Journal of Mathematical Psychology 113 (2023) 102754

Contents lists available at ScienceDirect

Journal of Mathematical Psychology

journal homepage: www.elsevier.com/locate/jmp

Semiorders and continuous Scott–Suppes representations. Debreu's Open Gap Lemma with a threshold

A. Estevan

Universidad Pública de Navarra. INARBE institute, Campus Arrosadía, 31006 Iruña-Pamplona, Navarra, Spain

ARTICLE INFO

Article history:
Received 12 January 2021
Received in revised form 20 December 2022
Accepted 27 January 2023
Available online 16 February 2023

Keywords:
Semiorders
Debreu's Open Gap Lemma
Continuity
Scott–Suppes representation

ABSTRACT

The problem of finding a utility function for a semiorder has been studied since 1956, when the notion of semiorder was introduced by Luce. But few results on continuity and no result like Debreu's Open Gap Lemma, but for semiorders, was found. In the present paper, we characterize semiorders that accept a continuous representation (in the sense of Scott–Suppes). Two weaker theorems are also proved, which provide a programmable approach to Open Gap Lemma, yield a Debreu's Lemma for semiorders, and enable us to remove the open-closed and closed-open gaps of a set of reals while keeping the threshold.

1. Introduction

In 1964 (and after a first attempt in 1954 that contained an error that was subsequently corrected Debreu, 1954) Gerard Debreu proved in his famous *Debreu's Open Gap Lemma* that, given any subset $S \subseteq \mathbb{R}$, there is a strictly increasing function $g : S \to \mathbb{R}$ such that all the gaps of $g(S)$ are open. This lemma quickly became a key result on decision theory when dealing with transitive relations, since it is essential to characterize the existence of a continuous utility function of any continuous (and representable) total preorder.

Theorem 1.1 (*Continuous Representation Theorem for Total Preorders*)**.** *A total preorder $\precsim$ on a topological space (X, τ) admits a continuous utility function u (such that $x \precsim y \iff u(x) \leq u(y), x, y \in X$) if and only if $\precsim$ is τ-continuous and admits a utility function.*

The importance of Debreu's Open Gap Lemma settles on the proof of the aforementioned Theorem. Given any utility u of a τ-continuous total preorder on X, this lemma provides a strictly increasing function g on $u(X)$ such that $g \circ u$ is now a continuous utility function of the total preorder (Bridges & Mehta, 1995; Debreu, 1964; Ok, 2007).

However, there are several situations in which the indifference associated to the preference of a decision maker fails to be transitive, as illustrated by Luce (1956) in his classical example of coffee and sugar (Luce, 1956). This example shows that a decision maker may exhibit an intransitive behavior on similar alternatives. This phenomenon has also been observed in psychological experiments, and in particular, some of them show that the agent does not distinguish alternatives until the difference of the corresponding magnitude is greater than a certain *just noticeable difference* (Amstrong, 1939, 1948; Fechner, 1860; Fishburn, 1970b, 1970c; Krantz, 1967; Pirlot & Vincke, 1997; Rubinstein, 1988; Tversky, 1969; Weber, 1834). Intransitive behaviors have been observed too when dealing with time preferences or with more than one criterion (Manzini & Mariotti, 2006, 2012; Masatlioglu & Ok, 2007).

In order to model these behaviors, in 1956 the notion of *semiorder* was defined in *Econometrica* by R.D. Luce. Although it is usually attributed to Luce, actually, it was first introduced by Wiener (as well as the concept of *interval order*) (Fishburn & Monjardet, 1992; Wiener, 1914, 1919). The notion of interval order generalizes the idea of semiorder, and it was studied exhaustively by Fishburn in the 1970's (Fishburn, 1970a, 1970b, 1970c, 1973). The use of both relations was due to the need of working with situations of intransitive indifference.

Since these relations give a better approach to those situations in which the decision maker shows an intransitive behavior, consumer preferences have been modeled by semiorders in the literature (Gilboa & Lapson, 1995; Jamison & Lau, 1973, 1977; Shafer, 1974; Sonnenschein, 1971). The structure of semiorder may appear too on time preferences or when working with more than one criterion, as shown in Manzini and Mariotti (2006, 2012), Masatlioglu and Ok (2007). Other applications of semiorders can be found in Pirlot and Vincke (1997).

Due to that *just noticeable difference* or threshold associated to semiorders, Scott and Suppes proposed to represent those ordered structures by means of a utility function u and a constant threshold k, which can be chosen as $k = 1$ without loss of

E-mail address: asier.mugertza@unavarra.es.

https://doi.org/10.1016/j.jmp.2023.102754

generality (Scott & Suppes, 1958). Thus, for a given semiorder $\prec$ on X, a *Scott–Suppes representation* is defined by means of a single function $u : X \to \mathbb{R}$ such that $x \prec y \Leftrightarrow u(x) + 1 < u(y)$, for every $x, y \in X$ (Candeal & Induráin, 2010; Estevan, Gutiérrez García, & Induráin, 2013b; Luce, 1956; Scott & Suppes, 1958; Vincke, 1980). This pair $(u, 1)$ is also known as *unit representation* (Bouyssou & Pirlot, 2021a, 2021b).

The representability problem for the finite case was solved by Scott and Suppes in 1958, but the general case was not resolved until the paper of Candeal et al. in 2010 (Candeal & Induráin, 2010) (see also Candeal, Estevan, Gutiérrez-García, & Induráin, 2012). In the meantime, Manders (1981) as well as Beja and Gilboa (1992) published important advances on the countably infinite case (Beja & Gilboa, 1992; Manders, 1981).

Although the *Open Gap Lemma* was formulated more than 60 years ago by Debreu (first in 1954 Debreu, 1954 and later in 1964 Debreu, 1964), it was still unknown if this strictly increasing function g was existing on $S \subseteq \mathbb{R}$ when we also impose to satisfy the geometrical condition $x + 1 < y \Leftrightarrow g(x) + 1 < g(y)$, for every $x, y \in S$. This result would provide an answer to the continuous representability (in the sense of Scott–Suppes) of semiorders. Thus, there is no *Continuous Representability Theorem* for intransitive relations (in particular, for semiorders).

Partial results related to the continuous representability of semiorders (and, hence, to the possible existence of that function g, which may not exist) were presented by Estevan et al. (2013) (Debreu, 1964; Estevan, Gutiérrez García, & Induráin, 2013a). An early work on continuous representations of semiorders was presented by Gensemer in 1987 (Gensemer, 1987). The finite case was solved by Estevan et al. (2013) in Candeal et al. (2012) (see Estevan, Schellekens, & Valero, 2017; Pirlot & Vincke, 1997, for instance, for studies on the finite case).

In the present paper we focus on semiorders and prove a characterization of the existence of a continuous representation for I-bounded semiorders in the sense of Scott–Suppes. That is, a big family of semiorders – that also includes the bounded semiorders – that admits a continuous unit representation is characterized. We call them *reasonable semiorders*, since the conditions satisfied by them seem quite reasonable from a decision maker's point of view. As a result, we achieve a version of Debreu's Open Gap Lemma with a threshold, so that given a set S in $\mathbb{R}$, now we know when a strictly increasing function exists such that $x + 1 < y$ if and only if $g(x) + 1 < g(y)$ and satisfying that all the gaps of $g(S)$ are open. The sets that accept this function g are called *R-sets*. As commented before, the representability problem for semiorders was finally solved by Candeal and Induráin (2010) (see also Candeal et al., 2012, as well as Bouyssou & Pirlot, 2021a, 2021b), and the present paper almost completely closes the continuity question.

In order to approach to the desired results and construct a theorem such as Debreu's Open Gap Lemma but for semiorders, the concept of ϵ-continuity was successfully introduced as a generalization on the idea of continuity for semiorders (Estevan, 2020). In the case of semiorders, there is an invariant threshold k in the representations (we may assume that $k = 1$) that allows us to compare the length of each jump-discontinuity with this value $k = 1$. Hence, it makes perfect sense to say that a semiorder is r-continuous (with $r > 0$) if a unit representation $(u, 1)$ exists such that the length of each jump-discontinuity is bounded by this constant r. Then, we can approach the idea of the usual continuity just letting r tend to 0. Through this idea we are able to present two weaker results with the advantage that approximately continuous representations can be computed. After that, the new *Debreu's Open Gap Lemma with a threshold* is introduced.

For more details on this subject we suggest some readings such as Aleskerov, Bouyssou, and Monjardet (2007), Bridges and Mehta (1995), Ok (2007), Pirlot and Vincke (1997). For those notions related to the continuous representability of semiorders and ϵ-continuity, we recommend to read (Estevan, 2020).

The structure of the paper goes as follows: First, in Section 2, the main concepts are presented, as well as some basic results on continuous representability of total preorders. Then, in Section 3, we focus on the special case of semiorders and recover those necessary conditions for the existence of a continuous unit representation introduced in Estevan et al. (2013a). The image subset $u(X)$ is studied for a given unit representation $(u, 1)$ of a semiorder that satisfies the aforementioned necessary conditions. In Section 4, the study is further explored in depth, the concept of 'adjoint net' is presented, improving those necessary conditions. The implications of these enhanced conditions on the appearance of image set $u(X)$ (for any unit representation $(u, 1)$) are described in a family of four theorems, by means of the so called 'compression intervals'. Finally, in Section 5, the idea of ϵ-continuity is presented and the continuous representability (in the sense of Scott–Suppes) for I-bounded semiorders is characterized, but before that, two weaker but more visible results are proved. This characterization is then abstracted from the context of semiorders, and presented just as a version of the *Debreu's Open Gap Lemma* but with additional component of a *threshold*. A constructive and programmable approach to Debreu's Open Gap Lemma ends the study.

2. Preliminaries

In this section we have compiled basic concepts related to semiorders, total preorders and continuity of functions. In the second and last subsection, we will focus on the continuous representations of total preorders, more specifically on the gaps.

2.1. Preliminaries on total preorders and continuity

Firstly, we introduce some definitions.

Definition 2.1. An asymmetric binary relation $\prec$ on X is said to be a *semiorder* if the following two conditions are satisfied:
(1) $(x \prec y) \wedge (z \prec t) \Rightarrow (x \prec t) \vee (z \prec y), \quad x, y, z, t \in X,$
(2) $(x \prec y) \wedge (y \prec z) \Rightarrow (x \prec w) \vee (w \prec z), \quad x, y, z, w \in X.$

A semiorder is said to be *bounded* if there are no strictly increasing or decreasing infinite sequences, i.e., there is no sequence $(x_n)_{n\in\mathbb{N}} \subseteq X$ such that $\cdots \prec x_{n+1} \prec x_n \prec \cdots \prec x_1$ or $x_1 \prec \cdots \prec x_n \prec x_{n+1} \prec \cdots$. And it is said to be *regular*[1] if there are no strictly increasing or decreasing infinite sequences $(x_n)_{n\in\mathbb{N}} \subseteq X$ such that $x \prec \cdots \prec x_{n+1} \prec x_n \prec \cdots \prec x_1$ or $x_1 \prec \cdots \prec x_n \prec x_{n+1} \prec \cdots \prec x$, for some $x \in X$.

Due to condition (1), a semiorder is a particular case of an *interval order*. Its symmetric complement is denoted by $\precsim$, so that $a \precsim b \Leftrightarrow \neg(b \prec a)$. The *indifference* relation $\sim$ associated to $\prec$ is defined by $a \sim b \Leftrightarrow (a \precsim b) \wedge (b \precsim a)$. It is well known that $\precsim$ and $\sim$ may fail to be transitive (Fishburn, 1970b, 1970c; Luce, 1956; Scott & Suppes, 1958).

A *preorder* $\precsim$ on X is a binary relation which is reflexive and transitive. An antisymmetric preorder is said to be an *order*. A *total preorder* $\precsim$ on a set X is a preorder such that if $x, y \in X$ then $(x \precsim y) \vee (y \precsim x)$ holds. In the case of preorders, it is well known that the corresponding indifference is transitive, in fact, it is an equivalence relation. Given a semiorder, we write $x \prec^0 y$ when there exists $z \in X$ such that $x \prec z \precsim y$ or $x \precsim z \prec y$. It is well known that the corresponding reflexive binary relation $\precsim^0$

[1] An equivalent condition was introduced by Beja and Gilboa (1992) as well as by Manders (1981). See also section 5 in Bouyssou and Pirlot (2021a).

(i.e., $x \precsim^0 y \iff \neg(y \prec^0 x)$) is a total preorder and it is called the *main trace* of the semiorder.

A semiorder $\prec$ defined on X is said to be *s-separable* if there is a countable subset $D \subseteq X$ with the following property: for every $x, y \in X$ such that $x \prec y$, there exist $d_1, d_2 \in D$ such that $x \prec d_1 \precsim^0 y$ and $x \precsim^0 d_2 \prec y$ (Candeal et al., 2012; Candeal & Induráin, 2010).

The *Scott–Suppes representation* (also known as *unit representation*) is defined by means of a single function $u : X \to \mathbb{R}$ and a threshold $k = 1$ such that $x \prec y$ if and only if $u(x) + 1 < u(y)$, for every $x, y \in X$ (Bouyssou & Pirlot, 2021a, 2021b; Candeal & Induráin, 2010; Estevan et al., 2013b; Luce, 1956; Scott & Suppes, 1958; Vincke, 1980). When the set X is endowed with a topology τ, the semicontinuity or continuity of the numerical representations (if any) is also studied (Campión, Candeal, Induráin, & Zudaire, 2008; Gensemer, 1987).

The analogous problem related to the existence of a continuous representation but now for total preorders was solved by Gerard Debreu in 1964 (Debreu, 1964). For this purpose, a *lacuna* of $S \subseteq \mathbb{R}$ was defined as a non-degenerate interval (that is, an interval that is not reduced to a single point) disjoint from S and having a lower bound and an upper bound in S, and a *gap* of S as a maximal lacuna of S. Two gaps $[a, b)$ and $(c, d]$ (also for the pairs (a, b) and $(c, d]$, $[a, b)$ and (c, d), and (a, b) and (c, d)) are said to be *adjacent* whenever $b = c$. The famous *Debreu's Open Gap Lemma* reads as follows (Debreu, 1964):

Lemma 2.2 (*Open Gap Lemma*)**.** *If $S \subseteq \mathbb{R}$, then there is a strictly increasing function $g : S \to \mathbb{R}$ such that all the gaps of $g(S)$ are open.*

Given a representation u of a total preorder (that is, a function $u : X \to \mathbb{R}$ such that $x \precsim y \iff u(x) \leq u(y)$), we will say that a gap of $u(X) \subseteq \mathbb{R}$ is a *bad gap*[2] if the function u is not continuous at the inverse images of the end-points of the gap. Notice that, since u is increasing on X, the discontinuities are in fact jump-discontinuities. We will refer to the length of the bad gap as the *length of the jump-discontinuity*. On the other hand, since the number of non degenerate and disjoint intervals contained in $\mathbb{R}$ is, at most, countably infinite, so is the number of bad gaps.

We recover these results in the following theorem. Although it is well-known, we will study it in more detail in Section 2.2.

Theorem 2.3. *Let $\precsim$ be a continuous and representable*[3] *total preorder on (X, τ). Let $(u, 1)$ be a representation of $\precsim$. Then, u has at most a countably infinite number of discontinuities (which are jump-discontinuities, and are points $a \in X$ such that $u(a)$ is an extreme of a open-closed or a closed-open gap).*

The characterization of the existence a Scott–Suppes representation is known and it is made by means of *s-separability* and *regularity* (Bouyssou & Pirlot, 2021a, 2021b; Candeal et al., 2012; Candeal & Induráin, 2010).

Theorem 2.4. *Let $\prec$ be a semiorder defined on X. Then, $\prec$ is representable in the sense of Scott and Suppes if and only if it is both s-separable and regular with respect to sequences. Furthermore, a Scott–Suppes representable semiorder also admits a representation $(u, 1)$ such that u also represents the main trace.*

Although they are quite common in the literature of the field, for the sake of completeness, we include some basic concepts and results related to continuity and semicontinuity.

[2] The name *bad gap* was already used by G. B. Mehta in Mehta (1997).

[3] The representability of a total preorder is characterized by means of *perfect separability* (Bridges & Mehta, 1995). For the sake of simplicity, we will not dwell on these representability questions.

Definition 2.5. Let $\prec$ be an asymmetric binary relation on (X, τ). Given $a \in X$, the sets $L_{\prec}(a) = \{t \in X \ : \ t \prec a\}$ and $U_{\prec}(a) = \{t \in X \ : \ a \prec t\}$ are called, respectively, the *strict lower* and *upper contours* of a relative to $\prec$. We say that $\prec$ is *τ-continuous* (or just *continuous*) if for each $a \in X$ the sets $L_{\prec}(a)$ and $U_{\prec}(a)$ are τ-open.

We will denote the *order topology* generated by $\prec$ as $\tau_{\prec}$, and it is defined by means of the subbasis provided by the lower and upper contour sets.

Let $\precsim$ be a reflexive binary relation on (X, τ). Given $a \in X$ the sets $L_{\precsim}(a) = \{t \in X \ : \ t \precsim a\}$ and $U_{\precsim}(a) = \{t \in X \ : \ a \precsim t\}$ are called, respectively, the *weak lower* and *upper contours* of a relative to $\precsim$. We say that $\precsim$ is *τ-lower semicontinuous* (*τ-upper semicontinuous*) if for each $a \in X$ the sets $L_{\precsim}(a)$ (resp. $U_{\precsim}(a)$) are τ-closed.

Definition 2.6. Let $\precsim$ be a preorder defined on X. The *upper topology* τ_u is obtained by choosing the closed sets to be the weak lower contour sets (as well as their finite unions and infinite intersections).

Dually, the *lower topology* τ_l is obtained by choosing the closed sets to be the weak upper contour sets (as well as their finite unions and infinite intersections).

Remark 2.7. If the binary relation $\precsim$ is in fact a total preorder, since $X \setminus L_{\precsim}(x) = U_{\prec}(x)$ and $X \setminus U_{\precsim}(x) = L_{\prec}(x)$ (for any $x \in X$), then the upper topology τ_u can be defined by choosing the open sets to be the strict upper contour sets (as well as their infinite unions and finite intersections). Dually, the lower topology τ_l can be defined by choosing the open sets to be the strict lower contour sets (as well as their infinite unions and finite intersections).

Definition 2.8. We say that $f : (X, \tau) \to \mathbb{R}$ is *lower semicontinuous* at x_0 if for every $\epsilon > 0$ there exists a neighborhood U of x_0 such that $f(x) > f(x_0) - \epsilon$ for all $x \in U$.

Dually, we say that $f : (X, \tau) \to \mathbb{R}$ is *upper semicontinuous* at x_0 if for every $\epsilon > 0$ there exists a neighborhood U of x_0 such that $f(x) < f(x_0) + \epsilon$ for all $x \in U$.

We say that f is *continuous* at x_0 whenever it is both lower semicontinuous and upper semicontinuous at x_0.

As usual, we say that f is continuous (also for lower semicontinuity and upper semicontinuity) if it is continuous (resp., lower semicontinuous and upper semicontinuous) at x for any $x \in X$.

For the sake of completeness we include the following characterization for semicontinuity, although it is well known in the literature (Bridges & Mehta, 1995; Dugundji, 1966; Engelkin, 1989).

Proposition 2.9. *Let (X, τ) be a topological space, x_0 a point in X and $f : (X, \tau) \to \mathbb{R}$ a function. The following statements are equivalent.*

1. *f is lower semicontinuous at x_0.*
2. *f is continuous at x_0 with respect to the upper topology on $\mathbb{R}$.*
3. *For any $\epsilon > 0$, $f^{-1}((f(x_0) - \epsilon, +\infty))$ is a neighborhood of x_0.*
4. $\lim_{x \to x_0} \inf f(x) \geq f(x_0)$.

We also include the dual of Proposition 2.9 for upper semicontinuity.

Proposition 2.10. *Let (X, τ) be a topological space, x_0 a point in X and $f : (X, \tau) \to \mathbb{R}$ a function. The following statements are equivalent.*

1. *f is upper semicontinuous at x_0.*
2. *f is continuous at x_0 with respect to the lower topology on $\mathbb{R}$.*
3. *For any $\epsilon > 0$, $f^{-1}((-\infty, f(x_0) + \epsilon))$ is a neighborhood of x_0.*
4. $\lim_{x \to x_0} \sup f(x) \leq f(x_0)$.

Next proposition can be found in Dugundji (1966).

Proposition 2.11. *Let (X, τ_X) be a topological space. A function f from (X, τ_X) onto $(\mathbb{R}, \tau_{\leq})$ is upper semicontinuous (resp. lower semicontinuous) if and only if $\{x \in X : f(x) < b\}$ (resp. $\{x \in X : f(x) > b\}$) is an open set for every $b \in \mathbb{R}$. Alternatively, a function f is upper semicontinuous (resp. lower semicontinuous) if and only if all of its upper level sets $\{x \in X : f(x) \geq b\}$ (resp. $\{x \in X : f(x) \leq b\}$) are closed, for every $b \in \mathbb{R}$.*

2.2. A little review on continuous utility representations for total preorders

The next Proposition 2.12 was proved in Estevan (2016), although this is a widely known result in the area, that may be found in more classical papers such as in Proposition 1 in Beardon and Mehta (1994) or in page 37 in Bridges and Mehta (1995), for instance.

Proposition 2.12. *Let $\precsim$ be a total preorder defined on X endowed with the order topology $\tau_{\prec}$. Given any representation u of the total preorder, then u is continuous at all points of X, excluding inverse images of the end-points of the gaps (of $u(X)$) that are neither closed nor open, that is, excluding $x \in X$ such that $(u(x) = a, b]$ or $[b, u(x) = a)$ is a gap of $u(X) \subseteq \mathbb{R}$.*

As commented before, since u is increasing on X, the discontinuities are in fact jump-discontinuities. Furthermore, since the number of non degenerate and disjoint intervals contained in $\mathbb{R}$ is, at most, countably infinite, so is the number of discontinuities.

