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ABOUT THE INTUITIONISTIC FUZZY SET GENERATORS

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Abstract: *In this paper form the definition of Intuitionistic Fuzzy Sets we analyze the Intuitionistic fuzzy Generators and the Complementation in these sets. We start by defining the intuitionistic fuzzy generators in order to then study the particular cases for which this definition coincides with the fuzzy complementation. Afterwards, we analyse the existence of equilibrium points, dual points and we present characterization theorems of intuitionistic fuzzy generators. Lastly, we study a manner of constructing intuitionistic fuzzy sets and analyse the structure of the complementary of intuitionistic fuzzy sets built.*

Keywords: *Intuitionistic fuzzy set, Intuitionistic fuzzy complementary, fuzzy negation, intuitionistic fuzzy generator.*

1. Introduction

Let $X \neq \phi$ be a given set (see [2]). An *intuitionistic fuzzy set* in X is an expression A given by

$$A = \{< x, \mu_A(x), \nu_A(x) > | x \in X\}$$

where

$$\mu_A : X \to [0,1]$$
$$\nu_A : X \to [0,1]$$

with the condition

$$0 \leq \mu_A(x) + \nu_A(x) \leq 1,$$

for all x in X.

The numbers $\mu_A(x)$ and $\nu_A(x)$ denote respectively the degree of membership and the degree of non-membership of the element x in the set A. We will denote $IFSs(X)$ the set of all the intuitionistic fuzzy sets in X. Obviously, when

$$\nu_A(x) = 1 - \mu_A(x)$$

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for every x in X, the set A is a fuzzy set. We will denote the set of all the fuzzy sets in X with $FSs(X)$

The following expressions are defined in [2, 3, 6, 7, 8, 9, 10] for every $A, B \in IFSs(X)$

1. $A \leq B$ if and only if $\mu_A(x) \leq \mu_B(x)$ and $\nu_A(x) \geq \nu_B(x)$ for all $x \in X$
2. $A \preceq B$ if and only if $\mu_A(x) \leq \mu_B(x)$ and $\nu_A(x) \leq \nu_B(x)$ for all $x \in X$
3. $A = B$ if and only if $A \leq B$ and $B \leq A$
4. $A_c = \{< x, \nu_A(x), \mu_A(x) > | x \in X\}$

We will call

$$\pi_A(x) = 1 - \mu_A(x) - \nu_A(x)$$

intuitionistic index of the element x in the set A.

In 1986, K. Atanassov established different ways of changing an intuitionistic fuzzy set into a fuzzy set and he defined the following operator:

If $A \in IFSs(X)$, then

$$D_\alpha(A) = \{< x, \mu_A(x) + \alpha \cdot \pi_A(x), 1 - \mu_A(x) - \alpha \cdot \pi_A(x) > | x \in X\} \tag{1}$$

with $\alpha \in [0, 1]$. Obviously $D_\alpha(A) \in FSs(X)$.

A study of the properties of this operator, (we will call it *Atanassov's operator*), is made in [4, 5].

It is known that if $A' = \{< x, \mu_A(x) > | x \in X\}$ is a fuzzy sets on a referential X and $\varphi : [0, 1] \to [0, 1]$ is a fuzzy complement, the set

$$A = \{< x, \mu_A(x), \varphi(\mu_A(x)) > | x \in X\}$$

is not always an intuitionistic fuzzy set. For example, if we take Sugeno's negation [12]

$$\varphi_\lambda(x) = \frac{1 - x}{1 + \lambda x} \quad \text{with} \quad -1 < \lambda < 0$$

or we take Yager's negation [15, 16]

$$\varphi_\omega(x) = (1 - x^\omega)^{\frac{1}{\omega}} \quad \text{with} \quad 1 < \omega$$

the set

$$A = \{< x, \mu_A(x), \varphi(\mu_A(x)) > | x \in X\}$$

is not an intuitionistic fuzzy set because

$$\mu_A(x) + \varphi(\mu_A(x)) > 1 \quad \text{for all} \quad x$$

A general study of the fuzzy negations is made in [1, 13, 15].

