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Intuitionistic fuzzy relations. (Part I)

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Abstract

This paper introduces the concept of intuitionistic fuzzy relation. We also study the choice of t-norms and t-conorms which must be done in order that the composition of intuitionistic fuzzy relations fulfils the largest number of properties. On the other hand, we also analyse the intuitionistic fuzzy relations in a set and their properties. Besides, we also study the properties of the intuitionistic fuzzy relations in a set and the properties of the composition with different t-norms and t-conorms.

Keywords: Intuitionistic fuzzy sets; Atanassov's operator; intuitionistic fuzzy relations; t-norm and t-conorm; intuitionistic fuzzy relations in a set.

1 Introduction

In this paper we will study the intuitionistic fuzzy relations, introduced by K. Atanassov in ([3]) and subsequently studied by Buhaescu in ([6]).

This paper is divided into two different parts. In the first part we will give the definition of intuitionistic fuzzy relation and we will study its main properties. Next, we will analyse the choice of t-norms and t-conorms which are more convenient in order that, firstly, the intuitionistim condition can be fulfilled and, secondly, the composition of intuitionistic fuzzy relations can satisfy the biggest possible number of properties. Afterwards, we will study the intuitionistic fuzzy relations in a set and we will also analyse the t-norms and t-conorms which have to be chosen in order that the intuitionistic fuzzy relations satisfy certain properties.

In the second part we will study the effect of Atanassov's operators on the properties of the intuitionistic fuzzy relations, that is to say, the conditions that an intuitionistic fuzzy relation must fulfill in order that the properties of reflexivity, symmetry, transitivity, etc... are maintained by means of operators.

2 Preliminaries

In order to define in the intuitionistic fuzzy relations some properties simial to the ones in the fuzzy relations, we will use the well-known triangular norms and conorms in [0,1], taking into account that as non- classical connectives, they do not satisfy the boolean standard identities.

We will call t-norm in [0,1] to every mapping

$$T : [0,1] \times [0,1] \to [0,1]$$

satisfying the properties

i) Boundary conditions, $T(x,1) = x$ and $T(x,0) = 0 \ \forall x \in [0,1]$

ii) Monotony, $T(x,y) \leq T(z,t)$ if $x \leq z$ and $y \leq t$

iii) Commutative, $T(x,y) = T(y,z) \ \forall x, y \in [0,1]$

iv) Associative, $T(T(x,y),z) = T(x,T(y,z)) \ \forall x, y, z \in [0,1]$

Given a t-norm T, we can consider the mapping

$$S : [0,1] \times [0,1] \to [0,1]$$
$$S(x,y) \equiv 1 - T(1-x, 1-y)$$

this mapping S, will be called dual t-conorm of T.

The more significant examples of t-norms and their associated dual t-conorms are the following ones:

i) $T(x,y) = \wedge(x,y), S(x,y) = \vee(x,y)$,

ii) $T(x,y) = x.y, S(x,y) = x\hat{+}y$

The most important properties of t-norms and t-conorms can be found in ([1], [9], [13], [14]).

Here we present the following Theorem with regard to the distributive property of t-norms and t-conorms. This Theorem will be used in different parts of the paper. In this paper, unless it is said in the opposite way, we will designate the t-norms and t-conorms with the Greek letters α, β, λ and ρ.

Let I be a finite family of indices and $\{a_i\}_{i\in I}$, $\{b_i\}_{i\in I}$ number collection of $[0,1]$. For every α t-norm or t-conorm and for every λ t-norm or t-conorm

i) $\underset{i}{\alpha}\,(a_i \vee b_i) \geq \underset{i}{\alpha}\,(a_i) \vee \underset{i}{\alpha}\,(b_i)$

ii) $\underset{i}{\lambda}\,(a_i \wedge b_i) \leq \lambda_i(a_i) \wedge \underset{i}{\lambda}\,(b_i)$

are verified.

With this result and wiht the result given by L. W. Fung and S.K. Ku ([10]) relative to the fact that α is an idempotent t-conorm (idempotent t-norm) if and only if $\alpha = \vee$ ($a = \wedge$), we get the:

Theorem 0 *Let $\{a_i\}_{i\in I}$, $\{b_i\}_{i\in I}$ be two finite number families of* $[0,1]$ *and α, λ t-norms or t-conorms not null. Then*

i) $\underset{i}{\alpha}\,(a_i \vee b_i) = \underset{i}{\alpha}\,(a_i) \vee \alpha_i(b_i)$ *if and only if* $\alpha = \vee$

ii) $\underset{i}{\lambda}\,(a_i \wedge b_i) = \underset{i}{\lambda}\,(a_i) \wedge \underset{i}{\lambda}\,(b_i)$ *if and only if* $\lambda = \wedge$.

Proof. i) $\Leftarrow$ It is enough to remember the associative property of the t-conorm $\vee$.

$\Rightarrow$) Supposing now that $\underset{i}{\alpha}\,(a_i \vee b_i) = (\underset{i}{\alpha}\,a_i) \vee (\underset{i}{\alpha}\,b_i)$, we will analyse all the posibilities for α.

Supposing that α is t-conorm, we will prove that the condition

$$\underset{i}{\alpha}\,(a_i \vee b_i) = \underset{i}{\alpha}\,(a_i) \vee \underset{i}{\alpha}\,(b_i)$$

is equal to the idempotency of α. In fact, if the condition is verified

$$\alpha(x,x) = \alpha(x \vee 0, 0 \vee x) = \alpha(x,0) \vee \alpha(0,x) = x \vee x = x,$$

then α is idempotent.

Reciprocally, if α is idempotent, $\alpha(x,x) = x \; \forall x \in [0,1]$, then thougt the monotony of α and of $\vee$ and taking into account that $\vee$ is the smallest in the t-conorms, we have

$$\begin{aligned} \underset{i}{\alpha}\,(a_i \vee b_i) &\leq \underset{i}{\alpha}\left(\left(\bigvee_i a_i\right) \vee \left(\bigvee_i b_i\right)\right) = \\ &= \left(\bigvee_i a_i\right) \vee \left(\bigvee_i b_i\right) < \underset{i}{\alpha}\,(a_i) \vee \underset{i}{\alpha}\,(b_i) \end{aligned}$$

and as a result of the property immediately previous to the enunciation of the Theorem $\underset{i}{\alpha}\,(a_i \vee b_i) \geq \underset{i}{\alpha}\,(a_i) \vee \underset{i}{\alpha}\,(b_i)$, the result is

$$\underset{i}{\alpha}\,(a_i \vee b_i) = \underset{i}{\alpha}\,(a_i) \vee \underset{i}{\alpha}\,(b_i).$$

Therefore, the condition i) of this Theorem is got from L. W. Fung and K. S. Fu's result.

Let's suppose now that α is t-norm; the hypothesis

$$\underset{i}{\alpha}\,(a_i \vee b_i) = \underset{i}{\alpha}\,(a_i) \vee \underset{i}{\alpha}\,(b_i)$$

allows us to write

$$\alpha(x,x) = \alpha(x \vee 0, 0 \vee x) = \alpha(x,0) \vee \alpha(0,x) = 0 \vee 0 = 0 \quad \forall x \in [0,1]$$

Let $(x,y) \in [0,1]^2$ and supposing that $x \leq y$ (respectively $y \leq x$), then $\alpha(x,y) \leq \alpha(y,y) = 0$ (respectively $\alpha(x,y) \leq \alpha(x,x) = 0$), so

$$\alpha(x,y) = 0 \quad \forall (x,y) \in [0,1]^2,$$

in opposition to $\alpha \neq 0$.

The item ii) of this Theorem is proved following a reasoning analogous to the one made in i). $\square$

Now we are going to remember the concept of intuitionistic fuzzy set and the definition of Atanassov's operators.

Let $X \neq \phi$ be a given set. ([2]) An *intuitionistic fuzzy set* in X is an expression A given by

$$A = \{< x, \mu_A(x), \nu_A(x) > | x \in X\} \text{ where} \\ \mu_A : X \to [0,1] \\ \nu_A : X \to [0,1]$$

with the condition $0 \leq \mu_A(x) + \nu_A(x) \leq 1 \ \forall x \in X$.

The numbers $\mu_A(x)$ and $\nu_A(x)$ denote respectively the degree of membership and the degree of non-membership of the element x in the set A. We will denote with IFSs the set of all the intuitionistic fuzzy sets in X. Obviously, when $\nu_A(x) = 1 - \mu_A(x)$ for every x in X, the set A is a fuzzy set. We will denote with FSs the set of all the fuzzy sets in X.

We will call $\pi_A(x) = 1 - \mu_A(x) - \nu_A(x)$ *intuitionistic index* of the element x in the set A.

The following expressions are defined in ([2], [4], [7], [8]) for every $A, B \in$ IFSs

1. $A \leq B \Leftrightarrow \mu_A(x) \leq \mu_B(x)$ and $\nu_A(x) \geq \nu_B(x) \ \forall x \in X$
2. $A \preceq B \Leftrightarrow \mu_A(x) \leq \mu_B(x)$ and $\nu_A(x) \leq \nu_B(x) \ \forall x \in X$
3. $A = B \Leftrightarrow A \leq B$ and $B \leq A$
4. $A_c = \{< x, \nu_A(x), \mu_A(x) > | x \in X\}$

Theorem 1 *([2], [7], [8]) Let T and S be t-norm and dual t-conorm in [0,1], then we define the expressions*

$$\begin{aligned} T(A,B) &\equiv \{< x, T(\mu_A(x), \mu_B(x)), S(\nu_A(x), \nu_B(x)) > | x \in X\} \\ S(A,B) &\equiv \{< x, S(\mu_A(x), \mu_B(x)), T(\nu_A(x), \nu_B(x)) > | x \in X\} \end{aligned}$$

for every $A, B \in$IFSs. Then, it is verified that

a) ([4]) If $S = \vee$ and $T = \wedge$, then $\{IFS, \wedge, \vee\}$ is a distributive lattice, which is bounded, not complemented and satisfies Morgan's laws.

b) For any S and T, the commutative, associative and $S(A_c, B_c) = (T(A,B))_c$, $T(A_c, B_c) = (S(A,B))_c$ properties are satisfied.

In 1986, K. Atanassov established different ways of changing and intuitionistic fuzzy set into a fuzzy set and defined the following operator:

If $E \in$ IFSs(X) then

$$D_p(E) = \{< x, \mu_E(x) + p \cdot \pi_E(x), 1 - \mu_E(x) - p \cdot \pi_E(x) > | x \in X\}$$

with $p \in [0,1]$. Obviously $D_p(E) \in$ FSs.

