Convergent and asymptotic expansions of the Pearcey integral

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Abstract

We consider the Pearcey integral $P(x,y)$ for large values of $|x|$, x, $y \in \mathbb{C}$. We can find in the literature several convergent or asymptotic expansions in terms of elementary and special functions, with different levels of complexity. Most of them are based in analytic, in particular asymptotic, techniques applied to the integral definition of $P(x,y)$. In this paper we consider a different method: the iterative technique used for differential equations in [Lopez, 2012]. Using this technique in a differential equation satisfied by $P(x,y)$ we obtain a new convergent expansion analytically simple that is valid for any complex x and y and has an asymptotic property when $|x| \to \infty$ uniformly for y in bounded sets. The accuracy of the approximation is illustrated with some numerical experiments and compared with other expansions given in the literature.

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1 Introduction

The mathematical models of many short wavelength phenomena, especially wave propagation and optical diffraction, contain, as a basic ingredient, oscillatory integrals with several nearly coincident stationary phase or saddle points. The uniform approximation of those integrals can be expressed in terms of certain canonical integrals and their derivatives [3], [13]. The importance of these canonical diffraction integrals is stressed in [9] by means of a very appropriate sentence: *The role played by these canonical diffraction integrals in the analysis of caustic wave fields is analogous to that played by complex exponentials in plane wave theory.*

Apart from their mathematical importance in the uniform asymptotic approximation of oscillatory integrals [7], the canonical diffraction integrals have physical applications in the description of surface gravity waves [6], [14], bifurcation sets, optics, quantum mechanics and acoustics (see [2, Sec. 36.14] and references there in).

In [2, Chap. 36] we can find a large amount of information about these integrals. First of all, they are classified according to the number of free independent parameters that describe the type of singularities arising in catastrophe theory, that also corresponds to the number of saddle points of the integral. The simplest integral with only one free parameter, that corresponding to the fold catastrophe, involves two coalescing stationary points: the well-known integral representation of the Airy function. The second one, depending on two free parameters corresponds to the cusp catastrophe and involves three coalescing stationary points. The canonical form of the oscillatory integral describing the cusp diffraction catastrophe is given by the cusp catastrophe or Pearcey integral [2, p.777, eq. 36.2.14]:

$$\bar{P}(x,y) := \int_{-\infty}^{\infty} e^{i(t^4+xt^2+yt)}dt. \tag{1}$$

This integral was first evaluated numerically by using quadrature formulas in [11] in the context of the investigation of the electromagnetic field near a cusp. The third integral of the hierarchy is the Swallowtail integral that depends on three free parameters and involves four coalescing stationary points. Apart from the classification of this family of integrals, in [2, Chap. 36] we can find many properties such as symmetries, illustrative pictures, bifurcation sets, scaling relations, zeros, convergent series expansions, differential equations and leading-order asymptotic approximations among others. But we cannot find many details about analytic approximation formulas, asymptotic expansions in particular.

The three first canonical integrals: Airy function, Pearcey integral and Swallowtail integral are the most important ones in applications. The first one is well-known and deeply investigated. In this paper we focus our attention in the second one and will consider the third one in a further research. The integral (1) exists only for $0 \le \arg x \le \pi$ and real y. As it is indicated in [9], after a rotation of the integration path through an angle of $\pi/8$ that removes the rapidly oscillatory term e^{it^4}, the Pearcey integral may be written in the form $\bar{P}(x,y) = 2e^{i\pi/8}P(xe^{-i\pi/4}, ye^{i\pi/8})$, with

$$P(x,y) := \int_0^{\infty} e^{-t^4-xt^2}\cos(yt)dt. \tag{2}$$

This integral is absolutely convergent for all complex values of x and y and represents the analytic continuation of the Pearcey integral $\bar{P}(x,y)$ to all complex values of x and y [9]. Therefore, it is more convenient to work with the representation (2) of the Pearcey integral.

We are interested in simple representations of $P(x,y)$ in terms of elementary or special functions that eventually may be used for its numerical evaluation. A convergent series expansion may be found in [2, p. 787, eqs. 36.8.1 and 36.8.2]:

$$P(x,y) = \frac{1}{4}\sum_{n=0}^{\infty}(-1)^n\Gamma\left(\frac{2n+1}{4}\right)a_{2n}(x,y), \tag{3}$$

with $a_0(x,y) = 1$, $a_1(x,y) = y$ and, for $n = 2, 3, 4, ...$,

$$a_n(x,y) = \frac{1}{n}[y\, a_{n-1}(x,y) + 2x\, a_{n-2}(x,y)].$$

Another convergent expansion of $P(x,y)$ may be obtained after an expansion of the cosine term in (2) and interchange of sum and integral [10]:

$$P(x,y) = \sum_{n=0}^{\infty}\frac{(-y^2)^n}{(2n)!}P_n(x), \qquad (x,y)\in\mathbb{C}^2, \tag{4}$$

with

$$P_n(x) := \begin{cases} \dfrac{1}{2^{n+3/2}}\Gamma\left(n+\dfrac{1}{2}\right)U\left(\dfrac{n}{2}+\dfrac{1}{4},\dfrac{1}{2};\dfrac{x^2}{4}\right) & \text{if } \Re x \geq 0, \\ \\ \dfrac{1}{4}\Gamma\left(\dfrac{n}{2}+\dfrac{1}{4}\right)M\left(\dfrac{n}{2}+\dfrac{1}{4},\dfrac{1}{2};\dfrac{x^2}{4}\right)-\dfrac{x}{4}\Gamma\left(\dfrac{n}{2}+\dfrac{3}{4}\right)M\left(\dfrac{n}{2}+\dfrac{3}{4},\dfrac{3}{2};\dfrac{x^2}{4}\right) & \text{if } \Re x < 0, \end{cases} \tag{5}$$

where $M(a,b;z)$ and $U(a,b;z)$ are confluent hypergeometric functions [8, Chap. 13]

Apart from convergent expansions, we can also find in the literature several asymptotic expansions of $P(x,y)$. In [4] we can find an asymptotic expansion of the Pearcey integral when (x,y) is near the caustic $8x^3-27y^2=0$ that remains valid as $|x|\to\infty$. The expansion is given in terms of Airy functions and its derivatives and the coefficients are computed recursively. We refer the reader to [4] for further details.

An exhaustive asymptotic analysis of this integral can be found in [9]. In particular, a complete asymptotic expansion is given in [9] by using asymptotic techniques for integrals applied to the integral (2). The asymptotic analysis of this integral for large $|x|$ is divided in two regions: $|\arg x|<\frac{\pi}{2}$ and $|\arg x|>\frac{\pi}{2}$. In the first region we find that [9]

$$P(x,y)\sim\frac{1}{2\sqrt{x}}e^{-y^2/(4x)}\sum_{n=0}^{\infty}\frac{(-1)^n\Gamma(2n+1/2)}{n!x^{2n}}M\left(-2n;1/2;\frac{y^2}{4x}\right). \tag{6}$$

The asymptotic expansion in the second region is a little bit more cumbersome, we refer to [9] for details. The integral $P(x,y)$ is also analyzed in [9] for large $|y|$, but the analysis is more cumbersome and only the first order term of the asymptotic expansion is given.

In [10] we can find the hyperasymptotic evaluation of the Pearcey integral for real values of x and y using Hadamard expansions. The Pearcey integral is written in terms of an infinite series whose terms are Hadamard series. The terms of these Hadamard series are incomplete gamma functions. The analytic expression is sophisticated and we refer the reader to [10] for details.

In this paper we show that, for any $(x,y)\in\mathbb{C}^2$,

$$P(x,y)=\lim_{n\to\infty}\left[P_0(x)V_n(x,y)-P_1(x)W_n(x,y)\right],$$

with $P_0(x)$ and $P_1(x)$ given in (5). The sequences of functions $V_n(x,y)$ and $W_n(x,y)$ are elementary functions that may be computed by means of a simple recursive algorithm and the limit is uniform for $x\in\mathbb{C}$ and y in bounded sets of $\mathbb{C}$. Moreover, the limits $\lim_{n\to\infty}V_n(x,y)$ and $\lim_{n\to\infty}W_n(x,y)$ may be arranged in the form of convergent expansions that have also an asymptotic property when $|x|\to\infty$ uniformly for y in bounded sets of $\mathbb{C}$. The starting point is the differential equation [2, p.788, eq. 36.10.4]

$$u'''-\frac{x}{2}u'-\frac{iy}{4}u=0 \tag{7}$$

satisfied by the Pearcey integral $\bar{P}(x,y)$. Here and in the remaining of the paper, x is considered a parameter and y the independent variable in the differential equations, where the primes denote derivatives with respect to y. In the following section we derive the two sequences of functions

$V_n(x,y)$ and $W_n(x,y)$. In Section 3 we show that this sequence may be rearranged in the form of an asymptotic expansion of $P(x,y)$ for large $|x|$ that is also convergent. Section 4 contains some numerical experiments and a few remarks.