Next corollary follows directly from Proposition 2.12 above .

Corollary 2.13. *Let (X, τ) be a topological space endowed with a τ-continuous total preorder $\precsim$. Any representation u of the total preorder is continuous at all the points of X, excluding the inverse images of the end-points of some gaps (of $u(X)$) that are neither closed nor open, that is, excluding some $x \in X$ such that $(u(x) = a, b]$ or $[b, u(x) = a)$ is a gap of $u(X) \subseteq \mathbb{R}$.*

Definition 2.14. Let (X, τ) be a topological space endowed with a τ-continuous total preorder $\precsim$. Given a representation u of the total preorder, we will say that a gap of $u(X) \subseteq \mathbb{R}$ is a *bad gap* if the function u is not continuous at the inverse images of the end-point of the gap.

Remark 2.15. With definition above and using Corollary 2.13, notice that given any representation u of a τ-continuous total preorder on (X, τ), then any closed or open gap ($[a, b], (a, b) \subseteq \mathbb{R}$) of $u(X)$ is not a bad gap.

When working with continuous representations of continuous total preorders, it is important to always keep in mind Corollary 2.13, as well as the following proposition.

Proposition 2.16. *Let (X, τ) be a topological space endowed with a τ-continuous total preorder $\precsim$. Let u be a representation of $\precsim$, $x \in X$ and $\varepsilon > 0$ such that $(u(x), u(x) + \varepsilon]$ is a gap of $u(X)$. If u fails to be continuous at x then there exists a net $(x_i)_{i \in I}$ in X convergent to x and such that:*

(i) $u(x) + \epsilon < u(x_i)$ for all $i \in I$, in case $u(x)$ is not the right end-point of a gap of $u(X)$. Thus, u fails to be upper semicontinuous.
(ii) $u(x) + \epsilon < u(x_i)$ for all $i \in I$, in case $u(x)$ is the right end-point of an open gap $(b, u(x))$ of $u(X)$. Thus, u fails to be upper semicontinuous.
(iii) $u(x) + \epsilon < u(x_i)$ for all $i \in I$, or $u(x_i) < u(x) - \sigma$ for all $i \in I$, in case $u(x)$ is the right end-point of a gap $[u(x) - \sigma, u(x))$ of $u(X)$. Thus, u fails to be upper semicontinuous, lower semicontinuous or both (respectively).

Proof. (i) Since u is not continuous at x, there exists a net $(x_i)_{i\in I}$ in X convergent to x and such that $(u(x_i))_{i\in I}$ does not converge to $u(x)$ in $\mathbb{R}$. The latter means that there exists $\delta > 0$ (we can take $\delta < \varepsilon$) and a subnet $(u(x_j))_{j\in J}$ such that

$$u(x_j) \notin (u(x) - \delta, u(x) + \delta) \qquad \text{for each } j \in J. \tag{1}$$

Since $u(x)$ is not the right end-point of a gap of $u(X)$ there exists some $y \in X$ such that $u(x) - \delta < u(y) < u(x)$ (in particular $y \prec x$).

Now using the fact that the total preorder $\precsim$ is τ-continuous we have that $U_{\prec}(y)$ is an open neighborhood of x and since the subnet $(x_j)_{j\in J}$ converges to x it follows that it is eventually in $U_{\prec}(y)$, i.e. there exists $j_0 \in J$ such that $x_j \in U_{\prec}(y)$ for all $j > j_0$. Consequently,

$$u(x) - \delta < u(y) < u(x_j) \qquad \text{for each } j > j_0. \tag{2}$$

Since $(u(x), u(x) + \varepsilon]$ is a gap of $u(X)$, it follows from (1) and (2) that $u(x) + \epsilon < u(x_j)$ for all $j > j_0$.

(ii) If $u(x)$ is the right end-point of an open gap $(b, u(x))$ of $u(X)$ (thus, there is $w \in X$ with $u(w) = b$), then $x \in U_{\prec}(w) = u^{-1}(u(x), +\infty) = U$, and by the continuity of the total preorder, U is open.

Since u is not continuous at x, there exists a net $(x_i)_{i\in I}$ in X convergent to x and such that $(u(x_i))_{i\in I}$ does not converge to $u(x)$ in $\mathbb{R}$. Thus, there exists $\delta > 0$(we can take $\delta < \varepsilon$) and a subnet $(u(x_j))_{j\in J}$ such that

$$u(x_j) \notin (u(x) - \delta, u(x) + \delta) \qquad \text{for each } j \in J. \tag{3}$$

On the other hand, since $(x_i)_{i\in I}$ converges to x, for any open neighborhood V of x there is an index i_0 such that $x_i \in V$ for any $i \geq i_0$. Therefore, since $U_{\prec}(w) = u^{-1}(u(x), +\infty) = U$ is open and $x \in U$, we can state that for any open neighborhood V of x there is an index i_0 such that $x_i \in V \cap U$, for any $i \geq i_0$. The latter means

$$u(x_i) \geq u(x) \qquad \text{for each } i \geq i_0. \tag{4}$$

Since $(u(x), u(x) + \varepsilon]$ is a gap of $u(X)$, it follows from (3) and (4) that $u(x_i) > u(x) + \varepsilon$.

(iii) If for any open neighborhood V of x there exists $x_i \in V$ such that $x_i \prec x$, that is, it holds that $V \cap L_{\prec}(x) \neq \emptyset$, then for any open neighborhood V_i of x there exists $x_i \in V_i$ such that $u(x_i) \leq u(x) - \sigma$. Therefore, $u^{-1}(u(x) - \frac{\sigma}{2}, +\infty)$ is not open and, hence, u fails to be lower semicontinuous at x. Thus, we can construct a net $(x_i)_{i\in I}$ converging to x such that $u(x_i) < u(x) - \sigma$ for all $i \in I$.

We reason dually if for any open neighborhood V of x there exists $x_i \in V$ such that $x \prec x_i$.

Otherwise, $\bar{x} = \{y \in X : y \sim x\}$ is open and, hence, $u^{-1}((u(x) - \delta, u(x) + \delta)) = \bar{x}$ is open for any $0 < \delta < \min\{\epsilon, \sigma\}$. Therefore, u is continuous at x, arriving to a contradiction. □

Remark 2.17. Proposition 2.16 for a gap $[u(x) - \epsilon, u(x))$ can be dually formulated. See Fig. 1 for an illustration of a gap $[u(x) - \epsilon, u(x))$.

3. The special case of semiorders

Forthwith we collect a few concepts, results and those necessary conditions for the existence of a continuous Scott–Suppes representation which are available in literature (Bosi, Estevan,

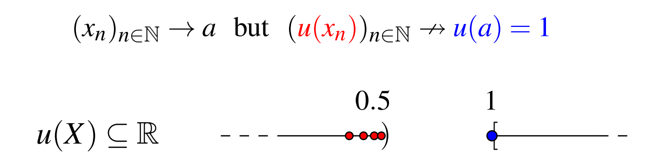

Fig. 1. Illustration of a discontinuity of a representation of a continuous total preorder.

Gutiérrez-García, & Induráin, 2015; Bridges & Mehta, 1995; Estevan, 2020; Estevan et al., 2013a). After that, in Section 5, we will summarize these concepts in order to present some representation theorems.

Given an interval order (e.g. a semiorder) on a set X, the indifference $\sim^0$ associated to the main trace is an equivalence relation. Furthermore, elements which are indistinguishable with respect to the indifference $\sim^0$ play exactly the same role in the order structure, and vice versa, that is, $x \sim^0 y$ if and only if $\{x \prec z \iff y \prec z\}$ as well as $\{w \prec x \iff w \prec y\}$ (Bosi et al., 2015).

Definition 3.1. Let $\prec$ be an interval order defined on a topological space (X, τ). The topology τ is said to be *compatible with respect to the indifference of the main trace of* $\prec$ (or *compatible*, for short) if $x \sim^0 y \Rightarrow (x \in \mathcal{O} \iff y \in \mathcal{O})$ holds true for every $x, y \in X$ and every τ-open subset $\mathcal{O} \in \tau$.

In particular, in the main case in which $x \sim_0 y \iff x = y$, the topology is always compatible. We want to highlight that, if we assume that the topology is compatible with the indifference $\sim^0$, then τ-continuity of the main trace is in fact a necessary condition for the continuity of the function, as stated in Proposition 3.2. And if we want to represent at the same time the trace, then its τ-continuity is, again, a necessary condition. This is also the case of the so-called *regular representations*, which assign the same value to equivalent elements (Bouyssou & Pirlot, 2021a, 2021b). Next proposition may be found in Bosi et al. (2015).

Proposition 3.2. *Let (X, τ) be a topological space endowed with a semiorder $\prec$. Assume that τ is compatible with respect to the indifference of the main trace of $\prec$. Suppose also that $\prec$ is representable in the sense of Scott and Suppes by means of a pair $(u, 1)$ with u continuous. Then, the total preorder $\precsim^0$ is τ-continuous.*

For the proofs of the main theorems we use Corollary 2.13, that guarantees that any representation u of a τ-continuous total preorder on (X, τ) is continuous at all the points of X, excluding the inverse images of the end-points of some gaps (of $u(X)$) that are neither closed nor open. Thus, when dealing with Scott–Suppes representations, τ-continuity of the main trace is required in order to achieve continuity. This may seem a very demanding condition, but it is not really so, since, as Proposition 3.2 states, when working with compatible topologies, the τ-continuity of the trace becomes a necessary condition for the existence of a continuous representation. And these compatible topologies are not strange at all, since this compatibility condition is implicitly satisfied in the most general cases such as when ($\triangle = \{(x, y) \in X \times X : x \sim^0 y\} = \{(x, x) : x \in X\}$) or when working on the quotient set $X/\sim^0$. If we refuse to use compatible topologies, then the continuity of the trace may fail to be necessary, but still sufficient, as shown in Corollary 5.17. Finally, it is worth remembering that, if a semiorder in a compatible topological space accepts a continuous representation, then it is continuously representable too with any other finer topology, compatible or not.

With respect to the continuous representability of semiorders, the following necessary conditions were proved by Estevan et al. (2013a).

Lemma 3.3. *Let (X, τ) be a topological space endowed with a semiorder $\prec$. Assume that $\prec$ is representable in the sense of Scott and Suppes by means of a pair $(u, 1)$ with u continuous. Then the following properties hold true:*

(a) *The semiorder $\prec$ is τ-continuous.*
(b) *If a net $(x_j)_{j\in J} \subseteq X$ converges to two points $a, b \in X$, then $a \sim^0 b$.*
(c) *If a net $(x_j)_{j\in J} \subseteq X$ converges to $a \in X$, and $b, c \in X$ are such that $x_j \prec b \precsim a$ and also $x_j \prec c \precsim a$ for every $j \in J$, then $b \sim^0 c$.*
(d) *If a net $(x_j)_{j\in J} \subseteq X$ converges to $a \in X$, and $b, c \in X$ are such that $a \precsim b \prec x_j$ and also $a \precsim c \prec x_j$ for every $j \in J$, then $b \sim^0 c$.*

When these necessary conditions (a)–(d) of Lemma 3.3 were introduced, it was already known that they were not sufficient in order to guarantee the existence of a continuous Scott–Suppes representation (see Estevan et al., 2013a for examples related to these conditions). However, these conditions place significant constraints on the appearance of a representation, as it is shown in Fig. 2 and described in Proposition 3.4 and Proposition 3.5.[4] Through this appearance we are able to recognize some semiorders that fail to be continuously representable.

Proposition 3.4. *Let $\prec$ be a representable and τ-continuous semiorder on (X, τ). Let $(u, 1)$ be a unit representation. Suppose that there is a discontinuity at a point a such that $[r, u(a))$ or $(u(a), r]$ is a gap of $S = u(X)$. Then, the length of the jump-discontinuity is strictly smaller than 1.*

Proof. Suppose that $(u(a), r]$ is the corresponding gap. If the length of the jump-discontinuity is at least 1, then for any $1 > \epsilon > 0$ there exists an element $b \in X$ with $u(b) \in (r, r + \epsilon)$ such that, by condition (a), $L_\prec(b) = u^{-1}((-\infty, u(a)])$ is open and it contains a. Therefore, for any net $(x_i)_{i\in I}$ such that $u(x_i) > r$ it holds that $a \in L_\prec(b)$ as well as $x_i \notin L_\prec(b)$ (for any $i \in I$). Thus, according to Proposition 2.16 and Remark 2.17, we deduce that there is no discontinuity at point a, arriving to the desired contradiction.

We argue dually for $[r, u(a))$. □

Proposition 3.5. *Let $\prec$ be a representable semiorder on (X, τ) satisfying conditions (a)–(d). Let $(u, 1)$ be a unit representation such that u also represents the trace.*

Suppose that there is a discontinuity at a point a such that $[r, u(a))$ is a closed-open gap of $S = u(X)$. Then, the following holds true:

(CO_l) $[r - 1, u(a) - 1) \cap S = \emptyset$.
(CO_r) $[r + 1, u(a) + 1] \cap S$ *contains, at most, one point.*

Dually, suppose that the discontinuity at a point a is such that $(u(a), r]$ is an open-closed gap of $S = u(X)$. Then, the following holds true:

(OC_l) $[r - 1, u(a) - 1] \cap S$ *contains, at most, one point.*
(OC_r) $[r + 1, u(a) + 1) \cap S = \emptyset$.

Proof. (CO_l): We argue by contradiction. If there was an element $u(b) \in [r - 1, u(a) - 1)$, then it would hold that $u(b) < u(a) - 1$, thus $b \prec a$. Since, by condition (a), $U_\prec(b)$ is open, that implies that (according to Proposition 2.16 and Remark 2.17) there is no discontinuity at a, arriving at a contradiction.

(CO_r): Suppose there is a gap $[r, u(a))$ such that u fails to be continuous at a. Then, by Proposition 2.16 and Remark 2.17, there

[4] This result was first presented in Estevan (2020). Nevertheless, we include it here (as well as its proof) for the sake of completeness.

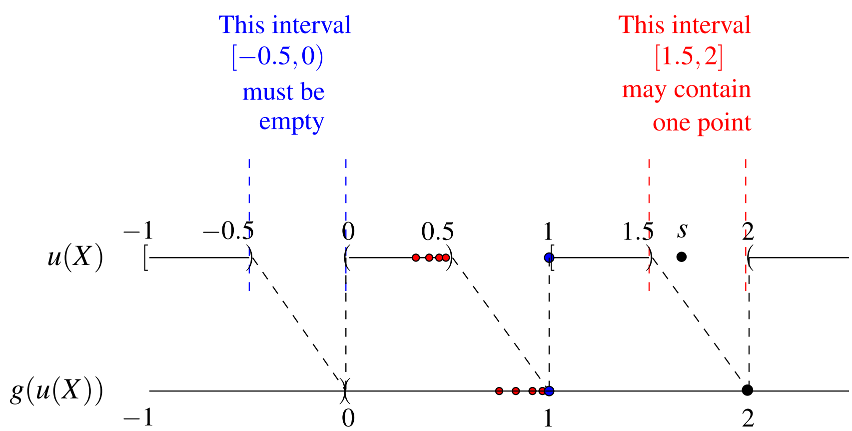

Fig. 2. Illustration of a Scott–Suppes representation of a semiorder $\prec$ on X satisfying the necessary conditions (a)–(d), and such that u also represents the trace. If u is discontinuous at $a = u^{-1}(1)$, such that $[0.5, 1)$ is a closed-open bad gap, then, by condition (a), there cannot be any point $b \in X$ such that $u(b) \in [-0.5, 0)$, that is, $u(X) \cap [-0.5, 0) = \emptyset$. Otherwise, $U_{\prec}(b) = u^{-1}([1, +\infty))$ would be open and, hence, there would be no discontinuity at $a = u^{-1}(1)$. Similarly, by condition (c) there is, at most, one point $b \in X$ such that $u(b) = s \in [1.5, 2]$. In this figure we also illustrate how we could modify u (lengthening proportionally the intervals, through g) in order to avoid that discontinuity at $a = u^{-1}(1)$, but keeping the representation (i.e., such that $x \prec y$ if and only if $g(u(x)) + 1 < g(u(y))$, $x, y \in X$). For that purpose, notice that the intervals $[-0.5, 0)$ and $[1.5, 2]$ must be compressed to a point.

is a net $(x_i)_{i \in I}$ in X, convergent to a and such that $u(x_i) < r < u(a)$ for all $i \in I$. For any two points $u(b), u(c) \in [r + 1, u(a) + 1]$, it holds that $u(x_i) + 1 < r + 1 < u(b) \leq u(a) + 1$ as well as $u(x_i) + 1 < r + 1 < u(c) \leq u(a) + 1$. Thus $x_i \prec b$ and $x_i \prec c$ for any $i \in I$, as well as $b \precsim a$ and $c \precsim a$. Hence, by condition (c), it holds that $b \sim^0 c$ and, since u also represents the trace $\precsim^0$, we conclude that $u(b) = u(c)$, that is $u(X) \cap [r + 1, u(a) + 1]$ has at most one point.

We argue dually for $(u(a), r]$. □

Therefore, given any Scott-Suppes representation $(u, 1)$ (such that u also represents the trace) of a semiorder $\prec$ on X, if $u(X)$ fails to satisfy the appearance described in Propositions 3.5 and 3.4 (see Fig. 2), then $\prec$ cannot be continuously representable.

At this stage, it is important to notice that, as illustrated in Fig. 2, a gap $[r, u(a))$ or $(u(a), r]$ generated by a discontinuity defines in turn two intervals $[r - 1, u(a) - 1]$ and $[r + 1, u(a) + 1]$ (dually, $[u(a) - 1, r - 1]$ and $[u(a) + 1, r + 1]$) that must be compressed to a point $s \in \mathbb{R}$ in order to construct a continuous representation at point a. Furthermore, $|u(a) - s| \leq 1$, that is, if there is any $b \in X$ such that $u(b)$ is in these compression intervals, then $b \sim a$ should be satisfied. This idea of *compression intervals* will be explored in detail in the next section.

4. The 'dragging' effect of semiorders

Given a Scott–Suppes representation $(u, 1)$ (such that u also represents the trace) of a semiorder on X, in the previous section we have studied the implications of a discontinuity of u at a point x in the intervals contiguous to the image $u(x)$. These implications on u, that model the appearance of the image set $u(X) \subseteq \mathbb{R}$, have been collected in Proposition 3.5.

In the following lines we present some other conditions that must be satisfied for the existence of a continuous Scott–Suppes representation. They are described by means of *n-adjoint* nets. Unlike the previous conditions (a)–(d), the consequences of these new conditions may extend beyond the intervals contiguous to the discontinuity, giving rise to what we will call the *dragging effect*. Thus, if we want to make changes on the function in the neighborhood of a point (but always keeping the representation), these changes can have implications not only on the contiguous intervals, but also on the successive ones.

The goal of the present section is to introduce these new necessary conditions as well as their implications on a representation $(u, 1)$, in order to describe the appearance of the image set $u(X) \subseteq \mathbb{R}$, as it was did in Proposition 3.5.

For the sake of completeness, we recover the concept of *n-adjoint* nets, that was first presented in Estevan (2020).

Definition 4.1. Let (X, τ) be a topological space endowed with a semiorder $\prec$. Let $(x_j)_{j \in J}$ and $(y_k)_{k \in K}$ be two nets on X. We shall say that these nets are *adjoint nets*, and we denote it by $(x_j) \preccurlyeq (y_k)$, if one of the following conditions holds:

Condition 1: If neither of these nets is constant, then the following two conditions both hold:

(1.i) for each $j_0 \in J$ there exists $k_0 \in K$ such that $x_{j_0} \prec y_k$ for any $k > k_0$,

(1.ii) for each $k_0 \in K$ there exists $j_0 \in J$ such that $y_{k_0} \precsim x_j$ for any $j > j_0$.

Condition 2: If one (and only one) of the nets is constant, that is $y_k = b$ for all $k \in K$ or $x_j = b$ for all $j \in J$, where b is called *adjoint point*, then any of the following conditions is satisfied:

(2.i) $x_j \prec b$ for each $j \in J$ and the net converges to $a \in X$ such that $b \precsim a$,

(2.ii) $b \prec y_k$ for each $k \in K$ and the net converges to $a \in X$ such that $a \precsim b$.

Analogously, for each $n \in \mathbb{N}$ we define the *n-adjoint nets*, and we denote them by $(x_j) \preccurlyeq^n (y_k)$, if there exists a chain of length n of adjoint nets: $(x_j) \preccurlyeq (a_{i_1}) \preccurlyeq \ldots \preccurlyeq (a_{i_{n-1}}) \preccurlyeq (y_k)$. For any m such that $-m \in \mathbb{N}$, we also say that $(x_j)_{j \in J}$ and $(y_k)_{k \in K}$ are *m-adjoint nets*, and we denote them by $(x_j) \preccurlyeq^m (y_k)$ if $(y_k) \preccurlyeq^{-m} (x_j)$. The following Fig. 3 tries to illustrate the idea of adjoint nets.

The following lemma shows the significance of n-adjoint nets, since they provide a rigid structure to the semiorder when dealing with (continuous) representations, as illustrated in Fig. 4. It was proved in Estevan (2020), but we include it here for the sake of completeness.

Lemma 4.2. *Let $\prec$ be a semiorder defined on a topological space (X, τ) and let $(u, 1)$ be a continuous representation. If $(x_j)_{j \in J}$ and $(y_k)_{k \in K}$ are n-adjoint nets, then $\lim_{j \in J} u(x_j) + n = \lim_{k \in K} u(y_k)$.*

Proof. We prove it by induction on $n \in \mathbb{N}$. For $n = 1$, if $(x_j) \preccurlyeq (y_k)$ and none of them is a constant, then for any $j_0 \in J$ there exists $k_0 \in K$ such that $x_{j_0} \prec y_k$ for each $k > k_0$, so $u(x_{j_0}) + 1 < \lim(u(y_k))$ for each $j_0 \in J$. Hence, $\lim(u(x_j)) + 1 \leq \lim(u(y_k))$.

