RANKING SETS ADDITIVELY IN DECISIONAL CONTEXTS: AN AXIOMATIC CHARACTERIZATION

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Ranking sets additively in decisional contexts: An axiomatic characterization.

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Abstract

Ranking finite subsets of a given set X of elements is the formal object of analysis in this paper. This problem has found a wide range of economic interpretations in the literature. The focus of the paper is on the family of rankings that are additively representable. Existing characterizations are too complex and hard to grasp in decisional contexts. Furthermore, Fishburn [13] showed that the number of sufficient and necessary conditions that are needed to characterize such a family has no upper bound as the cardinality of X increases. In turn, this paper proposes a way to overcome these difficulties and allows for the characterization of a meaningful (sub)family of additively representable rankings of sets by means of a few simple axioms. Pattanaik and Xu's [21] characterization of the *cardinality-based* rule will be derived from our main result, and other new rules that stem from our general proposal are discussed and characterized in even simpler terms. In particular, we analyze restricted-cardinality based rules, where the set of "focal" elements is not given *ex-ante*; but brought out by the axioms.

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1 Introduction

Let X be a set of elements (finite or infinite), and consider the problem of ranking all the possible finite subsets of X. This problem has been an object of research in a number of meaningful economic settings, according to different interpretations of the subsets of X and of the corresponding ranking over them. As a few noteworthy examples, we list the following settings[1]:

- In the literature about *freedom of choice* and *preference for flexibility*, the degree of freedom of choice (or flexibility for choice) enjoyed by an agent is evaluated by means of the available *opportunity set* offered to him, the feasible opportunity sets being the possible subsets of a given universal set of alternatives.

- In some social choice situations the social rule that aggregates individual preferences is a social choice *correspondence* that selects a subset from a feasible set of alternatives, rather than a social choice function (for example, selecting the members of a committee or club, or deciding on the qualifications required of candidates for a certain position). In this case, regarding the aggregation problem, what matters is the voter's preferences over subsets of the universal set of alternatives.

- The so-called *hedonic games* formalize coalition formation problems where agents are only concerned about the possible partners they can side with. Thus individual preferences are expressed as preferences over the possible subsets of the universal set of potential partners. Likewise, in *matching* theory, agents typically define their preferences over the possible sets of agents they can match.

- In some equity analyses a set describes the rights or liberties enjoyed by an individual (or group) in the society. More specifically, the elements of the sets could be *primary goods* à la Rawls, or *capacities* à la Sen. In this field, value judgments about the degree of liberty and well-being enjoyed by individuals are therefore made on the basis of rankings over sets of this kind.

- Finally, in the axiomatic analysis of qualitative (or subjective) probability, the events being evaluated for probability of occurrence are sets, the elements of an event being its possible incompatible outcomes. Then a set (event) is ranked over another in terms of probability if the former is (subjectively) judged to be more probable than the latter.

[1]For a survey on different models about ranking sets, including most of the interpretations considered herein, and plenty of related references, see Barberà et al. [4].

Surely there are other decision problems that admit a formalization in terms of set rankings. In any case, an apparently very natural way to evaluate a set would be to sum up the particular values of its elements. In fact, in all the fields of application mentioned above, additive rules for set evaluation are frequently found (see, as samples, Gravel et al. [15] in the freedom of choice context, Barberà et al. [5] in a voting framework, Bogomolnaia and Jackson [6] in hedonic games, or any of the references mentioned in the following paragraphs for the subjective probability measurement problem)[2]. The interpretation of the element values is straightforward under the different possible cases. In general terms, the value of an element measures its desirability except in the qualitative probability evaluation problem, where it is interpreted as the (subjective) probability of its occurrence.

In this perspective, one may wonder about the axiomatic structure of the family of rankings that admit such an additive representation. Formally speaking, our initial concern is the set of conditions that are necessary and sufficient for a complete preorder over the finite subsets of X in order to ensure that there exists a positive real-valued function v defined on X such that A is ranked over B if and only if $\sum_{a\in A} v(a) \geqslant \sum_{b\in B} v(b)$. In the qualitative probability setting, the evaluation of sets appeals to an (additive) measure of qualitative probability for the events in X that is compatible with the ranking.

It is precisely in the qualitative probability context where the problem has been most thoroughly explored, harking back to de Finetti [11] and Savage [27]. Savage [27] defines a relation of *qualitative probability* among sets –events– as a binary relation that besides being a complete preorder, satisfies the following three conditions: *a*) that no set is ever ranked below the empty set, *b*) that the empty set is ranked strictly below X, and *c*) that a set A is ranked over another set B if and only if for any set of new options, C, $A \cup C$ is ranked over $B \cup C$.

Since these conditions were originally introduced by de Finetti [11], they are sometimes called the *de Finetti axioms.* Despite having been conceived for the particular framework of qualitative probability, they can be easily extrapolated to the other contexts.

Later, Kraft et al. [17] proved that the conditions proposed by de Finetti and Savage are necessary for an additive representation of the complete preorder, though they are not sufficient unless X has fewer than five elements. They also proposed a new condition that is based on the theory of finite systems of linear inequalities. This condition, along with the de Finetti axioms, characterizes additively representable rankings of sets. It was reformulated by Scott [28] in a more tractable format. Admittedly, the drawback of imposing this condition is that it is somewhat complex, harsh, and hard to interpret in decisional contexts. Actually, as Roberts [25] points out, it is "really an infinite scheme of conditions",

[2] In the last section we include a brief discussion about the particular interpretation of sets as opportunity sets, where the suitability of an additive evaluation might be more arguable.

and Fishburn [12], [13] shows that, as the cardinality of X increases, there is no upper bound on the number of such necessary and sufficient conditions . Another drawback of that condition is that it only applies when X is finite.

