, and
- the spaces $\mathbb{S}_1$ and $\mathbb{S}_2$ are close to each other in the sense that

$$\Lambda_m[\mathbb{S}_1] \approx \Lambda_m[\mathbb{S}_2], \quad m \in \mathbb{N}.$$

If a basis is squeeze-symmetric then it is democratic, and its fundamental function is equivalent to the fundamental function of the symmetric sequence spaces that sandwich it. It is known that these symmetric sequence spaces can be chosen to be sequence Lorentz sequence spaces. Let us briefly recall their definition.

A *weight* will be a non-negative sequence $(w_n)_{n=1}^\infty$ with $w_1 > 0$. Given $0 < q < \infty$ and a weight $\boldsymbol{w} = (w_n)_{n=1}^\infty$, the Lorentz sequence space $d_q(\boldsymbol{w})$ consists of all sequences f in c_0 whose non-increasing rearrangement $(a_n^*)_{n=1}^\infty$ satisfies

$$\|f\|_{d_q(\boldsymbol{w})} = \left(\sum_{n=1}^\infty (s_n a_n^*)^q \frac{w_n}{s_n} \right)^{1/q} < \infty,$$

where $s_n = \sum_{k=1}^n w_k$. In turn, the weak Lorentz sequence space $d_\infty(\boldsymbol{w})$ consists of all sequences $f = (a_n)_{n=1}^\infty \in c_0$ whose non-increasing rearrangement $(a_n^*)_{n=1}^\infty$ satisfies

$$\|f\|_{d_\infty(\boldsymbol{w})} = \sup_n a_n^* s_n < \infty.$$

We have $\Lambda_m[d_q(\boldsymbol{w})] \approx s_m$ for $m \in \mathbb{N}$. Moreover if $0 < p \le q \le \infty$,

$$\|f\|_{d_q(\boldsymbol{w})} \lesssim \|f\|_{d_p(\boldsymbol{w})}, \quad f \in c_0.$$

Although Lorentz sequence spaces are named after the weight $\boldsymbol{w}$, they rather depend on the primitive sequence $(s_m)_{m=1}^\infty$. In fact, we have the following result.

Lemma 2.3 (see [5, §9])**.** *Let $\boldsymbol{w}(w_n)_{n=1}^\infty$ and $\boldsymbol{w}' = (w_n')_{n=1}^\infty$ be weights, and let $0 < q \le \infty$. Then $\|f\|_{d_q(\boldsymbol{w})} \approx \|f\|_{d_p(\boldsymbol{w}')}$ for $f \in c_0$ if and only if $\sum_{n=1}^m w_n \approx \sum_{n=1}^m w_n'$ for $m \in \mathbb{N}$.*

We refer the reader to [5, §9] for background on this kind of spaces.

A basis $\mathcal{X}$ of a quasi-Banach space $\mathbb{X}$ is squeeze-symmetric if and only if there are a weight $\boldsymbol{w}$ and $0 < q < \infty$ such that the series transform is a bounded operator from $d_q(\boldsymbol{w})$ into $\mathbb{X}$, and the coefficient transform is a bounded operator from $\mathbb{X}$ into $d_\infty(\boldsymbol{w})$. Moreover, if $\mathbb{X}$ is p-convex for some $0 < p \le 1$, we can pick $p = q$.

A basis $\mathcal{X}$ is said to be *bidemocratic* if and only if

$$\sup_m \frac{1}{m} \boldsymbol{\varphi_u}[\mathcal{X}, \mathbb{X}](m) \boldsymbol{\varphi_u}[\mathcal{X}^*, \mathbb{X}^*](m) < \infty.$$

It is known that bidemocratic bases are in particular squeeze-symmetric [5].

Usually, the TGA is studied for bases $\mathcal{X} = (\boldsymbol{x}_n)_{n=1}^\infty$ that are *semi-normalized*, i.e.,

$$0 < \inf_n \|\boldsymbol{x}_n\| \le \sup_n \|\boldsymbol{x}_n\| < \infty,$$

and M-bounded, i.e.,

$$\sup_n \|\boldsymbol{x}_n\| \, \|\boldsymbol{x}_n^*\| < \infty.$$

That is, it is usual to assume that both $\mathcal{X}$ and $\mathcal{X}^*$ are norm-bounded. For the purposes of this paper, however, it will be convenient not to take for granted these assumptions a priori.

Since the coefficient transform maps $\mathbb{X}$ into c_0 if and only if $\mathcal{X}^*$ is norm-bounded, there could be vectors $f \in \mathbb{X}$ for which the TGA $(\mathcal{G}_m(f))_{m=1}^\infty$ is not defined. To circumvent this initial drawback, we will consider greedy-type properties of $\mathcal{X}$ in terms of greedy sets and greedy projections.

A finite subset $A \subseteq \mathbb{N}$ is a *greedy set* of $f \in \mathbb{X}$ with respect to the basis $\mathcal{X}$ if $|\boldsymbol{x}_n^*(f)| \ge |\boldsymbol{x}_k^*(f)|$ whenever $n \in A$ and $k \notin A$.

Let $\operatorname{sign}(\cdot)$ be the sign function, defined for $\lambda \in \mathbb{F}\backslash\{0\}$ as $\operatorname{sign}(\lambda) = \lambda/|\lambda|$, and $\operatorname{sign}(0) = 1$. Given a basis $\mathcal{X}$ of a quasi-Banach space $\mathbb{X}$ we put

$$\varepsilon(f) = (\operatorname{sign}(\boldsymbol{x}_n^*(f)))_{n=1}^\infty \in \mathbb{E}^{\mathbb{N}}.$$

A basis $\mathcal{X}$ is said to be *truncation-quasi-greedy* if there is a constant C such that

$$\min_{n\in A} |\boldsymbol{x}_n^*(f)| \left\| \mathbb{1}_{\varepsilon(f),A}[\mathcal{X},\mathbb{X}] \right\| \le C \|f\|$$

for all $f \in \mathbb{X}$ and all greedy sets A of f. If the above holds for a given constant C, we say that $\mathcal{X}$ is truncation-quasi-greedy with constant C. It is known [5] that quasi-greedy bases are truncation-quasi-greedy. In turn, truncation-quasi-greedy bases are *unconditional for constant coefficients* (UCC for short), i.e.,

$$\|\mathbb{1}_{\varepsilon,A}[\mathcal{X},\mathbb{X}]\| \le C \|\mathbb{1}_{\varepsilon,B}[\mathcal{X},\mathbb{X}]\| \tag{2.2}$$

for all $B \subseteq \mathbb{N}$ finite, all $A \subseteq B$, and some constant C. We also point out that a basis is super-democratic if and only if it is democratic and UCC.

In consistency with the characterizations of greedy bases and almost greedy bases, squeeze-symmetric bases can be characterized as well as those democratic bases that satisfy an additional unconditionality-like condition, namely being truncation quasi-greedy.

Theorem 2.4 (see [5, Lemma 9.3 and Theorem 9.14])**.** *A basis is squeeze-symmetric if and only if it is truncation quasi-greedy and democratic.*

Given a basis $\mathcal{X}$ and a finite subset A of $\mathbb{N}$, the *coordinate projection* onto the subspace $[\boldsymbol{x}_n \colon n \in A]$ is the linear operator $S_A[\mathcal{X},\mathbb{X}] \colon \mathbb{X} \to \mathbb{X}$ given by

$$f \mapsto \sum_{n\in A} \boldsymbol{x}_n^*(f)\, \boldsymbol{x}_n.$$

Since the basis $\mathcal{X}$ is *unconditional* if and only if the operators $S_A[\mathcal{X},\mathbb{X}]$ are uniformly bounded, to quantify how far a basis is from being uncoditional it is customary to use the *unconditionality parameters*

$$\boldsymbol{k}_m[\mathcal{X},\mathbb{X}] = \sup\left\{\|S_A[\mathcal{X},\mathbb{X}]\| : A\subseteq\mathbb{N},\ |A|\le m\right\}, \quad m\in\mathbb{N}.$$

Notice that a basis is M-bounded if and only if $\sup_m \left\|S_{\{m\}}\right\| < \infty$. Hence, an M-bounded basis $\mathcal{X}$ of a Banach space $\mathbb{X}$ satisfies the estimate

$$\boldsymbol{k}_m[\mathcal{X},\mathbb{X}] \lesssim m, \quad m\in\mathbb{N}. \tag{2.3}$$

We will use other unconditionality-type parameters as instruments to obtain information on the growth of $(\boldsymbol{k}_m)_{m=1}^\infty$. To that end, for $m\in\mathbb{N}$ we put

$$\tilde{\boldsymbol{k}}_m[\mathcal{X},\mathbb{X}] = \sup\left\{\|S_A[\mathcal{X},\mathbb{X}](f)\| : f\in B_\mathbb{X}\cap[\boldsymbol{x}_j\colon 1\le j\le m],\ A\subseteq\mathbb{N}\right\},$$

where $B_\mathbb{X}$ denotes the closed unit ball of $\mathbb{X}$. Note that $\tilde{\boldsymbol{k}}_m \le \boldsymbol{k}_m$ for all $m\in\mathbb{N}$.

Given $m\in\mathbb{N}\cup\{0\}$, we set $S_m = S_{\{1,\dots,m\}}$. Note that $S_{\{m\}} = S_m - S_{m-1}$ for all $m\in\mathbb{N}$, and that $\mathcal{X}$ is a Schauder basis if and only if $\sup_m \|S_m\| < \infty$. Thus, any Schauder basis is M-bounded. Let us emphasize here that the celebrated theorem of Enflo [21] that proves the existence of a separable Banach space without a Schauder basis does not hold for M-bounded bases.

