UNCERTAINTY WITH ORDINAL LIKELIHOOD INFORMATION

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Uncertainty with ordinal likelihood information*

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Abstract

This paper proposes a new framework of choice under uncertainty, where the only information available to the decision maker is about the the ordinal likelihood of the different outcomes each action generates. This contrasts both with the classical models where the potential outcomes of each action have an associated probability distribution, and with the more recent *complete uncertainty* models, where the agent has no information whatever about the probability of the outcomes, even of an ordinal nature. We present an impossibility result in our framework, and some ways to circumvent it that result in different ranking rules.

Keywords: Uncertainty, Ordinal Likelihood, Nonprobabilistic models, Complete Uncertainty

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1 Introduction

In the conventional models of individual choice, when the outcomes of the alternatives are uncertain, they are assumed to be equipped with a probability distribution. There are, however, many decision problems in which it is hard to assume objective probabilities. Savage [22] provided a well-known solution to this problem by showing that, under certain conditions, individuals assign to the possible uncertain events probabilities of a subjective nature, which they then use to maximize the expected utility of the alternatives.

The assumption of the existence of a standard subjective probability measurement can, in many instances, be considered too strong a requirement, however. There are in fact some models in which the Ellsberg Paradox is explained by relaxing such requirement. One such model is that of Gilboa and Schmeidler [15] who explain it by relaxing the assumption of a unique subjective probability. Another is that of Schmeidler [23] who does so by assuming that it is not additive. From a technical point of view, a powerful reason for not considering conventional probabilities is that, for such probabilities to exist, Savage's [22] requirements, which are nontrivial, must necessarily be fulfilled (see also de Finetti [13], Kraft et al. [18], and Scott [24]).

The non-existence of a probability distribution for the outcomes of the alternatives has been approached, in a more radical way, under the so-called models of choice under "complete uncertainty". There, it is assumed that the decision maker has no information about the probabilities of the possible outcomes or about their relative likelihood. We find two approaches within the models of choice under complete uncertainty:

- The *vector-based* approach takes into account the set of possible states of nature and the corresponding outcomes of each alternative action under each state. Formally, therefore, an action (or alternative) is described by a function that assigns an outcome to each possible state. In other words, an action is a vector of outcomes contingent upon the possible states of nature. It is therefore assumed that the agent has no information about the probabilities or relative likelihood of the different states of nature. Examples of this approach are to be found in Arrow and Hurwicz [3], Maskin [19], Cohen and Jaffray [12], Barberà and Jackson [6], and Barret and Pattanaik [8].

- The *set-based* approach does not take into account any information about the possible states of nature. It simply considers the possible outcomes of each action. Therefore, each action is described, simply, by the set of outcomes it may generate (see, for example, Barberà et al. [4], Barberà and Pattanaik [7], Kannai and Peleg [16], Nitzan and Pattanaik [20], Pattanaik and Peleg [21], Bossert [9, 10], Bossert et al. [11] and Arlegi [1, 2]. For a survey, see Barberà et al. [5]). In these models the decision maker is assumed to lack information about the probabilities or relative likelihood of the outcomes directly. Thus, we can interpret that complete uncertainty under the set-based approach is even stronger than in the vector-based approach, since the connection between states of nature and outcomes is absent.

The authors that have developed the set-based approach invoke several relative advantages over the vector-based formulation. One is that the former might be more suitable for the tractability of overly complex problems, where it might be difficult to identify the particular states under which each outcome occurs. Such identification is sometimes unnecessary or simply impossible, and then the decision maker considers only the possible outcomes of each action. Secondly, in some situations the states of nature may be arbitrarily partitioned in different ways, making the vector-based approach dependent upon this arbitrariness. Finally, the set-based approach has been defended as a more suitable way to represent the Rawlsian problem of choice under the veil of ignorance (a deeper discussion of all these arguments can be found in Pattanaik and Peleg [21] and Bossert et al. [11]).

One shortcoming that is present in both approaches is that, from a descriptive point of view, the information on which they are based might be considered too vague. In general, human beings have some perception as to which states are more and less likely. Taking this idea, Kelsey [17] proposes a model in which the decision maker is unable to assign numerical probabilities to the possible states of nature, but knows which states are more and which are less likely. Thus, Kelsey's [17] model can be viewed as being midway between subjective expected utility and the vector-based models of choice under complete uncertainty.

We share Kelsey's [17] view that, in many contexts, individuals are unable to establish subjective probability distributions, and that the kind of comparisons they make are of an ordinal nature. In our paper, however, we apply this assumption to a set-based framework. That is, we do not take into account any information about the states of nature. We represent actions solely by means of the outcomes they might produce. In this way, the ordinal perception of likelihood is directly about outcomes, rather than about states of nature.[1] In our framework, therefore, we take the information associated to each individual action to have two components: the possible outcomes, and an (ordinal) ranking over them, which is made in terms of likelihood.

The differences with Kelsey's [17] modelling are in no way trivial. In addition to incorporating the relative advantages of the set-based approach, our model compares objects of a different nature (sets with possibly different cardinalities rather than contingent functions). Our results therefore belong to a different category.

The paper is organized as follows: Section 2 contains the basic notation and definitions. In Section 3 we present a first result showing that there is no preorder over the set of individual actions that at once satisfies three plausible axioms. We propose two solutions to overcome this negative result. First, in Section 4, we restrict the domain to rankings involving only individual actions with the same cardinality (equal number of possible outcomes). We then propose a new set of axioms for this case, renouncing one of the axioms of the impossibility result. In Section 5 we extend these criteria to the general domain, obtaining some rules that satisfy only two of the three axioms of

[1] We discovered Kelsey's article only after obtaining all our results, and the paper was practically finished. Reading his work has sharpened our perspective both on the problem and the scope of our contribution.

the impossibility theorem. We conclude in Section 6 with some comments concerning possible lines of further research within our general framework.

2 Notation and definitions

We define X as an infinite universal set of outcomes (or results), equipped with a linear order R (complete, transitive and antisymmetric), which reflects the agent's preferences over them.[2] The asymmetric factor, P, of R is defined as usual. We will say that X is *rich with respect to* R if for all $x, y \in X$ such that xPy, there exists $a, b, c \in X$ such that $aPxPbPyPc$. That is, in a rich domain, for any two outcomes, there is always another that is better, another that is worse, and another that is midway between the two. In general, richness will not be assumed throughout the paper, except when explicitly stated.

We assume that each individual action generates a set of possible outcomes, and that the final outcome is determined by a chance mechanism. Further, we assume that, for every action, the decision maker is able to assign a likelihood ranking over its possible outcomes. Given the above assumptions concerning the nature of the alternative actions, we represent each action, or alternative, by a certain non-empty finite ordered subset of X, where we adopt the convention that the elements are ordered from more to less likely. For example, $\vec{a} = (a_1, \ldots, a_n)$ represents an individual action that might produce the mutually exclusive outcomes $a_1, \ldots, a_n$ (and no others), such that the agent perceives outcome a_1 to be more likely than outcome a_2, the latter more likely than outcome a_3, and so on.

Formally, the difference between the above representation of the actions, and that used in the set-based approach to the choice under complete uncertainty problem is that in the latter the actions are represented by bare (non-ordered) sets, each representing the possible outcomes generated by a certain action. For example, in that framework the set $A = \{a_1, \ldots, a_n\}$

[2] The results of the paper can be easily adapted if we drop the antisymmetry requirement, introduced for the sake of fluency.

would represent an action under which the outcomes a_1 to a_n are possible, and, given that the choice under complete uncertainty framework makes no assumption about the likelihood or probability of the outcomes, the order of presentation is meaningless, and the same action could be represented by any permutation of them. Obviously, in our framework, any permutation of the elements within a set would represent a different action, since it would modify the relative likelihood of the outcomes.

We assume feasibility of every combination of the outcomes and, therefore, we study rankings over all the possible non-empty ordered subsets of X. We will denote the set of all the non-empty ordered subsets of X by Q, and for $k \in \mathbb{N}$, Q_k will be the set of all ordered subsets of X with cardinality k, being $Q = \bigcup_{k=1}^{\infty} Q_k$. Thus, our formal goal is to compare elements of Q by means of a binary relation $\succsim$ in order to reflect individual preferences over alternative actions. We will say that $\succsim$ is a *preorder* if it satisfies reflexivity and transitivity, and that it is a *complete preorder* if it is both a preorder and also satisfies completeness.

