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A Note on Bossert, Pattanaik and Xu's "Choice Under Complete Uncertainty: Axiomatic Characterization of Some Decision Rules"*

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Summary. Recent work by Bossert, Pattanaik and Xu provides axiomatic characterizations of some decision rules for individual decision making under complete uncertainty. This note shows that, in the case of two of these rules, they do not satisfy one of the axioms used for their characterization. A counterexample illustrating this fact is provided, as well as an alternative way to characterize the two rules under consideration, mantaining as far as possible the original axioms proposed by Bossert, Pattanaik and Xu.

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1 Introduction

Bossert, Pattanaik and Xu [1] characterize four new rules for ranking sets in a context of choice under complete uncertainty, where the elements of the sets are interpreted as possible uncertain outcomes. One of them is called *min-max* relation, other one is called *max-min* relation. The min-max relation compares any pair of sets by considering first the worst element (possible outcome) of each set. If the worst element of one set is better than that of the other, then the first set is considered to be better. But in the case when both worst elements are equal, unlike the standard maximin rule, the min-max criterion looks at the best element in both sets. Then, if the best element in one set is better than that of the other set, the first set is declared to be better, and if the respective best elements are equal, then both sets are considered to be indifferent. The max-min rule represents the dual case in relation with the min-max. According to the max-min rule, the decision maker looks first at the respective best possible outcomes, and in case they are equal, then he considers the respective worst ones.

As pointed out by the authors, these rules are plausible in contexts where the decision maker tends to concentrate on certain "focal" or "conspicuous" features of sets, for example their best and worst elements.

One of the axioms used for the characterization of both, the max-min rule and min-max rule is called *Independence* (IND). This axiom requires that if a set of possible outcomes A is strictly better than another set B, then, the addition of a new element to both sets never reverses the previous ranking between A and B. Bossert, Pattanaik and Xu [1] prove that this axiom, together with some others, are sufficient to assert that the only way to compare sets of outcomes is the min-max criterion. Afterwards, in combination with other different axioms, IND is used again to reach logically the max-min criterion. The authors leave the necessary part of the proof to the reader in the case of both rules. However, when we check whether or not they satisfy the axioms, both, the min-max and the max-min rule, violate IND.

This fact constitutes the main motivation for this Note. Section 2 presents the basic notation and axioms used in [1], as well as a counterexample showing that neither the max-min nor the min-max rule satisfy IND; also three new axioms are proposed. In Section 3 an alternative way to characterize both rules is proposed. This alternative proposal has been made trying to maintain as far as possible the original axioms of Bossert, Pattanaik and Xu [1]. Section 4 contains a brief description of another alternative way to reach, with different axioms, the same results.

2 The Basic Notation and Axioms

Throughout this work we will basically follow the original notation used by Bossert, Pattanaik and Xu in [1]. However, for the reader's convenience we will restate below the notation to be used.

Let X denote the non-empty and finite universal set of alternatives. Let K denote the set of all non-empty subsets of X. An element of K is interpreted as an *uncertain prospect* where the agent does not know the probability distribution, nor any likelihood ranking of the possible outcomes. K_2 denotes the class $\{A \in K \ s.t. \ \#A \leq 2\}$.

R will denote a given linear preference ordering over X, that is, a reflexive, transitive, complete and antisymmetric binary relation. P represents the asymmetric factor of R. For all $A \in K$ $\underline{a}$, $\overline{a}$ denote, respectively, the worst and best elements of A according to R, while for all $A, B \in K$, $min(A \cup B)$ and $max(A \cup B)$ denote, respectively, the worst and best elements of $(A \cup B)$ according to R. Note that the worst and best elements are both well-defined and unique for all $A \in K$ because X is finite and R is a linear ordering.