The awkwardness of Kraft et al.'s axiom motivated some authors to propose alternative simpler conditions (see for example, Luce [18], Suppes [31], Van Lier [34], or Fishburn and Roberts [14]), which nonetheless fail to be necessary for additive representability. Finally, we account for the characterization results by Villegas [35] and Chateauneuf [10]. Again in the context of probability measurement, they propose axiomatic characterizations of rankings over sets that conform a Boolean algebra, and for which there exists an additive representation in terms of a compatible probability measure $P : X \longrightarrow [0,1]$. Besides the lack of intuition, which is a handicap common to all this literature, from a technical standpoint also, these results do not apply to our case because our setting may not produce a Boolean algebra.

All in all, it becomes apparent that the existing results do not provide a completely satisfactory solution to the problem we have stated. Our major concern and motivation is that the existing proposals are based on conditions that are too complex and lacking in normative content or intuition to be applied in decisional contexts[3] .

At this point we will argue that *the complexity of the problem is drastically reduced if we consider a very suitable restriction on the codomain of the value function v.* This restriction is based on the well-known psychological principle of *categorization.* Thus we demonstrate that under such a constraint, a few axioms that can be easily interpreted within the different contexts listed above are necessary and sufficient for additive representation. Obviously we do not provide a solution to the general problem (i.e., where the codomain of c is unrestricted in $\mathbb{R}$). In turn we provide an axiomatic characterization in our particular setting, which is genuine in the *normative* sense usually adopted in decision theory and does not constrain the representation to any functional pattern other than addition. Moreover, a basic particularization of the model leads to a meaningful new family of rankings that includes Pattanaik and Xu's [21] *cardinality-based* rule as a particular case.

Our work is organized as follows. Section 2 contains the primitive elements of the model and presents, motivates and analyzes the codomain restriction. Section 3 presents the axiomatic characterization of additively representable rankings of opportunity sets that comply with that restriction (which we call weight-categorized), provided that a suitable necessary condition is fulfilled for expository convenience. This results in no loss of generality, but in an Appendix we

[3]Regarding the lack of intuition of Kraft et al.'s axiom, Gravel et al. [15] is a remarkable exception containing a reformulation and a new interpretation of the axiom. This interpretation, however, is narrowed to the particular context of ranking sets as a way to measure individual freedom of choice. In their model, moreover, the axiom only makes sense under a very particular perspective of the meaning of freedom of choice.

present the characterization result that offers a complete solution to the problem of identifying weight-categorized set rankings. Section 4 explores a meaningful particularization of our problem that is new in the literature, namely the *restricted-cardinality based* rule, which encompasses the aforementioned cardinality rule. We accomplish this independently of the prior development for the benefit of the reader. Section 5 presents the conclusive final remarks and poses some questions for future research.

2 The model

Throughout this Section, X will be any set, χ_0 will denote the set of finite subsets of X including the empty set, and $\succcurlyeq$ will be a complete and transitive binary relation defined on χ_0. Its strict part $\succ$ is the binary relation given by $A \succ B$ if and only if $A \succcurlyeq B$ and not $B \succcurlyeq A$. The indifference relation $\sim$ is derived according to: $A \sim B$ if and only if $A \succcurlyeq B$ and $B \succcurlyeq A$.

A *categorizing* function for X will be any function $c : X \longrightarrow \mathbb{N}$ such that either $Im(c) = \{1, 2, ..., n\}$ or $Im(c) = \{0, 1, 2, ..., n\}$ for some $n \in \mathbb{N} \cup \{+\infty\}$. The idea underlying a categorizing function is that the agent classifies the elements of X into *categories*, where the values of c are the corresponding labels associated to each category. This fits in with the prevailing principle in cognitive and social psychology that, especially since Allport [1], considers categorization as the natural mental process through which humans attach meaning to external information. In our case, n represents the number of categories used by the decision maker to classify the elements of X in terms of their desirability –or in terms of their likelihood in the appropriate context[4]. The categorical nature of the function makes it unacceptable to let the agent have the possibility of attaching a categorical value of 2 to one item when nothing is going to be assigned a value of 1, and so forth. We allow the agent whether or not to have elements valued at 0 within the universal set. In this respect, $\{(0,)\,1, ..., +\infty\}$ has an obvious meaning: regarding the categorization of the universal set X, the agent has a potentially infinite, successively increasing, number of labels at her disposal.

We will say that the ranking $\succcurlyeq$ is *weight-categorized* if there is a categorizing function c for X such that $A \succcurlyeq B$ if and only if $\sum_{a\in A} c(a) \geqslant \sum_{b\in B} c(b)$, where $\sum_{a\in\varnothing} c(a)$ is interpreted as 0. The interpretation of this rule is clear from the definition of categorizing function. Another feature that enhances its plausibility is the fact that cases where $c(x) < n$ for a fixed small $n \in \mathbb{N}$ admit yet more meaningful interpretations in certain instances that we are about to advance:

[4]The process by which n is determined lies beyond the scope of this paper, since it falls into the field of psychology, or even neurophysiology. What can be said here is that the value of n may depend on various aspects of the particular decision problem, such as the overall attractiveness of the alternatives, their similarity, or the decision maker's capacity for discrimination.

1. The study when $Im(c) = \{0\}$ is trivial: it corresponds to the *indifference rule* (i.e., $A \sim B$ for all possible $A, B \in \chi_0$).

2. When $c : X \longrightarrow \{0, 1\}$ different cases show up:

   (a) If $c(x) = 1$ for all $x \in X$ we have the *cardinality rule* proposed by Pattanaik and Xu [21] in the context of ranking opportunity sets in terms of freedom of choice.