For all $\vec{a} \in Q$, all $x \notin \vec{a}$ and all $m \leq (|\vec{a}| + 1)$, $I_m(\vec{a}, x)$ denotes the ordered set $(a_1, \ldots, a_{m-1}, x, a_m, \ldots, a_k)$. That is, $I_m(\vec{a}, x)$ denotes a new set of outcomes where x has been inserted in the m-th position of $\vec{a}$ without affecting the likelihood ordering of the remaining outcomes. We define the class of vectors $I(\vec{a}, x)$ as: $I(\vec{a}, x) = \{\vec{b} \in Q_{|\vec{a}|+1} \mid \text{there exists } m \leq (|\vec{a}| + 1) \text{ such that } \vec{b} = I_m(\vec{a}, x)\}$. That is, $I(\vec{a}, x)$ includes all the actions that can be obtained by inserting x in any position of $\vec{a}$.

$\Pi_{(i,j)}(\vec{a})$ denotes a permutation of $\vec{a}$ where a_i and a_j are the only permuted outcomes. That is, for $\vec{a} \in Q$, $\Pi_{(i,j)}(\vec{a}) = (a_{\pi(1)}, \ldots, a_{\pi(|\vec{a}|)})$, where Π is a permutation on $\{1, \ldots, |\vec{a}|\}$ such that $a_{\Pi(i)} = a_j$, $a_{\Pi(j)} = a_i$, and $a_{\Pi(l)} = a_l$ for all $l \notin \{i, j\}$.

For all $\vec{a}, \vec{b} \in Q$, we define the *non-ordered* sets $\vec{a} \cup \vec{b}$ and $\vec{a} \cap \vec{b}$ as follows: $\vec{a} \cup \vec{b} = \{x \in X \mid x \in \vec{a} \text{ or } x \in \vec{b}\}$ and $\vec{a} \cap \vec{b} = \{x \in X \mid x \in \vec{a} \text{ and } x \in \vec{b}\}$.

For all $\vec{a}, \vec{b} \in Q$ such that $\vec{a} \cap \vec{b} = \emptyset$, we define the ordered set $(\vec{a}, \vec{b}) \in Q_{|\vec{a}|+|\vec{b}|}$ as follows: $(\vec{a}, \vec{b}) = (a_1, \ldots, a_{|\vec{a}|}, b_1, \ldots, b_{|\vec{b}|})$.

Finally, for all finite $C \subset X$, $\max\{C\} = \{x \in C \mid xPy \text{ for all } y \in C\}$ and $\min\{C\} = \{x \in C \mid yPx \text{ for all } y \in C\}$. With a slight abuse of notation, we define the max and min operators for the elements of Q in the same way. That is, $\max\{\vec{a}\}$ ($\min\{\vec{a}\}$) represents the best (worst) outcome in $\vec{a}$. Note that, given the assumptions, for all $\vec{a} \in Q$, $\max\{\vec{a}\}$ and $\min\{\vec{a}\}$ are always singletons (the best and worst elements of any action are unique).

3 An impossibility result

Let us now consider the following axioms, which reflect some ideas that are intuitive to our framework.

Reordering (REO): For all $\vec{a} \in Q$ and $i < j$,

$$a_j P a_i \Rightarrow \Pi_{(i,j)}(\vec{a}) \succ \vec{a}$$

REO refers to the following intuition: Assume that a_i and a_j are both possible outcomes under a certain action $\vec{a}$, and that a_j is a better outcome than a_i, but is perceived as less likely by the agent. Then, if we consider another action with the same set of possible outcomes, but where a_j is perceived as more likely than a_i, without affecting the likelihood comparisons of the remaining elements, the latter action is strictly better than the former.

Under REO, all sets with the same associated outcomes are no longer indifferent, which is in contrast with the set-based approach. REO in fact shares the spirit of the *Interchange* axiom in Kelsey [17].

Dominance (DOM): For all $\vec{a}, \vec{b}, \vec{c} \in Q$ and for all $x, y \notin \vec{a}$ such that xPa_iPy for all $i \in \{1, \ldots, |\vec{a}|\}$, $\vec{b} \in I(\vec{a}, x)$ and $\vec{c} \in I(\vec{a}, y)$,

$$\vec{b} \succ \vec{a} \succ \vec{c}$$

DOM is a plausible extension of the Dominance axiom in the set-based approach to choice under complete uncertainty. It is related to Gärdenfors' principle [14], introduced by Kannai and Peleg [16] and later widely used in the related literature. The condition is very reasonable in our context also. In words, DOM states the following: assume that, at any position in the order of likelihood, we add to an action $\vec{a}$ a new outcome that is better (worse) than all the possible outcomes in $\vec{a}$, without altering the ordinal likelihood ordering of the original outcomes. Then, the final situation is better (worse) than the original one.

Composition (COM): For all $\vec{a}, \vec{b}, \vec{c}, \vec{d} \in Q$ such that $\vec{a} \succ \vec{b}$, $\vec{c} \succ \vec{d}$ and $\vec{a} \cap \vec{c} = \vec{b} \cap \vec{d} = \emptyset$,

$$(\vec{a}, \vec{c}) \succsim (\vec{b}, \vec{d})$$

Consider two alternative actions $\vec{a}$ and $\vec{c}$ that are better than another two, $\vec{b}$ and $\vec{d}$, respectively. Now consider an action that includes all the outcomes in $\vec{a}$ and $\vec{c}$, maintains the internal likelihood orders of both $\vec{a}$ and $\vec{c}$, and is such that any outcome in $\vec{a}$ is more likely than any outcome in $\vec{c}$. Take a similar composition of $\vec{b}$ and $\vec{d}$. Condition COM establishes that such a composition of $\vec{b}$ and $\vec{d}$ should not be strictly better than that of $\vec{a}$ and $\vec{c}$.

In order to defend the plausibility of COM, imagine that $\vec{a} \succ \vec{b}$, but $\vec{c}$ and $\vec{d}$ are indifferent. Then, it would be quite natural to assume that $(\vec{a}, \vec{c}) \succ (\vec{b}, \vec{d})$. In fact, this is the spirit of several very common conditions of Independence in ranking sets models. COM is an even weaker condition, in two senses: first, we assume $\vec{c}$ to be strictly better than $\vec{d}$, and second, we admit $(\vec{a}, \vec{c})$ to be indifferent to $(\vec{b}, \vec{d})$.

Another argument in favor of COM arises if we imagine a natural adaptation of this axiom to the choice under complete uncertainty framework under the set-based approach. Such an adaptation would claim that the union of two sets of outcomes, A and C, that are better than another two sets B and D, respectively, should be better than the union of B and D. To the best of our knowledge, all the rules proposed in the context of choice under complete uncertainty would satisfy this axiom, with the exceptions of the *median rule* by Pattanaik and Peleg [21] and certain rankings within the family characterized by Bossert [10].

The main result for this section is the following:

Theorem 3.1 *There is no preorder $\succsim$ satisfying REO, DOM and COM.*

[*Proof in the Appendix*]

Two remarks are in order concerning Theorem 3.1. First, it can be proved that the Theorem also applies for the case in which X is finite whenever $|X| \geq 3$. Secondly, completeness of $\succsim$ is not necessary for the impossibility result, even in the case of a finite domain.

Our impression is that COM may be controversial in some instances and that this in fact lies at the root of the impossibility result. In particular, this happens when the actions to be compared are of very different cardinality because a trade-off arises between the relative weight of adding certain outcomes at the end of an action and the desirability of such outcomes. That is, a slight difference in the desirability of $\vec{c}$ and $\vec{d}$ might be compensated by the small relative weight of the outcomes in $\vec{c}$ if the cardinality of $\vec{a}$ is considerably greater than that of $\vec{b}$. This motivates the direction of the rest of the paper. In Section 4, we restrict our analysis to the case in which the actions to be compared have the same cardinality, and in Section 5, we study the general case renouncing the COM condition when the actions to be compared are of different cardinality.

4 The equal cardinality case

We have already demonstrated the impossibility of combining axioms REO, DOM and COM in order to rank, even partially, sets of outcomes with an associated likelihood ranking. Therefore, in order to obtain some rules of comparison, we need to renounce at least one of the three conditions. In this dilemma, we do not consider the weakening or removal of REO, which is a referential axiom for our framework, an acceptable option to resolve this dilemma. In this section, we focus exclusively on comparisons of sets with the same cardinality, which means that we will ignore DOM. This enables us to overcome the impossibility result. Furthermore, we propose a new set of axioms, all relating to the comparison of sets with the same cardinality (actions with the same number of associated outcomes). First, we present some axioms that are adaptations to our framework of conditions that appear in the set-based approach to the problems of choice under complete uncertainty.

Extension (EXT): For all $x, y \in X$,

$$xPy \Rightarrow (x) \succ (y)$$

This is a very common axiom for the ranking of sets in many settings, and it is also straightforwardly plausible in our context.

