On a ‘philosophical’ question about Banach envelopes

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Abstract We show how to construct **non-locally** convex quasi-Banach spaces X whose dual separates the points of a dense subspace of X but does not separate the points of X . Our examples connect with a question raised by Pietsch [*About the Banach envelope of $l_{1,\infty}$* , Rev. Mat. Complut. 22 (2009), 209–226] and shed light into the unexplored class of quasi-Banach spaces with nontrivial dual which do not have sufficiently many functionals to separate the points of the space.

Keywords quasi-Banach space · Banach envelope · extension of operators · point separation property

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1 Introduction and background

The *Banach envelope* of a quasi-Banach space X over the real or complex field $\mathbb{F}$, consists of a Banach space $\widehat{X}$ together with a norm-one linear map $J_X: X \rightarrow \widehat{X}$ satisfying the following property: for every Banach space Y and every continuous linear map $T: X \rightarrow Y$ there is a unique continuous linear map $\widehat{T}: \widehat{X} \rightarrow Y$ such that $\widehat{T} \circ J_X = T$, and the “extension” $\widehat{T}$ has a norm bounded by that of T . Pictorially we have

$$\begin{array}{ccc} & \widehat{X} & \\ J_X \uparrow & \searrow \widehat{T} & \\ X & \xrightarrow{T} & Y \end{array}$$

In particular, X and $\widehat{X}$ have the same dual space. We will refer to J_X as the *envelope map* of X .

Since the Banach envelope of X is defined by means of a universal property, it is unique in the sense that if a Banach space Z and a bounded linear map $J: X \rightarrow Z$ satisfy the property, then there is a linear isometry $S: \widehat{X} \rightarrow Z$ with $S \circ J_X = J$. In general, if a Banach space Z and a bounded linear map $J: X \rightarrow Z$ are such that $\widehat{J}: \widehat{X} \rightarrow Z$ is a λ -isomorphism we say that Z is an *λ -isomorphic representation* of the Banach envelope of X via J . When the constant $\lambda \geq 1$ is irrelevant, we simply drop it. A 1-isomorphic representation will be called *isometric representation*.

A construction of the Banach envelope of a quasi-Banach space X goes as follows. Let $\text{co}(B_X)$ be the convex hull of the closed unit ball of X , and denote by $\|\cdot\|_e$ the Minkowski functional of $\text{co}(B_X)$, i.e.,

$$\|x\|_e = \inf \{t > 0: x/t \in \text{co}(B_X)\}, \quad x \in X. \quad (1)$$

Equation (1) yields a semi-norm on X . Moreover, it is easy to prove that, if $\|\cdot\|$ denotes the quasi-norm on X , then $\|\cdot\|_e$ is also given by the formula

$$\|x\|_e = \inf \left\{ \sum_{i=1}^n \|x_i\|: \sum_{i=1}^n x_i = x, x_i \in X, n \in \mathbb{N} \right\}, \quad x \in X,$$

which makes $\|\cdot\|_e$ the largest seminorm $\|\cdot\|$ on X for which $\|x\| \leq \|x\|_e$ for all $x \in X$. Passing to the quotient space X/N , with

$$N = \{x \in X: \|x\|_e = 0\}, \quad (2)$$

we obtain a normed linear space whose completion, $\widehat{X}$, is the required Banach envelope with associated envelope map $J_X := I \circ Q$, where $Q: X \rightarrow X/N$ is the canonical quotient map and $I: X/N \rightarrow \widehat{X}$ is the inclusion map. Note that,

by construction, $\text{Ker}(J_X) = N$ and $J_X(X)$ is dense in $\widehat{X}$. Note also that by the Hahn-Banach theorem,

$$\begin{aligned} N &= (X^*)^\perp := \{x \in X : x^*(x) = 0 \text{ for all } x^* \in X^*\} \\ &= \{x \in X : T(x) = 0 \text{ for all Banach spaces } Z \text{ and all } T \in \mathcal{L}(X, Z)\}, \end{aligned}$$

where $\mathcal{L}(X, Z)$ denotes the quasi-Banach space of all bounded linear operators from X to Z . This identity yields that the existence of nonzero elements in N can be equivalently re-stated in terms of the injectivity of the map J_X as follows.