   (b) The general case has remained unexplored until now and is the subject of a specific analysis in Section 4. This includes the cases above.

Any weight-categorized binary relation is bound to be a complete preorder, so we do not lose insight by dealing with such a class of relations throughout our study. Moreover, such a type of set orderings impose further restrictions. Particularly, in order to analyze the structure of a weight-categorized ranking of sets $\succcurlyeq$ we explore the implications regarding maximal chains of the restriction of its strict part $\succ$ to $\{\{x\} : x \in X\}$, which exist by Zorn's Lemma. They are associated with the sequence of indifference classes the decision maker is able to establish among individual elements. We will make thorough use of one of those fixed maximal chains $\mathcal{C}$, which inevitably agrees with the following Assumption if $\succcurlyeq$ is weight-categorized:

Assumption 1. $\mathcal{C}$ is countable and the number of elements that are below any element of the chain is finite.

This assumption on $\mathcal{C}$ sets a limit on the agent's capacity for discrimination by means of the strict preference among singletons, but it does not preclude the case where X is infinite and uncountable. Obviously, the assumption does not impose any restriction when X is finite. The family of complete preorders defined on χ_0 that satisfy Assumption 1 will be denoted by ξ. As argued earlier, it contains the whole class of weight-categorized rankings of sets. Thus *we will be concerned henceforth with complete preorders on χ_0 that satisfy Assumption 1.* By imposing this restriction, we avoid situations in which the number of indifference classes among singletons is uncountable, while cases such as $\{1\} \prec \{3\} \prec ... \prec \{2n-1\} \prec ... \prec \{2\} \prec \{4\} \prec ... \prec \{2n\} \prec ...$ are also banned, because each singleton must have an immediate predecessor. Other undesired instances, such as $.... \prec \{n\} \prec \{n-1\} \prec ... \prec \{2\} \prec \{1\}$ are also excluded.

Remark 2.1. *Assumption 1 seems plausible from a descriptive point of view of individual behavior. It fits very well into the bounded rationality literature and the very nature of categorization, meaning only that the capacity for refinement in human perception is not infinite. Nevertheless, at the cost of a loss of fluency,* it is possible to proceed without imposing such restrictions on the domain of admissible preferences. *The reader can check this in the Appendix.*

When $\succcurlyeq \in \xi$, because $\{\{x\} : x \in X\} = \{\{x\} : \{x\} \succ \varnothing\} \cup \{\{x\} : \{x\} \sim \varnothing\} \cup \{\{x\} : \{x\} \prec \varnothing\}$ a recursive argument shows that any maximal chain $\mathcal{C}$ of $\succ$ on $\{\{x\} : x \in X\}$ can be written as the union of the following three sets. The first has the form $\mathcal{C}_1 = \{\{x_n\} : n = 1, 2, ..., t\}$ if there is $\{x\} \succ \varnothing$, and $\mathcal{C}_1 = \varnothing$ otherwise. The second is defined as $\mathcal{C}_2 = \{\{x_0\}\}$ or $\mathcal{C}_2 = \varnothing$ according to whether or not there exists $\{x\} = \varnothing$. As for the third, it will be a certain $\mathcal{C}_3 = \{\{x_n\} : n = -r, -r+1, ..., -1\}$ for some natural $r > 0$ if there is $\{x\} \prec \varnothing$, and $\mathcal{C}_3 = \varnothing$ otherwise. Under this notation $\{x_1\}$ would be minimum in $\{y \in X : \{y\} \succ \varnothing, \{y\} \in \mathcal{C}\}$. When it exists it is interpreted as the first desirable or relevant element of the chain. It will be called *canonical* element and will play a crucial role in our model.

3 A characterization result

Next, we propose four axioms that will be used to characterize the family of orderings on ξ that are additively representable by means of a categorizing function (Theorem 3.1 below and the subsequent Remark 3.2). The two latter axioms are expressed in terms of $\mathcal{C}$ constructed as above.

- *Non-negativity* (NN). For all $A \in \chi_0$, $A \succcurlyeq \varnothing$
- *Independence* (IN). $\forall A, B \in \chi_0$ and $C \sim D$ such that $(A \cup B) \cap (C \cup D) = \varnothing$, $A \succcurlyeq B$ if and only if $A \cup C \succcurlyeq B \cup D$.
- *Decomposition* (DE). $\forall \{x_k\} \in \mathcal{C}$, $\{x_k\} \sim \{x_1, x_{k-1}\}$ whenever $k > 2$
- *Equivalent Singleton Expansions* (ESE).
  $\forall A, B \in \chi_0$, $A \succcurlyeq B \cup \{x_1\} \Leftrightarrow A \cup \{x_1\} \succcurlyeq B \cup \{x_2\}$ whenever $x_2, x_1 \notin B$ and $x_1 \notin A$.

As shown before, (NN) and (IN) are two of the three axioms that de Finetti proposes as necessary in the qualitative probability measuring problem. The former is rather standard and easily interpretable under the different possible contexts. The latter is also a common property in set ranking models in different scenarios. It simply says that the addition (or removal) of two indifferent sets C and D to (from) two given sets A and B does not affect the primitive relation over A and B.

(DE) allows any element of the chain to be expressed in relation to the first relevant one. In particular, (DE) says that any singleton $\{x\}$ is indifferent to a pair consisting of the canonical element and the one immediately inferior to x in the chain. The particular interpretation of (DE) in the qualitative probability framework is that the agent considers the event consisting of the only possible result $\{x_k\}$ (that he has put at the k-th level in terms of likelihood in the maximal chain) equally likely to occur as an event that includes two incompatible results: $\{x_1\}$, which he has placed at the lowest level among the probable outcomes, and $\{x_{k-1}\}$.