Responsiveness (RES): For all $j, k \in \mathbb{N}$ such that $j \leq k$, and all $\vec{a}, \vec{b} \in Q_k$ such that $a_j P b_j$ and $a_i = b_i$ for all $i \notin (\{1, \ldots, k\} \setminus \{j\})$,

$$\vec{a} \succ \vec{b}$$

This condition is stronger than EXT. It implies that the substitution of one outcome of an action with a better one results in a strict improvement. This axiom is related to the *Dominance* properties in Kelsey's [17] vector-based framework.

Independence (IND): For all $k \in \mathbb{N}$, all $\vec{a}, \vec{b} \in Q_k$, all $x \notin \vec{a} \cup \vec{b}$ and all $m \leq (k+1)$,

$$\vec{a} \succsim \vec{b} \Leftrightarrow I_m(\vec{a}, x) \succsim I_m(\vec{b}, x)$$

Independence-like conditions are very common across most of the set-ranking models, choice under complete uncertainty problems included (see Kannai and Peleg [16] or Pattanaik and Peleg [21], among others). More particularly, IND is a translation to our framework of Kelsey's [17] adaptation of Savage's [22] *Sure-Thing Principle.* Axiom IND says that if we add (remove) the same outcome to (from) *the same* position of two vectors, without changing the likelihood ordering of the remaining elements, the original evaluation remains invariant.

Neutrality (NEU): For all $k \in \mathbb{N}$, all $\vec{a}, \vec{b} \in Q_k$ and all one-to-one mapping $f : X \to X$ such that for all $x, y \in (\vec{a}, \vec{b})$, $xRy \Leftrightarrow f(x)Rf(y)$,

$$\vec{a} \succsim \vec{b} \Leftrightarrow (f(a_1), \ldots, f(a_k)) \succsim (f(b_1), \ldots, f(b_k))$$

NEU is a direct adaptation of an axiom with the same name that appears in Bossert [9], Nitzan and Pattanaik [20] and Pattanaik and Peleg [21], among others, and is related to Kelsey's [17] *Independence of Ranking of Irrelevant Outcomes*. According to NEU, the rule is immune to changes that do not affect either the likelihood ordering of the outcomes within each action or the desirability ordering of all the outcomes of the two actions to be compared. Usually, this axiom is defended as a requirement that the labelling of the outcomes should be irrelevant.

We are now going to introduce three axioms that state related ideas of robustness of the strict preference between two situations when new outcomes are added to them. These axioms are specific to this framework and have no direct links with any of the axioms of the literature of choice under complete uncertainty.

Likelihood sensitivity (LS): For all $k \in \mathbb{N}$, all $\vec{a}, \vec{b} \in Q_k$ such that $a_i \neq b_i$ for all $i \in \{1, \ldots, k\}$, and all $x \notin \vec{a}$, $y \notin \vec{b}$, there exists $\vec{c} \in Q$ such that

$$\vec{a} \succ \vec{b} \Rightarrow ((\vec{a}, \vec{c}), (x)) \succ ((\vec{b}, \vec{c}), (y))$$

Consider an action $\vec{a}$ that is better than another action $\vec{b}$ and suppose we add two new outcomes x and y to $\vec{a}$ and $\vec{b}$ respectively. The intuition behind LS is that the preference between $\vec{a}$ and $\vec{b}$ is maintained if x and y are added in a such a way as to be sufficiently unlikely. This is done by inserting a certain set of outcomes, as large as necessary, between the original sets and the new single outcomes. The real scope of the axiom arises when yPx. Then, what the condition establishes is that, in order to maintain the original preference for $\vec{a}$ versus $\vec{b}$, we can compensate any big difference in desirability for y versus x by making them sufficiently unlikely.

The following two axioms are weaker versions of LS.

Weak Likelihood sensitivity 1 (WLS1): For all $k \in \mathbb{N}$, all $\vec{a}, \vec{b} \in Q_k$ such that $a_i \neq b_i$ for all $i \in \{1, \ldots, k\}$, and all $x \notin \vec{a}$, $y \notin \vec{b}$ such that $\max\{\vec{a} \cup \vec{b}\} \, R \, \max\{x, y\}$, there exists $\vec{c} \in Q$ such that

$$\vec{a} \succ \vec{b} \Rightarrow ((\vec{a}, \vec{c}), (x)) \succ ((\vec{b}, \vec{c}), (y))$$

Weak Likelihood sensitivity 2 (WLS2): For all $k \in \mathbb{N}$, all $\vec{a}, \vec{b} \in Q_k$ such that $a_i \neq b_i$ for all $i \in \{1, \ldots, k\}$, and all $x \notin \vec{a}$, $y \notin \vec{b}$ such that $\min\{x, y\} \, R \, \min\{\vec{a} \cup \vec{b}\}$, there exists $\vec{c} \in Q$ such that

$$\vec{a} \succ \vec{b} \Rightarrow ((\vec{a}, \vec{c}), (x)) \succ ((\vec{b}, \vec{c}), (y))$$

Compared with LS, WLS1 and WLS2 state a similar idea, but the axioms are weaker because they apply to restricted domains of situations. In particular, WLS1 (WLS2) requires that the new outcomes to be added, x and y, are different and no better (worse) than the best (worst) outcome in the original sets.

Finally, we propose two conditions that reflect the idea that there always exist outcomes that are sufficiently good or bad as to reverse a given preference over two actions when added to one of them. These properties are also specific to this framework.

High Outcome Sensitivity (HOS): For all $k \in \mathbb{N}$ and all $\vec{a}, \vec{b} \in Q_k$ such that there exists $x \notin \vec{a} \cup \vec{b}$ with $xP \max\{\vec{a} \cup \vec{b}\}$, there exists $y \notin \vec{a} \cup \vec{b}$ such that

$$(a_1, \ldots, a_{k-1}, y) \succ \vec{b}$$

Low Outcome Sensitivity (LOS): For all $k \in \mathbb{N}$ and all $\vec{a}, \vec{b} \in Q_k$ such that there exists $x \notin \vec{a} \cup \vec{b}$ such that $\min\{\vec{a} \cup \vec{b}\}Px$, there exists $y \notin \vec{a} \cup \vec{b}$ such that

$$\vec{b} \succ (a_1, \ldots, a_{k-1}, y)$$

HOS (LOS) states that, when comparing two actions $\vec{a}$ and $\vec{b}$, there always exists an outcome y, such that we can make $\vec{a}$ better (worse) than $\vec{b}$ by substituing y for the least likely outcome in set $\vec{a}$. The intuitive idea is that we can always compensate the difference in the preference between $\vec{a}$ and $\vec{b}$ with an outcome y provided it is sufficiently good (bad). Note that the conditions only apply when there exists at least one outcome outside $\vec{a}$ and $\vec{b}$ that is better (worse) than all the outcomes inside.

Next, we are going to present some rules of comparison for the equal cardinality case. For this we need to introduce an additional piece of notation. For all $\vec{a} \in Q_k$, $\gamma(\vec{a})$ will denote a permutation of the outcomes in $\vec{a}$ such that $\gamma_i(\vec{a})\, P\, \gamma_{i+1}(\vec{a})$ for all $i < k$, where $\gamma_i(\vec{a})$ denotes the element of $\vec{a}$ that occupies the i-th position after the permutation. That is, γ reorders the elements of $\vec{a}$ from best to worst, according to R. In the same way, we define $\beta(\vec{a})$ as a permutation of the outcomes in $\vec{a}$ such that $\beta_{i+1}(\vec{a})\, P\, \beta_i(\vec{a})$ for all $i < k$, where $\beta_i(\vec{a})$ denotes the element of $\vec{a}$ that occupies the i-th position after the permutation. Furthermore, $L(\gamma_i(\vec{a}))$ will denote the position in likelihood terms that element $\gamma_i(\vec{a})$ occupies in $\vec{a}$. That is, $L(\gamma_i(\vec{a})) = k$ if $\gamma_i(\vec{a}) = a_k$. In a similar way, we define $L(\beta_i(\vec{a}))$ as the position in likelihood terms that the $i-$th worst outcome occupies in $\vec{a}$.

Definition 4.1 *A binary relation* $\succsim \subseteq \bigcup_{k\in\mathbb{N}}(Q_k \times Q_k)$ *is the* leximax-likelihood rule $\succsim_{LL}$ *if for all* $\vec{a}, \vec{b} \in Q_k$ *for any* $k \in \mathbb{N}$*:*

$\vec{a} \succsim_{LL} \vec{b} \Leftrightarrow \vec{a} = \vec{b}$ *or there is* $j \leq k$ *such that* $(a_j P b_j$ *and* $a_i = b_i$ *for all* $i < j)$.

