

Overtaker binary relations on complete completely distributive lattices related to the level sets of the L-fuzzy sets

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Abstract

The class of overtaker binary relations associated with the order in a lattice is defined and used to generalize the representations of L-fuzzy sets by means of level sets or fuzzy points.

Keywords: Distributive lattices, representation of L-fuzzy sets, fuzzy points, level sets.

Introduction

Thanks to the reflexive property of order, in ordered sets (L, \leq) every element α verifies $\alpha = \sup\{\beta / \beta \leq \alpha\}$. There are some relations that verify this property in spite of being proper subsets of \leq in $L \times L$. An example is the strict order $<$ associated with the order \leq in $L = [0, 1]$.

In the Fuzzy Set Theory there are theorems about the representation of fuzzy sets, such as Zadeh's level sets decomposition of a fuzzy set in level sets $\tilde{A}(x) = \sup\{\alpha \wedge A_\alpha(x) / \alpha \in L\}$ (See [6]). There are also results based on atoms or fuzzy points \tilde{x}_α (where $\tilde{x}_\alpha(x) = \alpha$, and $\tilde{x}_\alpha(y) = 0$, $y \neq x$), such as $\tilde{A} = \cup\{\tilde{x}_\alpha / \alpha < \tilde{A}(x)\}$.

In this work it is verified that in completely distributive lattices both kind of representation are particular cases of a more general one that is based on the family of overtaker relations.

The paper is organized in the following way. The first section introduces the concepts of additional and overtaker binary relations and some of their properties are analyzed. The second section takes into consideration some particular lattices that are interesting from the overtaker relations viewpoint. The last section shows the representation theorem.

1 Overtaker binary relations

Let $(L, \vee, \wedge, 0, 1)$ be a complete lattice with the induced order \leq .

Definition 1.1 *A relation $R \subseteq L \times L$ such that:*

$$A.1 \quad \alpha R \beta \Rightarrow \alpha \leq \beta.$$

$$A.2 \quad (\alpha \leq \beta, \gamma \leq \delta, \beta R \gamma) \Rightarrow \alpha R \delta.$$

$$A.3 \quad \forall \alpha \in L (0 R \alpha).$$

*is called **additional relation**. The set of additional relations is denoted $Ad(L)$.*

Every relation $R \in Ad(L)$ is antisymmetric and transitive and the relation R_1 equal to the order \leq on L is the maximum on $Ad(L)$, as it can be easily proved. Moreover, $R_0 \in Ad(L)$ given by: $\alpha R_0 \beta$ iff $\alpha = 0$ is the minimum.

Proposition 1.2 *If (\mathcal{M}, \leq) is the poset of maps m from L to $\mathcal{S}(L)$ such that $m(x) \subseteq \downarrow x \quad \forall x \in L$ (with the usual pointwise order given by $m \leq m'$ iff $m(x) \subseteq m'(x)$); $\downarrow x$ is the ideal $\{y/y \leq x\}$ of L ; $\mathcal{S}(L) \subseteq \mathcal{P}(L)$ is the set of semiideals in L and $I_M, I_{Ad(L)}$ are the identities in \mathcal{M} and in $Ad(L)$ then*

a) $(Ad(L), \subseteq, R_0, R_1)$ is a complete lattice.

b) The applications $\Phi : Ad(L) \Rightarrow \mathcal{M}$ and $\Psi : \mathcal{M} \Rightarrow Ad(L)$ defined

$$\begin{aligned} \Phi(R) &= m_R \text{ with } m_R(x) = \{y/y R x\} \subseteq \downarrow x \quad \forall x \in L \\ \Psi(m) &= \{(y, x)/y \in m(x)\} = R_m, \end{aligned}$$

are isomorphisms, that is $\Phi \circ \Psi = I_M, \Psi \circ \Phi = I_{Ad(L)}$.

Proof. It follows directly from definitions. ■

Observations.