Let $\succcurlyeq$ be an ordering over K, that is, $\succcurlyeq$ is a reflexive, transitive, and complete binary relation over the possible sets of outcomes, which represents the decision maker's preference ordering over the possible uncertain prospects. $\succ$ and $\sim$ denote, respectively, the asymmetric and the symmetric factors of $\succcurlyeq$.

The min-max relation (denoted by $\succcurlyeq_{mnx}$) and the max-min relation (denoted by $\succcurlyeq_{mxn}$) are respectively defined by:

For all $A, B{\in}K$, $A \succcurlyeq_{mnx} B$: $\Leftrightarrow$ $[(\underline{a}P\underline{b})$ or $((\underline{a}I\underline{b})$ and $(\overline{a}R\overline{b})]$.

For all $A, B{\in}K$, $A \succcurlyeq_{mxn} B$:$\Leftrightarrow$ $[(\overline{a}P\overline{b})$ $or((\overline{a}I\overline{b})$ and $(\underline{a}R\underline{b})]$.

In a first result, Bossert, Pattanaik and Xu prove that the two following axioms are sufficient (but not necessary) to assert that, for all $A{\in}K$, $A \sim \{\underline{a}, \overline{a}\}$ (see Theorem 1 in [1, pg.299]):

Simple Monotonicity (SM): for all $x, y{\in}X$ such that xPy, $\{x\} \succ \{x, y\} \succ \{y\}$.

Independence (IND): for all $A, B{\in}K$ and all $x{\in}X\backslash(A \cup B)$, $A \succ B$ implies $A \cup \{x\} \succcurlyeq B \cup \{x\}$

Afterwards, Bossert, Pattanaik and Xu present the following four additional axioms in order to characterize the min-max and the max-min rule,

Type 1 Simple Dominance (SD1): for all $x, y, z{\in}X$ such that $xPyPz$, $\{x, z\} \succ \{y, z\}$.

Type 2 Simple Dominance (SD2): for all $x, y, z{\in}X$ such that $xPyPz$, $\{x, y\} \succ \{x, z\}$.

Simple Uncertainty Aversion (SUA): for all $x, y, z{\in}X$ such that $xPyPz$, $\{y\} \succ \{x, z\}$.

Simple Uncertainty Appeal (SUP): for all $x, y, z{\in}X$ such that $xPyPz$, $\{x, z\} \succ \{y\}$.

By means of the previous axioms Bossert, Pattanaik and Xu prove the following result (see Lemma 2 in [1, pg.303]):

$$\succcurlyeq \; satisfies \; SM, SD1 \; and \; SUA \; iff \; for \; all A, B{\in}K_2, A \succcurlyeq B \Leftrightarrow A \succcurlyeq_{mnx} B. \tag{1}$$

$$\succcurlyeq \text{ satisfies } SM,\ SD2 \text{ and } SUP \text{ iff for all } A, B \in K_2, A \succcurlyeq B \Leftrightarrow A \succcurlyeq_{mxn} B. \tag{2}$$

Finally, by using (1), (2) and their Theorem 1, they propose the following Theorem (Theorem 3 in the original version, see [1, pg.304]):

$$\succcurlyeq \text{ satisfies } SM,\ IND,\ SD1 \text{ and } SUA \text{ if and only if } \succcurlyeq = \succcurlyeq_{mnx} \tag{3}$$

$$\succcurlyeq \text{ satisfies } SM,\ IND,\ SD2 \text{ and } SUP \text{ if and only if } \succcurlyeq = \succcurlyeq_{mxn} \tag{4}$$

However, neither $\succcurlyeq_{mnx}$ nor $\succcurlyeq_{mxn}$ satisfy axiom IND. Let us consider the following counterexamples:

Let $X = \{a, b, c, d, e\}$ such that $aPbPcPdPe$. Then $\{b, c\} \succ_{mnx} \{a, d\}$ but $\{a, d, e\} \succ_{mnx} \{b, c, e\}$

On the other hand, $\{b, e\} \succ_{mxn} \{c, d\}$ but $\{a, c, d\} \succ_{mxn} \{a, b, e\}$

This lack, as well as the plausability of both rules, motivates this Note in order to axiomatically characterize them. The following results try to fill this gap reasonably maintaining an important part of the axiomatic structure proposed originally by the authors in [1]. For that, three new axioms will be introduced:

Substitution (SUB): for all $A \in K$, for all $y \in A$ and $x \in X \backslash A$, xPy implies $(A \cup \{x\}) \backslash \{y\} \succcurlyeq A$.