Lemma 1 *Given a quasi-Banach space X and a subspace X_0 of X , the following are equivalent.*

- (i) *The semi-norm $\|\cdot\|_e$ given by (1) is a norm on X_0 .*
- (ii) *The envelope map $J_X : X \rightarrow \widehat{X}$ restricted to X_0 is one-to-one.*
- (iii) *There is a Banach space Z and a bounded linear operator from X into Z whose restriction to X_0 is one-to-one.*
- (iv) *$X_0 \cap (X^*)^\perp = \{0\}$, i.e., the dual space X^* separates the points of X_0 .*

In this paper we will refer to X as having the *point-separation property* if $(X^*)^\perp = \{0\}$, i.e., the dual space X^* separates the points of X ([27]). Applying Lemma 1 with $X = X_0$ we obtain the following.

Lemma 2 *Let X be a quasi-Banach space. The following are equivalent.*

- (i) *The envelope map J_X is one-to-one.*
- (ii) *There is a one-to-one bounded linear map from X into a Banach space.*
- (iii) *X has the point-separation property.*

Let us also bring up the connection between the existence of finite-dimensional uncomplemented subspaces and the existence of nonzero vectors in the space N defined as in (2).

Lemma 3 *Let X be a quasi-Banach space. We have*

$$X \setminus (X^*)^\perp = \{x \in X : \mathbb{F}x \text{ is complemented in } X\}.$$

Another way of introducing the Banach envelope of a quasi-Banach space X is by using the bidual. In this case the envelope map is the (not necessarily one-to-one) bidual map $h_X : X \rightarrow X^{**}$, defined as usual by $h_X(x)(f) = f(x)$, and $\widehat{X}$ is the closure of $h_X(X)$ in X^{**} . The more constructive approach we sketched above versus this functional approach has the advantage of inspiring useful generalizations of the notion of Banach envelope. For instance, we could define the p -Banach envelope of X for $0 < p < 1$ similarly to the case $p = 1$ (see, e.g., [4] for a precise definition, and [1, 2] for applications of p -envelopes).

The systematic study of quasi-Banach spaces dates back to the mid-20th century. Among the contributions to the theory from those early years we highlight the work of Hyers [20, 21], Day [14], Aoki [5], Burgin [8], Rolewicz [38], and Peck [33, 34]. The concept of Banach envelope was introduced in 1974

by Peetre [35, p. 125], and independently rediscovered by Shapiro in 1977 [40, p. 116]. Since then it has been considered by many authors in different spaces and settings. The Banach envelope of the spaces ℓ_p for $p < 1$ is (represented by) ℓ_1 via the inclusion map (see [27, p. 28]), whereas the Banach envelope of L_p for $p < 1$ is zero [14]. More generally, Lorentz spaces $L_{p,q}$ over $[0, 1]$ have trivial dual for $0 < p \leq 1$ and $0 < q \leq \infty$ with $(p, q) \neq (1, \infty)$ [19, 10]. As for Lorentz sequence spaces, a similar argument to the one used in the case of ℓ_p , yields that for $0 < p < 1$, $0 < q < \infty$, and $0 < r \leq 1$, the Banach envelopes of the spaces $\ell_{p,q}$, of the separable part of $\ell_{p,\infty}$ (which we denote $\ell_{p,\infty}^0$) and of $\ell_{1,r}$ is ℓ_1 via the inclusion map (see Theorem 1 below). Identifying the Banach envelope of the non-separable space $\ell_{p,\infty}$ is more challenging. Cwikel [10] admirably solved the problem by proving that the Banach envelope of $\ell_{p,\infty}$ for $0 < p < 1$ is ℓ_1 via the inclusion map. Note that, by Theorem 1, this result yields that the dual of $\ell_{p,\infty}/\ell_{p,\infty}^0$ is trivial. Combining [6, Theorem 3] with [9, Theorem 2.4.14] yields that for $1 < r < \infty$ the Banach envelope of $\ell_{1,r}$ is, via the inclusion map, the Banach space consisting of all sequences $f = (a_n)_{n=1}^\infty \in c_0$ such that

$$\sum_{n=1}^{\infty} \frac{n^{r-1}}{\log^r(1+n)} (f^{**}(n))^r < \infty,$$

where $f^{**}(n) = \max_{|A|=n} n^{-1} \sum_{k \in A} |a_k|$. Similarly, the Banach envelope of $\ell_{1,\infty}^0$ is the space known as the *little Marcinkiewicz space*, consisting of all sequences $f \in c_0$ such that

$$\lim_{n \rightarrow \infty} \frac{n}{\log(n+1)} f^{**}(n) = 0$$

(see [31, pp. 95–108], [32, p. 823] and [7]; see also [3, Example 10.25]). In contrast, no ‘tangible’ representation of the Banach envelope of $\ell_{1,\infty}$ is known (see [36]).