- 1) This proposition shows that (\mathcal{M}, \leq) is a complete lattice with the sup and inf operations defined $\xi(x) = \cup(m(x)/m \in \mathcal{H}) \in \mathcal{S}(L)$ and $\chi(x) = \cap(m(x)/m \in \mathcal{H}) \in \mathcal{S}(L) \forall x \in L$, where $\mathcal{H} \subseteq \mathcal{M}$, $\xi = \sup \mathcal{H}$ and $\chi = \inf \mathcal{H}$.
- 2) As a consequence of proposition 1.2, asociated to the zero R_0 and the unit R_1 relations in $\text{Ad}(L)$ we have the maps m_{R_0} and m_{R_1}

$$\begin{aligned} m_{R_0}(x) &= \{y/yR_0x\} = \{0\} \quad \forall x \in L. \\ m_{R_1}(x) &= \{y/y \leq x\} = \downarrow x \quad \forall x \in L. \end{aligned}$$

We consider now the relations that are associated with the class of maps $(m_s)_{S \in \mathcal{S}(L)}$ from L to $\mathcal{S}(L)$, that is, those defined by

$$m_s(x) = \begin{cases} x \wedge S = \downarrow x \cap S & \text{if } x \leq \sup S \\ \downarrow x & \text{otherwise} \end{cases} \quad \forall S \in \mathcal{S}(L) \quad (1)$$

where $x \wedge S = \{x \wedge y/y \in S\}$. We have:

Proposition 1.3 *Let $(m_s)_{S \in \mathcal{S}(L)}$ be the family of functions $m_s : L \rightarrow \mathcal{S}(L)$ defined in (1) and let $\mu = \inf(m_s/S \in \mathcal{S}(L))$, then*

- (i) $m_s \in \mathcal{M} \quad \forall S \in \mathcal{S}(L)$.
- (ii) $\mu(x) = \cap(S \in \mathcal{S}(L)/x \leq \sup S) \quad \forall x \in L$.
- (iii) $\mu = \inf(m_s/S \in \mathcal{S}(L))$ belongs to \mathcal{M} .

Proof. Direct from definitions. ■

With the functions m_s and μ are associated the additional relations R_{m_s} and R_μ through:

$$yR_{m_s}x \Leftrightarrow (y \in \downarrow x \cap S \text{ if } x \leq \sup S \text{ or } y \in \downarrow x \text{ otherwise}).$$

$$yR_\mu x \Leftrightarrow y \in \cap(S \in \mathcal{S}(L)/x \leq \sup S).$$

Next proposition provides alternative determinations for R_μ .

Proposition 1.4 *In a complete lattice L the following statements are equivalent:*

(i) $xR_\mu y$

(ii) $(S \text{ semiideal and } y \leq \sup S) \Rightarrow x \in S.$

(iii) $(\emptyset \neq N \subseteq L \text{ and } y \leq \sup N) \Rightarrow x \leq n \text{ for some } n \in N.$

Proof. (i) \Rightarrow (ii).

Suppose S semiideal and $y \leq \sup S$. Then

$$xR_\mu y \Rightarrow x \in \mu(y) = \cap(S/y \leq \sup S) \Rightarrow x \in S.$$

(ii) \Rightarrow (iii).

For $N \subseteq L$, we can consider the semi ideal $\downarrow N = \{z/\exists x \in N, z \leq x\}$. According to ref [1] $\sup N = \sup \downarrow N$. If $S = \downarrow N$ and $y \leq \sup S = \sup \downarrow N$ then $x \in \downarrow N$ and therefore there are some $n \in N$ verifying $x \leq n$.

(iii) \Rightarrow (i).