2 A sequence of convergent functions

When $\bar{P}(x,y)$ satisfies (7), we immediately derive that $P(x,y)$ is the unique solution of the following initial value problem:

$$\begin{cases} u''' - \dfrac{x}{2}u' - \dfrac{y}{4}u = 0 \quad \text{in } [0,Y], \\ u(0) = P(x,0), \quad u'(0) = 0, \quad u''(0) = P''(x,0), \end{cases} \tag{8}$$

with

$$P(x,0) = \int_0^\infty e^{-t^4 - xt^2} dt = P_0(x), \qquad P''(x,0) = -\int_0^\infty t^2 e^{-t^4 - xt^2} dt = -P_1(x),$$

with $P_n(x)$ given in (5). In the differential equation (8), we consider that the parameter $x \in \mathbb{C}$ and the argument of the independent variable y is fixed (indeed, the independent variable is $|y|$). Then, the symbol $[0,Y]$ stands for the segment of length $Y > 0$ of the ray that emanates from the origin at a fixed angle $\arg y \in [0, 2\pi)$ with respect to the positive real axis; therefore $0 \le |y| \le Y$.

It is clear that the unique solution $P(x,y)$ of (8) may be writen in the form

$$P(x,y) = P(x,0)V(x,y) + P''(x,0)W(x,y), \tag{9}$$

where $V(x,y)$ is the unique solution of the initial value problem:

$$\begin{cases} u''' - \dfrac{x}{2}u' - \dfrac{y}{4}u = 0 \quad \text{in } [0,Y], \\ u(0) = 1, \quad u'(0) = u''(0) = 0 \end{cases} \tag{10}$$

and $W(x,y)$ is the unique solution of the initial value problem:

$$\begin{cases} u''' - \dfrac{x}{2}u' - \dfrac{y}{4}u = 0 \quad \text{in } [0,Y], \\ u(0) = u'(0) = 0, \quad u''(0) = 1. \end{cases} \tag{11}$$

Our objective is to derive a couple of sequences of elementary functions that converge to $V(x,y)$ and $W(x,y)$ respectively. The starting point is the couple of initial value problems of the order three (10) and (11). Following the ideas introduced in [5] for initial value problems of the order two, we define

$$\phi_V(x,y) := 1, \qquad \phi_W(x,y) := \begin{cases} \dfrac{2}{x}\left[\cosh\left(y\sqrt{\dfrac{x}{2}}\right) - 1\right] & \text{if } x \ne 0, \\ \dfrac{y^2}{2} & \text{if } x = 0. \end{cases}$$

Then, $V(x,y) - \phi_V(x,y)$ and $W(x,y) - \phi_W(x,y)$ are the unique solutions of the respective initial value problems:

$$\begin{cases} u''' - \dfrac{x}{2}u' = F_\mu(x,y,u) := \dfrac{y}{4}[u + \phi_\mu(x,y)] & \text{in } [0,Y], \qquad \mu = V, W, \\ u(0) = u'(0) = u''(0) = 0. \end{cases} \tag{12}$$

We seek solutions of the equation $\mathbf{L}_\mu[u] := u''' - \frac{x}{2}u' - F_\mu(x,y,u) = 0$ in the Banach space $\mathcal{B} = \{u : [0,Y] \to \mathbb{C}; u \in \mathcal{C}[0,Y]; u(0) = 0\}$ equipped with the norm

$$\|u\|_\infty = \sup_{y\in[0,Y]} |u(y)|. \tag{13}$$

We write the equation $\mathbf{L}_\mu[u] = 0$ in the form $\mathbf{L}_\mu[u] = \mathbf{M}[u] - F_\mu(x,y,u)$, with $\mathbf{M}[u] := u''' - \frac{x}{2}u'$. Then we solve the equation $\mathbf{L}_\mu[u] = 0$ for u in the form $u = \mathbf{M}^{-1}[F_\mu[x,y,u]]$, where $\mathbf{M}^{-1}$ is the inverse of the operator $\mathbf{M}$ in the space $\mathcal{B}$. For any $v \in \mathcal{B}$, that inverse is given by $\mathbf{M}^{-1}(v) = \int_0^Y G(x,y,t)v(t)dt$, where $G(x,y,t)$ is the Green's function of the problem $\mathbf{M}[u] = v$ with homogeneous boundary conditions [12]. That is, $G(x,y,t)$ is the unique solution of the boundary value problem

$$\begin{cases} G_{yyy}(x,y,t) - \dfrac{x}{2}G_y(x,y,t) = \delta(y-t) & \text{in } (x,y,t) \in \mathbb{C} \times [0,Y]^2, \\ G(x,0,t) = G_y(x,0,t) = G_{yy}(x,0,t) = 0. \end{cases}$$

After a straightforward computation we obtain [12]

$$G(x,y,t) = G(x,y-t) = \phi_W(x,y-t)\chi_{[0,y]}(t), \tag{14}$$

where $\chi_{[0,y]}(t)$ is the characteristic function of the interval $[0,y]$ for $0 < |y| < Y$. Then, any solution $u(y)$ of (12) for $\mu = V, W$, is a solution of the integral equation

$$u(y) = \mathbf{M}^{-1}[F_\mu[x,y,u]] = \frac{1}{4}\int_0^y G(x,y-t)[u(t) + \phi_\mu(x,t)]t\,dt.$$