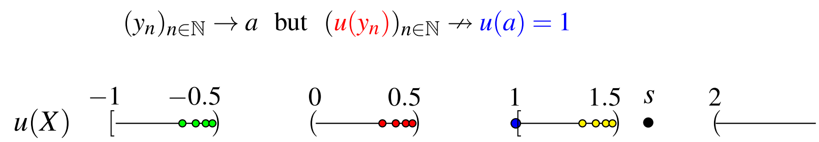

Fig. 3. Illustration of adjoint nets by means of a representation $(u, 1)$ of a semiorder on X. Here, the image of three nets are drawn: $(u(x_n))_{n\in\mathbb{N}} = (-0.5 - \frac{1}{n})_{n\in\mathbb{N}}$ in green, $(u(y_n))_{n\in\mathbb{N}} = (0.5 - \frac{1}{n})_{n\in\mathbb{N}}$ in red and $(u(z_n))_{n\in\mathbb{N}} = (1.5 - \frac{1}{n})_{n\in\mathbb{N}}$ in yellow. In particular, it is assumed that $(y_n)_{n\in\mathbb{N}}$ converges to $u^{-1}(1) = a$, thus, u is discontinuous at point $a \in X$. According to Definition 4.1, the net $(x_n)_{n\in\mathbb{N}}$ is a 1-adjoint net of $(y_n)_{n\in\mathbb{N}}$, as well as $(y_n)_{n\in\mathbb{N}}$ is 1-adjoint net of $(z_n)_{n\in\mathbb{N}}$. Thus, $(x_n)_{n\in\mathbb{N}} \preccurlyeq^2 (z_n)_{n\in\mathbb{N}}$. Notice too that, according to Condition 2 of Definition 4.1, $(y_n)_{n\in\mathbb{N}}$ is a 1-adjoint net of the constant net $(u^{-1}(s))_{n\in\mathbb{N}}$.

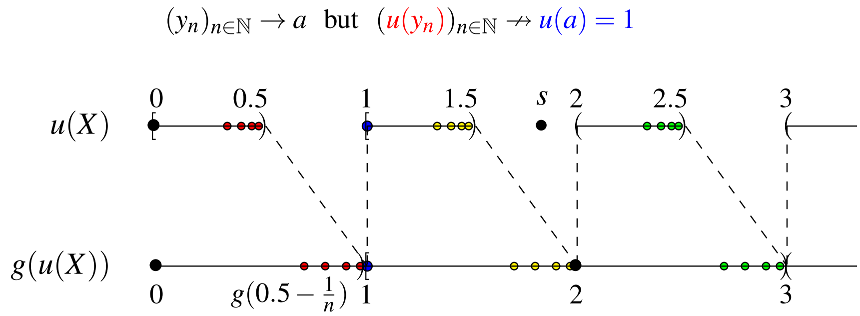

Fig. 4. Illustration of three adjoint nets by means of a representation $(u, 1)$ of a semiorder on X. Here, the image of three nets are drawn: $(u(y_n))_{n\in\mathbb{N}} = (0.5 - \frac{1}{n})_{n\in\mathbb{N}}$ in red, $(u(z_n))_{n\in\mathbb{N}} = (1.5 - \frac{1}{n})_{n\in\mathbb{N}}$ in yellow and $(u(w_n))_{n\in\mathbb{N}} = (2.5 - \frac{1}{n})_{n\in\mathbb{N}}$ in green. Again, $(y_n)_{n\in\mathbb{N}}$ converges to $u^{-1}(1) = a$, thus, u is discontinuous at point $a \in X$. Due to the existence of these adjoint nets, $(y_n)_{n\in\mathbb{N}} \preccurlyeq (z_n)_{n\in\mathbb{N}} \preccurlyeq (w_n)_{n\in\mathbb{N}}$, if we make changes on the interval $(0, 0.5)$ in order to avoid the discontinuity on $a = u^{-1}(1)$, these changes should be reproduced in the following intervals in order to keep the representation. In particular, if we stretch through g the interval $[0, 0.5)$ in order to avoid the jump discontinuity related to the gap $[0.5, 1)$, such that now $\lim_{n\to\infty} g(0.5 - \frac{1}{n}) = 1$, since $(y_n)_{n\in\mathbb{N}} \preccurlyeq (z_n)_{n\in\mathbb{N}}$, it holds that $\lim_{n\to\infty} g(u(y_n)) = \lim_{n\to\infty} g(1.5 - \frac{1}{n}) = 2$ whenever g keeps the representation. The same phenomenon holds with $(2.5 - \frac{1}{n})_{n\in\mathbb{N}}$, thus, the changes made on $[0, 1]$ due to the discontinuity of u at $a = u^{-1}(1)$ have implications on $[2, 3]$ too.

Similarly, for any $k_0 \in K$ there exists $j_0 \in J$ such that $y_{k_0} \precsim x_j$ for each $j > j_0$, so $u(y_{k_0}) \leq \lim(u(x_j)) + 1$ for each $k_0 \in K$. Hence, $\lim(u(y_k)) \leq \lim(u(x_j)) + 1$. So we have proved that $\lim(u(x_j)) + 1 = \lim(u(y_k))$.

If one of them is a constant net (suppose $y_k = b$ for all $k \in K$), then $x_j \prec b$ for any $j \in J$ and there exists $\lim(x_j) \in X$ such that $b \precsim \lim(x_j)$. So it holds that $\lim(u(x_j)) + 1 \leq \lim u(b) \leq (u(x_j)) + 1$. Hence, $\lim(u(x_j)) + 1 = \lim(u(y_k))$. Similarly if the case (2.*ii*) of Definition 4.1 holds.

Now, suppose that the lemma is true for a fixed $n \in \mathbb{N}$. If $(x_j)_{j\in J}$ and $(y_k)_{k\in K}$ are two $n+1$-adjoint nets, there exists another net $(z_r)_{r\in R}$ such that $(x_j) \preccurlyeq^n (z_r) \preccurlyeq (y_k)$. So, by the induction hypothesis, it holds that $\lim(u(x_j)) + n - 1 = \lim(u(z_r))$ and $\lim(u(z_r)) + 1 = \lim(u(y_k))$. Hence, it holds that $\lim(u(x_j)) + n = \lim(u(y_k))$.

We proceed analogously if one of them is a constant net. □

Remark 4.3. In fact, as proved in the proof, for any two n-adjoint nets $(x_j)_{j\in J}$ and $(y_k)_{k\in K}$ (connected by adjoint nets such that none of them is constant) it holds that $\lim_{j\in J} u(x_j) + n = \lim_{k\in K} u(y_k)$, even without requiring continuity for u.

Remark 4.4. By Lemma 4.2, there may be pair of nets such that their images converge to the same value on $\mathbb{R}$ when dealing with continuous representations, even without requiring any convergence condition on X.

For instance, given $(w_r)_{r\in R}$ and $(z_s)_{s\in S}$ two n-adjoint nets of $(x_j)_{j\in J}$ and $(y_k)_{k\in K}$, respectively, such that $(x_j)_{j\in J}$ and $(y_k)_{k\in K}$ converge to the same point, it is proved that $\lim u(w_r) = \lim u(z_s)$ for any continuous representation $(u, 1)$.

Remark 4.4 motivates the following definition.

Definition 4.5. Let (X, τ) be a topological space endowed with a continuously representable (in the sense of Scott and Suppes) semiorder $\prec$. Let $(x_j)_{j\in J}$ and $(y_k)_{k\in K}$ be two nets (one of them may be constant, i.e., $y_k = a$ for any $k \in K$) in X. We shall say that $(x_j)_{j\in J}$ and $(y_k)_{k\in K}$ *SS-converge* to each other (or that $(x_j)_{j\in J}$ *SS-converges* to a, for the constant case), and we denote it by $(x_j)_{j\in J} \leftrightarrow (y_k)_{k\in K}$, if $\lim u(x_j) = \lim u(y_k)$, for any continuous representation $(u, 1)$.

Definition 4.6. Let (X, τ) be a topological space endowed with a semiorder $\prec$ and $(u, 1)$ a representation. Let $(x_j)_{j\in J}$ and $(y_k)_{k\in K}$ be two nets that SS-converge to each other. Then, the interval $[\lim u(x_j), \lim u(y_k)]$ is said to be a *compression interval.*

Two compression intervals $[r_1, t_1]$ and $[r_2, t_2]$ (with $t_1 < r_2$) are said to be *contiguous* if $r_2 - r_1 \leq 1$ and $t_2 - t_1 \leq 1$.

Proposition 4.7. *Let (X, τ) be a topological space endowed with a continuously representable semiorder $\prec$. Let $(u, 1)$ be any Scott–Suppes representation and $[r, t]$ a compression interval. Then, $[r, t] \cap u(X)$ contains at most one point. Furthermore, given two contiguous compression intervals $[r_1, t_1]$ and $[r_2, t_2]$ (with $t_1 < r_2$) such that $s_1 = [r_1, t_1] \cap u(X)$ and $s_2 = [r_2, t_2] \cap u(X)$, then $s_2 \leq s_1 + 1$ is satisfied.*

Proof. If $[r, t]$ is a compression interval associated to a representation $(u, 1)$, assuming that it also represents the trace, then there are two nets $(x_i)_{i\in I}$ and $(y_j)_{j\in J}$ in X with $\lim u(x_i) = r$ and $\lim u(y_j) = t$ and such that $\lim v(x_i) = \lim v(y_j)$ for any continuous Scott–Suppes representation $(v, 1)$. Thus, for any $b, c \in X$ such that $r \leq u(b) \leq t$ and $r \leq u(c) \leq t$, since u and v also represent the trace, it holds that $\lim v(x_i) \leq v(b) \leq \lim v(y_j)$ as well as $\lim v(x_i) \leq v(c) \leq \lim v(y_j)$. Therefore, $v(b) = v(c)$, that is, $b \sim^0 c$ and, hence, $u(b) = u(c)$. So, we conclude that $[r, t] \cap u(X)$ contains at most one point.

Let $[r_1, t_1]$ and $[r_2, t_2]$ be (with $t_1 < r_2$) two contiguous compression intervals such that $u(c_1) = s_1 = [r_1, t_1] \cap u(X)$ and $u(c_2) = s_2 = [r_2, t_2] \cap u(X)$. Again, there are two pair of nets $(x_i)_{i\in I}$ and $(y_j)_{j\in J}$ and $(w_r)_{r\in R}$ and $(z_l)_{l\in L}$ in X with $\lim u(x_i) = r_1$ and $\lim u(y_j) = t_1$ and with $\lim u(w_r) = r_2$ and $\lim u(z_l) = t_2$

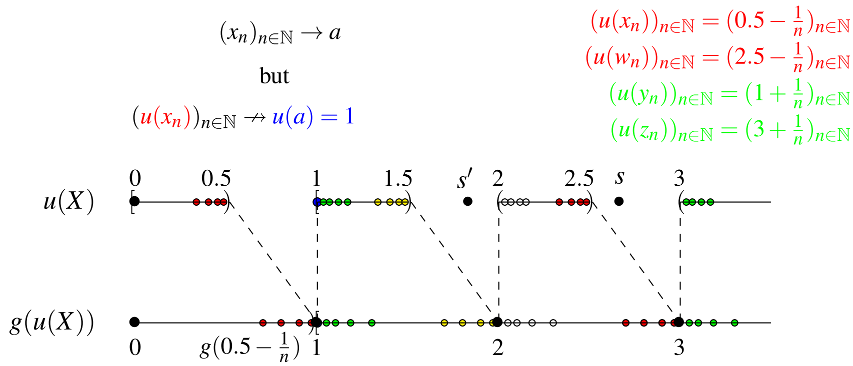

Fig. 5. This figure shows a Scott–Suppes representation $(u, 1)$ of a semiorder on X, such that u also represents the trace. There is a discontinuity at point $a = u^{-1}(1)$, since there is a sequence $(x_n)_{n\in\mathbb{N}}$ that converges to $a \in X$ but $(u(x_n))_{n\in\mathbb{N}} = (0.5 - \frac{1}{n})_{n\in\mathbb{N}}$ fails to converge to $u(a) = 1$. It is assumed that the sequence $(y_n)_{n\in\mathbb{N}}$ also converges to $a = u^{-1}(1)$. Thus, $(x_n)_{n\in\mathbb{N}} \leftrightarrow (y_n)_{n\in\mathbb{N}}$. There are two pairs of 2-adjoint nets, $(x_n)_{n\in\mathbb{N}} \precsim^2 (w_n)_{n\in\mathbb{N}}$ and $(y_n)_{n\in\mathbb{N}} \precsim^2 (z_n)_{n\in\mathbb{N}}$, such that $(x_n)_{n\in\mathbb{N}} \precsim (\alpha_n)_{n\in\mathbb{N}} \precsim (w_n)_{n\in\mathbb{N}}$ and $(y_n)_{n\in\mathbb{N}} \precsim (\beta_n)_{n\in\mathbb{N}} \precsim (z_n)_{n\in\mathbb{N}}$ (with $(\alpha_n)_{n\in\mathbb{N}} = (u^{-1}(1.5 - \frac{1}{n}))_{n\in\mathbb{N}}$ and $(\beta_n)_{n\in\mathbb{N}} = (u^{-1}(2 + \frac{1}{n}))_{n\in\mathbb{N}}$). Thus, $(x_n)_{n\in\mathbb{N}} \leftrightarrow (y_n)_{n\in\mathbb{N}}$, $(\alpha_n)_{n\in\mathbb{N}} \leftrightarrow (\beta_n)_{n\in\mathbb{N}}$ and $(w_n)_{n\in\mathbb{N}} \leftrightarrow (z_n)_{n\in\mathbb{N}}$. Therefore, $[0.5, 1]$, $[1.5, 2]$ and $[2.5, 3]$ are compression intervals, so that they may contain an unique point and always satisfying that the pairs of points belonging to contiguous intervals are indifferent.

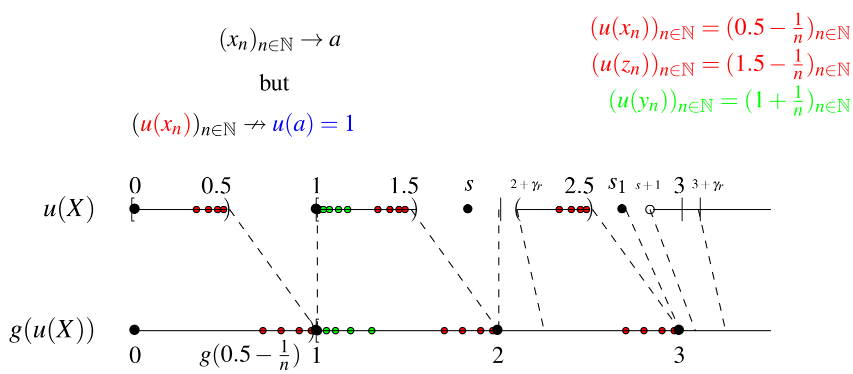

Fig. 6. This figure shows a Scott–Suppes representation $(u, 1)$ of a semiorder on X, such that u also represents the trace. There is a discontinuity at point $a = u^{-1}(1)$, since there is a sequence $(x_n)_{n\in\mathbb{N}}$ that converges to $a \in X$ but $(u(x_n))_{n\in\mathbb{N}} = (0.5 - \frac{1}{n})_{n\in\mathbb{N}}$ fails to converge to $u(a) = 1$. It is assumed that the sequence $(y_n)_{n\in\mathbb{N}}$ also converges to $a = u^{-1}(1)$. The nets (in red) $(x_n)_{n\in\mathbb{N}} \precsim^1 (z_n)_{n\in\mathbb{N}}$ are 1-adjoint nets, but also $(x_n)_{n\in\mathbb{N}} \precsim^1 (u^{-1}(s))_{n\in\mathbb{N}}$. Thus, it also holds true that $(z_n)_{n\in\mathbb{N}} \leftrightarrow (u^{-1}(s))_{n\in\mathbb{N}}$, so $[0.5, 1]$ and $[1.5, s]$ are compression intervals. Therefore, by condition (c), each interval $[1.5, 2]$ and $[2.5, s + 1]$ may contain – at most – one point of $u(X)$, which are (in this example) s and s_1, respectively. Finally, notice that according to the picture, $[2.5, s_1]$ is another compression interval.

such that $\lim v(x_i) = \lim v(y_j)$ and $\lim v(w_r) = \lim v(z_l)$for any continuous representation $(v, 1)$.

By contradiction, assume that $s_1 + 1 < s_2$. Then, $r_1 + 1 = \lim v(x_i) + 1 = v(c_1) + 1 < v(c_2) = \lim v(w_r) = r_2$ and, hence, $x_i \prec w_r$ for any $i \in I$ and any $r \in R$. Therefore, $r_1 + 1 = \lim u(x_i) + 1 < \lim u(w_r) = r_2$, so, the compression intervals fail to be contiguous. □

At this point, we are already able to introduce a conjecture just in order to show the objective of the present work.

Conjecture 4.8. *Let (X, τ) be a topological space endowed with a representable semiorder $\prec$ and let $(u, 1)$ be any Scott–Suppes representation. Then, the semiorder is continuously representable if and only if, each compression interval contains at most one point, and the pairs of points belonging to contiguous intervals are indifferent.*

However, it remains to be clarified how to detect those compression intervals, as we did in Lemma 3.3 and its corresponding Proposition 3.5. For that purpose, Lemma 4.9 and a corresponding family of four theorems are presented.

By means of the SS-convergence, we are able to generalize those necessary conditions $(a) - (d)$ for the existence of a continuous Scott–Suppes representation.[5] Through these conditions we identify the compression intervals (see Figs. 5, 6 and 7). Furthermore, from this Lemma 4.9, Proposition 4.7 arises now as a corollary.

Lemma 4.9. *Let (X, τ) be a topological space endowed with a semiorder $\prec$. Assume that $\prec$ is representable in the sense of Scott and Suppes by means of a pair $(u, 1)$ with u continuous. Suppose that the pair of nets $(x_j)_{j\in J}, (y_i)_{i\in I} \subseteq X$ SS-converge to each other. Then the following properties hold true:*

(a) *The semiorder $\prec$ is τ-continuous.*
(b) *If there are $a, b \in X$ such that $x_j \precsim^0 b \precsim^0 y_i$ and $x_j \precsim^0 a \precsim^0 y_i$ for any $i \in I$, $j \in J$, then $a \sim^0 b$.*
(c) *If there are $b, c \in X$ such that $x_j \prec b \precsim y_i$ and also $x_j \prec c \precsim y_i$ for every $j \in J$, $i \in I$, then $b \sim^0 c$. Furthermore, if there is $a \in X$ such that $x_j \precsim^0 a \precsim^0 y_i$ for any $i \in I$, $j \in J$, then $a \sim b$.*
(d) *If there are $b, c \in X$ such that $x_j \precsim b \prec y_i$ and also $x_j \precsim c \prec y_i$ for every $j \in J$, $i \in I$, then $b \sim^0 c$. Furthermore, if there is $a \in X$ such that $x_j \precsim^0 a \precsim^0 y_i$ for any $i \in I\, j \in J$, then $a \sim b$.*

Proof. The proof is reasoned as in the proof of Lemma 3.3 in Estevan et al. (2013a) but now, instead of argue on a net $(x_i)_{i\in I}$ that converges to a point a (so that $\lim u(x_i) = u(a)$), we have two nets $(x_i)_{i\in I}$ and $(y_j)_{j\in J}$ that SS-converge to each other (so, satisfying that $\lim u(x_i) = \lim u(y_j)$). □

[5] In fact, in Lemma 4.9, if the net $(x_j)_{j\in J}$ or $(y_i)_{i\in I}$ is the constant net defined by the element $a \in X$, then we recover the results presented in Lemma 3.3.

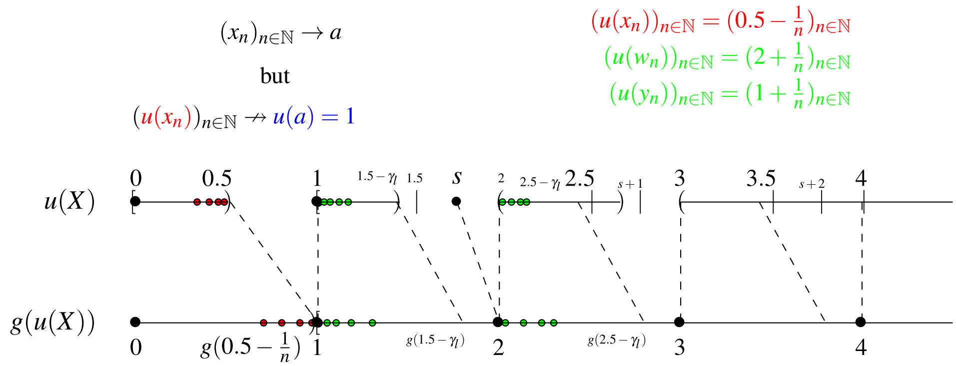

Fig. 7. This figure shows a Scott–Suppes representation $(u, 1)$ of a semiorder on X, such that u also represents the trace. There is a discontinuity at point $a = u^{-1}(1)$, since there is a sequence $(x_n)_{n\in\mathbb{N}}$ that converges to $a \in X$ but $(u(x_n))_{n\in\mathbb{N}} = (0.5 - \frac{1}{n})_{n\in\mathbb{N}}$ fails to converge to $u(a) = 1$. It is assumed that the sequence $(y_n)_{n\in\mathbb{N}}$ also converges to $a = u^{-1}(1)$. The nets (in green) $(y_n)_{n\in\mathbb{N}} \preccurlyeq^1 (w_n)_{n\in\mathbb{N}}$ are 1-adjoint nets, but also $(x_n)_{n\in\mathbb{N}} \preccurlyeq^1 (u^{-1}(s))_{n\in\mathbb{N}}$. Thus, since $(x_n)_{n\in\mathbb{N}} \leftrightarrow (y_n)_{n\in\mathbb{N}}$ it also holds true that $(u^{-1}(s))_{n\in\mathbb{N}} \leftrightarrow (w_n)_{n\in\mathbb{N}}$, so $[0.5, 1]$ and $[s, 2]$ are compression intervals. Therefore, by condition (c), each interval $[1.5, 2]$ and $[s + 1, 3]$ may contain – at most – one point of $u(X)$. Finally, notice that according to the picture, there are no more compression intervals.