A natural meaning of (DE) is that the canonical element is taken as a reference unit to establish the *distance* between one element in the chain and the next. That element becomes the *standard unit* that needs to be added to any singleton in order to make the resulting set indifferent to the singleton immediately above it. Thus, the axiom suggests some idea of equidistance between the steps in $\mathcal{C}$. In another sense, and considering that, by identifying $\mathcal{C}$, we are displaying the individual's maximal capacity of discrimination, (DE) stands for the idea that this capacity is *uniform* along the chain, that is, irrespective of the degree of desirability (likelihood in the probabilistic context) of the items under consideration.

Suppes [31], [32][5] uses a related property in his study of necessary conditions for the existence of a subjective probability measure. In particular he makes analogous use of his "unduly strong solvability axiom" when restricted to an equally-spaced standard sequence in the analysis of *finite approximate freedom structures* (see also [33, p. 252]): in a broad sense, $\mathcal{C}$ plays the role of this algebra of sets of decisions. Another parallelism is the fact that both in Suppes' and our approach, each minimal element of the respective special set is assigned the same value.

According to (ESE), if we add the canonical element x_1 to a set B that does not contain it, then when we compare the enlarged set $B \cup \{x_1\}$ with any other set A, the effect is the same as if we had enlarged A with x_1 and B with the second relevant element in the chain. Though it is an independent axiom, (ESE) expresses an idea of equidistance in $\mathcal{C}$ close to that in (DE). The particular interpretation of (ESE) in the different decisional contexts is immediate, now that it has been explained for the previous axiom.

Theorem 3.1. *Let $\succcurlyeq \in \xi$. Then $\succcurlyeq$ satisfies (NN), (IN), (DE) and (ESE) if and only if there exists a categorizing function $c : X \longrightarrow \mathbb{N}$ such that, for all $A, B \in \chi_0$, $A \succcurlyeq B$ if and only if $\sum_{a \in A} c(a) \geqslant \sum_{b \in B} c(b)$.*

Remark 3.2. *For the purpose of stating* sufficient *conditions only, the proof below permits us to check that (ESE) can be replaced by the simpler assumption of "Minimal Richness": this amounts to the existence of $a \in X$, $a \neq x_1$ such that $\{a\} \sim \{x_1\}$. In economic contexts, where scarcity applies especially to the most desired items, it is not unreasonable to admit that there are many items at the lowest level of desirability (or likelihood). The interested reader can therefore produce a simple Corollary under less dull assumptions.*

Proof of Theorem 3.1.

[5] We thank R. Duncan Luce for pointing out these references.

Necessity is proven as follows. If $Im(c) = \{0, ..., n\}$ then we select $x_0 = a$ with $c(a) = 0$. Now pick $x_1, ..., x_n$ with $c(x_i) = i$ for any possible $i = 1, ..., n$. Then $\mathcal{C}$ constituted by the union of the selected elements permits us to check our claim.

We now prove sufficiency. The maximal character of $\mathcal{C}$ together with transitivity and completeness of $\succcurlyeq$, ensure that for every $a \in X$ we can assign a unique label $l(a)$ such that $\{a\} \sim \{x_{l(a)}\}$, $\{x_{l(a)}\} \in \mathcal{C}$. Observe also that $\mathcal{C}_3 = \varnothing$ in our notation due to (NN).

Let $A, B \in \chi_0$. Write $A = \{a_1, ..., a_m\}$, $B = \{b_1, ..., b_n\}$. Then, for all $a_k \in A$ there exists a unique $x_{l(a_k)} \in \mathcal{C}$ such that $\{a_k\} \sim \{x_{l(a_k)}\}$ and for all $b_k \in B$ there is a unique $x_{l(b_k)} \in \mathcal{C}$ with $\{b_k\} \sim \{x_{l(b_k)}\}$. Our aim is to prove that

$$A \succcurlyeq B \text{ is equivalent to } \sum_{k=1,..,m} l(a_k) \geqslant \sum_{k=1,..,n} l(b_k) \tag{1}$$

This would mean that $\succcurlyeq$ is weight-categorized according to the weight assignment $c(a) = l(a)$ for each $a \in X$, where $l(a)$ is such that $\{a\} \sim \{x_{l(a)}\}$. We denote $w(A) = \sum_{a \in A} c(a)$ and $w(\varnothing) = 0$.

The conclusion is achieved through an algorithm that applies certain steps recursively until a final stage is reached. Each step transforms (1) into another equivalent statement. Preliminarily, two particular restrictions are granted. First, no generality is lost if we assume $l(a_1) \leqslant l(a_2) \leqslant ....$ and also $l(b_1) \leqslant l(b_2) \leqslant .....$ Second, due to (IN) we can further assume that $l(a_k) > 0$, $l(b_k) > 0$ for all possible k.

Step 1: We can assume, without loss of generality, that $a_k \sim b_{k'}$ fails to be true throughout.

By this we mean: if $a_k \sim b_{k'}$ then, by (IN) $A \succcurlyeq B$ if and only if $A' = A \setminus \{a_k\} \succcurlyeq B' = B \setminus \{b_{k'}\}$ and also $w(A) \geqslant w(B)$ if and only if $w(A') \geqslant w(B')$. Thus, the fact that $A' \succcurlyeq B' \Leftrightarrow w(A') \geqslant w(B')$ holds true is equivalent to the validity of (1). And this reduction can be iterated until we reach respective subsets A_1 and B_1 that fulfill our requirement.

Step 2: If either (the reduced subset A_1 obtained in Step 1 for) A or (the reduced subset B_1 obtained in Step 1 for) B is empty then we move to Step 3.

