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LORENZ AND LEXICOGRAPHIC MAXIMAL ALLOCATIONS FOR BANKRUPTCY PROBLEMS

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Javier Arin† and Juan Miguel Benito‡

Abstract

This paper investigates the use of egalitarian criteria to select allocations in bankruptcy problems. In our work, we characterize the sets of Lorenz maximal elements for these problems. We show that the allocation selected by the Proportional Rule is the only allocation that belongs to all these Lorenz maximal sets. We prove that the Talmud Rule selects the lexicographic maximal element within a certain set. We introduce and analyze a new sharing rule for bankruptcy problems that shares strong similarities with the Talmud Rule.

Keywords: bankruptcy problems, Lorenz criterion and lexicographic criterion.

1 Introduction

A bankruptcy problem consists of a set of claimants who must divide between them an infinitely divisible good, the endowment, that is not sufficient to satisfy their claims in full. The aim of this paper is to introduce egalitarian criteria to solve bankruptcy problems.

The use of egalitarian criteria to select outcomes from a given set has been widely analyzed in many different settings. For example, in the literature on

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coalitional games, is well-known that the most important solution concepts can be seen as selectors of egalitarian optimal outcomes from a certain set[1]. We focus on two egalitarian criteria: the Lorenz and lexicographic criteria.

In bankruptcy problems the Talmud Rule is one of the most important sharing rules. This solution coincides with the nucleolus of the associated bankruptcy games and the nucleolus can be seen as a solution that selects lexicographic maximal elements in a certain set.

It has been noted by different authors that if we consider the set of feasible allocations in a bankruptcy problem we need to conclude that the lexicographic maximal allocation coincides with the allocation selected by the sharing rule known as Constrained Equal Awards, a rule that seeks to give the same amount to any claimant whenever that amount does not exceed her\his claim.[2] Constrained Equal Losses (a rule that seeks to divide the loss equally whenever no claimant receives a negative payoff) can also be seen as a lexicographic maximizer since it selects the lexicographic maximal allocation in the set of vectors of losses. This paper proves that the Talmud Rule is also a lexicographic maximizer. It selects the lexicographic maximal allocation in the set of vectors of awards/losses considered in absolute terms. This analysis allows the definition of a new sharing rule, the Lexmax Rule. The new rule is based in the lexicographic ideas and can be seen as a natural counterpart of the Talmud Rule.

A second major contribution of the paper is to characterize the sets of Lorenz maximal vectors of awards/losses[3]. The existence of different Lorenz maximal sets is due to the fact that the vectors of awards/losses can be weighted and can be considered in real terms or absolute terms (See Subsection 2.3 for details). If we consider the intersection of the weighted Lorenz sets in absolute terms we find that it only contains the allocation provided by the Proportional Rule.

The rest of the paper is organized as follows: Section 2 introduces bankruptcy problems, sharing rules and egalitarian criteria. Section 3 deals with the different Lorenz maximal sets, Section 4 is devoted to the lexicographic rules: the Talmud Rule and the Lexmax Rule, and finally, Section 5 concludes.

[1] See for example Arin (2007).

[2] In Subsection 2.2 we introduce the two lexicographic criteria. The first criterion is based in the minmax principle and the second one is based in the maxmin principle.

[3] In the literature on egalitarianism it is agreed that an allocation should be maximal according to the Lorenz criterion as a minimal requirement for being called egalitarian. This fact motivates the study of Lorenz maximal allocations for bankruptcy problems.

2 Preliminaries

2.1 Bankruptcy problems

The tuple (N, d, E) is a bankruptcy *problem* if:

a) N is a finite nonempty set.

b) $\sum\limits_{i \in N} d_i > E$.

N represents the set of *agents or claimants*, $E \in \mathbb{R}_+$ represents the amount to be divided, and $d \in \mathbb{R}_+^N$ is a vector of claims whose i-th component is d_i. Then $i \preceq j$ means that we assume $d_i \leq d_j$ and $d_1 \geq 0$. We denote by Γ the class of bankruptcy problems.

An allocation to the claimants is represented by a real valued vector $x \in \mathbb{R}^N$ that satisfies $\sum_{i \in N} x_i = E$. The i-th coordinate of the vector x denotes the allocation given to claimant i.

We say that an allocation x satisfies claim boundedness and non negativity if $d_i \geq x_i \geq 0$ for all $i \in N$.

We denote by $F(N, d, E)$ the set of allocations that satisfy claim boundedness and non negativity.

A sharing rule ϕ in a set of problems Γ is a mapping that associates a vector $\phi\,(N, d, E) \in F\,(N, d, E)$ with every problem (N, d, E) in Γ.

Some well-known sharing rules are[4]:

Constrained Equal Awards (CEA). This solution divides the endowment equally among the agents under the constraint that no claimant receives more than his\her claim. Formally:

$$CEA(N, d, E) = (min(\beta, d_i))_{i \in N}$$

where β solves the equation $\sum_{i \in N} min(\beta, d_i) = E$.

Constrained Equal Losses (CEL). This solution divides the total loss $(\sum\limits_{i \in N} d_i - E)$ equally among the agents under the constraint that no claimant receives a negative amount. Formally:

$$CEL(N, d, E) = (max(0, d_i - \beta))_{i \in N}$$

and β solves the equation $\sum_{i \in N} max(0, d_i - \beta) = E$.

[4] A long list of rules can be found in a survey by Thomson (2003).

The **Proportional Rule (PR)**. This solution divides the endowment among the claimants proportionally to their claims. Formally:

$$PR(N, d, E) = \beta \cdot d$$

where $\beta \geq 0$ and $\beta \cdot (\sum_{i \in N} d_i) = E$.

Some convenient, well-known properties of a rule ϕ in Γ are the following.