Throughout the paper, we shall refer to all these necessary conditions (a)–(d) presented in Lemma 4.9 by (NC). We remark these conditions cannot be reduced even when working on the set $X/\sim^0$, in that case the only difference is that $a \sim^0 b$ means $a = b$. In the following family of theorems we summarize the structure of any (not necessarily continuous) unit representation of a semiorder that satisfies the necessary conditions (NC).

In order to describe the appearance of a Scott–Suppes representation of a semiorder that satisfies those necessary condition (NC), we present a family of four theorems (we call them *Gap Theorems for semiorders*) depending on:

Gap : The type of bad-gap we are focusing on:

- (CO) A closed-open gap $[r, u(a))$.
- (OC) An open-closed gap $(u(a), r]$.

Side : Which side we are looking at:

- (R) The right side: $(u(a), +\infty)$.
- (L) The left side: $(-\infty, u(a))$.

Starting from a gap $[r, u(a))$ generated by a discontinuity of u at $a \in X$, we can differentiate three zones as we move away (either to the right or to the left) from the gap. The *stable zone*, where there are nets attached in connection with the gap that cause the intervals $[r + n, u(a) + n]$ and $[r - n', u(a) - n']$ (with $n = 1, 2, \dots$ and $n' = 1, 2, \dots$) to compress to a point; the *inflection zone*, where one of the attached nets ceases due to the existence of a gap, but there are still compression intervals due to the existence of a number of points, and finally the *free zone*, where the discontinuity no longer has any effect (as regards to the necessary conditions).

Theorem 4.10 (*Closed-Open-Right Theorem*)**.** *Let $(u, 1)$ be a unit representation of a semiorder $\prec$ on (X, τ) that satisfies the necessary conditions (NC), and $S = u(X)$. Suppose that there is a discontinuity at a point $a \in X$ such that $[r, u(a))$ is a gap. Then:*

Stable zone: *If $\sup\{u(x) : u(x) < r+n\} = r+n$ and $\inf\{u(x) : u(x) > u(a)+n\} = u(a)+n$ for any $n \in \{1, \dots, m-1\}$ (for some $m \in \mathbb{N}$) or $n \in \mathbb{N}$, then $[r + n, u(a) + n] \cap S$ may contain one point s_n such that $s_n \leq s_{n-1} + 1$.*

Inflection zone: *If there exists m such that $\sup\{u(x) : u(x) < r + m\} = r + m - \gamma_l$ and $\inf\{u(x) : u(x) > u(a) + m\} = u(a) + m + \gamma_r$, with $\gamma_l, \gamma_r \geq 0$ and $\gamma_l + \gamma_r > 0$,*[6] *then $S \cap [r + m - \gamma_l, u(a) + m + \gamma_r]$ contains at most one point s_m.*

- *In case that point s_m exists:*
If $\gamma_l = 0$: Then $[r + m + 1, s_m + 1] \cap S$ may contain one point s_{m+1}, and so on with the following intervals $[r + n, s_n + 1]$ whenever $s_{n-1} \in S$ exists and $\sup\{u(x) : u(x) < r + n\} = r + n$ (with $n > m$), until arrive to a possible $m^* \in \mathbb{N}$ such that $[r+m^*, s_{m^*-1}+1] \cap S = \emptyset$ or $\sup\{u(x) : u(x) < r + n\} < r + n$. Then, the Inflection zone ends at $s_{m^*-1} + 1$. If $\gamma_r = 0$: Then $(s_m + 1, u(a) + m + 1] \cap S = \emptyset$, and so on with the following intervals $(s_n, u(a) + n]$ $(m < n \in \mathbb{N})$, whenever $s_n = s_m + (n - m) \in S$ exists as well as $\inf\{u(x) : u(x) > u(a) + n\} = u(a) + n$, until arrive to a possible $m^* \in \mathbb{N}$ such that $[s_{m^*}, u(a)+m^*] \cap S = \emptyset$ or $\inf\{u(x) : u(x) > u(a)+m^*\} > u(a)+m^*$. Then, the Inflection zone ends at $u(a) + m^*$.
- *If that point s_m does not exist or γ_r and γ_l are both bigger than 0, then the Inflection zone ends at $u(a) + m$.*

Proof. If there is a discontinuity at a point a such that $[r, u(a))$ is a gap, then there is a net $(u(y_i))_{i\in I}$ converging to r in $\mathbb{R}$ and there is another net $(u(x_j))_{j\in J}$ (it may be constant, i.e. $u(x_j) = u(a)$ for any $j \in J$) converging to $u(a)$ in $\mathbb{R}$.

First, in this Stable zone, notice that there exist n-adjoint nets (with $n < m$) $(z_t)_{t\in T}$ and $(w_r)_{r\in R}$ with $(y_i) \preccurlyeq^n (z_t)$ and $(x_j) \preccurlyeq^n (w_r)$ and such that they SS-converge to each other. Hence, by Lemma 4.2, $\lim u(w_r) = u(a) + n$ and $\lim u(z_t) = r + n$ and, furthermore, $[r + n, u(a) + n]$ is a compression interval. So, according to Proposition 4.7, $[r+n, u(a)+n] \cap S$ may contain one point s_n such that $s_{n+1} \leq s_n + 1$, and this holds true for each $n \in \mathbb{N}$ with $n < m$.

Secondly, in the Inflection zone, for that $m \in \mathbb{N}$ such that there exist $\gamma_l, \gamma_r \geq 0$ (with at least one of them different from 0) satisfying that $\sup\{u(x) : u(x) < r + m\} = r + m - \gamma_l$ or $\inf\{u(x) : u(x) > u(a) + m\} = u(a) + m + \gamma_r$, then from condition (c) of Lemma 4.9, it is deduced that $S \cap [a + m_r - \gamma_l, b + m_r + \gamma_r]$ contains at most one point s_m.

If that point $s_m = u(c_m)$ exists and $\gamma_l = 0$, then there is a net $(y_j)_{j\in J}$ such that $\lim u(y_j) = r + m$. Thus, notice that $[r + m, s_m]$ is a compression interval, since $(y_j)_{j\in J}$ SS-converge to c_m. Therefore, the following intervals $[r + n, s_n + 1]$ are also compression intervals whenever s_{n-1} exists as well as $\sup\{u(x) : u(x) < r+n\} = r+n$ is satisfied, and that holds true for any $n > m$ until arrive to a possible $m^* \in \mathbb{N}$ such that $[r + m^*, s_{m^*-1} + 1] \cap S = \emptyset$ or $\sup\{u(x) : u(x) < r + m^*\} < r + m^*$.

If that point $s_m = u(c_m)$ exists and $\gamma_r = 0$, then there is a net $(y_j)_{j\in J}$ such that $\lim u(y_j) = u(a) + m$. Thus, notice that $[s_m, u(a) + m]$ is a compression interval, since $(y_j)_{j\in J}$ SS-converge to c_m. Therefore, the next interval $[s_m + 1, u(a) + m + 1]$ contains, at most, the element image $s_{m+1} = s_m + 1 = u(c_{m+1})$ and, in case it exists and $\inf\{u(x) : u(x) > u(a) + m + 1\} = u(a) + m + 1$,

[6] In other words, with at least one of them different from 0.

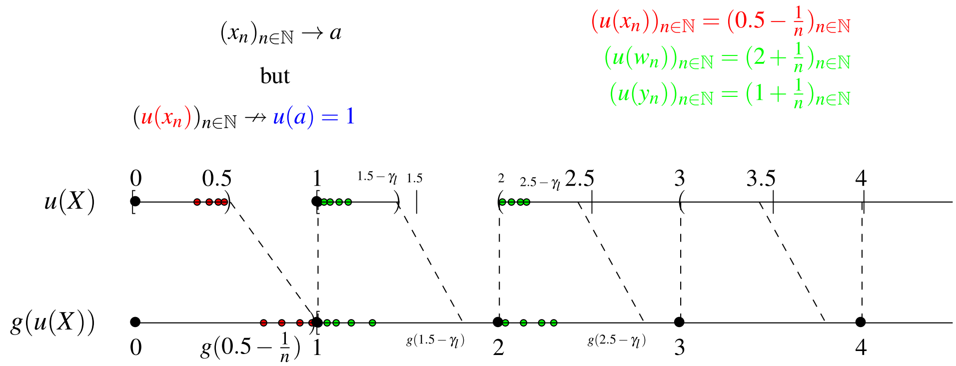

Fig. 8. This figure shows a Scott–Suppes representation $(u, 1)$ of a semiorder on X, such that u also represents the trace. There is a discontinuity at point $a = u^{-1}(1)$, since there is a sequence $(x_n)_{n\in\mathbb{N}}$ that converges to $a \in X$ but $(u(x_n))_{n\in\mathbb{N}} = (0.5 - \frac{1}{n})_{n\in\mathbb{N}}$ fails to converge to $u(a) = 1$. It is assumed that the sequence $(y_n)_{n\in\mathbb{N}}$ also converges to $a = u^{-1}(1)$. Since $(x_n)_{n\in\mathbb{N}} \leftrightarrow (y_n)_{n\in\mathbb{N}}$, $[0.5, 1]$ is a compression interval. But, notice that according to the picture, there are no more compression intervals.

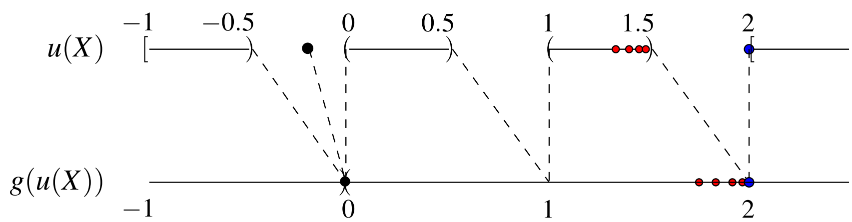

Fig. 9. Illustration of a Scott–Suppes representation of a semiorder $\prec$ on X satisfying the necessary conditions (a)–(d), and such that u also represents the trace. If u is discontinuous at $a = u^{-1}(2)$, such that $[1.5, 2)$ is a closed-open bad gap, then, by condition (a), there cannot be any point $b \in X$ such that $u(b) \in [0.5, 1)$, that is, $u(X) \cap [0.5, 1) = \emptyset$. If there is a point $u(s_1) = 1$, then the interval $[-0.5, 0]$ may contain at most the point $u(s_2) = 0$. But, notice that if $u(X) \cap [0.5, 1) = \emptyset$, then it is possible to construct a continuous Scott–Suppes representation even with the existence of a point $u(s)$ in $[-0.5, 0]$. In this figure we also illustrate how we could modify u (lengthening proportionally the intervals, through g) in order to avoid that discontinuity at $a = u^{-1}(2)$, but keeping the representation (i.e., such that $x \prec y$ if and only if $g(u(x)) + 1 < g(u(y))$, $x, y \in X$). For that purpose, notice that the intervals $[-0.5, 0)$ and $[0.5, 1]$ must be compressed to a point.

$[s_m + 1, u(a) + m + 1]$ is again a compression interval. This argument is repeated until an integer $m^* > m$ such that the point $s_{m^*} = s_{m^*-1} + 1$ in S fails to exists or until $\inf\{u(x) \colon u(x) > u(a) + m^*\} > u(a) + m^*$. □

Fig. 9 is intended to facilitate the understanding of the following theorem.

Theorem 4.11 (*Closed-Open-Left Theorem*). *Let $(u, 1)$ be a unit representation of a semiorder $\prec$ on (X, τ) that satisfies the necessary conditions (NC), and $S = u(X)$. Suppose that there is a discontinuity at a point a such that $[r, u(a))$ is a gap. Then:*

Stable zone: *If $\sup\{u(x) \colon u(x) < r - n\} = r - n$ and $\inf\{u(x) \colon u(x) > u(a) - n\} = u(a) - n$ for any $n \in \{1, \dots, m-1\}$ (for some $m \in \mathbb{N}$) or $n \in \mathbb{N}$, then $[r - n, u(a) - n] \cap S$ may contain one point s_n satisfying $s_n + 1 \geq s_{n-1}$ (in case s_{n-1} exists).*

Inflection zone: *If there exists that m such that $\sup\{u(x) \colon u(x) < r - m\} = r - m - \gamma_l$ and $\inf\{u(x) \colon u(x) > u(a) - m\} = u(a) - m + \gamma_r$, with $\gamma_l, \gamma_r \geq 0$ and $\gamma_l + \gamma_r > 0$, then $S \cap [r - m - \gamma_l, u(a) - m + \gamma_r]$ contains at most one point s_m.*

- *In case that point s_m exists:*

If $\gamma_l = 0$: Then $[r - m - 1, s_m - 1) \cap S = \emptyset$, and so on with the following intervals $([r - n - 1, s_n - 1])$ $(m < n \in \mathbb{N})$, whenever $s_n = s_m - (n - m) \in S$ exists as well as $\sup\{u(x) \colon u(x) < r - n\} = r - n$, until arrive to a possible $m^* \in \mathbb{N}$ such that $[r - m^*, s_{m^*}] \cap S = \emptyset$ or $\sup\{u(x) \colon u(x) < r - m^*\} < r - m^*$. Then, the Inflection zone ends at $r - m^*$.

If $\gamma_r = 0$: $[s_m - 1, u(a) - m - 1] \cap S$ may contain one point s_{m+1}, and so on with the following intervals $[s_n, u(a) - n]$ whenever $s_{n-1} \in S$ exists and $\inf\{u(x) \colon u(x) > u(a) - n\} = u(a) - n$ (with $n > m$), until arrive to a possible $m^* \in \mathbb{N}$ such that $[s_{m^*-1} - 1, u(a) - m^*] \cap S = \emptyset$ or $\inf\{u(x) \colon u(x) > u(a) - m^*\} > u(a) - m^*$. Then, the Inflection zone ends at $s_{m^*-1} - 1$.

- *If that point s_m does not exist or γ_r and γ_l are both bigger than 0, then the Inflection zone ends at $u(a) - m$.*

Proof. If there is a discontinuity at a point a such that $[r, u(a))$ is a gap, then there is a net $(u(y_i))_{i\in I}$ converging to r in $\mathbb{R}$ and there is another net $(u(x_j))_{j\in J}$ (it may be constant, i.e. $u(x_j) = u(a)$ for any $j \in J$) converging to $u(a)$ in $\mathbb{R}$.

First, in this Stable zone, notice that there exist n-adjoint nets (with $n < m$) $(z_t)_{t\in T}$ and $(w_r)_{r\in R}$ with $(z_t) \preccurlyeq^n (y_i)$ and $(w_r) \preccurlyeq^n (x_j)$ and such that they SS-converge to each other. Hence, by Lemma 4.2, $\lim u(w_r) = u(a) - n$ and $\lim u(z_t) = r - n$ and, furthermore, $[r - n, u(a) - n]$ is a compression interval. So, according to Proposition 4.7, $[r - n, u(a) - n] \cap S$ may contain one point s_n such that $s_{n+1} + 1 \geq s_n$, and this holds true for each $n \in \mathbb{N}$ with $n < m$.

Secondly, in the Inflection zone, for that $m \in \mathbb{N}$ such that there exist $\gamma_l, \gamma_r \geq 0$ (with at least one of them different from 0) satisfying that $\sup\{u(x) \colon u(x) < r - m\} = r - m - \gamma_l$ or $\inf\{u(x) \colon u(x) > u(a) - m\} = u(a) - m + \gamma_r$, then from condition (c) of Lemma 4.9, it is deduced that $S \cap [a - m_r - \gamma_l, b - m_r + \gamma_r]$ contains at most one point s_m.

If that point $s_m = u(c_m)$ exists and $\gamma_r = 0$, then there is a net $(y_j)_{j\in J}$ such that $\lim u(y_j) = u(a) - m$. Thus, notice that $[s_m, u(a) - m]$ is a compression interval, since $(y_j)_{j\in J}$ SS-converge to c_m. Therefore, the following intervals $[s_n - 1, u(a) - n]$ are also compression intervals whenever s_{n-1} exists as well as $\inf\{u(x) \colon u(x) > u(a) - n\} = u(a) - n$ is satisfied, and that holds true for any $n > m$ until arrive to a possible $m^* \in \mathbb{N}$ such that $[s_{m^*-1} - 1, u(a) - m^*] \cap S = \emptyset$ or $\inf\{u(x) \colon u(x) > u(a) - m^*\} > u(a) - m^*$.

If that point $s_m = u(c_m)$ exists and $\gamma_l = 0$, then there is a net $(y_j)_{j\in J}$ such that $\lim u(y_j) = r - m$. Thus, notice that $[r - m, s_m]$ is a compression interval, since $(y_j)_{j\in J}$ SS-converge to c_m. Therefore, the next interval $[r - m - 1, s_m - 1]$ contains, at most, the element image $s_{m+1} = s_m - 1 = u(c_{m+1})$ and, in case it exists and $\sup\{u(x): u(x) < r - m - 1\} = r - m - 1$, $[r - m - 1, s_m + 1]$ is again a compression interval. This argument is repeated until an integer $m^* > m$ such that the point $s_{m^*} = s_{m^*-1} - 1$ in S fails to exists or until $\sup\{u(x): u(x) < r - m^*\} < r - m^*$. □

The next two theorems are dually proved.

Theorem 4.12 (*Open-Closed-Left Theorem*)**.** *Let $(u, 1)$ be a unit representation of a semiorder $\prec$ on (X, τ) that satisfies the necessary conditions (NC), and $S = u(X)$. Suppose that there is a discontinuity at a point a such that $(u(a), r]$ is a gap. Then:*

Stable zone: *If $\sup\{u(x): u(x) < u(a) - n\} = u(a) - n$ and $\inf\{u(x): u(x) > r - n\} = r - n$ for any $n \in \{1, \ldots, m - 1\}$ (for some $m \in \mathbb{N}$) or $n \in \mathbb{N}$, then $[u(a) - n, r - n] \cap S$ may contain one point s_n such that $s_n + 1 \geq s_{n-1}$.*

Inflection zone: *If there exists that m such that $\sup\{u(x): u(x) < u(a) - m\} = u(a) - m - \gamma_l$ and $\inf\{u(x): u(x) > r - m\} = r - m + \gamma_r$, with $\gamma_l, \gamma_r \geq 0$ and $\gamma_l + \gamma_r > 0$, then $S \cap [u(a) - m - \gamma_l, r - m + \gamma_r]$ contains at most one point s_m.*

- *In case that point s_m exists:*

If $\gamma_r = 0$: Then $[s_m - 1, r - m - 1] \cap S$ may contain one point s_{m+1}, and so on with the following intervals $[s_n - 1, r - n - 1]$ whenever $s_{n-1} \in S$ exists and $\inf\{u(x): u(x) > r - n\} = r - n$ (with $n > m$), until arrive to a possible $m^* \in \mathbb{N}$ such that $[r - m^*, s_{m^*-1} - 1] \cap S = \emptyset$ or $\inf\{u(x): u(x) > r - n\} > r - n$. Then, the Inflection zone ends at $s_{m^*-1} - 1$. If $\gamma_l = 0$: Then $[u(a) - m - 1, s_m - 1) \cap S = \emptyset$, and so on with the following intervals $[u(a) - n, s_n)$ ($m < n \in \mathbb{N}$), whenever $s_n = s_m - (n - m) \in S$ exists as well as $\sup\{u(x): u(x) < u(a) - n\} = u(a) - n$, until arrive to a possible $m^* \in \mathbb{N}$ such that $[u(a) - m^*, s_{m^*}] \cap S = \emptyset$ or $\sup\{u(x): u(x) < u(a) - m^*\} < u(a) - m^*$. Then, the Inflection zone ends at $u(a) - m^*$.

- *If that point s_m does not exist or γ_r and γ_l are both bigger than 0, then the Inflection zone ends at $u(a) - m$.*

Theorem 4.13 (*Open-Closed-Right Theorem*)**.** *Let $(u, 1)$ be a unit representation of a semiorder $\prec$ on (X, τ) that satisfies the necessary conditions (NC), and $S = u(X)$. Suppose that there is a discontinuity at a point a such that $(u(a), r]$ is a gap. Then:*

Stable zone: *If $\inf\{u(x): u(x) > r + n\} = r + n$ and $\sup\{u(x): u(x) < u(a) + n\} = u(a) + n$ for any $n \in \{1, \ldots, m - 1\}$ (for some $m \in \mathbb{N}$) or $n \in \mathbb{N}$, then $[u(a) + n, r + n] \cap S$ may contain one point s_n satisfying $s_n \leq s_{n-1} + 1$ (in case s_{n-1} exists).*

Inflection zone: *If there exists that m such that $\inf\{u(x): u(x) > r + m\} = r + m + \gamma_r$ and $\sup\{u(x): u(x) < u(a) + m\} = u(a) + m - \gamma_l$, with $\gamma_l, \gamma_r \geq 0$ and $\gamma_l + \gamma_r > 0$, then $S \cap [u(a) + m - \gamma_l, r + m + \gamma_r,]$ contains at most one point s_m.*

- *In case that point s_m exists:*

If $\gamma_r = 0$: Then $(s_m + 1, r + m + 1] \cap S = \emptyset$, and so on with the following intervals $(s_n + 1, r + n + 1]$ ($m < n \in \mathbb{N}$), whenever $s_n = s_m + (n - m) \in S$ exists as well as $\inf\{u(x): u(x) > r + n\} = r + n$, until arrive to a possible $m^* \in \mathbb{N}$ such that $[s_{m^*}, r + m^*] \cap S = \emptyset$ or $\inf\{u(x): u(x) > r + m^*\} > r + m^*$. Then, the Inflection zone ends at $r + m^*$.

