Strategy-proof location of public facilities

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September 4, 2015

Abstract

Agents frequently have different opinions on where to locate a public facility. While some agents consider the facility a good and prefer to have it nearby, others dislike it and would like to see it built far away from their own locations. To aggregate agents' preferences in these situations, we propose a new preference domain according to which each agent is allowed to have single-peaked or single-dipped preferences on the location of the facility, but in such a way that the peak or dip is situated in her own location. We characterize all strategy-proof rules in this general framework and show that they are also group strategy-proof. Finally, we characterize for some focal cases the rules that additionally satisfy Pareto efficiency.

Keywords: Single-peaked preferences, single-dipped preferences, social choice rule, strategy-proofness, Pareto efficiency.

JEL-Numbers: D70, D71, D79.

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1 Introduction

Governments frequently decide where to locate public facilities like schools, hospitals, prisons, nuclear plants, or industrial parks. In order to select a location for a particular facility, the public decision makers have to take into account not only technical constraints but also the preferences of the agents in the society. Since preferences are private information and agents have incentives to reveal them truthfully or not depending on how this information is incorporated in the selection process, the objective is to construct social choice rules that induce agents to reveal their true preferences (a property known as strategy-proofness). However, Gibbard [7] and Satterthwaite [10] have shown in their famous impossibility result that if there are at least three feasible locations, strategy-proofness combined with a range of more than two alternatives or Pareto efficiency leads to dictatorial rules when preferences are unrestricted.

Given this negative result, the literature has aimed at constructing meaningful strategy-proof rules in situations when the preference domain can be restricted naturally (see Barberà [1] for a survey). The most well–known preference restrictions for the location of public facilities are the single-peaked and the single-dipped preferences, which appear when the considered facility is, respectively, a public good or a public bad. Formally, an agent has single-peaked (single-dipped) preferences on the real line if she has a best (worst) point and the further one moves away from this point maintaining the direction, the worse (better) the locations get. Moulin [9] and Barberà and Jackson [4] have shown that all strategy-proof rules on the domain of single-peaked preferences are generalized median voter rules. Similarly, Manjunath [8] has established for the domain of single-dipped preferences that all strategy-proof and unanimous rules have to always select an extreme location (see also Barberà et al. [3]).

However, the aforementioned frameworks cannot accommodate situations in which it is a priori unclear whether an agent considers a facility a good or a bad.[1] Therefore, a general model should include both single-peaked and single-dipped preferences as admissible preferences. Moreover,

[1]Paradigmatic examples include, among many others, dog parks, industrial parks, and soccer stadiums.

agents typically have single-peaked or single-dipped preferences where the peak or dip is situated in their own locations (their homes, their municipalities, ...). Since the location of the agents is frequently known to the social planner (for example, regional governments are aware of the location of each municipality and local governments know the residence of each person), we propose the domain where the set of admissible preferences of an agent corresponds to all single-peaked preferences with the peak in her location and all single-dipped preferences with the dip in her location.[2]

Our main result is a characterization of all strategy-proof rules on this preference domain. Thereby it will become clear that it is possible to escape from the Gibbard-Satterthwaite impossibilities in many instances. In particular, we will see that all strategy-proof rules share the following common structure: (*i*) first, all agents are only asked about their type of preferences (single-peaked or single-dipped) and depending on the set of agents that have single-peaked preferences, at most two locations are preselected; (*ii*) if only one alternative is preselected, then this alternative is finally chosen; and (*iii*) if two alternatives are preselected, then all agents that are situated strictly between the two preselected alternatives have to indicate their ordinal preference over them (and, in case of indifference, their entire preferences) and, depending on the answers, it is decided which of the two preselected alternatives is finally chosen. The particular ways to pick the preselected locations that pass to the second phase and the form in which the final location is chosen have to satisfy some monotonicity conditions. These conditions are generally not too restrictive and leave space for the construction of Pareto efficient and strategy-proof rules when there are agents that are situated at feasible locations and/or to the left or right of all feasible locations. Additionally, we show that the strategy-proof rules are also group strategy-proof.

Our findings can improve the way in which the decisions of where to locate some facilities are taken.

[2]Observe that the domain cannot be extended to include for each agent all single-peaked and all single-dipped preferences because this immediately leads to the Gibbard-Satterthwaite impossibilities. To see why, observe that an agent with single-dipped preferences will manipulate any generalized median voter rule in some instance. See Berga and Serizawa [5] for additional results showing the difficulty of expanding the domain of single-peaked preferences without arriving at the impossibilities.

It complements the classical domains of single-peaked and single-dipped preferences in such a way that the decision of which is the best domain restriction and, as a consequence, which are the most appropriate rules to implement depends on the structure of the particular facility. To see this, consider the following two alternative assumptions: (i) the facility is considered unanimously a good or a bad; or (ii) each agent can like or dislike the facility, but the peak/dip of her preferences is the point in which this agent is situated. If the second assumption is more (less) plausible, then this new framework is more (less) appropriate than the uniform single-peaked or single-dipped one.[3]

The remainder of the paper is organized as follows. Section 2 introduces the necessary notation, definitions and some examples that illustrate some of the insights of the results developed in the following sections. Section 3 introduces some conditions that define the main structure of the strategy-proof social choice rules. Section 4 provides a complete characterization of all strategy-proof rules and shows that group strategy-proofness is obtained for free. Finally, we explain when this characterized family allows us to escape from the Gibbard-Satterthwaite impossibilities and characterize for some focal cases the rules that additionally satisfy Pareto efficiency. All proofs are relegated to the Appendix.

2 Notation, definitions, and examples

Consider a social planner that wants to locate a public facility in a point on a set $T \subseteq \mathbb{R}$ of feasible locations. There is a finite group of agents N, and each agent belonging to N is situated at a point on the real line. We denote the agent situated at $i \in \mathbb{R}$ by $i \in N$.[4] For the moment, we do

[3] An alternative model would partition the set of agents into those that can only have single-peaked and those that can only have single-dipped preferences, but allows agents to declare any peak/dip. Feigenbaum and Sethuraman [6] study this model under the additional assumption that preferences are cardinally determined by the distance with respect to the peak/dip and study strategy-proof rules that maximize social welfare in this domain.

[4] This notation assumes implicitly that there is at most one agent at any point of the real line. As it can be seen from the proofs, our results also hold when multiple agents are situated at the same point (only Theorem 5 needs some minor adaptations).

not impose any restriction on N or T, or on the relation between these sets.

Let R_i be the weak preference relation of agent $i \in N$ on T. Formally, R_i is a complete and transitive binary relation. P_i and I_i are the strict and the indifference preference relation induced by R_i. We then say that R_i is a single-peaked preference with peak i if for all $x, y \in T$ such that $i \geq x > y$ or $i \leq x < y$, we have that $x \, P_i \, y$. Similarly, R_i is a single-dipped preference with dip i if for all $x, y \in T$ such that $i \geq x > y$ or $i \leq x < y$, we have that $y \, P_i \, x$.[5] The preference domain of agent i is $\mathcal{R}_i = \mathcal{R}_i^+ \cup \mathcal{R}_i^-$, where $\mathcal{R}_i^+$ ($\mathcal{R}_i^-$) is the set of all single-peaked (single-dipped) preferences with peak (dip) i. Observe that the individual preference domains are personalized.

A preference profile is a set of preferences $R = (R_i)_{i \in N}$. The domain of all admissible preference profiles is denoted by $\mathcal{R} = \times_{i \in N} \mathcal{R}_i$. Let $\mathcal{R}^A$ be the set of preference profiles such that only the agents in $A \subseteq N$ have single-peaked preferences. Sometimes we will write $\mathcal{R}_i^A$ to indicate $\mathcal{R}_i^+$ if $i \in A$, or $\mathcal{R}_i^-$ if $i \notin A$. Similarly, $R_S \in \mathcal{R}_S$ and $R_{-S} \in \mathcal{R}_{-S}$ are the restrictions of R to the agents in $S \subseteq N$ and $(N \setminus S)$, respectively. We will write R_{-i} instead of $R_{-\{i\}}$.

The following concepts will be useful in the course of our analysis. A non-ordered pair of alternatives $\{x, y\}$ is said to be a fixed pair for agent i if for all $a, b \in \{x, y\}$ such that $a < b$, we cannot have that $a < i < b$. Observe that given any type of preferences of agent i (single-peaked or single-dipped), if $\{x, y\}$ is a fixed pair for i, then this agent has always the same ordinal preferences over $\{x, y\}$. Similarly, the ordered pair of alternatives (x, y) is said to be a fixed pair for agent i at $\mathcal{R}_i^A$ if for all $R_i \in \mathcal{R}_i^A$, $x \, P_i \, y$. Or, to say it differently, (x, y) is a fixed pair for i at $\mathcal{R}_i^A$ if $i \in A$ and $[y < x \leq i$ or $y > x \geq i]$, or if $i \notin A$ and $[i \leq y < x$ or $i \geq y > x]$. Finally, we can see that if $\{x, y\}$ is a fixed pair for agent i, then (x, y) is a fixed pair for one type of preferences and (y, x) is a fixed pair for the other type.

The solution concept is a *social choice rule*, a function $f : \mathcal{R} \to T$ that selects for each preference profile $R \in \mathcal{R}$ a feasible location $f(R) \in T$. We denote the range of f in the domain $\mathcal{R}^A$ by $R_f(A)$. We say that f is *manipulable by group* $S \subseteq N$ if each agent belonging to S benefits from

[5] Technically speaking, these preferences only have a maximal/minimal alternative at i if $i \in T$. Otherwise, its most/less preferred feasible alternative is the closest one situated at i's left or right.

a simultaneous misrepresentation of preferences; that is, if there is a preference profile $R \in \mathcal{R}$ and some alternative preferences $R'_S \in \mathcal{R}_S$ such that $f(R'_S, R_{-S})\, P_i\, f(R)$ for all $i \in S$. Then, f is *strategy-proof* if it is not manipulable by any group $S \subseteq N$ with $|S| = 1$. Similarly, f is *group strategy-proof* if it is not manipulable by any group $S \subseteq N$. The social choice rule f is *Pareto efficient* if for all $R \in \mathcal{R}$, there is no $x \in T$ such that $x\, R_i\, f(R)$ for all $i \in N$ and $x\, P_j\, f(R)$ for some $j \in N$. Finally, f is *dictatorial* if there exists an agent $i \in N$ (called the dictator) such that for all $R \in \mathcal{R}$, $f(R)\, R_i\, x$ for all $x \in \bigcup_{A \subseteq N} R_f(A)$.

The remaining part of this section is devoted to three examples of different structures of N and T and their effects on the possibilities of constructing meaningful strategy-proof social choice rules. They provide some first insights on the results that will be developed in the following sections. In the first example each of the three agents is situated between two of the three feasible locations and no agent is situated at a feasible location.

Example 1 *Suppose that* $N = \{3, 4, 5\}$ *and* $T = \{1, 2, 6\}$. *Then, each agent* $i \in N$ *has six possible preferences over* T *under* $\mathcal{R}$. *If she has single-peaked preferences, her possible preferences are* $2P_i1P_i6$, $2P_i6P_i1$, *or* $6P_i2P_i1$. *Similarly, if she has single-dipped preferences, her possible preferences are* $1P_i2P_i6$, $1P_i6P_i2$, *or* $6P_i1P_i2$. *Thus,* $\mathcal{R}$ *coincides with the universal preference domain and the Gibbard-Satterthwaite impossibility results apply.*

Fortunately, Example 1 is an extreme case and mostly we are going to be able to construct meaningful rules. We show that for a particular case in which each agent is either situated to the left or to the right of all feasible locations.

Example 2 *Suppose that* $N = \{1, 2, 8, 9\}$ *and* $T = \{3, 4, 5, 6, 7\}$. *Let the social choice rule* f *be such that for all* $A \subseteq N$ *and all* $R \in \mathcal{R}^A$, $f(R) = 3 + |\{\{1, 2\} \cap (N \setminus A)\}| + |\{\{8, 9\} \cap A\}|$. *It can be checked that* f *is strategy-proof, group strategy-proof, Pareto efficient, non-dictatorial, and has more than two alternatives in its range.*

In the last example, the set of feasible locations is infinite and each agent is situated at a feasible location. Then, it is again possible to find rules that satisfy the desired properties.

Example 3 *Suppose that* $N = \{1, 2, 3\}$ *and* $T = [0, 4]$. *Consider the social choice rule* f *defined through the following procedure: (i) if the set of agents with single-peaked preferences is non-empty, choose the mean location of these agents; and (ii) otherwise, choose by simple majority with any tie-breaker between the extremes* 0 *and* 4. *It can be checked that* f *is strategy-proof, group strategy-proof, Pareto efficient, non-dictatorial, and has more than two alternatives in its range.*

Examples 1 to 3 show the importance of the structure of N and T. In Sections 3 and 4, we analyze the general problem without making any assumptions on N and T. Afterwards, in Section 5, we put some structure on N and T in order to obtain particular characterizations.

3 The main structure of the strategy-proof rules

Our first result is a necessary condition on the range of f that facilitates the further analysis. The condition states that if a social choice rule is strategy-proof, its range can include at most two alternatives for a given set of agents A with single-peaked preferences.

Proposition 1 *Suppose that* f *is strategy-proof. Then, for all* $A \subseteq N$, $|R_f(A)| \leq 2$.

Proposition 1 implies that any strategy-proof rule can be divided into two steps. In the first step, agents have to declare only their types of preferences and depending on the set of agents with single-peaked preferences A, one or two locations are preselected. If only one alternative is preselected, this alternative is finally implemented. If two alternatives are preselected, the alternative that is finally implemented has to be determined in the second step of the procedure. Example 3 illustrates that there are strategy-proof social choice rules that have two alternatives in their range for a given set of agents with single-peaked preferences. Since $R_f(A)$ contains at most two preselected locations, we indicate by $l_f(A)$ and $r_f(A)$ the elements of $R_f(A)$ such that $l_f(A) \leq r_f(A)$. Then, $S_f(A) = N \cap (l_f(A), r_f(A))$ corresponds to the set of agents that are situated strictly between the preselected alternatives.

Next, we derive several conditions the second step of a strategy-proof social choice rule has to satisfy. To do so, let $f_A : \mathcal{R}^A \to R_f(A)$ be the binary decision function that chooses between

$l_f(A)$ and $r_f(A)$ if the set of agents that declared to have single-peaked preferences is equal to A. The next proposition establishes that only the preferences of the agents belonging to $S_f(A)$ can affect the outcome of f_A. The intuition of this result is the following: since the two preselected locations form an ordered pair for all agents situated to the left of $l_f(A)$ or to the right of $r_f(A)$, the binary decision function f_A must be independent of these preferences in order to guarantee the strategy-proofness of f.