If $S \in \mathcal{S}(L)$ is a semiideal and $y \leq \sup S$ then for some $m_s \in S$ $x \leq m_s$. As S is semiideal, it follows that x is $\in S$, therefore x is in $\cap(S/y \leq \sup S) = \mu(y)$; hence $xR_\mu y$. ■

Definition 1.5 *A relation $R \in \text{Ad}(L)$ is a **overtaker relation** on L , if*

$$\forall \alpha (\alpha = \sup\{\beta/\beta R \alpha\}). \quad (2)$$

Let $\text{Ov}(L)$ be the set of overtaker relations; for $|L| \geq 2$ $\text{Ov}(L)$ is a proper subset of $\text{Ad}(L)$.

Some examples.

- 1) The order relation \leq of the lattice L belongs to $\text{Ad}(L)$ and verifies (3), so it is an overtaker one.
- 2) On the contrary, the relation $\alpha \neq 0 \Rightarrow \sup\{\beta/\beta R_0 \alpha\} < \alpha$ it is not.

Let $(R_{m_s})_{S \in \mathcal{S}(L)}$ be the family of binary relation associated with the family $(m_s)_{S \in \mathcal{S}(L)}$, that is $y R_{m_s} x$ iff $y \in m_s(x)$ and let $R_\mu \in \text{Ad}(L)$ be the relation associated with $\mu = \inf(m_s)_{S \in \mathcal{S}(L)} \in \mathcal{M}$; under these assumptions

Proposition 1.6 *In the case of a lattice L being Brouwerian and complete:*

- (i) $\forall x(x = \sup\{y/y R_{m_s} x\})$, that is, R_{m_s} is an overtaker relation for every s .
- (ii) The relation $R_\mu \in \text{Ad}(L)$ verifies $R_\mu = \inf_{\text{Ad}(L)} (\text{Ov}(L))$.

Proof. (i) As $m_s \in \mathcal{M} \forall S \in \mathcal{S}(L)$, R_{m_s} is an additional relation. Moreover, it is an overtaker one:

As

$$m_s(x) = \begin{cases} x \wedge S = \downarrow x \cap S & \text{if } x \leq \sup S \\ \downarrow x & \text{otherwise} \end{cases}$$

we have $x = \sup \downarrow x = \sup m_s(x)$ for $m_s(x) = \downarrow x$. In other case, when $x \leq \sup S$, then $m_s(x) = x \wedge S$ so as the lattice L is Brouwerian $\sup m_s(x) = \sup(x \wedge S) = x \wedge \sup S = x$.

(ii) Let $m_0 \in \mathcal{M}$, where \mathcal{M} is the set defined as in 1.2(b), then $m_0(x) = \cap\{m_R(x)/R \in \text{Ov}(L)\}$. As $R_\mu = \cap\{R_{m_s}/S \in \mathcal{S}(L)\}$ and $R_{m_s} \in \mathcal{M}$, we have $R_\mu \supseteq R_m$. In the reverse, for R in $\text{Ov}(L)$ we have that for all $y = \sup m_R(y)$, and, being $m_R(y)$ a semiideal (due to A.2 and 1.4(ii)), then $(x, y) \in R_\mu$ implies $x \in m_R(y)$, that is $(x, y) \in R$; as this occurs for every R in $\text{Ov}(L)$, $(x, y) \in R_{m_0}$, so $R_\mu \subseteq R_{m_0}$. ■

Note that μ is the infimum of $\text{Ov}(L)$ in $\text{Ad}(L)$, but not necessarily the minimum and therefore it is possible that $\mu \notin \text{Ov}(L)$, that is $\exists x(\sup\{y/y R_\mu x\} < x)$.

In a different context, the authors Balbes and Dwinger analyze the map μ and characterize the complete and completely distributive lattices with the following theorem

Theorem 1.7 ([1]) *If L is a complete lattice*

- (i) $\forall P(\mu(\sup P) = \cup(\mu(x) / x \in P))$.