- ϕ satisfies **order preservation for awards and losses** if for each (N, d, E) in Γ we have that $\phi(N, d, E)$ is order preserving for awards and losses. An allocation x is order preserving for awards and losses if $d_i \leq d_j$ implies that $x_i \leq x_j$ and $d_i - x_i \leq d_j - x_j$.

- ϕ satisfies **consistency** if for any problem (N, d, E) and any $S \subset N$ it holds that $\phi_i(S, (d_i)_{i \in S}, \sum_{i \in S} \phi_i(N, d, E)) = \phi_i(N, d, E)$ for all $i \in S$.

- ϕ satisfies **half claim boundedness (HCB)** if for any $(N, d, E) \in \Gamma$ we have that either $\phi_i(N, d, E) \geq \frac{d_l}{2}$ for all $i \in N$ or $\phi_i(N, d, E) \leq \frac{d_l}{2}$ for all $i \in N$.

- ϕ satisfies **λ-claim boundedness (λ-CB)** if for any $(N, d, E) \in \Gamma$ we have that either $\phi_i(N, d, E) \geq \lambda d_l$ for all $i \in N$ or $\phi_i(N, d, E) \leq \lambda d_l$ for all $i \in N$.

HCB is discussed in Aumann and Maschler (1985). This property is satisfied by the Talmud Rule and by the Proportional Rule. The λ-CB is clearly inspired by HCB and is satisfied by the Proportional Rule for any $\lambda \in [0, 1]$.

2.2 Egalitarian Criteria

For any vector $z \in \mathbb{R}^d$ we denote by $\theta(z)$ the vector that results from z by permuting the coordinates in such a way that $\theta_1(z) \leq \theta_2(z) \leq ... \leq \theta_d(z)$. Let $x, y \in \mathbb{R}^d$.

We say that the vector x *Lorenz dominates* the vector y (denoted by $x \succ_L y$) if $\sum\limits_{i=1}^{k} \theta_i(x) \geq \sum\limits_{i=1}^{k} \theta_i(y)$ for all $k \in \{1, 2, ..., d\}$ and if at least one of these inequalities is strict. The vector x *weakly Lorenz dominates* the vector y (denoted by $x \succeq_L y$) if $\sum\limits_{i=1}^{k} \theta_i(x) \geq \sum\limits_{i=1}^{k} \theta_i(y)$ for all $k \in \{1, 2, ..., d\}$.

We say that the vector x *lexicographically dominates* the vector y (denoted by $x \succ_{lex} y$) if there exists k such that $\theta_i(x) = \theta_i(y)$ for all $i \in \{1, 2, ..., k-1\}$ and $\theta_k(x) > \theta_k(y)$.

This lexicographic criterion provides Lorenz maximal allocations.

We say that the vector x *lexmax dominates* the vector y (denoted by $x \succ_{lm} y$) if there exists k such that $\theta_i(x) = \theta_i(y)$ for all $i \in \{k+1, k+2, ..., n\}$ and $\theta_k(x) < \theta_k(y)$.

The lexmax criterion provides Lorenz maximal allocations.

The last two criteria can be considered lexicographic criteria. The first one can be renamed as a maximin criterion and has been widely analyzed in many different models. The second criterion, a minimax criterion is a natural counterpart of the minimax criterion but has not received the same attention. The maximin criterion is also known as the Rawlsian criterion.

2.3 The set of awards-losses vectors

Let (N, d, E) be a problem and let x be an allocation. Each agent measures x_i in two ways. In one sense x_i measures how much he\she receives. In the other sense, $d_i - x_i$ measures how much he\she does not receive. Given the allocation x we define its associated ordered vector of awards-losses as follows:

$$x^{AL} = (x_1, ..., x_n, x_1 - d_1, ..., x_n - d_n).$$

We also use the following notation:

$$x^A = (x_1, ..., x_n) \text{ and } x^L = (x_1 - d_1, ..., x_n - d_n).$$

In this vector, awards and losses are equally weighted and equally treated. We also consider vectors where awards and losses are not equally treated. Given the allocation x we define its associated weighted vector of awards-losses as follows:

$$\lambda\text{-}x^{AL} = ((1-\lambda)x_1, ..., (1-\lambda)x_n, \lambda(x_1 - d_1), ..., \lambda(x_n - d_n))$$

where $\lambda \in [0, 1]$. Note that λ-x^{AL} with $\lambda = \frac{1}{2}$ is the vector of equal weights, which in our study is equivalent to considering x^{AL} or λ-x^{AL} with $\lambda = \frac{1}{2}$. The following set

$$|\lambda\text{-}AL(N, d, E)| = \left\{ \left|\lambda\text{-}x^{AL}\right| : x \in F(N, d, E) \right\}$$

is the set of vectors of awards-losses taken in absolute terms. Note that we use the notation λ-x^{AL} instead of λ-$x^{AL}(N, d, E)$. We consider there is no confusion, so we prefer the notation λ-x^{AL} for the sake of simplicity.

3 The Lorenz criterion

The first egalitarian criterion we consider is the Lorenz criterion. The Lorenz order is not complete and therefore by applying this criterion we do not, in general, obtain uniqueness. In this sense, the set of Lorenz maximal allocations (the set of Lorenz undominated allocations) can be seen as the maximal set of fair allocations. A Lorenz dominated allocation is not a candidate for selection when looking for fair allocations. The set of Lorenz undominated allocations is defined as follows:

$$L(N,d,E) = \left\{ \begin{array}{c} x \in F(N,d,E); \text{there is no } y \in F(N,d,E) \\ \text{such that } y^{AL} \succ_L x^{AL} \end{array} \right\}.$$

The Lorenz maximal set coincides with the set of allocations that satisfy order preservation in both ways, awards and losses Theorem 1). Therefore, order preservation emerges as a minimal requirement for a fair allocation.