Proposition 2 *If f is strategy-proof, then for all preference profiles $R, R' \in \mathcal{R}^A$ such that $R_{S_f(A)} = R'_{S_f(A)}$, $f_A(R) = f_A(R')$.*

As a corollary of Proposition 2, we can reduce the domain of f_A from $\mathcal{R}^A$ to $\mathcal{R}_{S_f(A)}$. Also, if $l_f(A) \neq r_f(A)$, then at least one agent has to be situated between the two preselected alternatives. Thus, we can partition $S_f(A)$ for a given profile $R \in \mathcal{R}^A$ into three groups depending on the ordinal preferences over the two preselected alternatives: $S^l_f(R) = \{i \in S_f(A) : l_f(A)\, P_i\, r_f(A)\}$, $S^r_f(R) = \{i \in S_f(A) : r_f(A)\, P_i\, l_f(A)\}$, and $S^i_f(R) = \{i \in S_f(A) : r_f(A)\, I_i\, l_f(A)\}$. With this notation at hand we can now describe some particular binary decision functions that indicate the sets of agents that are able to impose the left preselected alternative over the right one. These decisive sets depend on $R_{S^i_f(R)}$, the preferences of the agents belonging to $S_f(A)$ that are indifferent between $l_f(A)$ and $r_f(A)$ at R, and have the following structure.

Definition 1 *Given the subprofile $R_{S^i_f(R)}$ of a preference profile $R \in \mathcal{R}^A$, a family $\mathcal{G}(R_{S^i_f(R)})$ of $l_f(A)$–decisive sets over $R_f(A)$ is a family of subsets of $(S_f(A) \setminus S^i_f(R))$ such that:*

- *If $B \in \mathcal{G}(R_{S^i_f(R)})$ and $C \supseteq B$, then $C \in \mathcal{G}(R_{S^i_f(R)})$.*
- *If $j \notin B$ and $(B \cup \{j\}) \notin \mathcal{G}(R_{S^i_f(R)})$, then $B \notin \mathcal{G}(R'_{S^i_f(R')})$, where $R' \in \mathcal{R}^A$ is such that $R'_{S^i_f(R)} = R_{S^i_f(R)}$ and $S^i_f(R') = S^i_f(R) \cup \{j\}$.*
- *If $B \in \mathcal{G}(R_{S^i_f(R)})$ and $j \notin B \cup S^i_f(R)$, then $B \in \mathcal{G}(R'_{S^i_f(R')})$, where $R' \in \mathcal{R}^A$ is such that $R'_{S^i_f(R)} = R_{S^i_f(R)}$ and $S^i_f(R') = S^i_f(R) \cup \{j\}$.*
- *If $S^i_f(R) = \emptyset$, then $\emptyset \notin \mathcal{G}(R_{S^i_f(R)}) \neq \emptyset$.*

The definition shows that the decisive sets have to satisfy three intuitive monotonicity properties and a non-emptiness condition. First, all supersets of a decisive set are also decisive. Second, if a set of agents $B \cup \{j\}$ cannot impose $l_f(A)$ when only the agents of $B \cup \{j\}$ prefer $l_f(A)$ to $r_f(A)$, then the set B is also not able to impose $l_f(A)$ when agent j switches her preferences and becomes indifferent between the two preselected alternatives. Third, if a set of agents B can impose $l_f(A)$ when only the agents of B prefer $l_f(A)$ to $r_f(A)$ and agent j prefers $r_f(A)$ to $l_f(A)$, then B is also able to impose $l_f(A)$ when agent j becomes indifferent between the two preselected alternatives. Finally, the non-emptiness condition guarantees that each of the preselected alternatives is implemented at least once if all agents have strict preferences over $R_f(A)$. Still, a description of a family of $l_f(A)$–decisive sets is not sufficient to define a binary decision function f_A, because it does not provide a solution when all agents of $S_f(A)$ are indifferent between $l_f(A)$ and $r_f(A)$. In this case, we have to apply a tie-breaking rule $t_A : \mathcal{R}_{S_f(A)} \to R_f(A)$.

Definition 2 *A binary decision function* $f_A : \mathcal{R}_{S_f(A)} \to R_f(A)$ *is called a* voting by collections of $l_f(A)$–decisive sets with tie-breakers t_A *if for each subprofile* $R_{S^i_f(R)}$ *of a preference profile* $R \in \mathcal{R}^A$*, there exists a family of* $l_f(A)$*–decisive sets over* $R_f(A)$*,* $\mathcal{G}(R_{S^i_f(R)})$*, together with a tie-breaking rule* $t_A : \mathcal{R}_{S_f(A)} \to R_f(A)$ *such that:*

$$f_A(R_{S_f(A)}) = \begin{cases} t_A(R_{S_f(A)}) & \text{if } S^i_f(R) = S_f(A) \\ l_f(A) & \text{if } S^l_f(R) \in \mathcal{G}(R_{S^i_f(R)}) \\ r_f(A) & \text{otherwise.} \end{cases}$$

A voting by collections of $l_f(A)$–decisive sets with tie-breakers t_A is, although formally complicated to define, a relatively simple binary decision function. First, a collection of $l_f(A)$–decisive sets over $R_f(A)$ is defined for each subprofile $R_{S^i_f(R)}$ of agents belonging to $S_f(A)$ that are indifferent between the preselected alternatives. The outcome of f_A for a subprofile $R_{S_f(A)}$ is then $l_f(A)$ if the set of agents of $(S_f(A) \setminus S^i_f(R))$ that prefer $l_f(A)$ is a $l_f(A)$-decisive set for $R_{S^i_f(R)}$, and $r_f(A)$ otherwise. If $S_f(A) = S^i_f(R)$, the tie-breaking rule t_A is used to determine the alternative that is ultimately chosen.

We note that our family of binary decision functions is almost identical to the family introduced under the same name in Manjunath [8], where it is shown that these are the unique type of rules that are strategy-proof and unanimous in the domain of single-dipped preferences when the two alternatives to choose from are the extreme locations $\min T$ and $\max T$. The only difference is that the non-emptiness requirement of the family of decisive coalitions $\mathcal{G}(R_{S^i_f(R)})$ is always needed in Manjunath [8] and not only when $S^i_f(R) = \emptyset$. Our next result shows that the binary decision function f_A associated to any strategy-proof social choice rule f has to be a voting by collections of $l_f(A)$–decisive sets with tie-breakers t_A.

Proposition 3 *If f is strategy-proof, the family of binary decision functions $\{f_A : \mathcal{R}_{S_f(A)} \to R_f(A)\}_{A\subseteq N}$ is a family of voting by collections of $l_f(A)$–decisive sets with tie-breakers t_A.*

Propositions 1 and 3 describe the basic structure any strategy-proof social choice rule has to satisfy. The next corollary summarizes it.

Corollary 1 *Suppose that f is strategy-proof. Then, there is a decomposition of f into a function $R_f : 2^N \to T^2$ and a family $\{f_A : \mathcal{R}_{S_f(A)} \to R_f(A)\}_{A\subseteq N}$ of voting by collections of $l_f(A)$–decisive sets with tie-breakers t_A such that for all $A \subseteq N$ and all preference profiles $R \in \mathcal{R}^A$, $f(R) = f_A(R_{S_f(A)})$.*

4 A complete characterization of the strategy-proof rules

The structure of R_f

In order to obtain a complete characterization, we have to derive additional necessary conditions on top of Corollary 1. We first analyze the function R_f used in the first step of the two-step procedure. In particular, we are going to explain how the preselected alternatives could change as more agents declare to have single-peaked preferences.

Proposition 4 *Suppose that f is strategy-proof. Then, for all $A \subset N$ and all $i \in N$:*

- *If $r_f(A) \leq i$, then $l_f(A \cup \{i\}) \in [l_f(A), i]$ and $r_f(A \cup \{i\}) \in [r_f(A), i]$.*
- *If $l_f(A) \geq i$, then $l_f(A \cup \{i\}) \in [i, l_f(A)]$ and $r_f(A \cup \{i\}) \in [i, r_f(A)]$.*
- *If $i \in (l_f(A), r_f(A))$, $i \in [l_f(A \cup \{i\}), r_f(A \cup \{i\})] \subseteq [l_f(A), r_f(A)]$. Additionally, if $i \in R_f(A \cup \{i\})$, then $R_f(A \cup \{i\}) = \{i\}$ or $R_f(A \cup \{i\}) \cap R_f(A) \neq \emptyset$.*

Proposition 4 establishes that if agent i passes from having single-dipped to single-peaked preferences, the preselected alternatives in $R_f(A \cup \{i\})$ have to be *closer* to i than those in $R_f(A)$. In fact, if agent i is situated strictly between the two preselected alternatives of $R_f(A)$, then $l_f(A) \leq l_f(A \cup \{i\}) \leq i$ and $r_f(A) \geq r_f(A \cup \{i\}) \geq i$, that is, agent i is also situated weakly between the preselected alternatives of $R_f(A \cup \{i\})$, and the preselected alternatives weakly move into the direction of the location of agent i. Additionally, if this location i belongs to $R_f(A \cup \{i\})$, then either i is the unique preselected location or the second preselected alternative already belonged to $R_f(A)$. If, on the other hand, agent i is situated to the left or to the right of the preselected alternatives of $R_f(A)$, then each of the preselected alternatives of $R_f(A \cup \{i\})$ has to be between the location of agent i and the corresponding preselected location of $R_f(A)$.

The relation between decisive sets

Next, we study additional necessary conditions that arise in the second step of the rules. In particular, we analyze how the decisive sets change as an agent passes from having single-dipped to single-peaked preferences. We separate the analysis depending on the different cases that can appear due to Proposition 4. So, we assume first that agent i is situated strictly between the two preselected alternatives when she declares to have single-dipped preferences and the preselected alternatives change when agent i announces to have single-peaked preferences in such a way that agent i is also situated strictly between $l_f(A \cup \{i\})$ and $r_f(A \cup \{i\})$. The proposition then establishes that agent i is a dictator in the second step of the social choice rule; that is, agent i belongs to all decisive coalitions that can impose $l_f(A)$ or $l_f(A \cup \{i\})$ in any profile of $\mathcal{R}^A$ or $\mathcal{R}^{A \cup \{i\}}$ in which she is not indifferent.

Proposition 5 *Suppose that f is strategy-proof. Then, for all $A \subset N$ and all agents $i \notin A$ such that $i \in S_f(A) \cap S_f(A \cup \{i\})$, $R_f(A) \neq R_f(A \cup \{i\})$, and all preference profiles $R \in \mathcal{R}^{A\cup\{i\}}$ and $R' \in \mathcal{R}^A$ such that $i \notin S^i_f(R) \cup S^i_f(R')$,*

$$B \in \mathcal{G}(R_{S^i_f(R)}) \text{ if and only if } i \in B$$

$$B \in \mathcal{G}(R'_{S^i_f(R')}) \text{ if and only if } i \in B.$$

The next proposition focuses on the situations in which $R_f(A \cup \{i\}) = R_f(A)$. Then, if agent i is situated to the left (right) of both preselected locations of $R_f(A)$, it has to be *easier* (more difficult) to select the left point of $R_f(A)$ when agent i changes her type of preferences from single-dipped to single-peaked. Here, easier (more difficult) means that the set of coalitions that can impose the left point of $R_f(A)$ when agent i declares to have single-peaked preferences has to be a superset (subset) of the set when she declares to have single-dipped preferences. Also, if $R_f(A \cup \{i\}) = R_f(A)$ and agent i is situated between the two preselected alternatives, the set of coalitions that can impose the left point of the range has to be invariant to the type of preferences of agent i.

Proposition 6 *Suppose that f is strategy-proof. Then, for all $A \subset N$ and all agents $i \notin A$ such that $R_f(A) = R_f(A \cup \{i\})$ and all preference profiles $R \in \mathcal{R}^A$ and $R' = (R'_i, R_{-i}) \in \mathcal{R}^{A\cup\{i\}}$ such that $i \notin S^i_f(R) \cup S^i_f(R')$, $\mathcal{G}(R_{S^i_f(R)}) \subseteq \mathcal{G}(R'_{S^i_f(R')})$ whenever $i \leq r_f(A)$, and $\mathcal{G}(R'_{S^i_f(R')}) \subseteq \mathcal{G}(R_{S^i_f(R)})$ whenever $i \geq l_f(A)$.*

Next, we analyze what happens if agent i is situated to the left or to the right of the alternatives in $R_f(A)$ and at least one of these locations changes when agent i changes her type of preferences from single-dipped to single-peaked in such a way that $S_f(A) \cap S_f(A \cup \{i\})$ is non-empty. We then find that if agent i is situated to the right (left) of $R_f(A)$ and a coalition B is decisive when agent i declares to have single-peaked (single-dipped) preferences, then the intersection between B and all agents with single-dipped preferences that are situated between the two preselected alternatives is a decisive set once agent i changes preferences.

Proposition 7 *Suppose that f is strategy-proof. Then, for all $A \subset N$, all agents $i \notin A$, and all preference profiles $R \in \mathcal{R}^{A\cup\{i\}}$ and $R' = (R'_i, R_{-i}) \in \mathcal{R}^A$ such that $R_f(A) \neq R_f(A \cup \{i\})$:*

- *If $i \geq r_f(A) > l_f(A \cup \{i\})$, then for all $B \in \mathcal{G}(R_{S^i_f(R)})$, $(B \cap (S_f(A) \setminus A)) \in \mathcal{G}(R'_{S^i_f(R')})$.*
- *If $i \leq l_f(A) < r_f(A \cup \{i\})$, then for all $B \in \mathcal{G}(R'_{S^i_f(R')})$, $(B \cap (S_f(A \cup \{i\}) \setminus A)) \in \mathcal{G}(R_{S^i_f(R)})$.*

The last proposition studies what happens if i is situated strictly between the two preselected alternatives of $R_f(A)$ and i belongs to $R_f(A \cup \{i\})$. Then, by Proposition 4, the location of agent i is the unique preselected location or it is accompanied by $l_f(A)$ or $r_f(A)$. In the first case, there is no room for additional conditions in the second step of the rule. In the latter case, the result states that if a coalition is able to impose the point different from i when she has single-peaked preferences, the single-peaked agents of this coalition can impose the very same point if agent i has single-dipped preferences.