(ii) The complete lattice L is completely distributive iff

$$\forall x(x = \sup \mu(x)).$$

(iii) If L is completely distributive

$$\forall x(\mu(x) = \bigcup_{y \in \mu(x)} \mu(y)).$$

We are concerned with the following direct consequences of 1.7

Corollary 1.8 (i) $\mu \in \text{Ov}(L) \Leftrightarrow L$ is completely distributive.

(ii) For every complete lattice L the following are equivalent:

(a) $P \neq \emptyset$ and $xR_\mu \sup P$.

(b) $\exists m \in P : x \leq m$.

In lattice theory, a point $x \in L$ is called completely join-irreducible ([1]) iff $(x \leq \sup P \Rightarrow \exists m \in P : x \leq m)$. As a consequence of 1.8, the set $\mathcal{K} = \{(x, x) / x \in L, x \text{ is completely join-irreducible}\}$ included in $L \times L$ is a subset of R_μ , so in terms of R_μ the complete join-irreducibility of x is equivalent to $xR_\mu x$.

Corollary 1.9 (a) If L is a completely distributive lattice, then μ is the minimum of $\text{v}(L)$ in $\text{Ad}(L)$, i.e., $R_\mu \in \text{Ov}(L)$ and $(R \in \text{Ov}(L) \Rightarrow R_\mu \subseteq R)$.

(b) L is completely distributive lattice iff $\forall x(x = \sup\{y / y \in R_\mu x\})$.

Proof. (a) follows from proposition 1.6 and theorem 1.7 (ii). (b) is a consequence of (a). ■

The relation R_μ is a transitive one, that is $R_\mu \circ R_\mu \subseteq R_\mu$. For equality we have the following:

Proposition 1.10 If L is complete and completely distributive then

$$(i) R_\mu \circ R_\mu = R_\mu.$$

$$(ii) xR_\mu \sup P \Leftrightarrow R_\mu m \text{ for some } m \in P.$$

Proof. (i) It is sufficient to prove that $R_\mu \subseteq R_\mu \circ R_\mu$. From Corollary 1.9: $xR_\mu y \Leftrightarrow xR_\mu \sup(z/zR_\mu y)$ and from corollary 1.8 it follows that for some z , $xR_\mu zR_\mu y$.

(ii) If $xR_\mu(\sup P)$, then according to (i), $xR_\mu \circ R_\mu(\sup P)$ and in consequence there is some z such that $xR_\mu zR_\mu(\sup P)$, and 1.8 (b) guarantees the existence of some $m \in P$ such that $xR_\mu z \leq m$, so $xR_\mu m$. ■

2 Aspects of relation R_μ in special complete lattices

The binary relation R_μ depends on the order \leq in the lattice L . We look now for characterizations of R_μ in some types of lattices.

2.1 The relation R_μ in complete chains

Chains are lattices L where all elements are comparable under \leq . A totally ordered and complete lattice is a complete chain; complete chains are also completely distributive lattices. There we have the following

Proposition 2.1.1 *For $<$ the strict order binary relation in a complete chain L and $\mathcal{K} = \{(x, x) \mid x \in L, x \text{ is completely join-irreducible}\}$, we have*

$$R_\mu = \mathcal{K} \cup <$$

Proof. As $R_\mu \subseteq \leq$, $xR_\mu y$ implies $x < y$ or $x = y$, so $(x, y) \in < \cup \mathcal{K}$.

This is in the other sense for $(x, y) \in < \cup \mathcal{K}$, $(x, y) \in <$ or $(x, y) \in \mathcal{K}$; in the first case $x < y \leq \sup N$, and as L is a chain there exists some $n \in N$ such that $x \leq n$ and according with 1.4(iii), $xR_\mu y$; in the second case $(x, y) \in \mathcal{K} \subseteq R_\mu$. ■