The proof of this result relies on the following fact. For two elements k and l, a vector x, and a real number $\alpha > 0$, we say that (k,l,x,α) is an *equalizing bilateral transfer* (of size α from k to l with respect to x) if

$$x_k - \alpha \geq x_l + \alpha.$$

Now, Lemma 2 of Hardy, Littlewood and Polya (1952) implies that an allocation y Lorenz dominates another allocation x only if y can be obtained from x by a finite sequence of equalizing bilateral transfers.

Theorem 1 *The Lorenz maximal set coincides with the set of all allocations that satisfy order preservation in both ways: awards and losses.*

Proof. Let $x \in F(N,d,E)$ be such that x is not order preserving for awards. Therefore, there are claimants i,j such that $d_i \geq d_j$ and $x_i < x_j$. Then it also holds that $d_i - x_i > d_j - x_j$. Consider the following allocation z :

$$z_l = \begin{cases} x_l + \varepsilon & \text{if } l = i \\ x_l - \varepsilon & \text{if } l = j \\ x_l & \text{otherwise} \end{cases}$$

where $\varepsilon = \min(\frac{x_j - x_i}{2}, \frac{(d_i - x_i) - (d_j - x_j)}{2})$.

It is not difficult to check that $z^{AL} \succ_L x^{AL}$ since it still holds that $z_i \leq z_j$ and $d_i - z_i \geq d_j - z_j$. The proof is similar in the case where x violates order preservation for losses.

Let x be an allocation satisfying order preservation for awards and losses. Then

$$\sum_{1 \leq i \leq n} \theta_i(x^{AL}) = E - \sum_{1 \leq i \leq n} d_i$$

since the first n elements of the vector $\theta(x^{AL})$ are the ordered losses $(x_n - d_n, ..., x_1 - d_1)$[5]. Note also that

$$\sum_{i=n+1}^{2n} \theta_i(x^{AL}) = \sum_{1 \leq i \leq n} x_i = E$$

since the last n elements of the vector $\theta(x^{AL})$ are the ordered awards $(x_1, ..., x_n)$. Therefore, if there is an allocation z such that $z^{AL} \succ_L x^{AL}$ should be the case that $z^L \succ_L x^L$ and $z^A \succeq_L x^A$ or $z^L \succeq_L x^L$ and $z^A \succ_L x^A$. If $z^A \succ_L x^A$ then z^A can be obtained from x^A by a finite sequence of equalizing bilateral transfers.

Now consider a vector y^A resulting from x^A after a bilateral equalizing transfer. Let i, j two claimants such that $x_i < x_j$

$$y_l = \begin{cases} x_l + \varepsilon & \text{if } l = i \\ x_l - \varepsilon & \text{if } l = j \\ x_l & \text{otherwise} \end{cases}$$

where $0 < \varepsilon \leq \frac{x_j - x_i}{2}$.

It is clear that $y^A \succ_L x^A$ implies that $x^L \succ_L y^L$ and therefore y^{AL} does not Lorenz dominate x^{AL}.

A similar consideration follows for the case where we consider Lorenz domination with respect to the vector x^L. That is, if there exists an allocation y such that $y^L \succ_L x^L$ then $x^A \succ_L y^A$ and therefore y^{AL} does not Lorenz dominate x^{AL}. ■

Figure 1 shows the Lorenz maximal set when E moves from 0 to $d_1 + d_2$. As we know by theorem 1, figure 1 also is representing the set of all allocations that satisfy order preservation in awards and losses when E moves from 0 to $d_1 + d_2$.

The following corollary arises immediately since a convex combination of order preserving allocations is also order preserving.

Corollary 2 *The Lorenz maximal set is convex.*

[5] If there is any $x_1 < (x_i - d_i)$ we have the following contradiction: $x_1 < (x_i - d_i) \leq (x_1 - d_1) < x_1$.

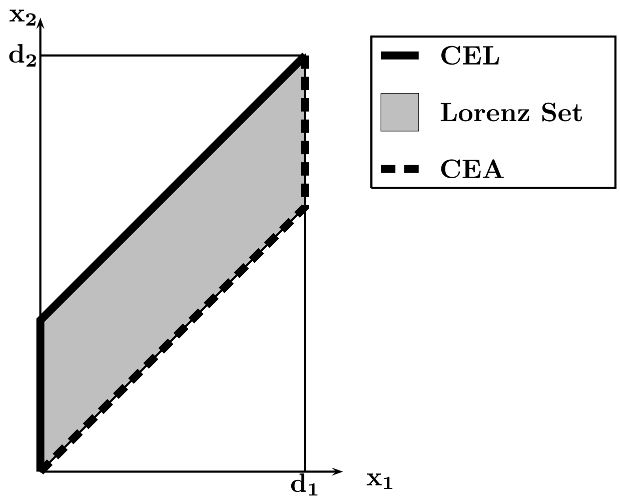


Figure 1: *Illustration of the **Lorenz maximal** set when E moves from 0 to $d_1 + d_2$.*

Note that if an allocation x is order preserving in (N, d, E) then $(x_i)_{i \in S}$ is also order preserving in $(S, (d_i)_{i \in S}, \sum_{i \in S} x_i)$ and therefore the Lorenz set satisfies the consistency principle.

Corollary 3 *Let $x \in L(N, d, E)$. Then $(x_i)_{i \in S} \in L(S, (d_i)_{i \in S}, \sum_{i \in S} x_i)$.*

We also define the weighted Lorenz maximal set for $\lambda \in [0, 1]$ as follows:

$$\lambda - L(N, d, E) = \left\{ \begin{array}{c} x \in F(N, d, E); \text{there is no } y \in F(N, d, E) \\ \text{such that } \lambda\text{-}y^{AL} \succ_L \lambda\text{-}x^{AL} \end{array} \right\}.$$

A direct consequence of the proof of Theorem 1 is that for $\lambda \in (0, 1)$ the weighted Lorenz maximal sets coincide. This is so because whenever $\lambda \in (0, 1)$ it is still true that $\sum_{1 \le i \le n} \theta_i(\lambda\text{-}x^{AL}) = \lambda(E - \sum_{1 \le i \le n} d_i)$ and $\sum_{n+1 \le i \le 2n} \theta_i(\lambda - x^{AL}) = (1 - \lambda)E$. Therefore the arguments of the proof can be repeated.