Definition 4.2 *A binary relation* $\succsim \subseteq \bigcup_{k\in\mathbb{N}}(Q_k \times Q_k)$ *is the* leximax-desirability rule $\succsim_{LD}$ *if for all* $\vec{a}, \vec{b} \in Q_k$ *for any* $k \in \mathbb{N}$*:*

$$\vec{a} \succsim_{LD} \vec{b} \Leftrightarrow \vec{a} = \vec{b} \textit{ or there is } j \leq k \textit{ such that:}$$

$$\textit{for all } i < j, \left[\gamma_i(\vec{a}) = \gamma_i(\vec{b}) \textit{ and } L(\gamma_i(\vec{a})) = L(\gamma_i(\vec{b}))\right] \textit{ and}$$

$$\left[\gamma_j(\vec{a}) P \gamma_j(\vec{b}) \textit{ or } \left(\gamma_j(\vec{a}) = \gamma_j(\vec{b}) \textit{ and } L(\gamma_j(\vec{a})) < L(\gamma_j(\vec{b}))\right)\right].$$

Definition 4.3 *A binary relation* $\succsim \subseteq \bigcup_{k\in\mathbb{N}}(Q_k\times Q_k)$ *is the* leximin-desirability rule $\succsim_{ld}$ *if for all* $\vec{a},\vec{b}\in Q_k$ *for any* $k\in\mathbb{N}$*:*

$$\vec{a}\succsim_{ld}\vec{b}\Leftrightarrow \vec{a}=\vec{b} \text{ or there is } j\leq k \text{ such that:}$$

$$\text{for all } i<j, \left[\beta_i(\vec{a})=\beta_i(\vec{b}) \text{ and } L(\beta_i(\vec{a}))=L(\beta_i(\vec{b}))\right] \text{ and}$$

$$\left[\beta_j(\vec{a})P\beta_j(\vec{b}) \text{ or } \left(\beta_j(\vec{a})=\beta_j(\vec{b}) \text{ and } L(\beta_j(\vec{a}))>L(\beta_j(\vec{b}))\right)\right].$$

The three rules above are well-defined, complete and linear along Q_k for any $k\in\mathbb{N}$. Note that they are not defined for pairs of sets with different cardinality.

The leximax-likelihood rule $\succsim_{LL}$ proceeds as follows in order to compare any two alternative actions: this rule first looks at the most likely outcome in each action. If one of them is strictly better than the other, then the action with the better most likely outcome is declared strictly preferred. In the event of a tie, the criterion looks at the second most likely outcomes in $\vec{a}$ and $\vec{b}$ respectively and proceeds analogously. If ties occur successively until both sets are exhausted, they are then declared indifferent. This occurs only when two actions are identical.

The leximax-desirability rule $\succsim_{LD}$ starts by looking, respectively, at the best outcome in each action. If there is a strict preference for one of the outcomes over the other according to R, then the set that contains the former is declared strictly better. In the event of indifference between the two outcomes, the rule proceeds to look at their positions in likelihood terms and declares a strict preference in favour of the set where the best outcome is most likely. Only in the event that the respective best outcomes occupy the same position does the criterion proceed to look, respectively, at the second-best outcome within each action and proceeds similarly. Note that, again, indifference between two actions arises only when they are identical.

The leximin-desirability rule $\succsim_{ld}$ is, in a sense, dual with respect to $\succsim_{LD}$. The rule first looks, respectively, at the worst element in each set. The set where the worst element is better is declared preferred, and in the event of a tie, the rule selects the action where the worst element is less likely. If both elements occupy the same position in their sets, then the rule looks at the second-worst element in each set and proceeds lexicographically in an analogous way.

The three rules above are related to other rules of a lexicographic nature in Pattanaik and Peleg [21] within the set-based approach to the problems of choice under complete uncertainty, and Kelsey [17] within the vector-based approach to the problems of choice under partial uncertainty.

The following proposition shows that the three rules satisfy most of the proposed axioms.

Proposition 4.1 *The next statements hold:*

1. *The leximax-likelihood rule $\succsim_{LL}$ satisfies EXT, RES, REO, IND, NEU, COM, LS, WLS1 and WLS2, but neither HOS nor LOS.*

2. *The leximax-desirability rule $\succsim_{LD}$ satisfies EXT, RES, REO, IND, NEU, COM, WLS1 and HOS, but neither LOS, WLS2 nor LS.*

3. *The leximin-desirability rule $\succsim_{ld}$ satisfies EXT, RES, REO, IND, NEU, COM, WLS2 and LOS, but neither HOS, WLS1 nor LS.*

(The proof is available upon request)

The results of Proposition 4.1 can be summarized in the following table:

Apart from the fact that the rules satisfy the axioms, we also have that some combinations of such axioms are enough to characterize the rules. All the proofs are shown in the Appendix.

Theorem 4.1 *Let any reflexive binary relation $\succsim \subseteq \bigcup_{k\in\mathbb{N}}(Q_k \times Q_k)$. Then, $\succsim$ satisfies EXT, IND and LS if and only if $\succsim = \succsim_{LL}$.*

Table 1: Results of Proposition 4.1.

	EXT	RES	REO	IND	NEU	COM	LS	WLS1	WLS2	HOS	LOS
$\succsim_{LL}$	Y	Y	Y	Y	Y	Y	Y	Y	Y	N	N
$\succsim_{LD}$	Y	Y	Y	Y	Y	Y	N	Y	N	Y	N
$\succsim_{ld}$	Y	Y	Y	Y	Y	Y	N	N	Y	N	Y

Note: Y indicates that the axiom is satisfied by the rule and N indicates that it is not satisfied.

Theorem 4.2 *Let any preorder $\succsim \subseteq \bigcup_{k\in\mathbb{N}}(Q_k \times Q_k)$. Then, $\succsim$ satisfies IND, NEU, HOS and WLS1 if and only if $\succsim = \succsim_{LD}$.*

Theorem 4.3 *Let any preorder $\succsim \subseteq \bigcup_{k\in\mathbb{N}}(Q_k \times Q_k)$. Then, $\succsim$ satisfies IND, NEU, LOS and WLS2 if and only if $\succsim = \succsim_{ld}$.*

As we saw in Table 1, axioms EXT, RES, REO, IND, NEU and COM constitute the core of the conditions that are satisfied by all the proposed rules. If one wants the rule to satisfy LS also, then the only possibility is the leximax-likelihood ranking $\succsim_{LL}$ (see Theorem 4.1). Otherwise, it is possible to take weaker versions of LS (WLS1 or WLS2), and then, by imposing HOS and LOS alternatively, we obtain rules that reflect a stronger sensitivity to the desirability of the outcomes (see Theorem 4.2 and Theorem 4.3).

5 The general case

In the previous section, condition DOM, which applies to different cardinality comparisons, was dropped as a necessary condition. This opened the way to obtain some positive results that will be put to use in this section in order to explore an alternative way to overcome the initial impossibility result produced by REO, DOM and COM. We now analyze the general case without imposing any further axioms on the binary relation $\succsim$. In turn, we reintroduce DOM in the above characterizations, and a domain restriction of richness. As a consequence, we obtain some partial extensions of the above criteria that fix certain different cardinality comparisons. Another effect of the generalization is that axiom COM, which was fulfilled for all the rules in the equal cardinality case, will be satisfied only for equal-cardinality comparisons.

Definition 5.1 *A preorder* $\succsim \subseteq Q \times Q$ *belongs to the family of* extended leximax-likelihood rules, $\succsim \in \succsim_{LL}^{e}$, *if for all* $\vec{a}, \vec{b} \in Q$,

- *If* $|\vec{a}| = |\vec{b}|$, *then* $\vec{a} \succsim_{LL} \vec{b} \Rightarrow \vec{a} \succsim \vec{b}$.
- *If* $|\vec{a}| > |\vec{b}|$, *then* $\{(a_1, \ldots, a_{|\vec{b}|}) \succ_{LL} \vec{b}$ *or* $[(a_1, \ldots, a_{|\vec{b}|}) \sim_{LL} \vec{b}$ *and* $a_j P a_i$ *for all* $j > |\vec{b}|$ *and* $i \leq |\vec{b}|]\} \Rightarrow \vec{a} \succ \vec{b}$.
- *If* $|\vec{a}| < |\vec{b}|$, *then* $\{\vec{a} \succ_{LL} (b_1, \ldots, b_{|\vec{a}|})$ *or* $[\vec{a} \sim_{LL} (b_1, \ldots, b_{|\vec{a}|})$ *and* $b_i P b_j$ *for all* $i \leq |\vec{a}|$ *and* $j > |\vec{a}|]\} \Rightarrow \vec{a} \succ \vec{b}$.

