Convergent expansions of the confluent hypergeometric functions in terms of elementary functions

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Abstract

We consider the confluent hypergeometric function $M(a,b;z)$ for $z\in\mathbb{C}$ and $\Re b > \Re a > 0$; and the the confluent hypergeometric function $U(a,b;z)$ for $b\in\mathbb{C}$, $\Re a > 0$ and $\Re z > 0$. We derive two convergent expansions of $M(a,b;z)$; one of them in terms of incomplete gamma functions $\gamma(a,z)$ and another one in terms of rational functions of e^z and z. We also derive a convergent expansion of $U(a,b;z)$ in terms of incomplete gamma functions $\gamma(a,z)$ and $\Gamma(a,z)$. The expansions of $M(a,b;z)$ hold uniformly in either $\Re z\ge 0$ or $\Re z\le 0$; the expansion of $U(a,b;z)$ holds uniformly in $\Re z>0$. The accuracy of the approximations is illustrated by means of some numerical experiments.

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1 Introduction

The power series expansion of the confluent hypergeometric function $M(a,b;z)$ is well known [6, Sec. 13.2, eq. 13.2.2]:

$$M(a,b;z) := \sum_{s=0}^{\infty}\frac{(a)_s}{(b)_s s!}z^s, \qquad a,b,z\in\mathbb{C}, \qquad b\neq 0,-1,-2,... \tag{1}$$

The power series expansion of the confluent hypergeometric function $U(a,b;z)$ may be derived from the above formula and the connection formula [6, Sec. 13.2, eq. 13.2.42]

$$U(a,b;z) := \frac{\Gamma(1-b)}{\Gamma(a-b+1)}M(a,b;z) + \frac{\Gamma(b-1)}{\Gamma(a)}z^{1-b}M(a-b+1,2-b;z). \tag{2}$$

The asymptotic expansion for large $|z|$ of the confluent hypergeometric function $U(a,b;z)$ is given in [6, Sec. 13.7, eq. 13.7.3],

$$U(a,b;z) \sim z^{-a}\sum_{s=0}^{\infty}\frac{(a)_s(a-b+1)_s}{s!}(-z)^{-s}, \qquad |\text{ph } z| \leq \frac{3}{2}\pi - \delta, \tag{3}$$

where δ denotes an arbitrarily small positive constant. The asymptotic expansion for large $|z|$ of the confluent hypergeometric function $M(a,b;z)$ is given in [6, Sec. 13.7, eq. 13.7.2] and may be derived from (3) and the connection formula [6, Sec. 13.2, eq. 13.2.41],

$$M(a,b;z) = \frac{\Gamma(b)}{\Gamma(b-a)}e^{\mp a\pi i}U(a,b;z) + \frac{\Gamma(b)}{\Gamma(a)}e^{\pm(b-a)\pi i}e^{z}U(b-a,b;e^{\pm\pi i}z). \tag{4}$$

These two Taylor expansions and two asymptotic expansions have the advantage of being given in terms of elementary functions of z. But they have the inconvenience of not being uniformly valid for all values of z; the power expansions fail for large values of $|z|$, whereas the asymptotic expansions fail for small values of $|z|$.

In this paper we derive convergent expansions of $M(a,b;z)$ and $U(a,b;z)$ in terms of elementary functions or of incomplete gamma functions that hold uniformly in large regions of the complex $z-$plane. The starting point is the following set of integral representations of the confluent hypergeometric functions [6, Chap. 13, eqs. 13.4.1 and 13.4.4],

$$M(a,b;z) = \frac{\Gamma(b)}{\Gamma(a)\Gamma(b-a)}\int_0^1 e^{zt}t^{a-1}(1-t)^{b-a-1}dt, \qquad \Re b > \Re a > 0, \tag{5}$$

and

$$U(a,b;z) = \frac{1}{\Gamma(a)}\int_0^{\infty} e^{-zt}t^{a-1}(1+t)^{b-a-1}dt, \qquad \Re a > 0, \qquad |\arg z| < \frac{\pi}{2}. \tag{6}$$

The power series expansion (1) may be derived from the integral (5) by replacing the exponential e^{zt} in the integrand by its Taylor expansion at the origin and interchanging series and integral. The Taylor expansion converges for $t \in [0,1]$, but the convergence is not uniform in $|z| \in [0,\infty)$. Therefore, the resulting expansion is convergent, but not uniformly in $|z| \in [0,\infty)$: the remainder is unbounded for large values of $|z|$. The power series expansion of $U(a,b;z)$ is not uniform in $|z| \in [0,\infty)$ either, as it is derived from (1).