Theorem 2.5 (See [22, Theorem 1.27])**.** *Every separable Banach space has an M-bounded total basis.*

It is clear that democratic bases are semi-normalized. In turn, as we next show, truncation-quasi-greedy bases are M-bounded. Recall that a *block basic sequence* of a basis $\mathcal{X} = (\boldsymbol{x}_n)_{n=1}^\infty$ of a quasi-Banach space $\mathbb{X}$ is a sequence $\mathcal{Y} = (\boldsymbol{y}_j)_{j=1}^\infty$ in $\mathbb{X}$ of the form

$$\boldsymbol{y}_j = \sum_{n\in D_j} a_n\,\boldsymbol{x}_n,$$

for some sequence $(D_j)_{j=1}^\infty$ of pairwise disjoint nonempty finite subsets of $\mathbb{N}$ and some sequence $(a_n)_{n=1}^\infty$ in $\mathbb{F}$ with $a_n\neq 0$ for all $n\in D_j$ and all $j\in\mathbb{N}$. If, for each $j\in\mathbb{N}$, $|a_n|$ is constant on D_j, we say that $\mathcal{Y}$ is a *constant-coefficient block basic sequence* of $\mathcal{X}$.

Lemma 2.6. *Any constant-coefficient block basic sequence of a truncation quasi-greedy basis is an M-bounded basis of its closed linear span.*

Proof. Let $\mathcal{X}$ be a truncation quasi-greedy basis of a quasi-Banach space $\mathbb{X}$ with constant $C\ge 1$. Let $(D_j)_{j=1}^\infty$ be a pairwise disjoint sequence of finite subsets of $\mathbb{N}$, and let $\varepsilon \in \mathbb{E}^{\mathbb{N}}$. We need to prove that the sequence

$$\boldsymbol{y}_j := \mathbb{1}_{\varepsilon, D_j}[\mathcal{X}, \mathbb{X}], \quad j \in \mathbb{N},$$

is an M-bounded basis of $[\boldsymbol{y}_j \colon j \in \mathbb{N}]$. Fix $(a_j)_{j=1}^\infty \in c_{00}$. Given $k \in \mathbb{N}$, there are greedy sets A and B of $f := \sum_{j=1}^\infty a_j \, \boldsymbol{y}_j$ such that $A \subseteq B$ and $B \setminus A = D_k$. We have

$$|a_k| = \min_{n \in B} |\boldsymbol{x}_n^*(f)| \le \min_{n \in A} |\boldsymbol{x}_n^*(f)| \, .$$

Hence, if κ denotes the modulus of concavity of $\mathbb{X}$,

$$\begin{aligned} |a_k| \, \|\boldsymbol{y}_k\| &= |a_k| \, \|\mathbb{1}_{\varepsilon, B}[\mathcal{X}, \mathbb{X}] - \mathbb{1}_{\varepsilon, A}[\mathcal{X}, \mathbb{X}]\| \\ &\le \kappa \, |a_k| \, \|\mathbb{1}_{\varepsilon, B}[\mathcal{X}, \mathbb{X}]\| + \kappa \, |a_k| \, \|\mathbb{1}_{\varepsilon, A}[\mathcal{X}, \mathbb{X}]\| \\ &\le 2\kappa C \, \|f\| \, . \end{aligned}$$

□

A sequence $(\boldsymbol{y}_n)_{n=1}^\infty$ of a quasi-Banach space $\mathbb{Y}$ is said to be an *affinity* of a sequence $(\boldsymbol{x}_n)_{n=1}^\infty$ if there is a sequence $(\lambda_n)_{n=1}^\infty$ in $\mathbb{F} \setminus \{0\}$ such that $\boldsymbol{y}_n = \lambda_n \, \boldsymbol{x}_n$ for all $n \in \mathbb{N}$. Suppose that $\mathcal{Y}$ is an affinity of $\mathcal{X}$. Then, if $\mathcal{X}$ is a basis, so is $\mathcal{Y}$. Morever, if $\mathcal{X}$ is M-bounded, so is $\mathcal{Y}$, and if $\mathcal{X}$ is a Schauder basis so is $\mathcal{Y}$. We also note that $\tilde{\boldsymbol{k}}_m[\mathcal{X}, \mathbb{X}] = \tilde{\boldsymbol{k}}_m[\mathcal{Y}, \mathbb{X}]$ and $\boldsymbol{k}_m[\mathcal{X}, \mathbb{X}] = \boldsymbol{k}_m[\mathcal{Y}, \mathbb{X}]$ for all $m \in \mathbb{N}$.

Given quasi-Banach spaces $\mathbb{X}$ and $\mathbb{Y}$, $\mathbb{X} \oplus \mathbb{Y}$ stands for its directed sum endowed with the quasi-norm

$$\|(f, g)\| = \max\{\|f\| \, , \|g\|\}, \quad f \in \mathbb{X}, \ g \in \mathbb{Y}.$$

O course, $\mathbb{X} \oplus \mathbb{Y}$ is a quasi-Banach space. Given a direct sum $\mathbb{X} \oplus \mathbb{Y}$ we denote by π_1 and π_2 the projections onto the first and the second components, respectively.

If $\mathcal{X} = (\boldsymbol{x}_n)_{n=1}^\infty$ and $\mathcal{Y} = (\boldsymbol{y}_n)_{n=1}^\infty$ are sequences in $\mathbb{X}$ and $\mathbb{Y}$, respectively, its direct sum is the sequence $\mathcal{X} \oplus \mathcal{Y} = (\boldsymbol{z}_n)_{n=1}^\infty$ in $\mathbb{X} \oplus \mathbb{Y}$ defined by

$$\boldsymbol{z}_{2n-1} = (\boldsymbol{x}_n, 0), \quad \boldsymbol{z}_{2n} = (0, \boldsymbol{y}_n), \quad n \in \mathbb{N}.$$

It is clear that if $\mathcal{X}$ and $\mathcal{Y}$ are bases, then $\mathcal{X} \oplus \mathcal{Y}$ is a basis with coordinate functionals $\mathcal{X}^* \oplus \mathcal{Y}^*$. We set $\mathbb{X}^2 = \mathbb{X} \oplus \mathbb{X}$ and $\mathcal{X}^2 = \mathcal{X} \oplus \mathcal{X}$.

3. Extension of the DKK method to general bases

Let $\mathbb{S}$ be a locally convex subsymmetric sequence space. Set $\Lambda_m = \Lambda_m[\mathbb{S}]$ for $m \in \mathbb{N}$. Let $\sigma = (\sigma_n)_{n=1}^\infty$ be an *ordered partition* of $\mathbb{N}$, i.e., a partition on $\mathbb{N}$ into nonempty integer intervals so that

$$\max(\sigma_n) < \min(\sigma_{n+1}), \quad n \in \mathbb{N}.$$

Consider the sequence $\mathcal{V}=\mathcal{V}[\mathbb{S},\sigma]=(\boldsymbol{v}_n)_{n=1}^\infty$ in $\mathbb{S}$ given by

$$\boldsymbol{v}_n=\frac{1}{\Lambda_{|\sigma_n|}}\mathbb{1}_{\sigma_n}[\mathcal{E},\mathbb{S}],\quad n\in\mathbb{N}.$$

The sequence $\mathcal{V}^*=\mathcal{V}^*[\mathbb{S},\sigma]=(\boldsymbol{v}_n^*)_{n=1}^\infty$ in $\mathbb{S}^*$ given by

$$\boldsymbol{v}_n^*=\frac{\Lambda_{|\sigma_n|}}{|\sigma_n|}\mathbb{1}_{\sigma_n}^*[\mathcal{E},\mathbb{S}_0^*],\quad n\in\mathbb{N}$$

is biorthogonal to $\mathcal{V}$. By construction, $\mathcal{V}$ is normalized. In turn, $\mathcal{V}^*$ is semi-normalized by (2.1).

Let $\operatorname{Ave}(f,A)$ denote the average of $f=(a_n)_{n=1}^\infty$ on a finite set $A\subseteq\mathbb{N}$, i.e.,

$$\operatorname{Ave}(f,A)=\frac{1}{|A|}\sum_{k\in A}a_k.$$

Consider the averaging projection $P_\sigma\colon\mathbb{F}^\mathbb{N}\to\mathbb{F}^\mathbb{N}$ defined by

$$P_\sigma(f)=(b_k)_{k=1}^\infty,\quad b_k=\operatorname{Ave}(f,\sigma_n)\text{ if }k\in\sigma_n,$$

and let Q_σ be its complementary projection, i.e., $Q_\sigma=\operatorname{Id}_{\mathbb{F}^\mathbb{N}}-P_\sigma$. By [24] or [6] we have $\|P_\sigma\|_{\mathbb{S}\to\mathbb{S}}\le 2$, so that $\|Q_\sigma\|_{\mathbb{S}\to\mathbb{S}}\le 3$. Note that

$$P_\sigma(f)=\sum_{n=1}^\infty\boldsymbol{v}_n^*(f)\,\boldsymbol{v}_n,\quad f\in\mathbb{F}^\mathbb{N}.$$

Let $\mathcal{X}=(\boldsymbol{x}_n)_{n=1}^\infty$ be a linearly independent sequence in a Banach $\mathbb{X}$ that generates the entire space $\mathbb{X}$. We define

$$\begin{aligned}\|f\|_{\mathcal{X},\mathbb{S},\sigma}&=\|Q_\sigma(f)\|_\mathbb{S}+\|L[\mathcal{V}[\mathbb{S},\sigma],\mathcal{X}](P_\sigma(f))\|,\\&=\|Q_\sigma(f)\|_\mathbb{S}+\left\|\sum_{n=1}^\infty\boldsymbol{v}_n^*(f)\,\boldsymbol{x}_n\right\|,\quad f\in c_{00},\end{aligned}$$

where

$$L[\mathcal{Y},\mathcal{X}]\colon\langle\mathcal{Y}\rangle\to\langle\mathcal{X}\rangle,\qquad\boldsymbol{y}_n\mapsto\boldsymbol{x}_n,$$

stands for the operator from the linear span of a basis $\mathcal{Y}=(\boldsymbol{y}_n)_{n=1}^\infty$ onto the linear span of a basis $\mathcal{X}=(\boldsymbol{x}_n)_{n=1}^\infty$. Note that $P_\sigma(f)\in\langle\mathcal{V}[\mathbb{S},\sigma]\rangle$ for all $f\in c_{00}$. Thus, $\|\cdot\|_{\mathcal{X},\mathbb{S},\sigma}$ is well-defined.