In [17], Duren, Romberg, and Shields identified the dual space of the Hardy space $H_p(\mathbb{D})$ for $0 < p < 1$ and thus they obtained (a representation of) its Banach envelope (cf. [39]). The Banach envelope of $H_p(\mathbb{D})$ is, via its inclusion into the space of analytic functions, a Bergman space which turns out to be isomorphic again to ℓ_1 (see [42, Theorem 8]). Hence, $\widehat{H_p(\mathbb{D})}$ can also be represented by the space ℓ_1 via a suitable mapping, namely, the coefficient transform with respect to the (unique, see [43]) unconditional basis of the space. In [22], Kalton gave an isomorphic representation of the Banach envelope of the separable part of **non-locally** convex Orlicz spaces over $[0, 1]$. Given a function $\alpha: I \rightarrow (0, \infty)$ on an interval I , we define $\hat{\alpha}: I \rightarrow [0, \infty)$ by

$$\hat{\alpha}(t) = \inf \left\{ \sum_{j=1}^m \lambda_j \alpha(t_j) : t_j \in I, \lambda_j \geq 0, \sum_{j=1}^m \lambda_j = 1, t = \sum_{j=1}^m \lambda_j t_j \right\}.$$

Let φ be an Orlicz function and put M_φ to denote the separable part of the **Orlicz** space $L_\varphi([0, 1])$. If $\liminf_{x \rightarrow \infty} \varphi(x)/x = 0$, then the Banach envelope of

M_φ is zero, while if $\liminf_{x \rightarrow \infty} \varphi(x)/x > 0$ then the Banach envelope of M_φ is M_ψ via the inclusion map, where the convex Orlicz function ψ is an extension of $\widehat{\varphi|_{[c, \infty)}}$, and $c > 0$ is such that $\inf_{x \geq c} \varphi(x)/x > 0$. Subsequently, Kalton obtained the analogous result for Orlicz sequence spaces [23]. To be more specific, Kalton proved that if ℓ_φ is separable, i.e., ℓ_φ and its separable part m_φ coincide, then its Banach envelope is ℓ_ψ under the inclusion map, where the convex Orlicz function ψ is obtained by extension of $\widehat{\varphi|_{[0, 1]}}$. This also inspired further research in [16], where a representation of the Banach envelope of non-locally convex Orlicz sequence spaces was obtained without the assumption of separability. In this article, Drewnowski and Nawrocki proved that if ψ is as above, then ℓ_ψ is the Banach envelope of ℓ_φ if and only if

(♣) $\varphi(2t) \leq c \max\{\varphi(t), \psi(bt)\}$ for small t , and some constants $b, c > 0$.

In other words the canonical extension

$$\widehat{j}: \widehat{\ell_\varphi} \rightarrow \ell_\psi \quad (3)$$

of the inclusion map $j: \ell_\varphi \rightarrow \ell_\psi$ is an isomorphism if and only if the Orlicz function φ satisfies (♣). They also proved that, in general, the Banach envelope of ℓ_φ is represented by the closure of $J(\ell_\varphi)$ in $\ell_\psi \oplus \ell_\varphi/m_\varphi$, where the envelope map J is given by $J(f) = (f, f + m_\varphi)$ for all $f \in \ell_\varphi$. Other more recent articles working specifically on Banach envelopes include, e.g., [3, 13, 15, 29, 30, 28].

Although Banach envelopes have played a key role in the understanding of the geometry of **non-locally** convex quasi-Banach spaces, in our opinion a systematic treatment of the concept is missing. The lack of a reliable reference makes the authors impose different assumptions in order to feel at ease working with them. For instance, the Banach envelope of a quasi-Banach space is often times thought of as the “smallest” Banach space containing X . In doing so we are taking for granted that the envelope map $J_X: X \rightarrow \widehat{X}$ is one-to-one (see, e.g., [39]), but as the examples above show, this is not always the case. Think of the space L_p for $p < 1$. One could argue that this is a pathological case, and that what happens is simply that L_p has no Banach envelope. In fact, in all the examples cited above, the envelope map happens to be either zero or one-to-one. In spite of that, we encounter intermediate situations like the spaces $\ell_p \oplus L_p$ ($0 < p < 1$) whose dual is neither trivial nor separating, i.e., whose envelope map is neither zero nor one-to-one. Of course, one could keep on considering this as a pathological example, or perhaps too simple to start paying attention to quasi-Banach spaces whose envelope map is not one-to-one.