Otherwise, two instances appear: either $0 < l(a_1) < l(b_1)$ or $0 < l(b_1) < l(a_1)$. By symmetry we argue under the first instance only. By switching elements or using (IN), we can assume $a_1 = x_{l(a_1)}$ and $b_1 = x_{l(b_1)}$. We insist that we have relabeled $A = A_1$ and $B = B_1$ only for notational convenience but the sets we are working on are subsets of the original ones, for which the property we are to prove means exactly the desired conclusion. Three separate cases stem from this:

Case 1: $1 = l(a_1) < l(b_1)$. In the event $2 < l(b_1)$, by (DE) and (IN) we can write $A \sim A' = \{x_1, a_2, ..., a_m\}$, $B \sim B' = \{x_1, x_{l(b_1)-1}, b_2, ..., b_n\}$. Then let $A_2 = A' \setminus \{x_1\}$ and $B_2 = B' \setminus \{x_1\}$.

When $2 = l(b_1)$, noting that $A = (A \setminus \{x_1\}) \cup \{x_1\}$ and $B = (B \setminus \{x_2\}) \cup \{x_2\}$, we can apply (ESE) obtaining $A \succcurlyeq B$ iff $A \setminus \{x_1\} \succcurlyeq (B \setminus \{x_2\}) \cup \{x_1\}$. Then we let $A_2 = A \setminus \{x_1\}$ and $B_2 = (B \setminus \{x_2\}) \cup \{x_1\}$.

Case 2: $2 = l(a_1) < l(b_1)$. By (IN) $A \sim A' = \{x_2, a_2, ..., a_m\}$.

If $l(b_1) = 3$, then by (IN) and (DE) $B \sim B' = \{x_1, x_2, b_2, ..., b_n\}$. Then, let $A_2 = A' \setminus \{x_2\}$ and $B_2 = B' \setminus \{x_2\}$.

If $l(b_1) > 3$, then again by (IN) and (DE) $B \sim B' = \{x_1, x_{j-1}, b_2, ..., b_n\}$. In such a case, let $A_2 = A'$ and $B_2 = B'$.

Case 3: $2 < l(a_1) < l(b_1)$. Using (DE) and (IN) we can write $A \sim A' = \{x_1, x_{l(a_1)-1}, a_2, ..., a_m\}$, $B \sim B' = \{x_1, x_{l(b_1)-1}, b_2, ..., b_n\}$. Let $A_2 = A' \setminus \{x_1\}$ and $B_2 = B' \setminus \{x_1\}$.

Whatever case arises, it is easy to convince oneself that (1) holds true if and only if the equivalence $A_2 \succcurlyeq B_2 \Leftrightarrow w(A_2) \geqslant w(B_2)$ is correct.

We return to Step 1, and re-label $A = A_2$, $B = B_2$ in order to avoid unnecessarily complex notation.

Step 3: At length, the algorithm stops because application of Step 2 reduces $w(A)$ and $w(B)$ strictly. This happens when (at least) one of the following circumstances arises:

Case 1: $A_1 = \varnothing$ after applying Step 1.

Under the conventions made this amounts to $w(A_1) = 0 \leqslant w(B_1)$ and $A_1 \preccurlyeq B_1$, or, equivalently, $w(A) \leqslant w(B)$ and $A \preccurlyeq B$.

Case 2: $B_1 = \varnothing$ after applying Step 1.

Under the conventions made this amounts to $w(A_1) \geqslant 0 = w(B_1)$ and $B_1 \preccurlyeq A_1$, or, equivalently, $w(A) \geqslant w(B)$ and $B \preccurlyeq A$.

Case 3: $A_1 = \varnothing$ and $B_1 = \varnothing$ after applying Step 1. This means the two prior cases at once.

We have concluded that w is a utility function for $\succcurlyeq$ as desired. ■

We conclude this section by checking that there is no redundant axiom in Theorem 3.1:

Proposition 3.3. *(NN), (IN), (DE) and (ESE) are independent axioms*

Proof.

- (NN). Let $X = \{x, y, z\}$ and let $\succcurlyeq$ such that $\{y, z\} \sim \{z\} \succ \{x, y, z\} \sim \{x, z\} \sim \{y\} \sim \varnothing \succ \{x, y\} \sim \{x\}$. Then, $\succcurlyeq$ satisfies (IN), (DE) and (ESE), but not (NN).
- (IN). Let $X = \{x, y, z\}$ and let $\succcurlyeq$ such that $\{x, y, z\} \succ \{y, z\} \succ \{x, z\} \succ \{x, y\} \succ \{x\} \sim \{y\} \sim \{z\} \succ \varnothing$. Then, $\succcurlyeq$ satisfies (NN), (DE) and (ESE) but not (IN), since $\{x\} \sim \{y\}$ but $\{x, z\} \succ \{x, y\}$.
- (DE). Let $X = \{1, 2, 3\}$, and let $\succcurlyeq$ be the *leximax* ordering on the set of subsets of X: $\{3, 2, 1\} \succ \{3, 2\} \succ \{3, 1\} \succ \{3\} \succ \{2, 1\} \succ \{2\} \succ \{1\} \succ \varnothing$. Then $\succcurlyeq$ satisfies (NN), (IN) and (ESE), but not (DE) since $\{3\} \succ \{2, 1\}$.
- (ESE). Let $X = \{1, 2, 3, 4\}$, and let $\succcurlyeq$ defined by $\{1, 2, 3, 4\} \succ \{2, 3, 4\} \succ \{1, 3, 4\} \succ \{3, 4\} \sim \{1, 2, 4\} \succ \{2, 4\} \sim \{1, 2, 3\} \succ \{2, 3\} \succ \{1, 4\} \succ \{1, 3\} \sim \{4\} \succ \{3\} \sim \{1, 2\} \succ \{2\} \succ \{1\} \succ \varnothing$. Then $\succcurlyeq$ satisfies (NN), (IN), and (DE), but not (ESE) since $\{2, 3\} \succ \{1, 4\}$ but $\{1, 2, 3\} \sim \{2, 4\}$.

