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A theorem for constructing interval-valued intuitionistic fuzzy sets from intuitionistic fuzzy sets

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Abstract: *In this paper we present a theorem that allows us to construct interval-valued intuitionistic fuzzy sets from intuitionistic fuzzy sets. We also study the way of recovering intuitionistic fuzzy sets used in the construction of the interval-valued intuitionistic fuzzy set from different operators. We analise the numerical measures of information of the interval-valued intuitionistic fuzzy set constructed in function with the numerical measures of information of the intuitionistic fuzzy sets used in its construction. We review the most important properties of intuitionistic fuzzy sets and of interval-valued intuitionistic fuzzy sets and we analise three operators among these sets along with their properties.*

Keywords: *Fuzzy set; intuitionistic fuzzy set; interval-valued intuitionistic fuzzy set; intuitionistic entropy; interval-valued intuitionistic entropy; operators on interval-valued intuitionistic fuzzy sets.*

1. Introduction

Let $D[0,1]$ be the set of closed subintervals of the interval $[0,1]$; we will represent the elements of the set with capital letters $M, N....$ It is known that $M = [M_L, M_U]$, where M_L and M_U are the lower and the upper extreme, respectively, having ([16]) $M = N$ if and only if $M_L = N_U$ and $M_U = N_U$ and $M \leq N$ if and only if $M_L \leq N_L$ and $M_U \leq N_U$. We will denote with $\bar{M}$ the *complementary* of M, that is, $\bar{M} = 1 - M = [1 - M_U, 1 - M_L]$, with 1 the interval $[1,1]$ and with 0 the interval $[0,0]$. We will denote with W_M the amplitude of M, that is $W_M = M_U - M_L$.

We will call t-norm in $[0,1]$ ([12], [14], [15]) every mapping

$$T : [0,1] \times [0,1] \longrightarrow [0,1]$$

satisfying the properties

i) Boundary conditions, $T(x,1) = x$ and for all $x \in [0,1]$

ii) Monotony, $T(x,y) \leq T(z,t)$ if $x \leq z$ and $y \leq t$

iii) Commutative, $T(x,y) = T(y,x)$ $\forall x,y \in [0,1]$

iv) Associative, $T(T(x,y),z) = T(x,T(y,z))$ for all $x,y,z \in [0,1]$

Given any t-norm T, we can consider the mapping

$$S : [0,1] \times [0,1] \longrightarrow [0,1]$$

$$S(x,y) \equiv 1 - T(1-x, 1-y)$$

this mapping S, will be called dual t-conorm of T.

The more significant examples of t-norms and their associated dual t-conorms are the following:

i) $T(x,y) = \wedge(x,y), S(x,y) = \vee(x,y)$, ii) $T(x,y) = x.y, S(x,y) = x\hat{+}y$

Let $X \neq \phi$ be a given set. ([1]) An intuitionistic fuzzy set on X is an expression A given by

$$A = \{< x, \mu_A(x), \nu_A(x) > |x \in X\} \qquad \text{where}$$

$$\mu_A : X \longrightarrow [0,1]$$

$$\nu_A : X \longrightarrow [0,1]$$

with the condition $0 \leq \mu_A(x) + \nu_A(x) \leq 1$ $\forall x \in X$

The numbers $\mu_A(x)$ and $\nu_A(x)$ denote respectively the degree of membership and the degree of non-membership of the elememt x to the set A. We will denote with $IFSs(X)$ the set of all the intuitionistic fuzzy sets in X.

For all $A \in IFSs(X)$, we will call the expression $\pi_A(x) = 1 - \mu_A(x) - \nu_A(x)$ the *intuitionistic index* of element x in set A.

If for all $x \in X$ it is verified that $\mu_A(x) = 1 - \nu_A(x)$, that is $\pi_A(x) = 0$, then set A is a fuzzy set. We will denote with $FSs(X)$ the set of all the fuzzy sets on the same referential X.