A study of the propierties of this operator, (we will call it *Atana ssov's operator*), is made in ([2], [7], [8]).

Let E be an intuitionistic fuzzy set and D_p the operator given in the previous definition, then the family of all fuzzy sets associated to E though the operator D_p, will be denoted by $\{D_p(E)\}_{p\in[0,1]}$. It is clear that $\{D_p(E)\}_{p\in[0,1]}$ is a totally ordered family of fuzzy sets.

3 Intuitionistic fuzzy relations

Let X, Y, Z and U be ordinary finite non-empty sets.

Definition 1 *We will call* intuitionistic fuzzy relation *to every intuitinoistic fuzzy subset of $X \times Y$, that is, to every expression R given by*

$$R = \{< (x,y), \mu_R(x,y), \nu_R(x,y) | x \in X, y \in Y\}$$

where

$$\mu_R : X \times Y \to [0,1]$$
$$\nu_R : X \times Y \to [0,1]$$

satisfy the condition $0 \leq \mu_R(x,y) + \nu_R(x,y) \leq 1$ for every $(x,y) \in X \times Y$.

We will denote with IFR$(X \times Y)$ the set of all the intuitionistic fuzzy subsets in $X \times Y$.

Concerning this definition it is worth pointing out the following aspect:

1. If $\mu_R(x,y) \neq 0$, then $\nu_R(x,y) \neq 1$
2. This definition of intuitinionistc fuzzy relation includes, as a particular case, binary relations and fuzzy relations.
3. In a way similar to the intuitionistic binary fuzzy relations, the n-arias intuitionistic fuzzy relations can be consiedered as elements of the set $[0,1]^{X\times\cdots\times Z}$.

Definition 2 *Given a binary intuitionistic fuzzy relation between X and Y we can define R^{-1} between Y and X by means of*

$$\mu_{R^{-1}}(y,x) = \mu_R(x,y), \quad \nu_{R^{-1}}(y,x) = \nu_R(x,y), \quad \forall (x,y) \in X \times Y$$

to which we will call inverse relation *of R.*

4 Operations with relations

4.1 Immediate properties

Let R and P be two intuitionistic fuzzy relations between X and Y, for every $(x,y) \in X \times Y$ we can define

a) $R \leq P \Leftrightarrow \mu_R(x,y) \leq \mu_P(x,y)$ and $\nu_R(x,y) \geq \nu_P(x,y)$.

b) $R \preceq P \Leftrightarrow \mu_R(x,y) \leq \mu_P(x,y)$ and $\nu_R(x,y) \leq \nu_P(x,y)$.

c) $R \vee P = \{< (x,y), \mu_R(x,y) \vee \mu_P(x,y), \nu_R(x,y) \wedge \nu_P(x,y) >\}$.

d) $R \wedge P = \{< (x,y), \mu_R(x,y) \wedge \mu_P(x,y), \nu_R(x,y) \vee \nu_P(x,y) >\}$.

e) $R_c = \{< (x,y), \nu_R(x,y), \mu_R(x,y) > | x \in X, y \in Y\}$.

Theorem 2 *Let R, P, Q be three elements of IFR $(X \times Y)$*

i) $R \leq P \Rightarrow R^{-1} \leq P^{-1}$

ii) $(R \vee P)^{-1} = R^{-1} \vee P^{-1}$

iii) $(R \wedge P)^{-1} = R^{-1} \wedge P^{-1}$

iv) $(R^{-1})^{-1} = R$

v) $R \wedge (P \vee Q) = (R \wedge P) \vee (R \wedge Q)$ *y* $R \vee (P \wedge Q) = (R \vee P) \wedge (R \vee Q)$

vi) $R \vee P \geq R$, $R \vee P \geq P$, $R \wedge P \leq R$, $R \wedge P \leq P$

vii) *If* $R \geq P$ *y* $R \geq Q$, *then* $R \geq P \vee Q$.
If $R \leq P$ *y* $R \leq Q$, *then* $R \leq P \wedge Q$.

Proof.

i) If $R \leq P$, then $\mu_{R^{-1}}(y,x) = \mu_R(x,y) \leq \mu_P(x,y) = \mu_{P^{-1}}(y,x)$ for every (x,y) of $X \times Y$, analogously

$$\nu_{R^{-1}}(y,x) = \nu_R(x,y) \geq \nu_P(x,y) = \nu_{P^{-1}}(y,x)$$

for every (x,y) in $X \times Y$.

ii)

$$\begin{aligned} \mu_{(R\vee P)^{-1}}(y,x) &= \mu_{(R\vee P)}(x,y) = \mu_R(x,y) \vee \mu_P(x,y) = \\ &= \mu_{R^{-1}}(y,x) \vee \mu_{P^{-1}}(y,x) = \mu_{R^{-1}\vee P^{-1}}(y,x). \end{aligned}$$

The proof for $\nu_{(R\vee P)^{-1}}(y,x) = \nu_{R^{-1}\vee P^{-1}}(y,x)$ is done in a similar way.

v) We will use the fact that the operators $\vee$ and $\wedge$ satisfy the distributive property when they are applied to elements of [0,1]

$$\begin{aligned} \mu_{R\wedge(P\vee Q)}(x,y) &= \mu_R(x,y) \wedge \{\mu_P(x,y) \vee \mu_Q(x,y)\} \\ &= \{\mu_R(x,y) \wedge \mu_P(x,y)\} \vee \\ &\vee \ \{\mu_R(x,y) \wedge \mu_Q(x,y)\} \\ &= \mu_{R\wedge P}(x,y) \vee \mu_{R\wedge Q}(x,y) \\ &= \mu_{(R\wedge P)\vee(R\wedge Q)}(x,y). \end{aligned}$$

The proof is analogous to the previous one, in the case of $\nu_{R\wedge(P\vee Q)}(x,y) = \nu_{(R\wedge P)\vee(R\wedge Q)}(x,y)$.

The rest of the items are proved in a way similar to the previous ones. □

We can generalize the operatins between binary intuitionistic fuzzy relations $R, Q \in$ IFR $(X \times Y)$, using the well-known triangular t-norms and t-conorms in [0,1]. For a triangular t-norm T and its dual t-conorm S, we get

$$\begin{aligned} T(R,Q) &= \{<(x,y), T(\mu_R(x,y), \mu_Q(x,y)), S(\nu_R(x,y), \nu_Q(x,y))>\} \\ S(R,Q) &= \{<(x,y), S(\mu_R(x,y), \mu_Q(x,y)), T(\nu_R(x,y), \nu_Q(x,y))>\}. \end{aligned}$$

4.2 Composition of IFR

Basing ourselves on the composition of binary fuzzy relations in [0,1] we can give the following definition:

Definition 3 *Let $\alpha, \beta, \lambda, \rho$ be t-norms or t-conorms not necessarily dual two-two, $R \in$IFR$(X \times Y)$ and $P \in$ IFR $(Y \times Z)$. We will call* composed relation $P \overset{\alpha,\beta}{\underset{\lambda,\rho}{\circ}} R \in$ *IFR $(X \times Z)$ to the one defined by*

$$P \overset{\alpha,\beta}{\underset{\lambda,\rho}{\circ}} R = \{<(x,z), \mu_{P \overset{\alpha,\beta}{\underset{\lambda,\rho}{\circ}} R}(x,z), \nu_{P \overset{\alpha,\beta}{\underset{\lambda,\rho}{\circ}} R}(x,z)> | x \in X, z \in Z\}$$

where

$$\begin{aligned} \mu_{P \overset{\alpha,\beta}{\underset{\lambda,\rho}{\circ}} R}(x,z) &= \underset{y}{\alpha} \{\beta[\mu_R(x,y), \mu_P(y,z)]\} \\ \nu_{P \overset{\alpha,\beta}{\underset{\lambda,\rho}{\circ}} R}(x,z) &= \underset{y}{\lambda} \{\rho[\nu_R(x,y), \nu_P(y,z)]\} \end{aligned}$$

whenever

$$0 \leq \mu_{P \overset{\alpha,\beta}{\underset{\lambda,\rho}{\circ}} R}(x,z) + \nu_{P \overset{\alpha,\beta}{\underset{\lambda,\rho}{\circ}} R}(x,z)(x,z) \leq 1 \qquad \forall (x,z) \in X \times Z.$$

The choice of the t-norms and t-conorms $\alpha, \beta, \lambda, \rho$ in the previous definition, is evidently conditioned by the fulfilment of

$$0 \leq \mu_{P \overset{\alpha,\beta}{\underset{\lambda,\rho}{\circ}} R}(x,z) + \nu_{P \overset{\alpha,\beta}{\underset{\lambda,\rho}{\circ}} R}(x,z) \leq 1 \qquad \forall (x,z) \in X \times Z.$$

In this direction:

Proposition 1 *In the conditions of the Definition 3, if λ^* and ρ^* are respectively the dual forms of λ and ρ and $\alpha \leq \lambda^*$, $\beta \leq \rho^*$, then*

$$0 \leq \mu_{P \underset{\lambda,\rho}{\overset{\alpha,\beta}{\circ}} R}(x,z) + \nu_{P \underset{\lambda,\rho}{\overset{\alpha,\beta}{\circ}} R}(x,z) \leq 1 \qquad \forall (x,z) \in X \times Z.$$

Proof. We know that

$$\begin{array}{ll} \mu_R(x,y) \leq 1 - \nu_R(x,y) & \forall (x,y) \in X \times Y \\ \mu_P(y,z) \leq 1 - \nu_P(y,z) & \forall (y,z) \in Y \times Z \end{array}$$

taking as hypothesis $\alpha \leq \lambda^*$ and $\beta \leq \rho^*$, we have

$$\beta[\mu_R(x,y), \mu_P(y,z)] \leq \rho^*[1 - \nu_R(x,y), 1 - \nu_P(y,z)]$$

$$\begin{array}{lcl} \underset{y}{\alpha} \{\beta[\mu_R(x,y), \mu_P(y,z)]\} & \leq & \underset{y}{\lambda^*} \{\rho^*[1 - \nu_R(x,y), 1 - \nu_P(y,z)]\} = \\ & = & 1 - \underset{y}{\lambda} \{1 - \rho^*[1 - \nu_R(x,y), 1 - \nu_P(y,z)]\} \\ & = & 1 - \underset{y}{\lambda} \{\rho[\nu_R(x,y), \nu_P(y,z)]\} \end{array}$$

therefore

$$\underset{y}{\alpha} \{\beta[\mu_R(x,y), \mu_P(y,z)]\} + \underset{y}{\lambda} \{\rho[\nu_R(x,y), \nu_P(y,z)]\} \leq 1. \qquad \square$$

Although we will see afterwards that the richest cases in properties correspond to the choice of $\alpha = \vee$, $\beta = \wedge$, $\lambda = \wedge$, $\rho = \vee$ and in this case the Proposition 1 is obviously verified, next theorem gives a necessary condition and it is sufficient for the verification of the restriction of the intuitionism, even with β and ρ whichsoever.