■

4 The RCB rule

In this section, the particular case where a *binary* categorizing function is available for $\succcurlyeq$ (that means $c(x) \in \{0, 1\}$ along X) is the subject of an additional study. This case leads to very intuitive interpretations and allows us to re-state the necessary and sufficient conditions of the general case in terms of other intuitive axioms. The treatment of this topic is independent of the proofs and arguments given for the general case.

We start defining $\succcurlyeq$ as *restricted-cardinality based* (henceforth RCB) if there exists $X_s \subseteq X$ such that $A \succcurlyeq B$ if and only if $|A \cap X_s| \geqslant |B \cap X_s|$. It is clear that RCB rules contain both the cardinality rule (when $X_s = X$) and the indifference rule (when $X_s = \varnothing$) as particular cases.

Thus any RCB ranking is requested to proceed in two steps: first *the elements in the set are classified into two classes*, and then the collections of elements are ranked according to the number of elements in one selected class. We interpret the X_s class as a selection of *significant* or *focal* elements in X. In most contexts they can also be interpreted as the subset of *satisfactory* alternatives[6]. We could cite many particular settings where this distinction is meaningful, and where the content of the terms "significant", "focal" or "satisfactory" varies depending on the particular interpretation of the set ranking problem.

Suppes [33] (p.248) already mentioned this kind of rule in the context of individual liberty evaluation, also making a first axiomatic approximation to it:

[6]In fact, our binary categorizing function coincides both formally and in spirit with Simon's [29] "*Simple Pay-off Functions*", which only distinguishes between "satisfactory" and "unsatisfactory" outcomes, and supports many of Simon's *satisficing* behavioural models.

"*There are many individuals who are primarily concerned about freedom in a particular domain, and therefore want the widest possible range of freedom but only 'qua' civil liberties, 'qua' economic choices or 'qua' something else*".

When freedom of choice is measured by means of opportunity sets (mutually exclusive options), we find a remarkable approach based on the so-called *reasonable preferences*, under which RCB rules are also suitable. From this perspective the social planner labels an opportunity as socially significant or "eligible" if, considering the full range of individual preferences that are reasonable in the choice situation under consideration, that opportunity is best for at least one such preference (see Jones and Sugden [16], Pattanaik and Xu [22] and Sugden [30] for details). Under this approach, the social planner evaluates the degree of freedom of choice enjoyed by an anonymous agent in the society (whose actual preferences are unknown by the social planner) through the number of potentially eligible opportunities available to him. Thus, from this position, the RCB rule would maximize the social provision of eligible options, and ignore those that no reasonable person would choose.

The RCB rule could also make sense in the qualitative probability measurement problem. Under such a context we would be assuming that for the purpose of the likelihood evaluation of a set of results, individuals only categorize between those that are *sufficiently likely* as to be taken into account, and those that are not.

In coalition or team formation problems, it is quite usual for the decision maker to maximize the number of a certain kind of partners (those of his own party or those who are going to vote alongside him; the number of qualified researchers in his team, etc.).

Regardless of the interpretation, a crucial feature of our model is that *the set of focal elements is not given ex-ante*; its existence and composition are instead induced from the axioms of the ranking over the sets. This departs from other models like Jones and Sugden [16], and Pattanaik and Xu [22], where the significant elements are determined *ex-ante* and for each set; Romero-Medina [26], where the significant elements are defined uniquely for X, but again in an *ex-ante* way; and Puppe [23], who inductively determines the significant element from the ranking of the sets, but taking into account the set to which they belong. Moreover, all these references focus only on the particular meaning of sets as *opportunity sets*.

It is simple to check that any RCB rule is weight-categorized since it can be derived from $c(x) = 1$ if $x \in X_s$ and $c(x) = 0$ otherwise. The converse is also true, that is, $\succcurlyeq$ stems from a *binary* categorizing function $c : X \longrightarrow \{0, 1\}$ exactly when it is an RCB rule.

Next we propose the following properties:

- *Symmetry between Significant Elements*: $\{x\} \succ \varnothing, \{y\} \succ \varnothing$ implies $\{x\} \sim \{y\}$, for each $x, y \in X$.

- *Simple Independence*: For every $A, B \in \chi_0$ and $x \notin A \cup B$, $A \succcurlyeq B$ if and only if $A \cup \{x\} \succcurlyeq B \cup \{x\}$.

Simple Independence says that adding/dropping the same element to/from two sets does not alter the initial ranking between them. This is just a weaker version of the condition of *Independence* already used in the previous section.

With respect to the first property, it says that any two elements that are desirable or relevant as singletons are ranked at the same level. This property implies that $\{x\} \succ \{y\} \succ \varnothing$ is impossible when $x \neq y$.

This property probably deserves a more detailed motivation, which would call for a suitable framing. The axiom is well suited to the "reasonable preferences" approach, for example. If we are social planners who accept the possibility that any anonymous member of the society might be either Christian or Muslim, we should not be judging whether allowing him the opportunity to worship in a church is socially better than allowing him the opportunity to worship in a mosque. In a similar vein, well-being analysis based on the amount of basic rights or primary goods tends to reject the idea of prioritizing certain particular rights (or primary goods) over others. In decision problems that involve evaluation of other people, individual preferences are very often based solely on binary judgments (whether or not candidates are qualified to be members of a committee, whether or not a partner in a coalition will use his vote to support certain issues, etc.). This kind of binary distinction between others also connects closely with the wide literature on categorization and stereotypes in social psychology.