The following expressions are defined in ([1], [2], [10]) for all $A, B \in X$

1. $A \leq B$ if and only if $\mu_A(x) \leq \mu_B(x)$ and $\nu_A(x) \geq \nu_B(x)$ $\forall x \in X$
2. $A \preceq B$ if and only if $\mu_A(x) \leq \mu_B(x)$ and $\nu_A(x) \leq \nu_B(x)$ $\forall x \in X$
3. $A = B$ if and only if $A \leq B$ and $B \leq A$
4. $A_c = \{< x, \nu_A(x), \mu_A(x) > |x \in X\}$

Theorem 1. ([1], [6], [10]) *Let T and S be t-norm and dual t-conorm in* $[0,1]$, *we define the expressions*

$$T(A,B) \equiv \{< x, T(\mu_A(x), \mu_B(x)), S(\nu_A(x), \nu_B(x)) > | x \in X\}$$
$$S(A,B) \equiv \{< x, S(\mu_A(x), \mu_B(x)), T(\nu_A(x), \nu_B(x)) > | x \in X\}.$$

for all $A, B \in IFSs(X)$. Then, it is verified that

a)([3]) If $S = \vee$ and $T = \wedge$ then $\{IFSs(X), \wedge, \vee\}$ is a distibutive lattice,which is bounded, not complemented and satisfies Morgans laws.

b) For any S and T, the commutative, associative and $S(A_c, B_c) = (T(A,B))_c$, $T(A_c, B_c) = (S(A,B))_c$ properties are satisfied.

Just as fuzzy entropy is a magnitude that measures the degree of fuzzyness of a fuzzy set, for intuitionistic fuzzy sets we need to define a magnitude that measures the degree of intuitionism of a set, that is, the extent of separation of intuitionistic fuzzy sets not from ordinary sets but from fuzzy sets. We shall call this magnitude *intuitionistic entropy*. A complete study of this magnitude is found in ([7], [10]).

Considering that in fuzzy entropy ordinary sets are taken as reference points, in intuitionistic entropy we will take fuzzy sets as reference points.

We shall demand the following conditions of an intuitionistic entropy:

a) It must measure the degree of difference from the fuzzy, i. e., it must be null for fuzzy sets.

b) It must reach a maximum when the degree of membership and the degree of non-membership of each element of the set are null.

c) The entropy of an intuitionistic fuzzy set must be equal to that of its complementary.

d) If the degree of membership and the degree of non-membership of each element of the set increase, the entropy should decrease.

Among all the expressions of intuitionistic entropy we shall take that given by

$$I_{IFS}(A) = \sum_{i=1}^{n} \pi_A(x_i)$$

because it satisfies the four previous conditions and besides it also verifies:

a) It coincides with the *indetermination index* introduced by R. Sambuc ([16]) when one is working with Φ-fuzzy sets.

b) In form it is very similar to the entropy of A. Kaufmann ([13]) for fuzzy sets when $\sum_{i=1}^{n} \pi_A(x_i)$ is expressed in terms of the intuitionistic fuzzy distance ([1]).

We know that K. Atanassov and G. Gargov introduced the concept of interval-valued intuitionistic fuzzy set in ([4]) in the following way

Let $X \neq \phi$ be a given set. ([4]) An interval valued intuitionistic fuzzy set in X is an expression given by

$$A = \{< x, M_A(x), N_A(x) > | x \in X\} \qquad \text{where}$$
$$M_A : X \longrightarrow D[0,1]$$
$$N_A : X \longrightarrow D[0,1]$$

with the condition $0 \leq M_{AU}(x) + N_{AU}(x) \leq 1 \ \forall x \in X$

The intervals $M_A(x)$ and $N_A(x)$ denote respectively the degree of membership and the degree of non-membership of element x to set A. We will denote with $IVIFSs(X)$ the set of all the intuitionistic interval-valued fuzzy sets in X.

The following expressions are defined in ([4], [10]) for all $A, B \in IVIFSs(X)$,

1. $A \leq B$ if and only if $M_A(x) \leq M_B(x)$ and $N_A(x) \geq N_B(x) \ \forall x \in X$
2. $A \preceq B$ if and only if $M_A(x) \leq M_B(x)$ and $N_A(x) \leq N_B(x) \ \forall x \in X$
3. $A = B$ if and only if $M_A(x) = M_B(x)$ and $N_A(x) = N_B(x) \ \forall x \in X$
4. $A_c = \{< x, N_A(x), M_A(x) > | x \in X\}$