However, there are quasi-Banach spaces whose Banach envelope provides valuable information about the linear structure of the space despite the fact that the envelope map is not one-to-one (or it is unclear whether it is one-to-one or not). A classical example is the Lorentz space $L_{1, \infty}$ over $[0, 1]$. It is known that even though every linear functional on $L_{1, \infty}$ vanishes on simple functions, the dual of $L_{1, \infty}$ is nontrivial (see [12]). The Banach envelope of $L_{1, \infty}$ was described in [11]. More recently, the authors of [4] used the Banach

envelopes of the Lipschitz free p -spaces $\mathcal{F}_p(\mathbb{R}^2)$ and $\mathcal{F}_p(\mathbb{R})$ for $0 < p < 1$ to show that they are not linearly isomorphic. And the same to tell apart $\mathcal{F}_p(\mathbb{R})$ from ℓ_p for $0 < p < 1$. And all that without knowing whether the envelope map of $\mathcal{F}_p(\mathbb{R}^d)$ is one-to-one or not!

Of course, assuming the point-separation property is essential in some situations. Suppose you want to determine whether a given Banach space Z can be the Banach envelope of some quasi-Banach space X . Since the Banach envelope of $Z \oplus L_p$ for $0 < p < 1$ is Z via the canonical projection, the problem is nonsensical (or trivial) unless we impose that X is non-locally convex and has the point-separation property. The answer to this question resides with the finite-dimensional structure of the space Z . Kalton proved in [25] that if X is a non-locally convex quasi-Banach space with the point-separation property then its Banach envelope has no type p for any $p > 1$. It follows that the spaces ℓ_p and L_p for $1 < p < \infty$, or the Orlicz spaces ℓ_φ and L_φ with the lower Orlicz-Matuszewska index $\alpha_\varphi > 1$, cannot be Banach envelopes of X .

Summing up, we must pay attention to the point-separation property of quasi-Banach spaces. Indeed, finding sufficient conditions that guarantee that the envelope map is one-to-one has both theoretical and practical interest. To substantiate our claim, let us explain a situation that occurs in applications. Some quasi-Banach spaces X (such as Lipschitz free p -spaces for $0 < p < 1$) are naturally constructed from a non-complete quasi-normed space X_0 that is dense in X . This definition leads to a precise description on how X_0 embeds into $\widehat{X}$ via J_X , but not on how X does! Thus, if we want to know that the character of the members of X_0 inside $\widehat{X}$ is preserved in passing to its completion, it is important to know whether we can transfer properties from $J_X|_{X_0}$ to J_X . In particular, the following question arises:

Question 1 Let X be a quasi-Banach space and let X_0 be a dense subspace of X . Assume that X^* separates the points of X_0 . Does it follow that X^* separates the points of X ? Equivalently, assuming that $J_X|_{X_0}: X_0 \rightarrow \widehat{X}$ is one-to-one, is $J_X: X \rightarrow \widehat{X}$ one-to-one?

The authors were glad to learn about other colleagues concern with this kind of problem. For instance, in his study of the Banach envelope of $\ell_{1,\infty}$ ([36]), Pietsch raised **and solved** the following question, which he elevated to the category of ‘philosophical’:

Question 2 ([36, p. 214]) Let X_0 and Z be normed spaces. Suppose that Z is complete and that $T: X_0 \rightarrow Z$ is a one-to-one linear operator. Is the extension of T to the completion of X_0 one-to-one?

We point out that the theory of Banach envelopes provided an earlier (non-separable!) negative answer to Question 2. Indeed, Drewnowski and Nawrocki showed in [16] that if an Orlicz function φ does not satisfy ($\clubsuit$), then the map $\widehat{j}$ defined as in (3) is not one-to-one. In spite of those examples, the answer to this question is positive in some important situations. A famous example also brought out by Pietsch is the canonical embedding, say T , of the ‘algebraic’ tensor product $X^* \otimes X$ into $\mathcal{L}(X, X)$, where X is a Banach space.

Grothendieck [18, §I, p. 165] proved that the extension of T to the projective tensor product $X^* \hat{\otimes} X$ is one-to-one if and only if X has the approximation property (AP for short). Hence, if X has the AP, we can regard the vectors in $X^* \hat{\otimes} X$ as operators.