The asymptotic expansion (3) may be derived from a direct application of Watson's lemma to the integral (6). Roughly speaking, Watson's lemma consists of a replacement of the factor $(1+t)^{b-a-1}$ in the integrand in the right hand side of (6) by its Taylor series at the origin and an interchange of sum and integral. Now, the situation with respect to the convergence is worse than in the case of the power series expansion. The Taylor expansion of the factor $(1+t)^{b-a-1}$ is not convergent in the whole integration interval, as it is unbounded [10, Chap. 1, p. 24, eq. (5.32)]. This translates into the fact that the expansion that we obtain after interchanging series and integral is not convergent; although it is asymptotic. As the remainder is unbounded for small $|z|$, the expansion is not uniform in $|z| \in [0, \infty)$. The asymptotic expansion of $M(a, b; z)$ is not uniform in $|z| \in [0, \infty)$ either, as it is derived from (3).

In this paper we propose a different technique that avoids the lack of uniformity in $|z|$: instead of the Taylor expansion of the exponential factor in (5), we consider the Taylor expansion of the other factor $t^{a-1}(1-t)^{b-a-1}$. Since this factor is not analytic at the origin (unless $a \in \mathbb{N}$), we investigate two alternatives: on the one hand, we consider the Taylor expansion of only the factor $(1-t)^{b-a-1}$ at the origin. On the other hand, we consider the Taylor expansion of the whole factor $t^{a-1}(1-t)^{b-a-1}$ at the middle point of the integration interval $t = 1/2$. This second choice is also motivated for a faster convergence of the Taylor expansion in the whole integration interval; for a similar discussion we refer to [1]. Either of these Taylor expansions are convergent for t in the integration interval of (5) and, obviously, they are independent of z. After the interchange of series and integral, the independence of z translates into a remainder that may be bounded independently of z in large regions of the complex plane. In the case of the integral representation of $U(a, b; z)$, a Taylor expansion of the factor $t^{a-1}(1+t)^{b-a-1}$ does not work as it is not convergent in the whole integration interval $(0, \infty)$. Then, we divide appropriately the integration interval in the right hand side of (6) in order to have convergent Taylor expansions of the factor $t^{a-1}(1+t)^{b-a-1}$ in both intervals.

Using the techniques described in the previous paragraph, in the following section we derive two expansions of $M(a, b; z)$. In Section 3 we derive an expansion of $U(a, b; z)$. These expansions are accompanied by realistic error bounds and are uniform in $|z|$ with either $\Re z \geq 0$ or $\Re z \leq 0$. Section 4 contains some numerical experiments and a few remarks. Through the paper we use the principal argument $\arg z \in (-\pi, \pi]$ for any complex number z.

2 Uniform convergent expansions of $M(a, b; z)$

2.1 An expansion in terms of incomplete gamma functions

Consider the Taylor expansion of $(1-t)^{b-a-1}$ at $t = 0$:

$$(1-t)^{b-a-1} = \sum_{k=0}^{n-1} \frac{(1+a-b)_k}{k!} t^k + r_n(a, b, t), \qquad t \in (0, 1), \tag{7}$$

where $r_n(a, b, t)$ is the Taylor remainder:

$$r_n(a, b, t) := \sum_{k=n}^{\infty} \frac{(1+a-b)_k}{k!} t^k, \qquad t \in (0, 1). \tag{8}$$

After suitable manipulations we obtain

$$r_n(a,b,t) = \frac{(1+a-b)_n}{n!}\, t^n \, {}_2F_1\left(\begin{array}{c} 1+a-b+n,\ 1 \\ \\ n+1 \end{array} \middle| t \right), \qquad t \in (0,1). \tag{9}$$

Replacing $(1-t)^{b-a-1}$ in (5) by the right hand side of (7) and interchanging sum and integral we obtain

$$M(a,b;z) = \frac{\Gamma(b)}{\Gamma(a)\Gamma(b-a)} \sum_{k=0}^{n-1} \frac{(1+a-b)_k}{k!\,(-z)^{a+k}} \gamma(a+k,-z) + R_n(a,b;z), \tag{10}$$

where $\gamma(a,z)$ is an incomplete gamma function and the remainder is

$$R_n(a,b;z) := \frac{\Gamma(b)}{\Gamma(a)\Gamma(b-a)} \int_0^1 e^{zt}\, t^{a-1}\, r_n(a,b,t)\, dt. \tag{11}$$