Theorem 2. ([4], [6]) *Let T and S be a t-norm and dual t-conorm, we define*

$$T(A,B) \equiv \{< x, [T(M_{AL}(x), M_{BL}(x)), T(M_{AU}(x), M_{BU}(x)],$$
$$[S(N_{AL}(x), N_{BL}(x)), S(N_{AU}(x), N_{BU}(x))] > | x \in X\}$$
$$S(A,B) \equiv \{< x, [S(M_{AL}(x), M_{BL}(x)), S(M_{AU}(x), M_{BU}(x)],$$
$$[T(N_{AL}(x), N_{BL}(x)), T(N_{AU}(x), N_{BU}(x))] > | x \in X\}$$

for all $A, B \in IVIFSs(X)$. Then it is verified that

a) If $T = \wedge$ and $S = \vee$ then, $\{IVIFSs(X), \wedge, \vee\}$ is a distributive lattice, which is bounded, not complementary and satisfies Morgan's laws.

b) For any T and S, the commutative, associative properties and $T(A_c, B_c) = (S(A,B))_c$, $S(A_c, B_c) = (T(A,B))_c$ are satisfied.

We want to study the concepts of distance and entropy for a finite universal set in interval-valued intuitionistic fuzzy sets. These magnitudes have been studied in ([10], [11]).

Definition 1. Let A and B be two interval valued intuitionistic fuzzy sets in X, which is finite and the cardinal of X is n, we can define the *Hamming distance* between them as follows

$$d_{H_{IVIFS}}(A,B) =$$
$$= \frac{1}{4}\Big(\sum_{i=1}^{n} |M_{AL}(x_i) - M_{BL}(x_i)| + |M_{AU}(x_i) - M_{BU}(x_i)| +$$
$$+ |N_{AL}(x_i) - N_{BL}(x_i)| + |N_{AU}(x_i) - N_{BU}(x_i)| \Big)$$

evidently the inequality $0 \leq d_{H_{IVIFS}}(A,B) \leq K$ is verified.

It is easy to prove that this definition satisfies the properties of a distance.

Clearly we can define the normalized Hamming distance as

$$\xi_{H_{IVIFS}}(A,B) = \frac{d_{H_{IVIFS}}(A,B)}{\sqrt{K}}$$

which verifies that $0 \leq \xi_{H_{IVIFS}}(A,B) \leq 1$.

Definition 2. We will define the so-called *euclidean distance* of the interval valued intuitionistic fuzzy sets A and B in X by the formula

$$d_{e_{IVIFS}}(A,B) =$$

$$= \left(\frac{1}{4}\sum_{i=1}^{n}((M_{AL}(x_i) - M_{BL}(x_i))^2 + (M_{AU}(x_i) - M_{BU}(x_i))^2 + \right.$$

$$\left. + (N_{AL}(x_i) - N_{BL}(x_i))^2 + (N_{AU}(x_i) - N_{BU}(x_i))^2)\right)^{1/2}$$

it is easy to prove that this definition satisfies distance's properties.

In the way that intuitionistic entropy is a magnitude that measures the degree of intuitionism of a set, we want to define a magnitude that measures the degree of interval-valuation of an $IVIFSs(X)$ that we will call *interval-valued intuitionistic entropy*. A complete study of the numerical measures of information in interval-valued intuitionistic fuzzy sets can be found in ([10], [11]).

This idea has lead us to establish the following conditions for such entropy.

a) It must be null when the set is intuitionistic fuzzy, so that it will measure the degree of separation of the intuitionistic fuzzy sets.

b) If the sum of the amplitudes of the intervals of membership and non-membership is one for all the elements of the universal set, then the entropy of intuitionistic interval-valuation must be maximun.

c) The entropy of an interval-valued intuitionistic fuzzy set must be the same as that of its complementary.

d) If the amplitude of the membership interval W_M and the amplitude of the non-membership interval W_N of each element in X decrease, the entropy must decrease.

The conditions above allow us to establish the following definition

Definition 3. A real function

$$I_{IVIFS} : IVIFSs(X) \rightarrow \mathbf{R}^+$$

is an *interval-valued intuitionistic entropy* on $IVIFSs(X)$ if I_{IVIFS} has the following properties:

1) $I_{IVIFS}(A) = 0$ if and only if $A \in IFSs(X)$
2) $I_{IVIFS}(A) = I_{IVIFS}(A_c)$ for all $A \in IVIFSs(X)$
3) $I_{IVIFS}(A) = K$ if and only if $M_{AU}(x) + N_{AU}(x) = 1$ and $M_{AL}(x) = N_{AL}(x) = 0$ for all x in X
4) If for all x in X, $W_{MA}(x) \leq W_{MB}(x)$ and $W_{NA}(x) \leq W_{NB}(x)$, then $I_{IVIFS}(A) \leq I_{IVIFS}(B)$.