Lemma 3.1. *Let $\mathbb{S}$ be a subsymmetric sequence space, σ be an ordered partition of $\mathbb{N}$ and $\mathcal{X}$ be a linearly independent sequence in a quasi-Banach space $\mathbb{X}$. Then $\|\cdot\|_{\mathcal{X},\mathbb{S},\sigma}$ is a quasi-norm on $\mathbb{X}$.*

Proof. It is clear from definition that $\|\cdot\|_{\mathcal{X},\mathbb{S},\sigma}$ is a semi-quasi-norm. Assume that $\|f\|_{\mathcal{X},\mathbb{S},\sigma}=0$. Then $Q_\sigma(f)=0$ and $\boldsymbol{v}_n^*(f)=0$ for all $n\in\mathbb{N}$. Then $P_\sigma(f)=0$ and so $f=P_\sigma(f)+Q_\sigma(f)=0$. ☐

We denote by $\mathbb{Y} = \mathbb{Y}[\mathcal{X}, \mathbb{S}, \sigma]$ the completion of the quasi-normed space $(c_{00}, \|\cdot\|_{\mathcal{X},\mathbb{S},\sigma})$.

Lemma 3.2. *Let $\mathbb{S}$ be a locally convex subsymmetric sequence space, σ be an ordered partition of $\mathbb{N}$, and $\mathcal{X}$ be a linearly independent sequence in a quasi-Banach space $\mathbb{X}$. Suppose that $\mathcal{X}$ generates $\mathbb{X}$. Then $\mathbb{Y}[\mathcal{X}, \mathbb{S}, \sigma] \simeq Q_\sigma(\mathbb{S}) \oplus \mathbb{X}$. To be precise $Q_\sigma(\mathbb{S}) \cap c_{00}$ is dense in $Q_\sigma(\mathbb{S})$, and the maps*

$$S := (Q_\sigma, L[\mathcal{V}[\mathbb{S}, \sigma], \mathcal{X}] \circ P_\sigma) \colon c_{00} \to (Q_\sigma(\mathbb{S}) \cap c_{00}) \oplus \langle \mathcal{X} \rangle$$
$$T := \pi_1 + L[\mathcal{X}, \mathcal{V}[\mathbb{S}, \sigma]] \circ \pi_2 \colon (Q_\sigma(\mathbb{S}) \cap c_{00}) \oplus \langle \mathcal{X} \rangle \to c_{00}$$

are linear bijections inverse of each other that extend to inverse isomorphisms, with $\|S\| \le 1$ and $\|T\| \le 2$.

Proof. The proof of the corresponding result from [6] holds in this general setting. Note that

$$S(f) = \left(Q_\sigma(f), \sum_{n=1}^\infty \boldsymbol{v}_n^*(f)\, \boldsymbol{x}_n \right), \quad f \in \mathbb{Y}[\mathcal{X}, \mathbb{S}, \sigma] \cap c_{00},$$
$$T(g, x) = g + \sum_{n=1}^\infty \boldsymbol{x}_n^*(x)\, \boldsymbol{v}_n, \quad g \in Q_\sigma(\mathbb{S}) \cap c_{00},\ x \in \langle \mathcal{X} \rangle. \qquad \square$$

Proposition 3.3. *Let $\mathbb{S}$ be a subsymmetric sequence space, let σ be an ordered partition of $\mathbb{N}$ with $|\sigma_n| \ge 2$ for all $n \in \mathbb{N}$, and let $\mathcal{X}$ be a linearly independent sequence of a quasi-Banach space $\mathbb{X}$ with $[\mathcal{X}] = \mathbb{X}$. Then the unit vector system $\mathcal{E}$ is a basis of $\mathbb{Y} = \mathbb{Y}[\mathcal{X}, \mathbb{S}, \sigma]$ if and only if $\mathcal{X}$ is a basis of $\mathbb{X}$. Moreover, in the case when $\mathcal{X}$ is a basis of $\mathbb{X}$ the following statements hold:*

(i) $\mathcal{E}$ is a total basis of $\mathbb{Y}$ if and only if $\mathcal{X}$ is a total basis of $\mathbb{X}$.

(ii) The dual basis of the unit vector system of $\mathbb{Y}$ is equivalent to the unit vector system of $\mathbb{Y}[\mathcal{B}, \mathbb{S}_0^*, \sigma]$, where $\mathcal{B} = (\boldsymbol{x}_n^* / \|\boldsymbol{v}_n^*\|)_{n=1}^\infty$.

(iii) $\|\boldsymbol{e}_n\|_{\mathbb{Y}} \approx \max\{1, \|\boldsymbol{x}_n\| \, \Lambda_{|\sigma_n|} / |\sigma_n|\}$ for $n \in \mathbb{N}$;

(iv) $\|\boldsymbol{e}_n^*\|_{\mathbb{Y}^*} \approx \max\{1, \|\boldsymbol{x}_n^*\| / \Lambda_{|\sigma_n|}\}$ for $n \in \mathbb{N}$;

(v) $\mathcal{E}$ is a semi-normalized and M-bounded basis of $\mathbb{Y}$ if and only if

$$\|\boldsymbol{x}_n\| \lesssim \frac{|\sigma_n|}{\Lambda_{|\sigma_n|}} \quad \text{and} \quad \|\boldsymbol{x}_n^*\| \lesssim \Lambda_{|\sigma_n|},\ n \in \mathbb{N}.$$

(vi) $\tilde{\boldsymbol{k}}_{M_n}[\mathcal{E}, \mathbb{Y}] \gtrsim \tilde{\boldsymbol{k}}_n[\mathcal{X}, \mathbb{X}]$, where $M_n = \sum_{k=1}^n |\sigma_k|$.

(vii) Set $\mathbb{Y}_n = [\boldsymbol{e}_k \colon k \in \sigma_n]$. Then, $\mathcal{X}$ is a Schauder basis of $\mathbb{X}$ if and only if $(\mathbb{Y}_n)_{n=1}^\infty$ is a Schauder decomposition of $\mathbb{Y}$.

(viii) If $\mathcal{E}$ is an UCC basis of $\mathbb{Y}$, then $\mathcal{X}$ is semi-normalized.

(ix) The block basic sequence $(\mathbb{1}_{\sigma_n}[\mathcal{E}, \mathbb{Y}])_{n=1}^\infty$ is isometrically equivalent to an affinity of $\mathcal{X}$.

(x) Set $\boldsymbol{w} = (\Lambda_n - \Lambda_{n-1})_{n=1}^{\infty}$ *and* $\boldsymbol{u} = (\Lambda_n/n)_{n=1}^{\infty}$*. Suppose that* $\mathbb{X}$ *is locally convex, that* $\mathcal{X}$ *is semi-normalized and* M*-bounded, and that* $1 + M_{n-1} \lesssim |\sigma_n|$ *for* $n \in \mathbb{N}$*. Then,*

$$d_1(\boldsymbol{u}) \subseteq \mathbb{Y} \subseteq d_\infty(\boldsymbol{w})$$

(with continuous embeddings).

(xi) Suppose that $\mathbb{X}$ *is locally convex,* $(\Lambda_n)_{n=1}^{\infty}$ *has the LRP, and* $M_n \lesssim |\sigma_n|$ *for* $n \in \mathbb{N}$*. The following are equivalent:*

(1) The unit vector system is a squeeze symmetric basis of $\mathbb{Y}$*.*

(2) The unit vector system is a truncation quasi-greedy basis of $\mathbb{Y}$*.*

(3) $\mathcal{X}$ *a is semi-normalized basis* M*-bounded basis of* $\mathbb{X}$*.*

Proof. Given $x \in \langle \mathcal{X} \rangle$ and $n \in \mathbb{N}$, let $\boldsymbol{x}_n^{\#}(x)$ be the nth coordinate of the expansion of x with respect to $\mathcal{X}$. For $k \in \mathbb{N}$, pick $n \in \mathbb{N}$ such that $k \in \sigma_n$. Via the isomorphism S provided by Lemma 3.2, the kth functional $\boldsymbol{e}_k^*$ corresponds with the map

$$\boldsymbol{z}_k^{\#} \colon (c_{00} \cap Q_\sigma(\mathbb{S})) \oplus \langle \mathcal{X} \rangle \to \mathbb{F}, \quad (g, x) \mapsto \boldsymbol{e}_k^*(g) + \frac{1}{\Lambda_{|\sigma_n|}} \boldsymbol{x}_n^{\#}(x).$$

That is, $\boldsymbol{z}_k^{\#} \circ S = \boldsymbol{e}_k^*|_{c_{00}}$. Since $|\boldsymbol{e}_k^*(g)| \le \|g\|_{\mathbb{S}}$ for all $g \in Q_\sigma(\mathbb{S})$, $\boldsymbol{e}_k^*$ defines a bounded operator on $\mathbb{Y}$, if and only if $\boldsymbol{x}_n^{\#}$ extends to a bounded operator on $\mathbb{X}$, in which case $\boldsymbol{e}_k^*|_{\mathbb{Y}}$ corresponds with the map

$$\boldsymbol{z}_k^* \colon Q_\sigma(\mathbb{S}) \oplus \mathbb{X} \to \mathbb{F}, \quad (g, x) = \boldsymbol{e}_k^*(g) + \frac{1}{\Lambda_{|\sigma_n|}} \boldsymbol{x}_n^*(x), \quad k \in \sigma_n.$$

This expression for $\boldsymbol{z}_k^*$ gives that the unit vector system is a total basis if and only if the map

$$(g, x) \mapsto F(g, x) = \left(\boldsymbol{e}_k^*(g) + \frac{1}{\Lambda_{|\sigma_n|}} \boldsymbol{x}_n^*(x) \right)_{k=1}^{\infty}, \quad g \in Q_\sigma(\mathbb{S}), \quad x \in \mathbb{X},$$

is one-to-one, where $k \in \sigma_n$ for each k. Notice that, for all $g \in Q_\sigma(\mathbb{S})$ and all $x \in \mathbb{X}$,

$$F(g, x) = g + \sum_{n=1}^{\infty} \frac{1}{\Lambda_{|\sigma_n|}} \boldsymbol{x}_n^*(x) \mathbb{1}_{\sigma_n}[\mathcal{E}, \mathbb{S}].$$

Choosing $g = 0$ we obtain the "only" if part of (i). Assume that $\mathcal{X}$ is total and that $F(g, x) = 0$. Then, averaging on σ_n we get $\boldsymbol{x}_n^*(x) = 0$. Hence $x = 0$ and so $g = F(g, x) = 0$.