Theorem 3 *For each $(x,z) \in X \times Z$, $\alpha = \vee$, $\lambda = \wedge$, β and ρ any t-norms or t-conorms, not necessarily dual, we have*

$$0 \leq \mu_{P \underset{\lambda,\rho}{\overset{\alpha,\beta}{\circ}} R}(x,z) + \nu_{P \underset{\lambda,\rho}{\overset{\alpha,\beta}{\circ}} R}(x,z) \leq 1 \Leftrightarrow \forall y \in Y \quad \exists y' \quad \textit{such that}$$

$$\beta[\mu_P(x,y), \mu_R(y,z)] \leq \rho^*[1 - \mu_P(x,y'), 1 - \nu_R(y',z)].$$

Proof. $\Rightarrow)$ $\mu_{P \overset{\alpha,\beta}{\underset{\lambda,\rho}{\circ}} R}(x,z) + \nu_{P \overset{\alpha,\beta}{\underset{\lambda,\rho}{\circ}} R}(x,z) \leq 1$ then $\mu_{P \overset{\alpha,\beta}{\underset{\lambda,\rho}{\circ}} R}(x,z) \leq$ $\leq 1 - \nu_{P \overset{\alpha,\beta}{\underset{\lambda,\rho}{\circ}} R}(x,z)$ therefore

$$\begin{aligned}
\bigvee_y \{\beta[\mu_R(x,y), \mu_P(y,z)]\} &\leq 1 - \bigwedge_y \{\rho[\nu_R(x,y), \nu_P(y,z)]\} = \\
&= \bigvee_y \{1 - \rho[\nu_R(x,y), \nu_P(y,z)]\} = \\
&= \bigvee_y \{\rho^*[1 - \nu_R(x,y), 1 - \nu_P(y,z)]\}
\end{aligned}$$

therefore $y' \in Y$ exists such that for every y

$$\beta[\mu_R(x,y), \mu_P(y,z)] \leq \rho^*[1 - \nu_R(x,y'), 1 - \nu_P(y',z)].$$

$\Leftarrow)$ $\forall y \in Y$, $\exists y' \in Y$ such that

$$\beta[\mu_R(x,y), \mu_P(y,z)] \leq \rho^*[1 - \nu_R(x,y'), 1 - \nu_P(y',z)],$$

then

$$\begin{aligned}
&\bigvee_y \{\beta[\mu_R(x,y), \mu_P(y,z)]\} \leq \bigvee_y \{\rho^*[1 - \nu_R(x,y), 1 - \nu_P(y,z)]\} \\
&= 1 - \bigwedge_y \{1 - \rho^*[1 - \nu_R(x,y), 1 - \nu_P(y,z)]\} = \\
&= 1 - \bigwedge_y \{\rho[\nu_R(x,y), \nu_P(y,z)]\},
\end{aligned}$$

therefore

$$\bigvee_y \{\beta[\mu_R(x,y), \mu_P(y,z)]\} + \bigwedge_y \{\rho[\nu_R(x,y), \nu_P(y,z)]\} \leq 1,$$

that is to say

$$0 \leq \mu_{P \overset{\alpha,\beta}{\underset{\lambda,\rho}{\circ}} R}(x,z) + \nu_{P \overset{\alpha,\beta}{\underset{\lambda,\rho}{\circ}} R}(x,z) \leq 1. \quad \square$$

Theorem 4 *For each $R \in IFR(X \times Y)$, $P \in IFR(Y \times Z)$ and $\alpha, \beta, \lambda, \rho$ any t-norms or t-conorms*

$$\left(P \overset{\alpha,\beta}{\underset{\lambda,\rho}{\circ}} R \right)^{-1} = R^{-1} \overset{\alpha,\beta}{\underset{\lambda,\rho}{\circ}} P^{-1}$$

is fulfilled.

Proof.

$$\begin{aligned}
\mu_{\left(P \overset{\alpha,\beta}{\underset{\lambda,\rho}{\circ}} R\right)^{-1}}(z,x) &= \mu_{\left(P \overset{\alpha,\beta}{\underset{\lambda,\rho}{\circ}} R\right)}(x,z) = \underset{y}{\alpha} \{\beta[\mu_R(x,y), \mu_P(y,z)]\} = \\
&= \underset{y}{\alpha} \{\beta[\mu_{R^{-1}}(y,x), \mu_{P^{-1}}(z,y)]\} = \\
&= \underset{y}{\alpha} \{\beta[\mu_{P^{-1}}(z,y), \mu_{R^{-1}}(y,x)]\} = \\
&= \mu_{R^{-1} \overset{\alpha,\beta}{\underset{\lambda,\rho}{\circ}} P^{-1}}(z,x).
\end{aligned}$$

Following the same reasoning, it is proved that

$$\nu_{\left(P \overset{\alpha,\beta}{\underset{\lambda,\rho}{\circ}} R\right)^{-1}}(z,x) = \nu_{R^{-1} \overset{\alpha,\beta}{\underset{\lambda,\rho}{\circ}} P^{-1}}(z,x) \qquad \forall (z,x) \in Z \times X.$$

Theorem 5 *In the conditions of the Definition 3,*

i) If $P_1 \leq P_2$ then $P_1 \overset{\alpha,\beta}{\underset{\lambda,\rho}{\circ}} R \leq P_2 \overset{\alpha,\beta}{\underset{\lambda,\rho}{\circ}} R$, for every $R \in IFR$

ii) If $R_1 \leq R_2$ then $P \overset{\alpha,\beta}{\underset{\lambda,\rho}{\circ}} R_1 \leq P \overset{\alpha,\beta}{\underset{\lambda,\rho}{\circ}} R_2$, for every $P \in IFR$

ii) If $P_1 \preceq S_2$ then $P_1 \overset{\alpha,\beta}{\underset{\lambda,\rho}{\circ}} R \preceq P_2 \overset{\alpha,\beta}{\underset{\lambda,\rho}{\circ}} R$, for every $R \in IFR$

iv) If $R_1 \preceq R_2$ then $P \overset{\alpha,\beta}{\underset{\lambda,\rho}{\circ}} R_1 \preceq P_2 \overset{\alpha,\beta}{\underset{\lambda,\rho}{\circ}} R$, for every $P \in IFR$

v) Let R, P be in IFR $(X \times X)$, If $P \leq R$ then $P \overset{\alpha,\beta}{\underset{\lambda,\rho}{\circ}} P \leq R \overset{\alpha,\beta}{\underset{\lambda,\rho}{\circ}} R$.

are verified.

Proof.

i) $P_1 \leq P_2$ then $\mu_{P_1}(y,z) \leq \mu_{P_2}(y,z)$ and $\nu_{P_1}(y,z) \geq \nu_{P_2}(y,z)$

$$\begin{aligned}
\mu_{P_1 \overset{\alpha,\beta}{\underset{\lambda,\rho}{\circ}} R}(x,z) &= \underset{y}{\alpha}\ \{\beta[\mu_R(x,y), \mu_{P_1}(y,z)]\} \leq \\
&\leq \underset{y}{\alpha}\ \{\beta[\mu_R(x,y), \mu_{P_2}(y,z)]\} = \mu_{P_2 \overset{\alpha,\beta}{\underset{\lambda,\rho}{\circ}} R}(x,z). \\
\nu_{P_1 \overset{\alpha,\beta}{\underset{\lambda,\rho}{\circ}} R}(x,z) &= \underset{y}{\lambda}\ \{\rho[\nu_R(x,y), \nu_{P_1}(y,z)]\} \leq \\
&\leq \underset{y}{\lambda}\ \{\rho[\nu_R(x,y), \nu_{P_2}(y,z)]\} = \nu_{P_2 \overset{\alpha,\beta}{\underset{\lambda,\rho}{\circ}} R}(x,z).
\end{aligned}$$

v) $P \leq R$ then though the item i) $P \overset{\alpha,\beta}{\underset{\lambda,\rho}{\circ}} P \leq R \overset{\alpha,\beta}{\underset{\lambda,\rho}{\circ}} P$ then though the item ii) $P \overset{\alpha,\beta}{\underset{\lambda,\rho}{\circ}} P \leq R \overset{\alpha,\beta}{\underset{\lambda,\rho}{\circ}} R$.

It is done in a similar way for the rest of the items. □

Going on with the properties of the composition of binary intuitionistic fuzzy relations, we are now going to study the distributivity.

Theorem 6 *For any α, β, λ and ρ t-norms or t-conorms, $R, P \in IFR$ $(Y \times Z)$ and $Q \in IFR$ $(X \times Y)$*

$$(R \vee P) \overset{\alpha,\beta}{\underset{\lambda,\rho}{\circ}} Q \geq (R \overset{\alpha,\beta}{\underset{\lambda,\rho}{\circ}} Q) \vee (P \overset{\alpha,\beta}{\underset{\lambda,\rho}{\circ}} Q)$$

holds.

Proof.

Starting from the points vi) and vii) of the theorem 1, we get

$$\left\{ \begin{array}{l} R \vee P \geq R \\ R \vee P \geq P \end{array} \right. \Rightarrow \left\{ \begin{array}{l} (R \vee P) \overset{\alpha,\beta}{\underset{\lambda,\rho}{\circ}} Q \geq R \overset{\alpha,\beta}{\underset{\lambda,\rho}{\circ}} Q \\ (R \vee P) \overset{\alpha,\beta}{\underset{\lambda,\rho}{\circ}} Q \geq P \overset{\alpha,\beta}{\underset{\lambda,\rho}{\circ}} Q \end{array} \right. \Rightarrow$$

$$\Rightarrow (R \vee P) \overset{\alpha,\beta}{\underset{\lambda,\rho}{\circ}} Q \geq (R \overset{\alpha,\beta}{\underset{\lambda,\rho}{\circ}} Q) \vee (P \overset{\alpha,\beta}{\underset{\lambda,\rho}{\circ}} Q). \qquad \square$$

The previous theorem determines the sign of the inequality for the distributive property of the composition respecting the union. Next theorem will give us a necessary and sufficient condition for the fulfilment of the equality.