Proposition 8 *Suppose that f is strategy-proof. Then, for all $A \subset N$, all agents $i \notin A$ such that $i \in (S_f(A) \setminus S_f(A \cup \{i\}))$, and all preference profiles $R \in \mathcal{R}^{A\cup\{i\}}$ and $R' = (R'_i, R_{-i}) \in \mathcal{R}^A$ such that $i \notin S^i_f(R')$:*

- *If $l_f(A) = l_f(A \cup \{i\})$, then for all $B \in \mathcal{G}(R_{S^i_f(R)})$, $(B \cap A) \in \mathcal{G}(R'_{S^i_f(R')})$.*
- *If $r_f(A) = r_f(A \cup \{i\})$, then for all $B \in \mathcal{G}(R'_{S^i_f(R')})$, $(B \cap S_f(A \cup \{i\}) \cap A) \in \mathcal{G}(R_{S^i_f(R)})$.*

The characterization result

So far, we have established a set of necessary conditions for a social choice rule f to be strategy-proof. Our main theorem shows that the union of these necessary conditions is also sufficient. This theorem also shows that the characterized rules are group strategy-proof. So, this stronger property is obtained for free.[6]

[6] Barberà et al. [2] show in their Theorem 1 that if a domain $\mathcal{D}$ satisfies a condition that they call *sequential inclusion*, then any strategy-proof social choice rule is also group strategy-proof on $\mathcal{D}$. It is also shown in their Theorem 4 that if a domain $\mathcal{D}$ allows for opposite preferences and any strategy-proof social choice rule on any subdomain of $\mathcal{D}$ is also group strategy-proof, then $\mathcal{D}$ satisfies sequential inclusion. Since our domain $\mathcal{R}$ allows for

Theorem 1 *The following statements are equivalent:*

1. *The social choice rule $f : \mathcal{R} \to T$ is strategy-proof.*

2. *The social choice rule $f : \mathcal{R} \to T$ is group strategy-proof.*

3. *There is a function $R_f : 2^N \to T^2$ satisfying Proposition 4 and a family $\{f_A : \mathcal{R}_{S_f(A)} \to R_f(A)\}_{A \subseteq N}$ of voting by collections of $l_f(A)$–decisive sets with tie-breakers t_A satisfying Propositions 5, 6, 7 and 8 such that for all $A \subseteq N$ and all preference profiles $R \in \mathcal{R}^A$, $f(R) = f_A(R_{S_f(A)})$.*

5 Discussion and additional characterizations

The structure of the characterized family and in particular the conditions imposed by Propositions 4 to 8 depend on the relation between N and T. In particular, there are situations, such as Example 1, in which all strategy-proof rules with at least three alternatives in the range are dictatorial, leading again to the Gibbard-Satterthwaite impossibility. Fortunately, Examples 2 and 3 indicate that this is generally not the case. To characterize all situations in which we can escape from this impossibility, we need to introduce some notation: a triple $\{x, y, z\} \subseteq T$ is said to be *full at N* if $N \cap \{x, y, z\} = \emptyset$ and $N \subset (\min\{x, y, z\}, \max\{x, y, z\})$.

Theorem 2 *There are non-dictatorial and strategy-proof social choice rules on $\mathcal{R}$ with at least three alternatives in the range if and only if there is a triple of T that is not full at N.*

The condition in Theorem 2 can be explained more intuitively depending on the size of T.

opposite preferences but does not satisfy sequential inclusion (the proof can be provided upon request), their results imply that there exists a subdomain of $\mathcal{R}$ where strategy-proofness does not imply group strategy-proofness.

Corollary 2 *The following statements hold:*

- *If* $|T| \geq 5$*, there are non-dictatorial and strategy-proof social choice rules on* $\mathcal{R}$ *with at least three alternatives in the range.*

- *If* $|T| = 4$*, there are non-dictatorial and strategy-proof social choice rules on* $\mathcal{R}$ *with at least three alternatives in the range if and only if* T *cannot be partitioned into* T_1 *and* T_2 *such that* $|T_1| = |T_2| = 2$*,* $\max T_1 < \min N$*, and* $\max N < \min T_2$*.*

- *If* $|T| = 3$*, there are non-dictatorial and strategy-proof social choice rules on* $\mathcal{R}$ *with at least three alternatives in the range if and only if* $N \cap T \neq \emptyset$ *and/or* $N \not\subset (\min T, \max T)$*.*

One can see that if there are more than 4 alternatives, then it is always possible to find strategy-proof (and, by Theorem 1, group strategy-proof) social choice rules that have at least three alternatives in their range. Otherwise, it is sufficient to have an agent that is situated at a feasible location or at the left (or right) of at least three alternatives.[7] However, even if the condition in Theorem 2 is met and there is a triple that is not full at N, it is not guaranteed that we can construct *meaningful* rules since it is possible that for some N and T the addition of Pareto efficiency to the condition of strategy-proofness leads to the dictatorial rules. Examples 2 and 3 indicate again that this is generally not the case. Our next proposition presents a first necessary condition on T for the existence of Pareto efficient rules.

Proposition 9 *Let* T *be a set of alternatives so that it is possible to find Pareto efficient rules on* $\mathcal{R}$*. Then,* $\min T$ *and* $\max T$ *exist.*

Proposition 9 shows that independently of the locations of the agents, it is necessary that T has a minimum and a maximum. So, we assume this from now on. Our next result characterizes the additional conditions the relation between N and T has to satisfy in order to be able to combine

[7] If $|T| = 3$ and none of these conditions is satisfied, $\mathcal{R}$ coincides with the universal domain (see Example 1). Thus, $\mathcal{R}$ restricts the universal domain but does not allow us to escape from the Gibbard-Satterthwaite impossibility only if $|T| = 4$ and the relation between N and T is the one described in the second point of Corollary 2.

Pareto efficiency and strategy-proofness without arriving at the dictatorial rules. To do so, let $l_j = \max\{x \in T : x \leq j\}$ and $r_j = \min\{x \in T : x \geq j\}$.

Theorem 3 *There are Pareto efficient and strategy-proof social choice rules on $\mathcal{R}$ different from the dictatorial ones if and only if there are two agents $i, j \in N$ such that each $k \in \{i, j\}$ satisfies one of the following conditions: (a) $k \notin (\min T, \max T)$, (b) $k \in T$ or (c) l_k and r_k exist; with at least one of them satisfying (a) or (b).*

According to Theorem 3 it is required that there is one agent situated at a feasible location or to the left (or right) of all feasible locations; and another agent that satisfies any of these characteristics or is situated at a location that has defined nearest feasible points to both its left and its right. Since a general characterization of all Pareto efficient and strategy-proof rules for all sets N and T that satisfy the properties of Theorem 3 is quite difficult to define in an intuitive way, we consider now two focal cases that reflect the two conditions that are sufficient to guarantee possibility results.

Agents situated outside the closure of feasible locations

The first condition that guarantees on its own the possibility of combining Pareto efficiency and strategy-proofness without arriving at dictatorial rules is that at least two agents are situated outside the closure of feasible locations. We are going to provide the characterization for the focal case when $N \cap [\min T, \max T] = \emptyset$. Observe that this framework covers Example 2. Let $N_l = \{i \in N : i < \min T\}$ and $N_r = \{i \in N : i > \max T\}$ be the sets of agents situated to the left and to the right of all feasible locations, respectively. Obviously, $N_l \cup N_r = N$.

Since no agent is situated between alternatives of T, only one alternative gets preselected in the first step of the two-step functions characterized in Theorem 1. Therefore, the range of R_f becomes T instead of T^2. This implies that the characterized rules are types-only; i.e., they only depend on the type of preferences (single-peaked or single-dipped) of each agent, but not on its particular structure. To introduce the characterization result, we need the following definition: a function

$g : 2^N \to T$ is said to be *monotone* if for all $A \subseteq N$ and $i \in N$, $g(A) \geq g(A \cup \{i\}) \geq i$ (respectively, $g(A) \leq g(A \cup \{i\}) \leq i$) whenever $g(A) \geq i$ (respectively, $g(A) \leq i$).

Theorem 4 *Suppose that $N \cap [\min T, \max T] = \emptyset$. A social choice rule $f : \mathcal{R} \to T$ is strategy-proof and Pareto efficient if and only if there is a monotone function $R_f : 2^N \to T$, where $R_f(N_l) = \min T$ and $R_f(N_r) = \max T$, such that for all $A \subseteq N$ and all $R \in \mathcal{R}^A$, $f(R) = R_f(A)$.*

Observe that the rule included in Example 2 belongs to the characterized family.

Agents situated at feasible locations

The second condition that guarantees on its own the possibility of combining Pareto efficiency and strategy-proofness without arriving at dictatorial rules is that there are at least two agents at feasible locations. We are thus interested in the focal case when $N \subseteq T$. Since agents situated at the extreme locations have exactly the same set of admissible preferences over T as agents at the left or right of all feasible alternatives, we assume for the sake of simplicity that $N \subset (\min T, \max T)$. This setting includes many of the natural frameworks studied in the literature; for example, when, like in Example 3, T is a closed interval and all agents are situated within T. Since the description of the Pareto efficient and strategy-proof rules is still quite complex, we impose tops-onliness as an additional condition. Formally, a social choice rule f is *tops-only* if for all $i \in N$, all $A \subseteq N$ and all preference profiles $R, R' \in \mathcal{R}^A$ such that $\{x \in T : x\, R_i\, y \text{ for all } y \in T\} = \{x \in T : x\, R'_i\, y \text{ for all } y \in T\}$, then $f(R) = f(R')$. Given that $N \subset T$, it is easy to see that for all $i \in N$ and all $R_i \in \mathcal{R}_i$, the top alternatives $t(R_i)$ satisfy that $t(R_i) \subset \{i, \min T, \max T\}$. For the sake of simplicity, we assume that the top of each agent is unique.

Before providing a formal definition of the rules, we are going to describe them intuitively. First, any rule f of the characterized family selects a non-empty set of *decisive* agents $D_f \subseteq N$ and a set of coalitions of decisive agents $\mathcal{G}_f \subset 2^{D_f}$ satisfying the properties of Definition 1. Then, if all decisive agents have single-dipped preferences, the outcome is one of the extreme points, and it depends on whether the set of decisive agents with top $\min T$ belongs to $\mathcal{G}_f$ (in which case, $\min T$

is selected) or not (in which case, $\max T$ is selected). If, however, at least one decisive agent has single-peaked preferences, an interior point situated between the locations of the decisive agents with single-peaked preferences is chosen by a monotone aggregator f_1 of the peaks of the agents with single-peaked preferences. The formal definition is as follows.

Definition 3 *The social choice rule f is said to be a* conditional two-step rule *if there is a non-empty set $D_f \subseteq N$ of decisive agents, a monotone function $f_1 : 2^N \to (\min T, \max T)$, and a non-empty set of decisive sets $\mathcal{G}_f \subset 2^{D_f}$ satisfying the properties of Definition 1 such that for all $A \subseteq N$ and all preference profiles $R \in \mathcal{R}^A$,*

$$f(R) = \begin{cases} f_1(A) \in [\min(A \cap D_f), \max(A \cap D_f)] & \textit{if } A \cap D_f \neq \emptyset \\ \min T & \textit{if } A \cap D_f = \emptyset \textit{ and } \{i \in N : t(R_i) = \min T\} \cap D_f \in \mathcal{G}_f \\ \max T & \textit{otherwise.} \end{cases}$$

The following result states the characterization.

Theorem 5 *Suppose that $N \subset T$ and $N \subset (\min T, \max T)$. A social choice rule f is strategy-proof, Pareto efficient and tops-only if and only if it is a conditional two-step rule.*

We can see that any conditional two-step rule depends on three characteristics: the set of decisive agents D_f, the coalitions of them that are decisive $\mathcal{G}_f$, and the aggregator f_1 used to choose an interior point. For instance, the rule proposed in Example 3 is obtained if D_f is the set of all agents, if $\mathcal{G}_f$ is the set of all coalitions that contain at least two agents, and if f_1 is the mean. Another example are dictatorships that are obviously included in the family. A dictatorship of agent i is associated with $D_f = \{i\}$ (observe that it is then not necessary to specify $\mathcal{G}_f$ and f_1).

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Appendix

Proof of Proposition 1

Let $O_i(A, R_{-i})$ be the option set of agent i given the preferences R_{-i} of the other agents and given that the set of agents with single-peaked preferences is equal to $A \subseteq N$. So, $x \in O_i(A, R_{-i})$ if there is a preference profile $R = (R_i, R_{-i}) \in \mathcal{R}^A$ such that $f(R) = x$. Our first lemma shows that if x and y belong to the option set of agent i, then i is situated between x and y.

Lemma 1 *Suppose that f is strategy-proof. Then, for all $A \subseteq N$, all agents $i \in N$, all profiles $R \in \mathcal{R}$, and all alternatives $x, y \in T$ such that $x < y$, if $x, y \in O_i(A, R_{-i})$, then $x < i < y$.*

Proof: Since $x, y \in O_i(A, R_{-i})$, there are two preferences $R_i, R'_i \in \mathcal{R}_i^A$ for agent i such that $f(R) = x$ and $f(R'_i, R_{-i}) = y$. If $i \leq x$, then agent i can manipulate f at (R'_i, R_{-i}) via R_i whenever $i \in A$ and at R via R'_i whenever $i \notin A$. Similarly, if $y \leq i$, then agent i can manipulate f at R via R'_i whenever $i \in A$ and at (R'_i, R_{-i}) via R_i whenever $i \notin A$. Hence, $x < i < y$. □

Lemma 1 directly implies that the option set of any agent contains at most two alternatives.

Corollary 3 *Suppose that f is strategy-proof. Then, for all $A \subseteq N$, all agents $i \in N$, and all profiles $R \in \mathcal{R}$, $|O_i(A, R_{-i})| \leq 2$.*

The next lemma shows that f always selects a maximal alternative of an agent's option set.

Lemma 2 *Suppose that f is strategy-proof. Then, for all $A \subseteq N$, all agents $i \in N$, and all profiles $R \in \mathcal{R}^A$, if $O_i(A, R_{-i}) = \{x, y\}$ and $f(R) = x$, then $x \, R_i \, y$.*

Proof: Suppose otherwise; that is, there is a preference profile R such that $f(R) = x$ and $y \, P_i \, x$. Since $y \in O_i(A, R_{-i})$ by assumption, there is a preference $R'_i \in \mathcal{R}_i^A$ such that $f(R'_i, R_{-i}) = y$. Then, agent i manipulates f at R via R'_i. Hence, $x \, R_i \, y$. □

The next lemma is crucial for the proof of the proposition. It shows that if the proposition was wrong (i.e. f is strategy-proof and $|R_f(A)| > 2$ for some $A \subseteq N$), then there would be a profile R and two preferences $R'_i \in \mathcal{R}_i^A$ and $R'_j \in \mathcal{R}_j^A$ so that the outcomes at R, (R'_i, R_{-i}), and (R'_j, R_{-j}) differ.