If $\gamma_l = 0$: $[u(a) + m + 1, s_m + 1] \cap S$ may contain one point s_{m+1}, and so on with the following intervals $[u(a) + n, s_n,]$ whenever $s_{n-1} \in S$ exists and $\sup\{u(x): u(x) < u(a) + n\} = u(a) + n$ (with $n > m$), until arrive to a possible $m^* \in \mathbb{N}$ such that $[u(a) + m^*, s_{m^*-1} + 1] \cap S = \emptyset$ or $\sup\{u(x): u(x) < u(a) + m^*\} < u(a) + m^*$. Then, the Inflection zone ends at $s_{m^*-1} + 1$.

- *If that point s_m does not exist or γ_r and γ_l are both bigger than 0, then the Inflection zone ends at $u(a) - m$.*

5. Debreu's open gap lemma with a threshold: Reasonable semiorders

Although the necessary conditions (a)–(d) collected in Lemma 3.3 may seem complicated or severe, they may make perfect sense from a decision maker point of view. Condition (a) (related to the continuity of the semiorder) means that (see Estevan et al., 2013a) if a net of alternatives $(x_i)_{i \in I}$ converges to a, then there is no point b such that $x_i \precsim b \prec a$ or $a \prec b \precsim x_i$, for all $i \in I$. In other words, if a decision maker may approach an alternative a by means of a sequence or net of alternatives $(x_i)_{i \in I}$ such that, in the limit, it is absolutely impossible to discern between them (topologically speaking), then it cannot exist another alternative b such that $x_i \precsim b \prec a$ or $a \prec b \precsim x_i$, for all $i \in I$. Thus, a coordination between topology and order is needed.

A similar situation holds with conditions (b), (c) and (d). Again, it seems *reasonable* to think that, if a decision maker may approach an alternative a by means of a sequence or net of alternatives $(x_i)_{i \in I}$, such that, in the limit, it is absolutely impossible to discern between them, then it cannot exist another alternative b such that $x_i \prec b \precsim a$ or $a \precsim b \prec x_i$, for all $i \in I$. Otherwise, there would be an incompatibility between the idea of proximity between alternatives (i.e. the topological space) and the preference of the decision maker. On the other hand, in this last case we will admit the possible existence of an unique limit or edge point b, since in that case, the existence of that unique (and only one) point is not incompatible with the existence of a continuous unit representation.

For instance, let us assume a customer interested in a car, where the variables to consider are price and color. Let us also assume that the price only affects the decision when the difference is greater than 50 euros. We may define then a sequence of red cars $\{r_1, r_2, \ldots\} = (r_n)_{n \in \mathbb{N}}$ with respective prices $(20.000 - \frac{50}{n})$. Thus, we may think that $(r_n)_{n \in \mathbb{N}}$ converges to the red car of 20.000 euros (denoted by r), as well as it seems reasonable to think that there cannot be a blue car b such that $r_n \precsim b \prec r$. We may think in a similar example with coffee, sugar and tea, in the spirit of Luce.

These remarks may be extended to the generalization made in Lemma 4.9.

These remarks on those conditions motivate the following Definition 5.1.

Definition 5.1. Let $\prec$ be a semiorder on a compatible topological space (X, τ). If it satisfies conditions (a)–(d) of Lemma 4.9, then we shall say that it is a *reasonable* semiorder, or *R-semiorder* for short. Otherwise, we say that it is a *non-reasonable* semiorder or *NR-semiorder*.

Definition 5.2. Let S be a subset of $\mathbb{R}$ and $\prec$ the usual semiorder on S defined by $x \prec y$ if and only if $x + 1 < y$, for any $x, y \in S$. We shall say that S is a *reasonable set*, or *R-set* for short, if $\prec$ satisfies conditions (a)–(d) of Lemma 4.9, that is, if S satisfies the conditions described in the Gap Theorems. Otherwise, we say that it is a *non-reasonable* set or a *NR-set*.

On the other hand, in order to simplify the proofs, in this section we shall argue on *irreducible* semiorders.

Definition 5.3. Let X be a nonempty set and $\prec$ a semiorder on X. We say that the semiorder is *irreducible* on X if there is no partition $A \cup B$ of X such that $a \prec b$ for any $a \in A$ and $b \in B$. This concept is also known as *connected with respect to the indifference* (the indifference relation of the semiorder) (Bouyssou & Pirlot, 2021a, 2021b).

When $\precsim$ is a total preorder, the pair (A, B) is also known as *ordered bipartition* (Bouyssou & Pirlot, 2021a).

Definition 5.4. Let X be a nonempty set and $\prec$ a semiorder on X. We say that the semiorder is *I-bounded* on X if there are no unbounded irreducible components. Also, we shall say that a real subset S is *I-bounded* as long as its usual semiorder on S is also I-bounded.

We reduce our study to those semiorders since any other one may be studied and represented through its irreducible components. In fact, given a representable semiorder $\prec$ on X such that $A \cup B$ is a partition of X satisfying that $a \prec b$ for any $a \in A$ and $b \in B$, if we know two unit representation $(u_1, 1)$ and $(u_2, 1)$ of $(A, \prec)$ and $(B, \prec)$ (respectively), then it is easy to construct a representation $(u, 1)$ on X as the following lemma shows.

Lemma 5.5. *Let $\prec$ be a representable semiorder on X such that $A \cup B$ is a partition of X satisfying that $a \prec b$ for any $a \in A$ and $b \in B$. Given two unit representations $(u_1, 1)$ and $(u_2, 1)$ of $(A, \prec)$ and $(B, \prec)$ (respectively), then the following function* $u(x) = \begin{cases} u_1(x) & ; \quad x \in A, \\ u_2(x) + m & ; \quad x \in B, \end{cases}$ *is a unit representation of $\prec$, where* $m = \sup u_1(A) - \inf u_2(B) + 2$.

Furthermore, if u_1 and u_2 are continuous, then the function u is a continuous unit representation if and only if $\prec$ is τ-continuous. Thus, A and B must be open.

Proof. Let us check that $x \prec y$ if and only if $u(x) + 1 < u(y)$, for any $x, y \in X$. First, notice that, since $a \prec b$ for any $a \in A$ and $b \in B$ and the semiorder is representable (in particular, regular), it can be concluded that $u_1(A)$ and $u_2(B)$ are bounded from above and below, respectively (otherwise we arrive to an absurd).

If $x, y \in A$ or $x, y \in B$, then it is trivial. If $x \in A$ and $y \in B$, then $x \prec y$, so it is enough to see that $u(x) + 1 < u(y)$. In fact, since $u_2(y) - \inf u_2(B) > 0$ and $\sup u_1(A) - u_1(x) > 0$, it holds true that $u(x) + 1 = u_1(x) + 1 \leq \sup u_1(A) + 1 < \sup u_1(A) + 2 + u_2(y) - \inf u_2(B) = u(y)$, for any $x \in A$ and any $y \in B$.

Second, with respect to the continuity, notice that, since $a \prec b$ for any $a \in A$ and $b \in B$ and the semiorder is representable, and hence, regular, there exist two elements $\overline{a} \in A$ and $\underline{b} \in B$ such that $A = L_{\prec}(\underline{b})$ and $B = U_{\prec}(\overline{a})$.

On one hand, the right implication of this second part is obvious, since τ-continuity is a necessary condition for the existence of a continuous Scott–Supper representation. Thus, A and B are open.

On the other hand, assume now that u_1 and u_2 are continuous as well as (since the semiorder is τ-continuous) A and B are open. As proved before, u is a unit representation, let us see that it is, in fact, a continuous representation. Given any $x \in X$ and any open neighborhood $U_x \in \tau$ of x, if $x \in A$ then $V_x = U_x \cap A \subseteq A$ is also an open neighborhood and, since u_1 is continuous, it holds that $u^{-1}(V_x) = u_1^{-1}(V_x) \in \tau$. Thus, u is continuous at point x. We argue dually if $x \in B$. Therefore, u is continuous and, hence, the semiorder is τ-continuous.

This concludes the proof. □

Hence, in order to simplify the present work and to avoid redundancies, from now, we shall assume that the semiorder studied is irreducible. Before we introduce our main results, we recover the following concept and proposition introduced in Estevan (2020).

Definition 5.6. Let (X, τ) be a topological space and $u: X \to \mathbb{R}$ a real function on X. Let $I = [a, b]$ be a bounded interval of the real line. A subset $\mathcal{C} = u(X) \cap I$ is said to be a *discontinuous Cantor set* if it satisfies the following properties:

(i) It has measure 0,
(ii) it has an infinite number of gaps,
(iii) every gap of $\mathcal{C}$ is a bad gap.

If there is a bounded interval I such that $\mathcal{C} = u(X) \cap I$ is a discontinuous Cantor set, then we will say that $u(X)$ *contains a discontinuous Cantor set*.

In the case $S \subseteq \mathbb{R}$, in an abuse of notation, we also say that S contains a discontinuous Cantor set if $i(S)$ contains a discontinuous Cantor set, where i is the inclusion function from $(S, \tau_<)$ to $(\mathbb{R}, \tau_<)$.

Remark 5.7. Notice that, given a discontinuous Cantor set $\mathcal{C} = I \cap u(X)$, then the sum of all the gaps of $\mathcal{C}$ is the length of the interval I.

The following Proposition 5.8 is already known Estevan (2020).

Proposition 5.8. *Let $\prec$ be a bounded semiorder on (X, τ). Let $(u, 1)$ be a unit representation. Then, there exists a maximal length gap and the gaps can be labeled in decreasing order $\{g_n\}_{n \in \mathbb{N}}$ of their length.*

Since we will argue on the length of gaps, we will use the concept of ϵ-continuity, introduced in Estevan (2020) for the first time. The concept of ϵ-continuity generalizes the idea of continuity for unit representations of semiorders and it is used for the proof of one the main theorems of the present work, the *Weak Theorem*.

Definition 5.9. Let $\prec$ be a semiorder on (X, τ). We shall say that the semiorder is *r-continuous* (for a positive value $r \in \mathbb{R}$) if there exists a unit representation $(u, 1)$ such that the length of each jump-discontinuity is strictly smaller than this constant r.

A semiorder is *ϵ-continuous* if for any $\epsilon > 0$ there exists a unit representation $(u_\epsilon, 1)$ such that the length of each jump-discontinuity is strictly smaller than the value ϵ.

This concept is weaker than the usual continuity and it is essential for the proof of Theorem 5.11. In Estevan (2020) it is shown that necessary conditions (NC) for the usual continuity are not needed for the existence of an ϵ-continuous unit representation. Thus, if a semiorder has a continuous unit representation, then it is ϵ-continuous, however, there exist ϵ-continuous semiorders that fail to be continuously representable. Furthermore, there exist semiorders that fail to be r-continuously representable, for a given $r > 0$ (with $r \leq 1$) (Estevan, 2020).

Now, we are ready to present our main theorems. The proofs of Theorems 5.10, 5.11 and 5.12 are included in Appendix A. Theorem 5.11 is just the continuity of Theorem 5.10.

First, we introduce the weakest one, which is interesting because the proof is constructive in a finite number of steps and, hence, it is programmable. The proofs of Theorems 5.10 and 5.11 consist of stretching segments of non-zero size to eliminate the largest of the gaps. For this method to work, we must guarantee that after stretching these segments, no successive gaps of greater or equal size will arise, moreover, the length of these will have to tend to 0. To make this happen, it is necessary and sufficient that there are no discontinuous Cantor sets. However, in Theorem 5.12 the technique used is different, and instead of stretching segments we use the Open Gap Lemma. Therefore, the absence of Cantor sets is not required here.

Again, we assume that the topology is compatible with the indifference.

Theorem 5.10 (*The Weakest Theorem*)**.** *Let $\prec$ be a Scott–Suppes representable and bounded semiorder on a compatible topological space (X, τ) and $(u, 1)$ a unit representation. If $\prec$ is reasonable, the main trace is continuous and there is no discontinuous Cantor set contained in $u(X)$, then it is ϵ-continuously representable.*

Furthermore, we may continue the finite process started in the proof of Theorem 5.10, applying infinite steps in the proof. Hence, it is possible to construct a sequence $\{u_n\}_{n\in\mathbb{N}}$ of $\frac{1}{n}$-continuous unit representations that converges to a limit u. The following theorem shows that this limit exists as well as it being, in fact, a continuous unit representation.

Theorem 5.11 (*The Weak Theorem*)**.** *Let $\prec$ be a Scott–Suppes representable and bounded semiorder on a compatible topological space (X, τ) and $(u, 1)$ a unit representation. If $\prec$ is reasonable, the main trace is τ-continuous and there is no discontinuous Cantor set contained in $u(X)$, then $\prec$ is continuously representable.*

Now, we present the *Strong Theorem*, where Debreu's Open Gap Lemma is needed for the proof. Here, the absence of Cantor subsets is not required.

Theorem 5.12 (*The Strong Theorem*)**.** *Let $\prec$ be a Scott–Suppes representable and bounded semiorder on a compatible topological space (X, τ). The semiorder $\prec$ is reasonable and the main trace is τ-continuous if and only if $\prec$ is continuously Scott–Suppes representable.*

By Lemma 5.5, Theorems 5.11 and 5.12 may be generalized to I-bounded semiorders as follows.

Corollary 5.13 (*The Weak Corollary*)**.** *Let $\prec$ be a Scott–Suppes representable and I-bounded semiorder on a topological space (X, τ) and $(u, 1)$ a unit representation. Assume that there is no discontinuous Cantor set contained in $u(X)$. Then, the semiorder is reasonable and the main trace is τ-continuous if and only if $\prec$ is continuously representable.*

Corollary 5.14 (*The Strong Corollary*)**.** *Let $\prec$ be a Scott–Suppes representable and I-bounded semiorder on a compatible topological space (X, τ). Then, the semiorder is reasonable and the main trace is τ-continuous if and only if $\prec$ is continuously representable.*

Then, we could conclude the following result, which is directly deduced from Theorem 5.12 and that we present as a *Debreu's Open Gap Lemma with a Threshold*.

Corollary 5.15 (*Debreu's Open Gap Lemma with a Threshold*)**.** *Let S be a subset of $\mathbb{R}$. Then, there exists a $\prec^0$-strictly increasing*[7] *function $g: S \to \mathbb{R}$ such that all the gaps of $g(S)$ are open or closed, and satisfying that $x+1<y \iff g(x)+1<g(y)$ if and only if S is a R-set.*

Proof. Let $\prec$ be the usual semiorder defined on S by $x \prec y$ if and only if $x+1 < y$, for any $x, y \in S$. Then, the inclusion function $i: S \to \mathbb{R}$ with the constant $k = 1$ is a Scott–Suppes representation of $\prec$.

Let us focus on the right implication. First, here we assume that if g is $\prec^0$-strictly increasing then it holds that $\{x \prec^0 y \implies g(x) < g(y)\}$ as well as $\{x \sim^0 y \implies g(x) = g(y)\}$, $x, y \in S$. Thus, g represents the trace. On the other hand, if all the gaps of $g(S)$ are open or closed, then this means (by Proposition 2.12) that g is continuous with respect to the order topology $\tau_{\prec^0}$ generated by the trace. And if $x+1<y \iff g(x)+1<g(y)$ is also satisfied, then g is in fact a continuous Scott–Suppes representation of the semiorder $\prec$ defined on S. Thus, by Lemma 4.9, we conclude that S is a R-set.

Now, we focus on the left implication. Since conditions (a)–(d) of Lemma 4.9 that define a R-set are given by means of the semiorder and its trace, notice that S is a R-set if and only $S/\sim^0$ is a R-set. We denote by j the quotient function, that is, $j(S) = S/\sim^0$. Thus, the semiorder $\prec$ defined on $S/\sim^0$ by $\bar{x} \prec \bar{y}$ if and only if $x+1<y$ (for any $\bar{x}, \bar{y} \in S/\sim^0$) is a reasonable semiorder. Furthermore, notice that on the quotient $S/\sim^0$ it holds that $\bar{x} \prec^0 \bar{y}$ if and only if $\bar{x} < \bar{y}$, $\bar{x}, \bar{y} \in S/\sim^0$. Therefore, if we endow $S/\sim^0$ with the order topology $\tau = \tau_{\prec^0}$ generated by the trace, then $\prec^0$ is trivially τ-continuous and – by Corollary 5.14 – there exists a continuous unit representation, let us call it $(v, 1)$. Thus, since v also represents the trace, all the gaps of $v(S/\sim^0)$ are open or closed (see Proposition 2.12) as well as $\bar{x}+1<\bar{y} \iff v(\bar{x})+1 < v(\bar{y})$ is satisfied, for any $\bar{x}, \bar{y} \in S/\sim^0$. Then, $g = v \circ j$. □

This section is closed with the following *Representation Theorem for Semiorders*, now given as a corollary, and enunciated in the spirit of *Debreu's Representation Theorem for Total Preorders*.

Corollary 5.16 (*First Continuous Representation Theorem for Semiorders*)**.**

Let $\mathcal{S}$ be the space of I-bounded semiordered sets endowed with a compatible topology. Then,

$\prec \in \mathcal{S}$ *is a representable R-semiorder if and only if*

$\prec \in \mathcal{S}$ *is continuously representable.*

As Corollary 5.17 shows, our results are still valid for any topological space, however, without assuming the compatibility, the continuity of the trace is a sufficient condition, but may fail to be necessary (see Example 2 and Example 3 in Bosi et al., 2015).

Corollary 5.17 (*Second Continuous Representation Theorem for Semiorders*)**.** *Let $\overline{\mathcal{S}}$ be the space of I-bounded semiordered sets endowed with a topology such that the main trace is continuous. Then,*

$\prec \in \overline{\mathcal{S}}$ *is a representable R-semiorder if and only if*

$\prec \in \overline{\mathcal{S}}$ *is continuously representable.*

Proof. Under the assumption of the continuity of the main trace, the proof of Theorem 5.12 (as well as Theorem 5.11 when adding the absence of Cantor sets) is still valid. For this proof we may also reason on the quotient set $X/\sim^0$, which satisfies the hypothesis of the theorems, in particular, the quotient topology is always compatible and the main trace is continuous as long as it is in the initial space (X, τ). □

6. Some applications of the present study

6.1. Continuous Scott–Suppes representability of semiorders

In the present work we focused on semiorders and, as a byproduct, we achieved an Open Gap Lemma with a Threshold. However, it is also possible to go the other way around, using this new lemma for the construction of a continuous Scott–Suppes representation. We collect these thoughts in the following corollaries.

Corollary 6.1. *Let $\prec$ be an I-bounded semiorder on $(X, \tau_{\prec^0})$, where $\tau_{\prec^0}$ denotes the topology generated by the main trace. Then, there exists a continuous unit representation $(v, 1)$ of $\prec$ if and only if, for any unit representation $(u, 1)$, $S = u(X)$ is a R-set.*

In that case, there exists a function g such that $(v = g \circ u, 1)$ is a continuous unit representation.

If the semiorder is defined on a compatible topological space finer than $\tau_{\prec^0}$, then there may be open-closed or closed-open gaps where the function u of the unit representation does not fail to be continuous. Thus, when checking the conditions of the Gap Theorems for a subset $u(X) \subseteq \mathbb{R}$, we have to interpret those open-closed and closed-open gaps as those gaps of $u(x)$ that imply a discontinuity, i.e., bad gaps.

[7] Here, we assume that $\{x \prec^0 y \implies g(x) < g(y)\}$ as well as $\{x \sim^0 y \implies g(x) = g(y)\}$, $x, y \in S$.

Corollary 6.2. *Let $\prec$ be an I-bounded and Scott–Suppes representable semiorder on a compatible topological space (X, τ). Then, there exists a continuous unit representation $(v, 1)$ of $\prec$ if and only if*

(i) the main trace is τ-continuous,
(ii) for any (not necessarily continuous) unit representation $(u, 1)$, $S = u(X)$ is a R-set.

If these conditions are satisfied, then there exists a function g such that $(g \circ u, 1)$ is a continuous unit representation.

6.2. A constructive weak version of Debreu's Open Gap Lemma

In the present work we also achieve as a byproduct a proof of a *weak* version of *Debreu's Open Gap Lemma*. We referred to it as *weak* because the absence of discontinuous Cantor set is assumed. But, it has a big remarkable benefit: the method is constructive, and finite when working with ϵ-continuity.

Although the notion of ϵ-continuity was defined for semiorders, it may be generalized to other kinds of representations of orderings under some adequate hypothesis. For example, in the case of total preorders or interval orders, assuming – without loss of generality – that the infimum and supremum of the representation are 0 and 1 -or any other values-, respectively. Hence, the length of the biggest jump-discontinuity may be compared with the diameter (or any other invariant) of the image of the representation.

Corollary 6.3. *Let S be a subset of the extended real line $\overline{\mathbb{R}}$. Assume that there is no discontinuous Cantor set contained in S. Then, there exists a strictly increasing function $g : S \to \mathbb{R}$ such that all the gaps of $g(S)$ are open or closed.*

Furthermore, this function g may be built as a composition of a family (may be infinite) of linear 2-piecewise functions. As a matter of a fact, for any $\epsilon > 0$, this function g can be constructed (in a finite number of steps) by a composition of a finite family of linear 2-piecewise functions, satisfying that the length of the bad gaps of $g(S)$ is less than ϵ.

Proof. First, since $\overline{\mathbb{R}}$ is homeomorphic to $[0, 1]$ (see Bridges & Mehta, 1995), without loss of generality we may assume that $S \subseteq [0, 1]$, with $\sup S = 1$ and $\inf S = 0$.

By Proposition 5.8, there is a maximal open-closed or closed-open gap (we shall refer to them as *bad gaps*): G_1. Let $\{\delta_n\}_{n\in\mathbb{N}}$ be the sequence of lengths corresponding to the bad gaps $\{G_n\}_{n\in\mathbb{N}}$ (ordered from biggest to smaller, see Proposition 5.8).