Question 2 makes perfect sense for quasi-normed spaces. In [41] we find an example of this (extended) question that lies at the core of the theory of **non-locally** convex quasi-Banach spaces. Given a measure space (Ω, μ) and a quasi-Banach space X , let $L_1(\mu, X)$ be the quasi-Banach space of all strongly measurable functions $f: \Omega \rightarrow X$ such that $\int_{\Omega} \|f\| d\mu < \infty$. Vogt [41] endowed the tensor product $L_1(\mu) \otimes X$ with a suitable quasi-norm so that both natural maps $T: L_1(\mu) \otimes X \rightarrow L_1(\mu, X)$ and $\mathcal{I}: L_1(\mu) \otimes X \rightarrow X$ are bounded. Then he proved that the extension of T to the completion, $L_{1,V}(\mu, X)$, of $(L_1(\mu) \otimes X, \|\cdot\|)$ is one-to-one. This way the elements of $L_{1,V}(\mu, X)$ ‘are functions’ and, so, the extension of $\mathcal{I}$ to $L_{1,V}(\mu, X)$ becomes an integral for such functions.

Question 1 might in turn be thought of as a particular case of Question 2 **when Question 2 is stated replacing normed for quasi-normed spaces**. Since those quasi-Banach spaces whose envelope map is not one-to-one can be regarded as ‘pathological’, one could be inclined to believe that Question 1 should have a positive answer. To our surprise, the experts on the subject were unable to either write down a proof or show us an example that disproved this conjecture. Our aim in this note is to fill this gap in the literature about Banach envelopes by solving in the negative Question 1. **Like Pietsch’s answer to Question 2, our answer to Question 1 lies within the setting of separable spaces.**

Background results

We conclude the introduction by making a few remarks which we hope will assist the non-specialist reader.

The orthogonal sets associated with a quasi-Banach space X are defined as follows. Given a subset B of X^* we put

$$B^{\perp} = \{x \in X: x^*(x) = 0 \quad \forall x^* \in B\}.$$

Similarly, given a subset A of X , we put

$$A^{\perp} = \{x^* \in X^*: x^*(x) = 0 \quad \forall x \in A\}.$$

We can naturally identify $A^{\perp}$ with the dual space of the quotient space X/V , where V is the closed subspace of X spanned by A . To be precise, the dual map $Q^*: (X/V)^* \rightarrow X^*$ of the canonical map $Q: X \rightarrow X/V$ is an isometric embedding whose range is $V^{\perp} = A^{\perp}$. As a consequence we obtain

$$((X/V)^*)^{\perp} = V^{\perp\perp}/V. \quad (4)$$

Suppose that Z is another quasi-Banach space. A bounded linear map $Q: X \rightarrow Z$ is said to be a *quotient map* if $Q(B_X)$ contains a neighbourhood of zero. Note that Q is a quotient map if and only if the canonical map

$\tilde{Q}: X/\text{Ker}(Q) \rightarrow Z$ is an isomorphism. In case that $\tilde{Q}$ is a λ -isomorphism, we say that Q is a λ -quotient map. By the Open Mapping theorem, the continuous linear map Q is a quotient map if and only if it is onto.

Proposition 1 *Suppose X and Z are quasi-Banach spaces. Let $Q: X \rightarrow Z$ be a λ -quotient map for some $\lambda \geq 1$. Then $\hat{Q}: \hat{X} \rightarrow \hat{Z}$ is a λ -quotient map. Moreover,*

$$\text{Ker}(\hat{Q}) = \overline{J_X(\text{Ker}(Q))}.$$

Proof Set $V = \overline{J_X(\text{Ker}(Q))}$. Since $\hat{Q} \circ J_X = J_Z \circ Q$, we have $\hat{Q}(x) = 0$ for all $x \in V = \overline{J_X(\text{Ker}(Q))}$. Hence, if

$$P: \hat{X} \rightarrow U := \hat{X}/V$$

is the canonical map, there is $T: U \rightarrow \hat{Z}$ such that $T \circ P = \hat{Q}$. By the Hahn-Banach theorem, it suffices to prove that $T^*: (\hat{Z})^* \rightarrow U^*$ is a λ -isomorphism.