Among all the expressions on interval-valued intuitionistic entropy we shall take for this paper the given in the following theorem.

Theorem 3. *Let $A \in IVIFSs(X)$. The expression*

$$I_{IVIFS}(A) = \sum_{i=1}^{n} W_{MA}(x_i) + W_{NA}(x_i)$$

is an interval-valued intuitionistic entropy.

Now we present two operators such that each $IVFSs(X)$ has assigned an $IFSs(X)$ or a family of these.

The first of them has been widely studied in ([8], [9]), of it we present its definition and its most important properties.

The determination of mappings which transform $IVIFSs(X)$ in $IFSs(X)$ can be analised through functions which assign for each interval of $D[0,1]$ (set of closed subintervals of the interval $[0,1]$) an inner point of this interval. In such a way we consider the g_p mapping

$$g_p : D[0,1] \longrightarrow [0,1] \qquad \text{such that}$$

$$g_p([M_L, M_U]) = M_L + p \cdot (M_U - M_L) = M_U - (1-p) \cdot (M_U - M_L)$$

with $0 \leq p \leq 1$.

We make the operators

$$H_{p,r} : IVIFSs(X) \longrightarrow IFSs(X)$$

$$H_{p,r}(A) = \{< x, g_p(M_A(x)), g_r(N_A(x)) > | x \in X\}$$

with $0 \leq p \leq 1$ y $0 \leq r \leq 1$, where

$$A = \{< x, M_A(x), N_A(x) > | x \in X\} \in IVIFSs(X)$$

$$M_A(x) = [M_{AL}(x), M_{AU}(x)]$$

$$N_A(x) = [N_{AL}(x), N_{AU}(x)],$$

in such a way that

$$H_{p,r}(A) = \{< x, M_{AL}(x) + p \cdot W_M(x), N_{AL}(x) + r \cdot W_{N_A}(x) > | x \in X\}$$

being $W_{M_A}(x) = M_{AU}(x) - M_{AL}(x)$ and $W_{N_A}(x) = N_{AU}(x) - N_{AL}(x)$.

Proposition 1. *For all* $A \in IVIFSs(X)$ *and* $p, r, p', r' \in [0,1]$, *it is verified that:*

i) $H_{p,r}(A) \in IFSs(X)$.

ii) If $p \leq p'$ *y* $r \leq r'$, *then* $H_{p,r}(A) \preceq H_{p',r'}(A)$.

iii) If $p \leq p'$ *y* $r \geq r'$, *then* $H_{p,r}(A) \leq H_{p',r'}(A)$.

In the remainder of this paper, each time we use these operators, we will do it for a value of p and the corresponding values of r to: $1-p$ or p, for this reason and to simplify the notation we will call

$$H_p = H_{p,1-p} \quad , \qquad H_p^{'} = H_{p,p}.$$

Proposition 2. *For all* $A \in IVIFSs(X)$ *and* $p, q \in [0,1]$.

i) If $p \leq q$, *then* $H_p(A) \leq H_q(A)$ *and* $H_p^{'}(A) \preceq H_q^{'}(A)$

ii) $(H_p(A_c))_c = H_{1-p}(A)$ *and* $(H_p^{'}(A_c))_c = H_p^{'}(A)$

iii) $H_p(H_q(A)) = H_q(A)$ *and* $H_p^{'}(H_q^{'}(A)) = H_q^{'}(A)$.

Theorem 4. *For all* $A, B \in IVIFSs(X)$.

i) $H_i(S(A,B)) = S(H_i(A), H_i(B))$
$H_i(T(A,B)) = T(H_i(A), H_i(B))$

ii) $H_i^{'}(S(A,B)) = S(H_i^{'}(A), H_i^{'}(B))$
$H_i^{'}(T(A,B)) = T(H_i^{'}(A), H_i^{'}(B))$

with $i = 0, 1$.

We will denote with

$$\{H_p(A)\}_{p\in[0,1]} \text{ and } \{H_p^{'}(A)\}_{p\in[0,1]}$$

the families of intuitionistic fuzzy sets which are obtained as result of applying H_p and $H_p^{'}$ with $p \in [0,1]$ to an interval-valued intuitionistic fuzzy set A respectively.