Now, focusing on G_1, we will construct a function f_1 on S that will remove this gap. We denote G_1 by $G_1 = [a_1, b_1)$ or $G_1 = (a_1, b_1]$. We define the following strictly increasing function:

$$f_1(x) = \begin{cases} x \cdot \frac{1}{1-\delta_1} & ; \quad x \leq a_1, \\ (x - \delta_1) \cdot \frac{1}{1-\delta_1} & ; \quad x \geq b_1, \end{cases}$$

Notice that f_1 keeps the length of S, i.e. $\sup f_1(S) - \inf f_1(S) = 1$. We repeat the process with the next gap, $f_1(G_2)$, (we may denote $f_1(G_2)$ by $[a_2, b_2)$ or $(a_2, b_2]$.

Therefore, we may define the sequence $\{l_n\}_{n\in\mathbb{N}}$ of lengths associated to the biggest bad gap after applying each functions $f_1, \dots, f_n$ on S, respectively, i.e. l_{n+1} denotes the length of the biggest gap of $f_n \circ \cdots \circ f_1(S)$ which corresponds to $f_n \circ \cdots \circ f_1(G_{n+1})$. This sequence is defined recursively as follows:

$$l_1 = \delta_1$$
$$l_2 = \delta_2 \cdot \frac{1}{1-\delta_1} = \frac{\delta_2}{1-l_1}$$
$$l_3 = \delta_3 \cdot \frac{1}{1-\delta_1} \cdot \frac{1}{1-\frac{\delta_2}{1-\delta_1}} = \frac{\delta_3}{1-\delta_1-\delta_2} = \frac{\delta_3}{(1-l_1)\cdot(1-l_2)}$$

...

So, it may be proved by induction that the length of the biggest gap after $n-1$ steps is

$$l_n = \frac{\delta_n}{1-\sum_{k=1}^{n-1}\delta_k} = \frac{\delta_n}{\Pi_{k=1}^{n-1}(1-l_k)}.$$

Thus, since there are no discontinuous Cantor sets, the sum $\sum_{k=1}^{+\infty}\delta_k$ is strictly smaller than 1, and $\{\delta_n\}_{n\in\mathbb{N}}$ tends to 0 when n tends to infinity, so we conclude that $\{l_n\}_{n\in\mathbb{N}}$ converges to 0 letting n tend to infinity.

Therefore, we have constructed a sequence of strictly increasing functions $\{g_n\}_{n\in\mathbb{N}}$, where $g_n = f_n \circ \cdots \circ f_1$, such that g_n is ϵ_n-continuous, for a family of positive values $\{\epsilon_n\}_{n\in\mathbb{N}}$ such that ϵ_n tend to 0 when n increases. As a matter of fact, since the sequence $\{\delta_n\}_{n\in\mathbb{N}}$ is decreasing and converges to 0, notice that the sequence of functions $\{g_n\}_{n\in\mathbb{N}}$ is a pointwise Cauchy sequence. To see that, notice that – since $\{\delta_n\}_{n\in\mathbb{N}}$ is decreasing and converges to 0 – the sequence of functions $\{f_n\}_{n\in\mathbb{N}}$ converge to the identity function, and since $g_{n+1} = f_{n+1} \circ g_n$, the Cauchy property is deduced. Hence, the limit function g of $\{g_n\}_{n\in\mathbb{N}}$ exists.

Let us see that g is continuous as well as strictly increasing.

On one hand, for any $\epsilon > 0$, there is $n_\epsilon \in \mathbb{N}$ such that the length of the bad gaps of $g_n(S)$ are smaller than ϵ, for any $n \geq n_\epsilon$. Hence, the bad gaps of $g(S)$ have no positive length, i.e. there are no bad gaps in $g(S)$.

On the other hand, g is still strictly increasing. To see that, first notice that if two point $x, y \in S$ are in the same side of a gap, i.e. $x, y \leq a_1$ or $x, y \geq b_1$ for a gap $G_1 = [a_1, b_1)$ or $G_1 = (a_1, b_1]$, then after applying f_1 the distance between x, y increases such that $d(f_1(x), f_1(y)) = \frac{d(x,y)}{1-\delta_1}$, where $1 > \delta_1 > 0$ is the length of the gap. The distance between x and y is reduced just in case $x \leq a_1$ and $y \geq b_1$ (or vice versa). In that case, it holds that $d(f_1(x), f_1(y)) = \frac{1}{1-\delta_1}\cdot(d(x, y) - \delta_1)$. It can be proved that after repeating this contraction process n times (i.e. after applying function $g_n = f_n \circ \cdots \circ f_1$) the distance achieved is

$$d(g_n(x), g_n(y)) = \frac{1}{1-\sum_{k=1}^{n}\delta_k}\cdot(d(x, y) - \sum_{k=1}^{n}\delta_k).$$

In fact, we may argue by induction and assume that after n steps, the distance is as described before. Hence, in the next step, for $n+1$, the distance would be

$$d(g_{n+1}(x), g_{n+1}(y)) = \frac{1}{1-l_{n+1}}\cdot(d(g_n(x), g_n(y)) - l_{n+1}),$$

where l_{n+1} is the length of the biggest gap between $g_n(x)$ and $g_n(y)$ and that comes from the transformation of the initial gap of length δ_n through the n previous steps. Hence, l_{n+1} is as described before, $l_{n+1} = \frac{\delta_{n+1}}{1-\sum_{k=1}^{n}\delta_k}$. Therefore, replacing l_{n+1} by $\frac{\delta_{n+1}}{1-\sum_{k=1}^{n}\delta_k}$ in the equation:

$$d(g_{n+1}(x), g_{n+1}(y)) = \frac{1}{1-\frac{\delta_{n+1}}{1-\sum_{k=1}^{n}\delta_k}}\cdot(d(g_n(x), g_n(y)) - \frac{\delta_{n+1}}{1-\sum_{k=1}^{n}\delta_k}).$$

Finally, after replacing the value $d(g_n(x), g_n(y)) = \frac{1}{1-\sum_{k=1}^{n}\delta_k}\cdot(d(x, y) - \sum_{k=1}^{n}\delta_k)$ and simplifying, we achieve the desired result:

$$d(g_{n+1}(x), g_{n+1}(y)) = \frac{1}{1-\sum_{k=1}^{n+1}\delta_k}\cdot(d(x, y) - \sum_{k=1}^{n+1}\delta_k).$$

Thus, given any two points $x < y$ in S, in the limit – i.e. after applying function g – the distance between $g(x)$ and $g(y)$ is at least, as big as

$$\frac{1}{1-\sum_{k=1}^{+\infty}\delta_k}\cdot(d(x, y) - \sum_{k=1}^{+\infty}\delta_k).$$

By hypothesis, there is no discontinuous Cantor set contained in S, hence, $\sum_{k=1}^{+\infty}\delta_k$ is strictly smaller than $d(x, y)$ (and, in particular, strictly smaller than 1). Thus, $g(x) < g(y)$ for any $x < y$. This concludes the proof. □

Remark 6.4. Given $\epsilon_0 > 0$, the previous proof can be limited to a finite number of steps just in order to achieve ϵ_0-continuity through a constructive and finite process. This may be interesting for programming purposes, as it is shown in Appendix B.

7. Concluding remarks

In the present paper we focused on semiorders and proved a characterization of the existence of a continuous Scott–Suppes representation for a big family of semiorders, that also includes the bounded semiorders. A version of Debreu's Open Gap Lemma with a threshold is achieved, so that given a S in $\mathbb{R}$, now we know when a strictly increasing function g exists such that $x+1 < y$ if and only if $g(x)+1 < g(y)$ and satisfying that now the gaps are not bad gaps. Thus, we almost completely close a problem that has been open since in 1956 the notion of a semiorder was introduced by Luce (1956).

Some other proof for these results may exist, in particular, by induction on the length of $u(X)$ (that is, on the length of the longest chain $x_1 \prec x_2 \prec \cdots \prec x_l$) and aggregating continuous representations $u_1 : (X_1, \tau_{|X_1}) \to (\mathbb{R}, \tau_u)$ and $u_2 : (X_2, \tau_{|X_2}) \to (\mathbb{R}, \tau_u)$ of a semiorder on (X, τ), with $X = X_1 \cup X_2$, in the line of Bouyssou and Pirlot (2021a, 2021b). However, it does not seem easy at all to know how to make this aggregation.

The case of unbounded and irreducible semiorders remains to be solved, which would bring the solution to the general case. Thus, Conjecture 4.8 is open for the general case.

This work also includes a programming tool (which was used by the author in order to check his calculations) that constructs ϵ-continuous utilities for total preorders. It can be complemented for semiorders in order to construct ϵ-continuous Scott–Suppes representations.

Several applications of the results obtained are expected to be published shortly.

Acknowledgments

Asier Estevan acknowledges financial support from the Ministry of Science and Innovation of Spain under grants PID2020-119703RB-I00 and PID2021-127799NB-I00 as well as from the UPNA, Spain under grant JIUPNA19-2022. Thanks are given to Denis Bouyssou for his comments and suggestions, as well as to Efe Ok and José Carlos R. Alcantud for their recommendations. I am also grateful to my thesis supervisors Esteban Induráin and Javier Gutiérrez-García.

I also thank the helpful work of the anonymous referees and the editor, to whom I am indebted for the detailed reading of previous versions of the manuscript and their very helpful suggestions and comments.

Appendix A. Proofs

Proof of Theorem 5.10

Proof. Let $\prec$ be a Scott–Suppes representable and bounded semiorder on a compatible topological space (X, τ), such that the trace is τ-continuous. Let $(u, 1)$ be a Scott–Suppes representation such that u also represents the trace. By Proposition 5.8, there is a maximal gap G_1 that generates a discontinuity at a point a_1. Let δ_1 be the length of G_1. By Corollary 2.13, G_1 is of the form $(u(a_1), r]$ or $[r, u(a_1))$. Without loss of generality, we may suppose that $G_1 = [r, u(a_1)) \subseteq (0, 1]$ with $u(a_1) = 1$ and $r = 1-\delta_1$, where $1 > \delta_1 > 0$ is the length of the jump-discontinuity (it would be proved dually for $(u(a_1), r]$), otherwise we should move the set by a function $t_1(x) = x + 1 - u(a_1)$.

Since $u(X)$ is bounded, we may divide it in T unit subintervals, such that $u(X) \subseteq I_{-M+1} \cup I_{-M+2} \cup \cdots \cup I_1 \cup \cdots \cup I_N$, with $M+N = T$ and such that $[r, u(a_1)) = [r, 1) \subseteq [0, 1] = I_1$.

Now, we focus on $I_1 = [0, 1]$, where the biggest gap $G_1 = [r, u(a_1))$ was found, and we denote the decreasing family of the lengths of the bad gaps in I_1 by $(\delta_n^1)_{n\in\mathbb{N}}$. First, we will construct a piecewise function f_1^1 on $\mathbb{R}$ (that is strictly increasing on $u(X)$) that will remove this gap G_1 of length δ_1^1, and repeat the process for the successive gap lengths $\delta_1^1, \ldots, \delta_{n_0}^1$, for some $n_0 \in \mathbb{N}$.[8]

After finishing the work on this subset I_1, we will repeat the process but now focusing on another unit interval I_{i_2} related to the next biggest bad gap. We will denote the decreasing family of the lengths of bad gaps in I_{i_2} (with respect to the initial function u) by $(\delta_n^{i_2})_{n\in\mathbb{N}}$, and so on, so that $(\delta_n^{i_k})_{n\in\mathbb{N}}$ denotes the decreasing family of the lengths of bad gaps in I_{i_k}, for each $k = 1, \ldots, T$.

In case of a bad gap $[k-\delta_l, k+\delta_r)$ or $(k-\delta_l, k+\delta_r]$ which is in the middle of two of those unit intervals I_k and I_{k-1}, we will consider it in our algorithm as two consecutive bad gaps $[k-\delta_l, k)$ and $[k, k+\delta_r)$ (dually, $(k-\delta_l, k]$ and $(k, k+\delta_r]$), so that δ_l and δ_r are elements of the sequences of lengths of $(\delta_n^{k-1})_{n\in\mathbb{N}}$ and $(\delta_n^k)_{n\in\mathbb{N}}$, respectively.

Coming back to I_1, firstly, we define the corresponding subfunctions $\lambda_1^1, \lambda_2^1, \lambda_3^1$ and c^1, which are linear functions that will be applied adequately in each threshold interval in order to keep the rigid structure of the semiorder, that is, in order to achieve another unit representation (but now without the gap G_1).

By the Gap Theorems, if $[r+1, u(a_1)+1]$ or $[r+1, u(a_1)+1)$ is a gap, then $u(X) \cap [r+2, u(a_1)+2]$ has at most one point s, and we will continue applying the reasoning corresponding to the stable zone (see Fig. 5), until arriving to a $m_r \in \mathbb{N}$ such that $u(X) \cap [r+m_r-\gamma_l', u(a_1)+m_r+\gamma_r']$ (with $\gamma_l', \gamma_r' \geq 0$ and, at least, one of them positive) has at most one point (as explained in the inflection zone, see Figs. 6, 7, 8, 10 and 11). We define the following *expansion* function:

1. $\lambda_1^1(x) = (x-n)\cdot\frac{1}{1-\delta_1^1} + n$, $x \in [n, n+r]$, $n \in \mathbb{N}$ with $0 < n < m_r$ (see Fig. 5),
2. $\lambda_2^1(x) = (x-n)\cdot\frac{1}{1-\delta_1^1} + n$, $x \in [1+n+\gamma_r', 1+n+r-\gamma_l']$, $n \in \mathbb{N}$ with $n \geq m_r$ (see Figs. 6, 7 and 8),
3. $\lambda_3^1(x) = n+1$, $x \in (n+r, n+1)$, $n \in \mathbb{N}$ with $0 < n < m_r$ (see Fig. 5).

Remember that in $[n+r, n+1]$ (for $0 < n < m_r$) there is – at most – one point s (see the Gap Theorems), so λ_3^1 does not imply a contraction on $u(X)\cap(n+r, n+1)$. And the *contraction* function on $[r+n-\gamma_l', n+1+\gamma_r']$, $n \in \mathbb{N}$ with $n \geq m_r$, that reduces the length of this interval from $\delta_1^1+\gamma_l'+\gamma_r'$ to $\alpha_l'+\alpha_r'$. Here, if there is no point s in $[r+m_r-\gamma_l', m_r+1+\gamma_r']$, then the contraction function is defined as follows (see Fig. 8):

1. $c^1(x) = (x-(r+n-\gamma_l'))\cdot\frac{\alpha_l'+\alpha_r'}{\delta_1^1+\gamma_l'+\gamma_r'} + \lambda_2^1(r+n-\gamma_l')$, where $\alpha_l' = -\lambda_2^1(r+n-\gamma_l')$ and $\alpha_r' = \lambda_2^1(1+n+\gamma_r')$.

However, if there is a point s in $[r+m_r-\gamma_l', m_r+1+\gamma_r']$ (see the Gap Theorems), then $c^1(x)$ is defined as follows (see Figs. 7, 6 and 11, respectively).:

[8] Here, the superscript 1 relates the function f_1^1 (the first function of a family of functions $(f_n^1)_{n\in\mathbb{N}}$) as well as the family of gaps $(\delta_n^1)_{n\in\mathbb{N}}$ to the interval I_1. By the way, notice that this function f_1^1 (as well as the upcoming functions f_n^i) will be continuous on $\mathbb{R}$.

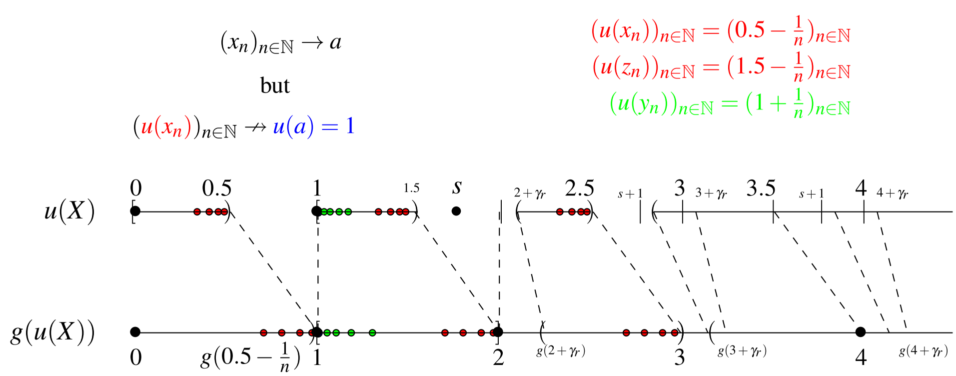

Fig. 10. Illustration of a bad gap for the particular representation, where $r = 0.5$ and $u(a) = 1$ and without containing a point. The way in which the jump-discontinuity may be avoided is illustrated too. Here notice that since there is no point s' in $[2.5, s+1]$, we do not have to contract any interval to 3. Hence, we may just proportionally identify the interval $[2.5, 3+\gamma_r]$ with $[3, g(3+\gamma_r)]$. We will continue applying this identifications in the successive intervals.

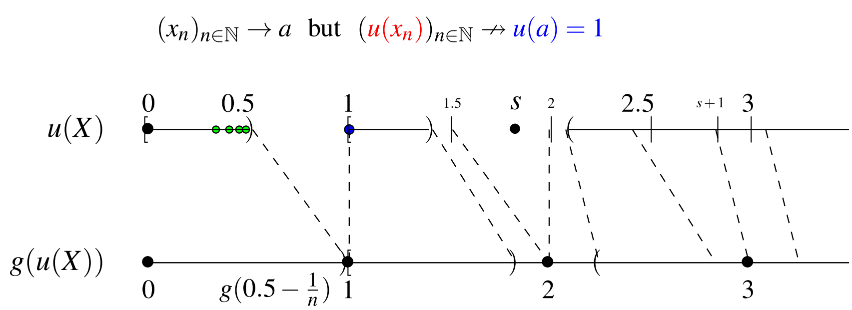

Fig. 11. Illustration of a bad gap for the particular representation where $r = 0.5$ and $u(a) = 1$. Since γ_l and γ_r both are strictly positive, the existence of a continuous representation is compatible with the presence of points whose images are in $[2.5, 3]$.

1. If $\delta_r = 0$, then $\delta_l > 0$ and $(s+1, m_r+2] \cap u(X) = \emptyset$. Then, $c_1^1(x) = (x - (r + m_r - \gamma_l')) \cdot \frac{\alpha_l'}{s-r-m_r+\gamma_l'} + \lambda_2^1(r + m_r - \gamma_l')$, for any $x \in [r + m_r - \gamma_l', s']$. The interval $(s, m_r+1]$ is contracted to $m_r + 1$.
   For any $n > m_r$, then we apply $c_1^1(x) = (x - (r + n - \gamma_l')) \cdot \frac{\alpha_l'}{1-r+\gamma_l'} + \lambda_2^1(r + n - \gamma_l')$, for any $x \in [r + n - \gamma_l', n]$.
2. If $\delta_l = 0$, then $\delta_r > 0$ and $[r + m_r + 1, s + 1] \cap u(X)$ may contain a unique point s_{m_r+1}. In case this point s_{m_r+1} exists, then, $c_2^1(x) = (x - (s+1)) \cdot \frac{\alpha_r'}{n+1+\gamma_r'-s} + n$ for any $x \in [s + n - m_r, n + \gamma_r']$, $n \geq m_r$, and the interval $[r + n, s + n - m_r]$ is contracted to $n + 1$, for any $n \geq m_r$ with $n < m_r^*$, where $m_r^* \in \mathbb{N}$ is such that there is no point in $[r + m_r^*, s_{m_r^*-1} + 1] \cap u(X)$.
   Then, for $n \geq m_r^*$, we apply $c_2^1(x) = (x - (r + n + 1)) \cdot \frac{\alpha_r'}{1+\gamma_r'-r} + n$ for any $x \in [r + n, n + \gamma_r']$. If there is no point s', then $m_r^* = m_r + 1$.
3. If $\delta_l > 0$ and $\delta_r > 0$, then we apply $c_1^1(x)$ in $[r + n - \gamma_l', s + n - m_r]$ and $c_2^1(x)$ in $[s + n - m_r, n + \gamma_r']$, for any $n \geq m_r$.

We continue the proof through a dual study on the left side of the gap.

By the Gap Theorems, if $[r-1, u(a_1)-1]$ or $[r-1, u(a_1)-1)$ is a gap, then $u(X) \cap [r-2, u(a_1)-2]$ has at most one point s (which is in fact the adjoint point $u(a_1)-2$ in case $u(a_1)-1$ exists), and we will continue applying the reasoning corresponding to the stable zone, until arriving to a $m_l \in \mathbb{N}$ such that $u(X) \cap [r - m_l - \gamma_l, u(a_1) - m_l + \gamma_r]$ (with $\gamma_l, \gamma_r \geq 0$ and, at least, one of them positive) has at most one point (as described in the inflection zone). The existence of this bigger gap (which does not suppose – by hypothesis, since the previous and smaller one G_1 we supposed to be the biggest – a discontinuity) may allow the existence of elements on the following intervals $[r - n, u(a_1) - n]$ for $n > m_l$.

So, now, we define the following *expansion* function on the left side of the gap (i.e., for $x < r$):

1. $\lambda_1^1(x) = (x + n) \cdot \frac{1}{1-\delta_1^1} - n$, $x \in [-n, r - n]$, $n \in \mathbb{N}$ with $0 \leq n < m_l$.
2. $\lambda_2^1(x) = (x + n) \cdot \frac{1}{1-\delta_1^1} - n$, $x \in [-n + \gamma_r, r - n - \gamma_l]$, $n \in \mathbb{N}$ with $n \geq m_l$.
3. $\lambda_3^1(x) = 1 - n$, $x \in (r - n, 1 - n)$, $n \in \mathbb{N}$ with $0 \leq n < m_l$.