The incomplete gamma functions in (10) may be computed recurrently in the form [5, Sec. 8.8, eq. (8.8.1)],

$$\gamma(a+k+1,z) = a\gamma(a+k,z) - z^{a+k}e^{-z}, \qquad k = 0,1,2,... \tag{12}$$

Expansion (10) was first obtained by K.E.Muller [4, eq. (15.6.1)], who proved the convergence of the expansion, but not its uniform properties. In the remaining of this section we prove the uniform character of expansion (10). The remainder $|r_n(a,b,t)|$ is integrable in $(0,1)$ and then we have that

$$|R_n(a,b;z)| \le \frac{|\Gamma(b)|\, H(z)}{|\Gamma(a)||\Gamma(b-a)|} \int_0^1 t^{\Re a-1}\, |r_n(a,b,t)| dt, \tag{13}$$

with

$$H(z) := \begin{cases} e^{\Re z} & \text{if } \Re z > 0, \\ \\ 1 & \text{if } \Re z \le 0. \end{cases} \tag{14}$$

More precisely, from the integral representation of the hypergeometric function [7, Sec. 15.6, eq. 15.6.1] we have that, for $t \in (0,1)$,

$$\left| {}_2F_1\left(\begin{array}{c} n+1+a-b,\ 1 \\ \\ n+1 \end{array} \middle| t \right) \right| \le {}_2F_1\left(\begin{array}{c} n+1+\Re a-\Re b,\ 1 \\ \\ n+1 \end{array} \middle| t \right).$$

Introducing this bound in (9), we obtain from (13):

$$|R_n(a,b;z)| \le \frac{H(z)\, |\Gamma(b)|\, |(1+a-b)_n|}{|\Gamma(a)||\Gamma(b-a)|\, n!\, (\Re a+n)} {}_3F_2\left(\begin{array}{c} n+1+\Re a-\Re b,\ 1,\ \Re a+n \\ \\ n+1,\ 1+\Re a+n \end{array} \middle| 1 \right). \tag{15}$$

From the series definition of the ${}_3F_2$ function [8, Sec. 16.2, eq. 16.2.1] and using the fact that

$$\frac{(\Re a+n)_k}{(\Re a+n+1)_k}=\frac{\Re a+n}{\Re a+n+k}<\frac{\Re a+n}{\Re a-\Re b+n+k}, \qquad n>\Re b-\Re a,$$

we obtain, after some manipulations, that the absolute value of the hypergeometric ${}_3F_2$ function in the right side of (15) may be bounded by

$$\frac{\Re a+n}{n+\Re a-\Re b}\,{}_2F_1\left(\begin{array}{c} n+\Re a-\Re b,\ 1 \\ n+1 \end{array}\middle|\, 1\right)=\frac{n(\Re a+n)}{(\Re b-\Re a)(n+\Re a-\Re b)}.$$

Therefore,

$$|R_n(a,b;z)|\le\frac{|\Gamma(b)|\,|(1+a-b)_n|}{|\Gamma(a)||\Gamma(b-a)|\,n!}\frac{n\,H(z)}{(\Re b-\Re a)(n+\Re a-\Re b)}. \tag{16}$$

Using Stirling's formula and this last bound we find that $R_n(a,b;z)\sim n^{\Re a-\Re b}$ when $n\to\infty$. This fact shows that, when $\Re b>\Re a>0$, expansion (10) is uniformly convergent in z for $\Re z\le\Re z_0$ for any fixed $z_0\in\mathbb{C}$.

An immediate consequence is the following expansion:

$$e^{-z}M(a,b;z)=\frac{e^{-z}\Gamma(b)}{\Gamma(a)\Gamma(b-a)}\sum_{k=0}^{n-1}\frac{(1+a-b)_k}{k!\,(-z)^{a+k}}\gamma(a+k,-z)+e^{-z}R_n(a,b;z). \tag{17}$$

For $n>\Re b-\Re a$, the absolute value of the remainder $e^{-z}R_n(a,b;z)$ is bounded by the right hand side of (16) with $H(z)$ replaced by $H(-z)$. Then, expansion (17) is uniformly convergent in z for $\Re z\ge\Re z_0$ for any fixed $z_0\in\mathbb{C}$.