The extended leximax-likelihood rules collapse with the leximax-likelihood rule when the actions to be compared have the same cardinality. Otherwise, they proceed as follows: they select the first outcomes of the set with greater cardinality such as to form a subset with the same number of outcomes as the smaller set. Then, the extended leximax-likelihood rules compare these sets by the leximax-likelihood rule. If there is a strict preference, then they replicate what the leximax-likelihood rule establishes. Otherwise, if the two selected sets are identical, they look at the remaining (less likely) outcomes of the larger set. If all of them are better than all the preceding outcomes, then the larger set is declared preferred. If they are worse, the smaller set is preferred. The remaining possible comparisons are not univocally determined, which is what distinguishes the different members of the family of extended leximax-likelihood rules.

Definition 5.2 *A preorder* $\succsim \subseteq Q \times Q$ *belongs to the family of* extended leximax-desirability rules, $\succsim \in \succsim_{LD}^{e}$, *if for all* $\vec{a}, \vec{b} \in Q$,

- *If* $|\vec{a}| = |\vec{b}|$, *then* $\vec{a} \succsim_{LD} \vec{b} \Rightarrow \vec{a} \succsim \vec{b}$.

- *If* $|\vec{a}| > |\vec{b}|$, *then* $[\max\{\vec{a}\} P \max\{\vec{b}\}] \Rightarrow \vec{a} \succ \vec{b}$.
- *If* $|\vec{a}| < |\vec{b}|$, $\vec{a} \succ \vec{b}$ *when there is no* $j \leq |\vec{a}|$ *such that:*

$$\textit{for all } i < j,\ \left[\gamma_i(\vec{b}) = \gamma_i(\vec{a}) \textit{ and } L(\gamma_i(\vec{b})) = L(\gamma_i(\vec{a}))\right] \textit{ and}$$

$$\left[\gamma_j(\vec{b}) P \gamma_j(\vec{a}) \textit{ or } \left(\gamma_j(\vec{b}) = \gamma_j(\vec{a}) \textit{ and } L(\gamma_j(\vec{b})) < L(\gamma_j(\vec{a}))\right)\right].$$

The extended leximax-desirability rules coincide with the leximax-desirability rule when the actions to be compared have the same cardinality. When the cardinalities are different, then what every rule of the family has in common is: a) that a preference for the larger set is established if its best outcome is better than the best outcome of the smaller set. In other words, the *intersection* of all the rules of the family prioritizes the larger set only if it contains the best of the outcomes among the two actions; b) that a preference for the smaller set is established by means of the following procedure: if the best element of the smaller set is better than the best element of the larger, then the smaller set is declared preferred. If they are equal, then the smaller is declared preferred if its best element is more likely. If the two are equal and both occupy the same likelihood position, the same process is applied with the respective second best elements. If this process exhausts the smaller set, then the smaller set is declared preferred.

In all other cases, comparisons are not univocally determined by all the members of the family.

As can be seen, when comparing actions with different cardinality, the extended leximax-desirability rules do not treat them symmetrically. In order to establish a preference for the smaller set, they follow a lexicographic procedure parallel to the leximax-desirability rule. However, to ensure that a strict preference for the larger set is declared by all the rules of the family, the conditions are more demanding. The intuition behind this is that, when the best outcomes of the larger and the smaller sets are indifferent and in the same likelihood position, this same likelihood position appear to have more weight when the number of outcomes is smaller.

Definition 5.3 *A preorder* $\succsim \subseteq Q \times Q$ *belongs to the family of* extended leximin-desirability rules, $\succsim \in \succsim_{ld}^{e}$, *if for all* $\vec{a}, \vec{b} \in Q$,

- *If* $|\vec{a}| = |\vec{b}|$, *then* $\vec{a} \succsim_{ld} \vec{b} \Rightarrow \vec{a} \succsim \vec{b}$.
- *If* $|\vec{a}| > |\vec{b}|$, $\vec{a} \succ \vec{b}$ *when there is no* $j \leq |\vec{b}|$ *such that:*

$$\textit{for all } i < j,\ \left[\beta_i(\vec{b}) = \beta_i(\vec{a}) \textit{ and } L(\beta_i(\vec{b})) = L(\beta_i(\vec{a}))\right] \textit{ and}$$

$$\left[\beta_j(\vec{b}) P \beta_j(\vec{a}) \textit{ or } \left(\beta_j(\vec{b}) = \beta_j(\vec{a}) \textit{ and } L(\gamma_j(\vec{b})) > L(\gamma_j(\vec{a}))\right)\right].$$

- *If* $|\vec{a}| < |\vec{b}|$, *then* $[\min\{\vec{a}\} P \min\{\vec{b}\}] \Rightarrow \vec{a} \succ \vec{b}$.

Again, the extended leximin-desirability rules coincide with the leximin-desirability rule when the actions to be compared have the same cardinality. Otherwise, they follow a comparison process that is dual to that of the extended leximax-desirability rules. In particular, a preference for the smaller set is now established unanimously only if its worst outcome is better than the worst outcome of the larger set. In turn, in order to ensure an unanimous preference for the larger set, the extended leximin-desirability rules apply the leximin-desirability procedure analogously to the way in which extended leximax-desirability rules apply the leximax-desirability procedure to establish a preference for a smaller set.

The above extensions can be identified by making use of the axiomatic battery from Section 4 and the extra assumptions of richness of the domain and DOM. We can also remove the necessary requirement of EXT in the first theorem, given that it is now implied by the remaining asssumptions.

Theorem 5.1

Let X *be rich with respect to* R. *Then, a preorder* $\succsim \subseteq Q \times Q$ *satisfies IND, LS and DOM if and only if* $\succsim \in \succsim_{LL}^{e}$.

Theorem 5.2

Let X be rich with respect to R. Then, a preorder $\succsim \subseteq Q \times Q$ satisfies IND, NEU, HOS, WLS1 and DOM if and only if $\succsim \in \succsim^{e}_{LD}$.

Theorem 5.3

Let X be rich with respect to R. Then, a preorder $\succsim \subseteq Q \times Q$ satisfies IND, NEU, LOS, WLS2 and DOM if and only if $\succsim \in \succsim^{e}_{ld}$.

One might think that the families described in Definitions 5.1, 5.2 and/or 5.3 are empty due to intransitivities and that, therefore, the corresponding characterization theorems are, in fact, impossibility theorems. However, it can be shown that there exist even linear orders within each of the families. In the Appendix, we show an example of a linear order for each of the families.

Remark 5.1 *Kannai and Peleg [16] proved the impossibility of combining certain ideas of Dominance and Independence in the set-based approach to choice under complete uncertainty (ranking sets of outcomes without any likelihood information) when $|X| \geq 6$. Also, Bossert [9] and Barberà et al. [4] proved that, when adding Neutrality to such ideas of Dominance and Independence, the impossibility stands for $|X| \geq 4$. A remarkable feature of Theorems 5.1, 5.2 and 5.3 is that, if we admit ordinal likelihood information, our proposed adaptations of the ideas of Dominance, Independence and Neutrality (axioms DOM, IND and NEU) become compatible, even in an infinite domain. One may conclude that it is the richness assumption that allows this compatibility. However, it can be shown that even without such assumption, rules can be found where the three axioms are compatible, though their description is rather tedious.*[3]

[3]Examples can be provided upon request.

6 Conclusions and further research

We have proposed a new formal framework to analyze problems of choice under uncertainty in plausible environments where the decision maker is unable to establish a complete probability distribution among the outcomes of each action, but is able to order them in terms of their likelihood. By imposing three intuitive axioms (*Reordering*, *Dominance* and *Composition*), we obtain that there is no preorder among actions that satisfies them all at once, either in the infinite case or in the finite one when $|X| \geq 3$.

With respect to the axioms, Reordering only applies to comparisons of equally-sized sets; Dominance only applies to comparisons of different-sized sets; and Composition can apply in both cases. Taking into account this and the fact that Composition might be arguable when actions of different cardinality are involved, we analyze in Section 4 the equal-cardinality case -implicitly renouncing Dominance. Then, by imposing other alternative axioms, we obtain axiomatic characterizations of different rules in this restricted domain. All these rules satisfy Reordering and Composition. They compare actions in a lexicographic way, maintaining the spirit of other lexicographic rules proposed in the related literature. Taking the results of Section 4 as a reference, we explore in Section 5 an alternative way to overcome the initial impossibility result. Now, Dominance is added as a further condition to the rules of Section 4 and, in turn, we renounce the need to fulfill Composition when comparing sets of different cardinality. The result is the characterization of three families that extend the rules presented in Section 4 to the general case.