Theorem 7 *Let R, P be two elements of $IFR(Y \times Z)$, $Q \in IFR\,(X \times Y)$, α and λ not null t-norms and t-conorms. Then*

$$(R \vee P) \overset{\alpha,\beta}{\underset{\lambda,\rho}{\circ}} Q = (R \overset{\alpha,\beta}{\underset{\lambda,\rho}{\circ}} Q) \vee (P \overset{\alpha,\beta}{\underset{\lambda,\rho}{\circ}} Q) \text{ if and only if } \alpha = \vee \text{ and } \lambda = \wedge.$$

Proof. $\Rightarrow$)

$$\begin{aligned} \mu_{(R\vee P)\overset{\alpha,\beta}{\underset{\lambda,\rho}{\circ}}Q}(x,z) &= \underset{y}{\alpha}\,\{\beta[\mu_Q(x,y), \mu_R(y,z) \vee \mu_P(y,z)]\} = \\ &= \underset{y}{\alpha}\,\{\beta[\mu_Q(x,y), \mu_R(y,z) \vee \beta[\mu_Q(x,y), \mu_P(y,z)]\} = \end{aligned}$$

because of the hypothesis of the theorem the result is

$$= \underset{y}{\alpha}\,\{\beta[\mu_Q(x,y), \mu_R(y,z))]\} \vee \underset{y}{\alpha}\,\{\beta[\mu_Q(x,y), \mu_P(y,z)]\}.$$

Let $\{a_y\}_{y\in Y}$, $\{b_y\}_{y\in Y}$ be two any finite family of numbers belonging to the interval [0,1].

a) If β is t-norm, we define (for x, z fixed and for every y)

$$\begin{aligned} \mu_Q(x,y) &= 1 \\ \mu_R(y,z) &= a_y \\ \mu_P(y,z) &= b_y, \end{aligned}$$

then it is known that

$$\begin{aligned} \beta[\mu_Q(x,y), \mu_R(y,z)] &= a_y \\ \beta[\mu_Q(x,y), \mu_S(y,z)] &= b_y. \end{aligned}$$

Besides, as $(R \vee P) \overset{\alpha,\beta}{\underset{\lambda,\rho}{\circ}} Q = (R \overset{\alpha,\beta}{\underset{\lambda,\rho}{\circ}} Q) \vee (P \overset{\alpha,\beta}{\underset{\lambda,\rho}{\circ}} Q)$ it is verified for every R, P and Q by means of hypothesis, we have

$$\underset{i}{\alpha}\,(a_i \vee b_i) = \underset{i}{\alpha}\,(a_i) \vee \underset{i}{\alpha}\,(b_i)$$

and this condition, verified for every $\{a_y\}_{y\in Y}$, $\{b_y\}_{y\in Y}$, we have proved in the Theorem 0, that is equivalent to $\alpha = \vee$.

b) If β is t-conorm, we define the degree of membership of R, P and Q as follows:

$$\begin{aligned} \mu_Q(x,y) &= 0 \\ \mu_R(y,z) &= a_y \\ \mu_P(y,z) &= b_y \end{aligned}$$

and with the same proceeding we conclude that verifying

$$\underset{i}{\alpha}\,(a_i \vee b_i) = \underset{i}{\alpha}\,(a_i) \vee \underset{i}{\alpha}\,(b_i)$$

as we have seen in the Theorem 0, if and only if $\alpha = \vee$.

This item of the demostration is finished following the same proceeding for the non-membership, concluding that $\lambda = \wedge$.

$\Leftarrow$) Let's take $\alpha = \vee$, $\lambda = \wedge, \beta$ and ρ any t-norms and t-conorms

$$\begin{aligned} \mu_{(R \vee P) \overset{\alpha,\beta}{\underset{\lambda,\rho}{\circ}} Q}(x,z) &= \bigvee_y \{\beta[\mu_Q(x,y), \mu_R(y,z) \vee \mu_P(y,z)]\} = \\ &= \bigvee_y \{\beta[\mu_Q(x,y), \mu_R(y,z)] \vee \beta[\mu_Q(x,y), \mu_P(y,z)]\} = \end{aligned}$$

using the associative property of the t-conorm $\vee$, we have

$$\begin{aligned} &= \bigvee_y \{\beta[\mu_Q(x,y), \mu_R(y,z)] \vee \beta[\mu_Q(x,y), \mu_P(y,z)]\} = \\ &= \bigvee_y \{\beta[\mu_Q(x,y), \mu_R(y,z)]\} \vee \bigvee_y \{\beta[\mu_Q(x,y), \mu_P(y,z)]\} = \\ &= \mu_{R \overset{\alpha,\beta}{\underset{\lambda,\rho}{\circ}} Q}(x,z) \vee \mu_{P \overset{\alpha,\beta}{\underset{\lambda,\rho}{\circ}} Q}(x,z) \qquad \forall (x,z) \in X \times Y. \end{aligned}$$

Following the same steps it is proved that

$$\nu_{(R \wedge P) \overset{\vee,\beta}{\underset{\wedge,\rho}{\circ}} Q}(x,z) = \nu_{(R \overset{\vee,\beta}{\underset{\wedge,\rho}{\circ}} Q)}(x,z) \wedge \nu_{(P \overset{\vee,\beta}{\underset{\wedge,\rho}{\circ}} Q)}(x,z). \quad \square$$

Theorem 8 *For eery $\alpha, \beta, \lambda, \rho$ any t-norms or t-conorms and $R, P \in$ IFR$(Y \times Z)$, $Q \in$ IFR$(X \times Y)$, it is verified that*

$$(R \wedge P) \overset{\alpha,\beta}{\underset{\lambda,\rho}{\circ}} Q \leq (R \overset{\alpha,\beta}{\underset{\lambda,\rho}{\circ}} Q) \wedge (P \overset{\alpha,\beta}{\underset{\lambda,\rho}{\circ}} Q).$$

Proof. Analogous to the one made in the Theorem 5. □

Theorem 9 *Let R, P be two elements of IFR$(Y \times Z)$, $Q \in$IFR$(X \times Y)$, α different from the null t-norm and λ different from the null t-conorm. Then*

$$(R \wedge P) \overset{\alpha,\beta}{\underset{\lambda,\rho}{\circ}} Q = (R \overset{\alpha,\beta}{\underset{\lambda,\rho}{\circ}} Q) \wedge (P \overset{\alpha,\beta}{\underset{\lambda,\rho}{\circ}} Q) \text{ if and only if } \alpha = \wedge \text{ and } \lambda = \vee.$$

Proof. It is done following the same proceeding to the one used in he proof of the Theorem 7. □

From the analysis of the previous Theorem it is deduced that the choice of the α, β, λ and ρ t-norms or t-conorms will depend on the problem traced on each case. However, the distributive equalites will demand the choice of $\vee$ and $\wedge$ for α and λ or λ and α respectively.

Theorem 10 *Let $Q \in$ IFR$(X \times Y)$, $P \in$ IFR$(Y \times Z)$ $R \in$ IFR$(Z \in U)$, β and ρ any t-norms or t-conorms.*

$$\text{If } \alpha = \vee \text{ and } \lambda = \wedge, \text{ then } (R \overset{\vee,\beta}{\underset{\wedge,\rho}{\circ}} P) \overset{\vee,\beta}{\underset{\wedge,\rho}{\circ}} Q = R \overset{\vee,\beta}{\underset{\wedge,\rho}{\circ}} (P \overset{\vee,\beta}{\underset{\wedge,\rho}{\circ}} Q).$$

Proof. In this proof we will use the facts that β is associative, that ([5], [11])

$$\begin{aligned} \bigvee_i \bigvee_j a_{i,j} &= \bigvee_j \bigvee_i a_{i,j}, \\ \beta\left(a, \bigvee_i b_i\right) &= \bigvee_i \left(\beta(a, b_i)\right), \\ \beta\left(\bigvee_i a_i, b\right) &= \bigvee_i \left(\beta(a_i, b)\right), \end{aligned}$$

and the same properties are applied to $\wedge$.

$$\mu_{(R \overset{\vee,\beta}{\underset{\wedge,\rho}{\circ}} P) \overset{\vee,\beta}{\underset{\wedge,\rho}{\circ}} Q)}(x, z) = \bigvee_y \left\{ \beta[\mu_Q(x, y), \mu_{R \overset{\vee,\beta}{\underset{\wedge,\rho}{\circ}} P}(y, z)] \right\} =$$

$$
\begin{aligned}
&= \bigvee_y \left\{ \beta[\mu_Q(x,z), \bigvee_t \{\beta[\mu_P(y,t), \mu_R(t,z)]\}\right\} = \\
&= \bigvee_y \left\{ \bigvee_t \{\beta[\mu_Q(x,z), \beta[\mu_P(y,t), \mu_R(t,z)]\}\right\} = \\
&= \bigvee_t \left\{ \bigvee_y [\beta\,[\beta[\mu_Q(x,y), \mu_P(y,t)]\mu_R(t,z)]]\right\} = \\
&= \bigvee_t \left\{ \beta \left[\bigvee_y \beta\,[\mu_Q(x,y), \mu_P(y,t)]\,, \mu_R(t,z)\right]\right\} = \\
&= \bigvee_t \left\{ \beta \left[\mu_{P \overset{\vee,\beta}{\underset{\wedge,\rho}{\circ}} Q}(x,t), \mu_R(t,z)\right]\right\} = \\
&= \mu_{R \overset{\vee,\beta}{\underset{\wedge,\rho}{\circ}} (P \overset{\vee,\beta}{\underset{\wedge,\rho}{\circ}} Q)}(x,z) \qquad \forall (x,z) \in X \times Z.
\end{aligned}
$$

The equality $\nu_{(R \overset{\vee,\beta}{\underset{\wedge,\rho}{\circ}} P) \overset{\vee,\beta}{\underset{\wedge,\rho}{\circ}} Q}(x,z) = \nu_{R \overset{\vee,\beta}{\underset{\wedge,\rho}{\circ}} (P \overset{\vee,\beta}{\underset{\wedge,\rho}{\circ}} Q)}(x,z)$ for every $(x,z) \in X \times Z$ corresponding to the non-membership is proved in analogous way. □

It is quite simple to prove that if $\alpha = \wedge$ and $\lambda = \vee$, the associative property is also fulfilled.

From the study made until now about the composition of the intuitionistic fuzzy relations, we can deduce the following conclusions:

1) If $\alpha = \vee$ and $\lambda = \wedge$, the composition $P \overset{\vee,\beta}{\underset{\wedge,\rho}{\circ}} R$ verifies the studied properties, except the distributive one of the composition respecting the intersection (Theorem 9).

2) If $\alpha = \wedge$ and $\lambda = \vee$, the composition $P \overset{\wedge,\beta}{\underset{\vee,\rho}{\circ}} R$ verifies all the properties, except the distributive one respecting the union (Theorem 7).