The space U^* is isometrically isomorphic to $V^\perp$. The same can be said about the spaces $(\hat{Z})^*$ and Z^* , and about $(\hat{X})^*$ and X^* . Besides, the latter isometry restricts to a natural identification between $W^\perp$ and $V^\perp$. Thus, we need only prove that the map $T': Z^* \rightarrow V^\perp$ given by

$$z^* \circ J_Z \mapsto z^* \circ T \circ P \circ J_X, \quad z^* \in (\hat{Z})^*,$$

is a λ -isomorphism. Since $T \circ P \circ J_X = J_Z \circ Q$, the map T' is given by $T'(z^*) = z^* \circ Q$ for all $z^* \in Z^*$. Since Q is a λ -quotient map with null space V , we are done. $\square$

Corollary 1 *Let X and Z be quasi-Banach spaces, and suppose that $Q: X \rightarrow Z$ is a quotient map. Then, $\hat{Q}: \hat{X} \rightarrow \hat{Z}$ is an isomorphism if and only if $\text{Ker}(Q) \subseteq (X^*)^\perp$.*

Proof By Proposition 1, $\hat{Q}$ is an isomorphism if and only if the closure of $W := J_X(\text{Ker}(Q))$ is null. Clearly, the closure of W is null if and only if W is null. In turn, $W = \{0\}$ if and only if $\text{Ker}(Q) \subseteq \text{Ker}(J_X)$. Since $\text{Ker}(J_X) = (X^*)^\perp$, we are done. $\square$

Our next result is just a re-formulation of the “if” part of Corollary 1.

Corollary 2 *Let X and Z be quasi-Banach spaces and suppose that $Q: X \rightarrow Z$ is a quotient map with $\text{Ker}(Q) \subseteq N := (X^*)^\perp$. Then, if $\lambda \geq 1$.*

- (i) $\hat{Z}$ is a λ -isomorphic representation of the Banach envelope of X via Q , and
- (ii) $\hat{X}$ is a λ -isomorphic representation of the Banach envelope of Z , via the unique linear bounded map $J: Z \rightarrow \hat{X}$ with $J \circ Q = J_X$.

In particular, $\hat{X}$ is an isometric representation of the Banach envelope of X/N via the inclusion map.

The following proposition is an immediate consequence of Corollary 2, and the subsequent lemma a partial converse of it.

Proposition 2 *Suppose X is a quasi-Banach space, and let $N = (X^*)^\perp$. Then X/N has the point-separation property.*

Lemma 4 *Let V be a closed subspace of a quasi-Banach space X . Suppose that X/V has the point-separation property. Then $(X^*)^\perp \subseteq V$.*

Proof We have $\{0\} = ((X/V)^*)^\perp$. Then, by (4), $V^{\perp\perp} = V$. Finally, the inclusion $(X^*)^\perp \subseteq V^{\perp\perp}$ follows from the mere definition of orthogonal set. $\square$

Let $0 < p \leq \infty$. A biorthogonal system $(\mathbf{x}_n, \mathbf{x}_n^*)_{n=1}^\infty$ is said to be *p-Besselian* if the coefficient transform $\mathcal{F}: X \rightarrow \mathbb{F}^\mathbb{N}$ given by $\mathcal{F}(f) = (\mathbf{x}_n^*(f))_{n=1}^\infty$ is a bounded operator from X into ℓ_p . The biorthogonal system $(\mathbf{x}_n, \mathbf{x}_n^*)_{n=1}^\infty$ is said to be semi-normalized if $0 < \inf_n \|\mathbf{x}_n\|$ and $\sup_n \|\mathbf{x}_n\| < \infty$.

$(\mathbf{e}_n)_{n=1}^\infty$ denotes the unit vector system of $\mathbb{F}^\mathbb{N}$, i.e., $\mathbf{e}_n = (\delta_{k,n})_{k=1}^\infty$, where $\delta_{k,n} = 1$ if $k = n$ and $\delta_{k,n} = 0$ otherwise.

Theorem 1 *Let X be a quasi-Banach space endowed with a semi-normalized 1-Besselian biorthogonal system $(\mathbf{x}_n, \mathbf{x}_n^*)_{n=1}^\infty$. Put*

$$Y = \overline{\text{span}(\mathbf{x}_n : n \in \mathbb{N})}.$$

Then the Banach envelope of X can be represented by $\ell_1 \oplus \widehat{X/Y}$ via the map

$$x \mapsto (\mathcal{F}(x), J_{X/Y}(x + Y)).$$

Proof Since $\sup_n \|J_X(\mathbf{x}_n)\| \leq \sup_n \|\mathbf{x}_n\| < \infty$, there is a bounded linear operator $S: \ell_1 \rightarrow \widehat{X}$ such that $S(\mathbf{e}_n) = J_X(\mathbf{x}_n)$ for all $n \in \mathbb{N}$. We also have a bounded linear operator $\widehat{\mathcal{F}}: \widehat{X} \rightarrow \ell_1$ such that $\widehat{\mathcal{F}} \circ J_X = \mathcal{F}$. We infer that $\widehat{\mathcal{F}} \circ S = \text{Id}_{\ell_1}$. Consequently, there is an isomorphism P from $\widehat{X}$ onto $\ell_1 \oplus \widehat{X}/S(\ell_1)$ such that