To finish the motivation of the Symmetry axiom within the different decisional contexts under consideration, even in the setting of probability measurement we find a very meaningful context, where the relevant issue is not to determine the degree of probability of an outcome but to judge whether it is probable or not. This is the case of the Rawlsian veil-under-ignorance scenario (Rawls [24]), in which an adequate development of the Rawlsian moral arguments suggests that any probability or likelihood information should be ignored.

The two axioms above allow us for the following characterization.

Theorem 4.1. *Let $\succcurlyeq$ be a complete and transitive relation on χ_0. Then:*
$\succcurlyeq$ is RCB if and only if it satisfies Non Negativity, Simple Independence and Symmetry between Significant Elements.

The next Lemma will be needed in our proof.

Lemma 4.2. *Let $\overline{\succcurlyeq}$ be a complete and transitive relation on the set of all finite subsets of a set Y. If it satisfies: $\{x\}\overline{\succ}\varnothing$ for all $x \in Y$, $\{x\}\overline{\sim}\{y\}$ for all $x, y \in Y$, and Simple Independence then it is cardinality-based. That is, $A\overline{\succcurlyeq}B$ if and only if $|A| \geqslant |B|$ for each $A, B \subseteq Y$.*

The proof of the Lemma is a simple variation on the characterization of the cardinality rule by Pattanaik and Xu [21] based on the following guides:

- Pattanaik and Xu's model considers the three following axioms to characterize the cardinality-based rule on the domain of finite subsets of Y excluding the empty set:
  - $\{x\} \eqsim \{y\}$ for all $x, y \in Y$
  - Simple Independence
  - $\{x, y\} \succ \{x\}$ for all $x, y \in Y$ (Simple Monotonicity)
- Considering the subsets of Y, included the empty set, $\{x\} \succ \varnothing$ for all $x \in X$ and Simple Independence together imply $\{x, y\} \succ \{x\}$ for all $x, y \in X$.
- Knowing that Simple Monotonicity is satisfied in our domain, which includes the empty set, the reader can check that the original proof by Pattanaik and Xu can be perfectly replicated for such a domain.

Proof of Theorem 4.1.
It is straightforward to check that $\succcurlyeq$ satisfies the required properties. Now let us assume that $\succcurlyeq$ is a complete and transitive relation on X that satisfies Independence and Symmetry between Significant Elements. We define $X_s = \{x \in X : \{x\} \succ \varnothing\}$. Then, Simple Independence yields

$$A \succcurlyeq B \text{ if and only if } A \cap X_s \succcurlyeq B \cap X_s \tag{2}$$

for each $A, B \in \chi_0$. The reason for this is that, since $\{a\} \sim \varnothing$ whenever $a \in D \backslash X_s$, transitivity of $\sim$ plus Simple Independence entail $D \backslash X_s \sim \varnothing$ and therefore $D = (D \setminus X_s) \cup (D \cap X_s) \sim D \cap X_s$ using Simple Independence recurrently.

The restriction of $\succcurlyeq$ to X_s satisfies Simple Independence and (NN) -since these properties are inherited by subsets-, and also $\{x\} \sim \{y\}$ for all $x, y \in X_s$. Thus Lemma 4.2 allows us to ensure that the restriction of $\succcurlyeq$ to X_s is cardinality-based because $\{x\} \succ \varnothing$ across X_s. We then get

$$A \cap X_s \succcurlyeq B \cap X_s \text{ if and only if } |A \cap X_s| \geqslant |B \cap X_s|$$

and this together with (2) finishes the argument. ■

5 Conclusions, remarks and topics for future research

We have seen in Sections 2 and 4 that *a common framework* incorporates some rules that have either been used in the literature or are both plausible and new. By doing so we have considered a simple particular specification of the additive model, which had been difficult to handle in the field under inspection due to its unfriendly behavior. The term "simple" means that instead of imposing an additional functional pattern on the representation, we specify a restriction only on its codomain.

It is worth discussing the suitability of an additive rule when sets are interpreted as opportunity sets. Within this framework, the elements of the set are mutually exclusive options, and the decision maker has the capability to choose the final outcome. Then, it is natural to assume that the values of all the alternatives that will not be ultimately chosen should not be taken into account. That is, the value of a set should equal the value of its best option, as in the indirect utility criterion of the standard consumer theory. However, we can put forward at least two arguments that suggest that additivity is also plausible in a freedom of choice context:

1.− The first argument relates to the view that freedom of choice has an *intrinsic value*. The position that the mere fact of being able to choose is valuable and independent of the final choice is philosophically well rooted (see, for example, Mill [19] or Nozick [20]). From such a perspective, counting the number of available opportunities in the opportunity set is a way to measure the degree of freedom of choice provided by it. This is precisely Pattanaik and Xu's [21] cardinalist criterion, which is additive and weight-categorized as we have shown. In this case, the weights attached to the elements should not be interpreted as their utilities, but rather as their contribution to the agent's capacity to choose[7].

2.− The second scenario, where freedom of choice admits an additive measurement, is the aforementioned *reasonable preferences* approach. In each situation therein, the social planner should arguably account for the value of all the "elegible" options offered to an anonymous agent. Furthermore, unlike in the cardinalist case, different alternatives in this framework might be plausibly have different values, depending, for example, on the probability of the option being chosen by the agent or on the social externalities it may generate.