Regarding further research, it would be of interest to investigate whether additional plausible conditions might constrain the families characterized in Section 5. Another line of research that, from our point of view, would make complete sense, would be to relax the linearity assumption about the likelihood relation among the outcomes within each action. We believe that, from a descriptive point of view, there are many situations where decision makers consider certain pairs of possible outcomes within an action to be equally likely. In our setting, this would mean that the likelihood relation among the outcomes should admit indifferences. It seems that this would affect the model in a nontrivial way, starting from the notational stage because actions could then no longer be described as ordered sets.

As a matter of fact, from a bounded rationality-like perspective, it would be reasonable to even further relax the structure of the binary likelihood relation among the outcomes within each action. A very appealing research exercise that occurs to us would be to analyze in our general framework the consequences of assuming that the likelihood relation is, for example, just an interval order, a semiorder, or a partial order.

Appendix

We show here the proofs of the theorems. We first present four lemmas that will be useful in the proofs of the results.

Lemma 6.1 *Let $\succsim$ be a binary relation on Q. If $\succsim$ satisfies EXT and IND, then it also satisfies RES.*

Proof: Let $\succsim \subseteq (Q \times Q)$ satisfying EXT and IND, and let $\vec{a}, \vec{b} \in Q_k$ for any $k \in \mathbb{N}$ such that $a_j P b_j$ and $a_i = b_i$ for all $i \neq j$. We apply IND $(k-1)$-times and we have that

$$\vec{a} \succsim \vec{b} \Leftrightarrow (a_j) \succsim (b_j)$$
$$\vec{b} \succsim \vec{a} \Leftrightarrow (b_j) \succsim (a_j)$$

By EXT, we have that $(a_j) \succ (b_j)$. Consequently, $\vec{a} \succ \vec{b}$, and $\succsim$ satisfies RES.

□

Lemma 6.2 *Let $\succsim$ be a binary relation on Q. Then, the next statements hold:*

1. *If $\succsim$ satisfies EXT, IND and WLS1, then it also satisfies REO.*

2. *If $\succsim$ satisfies EXT, IND and WLS2, then it also satisfies REO.*

Proof: We will prove both statements using the same reasoning. Let $\succsim \subseteq (Q \times Q)$ satisfying EXT, IND and WLS1, and $\vec{a} \in Q_k$ for any $k \in \mathbb{N}$ such that $a_j P a_i$. Note that $\Pi_{(i,j)}(\vec{a})$ and $\vec{a}$ have all positions equal, except i and j. Then, we apply IND $(k-2)$-times obtaining

$$\vec{a} \succsim \Pi_{(i,j)}(\vec{a}) \Leftrightarrow (a_i, a_j) \succsim (a_j, a_i)$$

$$\Pi_{(i,j)}(\vec{a}) \succsim \vec{a} \Leftrightarrow (a_j, a_i) \succsim (a_i, a_j)$$

Now, EXT ensures that $(a_j) \succ (a_i)$. The application of WLS1 or WLS2 implies that there exists $\vec{c} \in Q$ such that $(((a_j), \vec{c}), (a_i)) \succ (((a_i), \vec{c}), (a_j))$. Now, applying IND $|\vec{c}|$-times, obtaining that $(a_j, a_i) \succ (a_i, a_j)$. Consequently, $\Pi_{(i,j)}(\vec{a}) \succ \vec{a}$ and $\succsim$ satisfies REO. $\square$

Lemma 6.3 *Let $\succsim$ be a binary relation on Q. Then, the next statements hold:*

1. *If $\succsim$ satisfies HOS and NEU, then it also satisfies EXT.*
2. *If $\succsim$ satisfies LOS and NEU, then it also satisfies EXT.*

Proof: We are going to prove only the first case because the second is dual. Let $x, y \in X$ such that xPy. We have by HOS that there exists an outcome $z \in X$ such that zPy and $(z) \succ (y)$. Then, applying NEU, we conclude that $(x) \succ (y)$. Therefore, $\succsim$ satisfies also EXT. $\square$

Lemma 6.4 *Let X be rich with respect to R and let $\succsim$ be a preorder on Q. If $\succsim$ satisfies IND and DOM, then it also satisfies EXT.*

Proof: Let $x, y \in X$ such that xPy. Then, by the richness assumption, there exists $z \in X$ such that $xPzPy$. Then, by DOM, we have that $(x, z) \succ (z) \succ (y, z)$. By transitivity, $(x, z) \succ (y, z)$, and applying IND, we obtain $(x) \succ (y)$. $\square$

Proof of Theorem 3.1

Let $x, y, z \in X$ such that $xPyPz$, and consider the set $\vec{a} = (x, z, y)$. Then, if we apply the permutation $\Pi_{(2,3)}$ to $\vec{a}$, we obtain the set $\Pi_{(2,3)}(\vec{a}) = (x, y, z)$. Given that yPz, by REO we have that $(x, y, z) \succ (x, z, y)$. Furthermore, by DOM we can conclude that $(x) \succ (x, y)$ and $(z, y) \succ (z)$. Applying COM we have that $(x, z, y) \succsim (x, y, z)$, and by transitivity, $(x, y, z) \succ (x, y, z)$, contradicting reflexivity.

Proof of Theorem 4.1

The necessary part can be easily checked. To prove the sufficient part, let $\vec{a}, \vec{b} \in Q_k$. If $\vec{a} = \vec{b}$, we have by reflexivity that $\vec{a} \sim \vec{b}$. If $\vec{a} \neq \vec{b}$, we can assume by IND that $a_i \neq b_i$ for all $i \leq k$. We will proceed by induction on k. Let us start with $k = 1$. Suppose, without loss of generality, that a_1Pb_1. Then, by EXT we have that $(a_1) \succ (b_1)$ and it is proved for $k = 1$. Now, we will suppose that the statement is true for $k = t$ and we will prove the case $k = t + 1$. We have by the induction hypothesis that $(a_1, \ldots, a_t) \succ (b_1, \ldots, b_t)$ when a_1Pb_1. Then, LS says that there exists $\vec{c} \in Q$ such that $(((a_1, \ldots, a_t), \vec{c}), (a_{t+1})) \succ (((b_1, \ldots, b_t), \vec{c}), (b_{t+1}))$. Applying IND $|\vec{c}|$-times we obtain that $(a_1, \ldots, a_{t+1}) \succ (b_1, \ldots, b_{t+1})$, proving the result. Thus, $\succsim = \succsim_{LL}$.

Proof of Theorem 4.2

The necessary part is straightforward. To prove the sufficient part, let $\vec{a}, \vec{b} \in Q_k$. First, we know by Lemma 6.3 that EXT is satisfied. If $\vec{a} = \vec{b}$, we have by reflexivity that $\vec{a} \sim \vec{b}$. If $\vec{a} \neq \vec{b}$, by IND we can assume that $a_i \neq b_i$ for all $i \leq k$. If $k = 1$, we can apply EXT and we have that $a_1Pb_1 \Rightarrow \vec{a} \succ \vec{b}$. If $k > 1$, we need to prove, without loss of generality, the following two cases:

1. $\max\{\vec{a}\}P\max\{\vec{b}\}$. If $L(\gamma_1(\vec{a})) = k$, select $x \notin \vec{a}$ such that a_kPx, whose existence is guaranteed. Then, we construct the set $\vec{a}' = (a_1, \ldots, a_{k-1}, x)$. Now, we can apply HOS to sets $\vec{a}'$ and $\vec{b}$ and we have that there exists $y \notin \vec{a}'$ such that $(a_1, \ldots, a_{k-1}, y) \succ \vec{b}$. Now, if a_kPy, given the result of Lemma 6.1, we can apply RES obtaining $\vec{a} \succ (a_1, \ldots, a_{k-1}, y)$.

Transitivity concludes that $\vec{a} \succ \vec{b}$. If yPa_k, then, by NEU and transitivity, $\vec{a} \succ \vec{b}$. If, on the other hand, $L(\gamma_1(\vec{a})) < k$, we have by Lemma 6.2 that REO can be applied obtaining $\vec{a} \succ \Pi_{(i,k)}(\vec{a})$. Now, applying the previous reasoning to $\Pi_{(i,k)}(\vec{a})$ and $\vec{b}$, we have that $\Pi_{(i,k)}(\vec{a}) \succ \vec{b}$. Transitivity concludes that $\vec{a} \succ \vec{b}$.