Or equivalently, define

$$\tilde{u}(y) := H\left(y\sqrt{\frac{x}{2}}\right)u(y), \qquad \tilde{\phi}_\mu(x,y) := H\left(y\sqrt{\frac{x}{2}}\right)\phi_\mu(x,y), \qquad \mu = V, W, \tag{15}$$

with

$$H(z) := \begin{cases} e^{-z} & \text{if } \Re(z) \geq 0 \\ e^{z} & \text{if } \Re(z) < 0. \end{cases} \tag{16}$$

Then we have that for any solution $u(y)$ of (12), $\tilde{u}(y)$ is a solution of the integral equation

$$\tilde{u}(y) = [\mathbf{T}\tilde{u}](y), \tag{17}$$

where we have defined the operator

$$[\mathbf{T}\tilde{u}](y) := \frac{1}{4}\int_0^y K(x,y-t)[\tilde{u}(t) + \tilde{\phi}_\mu(x,t)]t\,dt, \tag{18}$$

with

$$K(x,y) := H\left(y\sqrt{\frac{x}{2}}\right) G(x,y). \tag{19}$$

From the fixed point theorem of Banach [1, p. 26, Theorem 3.1] it is well-known that, if any power of this operator is contractive in $\mathcal{B}$, then equation (17) has a unique solution $\tilde{u}(y)$ and the sequence $\tilde{u}_{n+1} = \mathbf{T}(\tilde{u}_n)$, $\tilde{u}_0 = 0$, converges to that solution $\tilde{u}(y)$. To show that $\mathbf{T}$ is contractive in $\mathcal{B}$ we only need to realize that the absolute value of the kernel $K(x, y-t)$ of the operator $\mathbf{T}$ is appropriately bounded. Using that for any complex z[1],

$$\left|\frac{H(z)(\cosh z - 1)}{z^2}\right| \le \frac{1}{2}, \tag{20}$$

we find that for $(y,t) \in [0,Y] \times [0,Y]$ and $x \in \mathbb{C}$:

$$|K(x, y-t)| \le \frac{|y-t|^2}{2}. \tag{21}$$

Then, from definition (19) and the bound (21), and using that, for $y > 0$,

$$\int_0^y t^{4m-3}(y-t)^2\, dt = \frac{y^{4m}}{4m(2m-1)(4m-1)} \le \frac{y^{4m}}{32m(m-1)^2}, \qquad m = 2,3,4,...,$$

we have that, for any couple v, $w \in \mathcal{B}$,

$$|[\mathbf{T}v](y) - [\mathbf{T}w](y)| \le \frac{1}{4}\int_0^y |K(x,y-t)||v(t)-w(t)||t\, dt| \le \frac{|y|^4}{8 \times 12} \|v - w\|_\infty . \tag{22}$$

We also have

$$\begin{aligned} |[\mathbf{T}^2 v](y) - [\mathbf{T}^2 w](y)| \le & \frac{1}{4}\int_0^y |K(x,y-t)||[\mathbf{T}v](t) - [\mathbf{T}w](t)||t\, dt| \le \\ & \frac{|y|^8}{8^2 \times 12 \times 32 \times 2 \times 1^2} \|v - w\|_\infty . \end{aligned} \tag{23}$$

and

$$\begin{aligned} |[\mathbf{T}^3 v](y) - [\mathbf{T}^3 w](y)| \le & \frac{1}{4}\int_0^y |K(x,y-t)||[\mathbf{T}^2 v](t) - [\mathbf{T}^2 w](t)||t\, dt| \le \\ & \frac{|y|^{12}}{8^3 \times 12 \times 32^2 \times (2\cdot 3) \times (1\cdot 2)^2} \|v - w\|_\infty . \end{aligned} \tag{24}$$

It is straightforward to prove, by means of induction over n, that for $n = 1, 2, 3, \ldots$,

$$|[\mathbf{T}^n v](y) - [\mathbf{T}^n w](y)| \le \frac{|y|^{4n}}{96 \cdot 256^{n-1} n!(n-1)!^2} \|v - w\|_\infty \tag{25}$$

and then

$$\|\mathbf{T}^n v - \mathbf{T}^n w\|_\infty \le \frac{|Y|^{4n}}{96 \cdot 256^{n-1} n!(n-1)!^2} \|v - w\|_\infty . \tag{26}$$

[1]When $z = 0$ this inequality is understood in the limit sense.