$$P(x) = (\widehat{\mathcal{F}}(x), x + S(\ell_1)), \quad x \in \widehat{X}.$$

Since $S(\ell_1) = \overline{J_X(Y)}$, applying Proposition 1 we obtain an isomorphism $T: \widehat{X}/S(\ell_1) \rightarrow \widehat{X/Y}$ such that

$$T(J_X(x) + S(\ell_1)) = J_{X/Y}(x + Y), \quad x \in X.$$

Therefore, the Banach envelope of X is $\ell_1 \oplus \widehat{X/Y}$ via the map $J = (\text{Id}_{\ell_1}, T) \circ P \circ J_X$. Finally, for $x \in X$ we have

$$J(x) = (\widehat{\mathcal{F}}(J_X(x)), T(J_X(x) + S(\ell_1))) = (\mathcal{F}(x), J_{X/Y}(x + Y)). \quad \square$$

2 Point separation property and density

Our proof in Theorem 2 below of the existence of quasi-Banach spaces whose envelope map is one-to-one on a dense subspace and yet do not have the point-separation property relies on the following proposition.

Proposition 3 *Let X and Z be quasi-Banach spaces, and let $Q: X \rightarrow Z$ be a quotient map. Suppose that X is separable and that Z is infinite-dimensional. Then there exists a dense subspace $X_0 \subseteq X$ such that Q is one-to-one on X_0 .*

Proof Without loss of generality we may assume that Q is a 1-quotient map, i.e., $\|z\| = \inf\{\|x\|: Q(x) = z\}$ for all $z \in Z$. Let $\kappa \geq 1$ be such that $\|z + z' + z''\| \leq \kappa(\|z\| + \|z'\| + \|z''\|)$ for all $z, z', z'' \in Z$.

Let $(z_n)_{n=1}^\infty$ be a linearly independent sequence in Z such that $Z_0 := \text{span}(z_n: n \in \mathbb{N})$ is dense in Z . Pick an algebraic complement Z_1 of Z_0 and let $(z'_n)_{n=1}^\infty$ be a linearly independent sequence in Z_1 such that $\lim_n z'_n = 0$. For each $n \in \mathbb{N}$, let $x_n \in Q^{-1}(z_n)$ and $x'_n \in Q^{-1}(z'_n)$ be such that $\lim_n x'_n = 0$.

Let $D \subseteq \text{Ker}(Q)$ be countable and dense. Choose a sequence $y = (y_n)_{n=1}^\infty$ in $\text{Ker}(Q)$ such that for every $k \in \mathbb{N}$ and every $d \in D$ there is $n \geq k$ such that $y_n = d$. For instance, if $D = \{d_n: n \in \mathbb{N}\}$, we could pick

$$y = (d_1, d_1, d_2, d_1, d_2, d_3, \dots, d_1, \dots, d_j, \dots, d_n, d_1, \dots, d_j, \dots, d_n, d_{n+1}, \dots).$$

Set

$$X_0 = \text{span} \left(\bigcup_{n=1}^\infty \{x_n, y_n + x'_n\} \right).$$

Let us see that $Q|_{X_0}$ is one-to-one. Suppose that $Q(x) = 0$ for some $x \in X_0$. Then, if $(a_n)_{n \in A}$ and $(b_n)_{n \in B}$ are finite families of scalars such that

$$x = \sum_{n \in A} a_n x_n + \sum_{n \in B} b_n (y_n + x'_n),$$

applying Q we obtain

$$0 = \sum_{n \in A} a_n z_n + \sum_{n \in B} b_n z'_n.$$

We infer that $a_n = b_k = 0$ for every $n \in A$ and $k \in B$ and so $x = 0$.