Monotone Consistency (MC): for all $A, B \in K$, $A \succcurlyeq B$ implies $A \cup B \succcurlyeq B$

Robustness (ROB) for all $A, B, C \in K$, $A \succcurlyeq B$ and $A \succcurlyeq C$ implies $A \succcurlyeq B \cup C$.

SUB simply states that replacing in any set of outcomes, one of them by another one which is better, leads to a prospect which is weakly prefered. Note that SUB does not imply axioms SD1 or SD2 because SD1 and SD2 deal with strict preferences. On the other hand, SD1 and/or SD2 neither imply SUB as long as SD1 and SD2 involve simple situations where only two possible outcomes are possible.

MC ensures that if an uncertain prospect A is weakly better than another prospect B, then the worst one cannot be strictly better than the union of both. The intuition behind MC is the following: after adding the possible outcomes in A to those in the worse prospect B, the decision maker maintains the same outcomes he had in B plus those in A which made him evaluate A as preferred to B. Therefore the new situation should not be strictly worse than in the case of having only B.

ROB establishes that if an uncertain prospect A is weakly better than a pair of prospects B and C, then the union of B and C cannot be strictly better than prospect A. ROB is closely related to an axiom called *Union* by Pattanaik and Peleg [2], also in a context of choice under complete uncertainty. According to Pattanaik and Peleg's axiom, if a singleton set $\{a\}$ is better than a pair of sets B and C, it is so in relation with the union of B and C. ROB is reasonable in a context of choice under complete uncertainty as far as the possible worse outcomes in B and in C which made the decision maker prefer the uncertain prospect A to both of them, remain after the union of B and C. Therefore that union should not be strictly better than prospect A, even if now $B \cup C$ contains more possible outcomes.

3 An Alternative characterization of the max-min and the min-max rules

We are now ready to propose the following results:

Theorem 1 *(In substitution of Theorem 1 in [1, pg.299])*

$$If \succcurlyeq satisfies\ SUB,\ MC\ and\ ROB,\ then,\ for\ all\ A{\in}K,\ A \sim \{\underline{a}, \overline{a}\} \quad (5)$$

Proof. Let $A{\in}K$ and let $A = \{a_1, a_2, \ldots, a_n\}$ denote the set A ordered according to P ($a_1 P a_2 P \ldots P a_n$). If $n \leq 2$ the proof is trivial. If $n > 2$, by reflexivity $\{a_1, a_n\} \succcurlyeq \{a_1, a_n\}$. By SUB $\{a_1, a_n\} \succcurlyeq \{a_2, a_n\}$. Therefore by ROB $\{a_1, a_n\} \succcurlyeq$

$\{a_1, a_2, a_n\}$. If $n > 3$, we apply again SUB to get $\{a_1, a_n\} \succcurlyeq \{a_3, a_n\}$, and as $\{a_1, a_n\} \succcurlyeq \{a_1, a_2, a_n\}$, again by ROB $\{a_1, a_n\} \succcurlyeq \{a_1, a_2, a_3, a_n\}$. Repeating as often as necessary we reach $\{\underline{a}, \overline{a}\} \succcurlyeq A$.