This means that the operator $\mathbf{T}^n$ is contractive in $\mathcal{B}$ for large enough n. From [1, p. 26, Theorem 3.1], we have that the sequence $\tilde{u}_{n+1} = \mathbf{T}(\tilde{u}_n)$, $n = 0, 1, 2, \ldots$, $\tilde{u}_0(y) = 0$, converges uniformly in $y \in [0, Y]$ to $\tilde{u}(y) := H(y\sqrt{x/2})u(y)$, where $u(y)$ is the unique solution of (12). Or equivalently, for $n = 0, 1, 2, \ldots$, the sequence

$$\begin{cases} V_0(x,y) = 1, \\ V_{n+1}(x,y) = 1 + \dfrac{1}{2x}\displaystyle\int_0^y \left\{\cosh\left[\sqrt{\frac{x}{2}}(y-t)\right] - 1\right\} t\, V_n(x,t)\, dt, \end{cases} \tag{27}$$

converges to $V(x, y)$ uniformly for $y \in [0, Y]$. And, for $n = 0, 1, 2, \ldots$, the sequence

$$\begin{cases} W_0(x,y) = \dfrac{2}{x}\left[\cosh\left(y\sqrt{\dfrac{x}{2}}\right) - 1\right], \\ \\ W_{n+1}(x,y) = \dfrac{2}{x}\left[\cosh\left(y\sqrt{\dfrac{x}{2}}\right) - 1\right] + \dfrac{1}{2x}\displaystyle\int_0^y \left\{\cosh\left[\sqrt{\frac{x}{2}}(y-t)\right] - 1\right\} t\, W_n(x,t)\, dt, \end{cases} \tag{28}$$

converges to $W(x, y)$ uniformly for $y \in [0, Y]$. In these formulas, when $x = 0$, the expressions

$$\frac{1}{x}\left[\cosh\left(y\sqrt{\frac{x}{2}}\right) - 1\right] \qquad \text{and} \qquad \frac{1}{x}\left[\cosh\left(\sqrt{\frac{x}{2}}(y-t)\right) - 1\right]$$

must be replaced by $y^2/4$ and $(y - t)^2/4$ respectively. The first few functions V_n and W_n are shown in the appendix.

Therefore, the Pearcey integral (2) may be written in the form

$$P(x,y) = \lim_{n\to\infty}[P_0(x)V_n(x,y) - P_1(x)W_n(x,y)], \tag{29}$$

where the functions $P_0(x)$ and $P_1(x)$ are defined in (5) and the sequences V_n and W_n are computed recursively using (27) and (28).

We can find an easy bound for the remainders

$$\begin{aligned} R_n^V(x,y) :=& [V(x,y) - V_n(x,y)]H(y\sqrt{x/2}), \\ R_n^W(x,y) :=& [W(x,y) - W_n(x,y)]H(y\sqrt{x/2}). \end{aligned} \tag{30}$$

To this end we set $v = \tilde{u}$ and $w = \tilde{u}_0 = 0$ in (25). Using that $\mathbf{T}^n\tilde{u} = \tilde{u}$ and $\mathbf{T}^n\tilde{u}_0 = \tilde{u}_n$ we find

$$|\tilde{u}(y) - \tilde{u}_n(y)| \leq \frac{|Y|^{4n}}{96 \cdot 256^{n-1} n!(n-1)!^2} \|\tilde{u}\|_\infty, \tag{31}$$

Using that $V(x, y) = \tilde{u}(y)/H(y\sqrt{x/2}) + \phi_V(x, y)$ and $V_n(x, y) = \tilde{u}_n(y)/H(y\sqrt{x/2}) + \phi_V(x, y)$ or $W(x, y) = \tilde{u}(y)/H(y\sqrt{x/2}) + \phi_W(x, y)$ and $W_n(x, y) = \tilde{u}_n(y)/H(y\sqrt{x/2}) + \phi_W(x, y)$ in (31) we find

$$\begin{aligned} |R_n^V(x,y)| \leq& \frac{|Y|^{4n}}{96 \cdot 256^{n-1} n!(n-1)!^2} \left\|[V - \phi_V]H(\cdot\sqrt{x/2})\right\|_\infty, \\ |R_n^W(x,y)| \leq& \frac{|Y|^{4n}}{96 \cdot 256^{n-1} n!(n-1)!^2} \left\|[W - \phi_W]H(\cdot\sqrt{x/2})\right\|_\infty, \end{aligned} \tag{32}$$

that shows the high speed of convergence of the above limits (of the sequences V_n and W_n).