However it is immediately apparent that if we take $\lambda = 0$

$$L(N, d, E) = \left\{ \begin{array}{c} x \in F(N, d, E); \text{there is no } y \in F(N, d, E) \\ \text{such that } y \succ_L x \end{array} \right\}$$

coincides with $CEA(N,d,E)$ and if we take $\lambda = 1$ the set

$$L(N,d,E) = \left\{ \begin{array}{c} x \in F(N,d,E); \text{there is no } y \in F(N,d,E) \\ \text{such that } y^L \succ_L x^L \end{array} \right\}$$

coincides with $CEL(N,d,E)$.

The last two results were noted by Bosmans *et al.* (2007) when studying Lorenz comparisons between vectors of n elements (being n the number of claimants). Many other authors have considered Lorenz comparisons of vectors of n elements in their works. For example, this type of analysis can be found in Thomson (2007).

This analysis points out a natural question: If we consider the vector of awards-losses in absolute terms does the new Lorenz set coincide with the set of allocations that satisfy order preservation in both ways? The answer is not.

We define the new **Lorenz maximal set** as a set of Lorenz undominated allocations in the following terms:

$$L_{AT}(N,d,E) = \left\{ \begin{array}{c} x \in F(N,d,E); \text{there is no } y \in F(N,d,E) \\ \text{such that } \left|y^{AL}\right| \succ_L \left|x^{AL}\right| \end{array} \right\}.$$

The new set is a subset of the Lorenz set defined above.

Theorem 4 *The set $L_{AT}(N,d,E)$ coincides with the set of all allocations that satisfy half claim boundedness and order preservation in both ways: awards and losses.*

Proof. Let $x \in F(N,d,E)$ be such that x is not order preserving for awards. Therefore, there are claimants i,j such that $d_i \geq d_j$ and $x_i < x_j$. Then it also holds that $d_i - x_i > d_j - x_j$. Consider the following allocation z :

$$z_l = \left\{ \begin{array}{cc} x_l + \varepsilon & \text{if } l = i \\ x_l - \varepsilon & \text{if } l = j \\ x_l & \text{otherwise} \end{array} \right.$$

where $\varepsilon = \min(\frac{x_j - x_i}{2}, \frac{(d_i - x_i) - (d_j - x_j)}{2})$.

It is not difficult to check that $\left|z^{AL}\right| \succ_L \left|x^{AL}\right|$ since it still holds that $z_i \leq z_j$ and $d_i - z_i \geq d_j - z_j$. The proof is similar in the case where x violates order preservation for losses.

Let $x \in F(N,d,E)$ be such that x does not satisfy HCB. Therefore, there are claimants i,j such that $x_i < \frac{di}{2}$ and $x_j > \frac{d_j}{2}$. Then it also holds that $d_i - x_i > x_i$

and $x_j > d_j - x_j$. Consider the following allocation z :

$$z_l = \begin{cases} x_l + \varepsilon & \text{if } l = i \\ x_l - \varepsilon & \text{if } l = j \\ x_l & \text{otherwise} \end{cases}$$

where ε is such that still holds that $d_i - z_i \geq z_i$ and $z_j \geq d_j - z_j$.

It is not difficult to check that $\left|z^{AL}\right| \succ_L \left|x^{AL}\right|$.

Assume that $E \leq \frac{1}{2}\sum_{i \in N} d_i$ and let x be an allocation satisfying HCB and order preservation in both ways. For any two claimants i, j (assuming $d_l \leq d_j$) it holds that (by HCB of x) $d_i - x_i \geq x_i$ and $d_j - x_j \geq x_j$. Since x also satisfies order preservation it also holds that $x_j \geq x_i$ and $d_j - x_j \geq d_l - x_l$.

Therefore we conclude that x_i is the minimum among the four numbers while $d_j - x_j$ is the maximum. Assume that allocation z results from a bilateral transfer made by claimant i to claimant j. That implies that $z_i < x_i$ and therefore $\left|z^{AL}\right|$ cannot Lorenz dominate $\left|x^{AL}\right|$. Assume that allocation z results from a bilateral transfer made by claimant j to claimant i. That implies that $d_j - z_j > d_j - x_j$ and therefore $\left|z^{AL}\right|$ cannot Lorenz dominate $\left|x^{AL}\right|$. The proof is almost identical if we consider $E > \frac{1}{2}\sum_{i \in N} d_i$. Therefore there is no bilateral transfer between claimants allowing a new allocation that can be used to claim that x is not an element of the set $L_{AT}(N, d, E)$. ■

Figure 2 illustrates the set $L_{AT}(N, d, E)$ in two-claimant problems when E moves from 0 to $d_1 + d_2$ which coincides with the set of allocations that satisfy HCB. Figure 2(a) shows the set of allocations that satisfy HCB when $\frac{d_2}{2} < d_2 - d_1$, and similarly, figure 2(b) shows the set of allocations that satisfy HCB when $\frac{d_2}{2} > d_2 - d_1$.

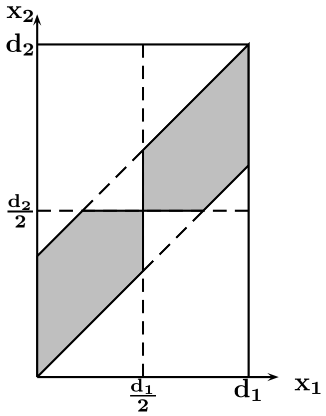
Following almost identical arguments as in Theorem 4, the following theorem can be proved.