And a *contraction* function on $[r - n - \gamma_l, u(a_1) - n + \gamma_r]$, $n \in \mathbb{N}$ with $n > m_l$, that reduces the length of this interval from $\delta_1^1 + \gamma_l + \gamma_r$ to $\alpha_l + \alpha_r$:

1. $c^1(x) = (x - (r - n - \gamma_l)) \cdot \frac{\alpha_l+\alpha_r}{\delta_1^1+\gamma_l+\gamma_2} + \lambda_2^1(r - n - \gamma_l)$, where $\alpha_l = -\lambda_2^1(r - n - \gamma_l)$ and $\alpha_r = \lambda_2^1(1 - n + \gamma_r)$.

However, if there is a point $s = s_m$ in $u(X) \cap [r - m_l - \gamma_l, u(a_1) - m_l + \gamma_r]$ (see the Gap Theorems), then $c^1(x)$ is defined as follows:

1. If $\delta_r = 0$, then $\delta_l > 0$ and $[s - 1, 1 - (m_l + 1)] \cap u(X)$ may contain a unique point s_{m+1}. In case this point s_{m+1} exists, then, $c_1^1(x) = (x - (r - n - \gamma_l)) \cdot \frac{\alpha_l}{s_n-(r-n)+\gamma_l} + \lambda_2^1(r - n - \gamma_l)$ for any $x \in [r - n - \gamma_l, s_n]$, $n \geq m_l$, and the interval $[s_n, 1 - n]$ is contracted to $1 - n$, for any $n \geq m_l$ with $n < m_l^*$, where $m_l^* \in \mathbb{N}$ is such that there is no point in $[s_{m_l^*-1} - 1, 1 - m_l^*] \cap u(X)$.
   Then, for $n \geq m_l^*$, we apply $c_1^1(x) = (x - (r - n - \gamma_l)) \cdot \frac{\alpha_l}{1+\gamma_l-r} + \lambda_2^1(r - n - \gamma_l)$ for any $x \in [r - n - \gamma_l, 1 - n]$.
2. If $\delta_l = 0$, then $\delta_r > 0$ and $[r - m_l - 1, s - 1) \cap u(X) = \emptyset$. Then, $c_2^1(x) = (x - s) \cdot \frac{\alpha_r}{1-m_l+\gamma_r-s} + \lambda_2^1(1 - m_l - \gamma_l')$, for any $x \in [s, 1 - m_l + \gamma_r]$. The interval $[r - m, s]$ is contracted to $1 - m$.
   For any $n > m_l$, then we apply $c_2^1(x) = (x - (r - n)) \cdot \frac{\alpha_r}{1-r+\gamma_r} + 1 - n$, for any $x \in [r - n, 1 - n + \gamma_r]$.

3. If $\delta_l > 0$ and $\delta_r > 0$, then we apply $c_1^1(x)$ for any $x \in [r-n-\gamma_l, s-n+m_l]$, and $c_2^1(x)$ for any $x \in [r-n, 1-n+\gamma_r]$, $n \geq m_l$.

Notice that, by the Gap Theorems, $(r-n, 1-n) \cap u(X)$ (for $0 \leq n < m_l$) would contain at most one point, so λ_3^1 does not imply a contraction on $(r-n, 1-n) \cap u(X)$.

We will argue similarly in the case of a bad gap of the form $G_1 = (u(a_1), r]$, constructing the corresponding functions.

First, notice that the function applied are strictly increasing on $u(X)$. Secondly, notice that $\lambda_i^1(t) + 1 = \lambda_i^1(t+1)$ (for any $i = 1, 2, 3$) as well as $c_k^1(t) + 1 = c_k^1(t+1)$ (for any $k = 1, 2$) for any $t \in u(X)$. Finally, $|u(x) - u(y)| \leq 1$ is satisfied if and only if $|f_1^1(u(x)) - f_1^1(u(y))| \leq 1$, $x, y \in X$ (see Figs. 5, 6, 7, 8, 10 and 11). Hence, after applying this piecewise function f_1^1 on $u(X)$, another unit representation $(u_1^1, 1)$ is achieved, but now without the aforementioned gap G_1.

By Proposition 5.8, the next biggest gap G_2 in I_1 – which length with respect to u is δ_2^1 – is selected and we continue the process, constructing the corresponding function f_2^1 and achieving another representation $(u_2^1, 1)$, where $u_2^1 = f_2^1 \circ f_1^1 \circ u = f_2^1 \circ u_1^1$.[9]

Let us see that, given any $\epsilon_0 > 0$, this process arrives to a point such that the representation $(u_{n_1}^1, 1)$ is ϵ_0-continuous in $I_1 = [0, 1]$.

Let $\{\delta_n^1\}_{n\in\mathbb{N}}$ be the sequence of lengths corresponding to the bad gaps $\{G_n\}_{n\in\mathbb{N}}$ (ordered from bigger to smaller, see Proposition 5.8) associated to the initial function u of the unit representation $(u, 1)$ on $I_1 = [0, 1]$. Obviously, letting n tend to infinity, δ_n^1 tends to 0 (see Proposition 5.8). Furthermore, notice that, since there is no discontinuous Cantor set contained in $u(X)$, the sum $\sum_{k=1}^{+\infty} \delta_k^1$ is strictly smaller than 1 (i.e. there exists $r_1 > 0$ such that $\sum_{k=1}^{+\infty} \delta_k^1 \leq 1 - r_1$).

We denote the family of bad gaps in I_1 corresponding to the representation $u_1^1 = f_1^1 \circ u$ by $\{f_1(G_n)\}_{n\in\mathbb{N}\setminus\{1\}} = \{G_n^1\}_{n\in\mathbb{N}}$. The length of the biggest gap G_1^1 corresponds to the possible expansion of the gap G_2 of $u(X)$, thus, the length of the gap G_1^1 is $l_2 = \frac{\delta_2^1}{1-\delta_1^1}$. This will be repeated again and again with each function $f_2^1, f_3^1, \ldots, f_{n_1}^1$.

Therefore, we are able to define the sequence $\{l_n\}_{n\in\mathbb{N}}$ of lengths associated to the biggest bad gap of each representation $(u_n^1, 1)$ in I_1. This sequence is defined recursively as follows:

$$l_1 = \delta_1^1$$

$$l_2 = \delta_2^1 \cdot \frac{1}{1-\delta_1^1} = \frac{\delta_2^1}{1-l_1}$$

$$l_3 = \delta_3^1 \cdot \frac{1}{1-\delta_1^1} \cdot \frac{1}{1-\frac{\delta_2^1}{1-\delta_1^1}} = \frac{\delta_3^1}{1-\delta_1^1-\delta_2^1} = \frac{\delta_3^1}{(1-l_1)\cdot(1-l_2)}$$

...

$$l_n = \frac{\delta_n^1}{1-\sum_{k=1}^{n-1}\delta_k^1} = \frac{\delta_n^1}{\Pi_{k=1}^{n-1}(1-l_k)}.$$

Thus, since (as said before) the sum $\sum_{k=1}^{+\infty} \delta_k^1$ is strictly smaller than 1 and $\{\delta_n^1\}_{n\in\mathbb{N}}$ converges to 0, we conclude that $\{l_n\}_{n\in\mathbb{N}}$ tends to 0 letting n tend to infinity.

Therefore, for any $\epsilon_1 > 0$ there is always a finite number n_1 such that the length of the biggest bad gap on $f_{n_1}^1 \circ f_{n_1-1}^1 \circ \cdots \circ f_1^1(u(X) \cap [0, 1])$ is smaller than ϵ_1 (we shall denote $f^1 = f_{n_1}^1 \circ f_{n_1-1}^1 \circ \cdots \circ f_1^1$ and $u^1 = f^1 \circ u$). In the limit, we reduce the measure of the union of bad gaps to 0, stretching $\frac{1}{1-\sum_{n\in\mathbb{N}}\delta_n^1}$ times the subset $u(X) \cap [0, 1]$.

[9] Again, we keep the subscript 1 in u_2^1 to refer to the first interval I_1.

After achieving the desired result on [0, 1], we now choose the next biggest gap (if it exists) which lies in I_{i_2}, for some $i_2 \in \{-M+1, \ldots, N\}$. Again (because of translations) we may assume that it is of the form $[u^1(a_2) - \delta_1^2, u^1(a_2)) \subseteq [0, 1]$ (or the dual $(u^1(a_2), u^1(a_2) + \delta_1^2 \subseteq [0, 1]]$), where u^1 now denotes the function $f^1 \circ u = f_{n_1}^1 \circ f_{n_1-1}^1 \circ \cdots \circ f_1^1 \circ u$.

Now, we repeat the same process on $[u^1(a_2) - 1, u^1(a_2))$ as before, achieving a family of functions that are applied on $u^1(X)$. After that, the aforementioned bad gaps of the first interval $I_1 = [0, 1]$ (where the biggest gap $G_1 = [r, u(a_1))$ was found) may increase $\frac{1}{1-\sum_{n\in\mathbb{N}}\delta_n^1} \cdot \frac{1}{1-\sum_{n\in\mathbb{N}}\delta_n^{i_2}}$ times, where now $\sum_{n\in\mathbb{N}}\delta_n^{i_2}$ is the sum of the length of the bad gaps in I_{i_2} with respect to the first function u. Hence, if the biggest discontinuity desired is ϵ_0, we should choose the ϵ_1 before smaller than $\epsilon_0 \cdot (1 - \sum_{n\in\mathbb{N}}\delta_n^1) \cdot (1 - \sum_{n\in\mathbb{N}}\delta_n^{i_2})$. As a matter of fact, since each kth step may increase the length of the gaps of the intervals corresponding to the steps before $\frac{1}{1-\sum_{n\in\mathbb{N}}\delta_n^{i_k}}$ times, we should choose the initial ϵ_1 such that

$$\epsilon_1 < \epsilon_0 \cdot \prod_{k=1}^{T}(1 - \sum_{n\in\mathbb{N}}\delta_n^{i_k}).$$

Once this second step has been completed, we have achieved a unit representation $(u^{1,i_2}, 1)$, where u^{1,i_2} denotes the function $f^{i_2} \circ f^1 \circ u = f^{i_2} \circ u^1$ and f^{i_2} is the function $f_{n_{i_2}}^{i_2} \circ f_{n_{i_2}-1}^{i_2} \circ \cdots \circ f_1^{i_2}$ constructed as before.

Thus, at the kth step the function $f^{i_k} = f_{n_{i_k}}^{i_k} \circ f_{n_{i_k}-1}^{i_k} \circ \cdots \circ f_1^{i_k}$ is constructed, achieving the representation $(u^{1,i_2,\ldots,i_k}, 1)$, where $u^{1,i_2,\ldots,i_k} = f^{i_k} \circ u^{1,i_2,\ldots,i_{k-1}}$). Since the semiorder is bounded, the image of X is contained in a bounded interval so, the process ends up after a finite number T of steps, achieving a unit representation $(u^{1,i_2,\ldots,i_T}, 1)$ where the length of the biggest bad gap is smaller than the desired value $\epsilon_0 > 0$. □

Proof of Theorem 5.11

Proof. By Theorem 5.10, the semiorder is ϵ-continuously representable. Hence, for any $n \in \mathbb{N}$ there exists a unit representation $(u_n, 1)$ such that the length of the biggest gap is less than $\frac{1}{n}$. As a matter of fact, given $\frac{1}{n_0}$, in the previous proof we used a method to construct a unit representation $(u_{n_0}, 1)$ which is $\frac{1}{n_0}$-continuous. Hence, for $\frac{1}{n_0+1}$, we may proceed analogously but now starting the process from u_{n_0}.

Furthermore, this is a pointwise Cauchy sequence (with respect to the supremum norm). To see this, first notice that we are able to construct a new function u_{n+1} from u_n just applying the corresponding functions $f^1, f^{i_2}, \ldots, f^{i_T}$ described in the proof of Theorem 5.11, thus, $u_{n+1} = f^{i_T} \circ f^{i_{T-1}} \circ \cdots \circ f^1 \circ u_n$. We denote by f_n the composition $f^{i_T} \circ f^{i_{T-1}} \circ \cdots \circ f^1$ used in order to pass from u_n to u_{n+1} and $f_n \circ \cdots \circ f_1 = F_n$. Thus, $u_{n+1} = f_n \circ u_n = F_n \circ u_1$.

So, $\|u_{n+1} - u_n\|_\infty = \sup\{u_{n+1}(x) - u_n(x)\}_{x\in X}$ is just

$$\|f_n \circ u_n - u_n\|_\infty = \sup\{f_n(u_n(x)) - u_n(x)\}_{x\in X} = \sup\{f_n(r) - r\}_{r\in u_n(X)}.$$

Now, remember that f_n has been defined by means of the linear functions λ_1^n, λ_2^n, λ_3^n and c^n (see the proof of Theorem 5.10). The slopes of those linear functions depend on the lengths of the gaps and, since the corresponding lengths $(l_n)_{n\in\mathbb{N}}$ converges to 0 (as proved in the proof of Theorem 5.10), the slopes tend to 1 when letting n tend to infinite. Thus, these linear functions converge to identity when n increases. Therefore, it holds true that $\lim_{n\to\infty} \sup\{f_n(r) - r\}_{r\in u_n(X)} = 0$ and, we conclude that $\{u_n\}_{n\in\mathbb{N}}$ is a pointwise Cauchy sequence of unit representations.

Therefore, the limit function $v = \lim_{n\to\infty} u_n$ exists and is, in fact, a unit representation. To see that, first notice that $x \prec y$ if and only if $u_n(x)+1<u_n(y)$, for any $n\in\mathbb{N}$, and then $v(x)+1\leq v(y)$. Let us see that the strict inequality is also conserved for the limit function u.

For any function f_n applied on $u_n(X)$, it holds that $f_n(u_n(x)+1)=f_n(u(x))+1$, for any $x\in X$. Therefore – and since f_n is strictly increasing on $u_n(X)$ –, the condition of being a unit representation is also satisfied by $f_n\circ u_n=u_{n+1}$. The only exception is the case of the point s (in case it exits) of the interval $I_{m_r+1}=(r+m_r-\gamma_l, u(a)+m_r+\gamma_r)$ and $I_{-m_l+1}=(r-m_l-\gamma_l, u(a)-m_l+\gamma_r)$ of the inflection zone, where the image defined for this point s does not differ in 1 units with the image of the point $s+1$, that is, $f_k^i(s)+1\neq f_k^i(s+1)$, (see Figs. 6 and 7). However, in that case we may find or define as a ghost[10] a successor $Suc(s)$ and an antecedent $Ant(s)$ for that point s such that, $Ant(s)\sim s\sim Suc(s)$ and $f_k^i(Ant(s))+1=f_k^i(s)=f_k^i(Suc(s))-1$. For example, in Fig. 6, $1=Ant(s)$ and $s_1=Suc(s)$, whereas in Fig. 7, $1=Ant(s)$ and the element 3 – which is a ghost, since it is not in $u(X)$ – would play the role of the antecedent of s.

Consequently, the only way for the inequality to be broken in the limit, such that $v(x)+1=v(y)$ with $x\prec y$, is through the contraction of an entire subinterval (which is not contained in a bad gap) to a point. Let us see now that this contraction to a point is impossible.

First, notice that, in order to contract an interval to a point, the number of contraction functions applied on the interval must be infinite. Let $J=\underline{[r,s]}$ be a non degenerate interval defined by two points $r,s\in\overline{u_1(X)}=\mathbb{R}-\{G_n\}_{n\in\mathbb{R}}$, where $\{G_n\}_{n\in\mathbb{R}}$ denotes the family of bad gaps of $u_1(X)$. Assume that $(u_n)_{n\in\mathbb{N}}=(F_n\circ u_1)_{n\in\mathbb{N}}$ is a family of representations such that, for each n, there is a new contraction on $F_n(J)$ when constructing $F_{n+1}=f_{n+1}\circ F_n$.

Let us focus first on the transformations carried out for the first interval I_1, that is, on function f^1.

Attending to the construction of $f^1=f^1_{n_1}\circ\cdots f^1_1$, notice that if a contraction function is applied on J by f^1_1, then there is bad gap $[r_1,u_1(a_1))$ (or $(u_1(a_1),r_1]$) with its corresponding inflection zone $I_k=(r_1+k-\gamma_l, u_1(a_1)+k+\gamma_r)$ such that $J\subseteq[r_1+k+(m-k)-\gamma_l, u_1(a_1)+k+(m-k)+\gamma_r]$, for some $m\in\mathbb{N}$. When applying f_2^1, if $f_1^1(J)$ lies again in a contraction zone, then there is bad gap $[r_2,u_2(a_2))$ (or $(u_2(a_2),r_2]$) with its corresponding inflection zone $I_{k_2}=(r_2+k_2-\gamma_l, u_2(a_2)+k_2+\gamma_r)$ such that $f_1^1(J)\subseteq[r_2+k_2+(m-k_2)-\gamma_l, u_2(a_2)+k_2+(m-k_2)+\gamma_r]$. Since the semiorder is bounded, the number of intervals I_k is finite, and we may reduce our argument to the previous inflection zone $I_k=(r_1+k-\gamma_l, u_1(a_1)+k+\gamma_r)$, such that $f_1^1(J)\subseteq f_1^1([r_1+m-\gamma_l, u_1(a_1)+m+\gamma_r])$. Thus, this second bad gap $[r_2,u_2(a_2))$ (or $(u_2(a_2),r_2]$) lies in $[u_1(a_1)-1,\gamma_r]$ or in $[r_1-\gamma_l,r_1]$ (see Fig. 12).

That is, the bad gaps needed to create contractions in the same area must be contained in $[u_1(a_1)-1,\gamma_r]$ or in $[r_1-\gamma_l,r_1]$. Therefore, since there are no discontinuous Cantor sets, the sum of their lengths is smaller than $\gamma=\gamma_l+\gamma_r$, thus, $\sum_{k=2}^{\infty}\delta_k<\gamma$ as well as $\sum_{k=1}^{\infty}\delta_k<1$.

Again, we use notation $(\delta_n)_{n\in\mathbb{N}}$ for the length of those bad gaps $(G_n)_{n\in\mathbb{N}}$ involved in this contraction process. As far as the size of the shrinkage is concerned, if we focus on the whole contracted interval, in the first step its distances goes from $L_1=\gamma+\delta_1$ to $L_2=\frac{\gamma}{1-\delta_1}$, as well as the length of the bad gap G_2 goes from δ_2 in $u_1(X)$ to $l_2=\frac{\delta_2}{1-\delta_1}$ in $f_1^1(u_1(X))$. In a next step, the distances L_2 is reduced to $L3=\frac{L_2-l_2}{1-l_2}=\frac{\gamma-l_2(1-l_1)}{(1-l_1)(1-l_2)}=\frac{\gamma-\delta_2}{(1-l_1)(1-l_2)}$, as well as the length of the bad gap G_3 goes from δ_3 in $u_1(X)$ to $l_3=\frac{\delta_3}{1-l_2}$ in $f_2^1(u_1(X))$. Thus, in the nth step, we have that

$$L_n=\frac{\gamma-\sum_{k=2}^{n}\delta_k}{\prod_{k=1}^{n-1}(1-l_k)}=\frac{\gamma-\sum_{k=2}^{n}\delta_k}{1-\sum_{k=1}^{n}\delta_k}.$$

Therefore, since $\sum_{k=2}^{\infty}\delta_k<\gamma$ as well as $\sum_{k=1}^{\infty}\delta_k<1$, we deduce that, in the limit, the length L_n is not reduced to a point, and since the transformations are proportional in the whole interval, there is no subinterval contracted to a point. This conclusion remains true when extended the process to the rest of the intervals I_k, since the semiorder being bounded, this process would be repeated a finite number T of times.

Thus, we conclude that $v=\lim_{n\to\infty}u_n$ is a unit representation too, as well as, since the length of the biggest gap of u is less than $\frac{1}{n}$ for any $n\in\mathbb{N}$, it is also continuous. □

Remark A.1. The proof before may be more visually appealing by putting it in the following form:

Let u_n^1 be the representation constructed by $f_n^1\circ\cdots\circ f_1^1\circ u$, reducing the lengths of the bad gaps contained in I_1. Then, we may prove (as we did in the proof of Theorem 5.10) that $(u_n^1)_{n\in\mathbb{N}}$ converges to a representation u^1, but now without those bad gaps that were contained in I_1.

Now, we call $u_1^{12}=u^1$ and repeat the argument on $f_n^2\circ\cdots\circ f_1^2\circ u_1^{12}$, achieving a representation u^{12}, but now without those bad gaps that were contained in I_1 and I_2.

Since the semiorder is bounder, the proof ends after T steps, achieving a representation $v=u^{12\ldots T}$, but now without bad gaps, thus, continuous.

Proof of Theorem 5.12

Proof. Let $(u,1)$ be a unit representation of the semiorder. We assume, without loss of generality, that u also represents the total preorder $\precsim_0$. We shall use function g of Debreu's Open Gap Lemma, which removes the bad gaps of S achieving another set $g(S)$ of the same length but now without bad gaps (that is, the function g that, given a continuous total preorder and a utility function u, allows to construct a continuous utility function $g\circ u$). This is needed in order to deal with the possible existence of discontinuous Cantor sets. However, the structure of the proof is similar to the proof of Theorem 5.11.

Since the semiorder is bounded, we may assume without loss of generality that $-M+w$ is the infimum of $\underline{u(X)}$ and N is the supremum, where $N,M\in\mathbb{N}$ and $w\in[0,1)$. Let $\overline{u(X)}$ be the set defined by $[-M+w,N]\setminus\bigcup_{n\in\mathbb{N}}G_n$, where $\{G_n\}_{n\in\mathbb{N}}$ is the family of bad gaps of $u(X)$.

By Proposition 5.8, there is a maximal gap G_1 on $u(X)$ of the form $(u(a_1),r]$ or $[r,u(a_1))$. Without loss of generality (and with the only purpose of simplify notation), we may suppose that $G_1=[r,u(a_1))$ with $u(a_1)=1$ and $r=u(a)-\delta_1$, otherwise we would move the set $u(X)$.