3) β and ρ can be any t-norms or t-conorms, not fixed by any condition.

5 Intuitionistic fuzzy relations in a set

We will study now the properties of the binary intuitionistic fuzzy relations $R \in \mathrm{IFR}(X \times X)$, defined in a set X. We will see that the exigency of the verification of certain ordinary properties, as it was expected, will allow us to determine the possible values of β and ρ.

It is convenient to state that in the notation used for the composition, $P \overset{\alpha,\beta}{\underset{\lambda,\rho}{\circ}} R$, the symbols α and β placed up are applied to the membership and the symbols λ and ρ placed down are applied to the non-membership. Therefore, the order of placement is very important, so this fact will have to taken into account in all what follows.

5.1 Identity relation

Definition 4 *1) The relation $\Delta \in IFR(X \times X)$ is called relation of identity if:*

$$\mu_\Delta(x,y) = \begin{cases} 1 & \text{if } x = y \\ 0 & \text{if } x \neq y \end{cases},$$

$$\nu_\Delta(x,y) = \begin{cases} 0 & \text{if } x = y \\ 1 & \text{if } x \neq y \end{cases} \quad \forall (x,y) \in X \times X.$$

2) The complementary relation Δ_c, defined by

$$\mu_\Delta(x,y) = \begin{cases} 0 & \text{if } x = y \\ 1 & \text{if } x \neq y \end{cases},$$

$$\nu_\Delta(x,y) = \begin{cases} 1 & \text{if } x = y \\ 0 & \text{if } x \neq y \end{cases} \quad \forall (x,y) \in X \times X.$$

will be represented by the symbol ∇.

Is evident that $\Delta = \Delta^{-1}$ and $\nabla = \nabla^{-1}$.

Theorem 11 *Let $\alpha, \beta, \lambda, \rho$ be any t-norms or t-conorms and $R \in IFR(X \times X)$.*

i) $R \overset{\alpha,\beta}{\underset{\lambda,\rho}{\circ}} \Delta = \Delta \overset{\alpha,\beta}{\underset{\lambda,\rho}{\circ}} R = R$ if and only if α is t-conorm, β is t-norm, λ is t-norm and ρ is t- conorm.

ii) $R \overset{\lambda,\rho}{\underset{\alpha,\beta}{\circ}} \nabla = \nabla \overset{\lambda,\rho}{\underset{\alpha,\beta}{\circ}} R = R$ *if and only if* α *is t-conorm,* β *is t-norm,* λ *is t-norm and* ρ *is t- conorm.*

Proof. i) $\Leftarrow$)

$$\begin{aligned}
\mu_{R \overset{\alpha,\beta}{\underset{\lambda,\rho}{\circ}} \Delta}(x,z) &= \underset{y}{\alpha} \{\beta[\mu_\Delta(x,y), \mu_R(y,z)]\} = \\
&= \underset{y\neq x}{\alpha} \{\beta[\mu_\Delta(x,x), \mu_R(x,z)], \beta[\mu_\Delta(x,y), \mu_R(y,z)]\} = \\
&= \underset{y\neq x}{\alpha} \{\beta[1, \mu_R(x,z)], \beta[0, \mu_R(y,z)]\} = \\
&= \underset{y\neq x}{\alpha} \{\mu_R(x,z), 0\} = \mu_R(x,z) \qquad (x,z) \in X \times X.
\end{aligned}$$

$$\begin{aligned}
\nu_{R \overset{\alpha,\beta}{\underset{\lambda,\rho}{\circ}} \Delta}(x,z) &= \underset{y}{\lambda} \{\rho[\nu_\Delta(x,y), \nu_R(y,z)]\} = \\
&= \underset{y\neq x}{\lambda} \{\rho[\nu_\Delta(x,x), \nu_R(x,z)], \rho[\nu_\Delta(x,y), \nu_R(y,z)]\} = \\
&= \underset{y\neq x}{\lambda} \{\rho[0, \nu_R(x,z)], \rho[1, \nu_R(y,z)]\} = \\
&= \underset{y\neq x}{\lambda} \{\nu_R(x,z), 1\} = \nu_R(x,z) \qquad (x,z) \in X \times X.
\end{aligned}$$

$\Rightarrow$) Now we suppose that $R \overset{\alpha,\beta}{\underset{\lambda,\rho}{\circ}} \Delta = \Delta \overset{\alpha,\beta}{\underset{\lambda,\rho}{\circ}} R = R$, it is fulfilled for each $R \in \mathrm{IFR}(X \times X)$. We will only deal with the membership functions (we will work in a similar way in the non-membership functions) and we will distinguish between four cases:

1) α t-norm and β t-norm. Taking $R = \Delta$, we get

$$\Delta \overset{\alpha,\beta}{\underset{\lambda,\rho}{\circ}} \Delta = \Delta \text{ then } \mu_{\Delta \overset{\alpha,\beta}{\underset{\lambda,\rho}{\circ}} \Delta}(x,z) = \mu_\Delta(x,z) \quad \forall (x,z) \in X \times X.$$

If $x = z$, then

$$\begin{aligned}
\mu_{\Delta \overset{\alpha,\beta}{\underset{\lambda,\rho}{\circ}} \Delta}(x,x) &= \underset{y}{\alpha} \{\beta[\mu_\Delta(x,y), \mu_\Delta(y,x)]\} = \\
&= \underset{y\neq x}{\alpha} \{\beta[\mu_\Delta(x,x), \mu_\Delta(x,x)], \beta[\mu_\Delta(x,y), \mu_\Delta(y,x)]\} = \\
&= \underset{y\neq x}{\alpha} \{\beta[1,1], \beta[0,0]\} = \underset{y\neq x}{\alpha} \{1,0\} = 0 \neq \mu_\Delta(x,z) = 1
\end{aligned}$$

for every $x \in X$ in opposition to the hypothesis.

2) α t-conorm and β t-conorm. Taking $R = \Delta$ and $x \neq z$, we have

$$\begin{aligned}
\mu_{\Delta \overset{\alpha,\beta}{\underset{\lambda,\rho}{\circ}} \Delta}(x,z) &= \underset{y}{\alpha} \{\beta[\mu_\Delta(x,y), \mu_\Delta(y,z)]\} = \\
&= \underset{y \neq x}{\alpha} \{\beta[\mu_\Delta(x,x), \mu_\Delta(x,z)], \beta[\mu_\Delta(x,y), \mu_\Delta(y,z)]\} = \\
&= \underset{y \neq x}{\alpha} \{\beta[1, \mu_\Delta(x,z)], \beta[0, \mu_\Delta(y,z)]\} = \\
&= \underset{\substack{y \neq x \\ y \neq z}}{\alpha} \{\beta[1,0], \beta[0, \mu_\Delta(z,z)], \beta[0, \mu_\Delta(y,z)]\} = \\
&= \underset{\substack{y \neq x \\ y \neq z}}{\alpha} \{\beta[1,0], \beta[0,1], \beta[0,0]\} = \\
&= \underset{\substack{y \neq x \\ y \neq z}}{\alpha} \{1,1,0\} = 1 \neq \mu_\Delta(x,z) = 0
\end{aligned}$$

for every $(x,z) \in X \times X$ different from $\mu_\Delta(x,z) = 0$ if $x \neq z$.

3) α t-norm and β t-conorm. Taking R in the following way

$$\mu_R(x,y) = \begin{cases} 1 & \text{if } x = y \\ \neq 1 & \text{if } x \neq y \end{cases}$$

by means of hypothesis $\mu_{R \overset{\alpha,\beta}{\underset{\lambda,\rho}{\circ}} \Delta}(x,x) = \mu_R(x,x) = 1 \ \forall x \in X$ have to be fulfilled.

$$\begin{aligned}
\mu_{R \overset{\alpha,\beta}{\underset{\lambda,\rho}{\circ}} \Delta}(x,x) &= \underset{y}{\alpha} \{\beta[\mu_\Delta(x,y), \mu_\Delta(y,x)]\} = \\
&= \underset{y \neq x}{\alpha} \{\beta[\mu_\Delta(x,x), \mu_R(x,x)], \beta[\mu_\Delta(x,y), \mu_R(y,x)]\} = \\
&= \underset{y \neq x}{\alpha} \{\beta[1,1], \beta[0, \mu_R(y,x)]\} = \\
&= \underset{y \neq x}{\alpha} \{1, \mu_R(y,x)\} = \mu_R(y,x) \neq 1 = \mu_R(x,x),
\end{aligned}$$

it is against the hypothesis. In consequence, it is proved that α is t-conorm and β is t-norm in the conditions of the Theorem.

Making a development, which is analogous to the previous one, for the non-membership functions, we deduce that λ is t-norm and ρ t-conorm.

ii) The proof of this item is similar to the one made in i), using ∇.
$\square$

5.2 Reflexivity and antireflexivity

Definition 5 *The relation $R \in IFR(X \times Y)$ is called:*

1) *Reflexive if for every $x \in X$ $\mu_R(x,x) = 1$. Just notice that $\nu_R(x,x) = 0 \ \forall x \in X$.*

2) *Antireflexive if for every $x \in X$ $\begin{cases} \mu_R(x,x) = 0 \\ \nu_R(x,x) = 1, \end{cases}$ that is to say, if its complementary R_c is reflexive.*

Theorem 12 *For every $R \in IFR(X)$, it is verified that*

i) If R is reflexive, then $\Delta \leq R$

ii) If R is antireflexive, then $\nabla \geq R$.

Proof. It is the consequence of the Definitions 4 and 5. □

Theorem 13 *For α t-conorm, β t-norm, λ t-norm and ρ t- conorm, it is verified that:*

i) If $R \in IFR(X \times Y)$ is reflexive, then $R \leq R \overset{\alpha,\beta}{\underset{\lambda,\rho}{\circ}} R$

ii) If $R \in IFR(X \times X)$ is antireflexive, then $R \geq R \overset{\lambda,\rho}{\underset{\alpha,\beta}{\circ}} R$

Let's notice the importance of the placement of α, β, λ and ρ. The second condition of the Theorem can be written in this way $R_c \leq R_c \overset{\alpha,\beta}{\underset{\lambda,\rho}{\circ}} R_c$ and, therefore, the sign of the inequality and the order of α, β, λ and ρ, with regard to the first condition, are maintained. However, we will use the first notation, taking always into account the placement that α, β, λ and ρ occupy.