Lemma 3 *Suppose that f is strategy-proof. Then, for all $A \subseteq N$ such that $|R_f(A)| > 2$, there are two agents $i, j \in N$ and three profiles $R, (R'_i, R_{-i}), (R'_j, R_{-j}) \in \mathcal{R}^A$ such that $f(R) \neq f(R'_i, R_{-i}) \neq f(R'_j, R_{-j}) \neq f(R)$.*

Proof: Consider any $A \subseteq N$ and suppose that $|R_f(A)| > 2$. Then, there are three preference profiles $D, D', D'' \in \mathcal{R}^A$ such that $f(D) \neq f(D') \neq f(D'') \neq f(D)$. Suppose without loss of generality that $f(D) = x$. Starting at D, change the preferences of all agents one-by-one so that we end up in profile D'. Since $f(D) \neq f(D')$, the function must have changed during this process. Let B be the profile where the outcome is still x and let (B'_i, B_{-i}) be the profile when the outcome switches the first time to another alternative, say y.

Next, construct the preference $\hat{D}_i \in \mathcal{R}_i^A$ in the following way: if $f(D'') \neq y$, $\hat{D}_i$ is set equal to D''_i, otherwise $\hat{D}_i$ is set equal to D'_i. We can see that if $f(\hat{D}_i, B_{-i}) \notin \{x, y\}$, then $|O_i(A, B_{-i})| > 2$ contradicting Corollary 3. So, $f(\hat{D}_i, B_{-i}) \in \{x, y\}$. Next, consider the profile $(\hat{D}_i, B_{-i})$ together with the profile from B and (B'_i, B_{-i}) that has not the same outcome as $(\hat{D}_i, B_{-i})$. Suppose without loss of generality that the alternative selected at B is not equal to the one selected at $(\hat{D}_i, B_{-i})$. Then, starting at B and $(\hat{D}_i, B_{-i})$, change the preferences of all agents except i one-by-one so that we end up in profile D'' if $f(D'') \neq y$ and in profile D' if $f(D'') = y$. The outcome at either of the two profiles has to change at some point of the process. Let C and (C'_i, C_{-i}) be the profiles where the outcomes are still x and y, respectively, and let (C'_S, C_{-S}) and $(C'_{S\cup\{i\}}, C_{-(S\cup\{i\})})$ be the first time in which one (or both) profiles have an outcome $z \notin \{x, y\}$. Assume without loss of generality that $f(C'_S, C_{-S}) = z$. Let $S = \{j_1, \ldots, j_s\}$ and observe that $s+1$ agents have changed their preferences in the sequence to produce the three different outcomes: first, agent i changes preferences from C'_i to C_i; then, s agents change preferences (from the ones in C_S to the ones in C'_S) to arrive at z. We complete the proof by induction on s.

- If $s = 1$, denote profile C by R, preference C'_i by R'_i, agent j_1 by j, and preference C'_{j_1} by R'_j. Then, we have established the result.

- Suppose that the statement of the proposition is correct for all $s < k$.

- We now prove the statement of the proposition for $s = k$. Consider any $V \subset S \cup \{i\}$ with $V \notin \{S, \{i\}\}$. If $f(C'_V, C_{-V}) \notin \{x, y\}$, consider the sequence that starts with (C'_i, C_{-i}), passes through C, and ends at (C'_V, C_{-V}) whenever $i \notin V$, and the sequence that starts with C, passes through (C'_i, C_{-i}), and ends at (C'_V, C_{-V}) whenever $i \in V$. We can see that the number of agents that have changed their preferences to produce three different results in these sequences is smaller than $k+1$. So, the result follows from the induction hypothesis. Suppose now that $f(C'_V, C_{-V}) = y$ for some $V \subset S$. Then, consider the sequence that starts at C, passes through (C'_V, C_{-V}), and ends at (C'_S, C_{-S}). In this sequence, the number of agents that have changed their preferences to produce the three different outcomes is $s = k < k+1$. The result follows then again from the induction hypothesis. Then, we deduce that $f(C'_V, C_{-V}) = x$ for all $V \subset S$. Suppose now that $f(C'_V, C_{-V}) = y$ for some $V \subset S \cup \{i\}$ with $i \in V \neq \{i\}$. Then, consider the sequence that starts at (C'_V, C_{-V}), passes through $(C'_{V\setminus\{i\}}, C_{-(V\setminus\{i\})})$ and ends at (C'_S, C_{-S}). In this sequence, the number of agents that have changed their preferences to produce three different outcomes is again less than $k+1$ and, thus, the result follows from the induction hypothesis. Consequently, we assume from now on that $f(C'_V, C_{-V}) = x$ for all $V \subset S \cup \{i\}$ with $V \notin \{S, \{i\}\}$.

Next, we concentrate on $f(C'_{S\cup\{i\}}, C_{-(S\cup\{i\})})$. If $f(C'_{S\cup\{i\}}, C_{-(S\cup\{i\})}) \notin \{x, z\}$, consider the sequence that starts at $(C'_{S\cup\{i\}\setminus\{j\}}, C_{-(S\cup\{i\}\setminus\{j\})})$, where agent j belongs to S, passes through $(C'_{S\cup\{i\}}, C_{-(S\cup\{i\})})$, and ends at (C'_S, C_{-S}). Then, denote $(C'_{S\cup\{i\}}, C_{-(S\cup\{i\})})$ by R, C_i by R'_i, and C_j by R'_j to establish the result. If, however, $f(C'_{S\cup\{i\}}, C_{-(S\cup\{i\})}) = z$, take the sequence that starts at (C'_i, C_{-i}) and then changes the preferences of all agents $j \in S$ one-by-one in order to end up at $(C'_{S\cup\{i\}}, C_{-(S\cup\{i\})})$. In this sequence, the number of agents that have changed their preferences to produce the three different outcomes is $s = k < k+1$. Then, the induction hypothesis applies again. Thus, we conclude that $f(C'_{S\cup\{i\}}, C_{-(S\cup\{i\})}) = x$.

Since $f(C'_V, C_{-V}) = x$ for all $V \subset S \cup \{i\}$ with $V \notin \{S, \{i\}\}$ and $f(C'_i, C_{-i}) = y$, we have that $O_j(A, (C'_i, C_{-\{i,j\}})) = \{x, y\}$ for all $j \in S$. It also follows from $f(C'_V, C_{-V}) = x$ for all $V \subset S \cup \{i\}$ with $V \notin \{S, \{i\}\}$ and $f(C'_S, C_{-S}) = z$ that $O_j(A, (C'_{S\setminus\{j\}}, C_{-S})) = \{x, z\}$ for all $j \in S$. Similarly, we obtain that $O_i(A, C_{-i}) = \{x, y\}$ and $O_i(A, (C'_S, C_{-S})) = \{x, z\}$. So, it follows from Lemma 1 that $\{y, z\}$ is a fixed pair of alternatives for all agents of $S \cup \{i\}$. Consider now any agent $m \in S$ and any preference $R''_m \in \mathcal{R}^A_m$ such that $y\, P''_m\, x$ and $z\, P''_m\, x$, which exists given the previous findings. It follows then from Lemma 2 together with $O_m(A, (C'_i, C_{-\{i,m\}})) = \{x, y\}$ and $O_m(A, (C'_{S\setminus\{m\}}, C_{-S})) = \{x, z\}$ that $f(R''_m, C'_i, C_{-\{i,m\}}) = y$ and $f(R''_m, C'_{S\setminus\{m\}}, C_{-S}) = z$, respectively.

Finally, take the sequence of profiles that starts with $(R''_m, C'_i, C_{-\{i,m\}})$, and change the preferences of all agents $l \in S \cup \{i\} \setminus \{m\}$ so that the sequence ends at $(R''_m, C'_{S\setminus\{m\}}, C_{-S})$. If only y and z are chosen along this sequence, then the agent at whom the outcome changes can manipulate f given that $\{y, z\}$ is a fixed pair for all agents belonging to $S \cup \{i\}$. So, there have to be at least three different outcomes along this sequence. However, observe that the number of agents that have changed their preferences to produce the three different outcomes is $s = k < k + 1$. So, the result follows from the induction hypothesis.

This concludes the proof. □

Now, we are ready to prove Proposition 1.

Proof of Proposition 1: Suppose that $|R_f(A)| > 2$. Then, by Lemma 3, there are three profiles $R, (R'_i, R_{-i}), (R'_j, R_{-j}) \in \mathcal{R}^A$ such that $f(R) = x$, $f(R'_i, R_{-i}) = y$, and $f(R'_j, R_{-j}) = z$. Let $f(R'_i, R'_j, R_{-\{i,j\}}) = w$ and observe that although x, y and z are different, it could be that $w \in \{x, y, z\}$. Also, assume without loss of generality that $x < y$. Next, we study the implications of the different option sets.

(1) Since $O_i(A, R_{-i}) = \{x, y\}$ and $x < y$ by assumption, Lemma 1 implies that $x < i < y$.

(2) Observe that $\{z, w\} \subseteq O_i(A, (R'_j, R_{-\{i,j\}}))$. So, if $w \neq z$, then $w < i < z$ or $z < i < w$ by Lemma 1.

(3) Since $O_j(A, R_{-j}) = \{x, z\}$, Lemma 1 implies that $x < j < z$ or $z < j < x$.

(4) Observe that $\{w, y\} \subseteq O_j(A, (R'_i, R_{-\{i,j\}}))$. So, if $w \neq y$, then $w < j < y$ or $y < j < w$ by Lemma 1.

We show that f is not strategy-proof by constructing manipulations depending on how w relates to the other alternatives.

Case 1: Suppose that $w = y$. Then, $y < i < z$ or $z < i < y$ by (2) and $x < j < z$ or $z < j < x$ by (3). Observe that if $y < i < z$, the fact that $x < i < y$ by (1) implies that $i < y < i$, which is impossible. So, we must have that $z < i < y$. This, together with $x < i < y$ by (1), implies that $\{x, z\}$ is a fixed pair for agent i. Since $x < i$, $z < i$, and j is between x and z by (3), we also have that either $\{x, y\}$ or $\{y, z\}$ is a fixed pair for agent j. Suppose that $\{x, y\}$ is a fixed pair for j (the other case is similar and thus omitted). We can then conclude that $z < j < x < i < y$. The remainder of this case is divided into two parts.

- If $i \in A$, consider any preference $R''_i \in \mathcal{R}^+_i$ such that $x \, P''_i \, y \, P''_i \, z$. Since $O_i(A, R_{-i}) = \{x, y\}$ and $O_i(A, (R'_j, R_{-\{i,j\}})) = \{y, z\}$, it follows from Lemma 2 that $f(R''_i, R_{-i}) = x$ and $f(R''_i, R'_j, R_{-\{i,j\}}) = y$. Thus, $O_j(A, (R''_i, R_{-\{i,j\}})) = \{x, y\}$ and, by Lemma 1, j lies between x and y. This is a contradiction because we have already seen before that $j < x < y$.

- If $i \notin A$, consider any preference $R''_i \in \mathcal{R}^-_i$ such that $z \, P''_i \, y \, P''_i \, x$. Since $O_i(A, R_{-i}) = \{x, y\}$ and $O_i(A, (R'_j, R_{-\{i,j\}})) = \{y, z\}$, it follows from Lemma 2 that $f(R''_i, R_{-i}) = y$ and $f(R''_i, R'_j, R_{-\{i,j\}}) = z$. Thus, $O_j(A, (R''_i, R_{-\{i,j\}})) = \{y, z\}$. Now, we separate the proof depending whether j belongs to A or not.

  If $j \notin A$, consider any preference $R''_j \in \mathcal{R}^-_j$ such that $y \, P''_j \, z \, P''_j \, x$. Since $O_j(A, R_{-j}) = \{x, z\}$, Lemma 2 implies that $f(R''_j, R_{-j}) = z$. We know that $O_j(A, (R''_i, R_{\{i,j\}})) = \{y, z\}$, so Lemma 2 also implies that $f(R''_i, R''_j, R_{-\{i,j\}}) = y$. Individual i will then manipulate f at this profile via R_i to obtain z. If $j \in A$, consider any preference $R''_j \in \mathcal{R}^+_j$ such that $x \, P''_j \, z \, P''_j \, y$. Since $O_j(A, R_{-j}) = \{x, z\}$, Lemma 2 implies that $f(R''_j, R_{-j}) = x$. Given that $O_j(A, (R''_i, R_{\{i,j\}})) = \{y, z\}$, Lemma 2 also implies that $f(R''_i, R''_j, R_{-\{i,j\}}) = z$. Agent i will then manipulate f at (R''_j, R_{-j}) via R''_i (observe that (x, z) is a fixed pair for i at $\mathcal{R}^A_i$).

Case 2: Suppose that $w = z$. The proof is similar to the one above and is thus omitted.

Case 3: Suppose that $w \notin \{y, z\}$. By (2), we have that $z < i < w$ or $w < i < z$. Let $z < i < w$. Then, since j lies between x and z by (3) and both x and z are smaller than i by (1), we conclude that $j < i$. Also observe that j lies between y and w by (4) and that both y and w are greater than i by (1). Consequently, $j > i$, which cannot be. Hence, we must have $w < i < z$. Next, by (4), we have that $y < j < w$ or $w < j < y$. Let $y < j < w$. Then, (1) and (4) imply that $x < i < y < j < w$. So, $i < w$, which contradicts that $w < i$. Hence, we must have that $w < j < y$. Similarly we have by (3) that $z < j < x$ or $x < j < z$. Let $z < j < x$. Then, $w < i < z < j < x$ by (2) and (3), which contradicts that $x < i$ by (1). So, we must have that $x < j < z$.

At this point, we can see that that the four conditions $x < i < y$, $w < i < z$, $w < j < y$, and $x < j < z$ are indeed compatible for the moment. In fact, it turns out that x and w are both smaller than each i and j, which are in turn both smaller than each y and z. This also implies that both $\{w, x\}$ and $\{y, z\}$ are fixed pairs for both agents i and j. Finally, consider any preference $\hat{R}_j \in \mathcal{R}_j^A$ such that $z\,\hat{P}_j\,x$ and $y\,\hat{P}_j\,w$. Since $O_j(A, R_{-j}) = \{x, z\}$ and $O_j(A, (R'_i, R_{-\{i,j\}})) = \{w, y\}$, $f(\hat{R}_j, R_{-j}) = z$ and $f(R'_i, \hat{R}_j, R_{-\{i,j\}}) = y$. Then, agent i will manipulate f at $(\hat{R}_j, R_{-j})$ via R'_i when her fixed pair at $\mathcal{R}_i^A$ is (y, z) and at $(R'_i, \hat{R}_j, R_{-\{i,j\}})$ via R_i when her fixed pair at $\mathcal{R}_i^A$ is (z, y). □

Proof of Proposition 2

We first establish the following lemma.