First, we focus on $I_0=[u(a_1)-1,u(a_1)]=\underline{[0,1]}$. Here, we apply function g_0^0 that removes the bad gaps of $\overline{u(X)}$ in I_0, returning a subset $S=g_0^0(\overline{u(X)}\cap[0,1])$ free of bad gaps and such that $\inf S=0$ and $\sup S=1$. Notice that given that biggest bad gap $G_1=[r,u(a_1))=[r,1)$, by the Gap Theorems, there is no bad gap containing 0. That is, there is no bad gap $(s,t]$ or $[s,t)$ with $s<0<t$. Thus, $\inf\{\overline{u(X)}\cap[0,1]\}=0$ and g_0^0 is defined from 0 to 1 (with some possible gaps in the middle in addition to G_1). The changes made on I_0 must be taking into account in $I_1=[1,2]$ and $I_{-1}=[-1,0]$ (if they exist) in order to keep the semiorder relation, so now we apply on $I_1=[1,2]$ and $I_{-1}=[-1,0]$ the functions[11] g_0^1 and g_0^{-1}.

Before define g_0^1 and g_0^{-1}, first notice that, by the Gap Theorems, if $[a,b)$ (dually $(a,b]$) is a bad gap in I_0, then $[a-1,b-1)$

[10] See Bouyssou & Pirlot, 2021a.

[11] This notation is devoted to help on the understanding of the meaning of the corresponding function. Hence, g_0^1 makes reference to the function g_0^0 (already defined on I_0) modified to be applied on I_1.

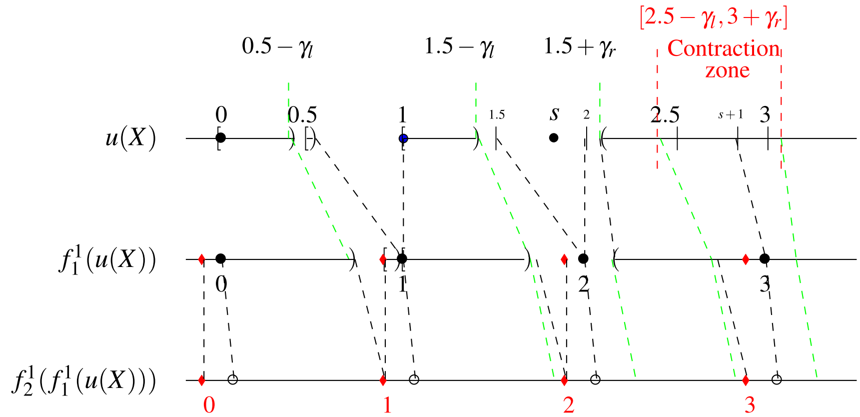

Fig. 12. Illustration of two consecutive contractions on the same interval.

(respectively $(a+1, b+1]$) is a lacuna, so the same function g_0^0 can be applied in I_{-1} (resp. I_1), since there is no point $u(x)$ in the interval that could be removed. On the other hand, $[a+1, b+1] \cap u(X)$ may contain a point s_1 (see the Gap Theorems), this point s_1 is identified with $u(a)+1 = 2$. Now, we are able to define $g_0^1(x) = g_0(x-1)+1$ (for any $x \in I_1 = [1, 2]$) and $g_0^{-1}(x) = g_0(x+1)-1$ (for any $x \in I_{-1} = [-1, 0]$). Notice that g_0^1 and g_0^{-1} are defined in the all set $I_1 \cap \overline{u(X)}$ and $I_{-1} \cap \overline{u(X)}$, respectively, and that, as a matter of a fact, they are strictly increasing on $\overline{u(X)} \cap I_1$ and $\overline{u(X)} \cap I_{-1}$, respectively. Thus, $g_0 \circ u$ (where we refer now to g_0 as the piecewise functions defined by g_0^1, g_0^0 and g_0^{-1}) keeps the semiorder relation on $u^{-1}([-1, 2]) \subseteq X$.

Reasoning analogously on I_2 and I_{-2} with g_0^1 and g_0^{-1}, respectively, we define the functions $g_0^{1,2}$ and $g_0^{-1,-2}$ on I_2 and I_{-2}, and so on, until arriving from the left to a $m_l \in \mathbb{N}$ such that $u(X) \cap [r - m_l - \gamma_l, u(a_1) - m_l + \gamma_r]$ (with $\gamma_l, \gamma_r \geq 0$ and, at least, one of them positive) has at most one point, as explained in the inflection zone (see the Gap Theorems as well as Figs. 4 and 5 for illustrations of the stable zone and Figs. 6, 7, 8, 10 and 11 for the inflection zone). Dually, from the right, until arriving to $m_r \in \mathbb{N}$ such that $u(X) \cap [r + m_r - \gamma'_l, u(a_1) + m_r + \gamma'_r]$ (with $\gamma'_l, \gamma'_r \geq 0$ and, at least, one of them positive) has at most one point. Here we apply the corresponding functions $g_0^{1,2,\ldots,m_r}$ and $g_0^{-1,-2,\ldots,-m_l}$. Before continuing to apply these functions in the successive intervals I_{-n} $(n > m_l)$ and I_n $(n > m_r)$, yet first we have to make some modifications on the set.

At this point, first we focus on the right side $u(X) \cap [r + n - \gamma'_l, u(a_1) + n + \gamma'_r]$ (for any $n > m_r$). Assume that there exists that point s in $u(X) \cap [r + m_r - \gamma'_l, u(a_1) + m_r + \gamma'_r]$. We distinguish three cases (as explained in the inflection zone, see Figs. 7, 6 and 11, respectively).

(a) If $\gamma'_r = 0$ (see Fig. 7), then $\gamma'_l > 0$ and $[s+1, u(a_1)+m_r+1]$ must be empty. In this case, we identify $[r+n-\gamma'_l, 1+n)$ with $[r+n-\gamma'_l, r+n)$.

(b) If $\gamma'_l = 0$, then $\gamma'_r > 0$ and $[r+m_r+1, s+1]$ may contain one point s_{m_r+1}. In this case we identify $[s+n-m_r, u(a_1)+n+\gamma'_r]$ with $[u(a_1)+n, u(a_1)+n+\gamma'_r]$, $n > m_r$ with $n < m_r^*$, where $m_r^* \in \mathbb{N}$ is such that there is no point in $[r + m_r^*, s_{m_r^*-1} + 1] \cap u(X)$. For $n \geq m_r^*$ (see Fig. 10) we identify $[r+n, u(a_1)+n+\gamma'_r]$ with $[u(a_1)+n, u(a_1)+n+\gamma'_r]$.

(c) Otherwise, $\gamma'_r > 0$ and $\gamma'_l > 0$ (see Fig. 11) and we apply both identifications described in (a) and (b). That is, we identify $[r+n-\gamma'_l, s+n-m_r]$ with $[r+n-\gamma'_l, r+n]$ and $[s+n-m_r, u(a_1)+n+\gamma'_r]$ with $[u(a_1)+n, u(a_1)+n+\gamma'_r]$, with $n > m_r$.

If there is no point s in $u(X) \cap [r+n-\gamma'_l, u(a_1)+n+\gamma'_r]$ (see Fig. 8, for instance), then we identify $[r+n-\gamma'_l, u(a_1)+n)$ with $[r+n-\gamma'_l, r+n]$ in case $\gamma'_l > 0$, otherwise we identify $[r+n, u(a_1)+n+\gamma'_r)$ with $[u(a_1)+n, u(a_1)+n+\gamma'_r]$, for any $n > m_r$.

Notice that these identifications conserve the semiorder structure, that is, if we denote by $Id(u(x))$ the identification made on $u(x)$, then it holds that $u(x)+1 < u(y)$ if and only if $Id(u(x))+1 < Id(u(y))$.

Now, we focus on the left side $u(X) \cap [r-n-\gamma_l, u(a_1)-n+\gamma_r]$ (for any $n > m_l$). Assume that there is a point $s \in u(X) \cap [r - m_l - \gamma_l, u(a_1) - m_l + \gamma_r]$. We distinguish three cases.

(a) If $\gamma_r = 0$, then $\gamma_l > 0$ and $[s-1, u(a_1)-n-1]$ may contain at most one point s_m. In this case, we identify $[r-n-\gamma_l, s_n-1)$ with $[r-n-\gamma_l, r-n)$ for any $n > m_l$ with $n < m_l^*$, where $m_l^* \in \mathbb{N}$ is such that there is no point in $[s_{m_l^*-1} - 1, 1 - m_l^*] \cap u(X)$. For $n \geq m_l^*$ we identify $[r-n-\gamma_l, 1-n)$ with $[r-n-\gamma_l, r-n)$.

(b) If $\gamma_l = 0$, then $\gamma_r > 0$ and $[s-1, u(a_1)-m_l-1]$ may contain at most one point s'. In this case we identify $[r-n, u(a_1)-n+\gamma_r]$ with $[u(a_1)-n, u(a_1)-n+\gamma_r]$, for any $n > m_l$.

(c) Otherwise, $\gamma_r > 0$ and $\gamma_l > 0$ and we apply both identifications described in (a) and (b). That is, we identify $[r-n-\gamma_l, s-n+m_l]$ with $[r-n-1-\gamma_l, r-n]$ and $[s-n+m_l, u(a_1)-m_l+\gamma_r]$ with $[u(a_1)-n, u(a_1)-n+\gamma_r]$, with $n > m_l$.

If there is no point s in $u(X) \cap [r - m_l - \gamma_l, u(a_1) - m_l + \gamma_r]$, then we identify $[r-n-\gamma_l, u(a_1)-n)$ with $[r-n-\gamma_l, r-n]$ in case $\gamma_l > 0$, otherwise we identify $[r-n, u(a_1)-n+\gamma_r)$ with $[u(a_1)-n, u(a_1)-n+\gamma'_r]$, for any $n > m_l$.

Now, we are able to successfully apply the corresponding functions $g_0^{1,2,\ldots,m_r}$ and $g_0^{-1,-2,\ldots,-m_l}$ on the successive intervals I_{-n}, I'_n (for $n > m_l$ and $n' > m_r$), until arriving to the last intervals $[-M, -M+1]$ and $[N-1, N]$. Hence, we have applied on $u(X)$ the piecewise function g_0 defined as $g_0(x) = g_0^0(x)$ if $x \in I_0$, $g_0(x) = g_0^{1,\ldots,k}(x)$ if $x \in I_k$ (for any $k = 1, \ldots, N$), and $g_0(x) = g_0^{-1,\ldots,-k}(x)$ if $x \in I_k$ (for any $k = -1, \ldots, -M$). We denote now by u_0 the function $g_0 \circ u$.[12]

By g_0^0 all the bad gaps on $[0, 1]$ have been removed and then, through functions $g_0^1, g_0^{1,2}, \ldots, g_0^{1,2,\ldots,N}$ and $g_0^{-1}, g_0^{-1,-2}, \ldots, g_0^{-1,-2,\ldots,-M}$ the changes made in $[0, 1]$ have been reproduced in $I_1, I_2, \ldots, I_N$ and $I_{-1}, \ldots, I_{-M}$ in order to keep the semiorder relation.

[12] For further results, notice that this function g_0 may be defined even on unbounded semiorders.

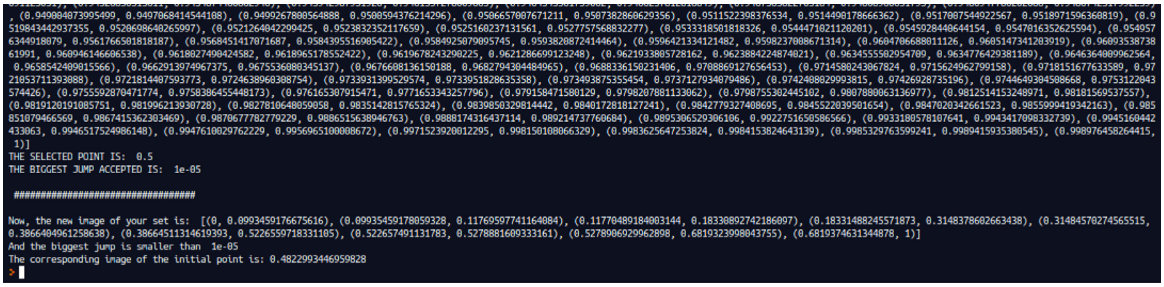

Fig. 13. Program's output for a set of 1000 gaps, point 0.5 and $r = 0.00001$.

Nevertheless, we have not removed the possible bad gaps in $g_0(\overline{u(X)}) \setminus [0, 1]$, thus, the process continues but now focusing on the biggest bad gap G_1^0 of $g_0(\overline{u(X)})$.[13]

Again, without loss of generality, we may suppose that $G_1^0 = [r, u_0(a_1))$ with $u_0(a_1) = 1$ and $r = u_0(a_1) - \delta_1^0$ (it would be proved dually for $(u_0(a_1), r]$), otherwise we should move the set by a function $t_1(x) = x + 1 - u_0(a_1)$. Hence, we would define again function g_1^0 on $G_1^0 = [u_0(a_1) - 1, u_0(a_1))$ and then the corresponding functions $g_1^1, g_1^{1,2}, \dots, g_1^{1,2,\dots,N_1}$ and $g_1^{-1}, g_1^{-1,-2}, \dots, g_1^{-1,-2,\dots,-M_1}$ (as we did before), constructing the piecewise function g_1.

Since the amount of intervals is finite, the process ends up after applying a last piecewise function g_T (where $T = M + N$) defined by means of a family of functions $g_T^1, g_T^{1,2}, \dots, g_T^{1,2,\dots,N_T}$ and $g_T^{-1}, g_T^{-1,-2}, \dots, g_T^{-1,-2,\dots,-M_T}$, achieving a continuous unit representation.

The other implication is already proved by Proposition 3.2 and Lemma 4.9. □

Appendix B. Python code

Here, in Fig. 13, we include a screenshot of a program written by the author in Python (the code is available if desired) that, given a real subset $S \subseteq [0, 1]$ and a value $r \geq 0$, reproduces a finite amount of steps of the proof of Corollary 6.3 until achieve a r-continuous representation (i.e., continuity in case $r = 0$). By the way, it also gives the final value of an initial point $x \in [0, 1]$, as well as it may show the outputs of each step (these intermediate outputs are not necessary and do not appear in the screenshot, but are useful to check the calculations of the proof of Corollary 6.3).

In particular, Fig. 13 shows the answer for a set with 1000 gaps (the description of this set is too long, so we cut the beginning of the image), the point is 0.5 and $r = 0.00001$.

References

Aleskerov, F., Bouyssou, D., & Monjardet, B. (2007). *Utility maximization, choice and preference* (2nd ed.). Berlin: Springer.

Amstrong, W. E. (1939). The determinateness of the utility functions. *The Economic Journal*, *49*, 453–467.

Amstrong, W. E. (1948). Uncertainty and the utility functions. *The Economic Journal*, *58*, 1–10.

Beardon, A. F., & Mehta, G. B. (1994). The utility theorems of Wold, Debreu, and Arrow-Hahn. *Econometrica*, *62*(1), 181–186.

Beja, A., & Gilboa, I. (1992). Numerical representations of imperfectly ordered preferences (A unified geometric exposition). *Journal of Mathematical Psychology*, *36*(3), 426–449.

Bosi, G., Estevan, A., Gutiérrez-García, J., & Induráin, E. (2015). Continuous representability of interval orders: The topological compatibility setting. *International Journal of Uncertainty, Fuzziness and Knowledge-Based Systems*, *23*(03), 345–365.

Bouyssou, D., & Pirlot, M. (2021a). Unit representation of semiorders I: Countable sets. *Journal of Mathematical Psychology*, *103*(2), Article 102566.

Bouyssou, D., & Pirlot, M. (2021b). Unit representation of semiorders II: The general case. *Journal of Mathematical Psychology*, *103*(2), Article 102568.

Bridges, D. S., & Mehta, G. B. (1995). *Representations of preference orderings*. Berlin, Heidelberg, New York: Springer-Verlag.

Campión, M. J., Candeal, J. C., Induráin, E., & Zudaire, M. (2008). Continuous representability of semiorders. *Journal of Mathematical Psychology*, *52*, 48–54.

Candeal, J. C., Estevan, A., Gutiérrez-García, J., & Induráin, E. (2012). Semiorders with separability properties. *Journal of Mathematical Psychology*, *56*, 444–451.

Candeal, J. C., & Induráin, E. (2010). Semiorders and thresholds of utility discrimination: solving the Scott–Suppes representability problem. *Journal of Mathematical Psychology*, *54*, 485–490.

Debreu, G. (1954). Representation of a preference ordering by a numerical function. In R. Thrall, C. Coombs, & R. Davies (Eds.), *Decision processes*. New York: John Wiley.

Debreu, G. (1964). Continuity properties of Paretian utility. *International Economic Review*, *5*, 285–293.

Dugundji, J. (1966). *Topology*. Allyn and Bacon.

Engelkin, R. (1989). *General topology*. Berlin: Heldermann Verlag.

Estevan, A. (2016). Generalized Debreu's Open Gap Lemma and continuous representability of biorders. *Order*, *33*(2), 213–229.

Estevan, A. (2020). Searching for a Debreu's Open Gap Lemma for semiorders. In G. Bosi, M. Campión, J. Candeal, & E. Indurain (Eds.), *Studies in systems, decision and control: vol. 263, Mathematical topics on representations of ordered structures and utility theory*. Springer.

Estevan, A., Gutiérrez García, J., & Induráin, E. (2013a). Further results on the continuous representability of semiorders. *International Journal of Uncertainty, Fuzziness and Knowledge-Based Systems*, *21*(5), 675–694.

Estevan, A., Gutiérrez García, J., & Induráin, E. (2013b). Numerical representations of semiorders. *Order*, *30*(2), 455–462.

Estevan, A., Schellekens, M., & Valero, O. (2017). Approximating SP-orders through total preorders: incomparability and transitivity through permutations. *Quaestiones Mathematicae*, *40*(3), 413–433.

Fechner, G. T. (1860). *Elements of psychophysics*. New York: Holt, Rinehart & Winston, Adler, H.E. (Trans.), 1966 reprinted.

Fishburn, P. C. (1970a). *Utility theory for decision-making*. New York: Wiley.

Fishburn, P. C. (1970b). Intransitive indifference in preference theory: a survey. *Operations Research*, *18*(2), 207–228.

Fishburn, P. C. (1970c). Intransitive indifference with unequal indifference intervals. *Journal of Mathematical Psychology*, *7*, 144–149.

Fishburn, P. C. (1973). Interval representations for interval orders and semiorders. *Journal of Mathematical Psychology*, *10*, 91–105.

Fishburn, P. C., & Monjardet, B. (1992). Norbert Wiener on the theory of measurement (1914, 1915, 1921). *Journal of Mathematical Psychology*, *36*, 165–184.

Gensemer, S. H. (1987). Continuous semiorder representations. *Journal of Mathematical Economics*, *16*, 275–289.

Gilboa, I., & Lapson, R. (1995). Aggregation of semiorders: intransitive indifference makes a difference. *Economic Theory*, *5*, 109–126.

Jamison, D. T., & Lau, L. J. (1973). Semiorders and the theory of choice. *Econometrica*, *41*, 901–911.

Jamison, D. T., & Lau, L. J. (1977). The nature if equilibrium with semiordered preferences. *Econometrica*, *45*, 1595–1605.

Krantz, D. H. (1967). Extensive measurement in semiorders. *Philosophy of Science*, *34*, 348–362.

Luce, R. D. (1956). Semiorders and a theory of utility discrimination. *Econometrica*, *24*, 178–191.

Manders, K. L. (1981). On JND representations of semiorders. *Journal of Mathematical Psychology*, *24*(3), 224–248.

Manzini, P., & Mariotti, M. (2006). A vague theory of choice over time. *The BE Journal of Theoretical Economics*, *6*(1).

Manzini, P., & Mariotti, M. (2012). Choice by lexicographic semiorders. *Theoretical Economics*, *7*, 1–23.

[13] Here the superscript 0 of G_1^0 refers to function u_0.

Masatlioglu, Y., & Ok, E. (2007). A theory of (relative) discounting. *Journal of Economic Theory*, *137*, 214–245.

Mehta, G. B. (1997). A remark on a utility representation theorem of rader. *Economic Theory*, *9*, 367–370.

Ok, E. (2007). *Real analysis with economic applications*. Princeton University.

Pirlot, M., & Vincke, Ph. (1997). *Series B: Mathematical and statistical methods*: *vol. 36*, *Semiorders. properties, representations, applications*. Kluwer Academic Publishers.

Rubinstein, A. (1988). Similarity and decision-making under risk. *Journal of Economic Theory*, *46*, 145–153.

Scott, D., & Suppes, P. (1958). Foundational aspects of theories of measurement. *The Journal of Symbolic Logic*, *23*, 113–128, 54 (2010) 485–490.

Shafer, W. J. (1974). The nontransitive consumer. *Econometrica*, *42*(5), 913–919.

Sonnenschein, H. (1971). Demand theory without transitive preferences, with applications to the theory of competitive equilibrium. In J. Chipman, & et al. (Eds.), *Preferences, utility, and demand*. New York: Harcourt-Brace-Jovanovich [ch. 10].

Tversky, A. (1969). Intransitivity of preferences. *Psychological Review*, *76*(1), 31–48.

Vincke, P. (1980). Linear utility functions on semiordered misture spaces. *Econometrica*, *48*(3), 771–775.

Weber, E. H. (1834). *Concerning touch*. New York: Academic Press, Ros, H. E. (Trans.), 1978 reprinted.

Wiener, N. (1914). Contribution to the theory of relative position. *Mathematical Proceedings of the Cambridge Philosophical Society*, *17*, 441–449.

Wiener, N. (1919). A new theory of measurement. *Proceedings of the London Mathematical Society*, *19*, 181–205.