In this paper we set out to study functions $\varphi : [0, 1] \to [0, 1]$ such that from a fuzzy set the sum

$$\mu_A(x) + \varphi(\mu_A(x)) \leq 1 \quad \text{for all} \quad x$$

We will conclude the paper studying the conditions in which from a fuzzy set $A' = \{< x, \mu_A(x) > | x \in X\}$ we build an intuitionistic fuzzy set $A = \{< x, \mu_A(x), \nu_A(x)) > | x \in X\}$.

2. Intuitionistic Fuzzy Generators

Definition 1. A function $\varphi : [0,1] \longrightarrow [0,1]$ will be called *intuitionistic fuzzy generator* iff

$$\varphi(x) \leq 1 - x, \quad \text{for all} \;\; x \in [0,1].$$

An intuitionistic fuzzy generator will be called *continuous, decreasing, increasing* iff φ is continuous, decreasing, increasing respectively.

Note that according to the definition above, $\varphi(0) \leq 1$ and $\varphi(1) = 0$.

Hereinafter we will denote the intuitionistic fuzzy generator IFG.

We have to keep in mind that there are fuzzy generators that are not IFG, for example Sugeno's

$$\varphi_\lambda(x) = \frac{1-x}{1+\lambda \cdot x}$$

for $-1 < \lambda < 0$, Yager's complement [15, 16]

$$\varphi_\omega(x) = (1 - a^\omega)^{\frac{1}{\omega}}$$

for $\omega > 1$.

Obiously, if we impose the conditions:

(i) $\varphi(0) = 1$

(ii) φ is non-increasing

on an intuitionistc fuzzy generator, we have a fuzzy negation.

Proposition 1. *The only continuous intuitionistic fuzzy complement constant for all* $x \in [0,1]$ *is the intuitionistic fuzzy complement* $\varphi(x) = 0$ *for all* $x \in [0,1]$.

Proof. Evident. □

Definition 2. Let us say that $x \in [0,1]$ is an *equilibrium point* for the intuitionistic fuzzy generator φ iff $\varphi(x) = x$.

Theorem 1. *Every continuous intuitionistic fuzzy generator has at most one equilibrium point.*

Proof. Let φ be a continuous intuitionistic fuzzy generator. An equilibrium point is a solution of the equation $\varphi(x) - x = 0$, where $x \in [0,1]$. Let us suppose that $\varphi(x) - x = b$, where b is a real number, and we take $F(x) = \varphi(x) - x = b$. As φ is an intuitionistic fuzzy generator we have $F(1) = \varphi(1) - 1 = -1$ and $F(0) = \varphi(0) \leq 0$. If $F(0) > 0$ then, by Rolle's theorem there is at least a $c \in [0,1]$ such that $F(c) = 0$, therefore $\varphi(c) = c$. If $\varphi(0) = 0$, obviously we have that the 0 is a point of equilibrium. □

Theorem 2. *Let $\varphi : [0,1] \to [0,1]$ an intuitionistic fuzzy generator.*

(i) If $x \in [0,1]$ is an equilibrium point, then $x \leq 1/2$.

(ii) If φ is continuous and decreasing in $[0,1]$, then φ has a unique equilibrium point.

Proof. i) $\varphi(x) = x \leq 1 - x$, then $x \leq 1/2$.
ii) Let us suppose that it is not true that there is a unique equilibrium point, that is, there are $x_1, x_2 \in [0,1]$ such that $x_1 < x_2$ and $\varphi(x_1) = x_1$ and $\varphi(x_2) = x_2$. Then, as φ is decreasing, we have that if $x_1 < x_2$, then $\varphi(x_1) \geq \varphi(x_2)$, therefore $x_1 \geq x_2$ in contradiction with the hipothesis that $x_1 < x_2$, so $x_1 = x_2$.

Definition 3. Let $\varphi : [0,1] \to [0,1]$ be an intuitionistic fuzzy generator, and let $x, y \in [0,1]$. We will say that x is the *dual point* of y with respect to φ or that y is the *dual point* of x with respect to φ if $\varphi(x) + \varphi(y) = x + y$ holds.