2.2 An expansion in terms of elementary functions

Expansion (10) is given in terms of incomplete gamma functions $\gamma(a,z)$. These approximant functions are simpler than the function $M(a,b;z)$, as they are functions of only two variables; but, admittedly, they are not elementary functions. We can go one step farther and derive an expansion of $M(a,b;z)$ in terms of elementary functions. To accomplish this task we consider the Taylor expansion at $t=1/2$ of the function $f(t):=t^{a-1}(1-t)^{b-a-1}$,

$$t^{a-1}(1-t)^{b-a-1}=\sum_{k=0}^{n-1}A_k(a,b)\left(t-\frac{1}{2}\right)^k+r_n(a,b,t), \qquad t\in(0,1), \tag{18}$$

where

$$r_n(a,b,t):=\sum_{k=n}^{\infty}A_k(a,b)\left(t-\frac{1}{2}\right)^k. \tag{19}$$

The Taylor coefficients $A_k(a,b)$ of this expansion are

$$\begin{aligned} A_n(a,b) := {} & 2^{n+2-b}\sum_{k=0}^{n}(-1)^k\frac{(1-a)_k(a+1-b)_{n-k}}{k!(n-k)!} \\ = {} & 2^{n+2-b}\frac{(a+1-b)_n}{n!}\,{}_2F_1\left(\begin{array}{c} 1-a,\ \ -n \\ \\ b-a-n \end{array}\middle| -1\right), \end{aligned} \tag{20}$$

and may be computed recursively in the form:

$$A_n(a,b) = \frac{2}{n}\left[(2a-b)A_{n-1}(a,b) + 2(n-b)A_{n-2}(a,b)\right],$$

with $A_0(a,b) = 2^{2-b}$ and $A_1(a,b) = (2a-b)2^{3-b}$. This recurrence relation may be derived from the differential equation $t(1-t)f' = [a-1+(2-b)t]f$ satisfied by the function $f(t)$.

When we insert the expansion (18) into the right hand side of (5) and interchange sum and integral we obtain

$$M(a,b;z) = \frac{\Gamma(b)}{\Gamma(a)\Gamma(b-a)}\sum_{k=0}^{n-1}A_k(a,b)F_k(z) + R_n(a,b;z), \tag{21}$$

with

$$\begin{aligned} F_n(z) := {} & \int_0^1 e^{zt}\left(t-\frac{1}{2}\right)^n dt = \left(\frac{d}{dz}-\frac{1}{2}\right)^n\frac{e^z-1}{z} \\ & = \frac{n!}{(-z)^{n+1}}\left[e_n\left(\frac{z}{2}\right) - e^z e_n\left(-\frac{z}{2}\right)\right], \quad e_n(z) := \sum_{k=0}^{n}\frac{z^k}{k!}, \end{aligned} \tag{22}$$

and

$$R_n(a,b;z) := \frac{\Gamma(b)}{\Gamma(a)\Gamma(b-a)}\sum_{k=n}^{\infty}A_k(a,b)F_k(z). \tag{23}$$

The functions $F_n(z)$ are defined for any $z\in\mathbb{C}$; although for $z=0$, the right hand side of (22) must be understood in the limit sense. It is straightforward to see that, for $n = 0,1,2,...,$ the coefficients $F_n(z)$ satisfy the recurrence relation:

$$F_{n+1}(z) = \frac{e^z+(-1)^n}{z\,2^{n+1}} - \frac{n+1}{z}F_n(z), \qquad F_0(z) = \frac{e^z-1}{z}. \tag{24}$$

To prove the convergence of (21) we observe that

$$|F_n(z)| \le \int_0^1 e^{t\,\Re z}\left|t-\frac{1}{2}\right|^n dt \le \frac{H(z)}{2^n\,(n+1)},$$

with $H(z)$ defined in (14).