Theorem 5. *Let* A *be an interval-valued intuitionistic fuzzy set and* $\{H_{p,1-p}(A)\} = \{A_p(A)\}_{p\in[0,1]}$ *the family of intuitionistic fuzzy sets associated to* A *by the operators* $H_{p,1-p}$ *defined above. Then:*

a) *For all* $A_k \in \{H_{p,1-p}(A)\}$,

$$I_{IVIFS}(A) = 4 \cdot d_{H_{IVIFS}}(A, A_k).$$

b) For all $A_i, A_k \in \{H_{p,1-p}(A)\}$,

$$d_{H_{IVIFS}}(A_i, A_k) = \frac{|k-i|}{2} \cdot I_{IVIFS}(A).$$

Corollary 1. *For all* $A \in IVIFSs(X)$ *and* $0 \leq p \leq 1$.

i) $\{H_p(A)\}_{p\in[0,1]}$ *is a family of intuitionistic fuzzy sets totally ordered with respect to the order* $\leq$.

ii) $\{H'_p(A)\}_{p\in[0,1]}$ *is a family of intuitionistic fuzzy sets ordered with respect to the order* $\preceq$.

We now propose a new operator, so that for each point $x \in X$ we take a value of p and a value of r corresponding to that point. By extension we will denote this new operator with $H_{p_x,r_x}(p_x, r_x \in [0,1]$ for all $x \in X)$.

Definition 4. For each $x \in X$, we take $p_x, r_x \in [0,1]$ and we consider

$$H_{p_x,r_x} : IVIFSs(X) \longrightarrow IFSs(X)$$

given by

$$H_{p_x,r_x}(A) = \{< x, M_{AL}(x) + p_x \cdot W_M(x), N_{AL}(x) + r_x \cdot W_{N_A}(x) > |x \in X\}$$

being $W_{M_A}(x) = M_{AU}(x) - M_{AL}(x)$ and $W_{N_A}(x) = N_{AU}(x) - N_{AL}(x)$.

Evidently $H_{p_x,r_x}(A) \in IFSs(X)$ for all $A \in IVIFSs(X)$. The most important properties of this operator can be found in ([10]).

2. Construction of an IVIFSs(X) from an IFSs(X)

Let $A \in IFSs(X)$. Let us consider mappings

$$X \longrightarrow [0,1] \times [0,1]$$

$$x \longrightarrow \quad (\lambda_x, \rho_x)$$

such that if $\pi_A(x) \neq 0$, λ_x and ρ_x satisfy $\lambda_x \leq \frac{\mu_A(x)}{\pi_A(x)}$ and $\rho_x \leq \frac{\nu_A(x)}{\pi_A(x)}$

Let $\alpha_x, \beta_x \in [0,1]$ such that $0 \leq \alpha_x + \beta_x \leq 1$, in these conditions:

Theorem 6. *Let* $\Gamma : IFSs(X) \longrightarrow IVIFSs(X)$, *with*

$$\Gamma(A) = \{< x, M_{\Gamma(A)}(x), N_{\Gamma(A)}(x) > |x \in X\}$$

such that

1) $M_{\Gamma(A)L}(x) = a + b\mu_A(x) - \lambda_x \pi_A(x)$ *with fixed* $a, b \in \mathbf{R}$ *for all* $A \in IFSs(X)$

2) $W_{M\Gamma(A)}(x) = (\alpha_x + \lambda_x)\pi_A(x)$ *for all* x *in* X

3) $N_{\Gamma(A)L}(x) = a' + b'\nu_A(x) - \rho_x \pi_A(x)$ *with fixed* $a', b' \in \mathbf{R}$ *for all* A *belonging to* $IFSs(X)$

4) $W_{N\Gamma(A)}(x) = (\beta_x + \rho_x)\pi_A(x)$

5) If $A \in FSs(X)$, *then* $\Gamma(A) = A$.

Then

i) $M_{\Gamma(A)L}(x) = \mu_A(x) - \lambda_x\pi_A(x)$, $\quad M_{\Gamma(A)U}(x) = \mu_A(x) + \alpha_x\pi_A(x)$

ii) $N_{\Gamma(A)L}(x) = \nu_A(x) - \rho_x\pi_A(x)$, $\quad N_{\Gamma(A)U}(x) = \nu_A(x) + \beta_x\pi_A(x)$

and conversely.