Theorem 3. *Let $\varphi : [0,1] \to [0,1]$ an intuitionistic fuzzy generator.*

(i) If x is the dual point of y with respect to φ, then $x + y \leq 1$.

(ii) The number 1 has dual point iff $\varphi(0) = 1$

(iii) Every equilibrium point is the dual point its self.

(iv) Each pair of equilibrium point of φ are dual points between each other.

Proof. i) $x + y = \varphi(y) + \varphi(x) \leq 1 - y + 1 - x = 2 - (x + y)$, then $x + y \leq 1$.

ii) $\Rightarrow$) $\varphi(1) + \varphi(y) = 1 + y$, as $\varphi(1) = 0$ then $\varphi(y) = 1 + y \leq 1$, therefore $y = 0$, then $\varphi(0) = 1$.

$\Leftarrow$) If $\varphi(0) = 1$, then $\varphi(0) + \varphi(1) = 1 + 0$.

iii) Evident.

iv) Let $\varphi(x) = x$ and $\varphi(y) = y$, then $\varphi(x) + \varphi(y) = x + y$. □

Definition 3. An intuitionistic fuzzy generator, $\varphi : [0,1] \to [0,1]$, will be called *involutive* in $[0,p]$, with $p \in [0,1]$ iff $\varphi(\varphi(x)) = x$ for all $x \in [0,p]$.

Theorem 4. *Let $\varphi : [0,1] \to [0,1]$ be an intuitionistic fuzzy generator. Then, $\varphi(x)$ is the dual point of x for all $x \in [0,p]$ with $p \in [0,1]$ iff φ is involutive in $[0,p]$.*

Proof. $\Rightarrow$) As $\varphi(x)$ is the dual point of x for all $x \in [0,p]$ we have that $\varphi(\varphi(x)) + \varphi(x) = x + \varphi(x)$ for all $x \in [0,p]$.

$\Leftarrow$) As φ is involutive in $[0,p]$ we have that for all $x \in [0,p]$, $\varphi(\varphi(x)) = x$, then $\varphi(\varphi(x)) + \varphi(x) = x + \varphi(x)$. □

Theorem 5. *Let $\varphi : [0,1] \to [0,1]$ be an intuitionistic fuzzy generator continuous, strictly decreasing and involutive in $[0,p]$ with $p \in [0,1]$ and $\varphi(x) = 0$ for all $x \in [p,1]$. Then $\varphi(0) = p$.*

Proof. As $p = \varphi(\varphi(p)) = \varphi(0)$. □

Obviously, in the same conditions as the previous theorem $\varphi^{-1} = \varphi$ in $[0,p]$.

3. Characterization Theorems of Intuitionistic Fuzzy Generators

In this section we will denote the standard negation N, $N : [0,1] \to [0,1]$ given by $N(x) = 1 - x$ for all $x \in [0,1]$.

Theorem 6. *Let $\varphi : [0,1] \to [0,1]$. Then, φ is a continuous intuitionistic fuzzy generator if and only if there exists a continuous function $f : [0,1] \to [0,1]$ such that*

(i) $f(x) \leq x$ for all $x \in [0,1]$

(ii) $\varphi(x) = (f \circ N)(x)$ for all $x \in [0,1]$.

Proof. $\Rightarrow$) φ is a continuous intuitionistic fuzzy generator. We will consider the function $f(x) = \varphi(1-x)$ for all $x \in [0,1]$. Evidently, φ is continuous, besides $f(x) = \varphi(1-x) \leq x$ for all $x \in [0,1]$. We can say that f satisfies iii), $(f \circ N)(x) = f(N(x)) = f(1-x) = \varphi(x)$ for all $x \in [0,1]$.