To prove (ii) we set

$$\mathcal{U} := \mathcal{V}[\mathbb{S}^*, \sigma] = \left(\frac{\boldsymbol{v}_n^*}{\|\boldsymbol{v}_n^*\|} \right)_{n=1}^{\infty}.$$

There is a natural isomorphism between the dual space of $Q_\sigma(\mathbb{S}) \oplus \mathbb{X}$ and $Q_\sigma(\mathbb{S}^*) \oplus \mathbb{X}^*$. In turn, since $\mathcal{B}$ is a basic sequence of $\mathbb{X}^*$ and $\mathbb{S}^*$ is a locally convex subsymmetric sequence space, $Q_\sigma(\mathbb{S}_0^*) \oplus [\mathcal{B}]$ can be identified with $\mathbb{Y}[\mathcal{B}, \mathbb{S}_0^*, \sigma]$. Via this identification we obtain a dual pairing between $\mathbb{Y}[\mathcal{B}, \mathbb{S}_0^*, \sigma]$ and $\mathbb{Y}[\mathcal{X}, \mathbb{S}, \sigma]$ given by

$$\begin{aligned}(g, f) \mapsto & \langle Q_\sigma(g), Q_\sigma(f)\rangle + L[\mathcal{U}, \mathcal{B}](P_\sigma(g))(L[\mathcal{V}, \mathcal{X}](P_\sigma(f))) \\ & \langle Q_\sigma(g), Q_\sigma(f)\rangle + L[\mathcal{V}^*, \mathcal{X}^*](P_\sigma(g))(L[\mathcal{V}, \mathcal{X}](P_\sigma(f))) \\ & = \langle Q_\sigma(g), Q_\sigma(f)\rangle + \langle P_\sigma(g), P_\sigma(f)\rangle \\ & = \langle f, g\rangle.\end{aligned}$$

To prove (iii) we note that the image of $\boldsymbol{e}_k$ by the isomorphism T provided by Lemma 3.2 is, if $k \in \sigma_n$,

$$\boldsymbol{z}_k := \left(\left(1 - \frac{1}{|\sigma_n|}\right) \boldsymbol{e}_k - \frac{1}{|\sigma_n|} \mathbb{1}_{\sigma_n \setminus \{k\}}[\mathcal{E}, \mathbb{S}], \frac{\Lambda_{|\sigma_n|}}{|\sigma_n|} \boldsymbol{x}_n \right).$$

Therefore,

$$\begin{aligned}\|\boldsymbol{e}_k\| \approx \|\boldsymbol{z}_k\| & \approx \max\left\{1 - \frac{1}{|\sigma_n|} + \frac{\Lambda_{|\sigma_n|-1}}{|\sigma_n|}, \frac{\Lambda_{|\sigma_n|}}{|\sigma_n|} \|\boldsymbol{x}_n\|\right\} \\ & \approx \max\left\{1, \frac{\Lambda_{|\sigma_n|}}{|\sigma_n|} \|\boldsymbol{x}_n\|\right\}.\end{aligned}$$

(iv) follows from combining (iii) and (ii). In turn, (v) is a consequence of (iii) and (iv). The statements (vi) and (vii) follow from the identity

$$S_{\cup_{k\in A}\sigma_k}[\mathcal{E}, \mathbb{Y}](f) = \left(S_{\cup_{k\in A}\sigma_k}[\mathcal{E}, \mathbb{S}](Q_\sigma(f)), S_A[\mathcal{X}, \mathbb{X}] \left(\sum_{n=1}^\infty \boldsymbol{v}_n^*(f)\, \boldsymbol{x}_n \right) \right). \tag{3.1}$$

The proof of (x) goes along the lines of the corresponding statement from [6], which also works in this more general setting.

To prove (viii), for $n \in \mathbb{N}$ we set

$$\alpha_n = \Lambda_{|\sigma_n|} \|\boldsymbol{x}_n\|, \quad \beta_n = \Lambda_{|\sigma_n| - \gamma_n},$$

where $\gamma_n = 0$ if $|\sigma_n|$ is even and $\gamma_n = 1$ if $|\sigma_n|$ is odd. Clearly, $\beta_n \approx \Lambda_{|\sigma_n|}$ for $n \in \mathbb{N}$.

Let (A_n, B_n, C_n) be a partition of σ_n such that $|A_n| = |B_n|$ and C_n is either empty or a singleton, and let (D_n, E_n) be disjoint subsets of σ_n with $C_n \subseteq D_n$ and $|D_n| = |E_n| = |A_n|$. If the unit vector basis of $\mathbb{Y}$ is UCC, then there is a constant C such that

$$\left\| \sum_{n\in A} a_n\, \boldsymbol{x}_n \right\|_{\mathcal{X}, \mathbb{S}, \sigma} \le C \left\| \mathbb{1}_{\varepsilon, A}[\mathcal{E}, \mathbb{Y}] \right\|_{\mathcal{X}, \mathbb{S}, \sigma}, \quad A \subseteq \mathbb{N},\ |a_n| \le 1,\ \varepsilon \in \mathbb{E}^A$$

(see [5, Lemma 3.2]). Therefore, if κ is the modulus of concavity of $\mathbb{Y}$,

$$\begin{aligned}\alpha_n &= \|\mathbb{1}_{A_n}[\mathcal{E},\mathbb{Y}] + \mathbb{1}_{B_n}[\mathcal{E},\mathbb{Y}] + \mathbb{1}_{C_n}[\mathcal{E},\mathbb{Y}]\|_{\mathcal{X},\mathbb{S},\sigma} \\ &\le \kappa\left(\|\mathbb{1}_{A_n}[\mathcal{E},\mathbb{Y}] + \mathbb{1}_{B_n}[\mathcal{E},\mathbb{Y}]\|_{\mathcal{X},\mathbb{S},\sigma} + \|\mathbb{1}_{C_n}[\mathcal{E},\mathbb{Y}]\|_{\mathcal{X},\mathbb{S},\sigma}\right) \\ &\le \kappa C\left(\|\mathbb{1}_{A_n}[\mathcal{E},\mathbb{Y}] - \mathbb{1}_{B_n}[\mathcal{E},\mathbb{Y}]\| + \|\mathbb{1}_{D_n}[\mathcal{E},\mathbb{Y}] - \mathbb{1}_{E_n}[\mathcal{E},\mathbb{Y}]\|\right) \\ &= 2\kappa C\beta_n,\end{aligned}$$

and, the other way around,

$$\beta_n = \|\mathbb{1}_{D_n}[\mathcal{E},\mathbb{Y}] - \mathbb{1}_{E_n}[\mathcal{E},\mathbb{Y}]\|_{\mathcal{X},\mathbb{S},\sigma} \le C\,\|\mathbb{1}_{\sigma_n}[\mathcal{E},\mathbb{Y}]\| = C\alpha_n.$$

We have proved that $\alpha_n \approx \beta_n$ for $n\in\mathbb{N}$. Hence, $\mathcal{X}$ is semi-normalized.

(ix) is clear. Finally, by Lemma 2.6, and taking into account that, by Lemma 2.2 and Lemma 2.3, $d_1(\boldsymbol{w}) = d_1(\boldsymbol{u})$ up to an equivalent norm, (xi) follows as an easy consequence of (viii), (ix), and (x). □

4. Existence of non M-bounded bases in Banach spaces

Our method to build a truncation quasi-greedy basis of ℓ_2 with

$$\boldsymbol{k}_m \approx 1 + \log m, \quad m\in\mathbb{N},$$

will consist of several steps. The first step is to transform a semi-normalized M-bounded basis into a basis $\mathcal{X}_1$ which is no longer M-bounded. The second step is to apply the DKK-method to transform $\mathcal{X}_1$ into a semi-normalized M-bounded basis $\mathcal{X}_2$ which turns out to have poor unconditionality constants. The third step is to apply the DKK-method again to obtain a squeeze-symmetric semi-normalized M-basis whose unconditionality constants depend on those of $\mathcal{X}_2$.

This section is geared towards manufacturing the aforementioned basis $\mathcal{X}_1$. We start with a bidimensional construction.

Given $R\ge\sqrt{2}$, we consider the pair of vectors of $\mathbb{F}^2$ given by

$$\boldsymbol{h}_{1,R} = (1,0), \quad \boldsymbol{h}_{2,R} = \left(1-\frac{2}{R^2}, \frac{2}{R}\sqrt{1-\frac{1}{R^2}}\right).$$

Notice that if $\alpha\in(0,\pi/4]$ is defined by $\sin(\alpha) = 1/R$, then $\boldsymbol{h}_{2,R} = (\cos(2\alpha),\sin(2\alpha))$. We consider $\mathbb{F}^2$ equipped with the Euclidean distance.