Consequences

- 1) In complete chains, R_μ is the “way below” relation \ll (defined in [2] and related with Fuzzy Set membership in [7]).
- 2) For finite chains L , $x \in L \Rightarrow x$ is completely join-irreducible and consequently $R_\mu = \leq$.
- 3) For $L = [0, 1] \subseteq \mathbb{R}$ the only completely join-irreducible element is 0, so we have

$$xR_\mu y \Leftrightarrow x < y \text{ or } x = y = 0.$$

2.2 The relation R_μ in product lattices

If $(L_x)_{x \in X}$ is a family of complete and completely distributive lattices, the product

$$\mathcal{L} = \times(L_x / x \in X) = \{\tilde{A} : X \rightarrow \bigcup_{x \in X} L_x / \forall x(\tilde{A}(x) \in L_x)\},$$

with the usual pointwise definitions of $\text{SUP}_{i \in I} \tilde{A}_i$ and $\text{INF}_{i \in I} \tilde{A}_i$:

$$\begin{aligned} (\text{SUP}_{i \in I} \tilde{A}_i)(x) &= \sup\{\tilde{A}_i(x) / i \in I\} \forall x \in X. \\ (\text{INF}_{i \in I} \tilde{A}_i)(x) &= \inf\{\tilde{A}_i(x) / i \in I\} \forall x \in X. \end{aligned}$$

is a complete and completely distributive lattice (See [1]).

Let (L_x, \leq_x) and (\mathcal{L}, \leq) be the posets associated with the complete lattices L_x and \mathcal{L} respectively; 0_x and $\tilde{0}$ the zeros in L_x and \mathcal{L} ; μ_x and μ the relations associated (as in 1.3) with L_x and \mathcal{L} ; $(R_{\mu_x})_{x \in X}$ the family of relations associated (as in 1.4(iii)) with $(L_x)_{x \in X}$. Finally, let R_μ be the analogous relation associated with product \mathcal{L} . We shall now consider the mutual relationships between $(R_{\mu_x})_{x \in X}$ and R_μ .

Definition 2.2.1 *Let $\tilde{A} \in \mathcal{L}$. The support [6] of \tilde{A} is*

$$\text{supp}(\tilde{A}) = \{x \in X / \tilde{A}(x) \neq 0_x\} \subseteq X.$$

Proposition 2.2.2 $\text{supp}(\text{SUP}_{i \in I} \tilde{A}_i) = \bigcup_{i \in I} \text{supp}((\tilde{A}_i))$ for every family $(\tilde{A}_i)_{i \in I}$ in \mathcal{L} .

Theorem 2.2.3 Let \mathcal{L} be the product lattice of the family $(L_x)_{x \in X}$ of complete lattices. The following conditions are equivalent:

- (i) $\tilde{0} \neq \tilde{A}R_\mu \tilde{B}$.
- (ii) $\text{supp}(A)$ is a singleton $\{x\}$ and $\tilde{A}(x)R_{\mu_x} \tilde{B}(x)$.

Proof. (i) \Rightarrow (ii): Suppose $\tilde{A} \in \mathcal{L}$ and $\tilde{A} \neq \tilde{0}$. If $\tilde{A}R_\mu \tilde{B}$, then from 1.4(iii) we have $\emptyset \neq \mathcal{H} \subseteq \mathcal{L}$ and $\tilde{B} \leq \text{SUP } \mathcal{H} \Rightarrow \exists \tilde{C} \in \mathcal{H} : \tilde{A} \leq \tilde{C}$.

We now consider the maps $\tilde{x}_\alpha : X \longrightarrow \mathcal{L}$ given by

$$\begin{cases} \tilde{x}_\alpha(x) = \alpha \in L_x. \\ \tilde{x}_\alpha(y) = 0_y \in L_y \quad \forall y \neq x. \end{cases} \quad (5)$$

defined for $x \in \text{supp}(B)$, $\alpha \in L_x$ and $0_x \neq \alpha \leq_x \tilde{B}(x)$.