Lemma 4 *Suppose that f is strategy-proof. Then, for any two profiles $R, (R'_S, R_{-S}) \in \mathcal{R}^A$ for some $A, S \subseteq N$ such that $f(R) \neq f(R'_S, R_{-S})$ there exists a set $D \subset S$ and an agent $i \in (S \setminus D)$ such that $f(R) = f(R'_D, R_{-D}) \neq f(R'_{D\cup\{i\}}, R_{-(D\cup\{i\})}) = f(R'_S, R_{-S})$.*

Proof: Starting at R, construct the sequence of profiles in which we change the preferences of all agents $j \in S$ one-by-one from R_j to R'_j. Since $f(R) \neq f(R'_S, R_{-S})$ by assumption, there is subset D of S and an agent $i \in (S \setminus D)$ such that $f(R'_D, R_{-D}) \neq f(R'_{D\cup\{i\}}, R_{-(D\cup\{i\})})$. Since $|R_f(A)| \leq 2$ by Proposition 1, it follows that $f(R) = f(R'_D, R_{-D})$ and $f(R'_{D\cup\{i\}}, R_{-(D\cup\{i\})}) = f(R'_S, R_{-S})$. □

Now, we are ready to prove the proposition.

Proof of Proposition 2: Suppose that there are two profiles $R, R' \in \mathcal{R}^A$ such that $R_{S_f(A)} = R'_{S_f(A)}$ and $f(R) \neq f(R')$. By Lemma 4, there is a set $D \subset N \setminus S_f(A)$ and an agent $i \in N \setminus (S_f(A) \cup D)$ such that $f(R) = f(R'_D, R_{-D}) \neq f(R'_{D\cup\{i\}}, R_{-(D\cup\{i\})}) = f(R')$. Thus, $O_i(A, (R'_D, R_{-(D\cup\{i\})})) = R_f(A)$ by Corollary 3. Since $i \notin S_f(A)$ by construction, this contradicts Lemma 1. $\square$

Proof of Proposition 3

Consider any $A \subseteq N$ and define for each subprofile $R_{S^i_f(R)}$ of a profile $R \in \mathcal{R}^A$ a set $\mathcal{G}(R_{S^i_f(R)}) \subseteq 2^{S_f(A)\setminus S^i_f(R)}$ of $l_f(A)$–decisive coalitions in the following way: the set B belongs to $\mathcal{G}(R_{S^i_f(R)})$ if there is a profile $R' \in \mathcal{R}^A$ such that $R'_{S^i_f(R')} = R_{S^i_f(R)}$, $S^l_f(R') = B$ and $f(R') = l_f(A)$.

We show first that a voting by these collections of $l_f(A)$-decisive sets with any tie-breakers t_A is a well–defined binary decision function. That is, if $B \in \mathcal{G}(R_{S^i_f(R)})$, then for all profiles $\bar{R} \in \mathcal{R}^A$ such that $B = S^l_f(\bar{R})$ and $\bar{R}_{S^i_f(\bar{R})} = R_{S^i_f(R)}$, $f(\bar{R}) = l_f(A)$. Suppose by contradiction that this is not the case and $f(\bar{R}) = r_f(A)$. By definition, there is a profile $R' \in \mathcal{R}^A$ such that $R'_{S^i_f(R')} = R_{S^i_f(R)}$, $S^l_f(R') = B$, and $f(R') = l_f(A)$. Then, by Lemma 4, there is a set $D \subset S_f(A)$ and an agent $j \in (S_f(A)\setminus D)$ such that $f(R') = f(\bar{R}_D, R'_{-D}) = l_f(A) \neq r_f(A) = f(\bar{R}_{D\cup\{j\}}, R'_{-(D\cup\{j\})}) = f(\bar{R})$. If $j \in S^l_f(\bar{R})$ (respectively, $j \in S^r_f(\bar{R})$), we have that $j \in S^l_f(R')$ (respectively, $j \in S^r_f(R')$) by assumption and agent j manipulates f at $(\bar{R}_{D\cup\{j\}}, R'_{-(D\cup\{j\})})$ (respectively, at $(\bar{R}_D, R'_{-D})$) via R'_j (respectively, via $\bar{R}_j$).

Now we show that if $B \in \mathcal{G}(R_{S^i_f(R)})$, $B \cup \{j\} \in \mathcal{G}(R_{S^i_f(R)})$ for all $j \in (S_f(A) \setminus S^i_f(R))$. Suppose otherwise; that is, there is a profile $\bar{R} \in \mathcal{R}^A$ such that $\bar{R}_{S^i_f(\bar{R})} = R_{S^i_f(R)}$, $S^l_f(\bar{R}) = B \cup \{j\}$ and $f(\bar{R}) = r_f(A)$. Then, agent j manipulates f at this profile via any $R'_j \in \mathcal{R}^A_j$ such that $r_f(A)P'_j l_f(A)$ to obtain $l_f(A)$.

Next, we show that if $S^i_f(R) = \emptyset$, then $\mathcal{G}(R_{S^i_f(R)}) \neq \emptyset$ or, what is the same, that if $S^l_f(R) = S_f(A)$, then $f(R) = l_f(A)$. Suppose to the contrary that, although $S^l_f(R) = S_f(A)$, $f(R) = r_f(A)$. Since $l_f(A) \in R_f(A)$, there is a profile $R' \in \mathcal{R}^A$ such that $f(R') = l_f(A)$. It follows from Proposition 2 that $f(R) = f(R_{S_f(A)}, R'_{-S_f(A)}) = r_f(A)$. Then, considering the profiles R' and $(R_{S_f(A)}, R'_{-S_f(A)})$ and applying Lemma 4, there is a set $D \subset S_f(A)$ and an agent $i \in (S_f(A) \setminus D)$ such that $f(R'_D, R_{-D}) = r_f(A) \neq l_f(A) = f(R'_{D\cup\{i\}}, R_{-(D\cup\{i\})})$. Since $l_f(A)\, P_i\, r_f(A)$ by assumption, agent i manipulates f at (R'_D, R_{-D}) via R'_i. It is possible to show in a similar way that if $S^i_f(R) = \emptyset$, then $\emptyset \notin \mathcal{G}(R_{S^i_f(R)})$ or, what is the same, that if $S^r_f(R) = S_f(A)$, then $f(R) = r_f(A)$.

Now, we establish that if $B \in \mathcal{G}(R_{S^i_f(R)})$ and $j \notin B \cup S^i_f(R)$, then $B \in \mathcal{G}(R'_{S^i_f(R')})$, where $R' \in \mathcal{R}^A$ is such that $R'_{S^i_f(R)} = R_{S^i_f(R)}$ and $S^i_f(R') = S^i_f(R) \cup \{j\}$. Suppose to the contrary that $B \notin \mathcal{G}(R'_{S^i_f(R')})$ so that $f(R') = r_f(A)$. Then, by Lemma 4, there is a set $D \subset S_f(A)$ and an agent $k \in (S_f(A) \backslash D)$ such that $f(R) = f(R'_D, R_{-D}) = l_f(A) \neq r_f(A) = f(R'_{D\cup\{k\}}, R_{-(D\cup\{k\})}) = f(R')$. If $k \in S^i_f(R)$, $R_k = R'_k$ by construction and, then, $f(R'_D, R_{-D}) = f(R'_{D\cup\{k\}}, R_{-(D\cup\{k\})})$, which contradicts the fact that they are different. If $k \in S^r_f(R)$, agent k manipulates f at (R'_D, R_{-D}) via R'_k. If, however, $k \in S^l_f(R)$, then $k \in S^l_f(R')$ by construction. So, agent k manipulates f at $(R'_{D\cup\{k\}}, R_{-(D\cup\{k\})})$ via R_k. Finally, observe that the proof that if $(B \cup \{j\}) \notin \mathcal{G}(R_{S^i_f(R)})$, then $B \notin \mathcal{G}(R'_{S^i_f(R')})$, where $R' \in \mathcal{R}^A$ is such that $R'_{S^i_f(R)} = R_{S^i_f(R)}$ and $S^i_f(R') = S^i_f(R) \cup \{j\}$, follows a similar reasoning.

Proof of Proposition 4

Since the first two statements are dual, we only consider the case $r_f(A) \leq i$. First, to see that $r_f(A \cup \{i\}) \leq i$ suppose otherwise. If $r_f(A) < i$, consider a profile $R \in \mathcal{R}^A$ such that $r_f(A\cup\{i\})\, P_j\, l_f(A\cup\{i\})$ for all $j \in S_f(A\cup\{i\})$ and $r_f(A\cup\{i\})\, P_i\, l_f(A)$. Agent i then manipulates f at R via any $R'_i \in \mathcal{R}^+_i$ such that $r_f(A \cup \{i\})\, P'_i\, l_f(A \cup \{i\})$ in order to obtain $r_f(A \cup \{i\})$ by Proposition 3. If, on the other hand, $r_f(A) = i$, consider a profile $R \in \mathcal{R}^{A\cup\{i\}}$ such that $r_f(A)\, P_j\, l_f(A)$ for all $j \in S_f(A)$ and $r_f(A \cup \{i\})\, P_k\, l_f(A \cup \{i\})$ for all $k \in S_f(A \cup \{i\})$. Then, $f(R) = r_f(A \cup \{i\})$ by Proposition 3, and agent i manipulates f at this profile via any $R'_i \in \mathcal{R}^-_i$.

Next, we show that $r_f(A \cup \{i\}) \geq r_f(A)$. Suppose that $r_f(A \cup \{i\}) < r_f(A)$ and consider the profile $R \in \mathcal{R}^{A\cup\{i\}}$ such that $r_f(A)\, P_j\, l_f(A)$ for all $j \in S_f(A)$ and $r_f(A\cup\{i\})\, P_k\, l_f(A\cup\{i\})$ for all $k \in S_f(A \cup \{i\})$. It follows from Proposition 3 that $f(R) = r_f(A \cup \{i\})$ and $f(R'_i, R_{-i}) = r_f(A)$

for all $R'_i \in \mathcal{R}_i^-$. Then, agent i can manipulate f at R via any $R'_i \in \mathcal{R}_i^-$. Observe that a similar argument can be used to show that $l_f(A \cup \{i\}) \geq l_f(A)$.

Finally, we consider the cases when $i \in (l_f(A), r_f(A))$. In order to see that $i \leq r_f(A \cup \{i\})$, assume by contradiction that $i > r_f(A \cup \{i\})$ and take a profile $R \in \mathcal{R}^{A\cup\{i\}}$ such that $r_f(A)\, P_j\, l_f(A)$ for all $j \in S_f(A)$ and $r_f(A)\, P_i\, r_f(A \cup \{i\})$. Then, agent i manipulates f at R via any $R'_i \in \mathcal{R}_i^-$ such that $r_f(A)\, P'_i\, l_f(A)$ in order to obtain $r_f(A)$ instead of any element of $R_f(A \cup \{i\})$. One can show in a similar way that $i \geq l_f(A \cup \{i\})$. Therefore, $i \in [l_f(A \cup \{i\}), r_f(A \cup \{i\})]$. We establish next that $l_f(A \cup \{i\}) \geq l_f(A)$ (the proof that $r_f(A \cup \{i\}) \leq r_f(A)$ is dual). Suppose otherwise; that is, $l_f(A \cup \{i\}) < l_f(A)$. Consider a profile $R \in \mathcal{R}^{A\cup\{i\}}$ such that $l_f(A \cup \{i\})\, P_j\, r_f(A \cup \{i\})$ for all $j \in S_f(A \cup \{i\})$ and $l_f(A)\, P_k\, r_f(A)$ for all $k \in S_f(A)$. Then, by Proposition 3, $f(R) = l_f(A \cup \{i\})$. Thus, agent i manipulates it via any $R'_i \in \mathcal{R}_i^-$ such that $l_f(A)\, P'_i\, r_f(A)$ in order to obtain $l_f(A)$ instead of $l_f(A \cup \{i\})$. We show next that if $l_f(A \cup \{i\}) = i$, then $r_f(A \cup \{i\}) \in \{i, r_f(A)\}$. Suppose otherwise; that is, $l_f(A \cup \{i\}) = i$, but $r_f(A \cup \{i\}) \in (i, r_f(A))$. Then, consider a profile $\bar{R} \in \mathcal{R}^{A\cup\{i\}}$ such that $r_f(A \cup \{i\})\, \bar{P}_j\, i$ for all $j \in S_f(A \cup \{i\})$, $l_f(A)\, \bar{P}_k\, r_f(A)$ for all $k \in S_f(A)$ and $l_f(A)\bar{P}_i r_f(A \cup \{i\})$. Then, by Proposition 3, we have that $f(R) = r_f(A \cup \{i\})$. However, agent i manipulates it via any $\hat{R}_i \in \mathcal{R}_i^-$ such that $l_f(A)\hat{P}_i r_f(A)$ to obtain $l_f(A)$ by Proposition 3. Finally, one can show in a similar way that if $r_f(A \cup \{i\}) = i$, then $l_f(A \cup \{i\}) \in \{l_f(A), i\}$.

Proof of Proposition 5

Consider first the case when $R_f(A) \cap R_f(A \cup \{i\}) = \emptyset$. We will only show that $B \in \mathcal{G}(R'_{S_f^i(R')})$ if and only if $i \in B$ for all $R' \in \mathcal{R}^A$ such that $i \notin S_f^i(R')$. Suppose otherwise; that is, there is some profile $\bar{R} \in \mathcal{R}^A$ such that $f(\bar{R}) = x$ and $y\, \bar{P}_i\, x$, where $x, y \in R_f(A)$. Assume without loss of generality that $x = r_f(A)$. Then, consider a profile $\hat{R} \in \mathcal{R}^A$ with $r_f(A)\, \hat{P}_j\, l_f(A)$ for all $j \in (S_f(A) \setminus \{i\})$, $l_f(A \cup \{i\})\, \hat{P}_k\, r_f(A \cup \{i\})$ for all $k \in S_f(A \cup \{i\})$, and $l_f(A \cup \{i\})\, \hat{P}_i\, r_f(A)$. Since $f(\bar{R}) = r_f(A)$, it follows from Proposition 3 that $f(\hat{R}) = r_f(A)$. However, agent i then manipulates f at $\hat{R}$ via any $R''_i \in \mathcal{R}_i^+$ such that $l_f(A \cup \{i\})\, P''_i\, r_f(A \cup \{i\})$ to obtain $l_f(A \cup \{i\})$.