On the other hand, by SUB $\{a_1, a_{n-1}\} \succcurlyeq \{a_1, a_n\}$. By MC that implies $\{a_1, a_{n-1}, a_n\} \succcurlyeq \{a_1, a_n\}$. If $n > 3$ we apply again SUB to get $\{a_1, a_{n-2}, a_n\} \succcurlyeq \{a_1, a_{n-1}, a_n\}$. By MC $\{a_1, a_{n-2}, a_{n-1}, a_n\} \succcurlyeq \{a_1, a_{n-1}, a_n\}$, and by transitivity $\{a_1, a_{n-2}, a_{n-1}, a_n\} \succcurlyeq \{a_1, a_n\}$. Repeating as often as necessary we reach $A \succcurlyeq \{\underline{a}, \overline{a}\}$, which together with $\{\underline{a}, \overline{a}\} \succcurlyeq A$ implies $\{\underline{a}, \overline{a}\} \sim A$. □

Independence of the axioms: Let $X = \{x, y, z\}$ and $xPyPz$.

- Let $\{z\} \succ \{y, z\} \succ \{x, z\} \sim \{x, y, z\} \succ \{y\} \succ \{x, y\} \succ \{x\}$. Then $\succcurlyeq$ satisfies ROB and MC, but not SUB.
- Let $\{x\} \succ \{x, y\} \succ \{y\} \succ \{x, y, z\} \succ \{x, z\} \succ \{y, z\} \succ \{z\}$. Then $\succcurlyeq$ satisfies SUB and MC, but not ROB (note that $\{x, z\} \sim \{x, z\}$ and $\{x, z\} \succ \{y, z\}$ but $\{x, y, z\} \succ \{x, z\}$).
- Let $\{x\} \succ \{x, y\} \succ \{y\} \succ \{x, z\} \succ \{x, y, z\} \succ \{y, z\} \succ \{z\}$. Then $\succcurlyeq$ satisfies SUB and ROB, but not MC ($\{x, y\} \succ \{x, z\}$ but $\{x, z\} \succ \{x, y, z\}$).

Theorem 2 *(In substitution of Theorem 3 in [1, pg.304])*

$$\succcurlyeq \textit{ satisfies SM, ROB, MC, SD1 and SUA if and only if } \succcurlyeq = \succcurlyeq_{mnx} \quad (6)$$

$$\succcurlyeq \textit{ satisfies SM, ROB, MC, SD2 and SUP if and only if } \succcurlyeq = \succcurlyeq_{mxn} \quad (7)$$

Note that in relation with the original Theorem 3 in [1], axiom IND is substituted by axioms ROB and MC, while the remaining axioms are the same.

Proof. : We will first show that $\succcurlyeq_{mnx}$ satisfies ROB and MC (it is straightforward to show that it satisfies SM, SD1 and SUA)

- **ROB**: for all $A, B, C \in K$, $A \succcurlyeq_{mnx} B$ and $A \succcurlyeq_{mnx} C$ implies one of the four following possibilites.

1. $\underline{a}P\underline{b}$ and $\underline{a}P\underline{c}$. In this case $\underline{a}Pmin(B \cup C)$. Therefore $A \succcurlyeq_{mnx} B \cup C$
2. $\underline{a} = \underline{b} = \underline{c}$ and $\overline{a}R\overline{b}, \overline{c}$. In this case $\underline{a} = min(B \cup C)$ and $\overline{a}Rmax(B \cup C)$. Therefore $A \succcurlyeq_{mnx} B \cup C$
3. $\underline{a} = \underline{b}P\underline{c}$ *and* $\overline{a}R\overline{b}$. Then $min(B \cup C) = \underline{c}$. Therefore $A \succcurlyeq_{mnx} B \cup C$.
4. $\underline{a} = \underline{c}P\underline{b}$ *and* $\overline{a}R\overline{c}$, which is analogous to the previous case.