The subset $\mathcal{H}_B = \{\tilde{x}_\alpha \mid x \in \text{supp}(B), \alpha \in L_x, 0_x \neq \alpha \leq_x \tilde{B}(x)\}$ is non empty and it is included in \mathcal{L} ; $\tilde{B} = \text{SUP } \mathcal{H}_B$, so there is an $\tilde{x}_\alpha \in \mathcal{H}_B$ such that $\tilde{A} \leq \tilde{x}_\alpha$. Thus, $\emptyset \neq \text{supp}(\tilde{A}) \subseteq \text{supp}(\tilde{x}_\alpha) = \{x\}$, so $\text{supp}(\tilde{A})$ is the singleton $\{x\}$.

In order to show $\tilde{A}(x)R_{\mu_x} \tilde{B}(x)$, we now consider a nonvoid subset H of L_x verifying $\tilde{B}_x \leq_x \text{sup } H$. It is sufficient to proving the existence of some h in H such that $\tilde{A}(x) \leq_x h$. Let $\mathcal{H} \subseteq \mathcal{L}$ be the subset of maps $\tilde{h} : X \longrightarrow \mathcal{L}$ associated with \tilde{B} as well as to H under the definitions

$$\begin{cases} \tilde{h}(x) = h \in H \subseteq L_x \\ \tilde{h}(y) = \tilde{B}(y) \in L_y \quad \text{if } y \neq x. \end{cases}$$

For these maps, when $y \neq x$ we have $(\text{SUP } \mathcal{H})(y) = \text{sup}\{\tilde{h}(y) \mid h \in H\} = \text{sup}\{\tilde{B}(y)\} = \tilde{B}(y)$, and $(\text{SUP } \mathcal{H})(x) = \text{sup}\{\tilde{h}(x) \mid h \in H\} = \text{sup}\{h \mid h \in H\} = \text{sup } H \geq_x \tilde{B}(x)$. So $\tilde{B} \leq \text{SUP } \mathcal{H}$. From 1.4(iii) follows the existence of a map $\tilde{h} \in \mathcal{H}$ such that $\tilde{A} \leq \tilde{h}$, so $\tilde{A}(x) \leq_x \tilde{h}(x) = h$.

In order to see (ii) \Rightarrow (i), let we suppose that $\text{supp}(\tilde{A}) = \{x\}$ and $\tilde{A}(x)R_{\mu_x} \tilde{B}(x)$. Let \mathcal{F} be such that $\tilde{B} \leq \text{SUP } \mathcal{F}$. Then $\tilde{B}(x) \leq_x (\text{SUP } \mathcal{F})(x) =$

$\sup\{\tilde{C}(x) / \tilde{C} \in \mathcal{F}\}$. By hypothesis, there is some $\tilde{C} \in \mathcal{F}$ that verifies $\tilde{A}(x) \leq_x \tilde{C}(x)$. This inequality and the obvious $\tilde{A}(y) = 0_y \leq_y \tilde{C}(y) \forall y \neq x$ show that $\tilde{A} \leq \tilde{C}$ and therefore $\tilde{A}R_\mu \tilde{B}$. ■

Corollary 2.2.4 *If $(L_x)_{x \in X}$ is a family of completely distributive lattices and \mathcal{L} is the completely distributive product $\mathcal{L} = \mathfrak{X}(L_x / x \in X)$, then*

$$\forall \tilde{B} \in \mathcal{L} (\tilde{B} = \text{SUP}\{\tilde{x}_\alpha / x \in \text{supp}(B), \alpha R_{\mu_x} \tilde{B}(x)\}).$$