2. $\max\{\vec{a}\} = \max\{\vec{b}\}$, with $L(\gamma_1(\vec{a})) = i < L(\gamma_1(\vec{b}))$. Consider the following sets: $(a_1, \ldots, a_i), (b_1, \ldots, b_i) \in Q_i$. Given that max is uniquely-valued, we can apply Case 1 obtaining $(a_1, \ldots, a_i) \succ (b_1, \ldots, b_i)$. We know, by WLS1, that there exists $\vec{c} \in Q$ such that $(((a_1, \ldots, a_i), \vec{c}), (a_{i+1})) \succ (((b_1, \ldots, b_i), \vec{c}), (b_{i+1}))$. Then, applying IND $|\vec{c}|$-times, we have that $(a_1, \ldots, a_i, a_{i+1}) \succ (b_1, \ldots, b_i, b_{i+1})$. Repeating this process $(k-i)$-times, we obtain $\vec{a} \succ \vec{b}$.

Therefore, $\succsim = \succsim_{LD}$.

Proof of Theorem 4.3

The necessary part is straightforward. To prove the sufficient part, let $\vec{a}, \vec{b} \in Q_k$. First, we know by Lemma 6.3 that EXT is satisfied. If $\vec{a} = \vec{b}$, we have by reflexivity that $\vec{a} \sim \vec{b}$. If $\vec{a} \neq \vec{b}$, by IND we can assume that $a_i \neq b_i$ for all $i \leq k$. If $k = 1$, we can apply EXT and we have that $a_1 P b_1 \Rightarrow \vec{a} \succ \vec{b}$. If $k > 1$, we need to prove, without loss of generality, the following cases:

1. $\min\{\vec{a}\} P \min\{\vec{b}\}$. If $L(\beta_1(\vec{b})) = k$, select $x \notin \vec{b}$ such that xPb_k, whose existence is guaranteed. Then, construct the set $\vec{b}' = (b_1, \ldots, b_{k-1}, x)$. Now, we can apply LOS to the sets $\vec{a}$ and $\vec{b}'$ and we have that there exists $y \notin \vec{b}'$ such that $\vec{a} \succ (b_1, \ldots, b_{k-1}, y)$. Now, if yPb_k, by Lemma 6.1 we can apply RES obtaining $(b_1, \ldots, b_{k-1}, y) \succ \vec{b}$. Transitivity concludes that $\vec{a} \succ \vec{b}$. If $b_k P y$, then, by NEU and transitivity, $\vec{a} \succ \vec{b}$. If, on the other hand, $L(\beta_1(\vec{b})) < k$, we have by Lemma 6.2 that we can apply REO, obtaining $\Pi_{(i,k)}(\vec{b}) \succ \vec{b}$. Now, applying the previous reasoning to $\vec{a}$ and $\Pi_{(i,k)}(\vec{b})$, we have that $\vec{a} \succ \Pi_{(i,k)}(\vec{b})$, and by transitivity $\vec{a} \succ \vec{b}$.

2. $\min\{\vec{a}\} = \min\{\vec{b}\}$, with $L(\beta_1(\vec{a})) = i > L(\beta_1(\vec{b}))$. Consider the following sets: $(a_1, \ldots, a_i), (b_1, \ldots, b_i) \in Q_i$. Given that min is uniquely-valued, we can apply Case 1 to obtain $(a_1, \ldots, a_i) \succ (b_1, \ldots, b_i)$. Now,

we know, by WLS2, that there exists $\vec{c} \in Q$ such that $(((a_1, \ldots, a_i), \vec{c}), (a_{i+1})) \succ (((b_1, \ldots, b_i), \vec{c}), (b_{i+1}))$. Then, applying IND $|\vec{c}|$-times, we have that $(a_1, \ldots, a_i, a_{i+1}) \succ (b_1, \ldots, b_i, b_{i+1})$. Repeating this process $(k-i)$-times, we obtain $\vec{a} \succ \vec{b}$.

Therefore, $\succsim = \succsim_{ld}$.

Proof of Theorem 5.1

Given Lemma 6.4, we have that EXT is satisfied. Then, we also have that EXT, IND and LS imply the desired result for all comparisons when the sets are of the same cardinality (see Theorem 4.1). For the remaining comparisons, let $\vec{a} \in Q_k$ and $\vec{b} \in Q_m$, with $k < m$. We have to prove the following cases:

1. $\vec{a} \succ_{LL} (b_1, \ldots, b_k)$. Then, by the richness of the domain, we can select $x_1, \ldots, x_{m-k} \in X$ such that $\min\{\vec{a}\} P \max\{x_1, \ldots, x_{m-k}\}$. We can apply DOM succesively in the appropriate order, obtaining $\vec{a} \succ (a_1, \ldots, a_k, x_1, \ldots, x_{m-k})$. Now, by the result of Theorem 4.1, we have that $(a_1, \ldots, a_k, x_1, \ldots, x_{m-k}) \succ \vec{b}$. Transitivity concludes that $\vec{a} \succ \vec{b}$.

2. $(b_1, \ldots, b_k) \succ_{LL} \vec{a}$. Then, by the richness assumption we can select $y_1, \ldots, y_{m-k} \in X$ such that $\min\{y_1, \ldots, y_{m-k}\} P \max\{\vec{a}\}$. Applying DOM succesively in the appropriate order, we obtain $(a_1, \ldots, a_k, y_1, \ldots, y_{m-k}) \succ \vec{a}$. Now, by the result of Theorem 4.1, we have that $\vec{b} \succ (a_1, \ldots, a_k, y_1, \ldots, y_{m-k})$. Transitivity concludes that $\vec{b} \succ \vec{a}$.

3. $(b_1, \ldots, b_k) \sim_{LL} \vec{a}$ and $b_i P \max\{\vec{a}\}$ for all $i > k$. By Theorem 4.1, we have that $(b_1, \ldots, b_k) \sim \vec{a}$. We apply DOM in the appropriate order to obtain $\vec{b} \succ (b_1, \ldots, b_k)$, and by transitivity, $\vec{b} \succ \vec{a}$.

4. $(b_1, \ldots, b_k) \sim_{LL} \vec{a}$ and $\min\{\vec{a}\} P b_i$ for all $i > k$. By Theorem 4.1, we have that $(b_1, \ldots, b_k) \sim \vec{a}$. Now, applying DOM in the appropriate order, we obtain $(b_1, \ldots, b_k) \succ \vec{b}$, and by transitivity, $\vec{a} \succ \vec{b}$.

5. It is not difficult to check that the remaining comparisons are not univocally determined by our axioms.

Therefore, $\succsim \in \succsim_{LL}^{e}$.

Proof of Theorem 5.2

We have that IND, NEU, HOS and WLS1 implies the result for the comparisons of sets with the same cardinality (see Theorem 4.2). For the remaining comparisons, let $\vec{a} \in Q_k$ and $\vec{b} \in Q_m$, with $k < m$. We have to prove the following cases:

1. $\max\{\vec{b}\} P \max\{\vec{a}\}$. Then, making use of the richness assumption, take $x_1, \dots, x_{m-k} \in X$ such that $\max\{\vec{b}\} P x_i P \max\{\vec{a}\}$ for all $i \in \{1, \dots, m-k\}$. Applying DOM succesively in the appropiate order, we obtain $(a_1, \dots, a_k, x_1, \dots, x_{m-k}) \succ \vec{a}$. Now, by the result of Theorem 4.2, we have that $\vec{b} \succ (a_1, \dots, a_k, x_1, \dots, x_{m-k})$, and by transitivity, $\vec{b} \succ \vec{a}$.

2. there is no $j \leq |\vec{a}|$ such that for all $i < j$, $\left[\gamma_i(\vec{b}) = \gamma_i(\vec{a}) \text{ and } L(\gamma_i(\vec{b})) = L(\gamma_i(\vec{a}))\right]$ and $\left[\gamma_j(\vec{b}) P \gamma_j(\vec{a}) \text{ or } \left(\gamma_j(\vec{b}) = \gamma_j(\vec{a}) \text{ and } L(\gamma_j(\vec{b})) < L(\gamma_j(\vec{a}))\right)\right]$. In this case, we divide the proof into the following two cases:

   - $\vec{a} \neq (b_1, \dots, b_k)$. Then, by the richness assumption, consider $x_1, \dots, x_{m-k} \in X$ such that $\min\{\vec{a}\} P \max\{x_1, \dots, x_{m-k}\}$. Then, applying DOM succesively in the correct order, we have that $\vec{a} \succ (a_1, \dots, a_k, x_1, \dots, x_{m-k})$. Note that $(a_1, \dots, a_k, x_1, \dots, x_{m-k})$ and $\vec{b}$ have the same cardinality and, therefore, Theorem 4.2 can be applied obtaining $(a_1, \dots, a_k, x_1, \dots, x_{m-k}) \succ \vec{b}$. Transitivity concludes that $\vec{a} \succ \vec{b}$.