Lemma 4.1. *Given $R\ge\sqrt{2}$, the vectors $\{\boldsymbol{h}_{1,R},\boldsymbol{h}_{2,R}\}$ form a basis of $\mathbb{F}^2$ whose biorthogonal functionals are*

$$\boldsymbol{h}^*_{1,R} = \frac{R^2}{2\sqrt{R^2-1}}(\sin(2\alpha),-\cos(2\alpha)), \quad \boldsymbol{h}^*_{2,R} = \frac{R^2}{2\sqrt{R^2-1}}(0,1),$$

where $\alpha\in(0,\pi/4]$ is given by $\sin(\alpha)=1/R$. Moreover,

$$\|\boldsymbol{h}_{1,R}\| = \|\boldsymbol{h}_{2,R}\| = 1, \left\|\boldsymbol{h}^*_{1,R}\right\| = \left\|\boldsymbol{h}^*_{2,R}\right\| \approx R, \|\boldsymbol{h}_{1,R}-\boldsymbol{h}_{2,R}\| = 2/R,$$

and

$$\sqrt{x^2+y^2} \le \|x\,\boldsymbol{h}_{1,R} + y\,\boldsymbol{h}_{2,R}\| \le x+y, \quad x,y\ge 0.$$

Proof. It is a routine computation. □

Let $\mathcal{X} = (\boldsymbol{x}_n)_{n=1}^\infty$ be a sequence in a quasi-Banach space $\mathbb{X}$. For each $n\in\mathbb{N}$ we consider the linear map

$$L_n\colon \mathbb{F}^2 \to \mathbb{X}, \quad L_n(1,0)=\boldsymbol{x}_{2n-1},\ L_n(0,1)=\boldsymbol{x}_{2n}. \tag{4.1}$$

Given a sequence $\eta=(\lambda_n,\mu_n)_{n=1}^\infty$ in $\mathbb{R}_+^2$ with $\lambda_n\mu_n\ge\sqrt{2}$ for all $n\in\mathbb{N}$, we define a sequence $\mathcal{X}_\eta=(\boldsymbol{y}_n)_{n=1}^\infty$ in $\mathbb{X}$ by

$$\boldsymbol{y}_{2n-1}=\lambda_n L_n(\boldsymbol{h}_{1,\lambda_n\mu_n}), \quad \boldsymbol{y}_{2n}=\lambda_n L_n(\boldsymbol{h}_{2,\lambda_n\mu_n}), \quad n\in\mathbb{N}.$$

Lemma 4.2. *Let $\mathcal{X}=(\boldsymbol{x}_n)_{n=1}^\infty$ be a semi-normalized M-bounded basis in a quasi-Banach space $\mathbb{X}$. Then the linear operators $(L_n)_{n=1}^\infty$ defined as in (4.1) are isomorphisms such that the norm of L_n and L_n^{-1} are uniformly bounded.*

Proof. Let κ be the modulus of concavity of $\mathbb{X}$. Let $c=\sup_n\|\boldsymbol{x}_n\|$ and $d=\sup_n\|\boldsymbol{x}_n^*\|$. We have

$$\frac{1}{d}\max\{|x|,|y|\}\le\|L_n(x,y)\|\le\kappa c(|x|+|y|) \quad x,\,y\in\mathbb{F}. \qquad \square.$$

Lemma 4.3. *Let $\eta=(\lambda_n,\mu_n)_{n=1}^\infty$ be a sequence in $\mathbb{R}_+^2$ with $\lambda_n\mu_n\ge\sqrt{2}$ for all $n\in\mathbb{N}$. Let $\mathcal{X}=(\boldsymbol{x}_n)_{n=1}^\infty$ be an M-bounded semi-normalized basis in a quasi-Banach space $\mathbb{X}$. Then $\mathcal{X}_\eta=(\boldsymbol{y}_n)_{n=1}^\infty$ is a basis of $\mathbb{X}$ such that, if $\mathcal{X}_\eta^*=(\boldsymbol{y}_n^*)_{n=1}^\infty$ denotes its dual basis,*

$$\begin{aligned}\left\|\boldsymbol{y}_{2n-1}^*\right\| \approx \|\boldsymbol{y}_{2n}^*\| &\approx \mu_n,\\ \|\boldsymbol{y}_{2n}-\boldsymbol{y}_{2n-1}\| &\approx \frac{1}{\mu_n}, \text{ and}\\ \|a\boldsymbol{y}_{2n-1}+b\boldsymbol{y}_{2n}\| &\approx \lambda_n(a+b)\end{aligned}$$

for $n\in\mathbb{N}$ and $a,\,b\ge 0$. Moreover,

(i) if $\mathcal{X}$ is total, so is $\mathcal{X}_\eta$, and

(ii) if $\mathcal{X}$ is equivalent to another basis $\mathcal{X}'$, then $\mathcal{X}_\eta$ is equivalent to $\mathcal{X}'_\eta$.

Proof. Let $(L_n)_{n=1}^\infty$ and $(L_n^*)_{n=1}^\infty$ be as in (4.1) with respect to $\mathcal{X}$ and $\mathcal{X}^*$, respectively. Since $L_n^*(g)(L_n(f))=\langle g,f\rangle$ for all $f,\,g\in\mathbb{F}^2$ and $n\in\mathbb{N}$, $\mathcal{X}_\eta$ is a basis of $\mathbb{X}$ whose biorthogonal functionals $(\boldsymbol{y}_n^*)_{n=1}^\infty$ are given by

$$\boldsymbol{y}_{2n-1}^*=\frac{1}{\lambda_n}L_n^*(\boldsymbol{h}_{1,\lambda_n\mu_n}^*), \quad \boldsymbol{y}_{2n}^*=\frac{1}{\lambda_n}L_n^*(\boldsymbol{h}_{2,\lambda_n\mu_n}^*), \quad n\in\mathbb{N}.$$

Combining Lemmas 4.1 and 4.2 yields the desired estimates for $\mathcal{X}_\eta$ and its dual basis. Since the vectors of $\mathcal{X}^*$ are linear combinations of $(\boldsymbol{y}_n^*)_{n=1}^\infty$, (i) holds. In turn, since the vectors in $\mathcal{X}_\eta$ are linear combinations of $\mathcal{X}$ with coefficients that do not depend of the given basis $\mathcal{X}$, (ii) holds. □

Lemma 4.4. *Let $\mathcal{U} = (\boldsymbol{u}_n)_{n=1}^\infty$ be a semi-normalized unconditional basis of a quasi-Banach space $\mathcal{R}$. Let $(\mu_n)_{n=1}^\infty$ be a positive sequence with $\mu := \inf_n \mu_n > 0$. Pick $\lambda \ge \sqrt{2}/\mu$ and set $\eta = (\lambda, \mu_n)_{n=1}^\infty$. Then $(\mathcal{U}^2)_\eta$ is equivalent to $\mathcal{U}^2$ for non-negative scalars.*

Proof. Denote $\mathcal{U}^2 = (\boldsymbol{x}_n)_{n=1}^\infty$ and $(\mathcal{U}^2)_\eta = (\boldsymbol{y}_n)_{n=1}^\infty$. Notice that

$$U_n(f) := \boldsymbol{x}_{2n-1}^*(f)\boldsymbol{x}_{2n-1} + \boldsymbol{x}_{2n}^*(f)\boldsymbol{x}_{2n} = \boldsymbol{y}_{2n-1}^*(f)\boldsymbol{y}_{2n-1} + \boldsymbol{y}_{2n}^*(f)\boldsymbol{y}_{2n}$$

for all $n \in \mathbb{N}$ and $f \in \mathcal{R}^2$. For $f = \sum_{n=1}^\infty b_n\, \boldsymbol{x}_n \in \mathcal{R}^2$ we have

$$\begin{aligned}\|f\| &= \max\left\{\left\|\sum_{n=1}^\infty b_{2n-1}\, \boldsymbol{u}_n\right\|, \left\|\sum_{n=1}^\infty b_{2n}\, \boldsymbol{u}_n\right\|\right\}\\ &\approx \left\|\sum_{n=1}^\infty (|b_{2n-1}| + |b_{2n}|)\, \boldsymbol{u}_n\right\|\\ &\approx \left\|\sum_{n=1}^\infty \|b_{2n-1}\, \boldsymbol{x}_{2n-1} + b_{2n}\, \boldsymbol{x}_{2n}\|\, \boldsymbol{u}_n\right\|\\ &= \left\|\sum_{n=1}^\infty \|U_n(f)\|\ \boldsymbol{u}_n\right\|.\end{aligned}$$

Let $(L_n)_{n=1}^\infty$ be as in (4.1) with respect to the basis $\mathcal{U}^2$. Pick an eventually null sequence of scalars $(a_n)_{n=1}^\infty$, and set $h = \sum_{n=1}^\infty a_n\, \boldsymbol{x}_n$ and $g = \sum_{n=1}^\infty a_n\, \boldsymbol{y}_n$. Taking into account Lemma 4.2 we obtain

$$\begin{aligned}\|g\| &\approx \left\|\sum_{n=1}^\infty \|U_n(g)\|\, \boldsymbol{u}_n\right\|\\ &= \left\|\sum_{n=1}^\infty \|\boldsymbol{y}_{2n-1}^*(g)\boldsymbol{y}_{2n-1} + \boldsymbol{y}_{2n}^*(g)\boldsymbol{y}_{2n}\|\, \boldsymbol{u}_n\right\|\\ &= \left\|\sum_{n=1}^\infty \|a_{2n-1}\boldsymbol{y}_{2n-1} + a_{2n}\boldsymbol{y}_{2n}\|\, \boldsymbol{u}_n\right\|.\end{aligned}$$

Therefore, if $a_n \ge 0$ for all $n \in \mathbb{N}$, applying Lemma 4.3 yields

$$\|g\| \approx \left\|\sum_{n=1}^\infty (a_{2n-1} + a_{2n})\boldsymbol{u}_n\right\| \approx \|h\|.$$ □

5. Squeeze-symmetric bases with large unconditonality parameters

We start this section by applying the DKK method, as developed in Section 3, to the bases we have constructed in Section 4. Within the scheme outlined at the beginning of Section 4, this construction corresponds to the second step in our route.