Proof. From 1.7 and 2.2.3 we deduce

$$\begin{aligned} \tilde{B} &= \text{SUP}\{\tilde{A} / \tilde{A}R_\mu \tilde{B}\} = \text{SUP}\{\tilde{A} / (\tilde{A} = \tilde{0}) \text{ or } (\tilde{0} \neq \tilde{A}R_\mu \tilde{B})\} = \\ &= \text{SUP}\{\tilde{A} / (\tilde{A} = \tilde{0}) \text{ or } (\tilde{A} = \tilde{x}_\alpha \text{ and } \alpha R_{\mu_x} \tilde{B}(x))\} \end{aligned}$$

and obviously we can leave out the map $\tilde{0}$ in the calculation of SUP. ■

Let be remarked that 2.2.4 gives us a representation of the maps $\tilde{B} \in \mathcal{L} = \mathfrak{X}(L_x / x \in X)$ by means of those $\tilde{x}_\alpha \in \mathfrak{X}(L_x / x \in X)$ defined by (5).

3 The relation R_μ and the L-fuzzy and fuzzy sets

The previous concepts and results can be applied to characterize some notions of Fuzzy Set Theory.

a) Characterization of fuzzy belonging of fuzzy points

In the case $L_x = L \forall x \in X$, \mathcal{L} is $\{\tilde{A} : X \rightarrow L\} = L^X$, that is the set of **L-fuzzy subsets of X** as defined in ref.[3]. In particular, when L is the chain $[0,1]$ \mathcal{L} comes of to be the set of **fuzzy sets of X** in Zadeh's classical sense of ref.[4]. According to 2.2.3 and 2.2.4 and the third consequence from 2.1.1, we can say for every pair \tilde{A} and \tilde{B} of fuzzy sets of X that

$$(\tilde{0} \neq \tilde{A}R_\mu \tilde{B}) \Leftrightarrow (\tilde{A} = \tilde{x}_\alpha \text{ and } \alpha < \tilde{B}(x)).$$

Let be remarked that the second member of the equivalence is exactly the definition of **fuzzy belonging of fuzzy-point** \tilde{x}_α , (see ref [8] and [9]), to the

fuzzy set \tilde{B} according to ref. [5]. Thus, fuzzy belonging is just a particular case of R_μ , and we have

$$\tilde{B} = \text{SUP} \{ \tilde{x}_\alpha / x \in \text{supp}(B), \alpha < \tilde{B}(x) \}.$$

for every fuzzy set $\tilde{B} \in [0, 1]^X$, that is, the representation of \tilde{B} through **fuzzy-points** \tilde{x}_α that appears in ref ([5]).

b) Level Sets

In the fuzzy sets theory, the fuzzy subsets \tilde{B} are often determined through the synthesis (see Zadeh [4]) of their families $(B_\alpha)_{\alpha \in [0,1]}$ of **strict level sets** $(B_\alpha) = \{x \in X / \alpha < \tilde{B}(x)\} \subseteq X \forall \alpha \in [0, 1]$.

Definition 3.1 Let $\tilde{B} \in L^X$. The μ -level sets of \tilde{B} are the ordinary subsets

$$B_\alpha^* = \{x \in X / \alpha R_\mu \tilde{B}(x)\} \subseteq X$$

for $\alpha \in L$. The family $(B_\alpha^*)_{\alpha \in L}$ verifies

Proposition 3.2 If \tilde{A} , \tilde{A}_i and \tilde{B} are **L-fuzzy sets** and $\alpha \in L$, $\beta \in L$ then

$$(i) \tilde{A} \leq \tilde{B} \Rightarrow A_\alpha^* \subseteq B_\alpha^* \quad \forall \alpha \in L.$$

$$(ii) \alpha \leq \beta \Rightarrow A_\beta^* \subseteq A_\alpha^* \quad \forall \tilde{A} \in L^X.$$

(iii) $\bigcup_{i \in I} (A_i)_\alpha^* \subseteq (\text{SUP}_{i \in I} \tilde{A}_i)_\alpha^* \forall \alpha \in L$, and if L is completely distributive lattice:

$$\bigcup_{i \in I} (A_i)_\alpha^* = (\text{SUP}_{i \in I} \tilde{A}_i)_\alpha^* \quad \forall \alpha \in L.$$

$$(iv) (\text{INF}_{i \in I} \tilde{A}_i)_\alpha^* \subseteq \bigcap_{i \in I} (A_i)_\alpha^* \quad \forall \alpha \in L. \blacksquare$$

Proof. (i) For $x \in A_\alpha^*$, $\alpha R_\mu \tilde{A}(x) \leq \tilde{B}(x)$, so $\alpha R_\mu \tilde{B}(x)$ and $x \in B_\alpha^*$.