Theorem 5 *The set λ-$L_{AT}(N, d, E)$ coincides with the set of all allocations that satisfy λ-claim boundedness and order preservation in both ways: awards and losses.*

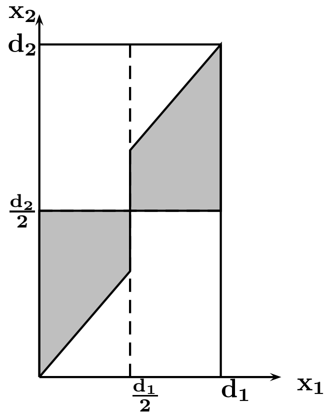
The set λ-$L_{AT}(N, d, E)$ is defined as follows:

$$\lambda\text{-}L_{AT}(N, d, E) = \left\{ \begin{array}{c} x \in F(N, d, E); \text{there is no } y \in F(N, d, E) \\ \text{such that } \left|\lambda - y^{AL}\right| \succ_L \left|\lambda - x^{AL}\right| \end{array} \right\}.$$

Since the Proportional Rule is the only sharing rule satisfying λ-CB for any $\lambda \in (0, 1)$ the following corollary is immediate.



2(a) L_{AT} when $\frac{d_2}{2} > d_2 - d_1$



2(b) L_{AT} when $\frac{d_2}{2} < d_2 - d_1$

Figure 2: *Illustration of the set* $L_{AT}(N, d, E)$ *in two-claimant problems when* E *moves from* 0 *to* $d_1 + d_2$.

Corollary 6 $\underset{\lambda\in(0,1)}{\cap} \lambda\text{-}L_{AT}(N, d, E) = \{PR(N, d, E)\}$.

Proof. Let (N, E, d) be a bankruptcy problem and let $\lambda = \frac{E}{\sum_{n\geq l\geq 1} d_l}$. Then $E = \lambda \sum_{n\geq l\geq 1} d_l$ and therefore $\lambda\text{-}L(N, d, E) = \{\lambda d\} = \{PR(N, d, E)\}$. ■

The Proportional Rule is the only rule that selects Lorenz maximal outcomes for any problem whenever awards and losses are simultaneously considered[6].

4 The Lexicographic criterion

4.1 The Maximin Principle

A central rule in the literature of bankruptcy problems is the Talmud Rule introduced by Aumann and Maschler (1985). This rule explains the resolution of three numerical examples that can be found in the Talmud. For many years was an open problem what rule was behind these examples. Aumann and Maschler prove that their rule prescribes the proposals of the examples in the Talmud.

[6]Note that if we only consider awards or losses the only allocation that is Lorenz maximal is the CEA allocation or the CEL allocation.

They also prove that the rule coincides with the nucleolus of a TU game associated with the bankruptcy problem. Given a bankrupycy problem (N, d, E) we define its associated bankruptcy game as a TU game (N, v) where N is the set of claimants and $v(S) = \max\left\{E - \sum_{l \notin S} d_l, 0\right\}$. See O'Neill (1982).

The nucleolus (Schmeidler, 1969) selects lexicographical maximal elements in the set of vectors of satisfactions of the coalitions. We prove that the Talmud Rule is also a Lexicographic rule. First we introduce the definition of the Talmud Rule.

Let (N, d, E) be a bankruptcy problem. Then

$$T_i(N, d, E) = \begin{cases} \min\left\{\frac{d_i}{2}, \alpha\right\} & \text{if } E \leq \frac{\sum_{n \geq l \geq 1} d_l}{2} \\ \frac{d_i}{2} + \max\left\{\frac{d_i}{2} - \alpha, 0\right\} & \text{otherwise} \end{cases}$$

where α is chosen such that $\sum_{n \geq i \geq 1} T_i(N, d, E) = E$.

This rule provides the allocation whose vector of awards-losses is the lexicographically maximal vector in the set $|AL(N, d, E)|$. That is,

Theorem 7 *Let (N, d, E) be a bankruptcy problem. Then*
$T(N, d, E) = \left\{x \in F(N, d, E); \left|x^{AL}\right| \succ_{lex} \left|y^{AL}\right|, \text{ for all } y \in F(N, d, E)\right\}$.

Proof. Let $z = T(N, d, E)$. We distinguish 4 cases:

a) $E \leq \frac{\sum_{n \geq l \geq 1} d_l}{2}$ and $z_i < \frac{d_i}{2}$ for all $i \in N$.

Then the first n elements of the vector $\theta(\left|z^{AL}\right|)$ are $(\frac{E}{n}, ..., \frac{E}{n})$ and clearly $\left|z^{AL}\right|$ lexicographically dominates any other vector $\left|y^{AL}\right|$ where y is an allocation.

b) $E \leq \frac{\sum_{n \geq l \geq 1} d_l}{2}$ and $z_l = \frac{d_i}{2}$ for all $l \in \{1, ..., k\}$. Then the first $2k$ elements of the vector $\theta(\left|z^{AL}\right|)$ are $(\frac{d_1}{2}, \frac{d_1}{2}, ..., \frac{d_k}{2}, \frac{d_k}{2})$ and the next $(n-k)$ elements are $(\frac{E - \frac{1}{2}\sum_{k \geq l \geq 1} d_l}{n-k}, ..., \frac{E - \frac{1}{2}\sum_{k \geq l \geq 1} d_l}{n-k})$. Clearly $\left|z^{AL}\right|$ lexicographically dominates any other vector $\left|y^{AL}\right|$ where y is an allocation.