– **MC**: for all $A, B \in K$, $A \succcurlyeq_{mnx} B$ implies $(\underline{a}P\underline{b})$ or $[(\underline{a} = \underline{b})$ *and* $(\overline{a}R\overline{b})$. Therefore $min(A \cup B)R\underline{b}$ and $max(A \cup B)R\overline{b}$, which implies $A \cup B \succcurlyeq_{mnx} B$

It is also straightforward to show that $\succcurlyeq_{mxn}$ satisfies SM, SD2 and SUP, and in order to prove that it also satisfies ROB and MC we would follow analogous steps as followed in the proof for $\succcurlyeq_{mnx}$.

Now, the converse part of the implications should be proved. That is, we shall start proving that if $\succcurlyeq$ satisfies SM, ROB, MC, SD1 and SUA, then $\succcurlyeq = \succcurlyeq_{mnx}$.

Step 1. As a first step we will prove that if $\succcurlyeq$ satisfies SM and SUA, then it satisfies SD2: Let $x, y, z \in X$, $xPyPz$. By SUA $\{y\} \succ \{x, z\}$, and by SM $\{x, y\} \succ \{y\}$. Then, by transitivity $\{x, y\} \succ \{x, z\}$.

Step 2. Secondly, we will prove that if $\succcurlyeq$ satisfies SM, ROB, MC, SD1 and SUA, then, for all $A \in K$, s.t. $\#A \leq 3$, $A \sim \{\underline{a}, \overline{a}\}$. Let $A \in K$, $\#A = n$, $n \leq 3$.

If $n \leq 2$ the proof is trivial.

If $n = 3$, let $A = \{a_1, a_2, a_3\}$ denote the set A ordered according to R $(a_1Pa_2Pa_3)$. By SD2 $\{a_1, a_2\} \succ \{a_1, a_3\}$. By MC $\{a_1, a_2, a_3\} \succcurlyeq \{a_1, a_3\}$. On the other hand, by reflexivity $\{a_1, a_3\} \succcurlyeq \{a_1, a_3\}$ and by SD1 $\{a_1, a_3\} \succ \{a_2, a_3\}$. Applying ROB we get $\{a_1, a_3\} \succcurlyeq \{a_1, a_2, a_3\}$, which together with $\{a_1, a_2, a_3\} \succcurlyeq \{a_1, a_3\}$ implies $\{a_1, a_2, a_3\} \sim \{a_1, a_3\}$.

Step 3. For any $m \in \mathbb{N}$ s.t. $m \geq 3$. If $\forall B \in K$ s.t. $\#B = m$, $B \sim \{\underline{b}, \overline{b}\}$, then, $\forall A \in K$ s.t. $\#A = m + 1$, $A \sim \{\underline{a}, \overline{a}\}$.

For any $A \in K$ s.t. $\#A = m + 1$, let $A = \{a_1, a_2, \ldots, a_{m+1}\}$ such that $(a_1Pa_2P \ldots a_{m+1})$. By hypothesis $A \backslash \{a_{m+1}\} \sim \{a_1, a_m\}$ and $A \backslash \{a_m\} \sim \{a_1, a_{m+1}\}$. By SD2 $\{a_1, a_m\} \succ \{a_1, a_{m+1}\}$. Then, by transitivity, $A \backslash \{a_{m+1}\} \succ A \backslash \{a_m\}$, and by MC, $A \succcurlyeq A\backslash\{a_m\}$.

On the other hand, by hypothesis $\{a_1, a_{m+1}\} \sim A\backslash\{a_m\} \sim A\backslash\{a_{m-1}\}$. By reflexivity $A\backslash\{a_m\} \sim A\backslash\{a_m\}$. Then, by ROB $A\backslash\{a_m\} \succcurlyeq A$, wich together with $A \succcurlyeq A\backslash\{a_m\}$ implies $A \sim A\backslash\{a_m\}$. By hypothesis $A\backslash\{a_m\} \sim \{a_1, a_{m+1}\}$. Then, by transitivity $A \sim \{a_1, a_{m+1}\}$

Step 4. $\forall A \in K,\ A \sim \{\underline{a}, \overline{a}\}$.