Proof. i)

$$\mu_{R \overset{\alpha,\beta}{\underset{\lambda,\rho}{\circ}} R}(x,z) \quad = \quad \underset{y}{\alpha} \{\beta[\mu_R(x,y), \mu_R(y,z)]\} =$$

$$= \underset{y\neq x}{\alpha} \{\beta[\mu_R(x,x),\mu_R(x,z)],\beta[\mu_R(x,y),\mu_R(y,z)]\} =$$
$$= \underset{y\neq x}{\alpha} \{\mu_R(x,z),\beta[\mu_R(x,y),\mu_R(y,z)]\} \geq \mu_R(x,z)$$

because α is t-conorm.

$$\begin{aligned}\nu_{R \underset{\lambda,\rho}{\overset{\alpha,\beta}{\circ}} R}(x,z) &= \underset{y}{\lambda} \{\rho[\nu_R(x,y),\nu_R(y,z)]\} = \\ &= \underset{y\neq x}{\lambda} \{\rho[\nu_R(x,x),\nu_R(x,z)],\rho[\nu_R(x,y),\nu_R(y,z)]\} = \\ &= \underset{y\neq x}{\lambda} \{\rho[0,\nu_R(x,z)],\rho[\nu_R(x,y),\nu_R(y,z)]\} = \\ &= \underset{y\neq x}{\lambda} \{\nu_R(x,z),\rho[\nu_R(x,y),\nu_R(y,z)]\} \leq \nu_R(x,z)\end{aligned}$$

because λ is t-norm.

The proof of the item ii) is analogous to the one made in the item i). □

Next example states the existence of intuitionistic fuzzy relations which satisfy the property $R \leq R \underset{\lambda,\rho}{\overset{\alpha,\beta}{\circ}} R$ and they are not reflexive.

Let X be the following set $X = \{x,y,z\}$ and $R \in \text{IFR}(X\times X)$ given by

$$\mu_R = \begin{pmatrix} & x & y & z \\ x & 0.3 & 0.7 & 0.2 \\ y & 0.5 & 0.8 & 0.5 \\ z & 0.1 & 0.4 & 0.1 \end{pmatrix} \quad \nu_R = \begin{pmatrix} & x & y & z \\ x & 0.6 & 0.1 & 0.8 \\ y & 0.2 & 0 & 0.4 \\ z & 0.6 & 0.2 & 0.7 \end{pmatrix}.$$

For $\alpha = \vee$, $\beta = \wedge$, $\lambda = \wedge$ and $\rho = \vee$, we have

$$\mu_{R \underset{\wedge,\vee}{\overset{\vee,\wedge}{\circ}} R} = \begin{pmatrix} & x & y & z \\ x & 0.5 & 0.7 & 0.5 \\ y & 0.5 & 0.8 & 0.5 \\ z & 0.4 & 0.4 & 0.4 \end{pmatrix} \quad \nu_{R \underset{\wedge,\vee}{\overset{\vee,\wedge}{\circ}} R} = \begin{pmatrix} & x & y & z \\ x & 0.2 & 0.1 & 0.4 \\ y & 0.2 & 0 & 0.4 \\ z & 0.2 & 0.2 & 0.4 \end{pmatrix}$$

resulting that $R \leq R \underset{\wedge,\vee}{\overset{\vee,\wedge}{\circ}} R$ not being R reflexive.

Theorem 14 *If $R \in IFR(X\times X)$ is reflexive, α,β are t-conorms and λ,ρ are t-norms, then* *i)* $R \leq R \overset{\alpha,\beta}{\underset{\lambda,\rho}{\circ}} R$

ii) $R \leq R \overset{\alpha,\beta}{\underset{\lambda,\rho}{\circ}} R$ *is the binary ordinary relation* $X \times X$.

Proof. i)

$$\begin{aligned}
\mu_{R \overset{\alpha,\beta}{\underset{\lambda,\rho}{\circ}} R}(x,z) &= \underset{y \neq x}{\alpha} \{\beta[1, \mu_R(x,z)], \beta[\mu_R(x,y), \mu_R(y,z)]\} = \\
&= \underset{y \neq x}{\alpha} \{1, \beta[\mu_R(x,y), \mu_R(y,z)]\} = 1 \geq \mu_R(x,z). \\
\nu_{R \overset{\alpha,\beta}{\underset{\lambda,\rho}{\circ}} R}(x,z) &= \underset{y \neq x}{\alpha} \{\rho[0, \nu_R(x,z)], \rho[\nu_R(x,y), \nu_R(y,z)]\} = \\
&= \underset{y \neq x}{\lambda} \{0, \rho[\nu_R(x,y), \nu_R(y,z)]\} = 0 \leq \nu_R(x,z),
\end{aligned}$$

so that $\mu_{R \overset{\alpha,\beta}{\underset{\lambda,\rho}{\circ}} R}(x,z) \geq \mu_R(x,z)$ and $\nu_{R \overset{\alpha,\beta}{\underset{\lambda,\rho}{\circ}} R}(x,z) \leq \nu_R(x,z)$ $\forall (x,z) \in X \times X$, therefore, $R \leq R \overset{\alpha,\beta}{\underset{\lambda,\rho}{\circ}} R$.

The item ii) is consequence of the item i). □

Theorem 15 *Given* $R \in IFR(X \times Y)$, *for* α *t-conorm and* λ *t-norm, it is verified that*

i) If R *is reflexive, then* $R \overset{\alpha,\beta}{\underset{\lambda,\rho}{\circ}} R$ *is reflexive.*

ii) If R *is antireflexive, then* $R \overset{\lambda,\rho}{\underset{\alpha,\beta}{\circ}} R$ *is antireflexive.*

Proof.

$$\begin{aligned}
\mu_{R \overset{\alpha,\beta}{\underset{\lambda,\rho}{\circ}} R}(x,x) &= \underset{y}{\alpha} \{\beta[\mu_R(x,y), \mu_R(y,x)]\} = \\
&= \underset{y \neq x}{\alpha} \{\beta[\mu_R(x,x), \mu_R(x,x)], \beta[\mu_R(x,y), \mu_R(y,x)]\} = \\
&= \underset{y \neq x}{\alpha} \{\beta[1,1], \beta[\mu_R(x,y), \mu_R(y,x)]\} = \\
&= \underset{y \neq x}{\alpha} \{1, \beta[\mu_R(x,y), \mu_R(y,z)]\} = 1.
\end{aligned}$$

In the same way, we can prove that $\nu_{R \overset{\alpha,\beta}{\underset{\lambda,\rho}{\circ}} R}(x,x) = 0 \; \forall x \in X$.

ii) The proof of this property is similar to the one made for the reflexivity. $\square$

Corollary 1 *If $R \in IFR(X \times X)$ is reflexive, α is t-norm and λ is t-conorm, then*

$$R^{n)} = \overbrace{R \overset{\alpha,\beta}{\underset{\lambda,\rho}{\circ}} R \overset{\alpha,\beta}{\underset{\lambda,\rho}{\circ}} R \cdots \overset{\alpha,\beta}{\underset{\lambda,\rho}{\circ}} R}^{n-times}$$

with $n = 1, 2, \ldots,$ it is reflexive.

Theorem 16 *Let R_1 be a reflexive intuitionistic fuzzy relation in $X \times X$. Then*

i) $(R_1)^{-1}$ is reflexive.

ii) $R_1 \vee R_2$ is reflexive for every $R_2 \in IFR(X \times X)$

iii) $R_1 \wedge R_2$ is reflexive $\Leftrightarrow R_2 \in IFR(X \times X)$ is reflexive.

Proof. Just notice that

$$\begin{aligned}
\mu_{R_1 \vee R_2}(x,x) &= \mu_{R_1}(x,x) \vee \mu_{R_2}(x,x) = 1 \vee \mu_{R_2}(x,x) = 1 \\
\nu_{R_1 \vee R_2}(x,x) &= \nu_{R_1}(x,x) \wedge \nu_{R_2}(x,x) = 0 \wedge \mu_{R_2}(x,x) = 0 \\
\mu_{R_1 \wedge R_2}(x,x) &= \mu_{R_1}(x,x) \wedge \mu_{R_2}(x,x) = 1 \wedge \mu_{R_2}(x,x) = \mu_{R_2}(x,x) \\
\nu_{R_1 \wedge R_2}(x,x) &= \nu_{R_1}(x,x) \vee \nu_{R_2}(x,x) = 0 \vee \mu_{R_2}(x,x) = \nu_{R_2}(x,x).
\end{aligned}$$ $\square$

As an immediate consequence, the result is that $R \overset{\alpha,\beta}{\underset{\lambda,\rho}{\circ}} \Delta$ is reflexive for every $R \in$ IFR$(X \times X)$. This observation gives place to the next Definition.

Definition 6 *We will call reflexive closure of a relation $R \in IFR(X \times X)$ to $R \vee \Delta$.*

5.3 Symmetry and antisymmetry.

Definition 7 *1) A relation $R \in IFR(X \times X)$ is called symmetric if $R = R^{-1}$, that is, if for every (x, y) of $X \times X$*

$$\begin{cases} \mu_R(x,y) & = \mu_R(y,x) \\ \nu_R(x,y) & = \nu_R(y,x) \end{cases}$$

in a contrary manner we will say that it is asymmetric.

2) Let R be an element of $IFR(X \times X)$ we will say that it is antysymmetrical intuitionistic relation if

$$\forall (x,y) \in X \times X, \quad x \neq y \Rightarrow \begin{cases} \mu_R(x,y) & \neq \mu_R(y,x) \\ \nu_R(x,y) & \neq \nu_R(y,x) \\ \pi_R(x,y) & = \pi_R(y,x) \end{cases}$$

The definition of antysymmetrical intuitionistic relation is justified because of the following argument:

The relation

$$x \preceq_R y \text{ if and only if } \begin{cases} \mu_R(x,y) & \leq \mu_R(y,x) \\ \nu_R(x,y) & \geq \nu_R(y,x) \end{cases}$$

is an order in the referential X if the intuitionistic fuzzy relation $R \in$ IFR$(X \times X)$ is reflexive, transitive and intuitionistic antisymmetrical fuzzy ([8]). This fact does not take place if we take, instead of the intuitionistic antisymmetrical previous property, the definition of antisymmetrical fuzzy property given by A. Kaufmann ([12]).

Theorem 17 *Let R be an element of $FR(X \times X)$. R is antisymmetrical intuitionistic if and only if*

$$\forall (x,y) \text{ with } x \neq y \text{ then } \mu_R(x,y) \neq \mu_R(y,x).$$

Proof. As $\nu_R(x,y) = 1 - \mu_R(x,y)$ and $\pi_R(x,y) = 0$ for every $(x,y) \in X \times X$, then

$$\mu_R(x,y) \neq \mu_R(y,x) \text{ if and only if } \begin{cases} \mu_R(x,y) & \neq \mu_R(y,x) \\ \nu_R(x,y) & \neq \nu_R(y,x) \\ \pi_R(x,y) & = \pi_R(y,x) \end{cases}$$

□

Theorem 18 *If $\alpha, \beta, \lambda, \rho$ are whichever t-norms or t-conorms and $R, P \in IFR(X \times X)$ are symmetrical, then* $R \overset{\alpha,\beta}{\underset{\lambda,\rho}{\circ}} P = \left(P \overset{\alpha,\beta}{\underset{\lambda,\rho}{\circ}} R\right)^{-1}$.