c) $E > \frac{\sum_{n \geq l \geq 1} d_l}{2}$ and $d_i > z_i > \frac{d_i}{2}$ for all $i \in N$. Then the first n elements of the vector $\theta(\left|z^{AL}\right|)$ are $(\frac{\sum_{n \geq l \geq 1} d_l - E}{n}, ..., \frac{\sum_{n \geq l \geq 1} d_l - E}{n})$ and clearly $\left|z^{AL}\right|$ lexicographically dominates any other vector $\left|y^{AL}\right|$ where y is an allocation.

d) $E > \frac{\sum_{n \geq l \geq 1} d_l}{2}$ and $z_l = \frac{d_i}{2}$ for all $l \in \{1, ..., k\}$. Then the first $2k$ elements of the vector $\theta(\left|z^{AL}\right|)$ are $(\frac{d_1}{2}, \frac{d_1}{2}, ..., \frac{d_k}{2}, \frac{d_k}{2})$ and the next $(n-k)$ elements are

$$(\frac{\frac{1}{2}\sum_{n \geq l \geq k+1} d_l - \frac{1}{2}E}{n-k}, ..., \frac{\frac{1}{2}\sum_{n \geq l \geq k+1} d_l - \frac{1}{2}E}{n-k}).$$

Clearly $\left|z^{AL}\right|$ lexicographically dominates any other vector $\left|y^{AL}\right|$ where y is an allocation. ■

Weighted Talmud Rules[7] are introduced and studied by Moreno-Ternero and Villar (2006). They call this family of rules the TAL-family.

Let (N, d, E) be a problem. Then

$$\lambda\text{-}T_i(N,d,E) = \begin{cases} \min\{\lambda d_i, \alpha\} & \text{if } E \leq \lambda \sum_{n\geq l\geq 1} d_l \\ \lambda d_i + \max\{(1-\lambda)d_i - \alpha, 0\} & \text{otherwise} \end{cases}$$

where α is chosen such that $\sum_{n\geq i\geq 1} \lambda\text{-}T_i(N,d,E) = E$.

It is not difficult to check that this rule provides the allocation whose vector of awards-losses is maximal in the set $\left|AL^{\lambda}(I(N,d,E))\right|$. That is for $\lambda \in (0,1)$ we have that

$$\lambda\text{-}T(N,d,E) = \left\{x \in F(N,d,E); \left|\lambda\text{-}x^{AL}\right| \succeq_{Lex} \left|\lambda\text{-}y^{AL}\right|, \text{ for all } y \in F(N,d,E)\right\}.$$

Given a bankruptcy problem (N, d, E) and $\lambda = \frac{E}{\sum_{n\geq l\geq 1} d_l}$ it holds that $E = \lambda \sum_{n\geq l\geq 1} d_l$ and therefore $\lambda\text{-}T_i(N,d,E) = \lambda d_i$. This fact is the proof of the following corollary.

Corollary 8 *Let (N, d, E) be a problem where $E = \lambda \sum_{n\geq l\geq 1} d_l$. Then λ-$T((N,d,E)) = PR((N,d,E))$.*

Aumann and Maschler (1985) characterize the Talmud Rule as the unique consistent rule for bankruptcy problems (Theorem A). In their work consistency is also called CG-consistency and explained as follows:

> Intuitively, a solution is consistent if any two claimants i,j use the *contested garment principle* to divide between them the total amount $x_i + x_j$ awarded to them by the solution.

The *contested garment principle* is a solution used to solve two-claimant problems. The solution coincides with the Talmud Rule and the theorem can be interpreted as follows; the Talmud Rule is the unique solution that consistently extends to n claimant problems the *contested garment principle*.

Replacing the *contested garment principle* by the solution prescribed by a Weighted Talmud Rule in two claimant problems we can characterize this Weighted Talmud Rule as the unique rule that consistently extends to n claimant problems this solution prescribed for two claimant problems.

[7] The term *Weighted Talmud Rule* is introduced by Hokari and Thomson (2003). In their case the weights refer to the claimants and not to awards\losses. We keep the term since we think there is no confusion and it is more consistent with the rest of the paper.

4.2 The Minimax Principle

This interpretation of the Talmud Rule, as a rule based in a *lexicographic maximin* criterion, suggests the definition of a new rule based on the *lexmax (lexicographic minimax)* criterion. In the literature of TU games this criterion inspires the definition of the Lexmax rule and the antinucleolus (see Arin (2007)). See also Luss (1999) for the application of the minimax principle in other models.

We call the new rule *Lexmax Rule*, and we denote it by *LM*, formally,

Definition 9 *Let* (N, d, E) *be a bankruptcy problem. Then* $LM(N, d, E) = \left\{x \in F(N, d, E); \left|x^{AL}\right| \succ_{lm} \left|y^{AL}\right|, \text{ for all } y \in F(N, d, E)\right\}$.

This rule satisfies order preservation (in both ways) and HCB since it provides allocations that belong to the set $L_{AT}(N, d, E)$. It is also quite immediately apparent that the new rule satisfies consistency.

The three facts can be used to define the following algorithm in order to compute the Lexmax Rule of a bankruptcy problem.