Lemma 5.1. *Let $\mathbb{S}$ be a subsymmetric sequence space, let σ be an ordered partition of $\mathbb{N}$ with $s_n := |\sigma_{2n-1}| = |\sigma_{2n}| \ge 2$ for all $n \in \mathbb{N}$, let $\mathcal{X}$ be a semi-normalized M-bounded basis of a quasi-Banach space $\mathbb{X}$. Let $(\lambda_n)_{n=1}^\infty$ in $(0,\infty)$ be such that $t_n := \lambda_n \Lambda_{s_n} \ge \sqrt{2}$ for all $n \in \mathbb{N}$ and*

$$\lambda_n \lesssim \frac{s_n}{\Lambda_{s_n}}, \quad n \in \mathbb{N}.$$

Set $\eta = (\lambda_n, \Lambda_{s_n})_{n=1}^\infty$ and $\mathbb{Y} := \mathbb{Y}[\mathcal{X}_\eta, \mathbb{S}, \sigma]$. Put

$$R_n(a,b) := \left\| a \mathbb{1}_{\sigma_{2n-1}}[\mathcal{E}, \mathbb{Y}] + b \mathbb{1}_{\sigma_{2n}}[\mathcal{E}, \mathbb{Y}] \right\|_{\mathcal{X}_\eta, \mathbb{S}, \sigma}, \quad a, b \in \mathbb{F}, \; n \in \mathbb{N},$$

and $M_n = 2\sum_{k=1}^n s_k$ for all $n \in \mathbb{N}$. Then,

(i) the unit vector system $\mathcal{E}$ is a semi-normalized M-bounded basis of $\mathbb{Y}$,
(ii) if $\mathcal{X}$ is total, so is $\mathcal{E}$,
(iii) $R_n(a,b) \approx (a+b) t_n$ for a, $b \ge 0$ and $n \in \mathbb{N}$,
(iv) $R_n(-1,1) \approx 1$ for $n \in \mathbb{N}$,
(v) $\tilde{\boldsymbol{k}}_{M_n}[\mathcal{E}, \mathbb{Y}] \gtrsim t_n$ for $n \in \mathbb{N}$.

Proof. Denote $\mathcal{X}_\eta = (\boldsymbol{y}_n)_{n=1}^\infty$. For all a, $b \in \mathbb{F}$ we have

$$R_n(a,b) = \left\| a \Lambda_{|\sigma_{2n-1}|} \boldsymbol{y}_{2n-1} + b \Lambda_{|\sigma_{2n}|} \boldsymbol{y}_{2n} \right\| = \Lambda_{s_n} \left\| a \boldsymbol{y}_{2n-1} + b \boldsymbol{y}_{2n} \right\|.$$

In light of this identity, we obtain (i), (ii), (iii) and (iv) by combining Lemma 4.3 with Proposition 3.3. Combining (iii) and (iv) gives (v). □

Combining conditions (iii) and (iv) in Lemma 5.1 gives a lower estimate for the unconditionality constants of the basis we build. Thus, if we plan to show the existence of bases with large unconditionality constants, we should apply the lemma with the members of the sequence $(t_n)_{n=1}^\infty$ in the hypotheses as large as possible.

Theorem 5.2. *Let $\mathbb{X}$ be a quasi-Banach space with an M-bounded basis $\mathcal{X}$. Suppose that a locally convex subsymmetric sequence space $\mathbb{S}$ is complemented in $\mathbb{X}$. Then $\mathbb{X}$ has a M-bounded basis $\mathcal{B}$ with*

$$\tilde{\boldsymbol{k}}_m[\mathcal{B}, \mathbb{X}] \gtrsim m, \quad m \in \mathbb{N}.$$

Moreover, if $\mathcal{X}$ is total, the basis $\mathcal{Y}$ is total.

Proof. Consider the ordered partition $\sigma = (\sigma_n)_{n=1}^\infty$ of $\mathbb{N}$ given by $|\sigma_{2n-1}| = |\sigma_{2n}| = 2^n$ for all $n \in \mathbb{N}$. The space $P_\sigma(\mathbb{S}) \oplus \mathbb{X}$ has an M-bounded semi-normalized basis, say $\mathcal{Z}$. Applying Lemma 5.1 to the basis $\mathcal{Z}$, the partition σ, and the sequence

$$\lambda_n = \frac{2^n}{\Lambda_{2^n}}, \quad n \in \mathbb{N},$$

yields a basis $\mathcal{Y}$ of

$$\mathbb{Y} := \mathbb{Y}[\mathcal{Z}_\eta, \mathbb{S}, \sigma] \simeq Q_\sigma(\mathbb{S}) \oplus P_\sigma(\mathbb{S}) \oplus \mathbb{X} \simeq \mathbb{S} \oplus \mathbb{X} \simeq \mathbb{X}$$

with

$$\tilde{\boldsymbol{k}}_{2^{n+2}-4}[\mathcal{Y}, \mathbb{Y}] \gtrsim 2^n, \quad n \in \mathbb{N}.$$

From here, the desired estimate follows in a routine way. □

To contextualize the next result we point out that any Schauder basis $\mathcal{X}$ of any superreflexive Banach space $\mathbb{X}$ satisfies the estimate

$$\boldsymbol{k}_m[\mathcal{X}, \mathbb{X}] \lesssim m^{1-\epsilon}, \quad m \in \mathbb{N},$$

for some $\epsilon > 0$ (see [8, Corollary 3.6]).

Corollary 5.3. *Let $\mathbb{X}$ be a separable Banach space. Suppose that a subsymmetric sequence space $\mathbb{S}$ is complemented in $\mathbb{X}$. Then $\mathbb{X}$ has an M-bounded total basis $\mathcal{B}$ with*

$$\tilde{\boldsymbol{k}}_m[\mathcal{B}, \mathbb{X}] \approx m, \quad m \in \mathbb{N}.$$

Proof. Combining Theorem 2.5 with Theorem 5.2 yields an M-bounded total basis $\mathcal{B}$ with $\tilde{\boldsymbol{k}}_m[\mathcal{B}] \gtrsim m$ for $m \in \mathbb{N}$. In light of (2.3), we are done. □

We next take our third step towards showing the existence of truncation quasi-greedy bases with poor unconditionality constants. To that end, we see applications of the DKK-method to the construction of greedy-like bases. While the DKK-method has been fed so far with Schauder bases, here we will drop this restriction.

Theorem 5.4. *Let $\mathbb{X}$ be a separable Banach space that has a subsymmetric sequence space $\mathbb{S}$ as a complemented subspace. Suppose also that $(\Lambda_m[\mathbb{S}])_{m=1}^\infty$ has the LRP. Then $\mathbb{X}$ has a squeeze-symmetric (bidemocratic if $(\Lambda_m[\mathbb{S}])_{m=1}^\infty$ has additionally the URP) total basis $\mathcal{B}$ with*

$$\tilde{\boldsymbol{k}}_m[\mathcal{B}, \mathbb{X}] \approx \log(1+m) \quad \text{and} \quad \boldsymbol{\varphi}_{\boldsymbol{u}}[\mathcal{B}, \mathbb{X}](m) \approx \Lambda_m[\mathbb{S}], \quad m \in \mathbb{N}.$$

Moreover, we can choose $\mathcal{B}$ to be non quasi-greedy.

Proof. Let $\sigma' = (\sigma'_n)_{n=1}^\infty$ be the partition of $\mathbb{N}$ given by $|\sigma'_{2n-1}| = |\sigma'_{2n}| = 2^n$ for all $n \in \mathbb{N}$, and let $(\sigma_n)_{n=1}^\infty$ be the partition of $\mathbb{N}$ given by $|\sigma_n| = 2^n$ for all $n \in \mathbb{N}$. By Theorem 2.5, the Banach space

$$\mathbb{X}_0 := P_\sigma(\mathbb{S}) \oplus P_{\sigma'}(\mathbb{S}) \oplus \mathbb{X}$$

has a semi-normalized M-bounded total basis, say $\mathcal{X}_0$. Set

$$\eta = (\lambda_n, \Lambda_{2^n})_{n=1}^\infty,$$

where $\lambda_n = 2^n/\Lambda_{2^n}$ for all $n \in \mathbb{N}$. By Lemma 3.2,

$$\mathbb{X}_1 := Q_{\sigma'}(\mathbb{S}) \oplus \mathbb{X}_0 \simeq \mathbb{Y}[(\mathcal{X}_0)_\eta, \mathbb{S}, \sigma'].$$

Therefore, applying Lemma 5.1 with the basis $\mathcal{X}_0$, the partition σ', and the sequence $(\lambda_n)_{n=1}^\infty$ gives a semi-normalized M-bounded total basis $\mathcal{X}_1$ of $\mathbb{X}_1$ such that $R_n(a,b) \approx 2^n(a+b)$ for a, $b \geq 0$ and $R_n(1,-1) \approx 1$, where

$$R_n(a,b) = \left\| a\mathbb{1}_{\sigma'_{2n-1}}[\mathcal{X}_1, \mathbb{X}_1] + b\mathbb{1}_{\sigma'_{2n}}[\mathcal{X}_1, \mathbb{X}_1] \right\|.$$

As noted in the proof of Theorem 5.2, this implies $\tilde{\boldsymbol{k}}_m[\mathcal{X}_1] \gtrsim m$. By Proposition 3.3, the unit vector system is a sequeeze-symmetric total basis of of $\mathbb{Y}_1 = \mathbb{Y}[\mathcal{X}_1, \mathbb{S}, \sigma]$ whose dual basis is equivalent to the unit vector system of $\mathbb{Y}_2 = \mathbb{Y}[\mathcal{X}_2, \mathbb{S}_0^*, \sigma]$ for a suitable semi-normalized M-bounded basis $\mathcal{X}_2$.