Proof.

$$R = R^{-1},\ P = P^{-1} \text{ then } R \overset{\alpha,\beta}{\underset{\lambda,\rho}{\circ}} P = R^{-1} \overset{\alpha,\beta}{\underset{\lambda,\rho}{\circ}} P^{-1} = \left(P \overset{\alpha,\beta}{\underset{\lambda,\rho}{\circ}} R\right)^{-1} \square$$

If R is evidently symmetrical, then $R \overset{\alpha,\beta}{\underset{\lambda,\rho}{\circ}} R$ is symmetrical. It is evident that the composition of two symmetrical relations will not always be symmetrical.

Let's notice that if $R, P \in$ IFR$(X \times X)$ are symmetrical, it is verified that

$$\begin{aligned} \mu_{R \overset{\vee,\beta}{\underset{\wedge,\rho}{\circ}} P}(x,z) &= \mu_{P \overset{\alpha,\beta}{\underset{\lambda,\rho}{\circ}} R}(z,x) \\ \nu_{R \overset{\vee,\beta}{\underset{\wedge,\rho}{\circ}} P}(x,z) &= \nu_{P \overset{\alpha,\beta}{\underset{\lambda,\rho}{\circ}} R}(z,x) \end{aligned}$$

for every $(x,z) \in X \times X$.

Definition 8 *A relation $R \in IFR(X \times X)$ is called perfect antisymmetrical intuitionistic relation if for every $(x,y) \in X \times X$ with $x \neq y$ and*

$$\begin{cases} \mu_R(x,y) > 0 \\ \ \textit{or} \\ \mu_R(x,y) = 0 \ \textit{and} \ \nu_R(x,y) < 1, \end{cases}$$

then

$$\begin{cases} \mu_R(y,x) = 0 \\ \ \textit{and} \\ \nu_R(y,x) = 1. \end{cases}$$

An example of an intuitionistic antisymmetrical perfect relation is the following one:

$$\mu_R = \begin{pmatrix} & x & y & z \\ x & 0.4 & 0.3 & 0.1 \\ y & 0 & 0.5 & 0 \\ z & 0 & 0 & 0.1 \end{pmatrix} \quad \nu_R = \begin{pmatrix} & x & y & z \\ x & 0.5 & 0.7 & 0.4 \\ y & 1 & 0.3 & 0.6 \\ z & 1 & 1 & 0.7 \end{pmatrix}.$$

Notice that if $R \in FR(X \times X)$ is perfect antisymmetrical intuitionistic, then it is antisymmetrical perfect fuzzy, because

$$\forall (x,y) \text{ with } x \neq y \text{ and } \mu_R(x,y) > 0 \Rightarrow \begin{cases} \mu_R(y,x) = 0 \\ \text{and} \\ \nu_R(y,x) = 1, \end{cases}$$

because the possibility $\mu_R(x,y) = 0$ and $\nu_R(x,y) < 1$ can never appear in fuzzy sets.

5.4 Transitivity and c-transitivity.

Definition 9 *Let's take α t-conorm, β t-norm, λ t-norm and ρ t-conorm.*

1) We will say that $R \in IFR\ (X \times X)$ is transitive if $R \geq R \overset{\alpha,\beta}{\underset{\lambda,\rho}{\circ}} R$.

2) We will say that $R \in IFR\ (X \times X)$ is c-transitive if $R \leq R \overset{\lambda,\rho}{\underset{\alpha,\beta}{\circ}} R$.

Notice that not only the sign of inequality changes in the items 1) and 2), but also the order of α, β, λ and ρ.

Definition 10 *Let R be an element of $IFR(X \times X)$, α t-conorm, β t-norm, λ t-norm and ρ t-conorm.*

1) We will call transitive closure of R, to the minimum intuitionistic fuzzy relation $\hat{R}$ on $X \times X$ which contains R and it is transitive, that is to say

a) $R \leq \hat{R}$

b) $\hat{R} \overset{\alpha,\beta}{\underset{\lambda,\rho}{\circ}} \hat{R} \leq \hat{R}$

c) If $R, P \in IFR(X \times X)$, $R \leq P$ and P is transitive, then $\hat{R} \leq P$.

2) We will call c-transitive closure of R to the biggest c-transitive relation $\check{R} \in IFR(X \times X)$ contained in R.

Theorem 19 *For every $R \in IFR(X \times X)$, it is verified that:*
If $\alpha = \vee$, $\lambda = \rho$, $\beta = \wedge$ and $\rho = \vee$, then

i) $\hat{R} = R \vee R \underset{\wedge,\vee}{\overset{\vee,\wedge}{\circ}} R \vee R \underset{\wedge,\vee}{\overset{\vee,\wedge}{\circ}} R \underset{\wedge,\vee}{\overset{\vee,\wedge}{\circ}} R \vee \ldots \vee R^n$

ii) $\overset{\vee}{R} = R \wedge R \underset{\vee,\wedge}{\overset{\wedge,\vee}{\circ}} R \wedge R \underset{\vee,\wedge}{\overset{\wedge,\vee}{\circ}} R \underset{\vee,\wedge}{\overset{\wedge,\vee}{\circ}} R \wedge \ldots \wedge R^n$

begin $n = Cardinal(X)$.

Proof. i)

a) $R \leq \hat{R}$ is evident.

b) We will use now the distributive property of the composition respecting the union, $(R \vee S) \underset{\lambda,\rho}{\overset{\alpha,\beta}{\circ}} Q = \left(R \underset{\lambda,\rho}{\overset{\alpha,\beta}{\circ}} Q\right) \vee \left(S \underset{\lambda,\rho}{\overset{\alpha,\beta}{\circ}} Q\right) \Leftrightarrow \alpha = \vee$ and $\lambda = \wedge$, studied in the Theorem 7.

$$\begin{aligned} &\left(R \vee R \underset{\wedge,\vee}{\overset{\vee,\wedge}{\circ}} R \vee R \underset{\wedge,\vee}{\overset{\vee,\wedge}{\circ}} R \underset{\wedge,\vee}{\overset{\vee,\wedge}{\circ}} R \vee \ldots \vee R^n\right) \underset{\wedge,\vee}{\overset{\vee,\wedge}{\circ}} \\ &\left(R \vee R \underset{\wedge,\vee}{\overset{\vee,\wedge}{\circ}} R \vee R \underset{\wedge,\vee}{\overset{\vee,\wedge}{\circ}} R \underset{\wedge,\vee}{\overset{\vee,\wedge}{\circ}} R \vee \ldots \vee R^n\right) \underset{\wedge,\vee}{\overset{\vee,\wedge}{\circ}} = \\ &= R \underset{\wedge,\vee}{\overset{\vee,\wedge}{\circ}} \left(R \vee R \underset{\wedge,\vee}{\overset{\vee,\wedge}{\circ}} R \vee R \underset{\wedge,\vee}{\overset{\vee,\wedge}{\circ}} R \underset{\wedge,\vee}{\overset{\vee,\wedge}{\circ}} R \vee \ldots \vee R^n\right) \vee \\ &\vee R \underset{\wedge,\vee}{\overset{\vee,\wedge}{\circ}} R \underset{\wedge,\vee}{\overset{\vee,\wedge}{\circ}} \left(R \vee R \underset{\wedge,\vee}{\overset{\vee,\wedge}{\circ}} R \vee R \underset{\wedge,\vee}{\overset{\vee,\wedge}{\circ}} R \underset{\wedge,\vee}{\overset{\vee,\wedge}{\circ}} R \vee \ldots \vee \ldots\right) \ldots \leq \\ &\leq \left(R \vee R \underset{\wedge,\vee}{\overset{\vee,\wedge}{\circ}} R \vee R \underset{\wedge,\vee}{\overset{\vee,\wedge}{\circ}} R \underset{\wedge,\vee}{\overset{\vee,\wedge}{\circ}} R \vee \ldots \vee R^n\right). \end{aligned}$$

c) We will see now that is the minimum transitive relation which contains R; so we will use the following notation

$$R^2 = R \underset{\wedge,\vee}{\overset{\vee\wedge}{\circ}} R, \quad R^3 = R \underset{\wedge,\vee}{\overset{\vee\wedge}{\circ}} R \underset{\wedge,\vee}{\overset{\vee\wedge}{\circ}} R, \ldots$$

Let's take $R \leq P$, being P transitive, that is $P \underset{\wedge,\vee}{\overset{\vee\wedge}{\circ}} P \leq P$, because of the monotony of the composition (Theorem 5), we get

$$\begin{aligned} R &\leq P \\ R^2 &\leq P^2 \leq P \\ R^3 &\leq P \end{aligned}$$

therefore $\bigvee\limits_{n=1} R^{n)} = P \Rightarrow \hat{R} \leq P$, then $\hat{R} = R \vee R \overset{\vee\wedge}{\underset{\wedge,\vee}{\circ}} R \vee R \overset{\vee\wedge}{\underset{\wedge,\vee}{\circ}} R \overset{\vee\wedge}{\underset{\wedge,\vee}{\circ}} R \vee \ldots \vee R^n$

ii)

a) $R \geq \overset{\vee}{R}$ is evident.

b) In order to see that $\overset{\vee}{R} \overset{\wedge,\vee}{\underset{\vee,\wedge}{\circ}} \overset{\vee}{R} \geq \overset{\vee}{R}$ we will use the distributive property of the composition respecting the intersection (Theorem 9).