   - $\vec{a} = (b_1, \dots, b_k)$. Then, consider $y_1, \dots, y_{m-k} \in X$ such that $\min\{\vec{a}\} P y_i P \max\{b_{k+1}, \dots, b_m\}$ for all $i \in \{1, \dots, m-k\}$. Then, applying DOM succesively in the appropiate order, we have that $\vec{a} \succ (a_1, \dots, a_k, y_1, \dots, y_{m-k})$. Then, by Theorem 4.2 we obtain that $(a_1, \dots, a_k, y_1, \dots, y_{m-k}) \succ \vec{b}$ and, by transitivity, $\vec{a} \succ \vec{b}$.

3. It is not difficult to check that the remaining comparisons are not univocally determined by our axioms.

Therefore, $\succsim \in \succsim_{LD}^{e}$

Proof of Theorem 5.3

We have that IND, NEU, LOS and WLS2 implies the result for the comparisons of sets with the same cardinality (see Theorem 4.3). For the remaining comparisons, let $\vec{a} \in Q_k$ and $\vec{b} \in Q_m$, with $k < m$. We have to prove the following cases:

1. $\min\{\vec{a}\} P \min\{\vec{b}\}$. Then, by the richness assumption, we can select $x_1, \ldots, x_{m-k} \in X$ such that $\min\{\vec{a}\} P x_i P \min\{\vec{b}\}$ for all $i \in \{1, \ldots, m-k\}$. By DOM, we have that $\vec{a} \succ (a_1, \ldots, a_k, x_1, \ldots, x_{m-k})$. Now, by the result of Theorem 4.3, we have that $(a_1, \ldots, a_k, x_1, \ldots, x_{m-k}) \succ \vec{b}$. Transitivity concludes that $\vec{a} \succ \vec{b}$.

2. there is no $j \leq |\vec{b}|$ such that for all $i < j$, $\left[\beta_i(\vec{b}) = \beta_i(\vec{a}) \text{ and } L(\beta_i(\vec{b})) = L(\beta_i(\vec{a}))\right]$ and $\left[\beta_j(\vec{b}) P \beta_j(\vec{a}) \text{ or } \left(\beta_j(\vec{b}) = \beta_j(\vec{a}) \text{ and } L(\gamma_j(\vec{b})) > L(\gamma_j(\vec{a}))\right)\right]$. In this case, consider the following two situations:

   - $\vec{a} \neq (b_{m-k+1}, \ldots, b_m)$. Then, by the richnesss assumption, consider $x_1, \ldots, x_{m-k} \in X$ such that $\min\{x_1, \ldots, x_{m-k}\} P \max\{\vec{a}\}$. Then, applying DOM succesively in the appropiate order, we obtain $(x_1, \ldots, x_{m-k}, a_1, \ldots, a_k) \succ \vec{a}$. Note that $(x_1, \ldots, x_{m-k}, a_1, \ldots, a_k)$ and $\vec{b}$ have the same cardinality and therefore Theorem 4.3 can be applied obtaining $\vec{b} \succ (x_1, \ldots, x_{m-k}, a_1, \ldots, a_k)$. Transitivity concludes that $\vec{b} \succ \vec{a}$.
   - $\vec{a} = (b_{m-k+1}, \ldots, b_m)$. Then, by the richnesss assumption, consider $y_1, \ldots, y_{m-k} \in X$ such that $\min\{b_1, \ldots, b_{m-k}\} P y_i P \max\{\vec{a}\}$ for all $i \in \{1, \ldots, m-k\}$. Succesive applications of DOM in the appropiate order implies that $(y_1, \ldots, y_{m-k}, a_1, \ldots, a_k) \succ \vec{a}$. Theorem 4.3 implies that $\vec{b} \succ (y_1, \ldots, y_{m-k}, a_1, \ldots, a_k)$. Transitivity concludes that $\vec{b} \succ \vec{a}$.

3. It is not difficult to check that the remaining comparisons are not univocally determined by our axioms.

Therefore, $\succsim \in \succsim_{ld}^{e}$

Example of linear orders in the extended families

We provide three linear orders that belong to each of the extended families ($\succsim^e_{LL}$, $\succsim^e_{LD}$ and $\succsim^e_{ld}$, respectively).

- $\succsim^e_{LL}$: Consider $\succsim_1 \in \succsim^e_{LL}$, which compares any two actions $\vec{a}, \vec{b} \in Q$ as follows:

  - If $|\vec{a}| = |\vec{b}|$, then $\vec{a} \succsim_{LL} \vec{b} \Rightarrow \vec{a} \succsim_1 \vec{b}$.
  - If $|\vec{a}| > |\vec{b}|$, then $\vec{a} \succ_1 \vec{b} \Leftrightarrow \{(a_1, \ldots, a_{|\vec{b}|}) \succ_{LL} \vec{b}$ or $[(a_1, \ldots, a_{|\vec{b}|}) \sim_{LL} \vec{b}$ and $a_{|\vec{b}|+1} P \max\{\vec{b}\}]\}$
    and
    $\vec{b} \succ_1 \vec{a} \Leftrightarrow \{\vec{b} \succ_{LL} (a_1, \ldots, a_{|\vec{b}|})$ or $[\vec{b} \sim_{LL} (a_1, \ldots, a_{|\vec{b}|})$ and $\max\{\vec{b}\}$ $P a_{|\vec{b}|+1}]\}$.

- $\succsim^e_{LD}$: Consider $\succsim_2 \in \succsim^e_{LD}$, which compares any two actions $\vec{a}, \vec{b} \in Q$ as follows:

  - If $|\vec{a}| = |\vec{b}|$, then $\vec{a} \succsim_{LD} \vec{b} \Rightarrow \vec{a} \succsim_2 \vec{b}$.
  - If $|\vec{a}| > |\vec{b}|$, then $\vec{a} \succ_2 \vec{b} \Leftrightarrow$ there is $j \leq |\vec{b}|$ such that:

    $$\text{for all } i < j, \left[\gamma_i(\vec{a}) = \gamma_i(\vec{b}) \text{ and } L(\gamma_i(\vec{a})) = L(\gamma_i(\vec{b}))\right] \text{ and}$$

    $$\left[\gamma_j(\vec{a}) P \gamma_j(\vec{b}) \text{ or } \left(\gamma_j(\vec{a}) = \gamma_j(\vec{b}) \text{ and } L(\gamma_j(\vec{a})) < L(\gamma_j(\vec{b}))\right)\right]$$

    and
    $\vec{b} \succ_2 \vec{a} \Leftrightarrow$ there is no $j \leq |\vec{b}|$ such that:

    $$\text{for all } i < j, \left[\gamma_i(\vec{a}) = \gamma_i(\vec{b}) \text{ and } L(\gamma_i(\vec{a})) = L(\gamma_i(\vec{b}))\right] \text{ and}$$

    $$\left[\gamma_j(\vec{a}) P \gamma_j(\vec{b}) \text{ or } \left(\gamma_j(\vec{a}) = \gamma_j(\vec{b}) \text{ and } L(\gamma_j(\vec{a})) < L(\gamma_j(\vec{b}))\right)\right].$$

- $\succsim^e_{ld}$: Consider $\succsim_3 \in \succsim^e_{ld}$, which compares any two actions $\vec{a}, \vec{b} \in Q$ as follows:

  - If $|\vec{a}| = |\vec{b}|$, then $\vec{a} \succsim_{ld} \vec{b} \Rightarrow \vec{a} \succsim_3 \vec{b}$.

– If $|\vec{a}| > |\vec{b}|$, then $\vec{a} \succ_3 \vec{b} \Leftrightarrow$ there is no $j \leq |\vec{b}|$ such that:

$$\text{for all } i < j,\ \big[\beta_i(\vec{b}) = \beta_i(\vec{a}) \text{ and } L(\beta_i(\vec{b})) = L(\beta_i(\vec{a}))\big] \text{ and}$$

$$\big[\beta_j(\vec{b}) P \beta_j(\vec{a}) \text{ or } \big(\beta_j(\vec{b}) = \beta_j(\vec{a}) \text{ and } L(\beta_j(\vec{b})) > L(\beta_j(\vec{a}))\big)\big].$$

and

$\vec{b} \succ_3 \vec{a} \Leftrightarrow$ there is $j \leq |\vec{b}|$ such that:

$$\text{for all } i < j,\ \big[\beta_i(\vec{b}) = \beta_i(\vec{a}) \text{ and } L(\beta_i(\vec{b})) = L(\beta_i(\vec{a}))\big] \text{ and}$$

$$\big[\beta_j(\vec{b}) P \beta_j(\vec{a}) \text{ or } \big(\beta_j(\vec{b}) = \beta_j(\vec{a}) \text{ and } L(\beta_j(\vec{b})) > L(\beta_j(\vec{a}))\big)\big].$$

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