Regarding possibilities for further research, our approach poses some natural questions. On the one hand, our specification might ask for interpretations of cases that we have not considered here, since we have limited ourselves to studying the binary instance (0 vs. 1) in depth. On the other hand, some other specifications of the codomain could eventually give rise to meaningful interpretations while retaining an axiomatizable character. We are particularly intrigued by the possibility of extending our general model in Section 2 in such a way as to allow c also to take negative values. A particularly meaningful question would be the following: What axioms characterize *Good-Neutral-Bad* rankings of sets? By this we mean the existence of a GNB function $c : X \longrightarrow \{1, 0, -1\}$ such that $A \succcurlyeq B$ if and only if $\sum_{a\in A} c(a) \geqslant \sum_{b\in B} c(b)$, where $\sum_{a\in\varnothing} c(a)$ is to be interpreted as 0. Such a specification is lacking in the general model in Section 2, yet it is

[7]See Gravel et al. [15] for more details about the discussion on the plausibility of additivity in a freedom of choice setting.

a simple generalization of RCB rules in the same spirit. Then, what is required in this ranking method is, first, that a classification into 3 classes -good, neutral, bad- is performed by the agent, and then a "binary" weight assignment is used to rank the subsets. The interpretation of this rule for ranking sets is clear and it strikes us as having psychological appeal in many settings, such as coalitions where there exists the possibility of undesirable partners, choice among opportunity sets where there are "noisy" opportunities, or a voting problem where the voter not only tries to maximize the number of qualified candidates but also tries to avoid the presence of undesirable candidates.

Further, one may wonder whether there are certain known particular rules that can be represented by means of a weight-categorized function. We have seen, for example, that not only the cardinality rule or the indifference rule, but also RCB rules and GNB rules, do behave like additively representable rules (specifically, of the weight-categorical type) and thus the question, in our view, appears relevant. What can be said about other related rules? For example, the interested reader can check by means of our characterization that referential rules from the ranking sets literature, such as the *cardinality-first-lexicographic* and the *indirect-utility-first* rules (see Bossert et al. [7]) are not additively representable by any weight-categorized function.

Finally, there is some literature on a different interpretation of the set ranking problem that has not so far been mentioned. This is the so-called problem of choice *under complete uncertainty*, where sets represent mutually exclusive outcomes, without any associated probability distribution. Then, any individual action generates a set of possible outcomes; and the final result depends on external factors such as nature, chance, or the strategies of other agents. Therefore, ranking actions is equivalent to ranking their corresponding sets of possible outcomes. Examples of works that have characterized rules for such a decision situation are Bossert et al. [8] and Arlegi [2, 3]. However, the additive evaluation of sets proposed herein is inconsistent with this kind of problems. The intuitive reason is that the desirability of an outcome (say, "winning 10 euros" (10)), depends on the set where it is included: in this context, the set of possible outcomes $\{100, 10\}$ is worse than the sure-result set $\{100\}$, so "winning 10" should have a negative value. But $\{1, 10\}$ is better than $\{1\}$, which would only be possible if "winning 10" computes positively. Thus, if one wishes to maintain the additive flavor of the rule in this context, some changes need to be introduced. Bossert and Slinko [9], for example, propose an additive rule that accounts only for the values of the best and worst possible results in the set. Another natural approach to the problem might be to introduce other information about the set in the evaluation function (e.g., the number of elements in the set, as in an *average* rule).

6 Appendix

Here we present the characterization result that offers a complete solution to the problem of identifying weight-categorized rankings of sets:

Axioms:

- *Non-negativity* (NN). For all $A \in \chi_0$, $A \succcurlyeq \varnothing$
- *Independence* (IN). $\forall A, B \in \chi_0$ and $C \sim D$ such that $(A \cup B) \cap (C \cup D) = \varnothing$, $A \succcurlyeq B$ if and only if $A \cup C \succcurlyeq B \cup D$.
- *Countability* (CO). Any maximal chain $\mathcal{C}$ of $\succ$ on $\{\{x\} : x \in X\}$ is countable and for all $\{x\} \in \mathcal{C}$ the set $\{y \in X : \{x\} \succ \{y\}, \{y\} \in \mathcal{C}\}$ is finite.
- *Decomposition** (DE*). For any maximal chain $\mathcal{C}$ of $\succ$ on $\{\{x\} : x \in X\}$ and $\{x_c\} \in \mathcal{C}$ minimum in $\{y \in X : \{y\} \succ \varnothing, \{y\} \in \mathcal{C}\}$, then $\{x\} \succ \{y\}$, $y \neq x_c$ and there is no $z \in X$ with $\{x\} \succ \{z\} \succ \{y\}$ together yield $\{x\} \sim \{x_c, y\}$.
- *Equivalent Singleton Expansions** (ESE*) For any maximal chain $\mathcal{C}$ of $\succ$ on $\{\{x\} : x \in X\}$ and $\{x_c\} \in \mathcal{C}$ minimum in $\{y \in X : \{y\} \succ \varnothing, \{y\} \in \mathcal{C}\}$, then $\{x\} \succ \{x_c\}$ and there is no $z \in X$ with $\{x\} \succ \{z\} \succ \{x_c\}$ together yield the equivalence $A \succcurlyeq B \cup \{x_c\} \Leftrightarrow A \cup \{x_c\} \succcurlyeq B \cup \{x\}$ whenever $x, x_c \notin B$ and $x_c \notin A$.

Theorem 6.1. *A binary relation $\succcurlyeq$ defined on χ_0 satisfies (NN), (CO) (IN*), (DE*) and (ESE*) if and only if there exists a categorizing function $c : X \longrightarrow \mathbb{N}$ such that for all $A, B \in \chi_0$, $A \succcurlyeq B$ if and only if $\sum_{a \in A} c(a) \geqslant \sum_{b \in B} c(b)$.*

Sketch of proof.

For sufficiency, the proof is the same as for Theorem 3.1 once one notes that, together with (CO), (DE*) implies (DE) and (ESE*) implies (ESE). For necessity, note that any weight-categorized ranking satisfies (CO). ■

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