Suppose now that $|R_f(A) \cap R_f(A \cup \{i\})| = 1$. It is assumed without loss of generality that

$R_f(A) \cap R_f(A \cup \{i\}) = r_f(A) = r_f(A \cup \{i\})$. We first show that $[B \in \mathcal{G}(R_{S_f^i(R)})$ iff $i \in B$ for all $R \in \mathcal{R}^{A\cup\{i\}}$ with $i \notin S_f^i(R)]$ if and only if $[B \in \mathcal{G}(R'_{S_f^i(R')})$ iff $i \in B$ for all $R' \in \mathcal{R}^A$ with $i \notin S_f^i(R')]$. So, suppose first that $B \in \mathcal{G}(R_{S_f^i(R)})$ iff $i \in B$ for all $R \in \mathcal{R}^{A\cup\{i\}}$ with $i \notin S_f^i(R)$, but, by contradiction, that there is some $R' \in \mathcal{R}^A$ such that $f(R') = x$ and $y\, P'_i\, x$, where $x, y \in R_f(A)$. Let $x = r_f(A)$ –the case when $x = l_f(A)$ is similar and thus omitted– and consider a preference $R''_i \in \mathcal{R}_i^-$ such that $l_f(A \cup \{i\})\, P''_i\, r_f(A)$. If it was the case that $f(R''_i, R'_{-i}) = l_f(A)$, then individual i could manipulate f at R' via R''_i. Hence, $f(R''_i, R'_{-i}) = r_f(A)$. Since $B \in \mathcal{G}(R_{S_f^i(R)})$ iff $i \in B$ for all $R \in \mathcal{R}^{A\cup\{i\}}$ with $i \notin S_f^i(R)$ by assumption, we also have that for all $\bar{R}_i \in \mathcal{R}_i^+$ with $l_f(A \cup \{i\})\, \bar{P}_i\, r_f(A)$, $f(\bar{R}_i, R'_{-i}) = l_f(A \cup \{i\})$. But agent i can then manipulate f at (R''_i, R'_{-i}) via any $\bar{R}_i \in \mathcal{R}_i^+$ with $l_f(A \cup \{i\})\, \bar{P}_i\, r_f(A)$. Observe finally that the proof of the other implication is similar and thus omitted. Consequently, the two implications are equivalent.

Consider now any profile $\hat{R} \in \mathcal{R}^A$ with $i \notin S_f^i(\hat{R})$ such that $f(\hat{R}) = l_f(A)$. If there is a preference $R'_i \in \mathcal{R}_i^+$ such that $f(R'_i, \hat{R}_{-i}) = r_f(A \cup \{i\}) = r_f(A)$, then strategy-proofness implies that $O_i(A, \hat{R}_{-i}) = R_f(A)$; in fact, if it was the case that $f(R''_i, \hat{R}_{-i}) = l_f(A)$ for all $R''_i \in \mathcal{R}_i^-$, then agent i would be able to manipulate f at any profile $(\bar{R}_i, \hat{R}_{-i}) \in \mathcal{R}^A$ such that $r_f(A)\, \bar{R}_i\, l_f(A)$ via R'_i. Similarly, if there is a preference $\bar{R}_i \in \mathcal{R}_i^-$ such that $f(\bar{R}_i, \hat{R}_{-i}) = r_f(A)$, we can also conclude that $O_i(A, \hat{R}_{-i}) = R_f(A)$. In any of these cases, given that $O_i(A, \hat{R}_{-i}) = R_f(A)$, Lemma 2 implies that $B \in \mathcal{G}(\hat{R}_{S_f^i(\hat{R})})$ iff $i \in B$. If this occurs with all profiles of $\mathcal{R}^A$, we would obtain that i is a dictator in all profiles of $\mathcal{R}^A$ and, by the equivalence obtained in the previous paragraph, i is also a dictator in all profiles of $\mathcal{R}^{A\cup\{i\}}$ and the proof is complete. So, suppose from now on that there is a profile $\hat{R} \in \mathcal{R}^A$ with $i \notin S_f^i(\hat{R})$ such that $f(\hat{R}) = l_f(A)$, $f(R'_i, \hat{R}_{-i}) = l_f(A \cup \{i\})$ for all $R'_i \in \mathcal{R}_i^+$ and $f(\bar{R}_i, \hat{R}_{-i}) = l_f(A)$ for all $\bar{R}_i \in \mathcal{R}_i^-$. Then, $\mathcal{G}^*(\hat{R}_{S_f^i(\hat{R})}) = \mathcal{G}^*((R'_i, \hat{R}_{-i})_{S_f^i(R'_i, \hat{R}_{-i})})$, where $\mathcal{G}^*(R_{S_f^i(R)})$ includes all minimal coalitions of $\mathcal{G}(R_{S_f^i(R)})$.

Suppose now that the proposition is wrong. Then, there is some coalition $D \in \mathcal{G}^*(\hat{R}_{S_f^i(\hat{R})}) = \mathcal{G}^*((R'_i, \hat{R}_{-i})_{S_f^i(R'_i, \hat{R}_{-i})})$ such that $D \neq \{i\}$. If $i \in D$ for all $D \in \mathcal{G}^*(\hat{R}_{S_f^i(\hat{R})})$, consider any preference $\bar{R}_i \in \mathcal{R}_i^+$ such that $r_f(A)\, \bar{P}_i\, l_f(A \cup \{i\})$. Then, $f(\bar{R}_i, \hat{R}_{-i}) = r_f(A)$. Since this contradicts that $f(R'_i, \hat{R}_i) = l_f(A \cup \{i\})$ for all $R'_i \in \mathcal{R}_i^+$, we can conclude that there is some $D \in \mathcal{G}^*(\hat{R}_{S_f^i(\hat{R})})$

such that $i \notin D$. We are going to show next that $D \subseteq S_f(A) \cap A$. To do this, suppose otherwise and consider the profile $\bar{R} \in \mathcal{R}^A$ such that $\bar{R}_{S^i_f(\bar{R})} = \hat{R}_{S^i_f(\hat{R})}$, $l_f(A)\, \bar{P}_j\, r_f(A)$ for all $j \in D \cap A \cap S_f(A)$, $l_f(A)\, \bar{P}_k\, r_f(A)\, \bar{P}_k\, l_f(A \cup \{i\})$ for all $k \in D \setminus (A \cap S_f(A))$, and both $r_f(A)\, \bar{P}_l\, l_f(A)$ and $r_f(A)\, \bar{P}_l\, l_f(A \cup \{i\})$ for all $l \in (S_f(A \cup \{i\}) \setminus D)$. Since $D \in \mathcal{G}^*(\hat{R}_{S^i_f(\hat{R})}) = \mathcal{G}^*((R'_i, \hat{R}_{-i})_{S^i_f(R'_i, \hat{R}_{-i})})$ by assumption, $f(\bar{R}) = l_f(A)$ and $f(R'_i, \bar{R}_{-i}) = r_f(A \cup \{i\}) = r_f(A)$ for any $R'_i \in \mathcal{R}^+_i$. However, agent i will then manipulate f at $\bar{R}$ via R'_i. Consequently, D only contains agents from $S_f(A)$ with single-peaked preferences at $\hat{R} \in \mathcal{R}^A$. It can be shown in a very similar way that $D \subseteq S_f(A \cup \{i\}) \cap (N \setminus A)$. Thus, $D = \emptyset$. Since $\emptyset$ is not a $l_f(A)$–decisive set over $R_f(A)$ when the set of agents belonging to $S_f(A)$ that are indifferent between $l_f(A)$ and $r_f(A)$ is empty, we have reached a contradiction for this case. Hence, i is a dictator whenever $S^i_f(\hat{R}) = \emptyset$. It follows then from the iterative application of Definition 1 (third point) that i is always a dictator, independently of $S^i_f(\hat{R})$.

Proof of Proposition 6

We will only show that for all preference profiles $R \in \mathcal{R}^A$ and all $R' = (R'_i, R_{-i}) \in \mathcal{R}^{A \cup \{i\}}$, $\mathcal{G}(R_{S^i_f(R)}) \subseteq \mathcal{G}(R'_{S^i_f(R')})$ whenever $i \leq r_f(A)$; the other implication can be proved similarly. Suppose there is a set C of agents that is a decisive set when i has single-dipped preferences but not when i has single-peaked preferences; that is, $C \in (\mathcal{G}(R_{S^i_f(R)}) \setminus \mathcal{G}(R'_{S^i_f(R')}))$. If $i \notin C$, consider a profile $R'' \in \mathcal{R}^A$ such that $R''_{S^i_f(R'')} = R_{S^i_f(R)}$, $r_f(A)\, P''_i\, l_f(A)$, and for all agents $k \in (S_f(A) \setminus S^i_f(R''))$, $l_f(A)\, P''_k\, r_f(A)$ if and only if $k \in C$. Then, $f(R'') = l_f(A)$ and $f(R'_i, R''_{-i}) = r_f(A)$. Thus, agent i manipulates f at R'' via R'_i. If, on the other hand, $i \in C$, consider a profile $\bar{R} \in \mathcal{R}^{A \cup \{i\}}$ such that $\bar{R}_{S^i_f(\bar{R})} = R'_{S^i_f(R')}$ and for all $k \in (S_f(A) \setminus S^i_f(\bar{R}))$, $l_f(A)\, \bar{P}_k\, r_f(A)$ if and only if $k \in C$. Then, $f(\bar{R}) = r_f(A)$, but agent i can manipulate f at this profile via any $\hat{R}_i \in \mathcal{R}^-_i$ such that $l_f(A)\, \hat{P}_i\, r_f(A)$ to obtain $l_f(A)$.

Proof of Proposition 7

We will only consider the case when $i \geq r_f(A) > l_f(A \cup \{i\})$, the other situation is similar. So suppose that $B \in \mathcal{G}(R_{S_f^i(R)})$ for some $R \in \mathcal{R}^{A\cup\{i\}}$ and assume by contradiction that there is a preference $R_i' \in \mathcal{R}_i^-$ such that $(B \cap (S_f(A) \setminus A)) \notin \mathcal{G}((R_i', R_{-i})_{S_f^i(R_i', R_{-i})})$. Consider a profile $R'' \in \mathcal{R}^{A\cup\{i\}}$ such that $R''_{S_f^i(R'')} = R_{S_f^i(R)}$, $l_f(A\cup\{i\})\, P_k''\, r_f(A\cup\{i\})$ for $k \in S_f(A\cup\{i\})$ if and only if $k \in B$, and $r_f(A)\, P_l''\, l_f(A)$ for all $l \in (B \cap S_f(A) \cap A) \cup (S_f(A) \setminus (B \cup A)) \cup (S_f(A) \setminus S_f(A \cup \{i\}))$. Then, $f(R'') = l_f(A)$ and $f(R_i', R_{-i}'') = r_f(A)$. Thus, agent i can manipulate f at R'' via R_i'.

Proof of Proposition 8

We will only consider the case when $l_f(A) = l_f(A \cup \{i\})$ because the other is similar. By contradition, suppose that there is some $B \in \mathcal{G}(R_{S_f^i(R)})$ for some $R \in \mathcal{R}^{A\cup\{i\}}$, but $(B \cap A) \notin \mathcal{G}(R'_{S_f^i(R')})$ for some $R' = (R_i', R_{-i}) \in \mathcal{R}^A$. Then, consider a profile $\bar{R} \in \mathcal{R}^{A\cup\{i\}}$ such that $\bar{R}_{S_f^i(\bar{R})} = R_{S_f^i(R)}$, $l_f(A\cup\{i\})\bar{P}_j i$ for $j \in S_f(A\cup\{i\})$ if and only if $j \in B$ and $r_f(A)\bar{P}_k l_f(A)$ for all $k \in (S_f(A) \setminus (B \cap A))$. Then, $f(\bar{R}) = l_f(A)$, but agent i can manipulate it via any $\hat{R}_i \in \mathcal{R}_i^-$ such that $r_f(A)\hat{P}_i l_f(A)$ to obtain $r_f(A)$.

Proof of Theorem 1

[2] $\Rightarrow$ [1]: Obvious.

[1] $\Rightarrow$ [3]: This implication has been shown throughout the paper.

[3] $\Rightarrow$ [2]: Take any of these social choice rules f. Assume, by contradiction, that there is a profile $R \in \mathcal{R}$, a group of agents $S \subseteq N$ and a subprofile R_S' of a profile $R' \in \mathcal{R}$ such that $f(R_S', R_{-S})\, P_i\, f(R)$ for all $i \in S$. We can assume without loss of generality that $f(R_S', R_{-S}) > f(R)$. Then, all agents $i \leq f(R)$ of S must have single-dipped preferences at R and all agents $i \geq f(R_S', R_{-S})$ of S must have single-peaked preferences at R for the manipulation to be effective. Let $A \subseteq N$ be the set of agents with single-peaked preferences at R and let $C \subseteq N$ be the set of agents with single-peaked preferences at (R_S', R_{-S}). Denote by B be the set of agents with single-

peaked preferences if only agents belonging to $S_f(A)$ have changed their preferences. Change the preferences of all agents $i \in S$ one-by-one from R_i to R'_i in the following order: (i) start with the agents belonging to $S_f(A)$ in any arbitrary order; (ii) continue in any arbitrary order with all agents of $(S_f(B) \setminus S_f(A))$; (iii) continue with the agents situated to the left of $l_f(B)$ taking in each step the agent situated most to the right; (iv) continue with the single-peaked agents situated to the right of $r_f(B)$ taking in each step the agent situated most to the right; and (v) complete the process with the single-dipped agents situated to the right of $r_f(B)$ taking in each step the agent situated most to the left. By the successive application of Proposition 4, we can deduce that $[l_f(A), r_f(A)] \cap [l_f(B), r_f(B)] \neq \emptyset$.

If $l_f(B) < l_f(C)$ and/or $r_f(B) < r_f(C)$, then there is at least one agent $i \in (S \setminus S_f(A))$ that moves one or both preselected locations to the right when changing her preferences from R_i to R'_i.

Suppose first that the agents in $(S \setminus S_f(A))$ that move one or both preselected locations to the right are only located in $(S_f(B) \setminus S_f(A))$. Let i be one of these agents. The fact that $(S_f(B) \setminus S_f(A)) \neq \emptyset$ implies that there is at least one agent $j \in S_f(A) \cap S \cap A$ such that $[l_f(A), r_f(A)] \subset [l_f(A \setminus \{j\}), r_f(A \setminus \{j\})]$ and $R'_j \in \mathcal{R}_j^-$. By Proposition 5, j is a dictator in $R_f(A)$ and in $R_f(A \setminus \{j\})$. When the rest of the single-peaked agents (that is, the agents of $S_f(A) \cap S \cap A$) change their preferences, Proposition 5 implies that $R_f(A \setminus \{j\}) = R_f(E)$, where E is the set of agents with single-peaked preferences in that moment. By Proposition 6, j is a dictator in $R_f(E)$.