If $\#A \leq 3$ we apply directly Step 2. If $\#A = l > 3$, from Step 2, and applying Step 3 $(l-3)$ successive times, we reach $A \sim \{\underline{a}, \overline{a}\}$.

At this point, Step 4, together with (1) and transitivity of $\succcurlyeq$, prove directly that SM, ROB, MC, SD1 and SUA imply $\succcurlyeq = \succcurlyeq_{mnx}$.

To prove that SM, ROB, MC, SD2 and SUP imply $\succcurlyeq = \succcurlyeq_{mxn}$ we proceed analogously: As a first step it is easy to prove that SM and SUP implies SD1. Therefore, we can obtain the same result of the previous Step 2 and apply it together with (2) (where only axioms SM, SD2 and SUP are required).

□

The following examples show that the axioms used respectively in (6) and in (7) are independent. Let $X = \{x, y, z\}$ and $xPyPz$.

Independence of SM, ROB, MC, SD1 and SUA:

- Let $\{x\} \sim \{x, y\} \succ \{y\} \succ \{x, z\} \sim \{x, y, z\} \succ \{y, z\} \succ \{z\}$. Then $\succcurlyeq$ satisfies ROB, MC, SD1 and SUA, but not SM.
- Let $\{x\} \succ \{x, y\} \succ \{y\} \succ \{x, y, z\} \sim \{x, z\} \sim \{y, z\} \succ \{z\}$. Then $\succcurlyeq$ satisfies SM, ROB, MC and SUA, but not SD1.
- $\succcurlyeq_{mxn}$ satisfies SM, ROB, MC, and SD1, but not SUA.
- For the independence of ROB and MC see the corresponding examples after the proof of this Note's Theorem 1.

Independence of SM, ROB, MC, SD2 and SUP:

- Let $\{x\} \sim \{x, y\} \succ \{x, z\} \sim \{x, y, z\} \succ \{y\} \succ \{y, z\} \succ \{z\}$. Then $\succcurlyeq$ satisfies ROB, MC, SD2 and SUP, but not SM.

- Let $\{x\} \succ \{x,y\} \succ \{x,y,z\} \succ \{x,z\} \succ \{y\} \succ \{y,z\} \succ \{z\}$. Then $\succcurlyeq$ satisfies SM, MC, SD2 and SUP, but not ROB.
- Let $\{x\} \succ \{x,y\} \succ \{x,z\} \succ \{y\} \succ \{x,y,z\} \succ \{y,z\} \succ \{z\}$. Then $\succcurlyeq$ satisfies SM, ROB, SD2 and SUP, but not MC.
- Let $\{x\} \succ \{x,y\} \sim \{x,z\} \sim \{x,y,z\} \succ \{y\} \succ \{y,z\} \succ \{z\}$. Then $\succcurlyeq$ satisfies SM, ROB, MC and SUP, but not SD2.
- $\succcurlyeq_{mnx}$ satisfies SM, ROB, MC, and SD2, but not SUP.

4 Final Remark

We could perfectly keep Bossert, Pattanaik and Xu's Theorem 1 as they propose; that is, using axioms SM and IND. (Note that, although IND is not satisfied by the max-min and the min-max rule, their Theorem 1 is totally correct). On the other hand, we could characterize the max-min and the min-max rule as made in the previous section of this Note; that is, by means of axioms SM, ROB, MC plus SD1 and SUA in the case of the min-max, and plus SD2 and SUP in the case of the max-min.

This alternative leads to a slighter modification of the original work by Bossert, Pattanaik and Xu, and would be more justifiable if we conceive Theorem 1 on the one hand, and the characterization of the min-max and the max-min on the other hand, as separate results. However, if we want to preserve a certain axiomatic coherence between both Theorems, the modifications proposed in the previous section seem to be the more plausible.

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