(ii) For $x \in A_\beta^*$, we have $\beta R_\mu \tilde{A}(x)$ and then $\alpha \leq \beta R_\mu \tilde{A}(x)$. This implies $\alpha R_\mu \tilde{A}(x)$ and $x \in A_\alpha^*$.

(iii) As $\tilde{A}_j \leq (\text{SUP}_{i \in I} \tilde{A}_i)$ for each j and according the first part

$$(A_j)_\alpha^* \subseteq (\text{SUP}_{i \in I} \tilde{A}_i)_\alpha^* \text{ for each } j, \text{ so } \bigcup_{j \in I} (A_j)_\alpha^* \subseteq (\text{SUP}_{i \in I} \tilde{A}_i)_\alpha^*.$$

In the case of L completely distributive, $x \in (\text{SUP}_{i \in I} \tilde{A}_i)_\alpha^*$ implies $\alpha R_\mu(\text{SUP}_{i \in I}(x))$, that is $\alpha R_\mu \sup_{i \in I} \tilde{A}_i(x)$ which together with proposition 1.10 (ii), guarantees $\alpha R_\mu \tilde{A}_i(x)$ for some $i \in I$, that is x belongs to $\bigcup_{j \in I} (A_j)_\alpha^*$.

(iv) The proof is similar to that of the first part of (iii). ■

Proposition 3.3 *Let L be completely distributive, $\tilde{A} \in L^X$ a L -fuzzy set and $A_\alpha^*(x)$ the membership function of ordinary set A_α^* . Then; for every $x \in X$*

$$\{\beta / \beta R_\mu \tilde{A}(x)\} = \{\alpha \wedge A_\alpha^*(x) / \alpha \in L\}.$$

Proof. Let β be an element of the left set, then $x \in A_\beta^*$, $A_\beta^*(x) = 1$ and $\beta = \beta \wedge 1 = \beta \wedge A_\beta^*(x)$, so β belongs to the right set.

Conversely every β of the right set either equals or does not 0. If $\beta = 0$, it trivially satisfies $0 \in \{\beta / \beta R_\mu \tilde{A}(x)\}$. If $\beta \neq 0$ then $\beta = \alpha \wedge A_\alpha^*(x)$, where $\alpha \neq 0$ and $0 \neq A_\alpha^*(x) = 1$ and $\alpha = \beta$, that is $\beta R_\mu \tilde{A}(x)$, thus implying that β belongs to the right set. ■

Corollary 3.4 *(Representation of L -fuzzy sets). For every $\tilde{A} \in L^X$, $\tilde{A} = \text{SUP}\{\alpha \wedge A_\alpha^* / \alpha \in L\}$.*

Proof. Just by Zadeh's decomposition and 3.3. ■

Conclusions and future works

The need for representation theorems of fuzzy sets concerning the possibility of building fuzzy sets from families of ordinary sets, has been considered in the literature of the field since its inception (see [11], [12] and [13]).

Our representation is relevant to L -fuzzy sets when L is a product lattice. There it brings advantage over the usual one because only the “coefficients” α placed over the axis need to be accounted for, as shows Th. 2.2.3.

We have in mind to continue working on extending the concepts of fuzzy membership from this point of view. This can be interesting for fuzzy topology.

Acknowledgements

The authors are grateful to the editor and to the reviewer of the first version of the paper for his criticism and comments.

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