On the other hand, from [9, eqs. (16) and (25)] and the integral representation of ${}_2F_1$ [7, Sec. 15.6, eq. 15.6.1], we find that, for $n+2>\Re b$, the coefficients $A_n(a,b)$ may be written in the form

$$A_n(a,b)=\frac{\sin[(b-a)\pi]}{\pi}\int_0^1 t^{n+1-b}(1-t)^{b-a-1}(1-t/2)^{-n-1}dt$$

$$+\frac{(-1)^n\sin(a\,\pi)}{\pi}\int_0^1 t^{n+1-b}(1-t)^{a-1}(1-t/2)^{-n-1}dt.$$

Using that $(1-t/2)^{-n-1}\leq 2^{n+1}$ for $t\in(0,1)$, we have the following bound for coefficients $A_n(a,b)$ when $n+2>\Re b$:

$$|A_n(a,b)|\leq\frac{2^{n+1}}{\pi}\left(|\sin[(b-a)\pi]|\frac{\Gamma(\Re b-\Re a)\Gamma(2-\Re b+n)}{\Gamma(2-\Re a+n)}+|\sin(a\,\pi)|\frac{\Gamma(\Re a)\Gamma(2-\Re b+n)}{\Gamma(2+\Re a-\Re b+n)}\right). \tag{25}$$

Using these bounds in (23) we find

$$|R_n(a,b;z)|\leq\frac{2|\Gamma(b)|\,H(z)|\sin[(b-a)\pi]|\Gamma(\Re b-\Re a)}{\pi\,|\Gamma(a)||\Gamma(b-a)|}\sum_{k=n}^{\infty}\frac{\Gamma(2-\Re b+k)}{(k+1)\Gamma(2-\Re a+k)}+ \tag{26}$$

$$\frac{2|\Gamma(b)|\,H(z)|\sin(a\,\pi)|\Gamma(\Re a)}{\pi\,|\Gamma(a)||\Gamma(b-a)|}\sum_{k=n}^{\infty}\frac{\Gamma(2-\Re b+k)}{(k+1)\Gamma(2+\Re a-\Re b+k)}.$$

From $1-\Re b+n>0$ (and then $1-\Re b+k>0$) we have that

$$\frac{\Gamma(2-\Re b+k)}{k+1}<\frac{\Gamma(2-\Re b+k)}{k+1-\Re b}=\Gamma(1-\Re b+k),$$

and then (26) may be written in the form

$$|R_n(a,b;z)|\leq\frac{2|\Gamma(b)|\,H(z)|\sin[(b-a)\pi]|\Gamma(\Re b-\Re a)}{\pi\,|\Gamma(a)||\Gamma(b-a)|}\frac{\Gamma(1-\Re b+n)}{\Gamma(2-\Re a+n)}{}_2F_1\left(\begin{array}{c}1,\ 1-\Re b+n\\ \\ 2-\Re a+n\end{array}\middle|\,1\right)+$$

$$\frac{2|\Gamma(b)|\,H(z)|\sin(a\,\pi)|\Gamma(\Re a)}{\pi\,|\Gamma(a)||\Gamma(b-a)|}\frac{\Gamma(1-\Re b+n)}{\Gamma(2+\Re a-\Re b+n)}{}_2F_1\left(\begin{array}{c}1,\ 1-\Re b+n\\ \\ 2+\Re a-\Re b+n\end{array}\middle|\,1\right). \tag{27}$$

Finally, using formula [7, Sec. 15.4, eq. 15.4.20] for argument unity of ${}_2F_1$, we find the following bound for the remainder $R_n(a,b;z)$:

$$|R_n(a,b;z)|\leq H(z)\frac{2|\Gamma(b)|\,\Gamma(1-\Re b+n)}{\pi\,|\Gamma(a)||\Gamma(b-a)|}\left(\frac{|\sin[(b-a)\pi]|}{\Gamma(1-\Re a+n)}+\frac{|\sin(a\pi)|}{\Gamma(1+\Re a-\Re b+n)}\right). \tag{28}$$

From the right hand side of (28) and the Stirling formula for the gamma function we find that $R_n(a,b;z) \sim n^{-\min\{\Re a, \Re b-\Re a\}}$ when $n \to \infty$. Then, the series (21) is convergent for $\Re b > \Re a > 0$. Moreover, the expansion (21) is uniformly convergent in z for $\Re z \le \Re z_0$ with fixed $z_0 \in \mathbb{C}$.