Proof. $\Rightarrow$) As $M_{\Gamma(A)L}(x) = a + b\mu_A(x) - \lambda_x\pi_A(x)$, we have that $M_{\Gamma(A)U}(x) = M_{\Gamma(A)L}(x) + W_{M\Gamma(A)}(x) = a + b\mu_A(x) + \alpha_x\pi_A(x)$, and as $N_{\Gamma(A)L}(x) = a' + b'\mu_A(x) - \rho_x\pi_A(x)$, resulting that $N_{\Gamma(A)_U}(x) = N_{\Gamma(A)L}(x) + W_{N\Gamma(A)}(x) = a' + b'\nu_A(x) + \beta_x\pi_A(x)$.

In these conditions, taking $A = \{< x, 1, 0 > | x \in X\}$ we have that $\Gamma(A) = A$ by 5), that is

$$M_{\Gamma(A)L}(x) = a + b = 1$$
$$N_{\Gamma(A)L}(x) = a' = 0$$

Likewise, taking $A = \{< x, 0, 1 > | x \in X\}$ by 5) we obtain that $\Gamma(A) = A$, that is

$$M_{\Gamma(A)L}(x) = a = 0$$
$$N_{\Gamma(A)L}(x) = a' + b' = 1.$$

From where $a = 0, b = 1, a' = 0$ and $b' = 1$ and therefore we have i) and ii).

Now we will prove that $\Gamma(A)$ is an interval-valued intuitionistic fuzzy set, that is

$a)0 \le M_{\Gamma(A)L}(x) \le 1$, $\quad b)0 \le M_{\Gamma(A)U}(x) \le 1$, $\quad c)M_{\Gamma(A)L}(x) \le M_{\Gamma(A)U}(x)$

$d)0 \le N_{\Gamma(A)L}(x) \le 1$, $\quad e)0 \le N_{\Gamma(A)U}(x) \le 1$, $\quad f)N_{\Gamma(A)L}(x) \le N_{\Gamma(A)U}(x)$

$g)0 \le M_{\Gamma(A)L}(x) + N_{\Gamma(A)L}(x) \le 1$

for all x in X.

a) $M_{\Gamma(A)L}(x) = \mu_A(x) - \lambda_x\pi_A(x) \le 1$. As $\lambda_x \le \frac{\mu_A(x)}{\pi_A(x)}$, we have $\mu_A(x) - \lambda_x\pi_A(x) \ge \mu_A(x) - \frac{\mu_A(x)}{\pi_A(x)}\pi_A(x) = 0$.

b) $M_{\Gamma(A)U}(x) = \mu_A(x) + \alpha_x\pi_A(x) \le 0$. We know that $\alpha_x \le 1$, therefore $M_{\Gamma(A)U}(x) = \mu_A(x) + \alpha_x\pi_A(x) \le \mu_A(x) + \pi_A(x) = 1 - \nu_A(x) \le 1$.

c) $M_{\Gamma(A)L}(x) = \mu_A(x) - \lambda_x\pi_A(x) \le \mu_A(x) \le \mu_A(x) + \alpha_x\pi_A(x) = M_{\Gamma(A)U}(x)$.

d) $N_{\Gamma(A)L}(x) = \nu_A(x) - \rho_x\pi_A(x) \leq \nu_A(x) \leq 1$. As $\rho_x \leq \frac{\nu_A(x)}{\pi_A(x)}$ we have $\nu_A(x) - \rho_x\pi_A(x) \geq \nu_A(x) - \frac{\nu_A(x)}{\pi_A(x)}\pi_A(x) = 0$.

e) $N_{\Gamma(A)U}(x) = \nu_A(x) + \beta_x\pi_A(x) \geq 0$. As $\beta_x \leq 1$ we have, $\nu_A(x) + \beta_x\pi_A(x) \leq \nu_A(x) + \pi_A(x) = 1 - \mu_A(x) \leq 1$.

f) $N_{\Gamma(A)L}(x) = \nu_A(x) - \rho_x\pi_A(x) \leq \nu_A(x) + \beta_x\pi_A(x) = N_{\Gamma(A)U}(x)$.

g) As $0 \leq M_{\Gamma(A)U}(x)$ and $0 \leq N_{\Gamma(A)U}$, we have that $0 \leq M_{\Gamma(A)U}(x) + N_{\Gamma(A)U}(x)$. Therefore $0 \leq M_{\Gamma(A)U}(x) + N_{\Gamma(A)U}(x) = \mu_A(x) + \alpha_x\pi_A(x) + \nu_A(x) + \beta_x\pi_A(x) = \mu_A(x) + \nu_A(x) + (\alpha_x + \beta_x)\pi_A(x) \leq \mu_A(x) + \nu_A(x) + \pi_A(x) = 1$ since by hypothesis $\alpha_x + \beta_x \leq 1$.