$\Leftarrow$) $\varphi(x) = (f \circ N)(x) = f(N(x)) = f(1-x) \leq 1 - x$ for all $x \in [0,1]$. Evidently φ is a continuous function. □

An example of this theorem is the following:

$$f(x) = \begin{cases} \frac{1}{2}x & \text{if } x \leq \frac{1}{4} \\ \frac{1}{6} - \frac{1}{6}x & \text{if } x \geq \frac{1}{4} \end{cases}$$

the corresponding IFN is

$$\varphi(x) = \begin{cases} \frac{1}{6}x & \text{if } x \leq \frac{3}{4} \\ \frac{1}{2}(1-x) & \text{if } x \geq \frac{3}{4} \end{cases}$$

Theorem 7. *Let $\varphi : [0,1] \to [0,1]$. Then,*

φ is a continuous, monotone decreasing intuitionistic fuzzy generator if and only if there exists a continuous function $f : [0,1] \to [0,1]$ such that

(i) monotone increasing

(ii) $f(x) \leq x$ for all $x \in [0,1]$

(iii) $\varphi(x) = (f \circ N)(x)$ for all $x \in [0,1]$.

Proof. $\Rightarrow$), Similar to above, we will now say that f is increasing, let $x_1, x_2 \in [0,1]$ such that $x_1 < x_2$, as φ is decreasing we have $\varphi(x_1) \geq \varphi(x_2)$, therefore $f(1-x_1) \geq (1-x_2)$ and besides $1 - x_1 \geq 1 - x_2$ we have that f is increasing.

$\Leftarrow$) Similar to what was done previous theorem, we will now see that φ is decreasing, let $x_1, x_2 \in [0,1]$ such that $x_1 < x_2$, therefore $1 - x_1 > 1 - x_2$ as f is increasing we have that $f(1-x_1) \geq f(1-x_2)$ therefore $\varphi(x_1) \geq \varphi(x_2)$. □

An example of this theorem is the following:

$$f(x) = \begin{cases} 0 & \text{if } x \leq \frac{1}{4} \\ (\frac{2}{3}x - \frac{1}{6})^2 & \text{if } x \geq \frac{1}{4} \end{cases}$$

the corresponding IFC is

$$\varphi(x) = \begin{cases} 0 & \text{if } x \geq \frac{3}{4} \\ (\frac{1}{2} - \frac{2}{3}x)^2 & \text{if } x \leq \frac{3}{4} \end{cases}$$

6. Construction of IFSs from FSs.

In this section we present a method of construction of intuitionistic fuzzy sets from fuzzy sets. This method is basically constructing an IFS from the membership functions of the fuzzy set considered and the intuitionistic fuzzy generator analysed in the sections above. Afterwards we will see the general structure of the generator of the fuzzy sets built.

Theorem 8. *Let $A' = \{< x, \mu_A(x) > | x \in X\}$ be an fuzzy set on the referential set $X \neq \phi$, and let φ an intuitionistic fuzzy generator. The set*

$$\{< x, \mu_A(x), \varphi(\mu_A(x)) > | x \in X\}$$

is an intuitionistic fuzzy set on X.

Proof. We only need to recall definition 1. □

We will denote the intuitionistic fuzzy sets built in the way indicated in the theorem above $\varphi(A)$.

It is clear that from functions f that satisfy the conditions of the theorems exposed in the section above we can build intuitionistic fuzzy generators that when applied to the membership functions of fuzzy sets we get intuitionistic fuzzy sets satisfying the conditions that we have imposed on the functions f.

It is interesting to note that intuitionistic index of each element of this set is the following: $\pi_{\varphi(A)}(x) = 1 - \mu_A(x) - \varphi(\mu_A(x))$, obviously $0 \leq \pi_{\varphi(A)}(x) \leq 1$ for all $x \in X$. Besides, taking into account the definition of D_α, we have that

$$D_0(\varphi(A)) = A' = \{< x, \mu_A(x) > | x \in X\},$$

$$D_1(\varphi(A)) = \{< x, 1 - \varphi(\mu_A(x)) > | x \in X\}.$$

It is interesting to point out a particular case in which the functions f of the theorems of the section above coincide with the membership function μ.

Corollary 1.

Let $A' \in FSs(X)$ such that $\mu_A : [0,1] \to [0,1]$ continuous.

If $\mu_A(x) \leq x$ for all $x \in [0,1]$, then $\{< x, \mu_A(x), \nu_A(x) = \mu_A(1-x) > | x \in [0,1]$ is an intuitionistic fuzzy set.

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