$$(R \wedge P) \overset{\alpha,\beta}{\underset{\lambda,\rho}{\circ}} Q = \left(R \overset{\alpha,\beta}{\underset{\lambda,\rho}{\circ}} Q\right) \wedge \left(P \overset{\alpha,\beta}{\underset{\lambda,\rho}{\circ}} Q\right) \Leftrightarrow \alpha = \wedge,\ \lambda = \vee.$$

$$\left(R \wedge R \overset{\wedge,\vee}{\underset{\vee,\wedge}{\circ}} R \wedge \overset{\wedge,\vee}{\underset{\vee,\wedge}{\circ}} R \overset{\wedge,\vee}{\underset{\vee,\wedge}{\circ}} R \wedge \ldots \wedge R^n\right) \overset{\wedge,\vee}{\underset{\vee,\wedge}{\circ}}$$
$$\left(R \wedge R \overset{\wedge,\vee}{\underset{\vee,\wedge}{\circ}} R \wedge \overset{\wedge,\vee}{\underset{\vee,\wedge}{\circ}} R \overset{\wedge,\vee}{\underset{\vee,\wedge}{\circ}} R \wedge \ldots \wedge R^n\right) =$$
$$R \overset{\wedge,\vee}{\underset{\vee,\wedge}{\circ}} \left(R \wedge R \overset{\wedge,\vee}{\underset{\vee,\wedge}{\circ}} R \wedge \overset{\wedge,\vee}{\underset{\vee,\wedge}{\circ}} R \overset{\wedge,\vee}{\underset{\vee,\wedge}{\circ}} R \wedge \ldots \wedge R^n\right) \wedge$$
$$\wedge \left(R \overset{\wedge,\vee}{\underset{\vee,\wedge}{\circ}} R \wedge \overset{\wedge,\vee}{\underset{\vee,\wedge}{\circ}} R\right) \wedge \ldots \wedge R^n =$$
$$= R \overset{\wedge,\vee}{\underset{\vee,\wedge}{\circ}} R \wedge R \overset{\wedge,\vee}{\underset{\vee,\wedge}{\circ}} R \overset{\wedge,\vee}{\underset{\vee,\wedge}{\circ}} R \wedge \ldots \wedge R^n \geq \overset{\vee}{R}.$$

c) Finally, we will see that is the biggest c-transitive relation contained in R. Let's take $P \leq R$, P c-transitive, that is $P \leq P \overset{\wedge,\vee}{\underset{\vee,\wedge}{\circ}} P$

$$\begin{array}{lcl} P & \leq & R \\ P & \leq & P \overset{\wedge,\vee}{\underset{\vee,\wedge}{\circ}} P \leq R \overset{\wedge,\vee}{\underset{\vee,\wedge}{\circ}} R \\ & \ldots & \end{array}$$

therefore, $P \leq \bigwedge\limits_{n=1} R^{n)} = \overset{\vee}{R} \leq R$. $\square$

Theorem 20 *Let R, P be two elements of $IFR(X \times X)$ and let's have $\alpha = \vee$, $\lambda = \wedge$, $\beta = \wedge$ and $\rho = \vee$. Then*

$$R \leq P \textit{ then } \hat{R} \leq \hat{P} \textit{ and } \overset{\vee}{R} \leq \overset{\vee}{P}.$$

Proof. It is similar to the ones made in the items c) of i) and ii) of the previous Theorem. $\square$

Corollary 2 *For every* $R \in IFR(X \times X)$, $\overset{\vee}{R} \leq r \leq \hat{R}$ *holds.*

Starting from the reflexivity and transitivity of the intuitionistic fuzzy relations, the following result can be established

Corollary 3 *For* α *t-conorm,* β *t-norm* λ *t-norm and* ρ *t-conorm, it is verified that*

i) If $R \in IFR(X \times X)$ *is reflexive and transitive, then* $R = R \overset{\alpha,\beta}{\underset{\lambda,\rho}{\circ}} R$.

ii) If $R \in IFR(X \times X)$ *is antireflexive and c-transitive, then* $R = R \overset{\lambda,\rho}{\underset{\alpha,\beta}{\circ}} R$.

From the analysis made about the intuitionistic fuzzy relations in a set, now we can extract the following conclusions:

A) The fulfilment of the properties $R \overset{\alpha,\beta}{\underset{\lambda,\rho}{\circ}} \Delta = \Delta \overset{\alpha,\beta}{\underset{\lambda,\rho}{\circ}} R = R$ y $R \overset{\lambda,\rho}{\underset{\alpha,\beta}{\circ}} \nabla = \nabla \overset{\lambda,\rho}{\underset{\alpha,\beta}{\circ}} R = R$ (Theorem 11), requires that β should be t-norm, ρ t-conorm and $\alpha = \vee$, $\lambda = \wedge$.

B) For every intuitionistic fuzzy relation R of IFR$(X \times X)$, we have proved in the Theorem 13 that if α is t-conorm, β, t-norm, λ t-norm and ρ t-conorm, it is verified that:

1) If R is reflexive, then $R \leq R \overset{\alpha,\beta}{\underset{\lambda,\rho}{\circ}} R$.

2) If R is antireflexive, then $R \geq R \overset{\lambda,\rho}{\underset{\alpha,\beta}{\circ}} R$.

C) In the Theorem 19, we have proved that if $\alpha = \vee$, $\lambda = \wedge$, $\beta = \wedge$ and $\rho = \vee$, then

1) $\hat{R} = R \vee R \overset{\vee,\wedge}{\underset{\wedge,\vee}{\circ}} R \vee R \overset{\vee,\wedge}{\underset{\wedge,\vee}{\circ}} R \overset{\vee,\wedge}{\underset{\wedge,\vee}{\circ}} R \vee \ldots \vee R^n$

2) $\overset{\vee}{R} = R \wedge R \overset{\wedge,\vee}{\underset{\vee,\wedge}{\circ}} R \wedge R \overset{\wedge,\vee}{\underset{\vee,\wedge}{\circ}} R \overset{\wedge,\vee}{\underset{\vee,\wedge}{\circ}} R \wedge \ldots \wedge R^n$.

The previous reasonings justify the fact that since this moment we are going to do the study of the composition of relations for the particular case $\alpha = \vee$, β t-norm, $\lambda = \wedge$ and ρ t-conorm. It is evidnet that $\vee$ and $\wedge$ satisfy the restriction of the Proposition 1, ($\alpha \leq \lambda^*$). It is also evident that the richest case in properties is $\alpha = \vee$, $\beta = \wedge$, $\lambda = \wedge$

and $\rho = \vee$. In spite of all these arguments, we will always indicate the choice of t-norms and t-conorms made.

Finally, we will insist on the fact that there is a difference between $R \overset{\vee,\beta}{\underset{\wedge,\rho}{\circ}} P$ and $R \overset{\wedge,\rho}{\underset{\vee,\beta}{\circ}} P$, because, although we will use the first one more often, the second one will be very important in the next part.

6 Relations partially included

Now we are going to present a new type of relations (partially included relations). The justification for the introduction of this new type of relations is the following one: Atanassov's operators do not generally keep the transitive property of the intuitionistic fuzzy relations. In the second part of this paper we will see that, if the intuitionistic fuzzy relation is transitive and partially included, in this case, Atanassov's operators keep the transitive property, and for this reason the study of these relations is justified.

Definition 11 *We will say that a relation $R \in IFR(X \times X)$ is partially included, if for every $x, y, z \in X$ with $\mu_R(x,z) \neq \mu_R(z,y)$,*

$$Sign\ (\mu_R(x,z) - \mu_R(z,y)) = Sign\ (\nu_R(z,y) - \nu_R(x,z)).$$

Just notice that htis condition is equal to

$$\begin{array}{c} if\ \mu_R(x,z) < \mu_R(z,y),\ then\ 1 - \nu_R(x,z) \leq 1 - \nu_R(z,y)\ or \\ if\ \mu_R(x,z) > \mu_R(z,y),\ then\ 1 - \nu_R(x,z) \geq 1 - \nu_R(z,y). \end{array}$$

Let's take $X = \{x, y, z\}$, the relation $R \in \text{IFR}(X \times X)$ given by

$$\mu_R = \begin{pmatrix} & x & y & z \\ x & 0.4 & 0.7 & 0.1 \\ y & 0.4 & 0 & 0.2 \\ z & 0.9 & 0.7 & 1 \end{pmatrix} \qquad \nu_R = \begin{pmatrix} & x & y & z \\ x & 0.3 & 0.1 & 0.7 \\ y & 0.3 & 0.4 & 0.4 \\ z & 0 & 0.1 & 0 \end{pmatrix}$$

is partially included.

The previous definition only affects to intuitionistic fuzzy relations, because it is verified by all the fuzzy relations, as we can prove it in the

Theorem 21 *If $R \in FR(X \times X)$, then is partially included.*

Proof. As $\nu_R(x,y) = 1 - \mu_R(x,y)$ for every $x, y \in X$, we have:

$$\begin{aligned} \text{Sign } (\nu_R(z,y) - \nu_R(x,z)) &= \text{Sign } (1 - \mu_R(z,y) - 1 + \mu_R(x,z)) = \\ &= \text{Sign } (\mu_R(x,z) - \mu_R(z,y)). \quad \square \end{aligned}$$

Theorem 22 *Let R be an element of IFR$(X \times X)$, if for every $(x,y) \in X \times X$ $\pi_R(x,y) = constant$, then R is partially included.*

Proof.

$$\begin{aligned} \pi_R(x,y) &= 1 - \mu_R(x,y) - \nu_R(x,y) = \text{constant, then} \\ \nu_R(x,y) &= 1 - \mu_R(x,y) - \text{constant.} \end{aligned}$$

If $\mu_R(x,y) > \mu_R(y,z)$, then $\nu_R(x,y) < \nu_R(y,z)$
If $\mu_R(x,y) < \mu_R(y,z)$, then $\nu_R(x,y) > \nu_R(y,z)$
for every $x, y, z \in X$. $\square$

Theorem 23 *Let R be an element of IFR$(X \times X)$ such that for every tern $x, y, z \in X$*

$$\textit{Sign } (\mu_R(x,y) - \mu_R(y,z)) = \textit{Sign } (\pi_R(x,y) - \pi_R(y,z)),$$

then R is partially included.

Proof. Supposing that R is not partially included, that is, a tern $x, y, z \in X$ exists in such a way that, for example, it satisfies

$$\begin{aligned} \mu_R(x,y) &> \mu_R(y,z) \\ \nu_R(x,y) &> \nu_R(y,z), \end{aligned}$$

it is verified in this case that $\mu_R(x,y) + \nu_R(x,y) > \mu_R(y,z) + \nu_R(y,z)$ and therefore

$$\pi_R(x,y) < \pi_R(y,z),$$

from which we have got for this term:

$$\text{Sign }\ (\mu_R(x,y) - \mu_R(y,z)) \neq \text{Sign }\ (\pi_R(x,y) - \pi_R(y,z))$$

which is in opposition to the hypothesis. □

Theorem 24 *If Cardinal*$(X) = 2$ *and* $R \in IFR(X \times X)$ *is reflexive and symmetric, then* R *is partially included..*

Proof. It is enough to apply the definition of partially included relations. □

7 Remark

The theory developped in this work is used in the paper Intuitionistic fuzzy relations, (Part II), where the effect of Atanassov's operators on the properties of intuitionistic fuzzy relations in a set is studied.

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