A procedure for computing the Lexmax Rule of a bankruptcy problem

Let (N, d, E) be a problem. In order to obtain $LM((N, d, E))$ consider the following 4 cases:

a) Let $E < \frac{\sum_{n \geq i \geq 1} d_i}{2}$ and $\frac{d_n}{2} < d_n - CEL_n(N, d, E)$. Then

$$LM(N, d, E) = CEL(N, d, E).$$

b) Let $E < \frac{\sum_{n \geq i \geq 1} d_i}{2}$ and $\frac{d_n}{2} \geq d_n - CEL_n(N, d, E)$. Then

$$LM_n(N, d, E) = \frac{d_n}{2}.$$

To obtain the allocation for the rest of the claimants consider the problem $A_{n-1} = (N \backslash \{n\}, (d_i)_{i \in \{1,\ldots,n-1\}}, E - \frac{d_n}{2})$. If $\frac{d_n - 1}{2} < d_{n-1} - CEL_{n-1}(A_{n-1})$ then

$$LM(A_{n-1}) = CEL(A_{n-1}).$$

If $\frac{d_{n-1}}{2} \geq d_{n-1} - CEL_{n-1}(A_{n-1})$ then

$$LM_{n-1}(A_{n-1}) = \frac{d_n - 1}{2}.$$

To obtain the allocation for the rest of the claimants consider the problem $A_{n-2} = (N \backslash \{n, n-1\}, (d_i)_{i \in \{1,\ldots,n-2\}}, E - \frac{d_n}{2} - \frac{d_{n-1}}{2})$ and continue with this procedure until an allocation for all claimants is obtained.

c) Let $E \geq \frac{\sum_{n \geq i \geq 1} d_i}{2}$ and $\frac{d_n}{2} < d_n - CEA_n(N,d,E)$. Then

$$LM(N,d,E) = CEA(N,d,E).$$

d) Let $E \geq \frac{\sum_{n \geq i \geq 1} d_i}{2}$ and $\frac{d_n}{2} \geq d_n - CEA_n(N,d,E)$. Then

$$LM_n(N,d,E) = \frac{d_n}{2}.$$

To obtain the allocation for the rest of the claimants consider the problem $A_{n-1} = (N \backslash \{n\}, (d_i)_{i \in \{1,\ldots,n-1\}}, E - \frac{d_n}{2})$. If $\frac{d_n - 1}{2} < d_{n-1} - CEA_{n-1}(A_{n-1})$ then

$$LM(A_{n-1}) = CEA(A_{n-1}).$$

If $\frac{d_{n-1}}{2} \geq d_{n-1} - CEL_{n-1}(A_{n-1})$ then

$$LM_{n-1}(A_{n-1}) = \frac{d_n - 1}{2}.$$

To obtain the allocation for the rest of the claimants consider the problem $A_{n-2} = (N \backslash \{n, n-1\}, (d_i)_{i \in \{1,\ldots,n-2\}}, E - \frac{d_n}{2} - \frac{d_{n-1}}{2})$ and continue with this procedure until an allocation for all claimants is obtained.

In case a) CEL satisfies HCB and losses are higher than awards. In case b) CEL of the original problem violates HCB and therefore we fix the allocation of claimant n in order to preserve HCB. The consistency of the Lexmax Rule allows us to seek the allocation of the rest of the claimants in a new reduced problem where again losses are higher than awards. If in the new case CEL satisfies HCB this is the allocation for the rest of the claimants and otherwise we fix the allocation of claimant $n-1$ and we continue with a new reduced problem where again losses are higher than awards.

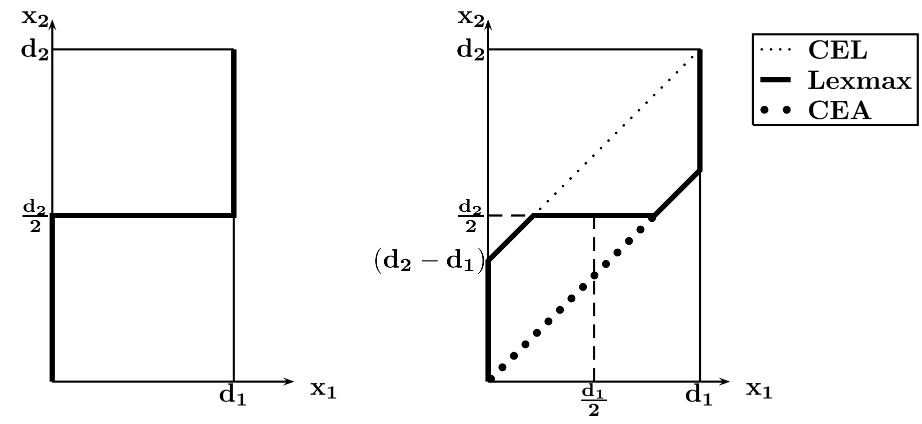
Cases c) and d) are the reverse of cases a and b when awards are higher than losses and the reference is CEA instead of CEL.

Figure 3 illustrates how these rules perform in two-claimant problems when E moves from 0 to $d_1 + d_2$. Figure 3(a) shows the Lexmax Rule when $\frac{d_2}{2} \leq (d_2 - d_1)$. Similarly, figure 3(b) shows the case $\frac{d_2}{2} > (d_2 - d_1)$.

Similarly, given a bankruptcy problem (N,d,E), the λ-Lexmax Rules are defined as follows:

$$\lambda\text{-}LM(N,d,E) = \left\{ x \in F(N,d,E); \left| \lambda\text{-}x^{AL} \right| \succ_{lm} \left| \lambda\text{-}y^{AL} \right|, \text{ for all } y \in F(N,d,E) \right\}.$$

The computation of the Weighted Lexmax Rules results from replacing the parameter $\frac{1}{2}$ by λ in the procedure above. The procedure can be used to provide



$2(a)$ LM when $\frac{d_2}{2} \leq d_2 - d_1$ $\qquad$ $2(b)$ LM when $\frac{d_2}{2} > d_2 - d_1$

Figure 3: *Illustration of the* ***Lexmax Rule*** *when E moves from 0 to $d_1 + d_2$.*

the following alternative definition of the Lexmax Rule that shares similarities with the definition of the Talmud Rule.

We introduce the definition of the Talmud Rule.