Suppose that $(\Lambda_m[\mathbb{S}])_{m=1}^\infty$ has the URP. Then $(\Lambda_m[\mathbb{S}_0^*])_{m=1}^\infty$ has the LRP and, hence, the unit vector system of $\mathbb{Y}_2$ is squeeze-symmetric as well. In particular,

$$\boldsymbol{\varphi_u}[\mathcal{E}, \mathbb{Y}_1](m)\boldsymbol{\varphi_u}[\mathcal{E}, \mathbb{Y}_2](m) \approx \Lambda_m[\mathbb{S}]\Lambda_m[\mathbb{S}_0^*] \approx m, \quad m \in \mathbb{N}.$$

Hence, the unit vector system of $\mathbb{Y}_1$ is bidemocratic.

By Lemma 3.2,

$$\mathbb{Y}_1 \simeq Q_\sigma(\mathbb{S}) \oplus \mathbb{X}_1 \simeq \mathbb{S} \oplus \mathbb{S} \oplus \mathbb{X} \simeq \mathbb{X}.$$

Proposition 3.3 also gives

$$\tilde{\boldsymbol{k}}_{2^{n+1}-2}[\mathcal{E}, \mathbb{Y}_1] \gtrsim n, \quad n \in \mathbb{N}.$$

Therefore, $\tilde{\boldsymbol{k}}_m[\mathcal{E}, \mathbb{Y}_1] \gtrsim \log m$ for $m \geq 2$.

To prove that $\mathcal{E}$ is not a quasi-greedy basis of $\mathbb{Y}_1$, we consider the vectors

$$f = \sum_{k=1+M_{2n-2}}^{M_{2n-1}} \frac{1}{\Lambda_{|\sigma_k|}} \mathbb{1}_{\sigma_k}[\mathcal{E}, \mathbb{Y}_1], \quad g = \sum_{k=1+M_{2n-1}}^{M_{2n}} \frac{1}{\Lambda_{|\sigma_k|}} \mathbb{1}_{\sigma_k}[\mathcal{E}, \mathbb{Y}_1],$$

where $M_n = \sum_{k=1}^n |\sigma'_k|$. We have $\|-f+g\| = R_n(-1,1) \approx 1$ and $\|g\| = R_n(0,1) \approx 2^n$. Since $-f$ is a greedy projection of $-f+g$, we are done. □

We are now in a position to prove that separable Hilbert spaces have squeeze-symmetric bases with large unconditionality constants. In fact, Theorem 5.4 allows us to state a mild condition on a Banach space which ensures the existence of a truncation quasi-greedy basis for which the estimate (1.1) is optimal.

Corollary 5.5. *Let $\mathbb{X}$ be a separable Banach space that has a subsymmetric sequence space $\mathbb{S}$ as a complemented subspace. Suppose also that $\mathbb{X}$ has some* *nontrivial type (resp., cotype). Then, $\mathbb{X}$ has a bidemocratic (resp., sequeeze-symmetric) total basis $\mathcal{B}$ with*

$$\tilde{\boldsymbol{k}}_m[\mathcal{B},\mathbb{X}] \approx \log(1+m), \quad m\in\mathbb{N}.$$

Moreover, we can choose $\mathcal{B}$ so that is not quasi-greedy.

Proof. Just combine Proposition 2.1 with Theorem 5.4. □

Note that Corollary 5.5 yields, in particular, the existence of bidemocratic non-quasi-greedy bases for a wide class of Banach spaces. In this regard, it improves the results in this direction obtained in [2].

Remark 5.6. The basis $\mathcal{X}_1$ we have used to prove Theorem 5.4 may not be a Schauder basis. In fact, it is not a Schauder basis if $\mathbb{X}$ is superreflexive (see comments preceding Corollary 5.3). Then the basis $\mathcal{B}$ provided by Corollary 5.5 may not be a Schauder basis. So, we do not have an example of a truncation quasi-greedy Schauder basis of a super-reflexive space with bad unconditionality constants. We wonder whether this is due to a limitation of our method or, oppositely, the unconditionality constants of such bases satisfy better estimates.

Remark 5.7. Let $\boldsymbol{w}$ be a weight whose primitive sequence is doubling. Let $1\le q\le r\le\infty$ be such that either $q>1$ or $r<\infty$. Suppose that a basis $\mathcal{X}$ of a quasi-Banach space $\mathbb{X}$ is squeezed between $d_q(\boldsymbol{w})$ and $d_r(\boldsymbol{w})$. Then, by [13, Lemma 2.5 and Equation (2.7)] and [4, Remark 5.2],

$$\boldsymbol{k}_m[\mathcal{X},\mathbb{X}] \lesssim (1+\log m)^{1/q-1/r}.$$

We infer that the bidemocratic basis $\mathcal{B}$ obtained in Corollary 5.5 can not be squeezed in such a way. So, if we choose $\mathbb{X}$ to be a Hilbert space, $\mathcal{B}$ is a counterexample that solves in the negative [14, Question 5.3].

We close this section by using the machinery we have developed to generalize the construction of a democratic non-UCC basis of ℓ_2 carried out in [5, Example 11.21].

Proposition 5.8. *Let $\mathbb{S}$ be a locally convex subsymmetric sequence space. Then $\mathbb{S}$ has a democratic M-bounded total basis $\mathcal{X}$ with*

$$\boldsymbol{\varphi_u}[\mathcal{X},\mathbb{X}](m) \approx \boldsymbol{\varphi_l}[\mathcal{X},\mathbb{X}](m) \approx \Lambda_m[\mathbb{S}], \quad m\in\mathbb{N},$$

and $\boldsymbol{\varphi_l^s}[\mathcal{X},\mathbb{X}](m)\approx 1$ for $m\in\mathbb{N}$.

Proof. Let σ the an ordered partition of $\mathbb{S}$ with $|\sigma_{2n-1}|=|\sigma_{2n}|=n+1$ for all $n\in\mathbb{N}$. Set $\lambda=\sqrt{2}/\Lambda_2$ and $\eta=(\lambda,\Lambda_{n+1})_{n=1}^\infty$. Consider the unit vector system $\mathcal{E}$ of the space

$$\mathbb{Y} := \mathbb{Y}[(\mathcal{V}[\mathbb{S},\sigma])_\eta, \mathbb{S}, \sigma].$$

By Lemma 5.1, $\boldsymbol{\varphi_l^s}[\mathcal{E},\mathbb{Y}](2n) \approx 1$ for $n \in \mathbb{N}$. Let τ be the ordered partition of $\mathbb{N}$ given by $|\tau_n| = n+1$ for all $n \in \mathbb{N}$. By subsymmetry, $(\mathcal{V}[\mathbb{S},\tau])^2$ is equivalent to $\mathcal{V}[\mathbb{S},\sigma]$. Then, by Lemma 4.3(ii) and Lemma 4.4, $\mathcal{V}[\mathbb{S},\sigma]$ and $(\mathcal{V}[\mathbb{S},\sigma])_\eta$ are equivalent for positive scalars. Consequently, the unit vector systems of $\mathbb{Y}$ and

$$\mathbb{Y}_1 := \mathbb{Y}[(\mathcal{V}[\mathbb{S},\sigma], \mathbb{S}, \sigma])$$

are equivalent for positive scalars. In turn, the unit vector system of $\mathbb{Y}_1$ is equivalent to that of $\mathbb{S}$. Thus,

$$\|\mathbb{1}_B[\mathcal{E},\mathbb{Y}]\| \approx \Lambda_m[\mathbb{S}], \quad B \subseteq \mathbb{N},\, m \in \mathbb{N},\, |B| = m < \infty.$$

By Lemma 3.2,

$$\mathbb{Y} \simeq Q_\sigma(\mathbb{S}) \oplus P_\sigma(\mathbb{S}) \simeq \mathbb{S}, \tag{5.1}$$

and so the basis $\mathcal{X}$ of $\mathbb{S}$ that corresponds with $\mathcal{E}$ via the isomorphism between $\mathbb{Y}$ and $\mathbb{S}$ provided by (5.1) satisfies the desired properties. □

6. Measuring conditionality via the near unconditionality functions

In Section 5, we have used the DKK method to obtain greedy-like bases with large unconditionality constants. In general, the conditionality of such bases in terms of their unconditionality constants has been extensively studied in the literature (see, e.g., [4], [8], [6], [13], [15], [16]). In this section, we look at their conditionality using a different meausuring tool. First, we need some additional notation and terminology. Given a basis $\mathcal{X} = (\boldsymbol{x}_n)_{n=1}^\infty$ of a quasi-Banach space $\mathbb{X}$ we define

$$\mathcal{Q} = \mathcal{Q}[\mathcal{X},\mathbb{X}] = \{f \in \mathbb{X}\colon \|\mathcal{F}(f)\|_\infty = \sup_n |\boldsymbol{x}_n^*(f)| \le 1\}, \text{ and}$$
$$A(f,a) = \{n \in \mathbb{N}\colon |\boldsymbol{x}_n^*(f)| \ge a\}, \quad f \in \mathbb{X},\, a > 0.$$

We say that basis $\mathcal{X}$ is *nearly unconditional* if for each $a \in (0,1]$ there is a constant C such that, for any $f \in \mathcal{Q}$,

$$\|S_A(f)\| \le C\,\|f\|, \tag{6.1}$$

whenever $A \subseteq A(f,a)$. The *near unconditionality function of the basis* is the function $\phi\colon (0,1] \to [1,+\infty)$ that gives the optimal C for which (6.1) holds.