Let us next prove that X_0 is a dense subspace of X . Let $x \in X$ and $\varepsilon > 0$. Pick $(a_n)_{n=1}^m$ such that

$$\left\| Q(x) - \sum_{n=1}^m a_n z_n \right\| < \frac{\varepsilon}{3\kappa}.$$

Since

$$Q \left(x - \sum_{n=1}^m a_n x_n \right) = Q(x) - \sum_{n=1}^m a_n z_n,$$

there is $y \in Y$ such that

$$\left\| -y + x - \sum_{n=1}^m a_n x_n \right\| < \frac{\varepsilon}{3\kappa}.$$

Let $y' \in D$ be such that $\|y - y'\| \leq 3^{-1}\kappa^{-1}\varepsilon$. Now let $r \in \mathbb{N}$ be such that $y_r = y'$ and $\|x'_r\| \leq 3^{-1}\kappa^{-1}\varepsilon$. We have

$$x' := y_r + x'_r + \sum_{n=1}^m a_n x_n \in X_0,$$

and

$$\|x - x'\| \leq \kappa \left(\|y - y_r\| + \|x'_r\| + \left\| -y + x - \sum_{n=1}^m a_n x_n \right\| \right) < \varepsilon. \quad \square$$

Theorem 2 *Let X be a separable quasi-Banach space with $N = (X^*)^\perp \neq \{0\}$ and X/N infinite-dimensional. Then:*

- (i) J_X is not one-to-one, and
- (ii) there exists a dense subspace $X_0 \subseteq X$ such that J_X is one-to-one on X_0 .

Proof (i) is a consequence of Lemma 2. In order to prove (ii) we use Proposition 3 to pick a dense subspace $X_0 \subseteq X$ on which the canonical map $Q: X \rightarrow X/N$ is one-to-one. Then $I \circ Q$ is one-to-one on X_0 , where $I: X/N \rightarrow \widehat{X}$ is the inclusion map. Since $J_X = I \circ Q$, we are done. $\square$

Given a quasi-Banach space X , the closed linear span of the union of all subspaces of X with trivial dual has trivial dual. Hence X has a maximum subspace V with trivial dual, and V is closed. Following [24] we call V the *core* of X , and we put $V = \text{core}(X)$. Clearly we have

$$\text{core}(X) \subseteq (X^*)^\perp. \quad (5)$$

In particular, if X has the point-separation property its core is null.

Example 1 Let Y be a nonzero separable quasi-Banach space with trivial dual and Z be an infinite-dimensional space with the point-separation property. Suppose that X is a twisted sum of Y and Z , i.e., X contains a closed subspace V isomorphic to Y with X/V isomorphic to Z . Then, $V \subseteq \text{core}(X)$ and, by Lemma 4, $(X^*)^\perp \subseteq V$. Taking into account the inclusion (5) we obtain

$$V = \text{core}(X) = (X^*)^\perp.$$

Therefore, Theorem 2 applies to X . Note that X is a twisted sum of Y and Z if and only if X is isomorphic to $Y \oplus_F Z$ for some quasi-linear map $F: Z \rightarrow Y$ (see [26, Theorem 2.4]). In particular Theorem 2 applies to the direct sum $Y \oplus Z$. To give a definite example, let us mention that although the envelope map of $L_p \oplus \ell_p$, $0 < p < 1$, is not one-to-one, it is one-to-one on a dense subspace.

If X is as in Example 1, then its core is nonzero. We close by exhibiting that we can also apply Theorem 2 to quasi-Banach spaces with null core. Note that, surprisingly, this yields a quasi-Banach space X with $\text{core}(X) \subsetneq (X^*)^\perp$.

Example 2 In [37] (see also [27, §5.4]) Ribe constructed a quasi-Banach space R with a one-dimensional subspace $L \subseteq R$ such that R/L is isomorphic to ℓ_1 , i.e., R is a twisted sum of $\mathbb{F}$ and ℓ_1 , and L is uncomplemented in R . In particular R/L has the point-separation property. By Lemma 4, $(R^*)^\perp \subseteq L$ and, by Lemma 3, $L \subseteq (R^*)^\perp$. Therefore, $(R^*)^\perp = L$. As far as the core is concerned, since $L^* \neq \{0\}$, applying the inclusion (5) yields $\text{core}(R) \subsetneq L$. Consequently, $\text{core}(R) = \{0\}$.

By Lemma 1, J_X is not one-to-one. Since $R/(R^*)^\perp$ is isomorphic to ℓ_1 , it is infinite-dimensional. Therefore, Theorem 2 applies to R . We conclude by identifying the Banach envelope of R . The properties of the Ribe space yield a quotient map $Q: R \rightarrow \ell_1$ with $\text{Ker}(Q) = L$. By Corollary 2, ℓ_1 is a representation of the Banach envelope of R via Q .

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Conflict of interest

The authors declare that they have no conflict of interest.

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