Now, consider any agent $k \in (S_f(A) \cap (S \setminus A))$ such that $R'_k \in \mathcal{R}_k^+$. We would like to show that $R_f(E \cup \{k\}) = R_f(E)$. Suppose by contradiction that this is not the case. Assume without loss of generality that $k > j$. Given that k cannot be a dictator in $R_f(E)$, Propositions 4 (third case) and 5 implies that $R_f(E \cup \{k\}) \in \{\{k\}, \{k, l_f(E)\}, \{k, r_f(E)\}\}$. It is not possible that $R_f(E \cup \{k\}) = \{k\}$ because if we continue changing the preferences of the rest of agents of $(S_f(A) \cap (S \setminus A))$, Proposition 4 would imply that $S_f(B) \subseteq S_f(A)$ contradicting that $i \in (S_f(B) \setminus S_f(A))$. If $R_f(E \cup \{k\}) \in \{\{k, l_f(E)\}, \{k, r_f(E)\}\}$, we would have by Proposition 8 that the empty coalition would belong to $\mathcal{G}(R_{S_f^i(R)})$ for any $R \in \mathcal{R}^{E \cup \{k\}}$ such that j is not indifferent between $l_f(E)$

and $r_f(E)$, violating Proposition 3. Therefore, $R_f(E \cup \{k\}) = R_f(E)$ and, by Proposition 6, we also have that j is a dictator in $R_f(E \cup \{k\})$. Repeating the same arguments we arrive at $R_f(A \setminus \{j\}) = R_f(B)$ and j being a dictator in $R_f(B)$. Then, $l_f(B) \leq l_f(A)$ and $r_f(A) \leq r_f(B)$. Given that $j \in A$ and that she was a dictator in $R_f(A)$, we necessarily need that $f(R'_S, R_{-S}) \in (l_f(A), r_f(A))$ for the manipulation to be effective. Since $f(R) < f(R'_S, R_{-S})$, it follows that $f(R) = l_f(A)$. If $r_f(C) > r_f(B)$, by Proposition 4 (third case) we have that $R_i \in \mathcal{R}_i^+$ and $R'_i \in \mathcal{R}_i^-$. Since all agents of S at the left of $f(R)$ have single-dipped preferences at R, we must have that $i \in [r_f(A), r_f(B))$. Then, by Proposition 5, i is a dictator for $R_f(B)$. This contradicts that j is a dictator for the same set. On the other hand, if $l_f(C) > l_f(B)$, by Proposition 4 (third case) we have that $R_i \in \mathcal{R}_i^-$ and $R'_i \in \mathcal{R}_i^+$. Since all agents of S at the right of $r_f(A)$ have single-peaked preferences at R, we must have that $i \in (l_f(B), l_f(A)]$. By Proposition 4 (third case), no agent in $(l_f(B), l_f(A)]$ can move the left preselected alternative to the right of $l_f(A)$ and, therefore, $f(R'_S, R_{-S}) = r_f(C) < r_f(A) \leq r_f(B)$. Then, by Proposition 5, i is a dictator for $R_f(B)$, contradicting that j is a dictator for the same set.

Suppose now that there is an agent $i \in (S \setminus (S_f(A) \cup S_f(B))$ such that $i \leq l_f(B)$ that moves one or both preselected locations to the right. It can be checked that i is situated to the left of both preselected alternatives when she is called to change preferences. Then, $R_i \in \mathcal{R}_i^+$ and $R'_i \in \mathcal{R}_i^-$ by Proposition 4 (second case). Since $i \notin S_f(A)$ and $[l_f(A), r_f(A)] \cap [l_f(B), r_f(B)] \neq \emptyset$, we can also deduce that $i \leq l_f(A)$ and, therefore, $i \leq f(R)$. This contradicts that all agents of S to the left of $f(R)$ have single-dipped preferences at R.

Suppose finally that that there is an agent $i \in (S \setminus (S_f(A) \cup S_f(B))$ such that $i \geq r_f(B)$ that moves one or both preselected locations to the right. It can be checked that i is situated to the right of both preselected alternatives when she is called to change preferences. Then, $R_i \in \mathcal{R}_i^-$ and $R'_i \in \mathcal{R}_i^+$ by Proposition 4 (first case). Let $S_r^d = \{i \in S \,|\, i \geq r_f(B) \text{ and } R_i \in \mathcal{R}_i^-\}$. Given the order of changes that we have established, it follows from the iterated application of Proposition 4 (all cases) that $\max S_r^d \geq r_f(C)$ and, therefore, $f(R) \, P_{\max S_r^d} \, f(R'_S, R_{-S})$. This contradicts that

S can manipulate f at R via R'_S.

Consequently, we assume from now on that $l_f(B) \geq l_f(C)$ and $r_f(B) \geq r_f(C)$. Suppose first that $f(R'_S, R_{-S}) = l_f(C)$. Since $l_f(B) \geq l_f(C) = f(R'_S, R_{-S})$ and $f(R'_S, R_{-S}) > f(R)$, we must have that $l_f(B) > l_f(A)$. By Proposition 4, there exists an agent $i \in S_f(A) \cap S$ such that $i \geq l_f(B)$ that changes her preferences from $R_i \in \mathcal{R}_i^-$ to $R'_i \in \mathcal{R}_i^+$. It follows then that $i \geq f(R'_S, R_{-S})$. Given that i has single-dipped preferences at R, we have reached a contradiction. Then, we assume from now on that $f(R'_S, R_{-S}) = r_f(C)$. Suppose additionally that $f(R) = r_f(A)$. Since $r_f(B) \geq r_f(C) = f(R'_S, R_{-S})$ and $f(R'_S, R_{-S}) > f(R)$, we have that $r_f(B) > r_f(A)$. By Proposition 4, there exists an agent $i \in S_f(A) \cap S$ that changes her preferences from $R_i \in \mathcal{R}_i^+$ to $R'_i \in \mathcal{R}_i^-$. Given that $i \leq f(R)$ and she has single-peaked preferences at R, we have a contradiction. So, we assume from now on that $f(R) = l_f(A)$. We divide the analysis into three cases.

1) Suppose that $r_f(B) < r_f(A)$ and/or $l_f(B) > l_f(A)$. We will only prove the case when $r_f(B) < r_f(A)$ because the other is similar and thus omitted. Since $r_f(C) \leq r_f(B)$, we have that $f(R'_S, R_{-S}) = r_f(C) \in (l_f(A), r_f(B)]$. By Proposition 4, there exists an agent $i \in S_f(A) \cap S$ such that $i \leq r_f(B)$ that changes her preferences from $R_i \in \mathcal{R}_i^-$ to $R'_i \in \mathcal{R}_i^+$. Suppose first that $i < r_f(B)$. Then, Proposition 5 implies that i is a dictator at $R_f(A)$ or, what is the same, that $l_f(A) R_i r_f(A)$. Then, given that $R_i \in \mathcal{R}_i^-$, we have that $l_f(A) P_i x$ for all $x \in (l_f(A), r_f(A))$ and, in particular, $f(R) P_i f(R'_S, R_{-S})$, contradicting the fact that S can manipulate f. Suppose finally that $i = r_f(B)$. Then, $i \geq f(R'_S, R_{-S})$ and has single-dipped preferences at R, which is not possible.

2) Suppose that $r_f(B) > r_f(A)$ and/or $l_f(A) > l_f(B)$. We will only prove the case when $r_f(B) > r_f(A)$ because the other is similar and thus omitted. By Proposition 4, there exists an agent $j \in S_f(A) \cap S$ that changes her preferences from $R_j \in \mathcal{R}_j^+$ to $R'_j \in \mathcal{R}_j^-$. Then, Proposition 5 implies that j is a dictator in $R_f(A)$. Thus, $f(R) = l_f(A)\, R_j\, r_f(A)$. It follows from similar arguments as employed previously in the proof that j is also a dictator in $R_f(B)$. If $r_f(C) \geq r_f(A)$, we have that $f(R)\, R_j\, f(R'_S, R_{-S})$. So, this cannot be.

If $r_f(C) < r_f(A)$, we have that $r_f(C) < r_f(A) < r_f(B)$ and $l_f(B) \leq l_f(A)$. This is only possible if there is an agent $i \in S$ such that $[i \leq l_f(C)$, $R_i \in \mathcal{R}_i^-$ and $R_i' \in \mathcal{R}_i^+]$ or $[i \geq r_f(B)$, $R_i \in \mathcal{R}_i^+$ and $R_i' \in \mathcal{R}_i^-]$. Suppose that $i \leq l_f(C)$, $R_i \in \mathcal{R}_i^-$ and $R_i' \in \mathcal{R}_i^+$ (the other case is similar and thus omitted) and, by simplicity, that there is no agent satisfying the other case. Then, if $R_f(B \cup \{i\}) = R_f(B)$, j is a dictator in $R_f(B \cup \{i\})$ by Proposition 6. If, however, $R_f(B \cup \{i\}) \neq R_f(B)$, we have that $r_f(B \cup \{i\}) > l_f(B)$ because, otherwise, it is not possible that $r_f(C) > l_f(B)$. Then, by Proposition 7 we obtain that j is a dictator in $R_f(B \cup \{i\})$ or that the emptyset is a decisive coalition, which is not possible by Proposition 3. Repeating this process with the remaining agents, the unique option is that j is also a dictator in $R_f(C)$. Thus, $j < r_f(C)$. By the same argument, j is also a dictator when only i misses to change preferences; that is, j is a dictator for $R_f(C \setminus \{i\})$.

Consider now the profile $(R'_{S\setminus\{j\}}, R_{-(S\setminus\{j\})}) \in \mathcal{R}^{C\cup\{j\}}$; that is, all agents $i \in S$ apart from j have changed their preferences from R_i to R_i'. By Proposition 4 (third case), $r_f(C \cup \{j\}) \in [j, r_f(C)]$. If $r_f(C\cup\{j\}) \in (j, r_f(C))$ or if $[r_f(C\cup\{j\}) = r_f(C)$ and $l_f(C\cup\{j\}) < j]$, then j is still a dictator in $R_f(C\cup\{j\})$ and $i \leq l_f(C) \leq l_f(C\cup\{j\})$. Suppose now that $l_f(C\cup\{j\}\setminus\{i\}) \leq j$. If individual i changes her preferences at $(R'_{S\setminus\{i\cup j\}}, R_{-(S\setminus\{i\cup j\})})$ from $R_i \in \mathcal{R}_i^-$ to $R_i' \in \mathcal{R}_i^+$, then, by Proposition 7, the decisive coalitions at $R_f(C \cup \{j\})$ consist of agents with single-dipped preferences. This contradicts that j is a dictator with single-peaked preferences at $R_f(C\cup\{j\})$. On the other hand, if $l_f(C\cup\{j\}\setminus\{i\}) > j$, then $l_f(C\setminus\{i\}) > j$ by Proposition 4 (second case). This contradicts that j is a dictator at $R_f(C\setminus\{i\})$. If $[r_f(C\cup\{j\}) = r_f(C)$ and $l_f(C\cup\{j\}) = j]$ or $[r_f(C\cup\{j\}) = j$ and $l_f(C\cup\{j\}) < j]$, then Proposition 8 implies that the emptyset is a decisive coalition, which violates Proposition 3. Finally, if $r_f(C\cup\{j\}) = l_f(C\cup\{j\}) = j$, then $j \leq l_f(C\cup\{j\}\setminus\{i\}) \leq r_f(C\cup\{j\}\setminus\{i\})$ by Proposition 4. Then, also by Proposition 4, $j \leq l_f(C \setminus \{i\})$ which contradicts the fact that j is a dictator for $R_f(C \setminus \{i\})$.

3) Suppose that $r_f(B) = r_f(A)$ and $l_f(B) = l_f(A)$. If $R_f(C) = R_f(A)$, by the iterated application of Proposition 6, the set of decisive sets of agents for $R_f(C)$ is equal to the set of decisive sets of agents for $R_f(A)$. It follows then from $f(R) = l_f(A)$ and $f(R_S', R_{-S}) = r_f(C)$ that there is some

agent $i \in S \cap S_f(A)$ that weakly prefers $l_f(A)$ to $r_f(A)$ at R but the other way around at (R'_S, R_{-S}). Then, $f(R)\, R_i\, f(R'_S, R_{-S})$, which contradicts the fact that S can manipulate f. On the other hand, if $R_f(C) \neq R_f(A)$, then there is an agent $i \in S \setminus S_f(A)$ that changes her type of preferences such that she has single-dipped preferences at R and is situated to the left of $l_f(A)$ or has single-peaked preferences at R and is situated to the right of $r_f(A)$. Suppose without loss of generality that $B \cup \{i\} = C$, $i \leq l_f(A)$, $R_i \in \mathcal{R}_i^-$ and $R'_i \in \mathcal{R}_i^+$. Then, $l_f(C) \leq l_f(B) = l_f(A)$. Since $r_f(C) \leq r_f(B) = r_f(A)$ by assumption, $r_f(C) = f(R'_S, R_{-S}) \in (l_f(A), r_f(A)]$ for the manipulation to be successful. It follows then from Proposition 7 that $f(R) = l_f(A)$ and $f(R'_S, R_{-S}) = r_f(C)$ implies that there is an agent $k \in S_f(A) \cap S_f(C) \cap S$ with $R_k \in \mathcal{R}_k^-$ such that $l_f(A) R_k r_f(A)$ and $r_f(C) R'_k l_f(C)$. Therefore, $f(R) R_k f(R'_S, R_{-S})$, contradicting the fact that S manipulates f.

Proof of Theorem 2

$\Leftarrow$]: Suppose that there is a triple $\{x, y, z\} \subseteq T$ that is not full at N. Consider first the case in which $N \cap \{x, y, z\} \neq \emptyset$. Suppose without loss of generality that $x \in N$ and $y < z$. If $(N \setminus \{x\}) \not\subset (y, z)$, consider an agent $j \notin (y, z)$. Then, a rule f such that $R_f(A) = x$ if $x \in A$, $R_f(A) = y$ if $[A \cap \{x, j\} = \{j\}$ and $j < y]$ or $[A \cap \{x, j\} = \emptyset$ and $j > z]$, and $R_f(A) = z$ otherwise is strategy-proof, has range 3, but is not dictatorial. If, however, $(N \setminus \{x\}) \subset (y, z)$, any rule f such that $R_f(A) = x$ if $x \in A$, $R_f(A) = \{y, z\}$ otherwise, where $\mathcal{G}(R_{S^i_f(R)})$ contains all non-empty coalitions for all $R \in \mathcal{R}^A$ with $x \notin A$, is strategy-proof, has range 3, but is not dictatorial.

Suppose next that $N \cap \{x, y, z\} = \emptyset$ and $N \not\subset (\min\{x, y, z\}, \max\{x, y, z\})$. Assume without loss of generality that $x < y < z$ and that there is one agent $i \in N$ with $i < x$. If $(N \setminus \{i\}) \not\subset (y, z)$, consider an agent $j \notin (y, z)$. Then, a rule f such that $R_f(A) = x$ if $i \in A$, $R_f(A) = y$ if $[A \cap \{i, j\} = \{j\}$ and $j < y]$ or $[A \cap \{i, j\} = \emptyset$ and $j > z]$, and $R_f(A) = z$ otherwise is strategy-proof, has range 3, but is not dictatorial. If, however, $(N \setminus \{i\}) \subset (y, z)$, any rule f such that $R_f(A) = x$ if $i \in A$, $R_f(A) = \{y, z\}$ otherwise, where $\mathcal{G}(R_{S^i_f(R)})$ contains all non-empty coalitions for all $R \in \mathcal{R}^A$ with $i \notin A$, is strategy-proof, has range 3, but is not dictatorial.