3 Asymptotic character of the expansion

We have seen that $V(x,y)$ may be obtained from the limit $V(x,y) = \lim_{n\to\infty} V_n(x,y)$ and $W(x,y)$ from the limit $W(x,y) = \lim_{n\to\infty} W_n(x,y)$ uniformly in $[0,Y]$, where $V_n(x,y)$ and $W_n(x,y)$ are the recurrences defined in (27) and (28) respectively. In other words, $V(x,y)$ admits the series expansion

$$V(x,y) = 1 + \sum_{k=0}^{\infty}[V_{k+1}(x,y) - V_k(x,y)] = 1 + \frac{1}{H\left(y\sqrt{\frac{x}{2}}\right)}\sum_{k=0}^{\infty}[\tilde{V}_{k+1}(x,y) - \tilde{V}_k(x,y)], \qquad (33)$$

with

$$\tilde{V}_n(x,y) := [V_n(x,y) - 1]H(y\sqrt{x/2}), \qquad n = 0,1,2,\ldots \qquad (34)$$

Then, using the definition (30) we may write (33) in the form

$$\begin{aligned} V(x,y) =& 1 + \sum_{k=0}^{n-1}[V_{k+1}(x,y) - V_k(x,y)] + \frac{R_n^V(x,y)}{H\left(y\sqrt{\frac{x}{2}}\right)} = \\ & 1 + \frac{1}{H\left(y\sqrt{\frac{x}{2}}\right)}\left[\sum_{k=0}^{n-1}[\tilde{V}_{k+1}(x,y) - \tilde{V}_k(x,y)] + R_n^V(x,y)\right]. \end{aligned} \qquad (35)$$

Similarly we may write

$$W(x,y) = \phi_W(x,y) + \frac{1}{H\left(y\sqrt{\frac{x}{2}}\right)}\left[\sum_{k=0}^{n-1}[\tilde{W}_{k+1}(x,y) - \tilde{W}_k(x,y)] + R_n^W(x,y)\right]. \qquad (36)$$

with

$$\tilde{W}_n(x,y) := [W_n(x,y) - \phi_W(x,y)]H(y\sqrt{x/2}), \qquad n = 0,1,2,\ldots \qquad (37)$$

We are going to show that (35) and (36) are asymptotic expansions, for large $|x|$, of $V(x,y)$ and $W(x,y)$ respectively, uniformly in $y\in[0,Y]$. From definition (18) we have

$$\tilde{V}_n(x,y) = [\mathbf{T}\tilde{V}_{n-1}](x,y) = \frac{1}{4}\int_0^y K(x,y-t)[\tilde{V}_{n-1}(x,t) + \tilde{\phi}_V(x,t)]t\,dt. \qquad (38)$$

and

$$\tilde{V}_{n+1}(x,y) = [\mathbf{T}\tilde{V}_n](x,y) = \frac{1}{4}\int_0^y K(x,y-t)[\tilde{V}_n(x,t) + \tilde{\phi}_V(x,t)]t\,dt, \qquad (39)$$

with $K(x,y,t)$ defined in (19). Subtracting (38) and (39) and using the bound

$$|H(z)(\cosh z - 1)| \le 2, \qquad z\in\mathbb{C}. \qquad (40)$$

we find that, for $x\neq 0$,

$$\left\|\tilde{V}_{n+1} - \tilde{V}_n\right\|_\infty \le \left|\frac{Y^2}{2x}\right| \left\|\tilde{V}_n - \tilde{V}_{n-1}\right\|_\infty. \qquad (41)$$

We have $\tilde{V}_0(x,y) = 0$ and $\tilde{V}_1(x,y) = [\mathbf{T}\tilde{V}_0](x,y) = \mathcal{O}(x^{-1})$ uniformly for $y\in[0,Y]$. Using this and (41) we find, by induction over n,

$$\tilde{V}_n(x,y) - \tilde{V}_{n-1}(x,y) = \mathcal{O}(x^{-n}), \qquad n = 1,2,3,\ldots, \qquad (42)$$

uniformly for $y \in [0, Y]$. And similarly,

$$\tilde{W}_n(x, y) - \tilde{W}_{n-1}(x, y) = \mathcal{O}(x^{-n}), \qquad n = 1, 2, 3, \ldots. \tag{43}$$

We have derived the bounds (32) that show the convergence of the sequences V_n and W_n from the bound (20). Repeating exactly the same arguments, but using (40) instead of (20) we obtain

$$\begin{aligned} |R_n^V(x, y)| &\le \frac{1}{2^n n!}\left|\frac{Y^2}{x}\right|^n \left\|[V - \phi_V]H(\cdot\sqrt{x/2})\right\|_\infty, \\ |R_n^W(x, y)| &\le \frac{1}{2^n n!}\left|\frac{Y^2}{x}\right|^n \left\|[W - \phi_W]H(\cdot\sqrt{x/2})\right\|_\infty, \end{aligned} \tag{44}$$