$\Leftarrow$) If

i) $M_{\Gamma(A)L}(x) = \mu_A(x) - \lambda_x\pi_A(x), \quad M_{\Gamma(A)U}(x) = \mu_A(x) + \alpha_x\pi_A(x)$

ii) $N_{\Gamma(A)L}(x) = \nu_A(x) - \rho_x\pi_A(x), \quad N_{\Gamma(A)U}(x) = \nu_A(x) + \beta_x\pi_A(x)$,

with these conditions we have just seen that $\Gamma(A)$ is an interval-valued intuitionistic fuzzy set. It is clear that i) satisfies conditions 1) and 2), and that ii) satisfies 3) and 4). Lastly, If $\forall x \in X \ \pi_A(x) = 0$, that is A is fuzzy set, we have that $M_{\Gamma(A)_L}(x) = M_{\Gamma(A)U}(x) = \mu_A(x)$ and $N_{\Gamma(A)_L}(x) = N_{\Gamma(A)U}(x) = \nu_A(x)$, therefore $\Gamma(A) = A$, hence 5) holds. $\square$

The conditions required in this theorem stand justified by the following:

1.- Conditions 1) and 3) are imposed because we want algorithmic constructions of easy implementation.

2.- Conditions 2) and 4) are imposed because we want to build $IVIFSs(X)$ such that the amplitude of the intervals of membership and non-membership of each element never surpass its intuitionistic index.

3.- The condition, If $A \in FSs(X)$ then $\Gamma(A) = A$, is due to the fact that with this construction we want to generate $IVIFSs(X)$ from $IFSs(X)$ not from FSs, so that if the intuitionistic index chosen is null for all x in X, the set that we obtain is the initial fuzzy set and not another fuzzy set.

Now we will study the way of to recover the intuitionistic fuzzy set A used in the construction of $\Gamma(A) \in IVIFSs(X)$ with the above theorem, by means of the H_{p_x,r_x} operators.

Theorem 7. *Let $A \in IFSs(X)$ and $\Gamma(A)$ the interval-valued intuitionistic fuzzy set constructed in the previous theorem, such that $0 < \alpha_x + \lambda_x \leq 1$ and $0 < \beta_x + \rho_x \leq 1$ for all x in X. Then*

$$H_{\frac{\lambda_x}{\alpha_x+\lambda_x}, \frac{\rho_x}{\beta_x+\rho_x}}(\Gamma(A)) = A.$$

Proof. Obvious. □

Let us observe that with this construction, the intervals of memebership and non-memebership of each element do not need to have the same amplitude. Only in the particular case in which $\alpha_x + \lambda_x = \beta_x + \rho_x$ does this occur, besides, if $\alpha_x + \lambda_x = \beta_x + \rho_x = 1$, then the amplitude of the interval of membership and the amplitude of the interval of non-membership are the same and is equal to the intuitionistic index $\pi_A(x)$ of the element considered.

Theorem 8. *Let $\Gamma(A) \in IVIFSs(X)$ be the interval-valued intuitionistic fuzzy set constructed with Theorem 6 from of the intuitionistic fuzzy set A.*

$$I_{IVIFS}(\Gamma(A)) = I_{IFS}(A) \cdot \sum_{i=1}^{n} \alpha_{x_i} + \lambda_{x_i} + \beta_{x_i} + \rho_{x_i}.$$

Proof. We only need to remember the form of the expressions of I_{IVIFS} and I_{IFS}, as well as conditions 2) and 4) of the statement of theorem 6. □

3. Future research

In this paper we have presented a theorem for the construction of interval-valued intuitionistic fuzzy sets from an intuitionistic fuzzy set. In future papers will appear different theorems of construction of $IVIFSs(X)$ from two intuitionistic fuzzy sets. Also be the relation that exists between the different $IVIFSs(X)$ constructed with different methods and a general Theorem for constructing interval-valued intuitionistic fuzzy sets from general functions. These new methods of construction have already been carried out and will appear in future papers.

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