Let (N, d, E) be a bankruptcy problem. Then

$$LM_i(N, d, E) = \begin{cases} \max\left\{\min\left\{\frac{d_i}{2}, d_i - \alpha\right\}, 0\right\} & \text{if } E \leq \frac{\sum_{n \geq l \geq 1} d_l}{2} \\ \min\left\{\max\left\{\alpha, \frac{d_i}{2}\right\}, d_i\right\} & \text{otherwise} \end{cases}$$

where α is chosen such that $\sum_{n \geq i \geq 1} T_i(N, d, E) = E$.

If the Estate is less than half of the total claims the Talmud Rule provides the CEA allocation whenever this allocation satisfies HCB (See Chun et al. (2001)). In the other case the Talmud Rule assigns the CEL allocation whenever this allocation satisfies HCB. The Lexmax Rule replaces CEA by CEL in the first case and CEL by CEA in the second case. Therefore the Lexmax Rule is a natural counterpart of the Talmud Rule. The following table represents different bankruptcy problems all of them with the same set of claimants and claims $((100, 200, 300))$.

E	Talmud Awards	Lexmax Awards	Talmud Losses	Lexmax Losses
100	$(33\frac{1}{3}, 33\frac{1}{3}, 33\frac{1}{3})$	$(0, 0, 100)$	$(66\frac{2}{3}, 166\frac{2}{3}, 266\frac{2}{3})$	$(100, 200, 200)$
200	$(50, 75, 75)$	$(0, 50, 150)$	$(50, 125, 225)$	$(100, 150, 150)$
300	$(50, 100, 150)$	$(50, 100, 150)$	$(50, 100, 150)$	$(50, 100, 150)$
400	$(50, 125, 225)$	$(100, 150, 150)$	$(50, 75, 75)$	$(0, 50, 150)$
500	$(66\frac{2}{3}, 166\frac{2}{3}, 266\frac{2}{3})$	$(100, 200, 200)$	$(33\frac{1}{3}, 33\frac{1}{3}, 33\frac{1}{3})$	$(0, 0, 100)$

The first three problems, mentioned in the Talmud, motivate the paper by Aumann and Maschler (1985).

Note that if $E < \frac{\sum_{n \geq l \geq 1} d_l}{2}$ then $T(N, d, E) \succ_L LM(N, d, E)$.

If $E > \frac{\sum_{n \geq l \geq 1} d_l}{2}$ then $LM(N, d, E) \succ_L T(N, d, E)$. The situation is reversed if we compare losses, that is, if $E < \frac{\sum_{n \geq l \geq 1} d_l}{2}$ then $(d - LM(N, d, E)) \succ_L (d - T(N, d, E))$ and if $E > \frac{\sum_{n \geq l \geq 1} d_l}{2}$ then $(d - T(N, d, E)) \succ_L (d - LM(N, d, E))$.

Finally, we add that in the literature of bankruptcy problems the Reverse Talmus Rulw has been defined.

The Reverse Talmud Rule is defined as follows:

$$RT_i(N, d, E) = \begin{cases} \max\left\{\frac{d_i}{2} - \alpha, 0\right\} & \text{if } E \leq \frac{\sum_{n \geq l \geq 1} d_l}{2} \\ \frac{d_i}{2} + \min\left\{\frac{d_i}{2}, \alpha\right\} & \text{otherwise} \end{cases}$$

Arin and Benito (2010) show that the Reverse Talmud Rule is a Least Square value. The Talmud Rule allows two different interpretations:

1.- If $E \leq \frac{\sum_{n \geq l \geq 1} d_l}{2}$ the rule provides the CEA allocation whenever the allocation satisfies HCB. If $E > \frac{\sum_{n \geq l \geq 1} d_l}{2}$ the rule provides the CEL allocation whomever the allocation satisfies HCB.

2.- If $E \leq \frac{\sum_{n \geq l \geq 1} d_l}{2}$ the rule provides the CEA allocation of a new problem where the claims are half of the original claims. If $E > \frac{\sum_{n \geq l \geq 1} d_l}{2}$ the rule provides to each claimant half of his\her claim plus the CEL allocation of a new problem where the claims are half of the original claims and the Estate results $E - \frac{\sum_{n \geq l \geq 1} d_l}{2}$.

Replacing CEA by CEL and CEL by CEA the first interpretation provides the Lexmax rule. In the second interpretation the same replacement originates the Reverse Talmud rule.

5 Conclusions

This research can be summarized with the table below. In the table rules are linked with egalitarian criteria and sets and can be interpreted as answers to the following two questions:

1. What egalitarian criterion is used to make egalitarian comparisons between elements?
2. From what set are those elements taken?

Rule	**Criterion**	**Set**	**Weight** : λ
CEA	Lex and Lexmax	$A(N, d, E)$	
CEL	Lex and Lexmax	$L(N, d, E)$	
LM	lexmax	$\lvert\lambda\text{-}AL(N, d, E)\rvert$	$\frac{1}{2}$
T	Lex	$\lvert\lambda\text{-}AL(N, d, E)\rvert$	$\frac{1}{2}$
λ-T	Lex	$\lvert\lambda\text{-}AL(N, d, E)\rvert$	λ
PR	Lexmax and Lex	$\lvert\lambda\text{-}AL(N, d, E)\rvert$	$\frac{E}{\sum_{n \geq l \geq 1} d_l}$

The table gives a unified framework to place many different rules that have been introduced and analyzed by several authors.

The table[8] also indicates how to extend this type of solutions to other different settings. In particular, in airport problems (Littlechild, 1974) it is generally

[8]The Reverse Talmud rule and its associated Reverse λ-Talmud rules also solve the same questions. The egalitarian criterion used to generate the rules is the Least Square criterion. See Arin and Benito /2010) for details.

accepted that solutions must select core allocations and not merely imputations. Therefore, the search for egalitarian maximal elements should be restricted to the core of the airport problem. In other settings, other constraints may exist and solutions are required to satisfy them. This is a restriction of the set where egalitarian maximal elements are sought. Also in claim problems different constraints could be considered.

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