that prove the asymptotic character of the expansions (35) and (36). These bounds show that $R_n^V(x) = \mathcal{O}(x^{-n})$ and $R_n^W(x) = \mathcal{O}(x^{-n})$. But indeed the order is one power of x smaller: observe that $\tilde{V} = \lim_{n\to\infty} \tilde{V}_n = \sum_{k=0}^{\infty}[\tilde{V}_{k+1} - \tilde{V}_k] = \sum_{k=0}^{\infty} \mathcal{O}(x^{-k-1}) = \mathcal{O}(x^{-1})$. This and (34) prove that, for $n = 1, 2, 3, \ldots$, $R_n^V(x) = \mathcal{O}(x^{-n-1})$ uniformly for $y \in [0, Y]$. Similarly, $R_n^W(x) = \mathcal{O}(x^{-n-1})$.

4 Numerical experiments and final remarks

In this section we illustrate the accuracy of what we consider the most simple analytical algorithms (3), (4), (6) and (27)-(28)-(29). In the following tables we show the relative error obtained with these algorithms for several values of (x, y) and different orders n of the approximation. As we do not have at our disposal the exact value of the Pearcey integral, we have taken the $n = 100$ approximation (4) as the exact value of $P(x, y)$. For all the values (x, y) considered in the tables, this "exact value" contains more than one hundred correct decimal digits, that is enough for the approximations analyzed in the tables. For simplicity, we do not reproduce the exact value of the relative error but only its order of magnitude.

Expansion (3) is a simple expansion in terms of polynomials in x and y whose terms are computed recursively. The drawback is its slow speed of convergence for moderate or large values of x or y, as the terms of the expansion grow as $(|x| + |y|)^{2n}(n/2)!/(2n)!$.

Expansion (4) is analytically more complicated as the terms of the expansion are special functions of two variables: confluent hypergeometric functions $U(n/2+1/4, 1/2; x^2/4)$. On the other hand it converges faster than (3) as the terms of the expansion grow as $y^{2n}n!/[2^n(n/2)!(2n)!]$.

The asymptotic expansion (6) has also a simple form and the coefficients are explicit, although the terms of the expansion are given in terms of confluent hypergeometric functions of the variable $y^2/(4x)$. Of course, as we can check in the tables, it only gives good approximations for large values of $|x|$ and moderate values of y.

From (29) we see that the Pearcey integral $P(x, y)$ may be written in the form of a linear combination of two confluent hypergeometric functions of only the variable x. Both, the sequence $V_n(x, y)$ and the sequence $W_n(x, y)$, derived recursively from (27) and (28) respectively, are elementary functions of x and y. Then, the coefficients of the above mentioned linear combination may be approximated by means of elementary functions (powers and exponentials of x and y). The speed of convergence of the limit involved in (29) is of the order $y^{4n}/[256^{n-1}n!(n-1)!^2]$, uniformly in $x \in \mathbb{C}$.

For the values of (x, y) considered in the tables, we see that (6) is more accurate when $|x|$ is large and y moderate. Expansion (3) is accurate only for small values of x and y. Expansions (4) and (29) are more accurate in a wider region of (x, y): expansion (29) converges faster for small or moderate values of y and (4) converges faster for large values of y.

Other approximations that we have not checked numerically because they are analytically more sophisticated are the expansions derived in [4] and [10]. The asymptotic expansion given in [4] is a good approximation for large $|x|$ and $|y|$ near the caustic. The expansion involves Airy functions and the analytic computation of the coefficients is a little bit cumbersome. Perhaps, the most accurate numerical computations of $P(x, y)$ are given in [10] using the Hadamard technique, although the formulas are restricted to real values of x and y. The analytical expressions are certainly complicated, as the Pearcey integral is written in terms of an infinite series whose terms are also series: Hadamard series of incomplete gamma functions. For example, for $x = 7$, $y = 1$ we obtain relative errors of the order 10^{-7} at the first order of the approximation or of the order 10^{-12} at the second order of the approximation. For $x = -4$, $y = 12$ we obtain relative errors of the order 10^{-7} at the first order of the approximation or of the order 10^{-11} at the second order of the approximation.

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6 Appendix

The functions $V_n(x, y)$ defined in the recurrence (27) may be written in the form

$$V_n(x,y) = 1 + \frac{1}{4^n n! x^{2n}} \left\{ p_n(x,y) \cosh\left[y\sqrt{\frac{x}{2}}\right] - q_n(x,y)\, y\sqrt{2x}\, \sinh\left[y\sqrt{\frac{x}{2}}\right] + r_n(x,y) \right\},$$

where $p_n(x, y)$, $q_n(x, y)$ and $r_n(x, y)$ are polynomials of x and y. The first few ones are

$$\begin{cases} p_1(x,y) = 4, \\ q_1(x,y) = 0, \\ r_1(x,y) = -xy^2 - 4, \\ p_2(x,y) = 4[-24 + x(8x + y^2)], \\ q_2(x,y) = 12, \\ r_2(x,y) = 96 + x[-32x - 8(-4 + x^2)y^2 + xy^4], \\ p_3(x,y) = 3\{3840 + x[128x(-3 + x^2) + 2(15 + 8x^2)y^2 + xy^4]\}, \\ q_3(x,y) = 6(15 + 24x^2 + 5xy^2), \\ r_3(x,y) = -384(30 - 3x^2 + x^4) - 96x(30 - 4x^2 + x^4)y^2 + 12x^2(-9 + x^2)y^4 - x^3y^6. \end{cases}$$