$\Rightarrow$]: Suppose that all triples are full at N and that, by contradiction, there is a strategy-proof rule f with range greater than or equal to 3 that is not dictatorial. Consider a triple $\{x, y, z\}$, $x < y < z$, contained in the range of f. Then, $N \subset ((x, z) \setminus \{y\})$. We distinguish two cases:

Let $|R_f(\emptyset)| = 1$. If $R_f(\emptyset) < z$, it follows from the iterated application of Proposition 4 that $z \notin R_f(A)$ for any $A \subseteq N$, contradicting the fact that z belongs to the range of f. Similarly, if $R_f(\emptyset) > x$, the iterated application of Proposition 4 implies that $x \notin R_f(A)$ for any $A \subseteq N$, contradicting the fact that x belongs to the range of f.

Let $|R_f(\emptyset)| = 2$. If $r_f(\emptyset) < z$ (respectively, $l_f(\emptyset) > x$), it follows from the iterated application of Proposition 4 that $z \notin R_f(A)$ (respectively, $x \notin R_f(A)$) for any $A \subseteq N$, contradicting the fact that z (respectively, x) belongs to the range of f. Thus, $l_f(\emptyset) \leq x$ and $r_f(\emptyset) \geq z$. Since no agent $i \in N$ is situated at a feasible point, we have by Proposition 4 (third case) that $R_f(\{i\}) = R_f(\emptyset)$ (we will say that i is *indecisive*) or $i \in (l_f(\{i\}), r_f(\{i\})) \subset [l_f(\emptyset), r_f(\emptyset)]$ (we will say that i is *decisive*). By Proposition 5, if an agent i is decisive, she is a dictator in $R_f(\{i\})$ and in $R_f(\emptyset)$. Since it is not possible to have more than one dictator for $R_f(\emptyset)$, we conclude that there is at most one decisive agent that will be denoted, if it exists, by d. Consider now any set A with $|A| > 1$.

- Suppose that $d \notin A$. We are going to prove by induction that $R_f(A) = R_f(\emptyset)$. So, suppose that $R_f(D) = R_f(\emptyset)$ for all $D \subset A$. Take any profile $R \in \mathcal{R}^A$ and suppose that $f(R) \notin \{l_f(\emptyset), r_f(\emptyset)\}$, contrary to what we intend to show. By construction, $f(R) \in (l_f(\emptyset), r_f(\emptyset))$ and, by assumption, $f(R) \neq j$ for all $j \in A$. Then, $f(R) \in R_f(A)$ and $|R_f(A)| = 2$. Since $R_f(D) = \{l_f(\emptyset), r_f(\emptyset)\}$ for all $D \subset A$, we have by Proposition 5 that all agents of A are dictators in $R_f(A)$, which is impossible. Therefore, $R_f(A) = R_f(\emptyset)$.

- Suppose next that $d \in A$. We are going to prove by induction that $R_f(A) = R_f(\{d\})$. So, suppose that $R_f(D) = R_f(\{d\})$ for all $D \subset A$ with $d \in D$, but $R_f(A) \neq R_f(\{d\})$. Since $R_f(E) = R_f(\emptyset)$ for all $E \subseteq (A \setminus \{d\})$ and $R_f(D) = R_f(\{d\})$ for all $D \subset A$ with $d \in D$, we have by Proposition 6 that d is a dictator at all $R_f(B)$ with $B \subset A$.

  Suppose that $|R_f(A)| = 2$. Then, if $R_f(A) \neq R_f(\{d\})$, we have by Proposition 5 that any agent $j \in A$ is a dictator at $R_f(A \setminus \{j\})$, which contradicts the fact that d is a dictator there.

So, suppose now that $|R_f(A)| = 1$. Then, $S_f(A) = \emptyset$ and we conclude that $d \in S_f(A \setminus \{d\})$ and $d \notin S_f(A)$. Since $d \notin R_f(A)$ by construction, we have reached a contradiction with Proposition 4 (third case). Therefore, $R_f(A) = R_f(\{d\})$.

We have seen that $R_f(A) = R_f(\{d\})$ if $d \in A$ and that $R_f(A) = R_f(\emptyset)$ if $d \notin A$. If there is no decisive agent d, the range of f only contains $l_f(\emptyset)$ and $r_f(\emptyset)$. This contradicts that f has a range of at least 3. If there is a decisive agent d, by Proposition 6, we have that d is a dictator for all $R_f(A)$. Since $l_f(\emptyset) \leq l_f(\{d\}) \leq d \leq r_f(\{d\}) \leq r_f(\emptyset)$, this rule is dictatorial, being d the dictator.

Proof of Proposition 9

We will prove that if $\min T$ or $\max T$ does not exist, then there are no Pareto efficient rules. To do that we need the following notation: $N_l = \{i \in N \mid i \leq \inf T\}$, $N_r = \{i \in N \mid i \geq \sup T\}$, and $N_c = N \setminus (N_l \cup N_r)$.[8]

Suppose that $\min T$ does not exist (the proof when $\max T$ does not exist is similar). Consider a profile $R \in \mathcal{R}^{N_l}$ such that for all $i \in N_c$ and all $y > i$, there exists $x \leq i$ with xP_iy. Then, there is no Pareto efficient alternative in this profile and, therefore, it is not possible to construct a Pareto efficient social choice rule.

Proof of Theorem 3

$\Leftarrow$]: Suppose that $i \in N \cap T$ and that l_j and r_j exist (this allows j to satisfy (a) or (c)). Consider any social choice rule f such that $R_f(A) = \{i\}$ if $i \in A$, $R_f(A) = \{l_j, r_j\}$ if $A \cap \{i,j\} = \{j\}$, and $R_f(A) = \{\min T, \max T\}$ otherwise, and where j is a dictator in all $R_f(A)$ such that $|R_f(A)| = 2$. This rule is strategy-proof and Pareto efficient. Suppose next that $i \notin (\min T, \max T)$ and that l_j and r_j exist (this allows j to satisfy (a) or (c)). Assume without loss of generality that $i < \min T$. Consider any social choice rule f such that $R_f(A) = \{\min T\}$ if $i \in A$, $R_f(A) = \{l_j, r_j\}$ if $A \cap \{i,j\} = \{j\}$, and $R_f(A) = \{\min T, \max T\}$, and where j is a dictator in all $R_f(A)$ such that $|R_f(A)| = 2$. This rule is strategy-proof and Pareto efficient. Suppose finally that $i, j \notin$

[8] If $\inf T$ or $\sup T$ does not exist, the corresponding sets are empty.

$(\min T, \max T)$. Consider any social choice rule f such that $R_f(A) = \{\min T\}$ if $k < \min T$ for all $k \in (A \cap \{i, j\})$, and $R_f(A) = \{\max T\}$ otherwise. Any of these rules is strategy-proof and Pareto-efficient.

$\Rightarrow$]: Suppose that $N \cap T = \emptyset$ and $N \subset (\min T, \max T)$. Then, using similar arguments as in the proof of Proposition 9, we have that $R_f(\emptyset) = \{\min T, \max T\}$. By similar arguments as in the proof of Theorem 2, we also have that $R_f(\{i\}) \neq R_f(\emptyset)$ for at most one agent $i \in N$.

If $R_f(\{i\}) = R_f(\emptyset)$ for all $i \in N$, it follows from the same arguments as those in the proof of Theorem 2 that $R_f(A) = R_f(\emptyset)$ for all $A \subseteq N$. Consider a profile $R \in \mathcal{R}^N$ such that there is some $x \in (T \setminus \{\min T, \max T\})$ with $x \, P_i \, \min T \, P_i \, \max T$ for all $i \in N$, which exists given the assumptions. Then, $f(R) = \min T$, but this location is Pareto dominated by x. If, on the other hand, $R_f(\{j\}) \neq R_f(\emptyset)$ for only one agent $j \in N$, it follows from the same arguments as those in the proof of Theorem 2 that $R_f(A) = R_f(\{j\})$ whenever $j \in A$, and that $R_f(A) = \{\min T, \max T\}$ otherwise. By Propositions 5 and 6, j is a dictator at $R_f(A)$ for all $A \subseteq N$. Since $l_f(\emptyset) \leq l_f(\{j\}) \leq j \leq r_f(\{j\}) \leq r_f(\emptyset)$, this rule is dictatorial, being j the dictator.

Proof of Theorem 4

$\Leftarrow$]: The assumption $N \cap [\min T, \max T] = \emptyset$ implies that the range of R_f is T instead of T^2. Hence, Propositions 5 to 8 are satisfied. Similarly, the condition that R_f is monotone guarantees that Proposition 4 holds. Then, strategy-proofness is guaranteed by Theorem 1. To show that these rules are Pareto efficient, take any profile $R \in \mathcal{R}^A$ for some $A \subseteq N$. If $A = N_l$ (respectively, $A = N_r$), all agents prefer $\min T$ (respectively, $\max T$) to any other alternative, and precisely this unanimously best alternative is chosen. In all other cases, observe that for all $x, y \in T$, there are two agents $i, j \in N$ such that $xP_i \, y$ and $y \, P_j \, x$. This guarantees that any choice is Pareto efficient in these cases.

$\Rightarrow$]: Consider now any strategy-proof and Pareto efficient social choice rule. Then, the rule belongs to the one characterized in Theorem 1. Since the range of R_f is T by the structure of N and T, we can concentrate directly on this function. This function is obviously monotone given that

Proposition 4 is implied by strategy-proofness. Similarly, we can derive that $R_f(N_l) = \min T$ and $R_f(N_r) = \max T$ by Pareto efficiency, using the same arguments as in the previous paragraph.

Proof of Theorem 5

It can be checked easily that all conditional two-step rules f are strategy-proof, Pareto efficient, and tops-only. To see the other implication, we start with the strategy-proof rules characterized in Theorem 1 and investigate the possible values R_f can take. First, we have by tops-onliness that if $|R_f(A)| = 2$ for some $A \subseteq N$, $R_f(A) = \{\min T, \max T\}$. Using similar arguments as those in the proof of Proposition 9, we can deduce that $R_f(\emptyset) = \{\min T, \max T\}$. Consider now any $i \in N$ and we are going to show that $R_f(\{i\}) \in \{\{i\}, R_f(\emptyset)\}$. Suppose that $R_f(\{i\}) \neq R_f(\emptyset)$. Then, $|R_f(\{i\})| = 1$ and, by Proposition 4 (third case), we have that $R_f(\{i\}) = i$.

Then, we can divide N into the group of decisive agents D_f (those $i \in N$ such that $R_f(\{i\}) = i$) and the group of indecisive agents $N \setminus D_f$ (those $i \in N$ such that $R_f(\{i\}) = \{\min T, \max T\}$). We can show applying the same arguments as those in the proof of Theorem 2 that $R_f(A) = R_f(\emptyset)$ for all $A \subseteq (N \setminus D_f)$. We show now that there is at least one decisive agent. Suppose, by contradiction, that $D_f = \emptyset$. Then, $R_f(A) = R_f(\emptyset)$ for all $A \subseteq N$. However, it is easy to see that for all $R \in \mathcal{R}^N$, $\min N$ (respectively, $\max N$) Pareto dominates $\min T$ (respectively, $\max T$). Since $R_f(\emptyset) = \{\min T, \max T\}$, this contradicts Pareto efficiency and, therefore, $D_f \neq \emptyset$.

Let us now concentrate on $R_f(A)$ when $A \cap D_f \neq \emptyset$ and we will prove that $R_f(A) \in [\min(A \cap D_f), \max(A \cap D_f)]$. The proof proceeds by double induction.

1. We first consider the case when only one decisive agent i has single-peaked preferences. We have to show that for all $A \subseteq N$ such that $A \cap D_f = \{i\}$, $R_f(A) = i$.

   (a) Let $A = \{i\}$ and $i \in D_f$. Since i is a decisive agent, $R_f(A) = i$ by definition.

   (b) Suppose that for all $B \subset A$ and $B \cap D_f = \{i\}$, $R_f(B) = i$. Since $A \cap D_f = \{i\}$ by assumption and the set of preselected alternatives is equal to $\{\min T, \max T\}$ whenever all decisive agents have single-dipped preferences, $R_f(A \setminus \{i\}) = \{\min T, \max T\}$. Thus, by Proposition 4 (third case), $R_f(A) \in \{\{i\}, \{\min T, \max T\}\}$. Moreover, by the induction hypothesis, $R_f(A \setminus \{j\}) = i$ for all $j \in (A \setminus \{i\})$. It follows then from Proposition 4 (third case) that $R_f(A)$ is not equal to $\{\min T, \max T\}$. Thus, $R_f(A) = i$.

2. We now move to the case when a subset C of decisive agents of size greater than one has single-peaked preferences. So, we consider a set $A \subseteq N$ such that $A \cap D_f = C$, $|C| > 1$, and we have to show that $R_f(A) \in [\min C, \max C]$. The induction hypothesis states that for all $B \subset A$ with $B \cap D_f \neq \emptyset$, $R_f(B) \in [\min(B \cap D_f), \max(B \cap D_f)]$. Suppose to the contrary that $R_f(A) < \min C$ (the proof when $R_f(C) > \max C$ is similar and thus omitted). Then, by the induction hypothesis, we obtain that $R_f(A \setminus \{\min C\}) \in [\min C, \max C]$. Then, $R_f(A) \in [\min C, \max C]$ by Proposition 4.

Then, we have deduced that $R_f(A) = \{\min T, \max T\}$ if $A \cap D_f = \emptyset$ and $|R_f(A)| = 1$ with $R_f(A) \in [\min(A \cap D_f), \max(A \cap D_f)]$ if $A \cap D_f \neq \emptyset$. Then, we can consider for these latter cases a function $f_1 : 2^N \to (\min T, \max T)$ such that $f_1(A) = R_f(A)$ for all $A \subseteq N$ such that $A \cap D_f \neq \emptyset$. The monotonicity of f_1 follows from Proposition 4.

It only remains to be shown how to choose between the preselected alternatives $\min T$ and $\max T$ in all $A \subseteq N$ such that $A \cap D_f = \emptyset$. The decisive coalitions $\mathcal{G}_f$ that can implement $\min T$ over $\max T$ can be formed only by single-dipped agents, given tops-onliness. By Proposition 6 we have that these decisive coalitions are the same for all $R_f(A)$ with $A \subseteq N \setminus D_f$ and, then, they are formed only by decisive agents.