The notion of near unconditionality was introduced by Elton in [20], and imported to greedy approximation theory by Dilworth et al. in [17]. It is known that truncation quasi-greedy bases are nearly unconditional (see [3, Corollary 3.5] and [17, Proposition 4.5]), so their conditionality can be studied by means of the near unconditionality function ϕ. In this context, it is natural to ask how this approach compares with the standard way of studying it via the unconditionality parameters $(\boldsymbol{k}_m)_{m=1}^\infty$. As we shall see, despite the fact that the cardinality of the sets in the projections involved in (6.1) is unrestricted, the results obtained for the unconditionality constants of quasi-greedy and truncation quasi-greedy bases also hold for ϕ, in a sense that will become clear below. We begin with a lemma that allows us to compare both measures of conditionality.

Lemma 6.1. *Let $\mathcal{X}$ be a semi-normalized M-bounded nearly unconditional basis of a p-Banach space $\mathbb{X}$, $0 < p \le 1$. Let $F\colon (0,\infty) \to [1,\infty)$ be a non-decreasing function such that*

$$\boldsymbol{k}_m \ge F(m), \quad m \in \mathbb{N}.$$

Then, if $\alpha_1 = \sup_n \|\boldsymbol{x}_n\|$ and $\alpha_2 = \sup_n \|\boldsymbol{x}_n^*\|$,

$$\phi(a) \ge \frac{1}{4^{1/p}\alpha_1\alpha_2} F\left(a^{-p}\right), \quad 0 < a \le 1.$$

Proof. Pick $m \in \mathbb{N}$ such that $m \le a^{-p} < m+1$. Then, fix $f \in \mathbb{X}$ with $\|f\| = 1$, and $A \subseteq \mathbb{N}$ with $|A| \le m$. Set $g = f/\|\mathcal{F}(f)\|_\infty$, so that $g \in \mathcal{Q}$. Let (A_1, A_2) be the partition of A defined by

$$A_1 := \left\{n \in A\colon\ |\boldsymbol{x}_n^*(g)| \le m^{-1/p}\right\}.$$

We have

$$\begin{aligned}\|S_A(g)\|^p &\le \|S_{A_1}(g)\|^p + \|S_{A_2}(g)\|^p \\ &\le \sum_{n\in A_1} |\boldsymbol{x}_n^*(g)|^p \|\boldsymbol{x}_n\|^p + \phi^p\left(m^{-1/p}\right)\|g\|^p \\ &\le \frac{|A_1|\,\alpha_1^p}{m} + \phi^p\left(m^{-1/p}\right)\|g\|^p.\end{aligned}$$

Thus, since $|A_1| \le m$, $\|g\|\,\|\mathcal{F}(f)\|_\infty = 1$, and $\|\mathcal{F}(f)\|_\infty \le \alpha_2$,

$$\|S_A(f)\|^p \le \alpha_1^p\alpha_2^p + \phi^p\left(m^{-1/p}\right) \le 2\alpha_1^p\alpha_2^p\phi^p\left(m^{-1/p}\right).$$

Taking the supremum over f and A we obtain

$$\boldsymbol{k}_m \le 2^{1/p}\alpha_1\alpha_2\phi\left(m^{-1/p}\right). \tag{6.2}$$

Since ϕ is non-increasing, combining (6.2) with the inequality $\boldsymbol{k}_{m+1}^p \le \boldsymbol{k}_{2m}^p \le 2\boldsymbol{k}_m^p$ gives

$$F(a^{-p}) \le F(m+1) \le \boldsymbol{k}_{m+1} \le 4^{1/p}\alpha_1\alpha_2\phi\left(m^{-1/p}\right) \le 4^{1/p}\alpha_1\alpha_2\phi(a). \qquad \square$$

Lemma 6.1 tells us that, in a natural sense, lower bounds for $\boldsymbol{k}_m$ are stronger than lower bounds for ϕ. Thus, it allows us to transfer to the language of near unconditionality functions any result that guarantees the existence of bases with large unconditionality constants. For instance, we infer the following results.

Corollary 6.2. *Let $\mathbb{X}$ be a separable Banach space. Suppose that a symmetric sequence space $\mathbb{S}$ is complemented in $\mathbb{X}$. Suppose also that $(\Lambda_m[\mathbb{S}])_{m=1}^\infty$ has the LRP. Then $\mathbb{X}$ has a squeeze symmetric (bidemocratic if $(\Lambda_m[\mathbb{S}])_{m=1}^\infty$ also has the URP) total basis $\mathcal{B}$ with*

$$\phi(a) \gtrsim 1 - \log a, \quad 0 < a \le 1,$$

and $\boldsymbol{\varphi_u}[\mathcal{B}, \mathbb{X}](m) \approx \Lambda_m[\mathbb{S}]$ for $m \in \mathbb{N}$. Moreover, $\mathcal{B}$ is not quasi-greedy.

Corollary 6.3. *There is a reflexive Banach space $\mathbb{X}$ with an almost greedy Schauder basis $\mathcal{X}$ such that*

$$\phi(a) \gtrsim 1 - \log a, \quad 0 < a \le 1.$$

Corollary 6.4. *Let $\mathbb{X}$ be a Banach space with a Schauder basis. Suppose that ℓ_p, $1 < p < \infty$, is complemented in $\mathbb{X}$. Then $\mathbb{X}$ has, for any $0 < \epsilon < 1$, an almost greedy Schauder basis $\mathcal{B}$ with*

$$\phi(a) \gtrsim (1 - \log a)^{1-\epsilon}, \quad 0 < a \le 1,$$

and $\boldsymbol{\varphi_u}[\mathcal{B}, \mathbb{X}](m) \approx m^{1/p}$ for $m \in \mathbb{N}$.

In fact, Corollary 6.2, Corollary 6.3 and Corollary 6.4 follow from combining Lemma 6.1 with Theorem 5.4, [6, Theorem 4.18] and [6, Theorem 4.5], respectively. We refer the reader to [6] for more examples of Banach spaces with large unconditionality constants, then large near unconditionality functions.

Theorem 6.5 and Theorem 6.6 below, which improve [9, Theorem 5.1] and [7, Theorem 1.1], respectively, prove that Corollaries 6.2, 6.3 and 6.4 are optimal in the sense that the near unconditionality functions of the bases we obtain are such large as possible.

Theorem 6.5. *Let $\mathcal{X}$ be a truncation quasi-greedy basis of a p-Banach space $\mathbb{X}$. Then,*

$$\phi(a) \lesssim (1 - \log a)^{1/p}, \quad 0 < a \le 1.$$

Theorem 6.6. *Let $\mathcal{X}$ be a quasi-greedy basis of a superreflexive Banach space $\mathbb{X}$. Then, there is $0<\epsilon<1$ such that*

$$\phi(a) \lesssim (1-\log a)^{1-\epsilon}, \quad 0<a\le 1.$$

Before facing the proof of these results, we bring up an auxiliary lemma.

Lemma 6.7 (See [2, Lemma 2.3], [5, Proposition 4.16] or [9, Lemma 5.2])**.** *Suppose that $\mathcal{X}=(\boldsymbol{x}_n)_{n=1}^\infty$ is a semi-normalized truncation quasi-greedy basis of a quasi-Banach space $\mathbb{X}$. Then there is a constant C such that*

$$\left\| \sum_{n\in A} a_n\, \boldsymbol{x}_n \right\| \le C\,\|f\|$$

for every $f\in\mathbb{X}$, every finite set $A\subseteq\mathbb{N}$, and every finite family $(a_n)_{n\in A}$ such that $\max_{n\in A}|a_n| \le \min_{n\in A}|\boldsymbol{x}_n^*(f)|$.

Proof of Theorems 6.5 and 6.6. In the superreflexive case, an application of [14, Theorem 2.3] gives $C_0\in[1,\infty)$ and $p\in(1,\infty)$ such that

$$\left\| \sum_{k=1}^m f_k \right\|^p \le C_0^p \sum_{k=1}^m \|f_k\|^p \tag{6.3}$$

for all $m\in\mathbb{N}$ and all disjointly supported families $(f_k)_{k=1}^m$ in $\langle\mathcal{X}\rangle$ such that

$$\max_{n\in\operatorname{supp}(f_k)} |\boldsymbol{x}_n^*(f_k)| \le \min_{n\in\operatorname{supp}(f_{k-1})} |\boldsymbol{x}_n^*(f_{k-1})|, \quad 2\le k\le m.$$

In turn, in the locally p-convex case for all values of $0<p\le 1$, (6.3) holds with $C_0=1$. Fix $f\in\mathcal{Q}$, $0<a\le 1$ and $A\subseteq A(f,a)$. Pick $n\in\mathbb{N}$ such that $2^{-n}<a\le 2^{1-n}$. Consider the partition $(A_k)_{k=0}^n$ of A given by $A_0=A\cap A(f,1)$,

$$A_k = A\cap\left(A(f,2^{-k})\setminus A(f,2^{1-k})\right), \quad 1\le k\le n-1,$$

and $A_n=A\setminus A(f,2^{1-n})$. In both cases we have

$$\|S_{A_k}(f)\| = 2\,\|S_{A_k}(f/2)\| \le 2C\,\|f\|, \quad 0\le k\le n,$$

where C is the constant provided by Lemma 6.7. Hence,

$$\begin{aligned}
\|S_A(f)\|^p &= \left\| \sum_{k=0}^n S_{A_k}(f) \right\|^p \\
&\le C_0^p \sum_{k=0}^n \|S_{A_k}(f)\|^p \\
&\le (1+n)2^p C^p C_0^p\,\|f\|^p \\
&\le (2-\log_2 a)2^p C^p C_0^p\,\|f\|^p.
\end{aligned}$$

Taking the supremum over A and f we obtain

$$\phi(a) \le 2CC_0(2-\log_2 a)^{1/p}, \quad 0<a\le 1. \qquad \square$$

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