The functions $W_n(x,y)$ defined in the recurrence (28) may be written in the form

$$\begin{cases} W_n(x,y) = & \dfrac{2}{x}\left(\cosh\left[y\sqrt{\dfrac{x}{2}}\right]-1\right)+ \\ & \dfrac{1}{2^{3n-1}n!x^{2n+1}}\left\{\bar{p}_n(x,y)\cosh\left[y\sqrt{\dfrac{x}{2}}\right]-\bar{q}_n(x,y)\,y\sqrt{2x}\,\sinh\left[y\sqrt{\dfrac{x}{2}}\right]+\bar{r}_n(x,y)\right\}, \end{cases}$$

where $\bar{p}_n(x,y)$, $\bar{q}_n(x,y)$ and $\bar{r}_n(x,y)$ are polynomials of x and y. The first few ones are

$$\begin{cases} \bar{p}_1(x,y) = xy^2, \\ \bar{q}_1(x,y) = 3, \\ \bar{r}_1(x,y) = 2xy^2, \\ \bar{p}_2(x,y) = 768 + 78xy^2 + 16x^3y^2 + x^2y^4, \\ \bar{q}_2(x,y) = 2(87 + 24x^2 + 5xy^2), \\ \bar{r}_2(x,y) = -2(384 + 48xy^2 - 16x^3y^2 + 2x^2y^4), \\ \bar{p}_3(x,y) = x\{18540y^2 + x[18432 + 48x(39 + 8x^2)y^2 + 6(67 + 4x^2)y^4 + xy^6]\}, \\ \bar{q}_3(x,y) = 3\{13860 + x[48x(29 + 8x^2) + 20(39 + 4x^2)y^2 + 7xy^4]\}, \\ \bar{r}_3(x,y) = 8x[-2304x + 96(30 - 3x^2 + x^4)y^2 - 12x(-8 + x^2)y^4 + x^2y^6]. \end{cases}$$

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$(x, y) = (1, 1)$

n	Formula (3)	Formula (6)	Formula (4)	Formula (29)
1	10^{-1}	10^{-2}	10^{-3}	10^{-6}
5	10^{-3}	10^{+3}	10^{-10}	10^{-23}
10	10^{-6}	10^{+10}	10^{-21}	10^{-49}

$(x, y) = (10, 1)$

n	Formula (3)	Formula (6)	Formula (4)	Formula (29)
1	10^{+1}	10^{-4}	10^{-4}	10^{-6}
5	10^{+4}	10^{-8}	10^{-13}	10^{-23}
10	10^{+5}	10^{-11}	10^{-26}	10^{-49}

$(x, y) = (-1, 5)$

n	Formula (3)	Formula (6)	Formula (4)	Formula (29)
1	10^{+2}	-	10^{+2}	10^{+2}
5	10^{-1}	-	10^{+2}	10^{-5}
10	10^{-5}	-	10^{-3}	10^{-17}

$(x, y) = (10, 5)$

n	Formula (3)	Formula (6)	Formula (4)	Formula (29)
1	10^{+2}	10^{-4}	10^{-1}	10^{+3}
5	10^{+5}	10^{-8}	10^{-5}	10^{-5}
10	10^{+6}	10^{-10}	10^{-11}	10^{-17}

$(x, y) = (100, i)$

n	Formula (3)	Formula (6)	Formula (4)	Formula (29)
1	10^{+3}	10^{-8}	10^{-6}	10^{-5}
5	10^{+8}	10^{-19}	10^{-19}	10^{-22}
10	10^{+15}	10^{-32}	10^{-37}	10^{-48}

$(x, y) = (100, 10i)$

n	Formula (3)	Formula (6)	Formula (4)	Formula (29)
1	10^{+3}	10^{-7}	10^{-2}	10^{-2}
5	10^{+8}	10^{-18}	10^{-7}	10^{-7}
10	10^{+15}	10^{-30}	10^{-15}	10^{-15}

Table 1: Numerical experiments about the relative errors in the approximation of the normalized Pearcey integral $P(x, y)$ for several values of x and y and several values of the degree of approximation n.