An immediate consequence is the following expansion:

$$e^{-z}M(a,b;z) = \frac{e^{-z}\Gamma(b)}{\Gamma(a)\Gamma(b-a)}\sum_{k=0}^{n-1} A_k(a,b)F_k(z) + e^{-z}R_n(a,b;z). \tag{29}$$

For $n+1 > \Re b$, the absolute value of the remainder $e^{-z}R_n(a,b;z)$ is bounded by the right hand side of (28) with $H(z)$ replaced by $H(-z)$. Then, expansion (29) is uniformly convergent in z for $\Re z \ge \Re z_0$ for any fixed $z_0 \in \mathbb{C}$.

3 A uniform convergent expansion of $U(a,b;z)$

From the connection formula (2) and either of the expansions derived in the previous section for the function $M(a,b;z)$, it is possible to derive uniform expansions for the function $U(a,b;z)$ similar to those of $M(a,b;z)$, either in terms of incomplete gamma functions or in terms of elementary functions. The drawback is that these expansions are only valid in a restricted region of the parameters a and b: $0 < \Re a < 1$ and $\Re a < \Re b < \Re a + 1$.

The derivation of a uniform expansion of the function $U(a,b;z)$ in a large region of the parameters a and b requires the use of similar techniques to those used in the previous section for the function $M(a,b;z)$. Although it is a little bit more cumbersome now: the integration interval in the integral representation (6) is unbounded and then, just the use of the Taylor expansion of $f(t) = t^{a-1}(1+t)^{b-a-1}$ or of $f(t) = (1+t)^{b-a-1}$ does not work. Then, we separate the integral in the right hand side of (6) into two integrals: $U(a,b;z) = U_1(a,b;z) + U_2(a,b;z)$, with

$$U_1(a,b;z) := \frac{1}{\Gamma(a)}\int_0^1 e^{-zt}t^{a-1}(1+t)^{b-a-1}dt \qquad \text{and}$$

$$U_2(a,b;z) := \frac{1}{\Gamma(a)}\int_1^\infty e^{-zt}t^{a-1}(1+t)^{b-a-1}dt. \tag{30}$$

In the following two subsections we investigate uniform expansions of these two functions separately.

3.1 A uniform convergent expansion of $U_1(a,b;z)$

Consider the Taylor expansion at $t=0$ of the factor $(1+t)^{b-a-1}$,

$$(1+t)^{b-a-1} = \sum_{k=0}^{n-1} \frac{(1+a-b)_k}{k!}(-t)^k + r_n^1(a,b,t), \qquad t \in (0,1), \tag{31}$$

where $r_n^1(a,b,t)$ is the Taylor remainder:

$$r_n^1(a,b,t) := \sum_{k=n}^{\infty} \frac{(1+a-b)_k}{k!}(-t)^k, \qquad t \in (0,1).$$

After straightforward manipulations we obtain

$$r_n^1(a,b,t) = \frac{(1+a-b)_n}{n!}(-t)^n {}_2F_1\left(\begin{array}{c} n+1+a-b,\ 1 \\ n+1 \end{array}\middle| -t\right), \qquad t \in (0,1).$$

Inserting (31) into first integral of (30) and interchanging sum and integral we obtain

$$U_1(a,b;z) = \frac{1}{\Gamma(a)} \sum_{k=0}^{n-1} \frac{(-1)^k (1+a-b)_k}{k!\, z^{a+k}} \gamma(a+k,z) + R_n^1(a,b;z), \tag{32}$$

with

$$R_n^1(a,b;z) := \frac{1}{\Gamma(a)} \int_0^1 e^{-zt} t^{a-1} r_n^1(a,b,t) dt.$$

As the remainder $t^{a-1} r_n^1(a,b,t)$ is an integrable function in $(0,1)$ we have that

$$|R_n^1(a,b;z)| \leq \frac{e^{-\Re z}}{|\Gamma(a)|} \int_0^1 t^{\Re a-1} |r_n^1(a,b,t)| dt.$$

This means that the expansion (32) is uniform in z for $\Re z \geq \Re z_0$ and fixed $z_0 \in \mathbb{C}$. Moreover, we can give a bound for the remainder $R_n^1(a,b;z)$. Using the integral representation of the hypergeometric function [7, Sec. 15.6, eq. 15.6.1] we have that

$$|r_n^1(a,b,t)| \leq \frac{t^n |(1+a-b)_n|}{n!} {}_2F_1\left(\begin{array}{c} n+1+\Re a-\Re b,\ 1 \\ n+1 \end{array}\middle| -t\right), \tag{33}$$

and then

$$|R_n^1(a,b;z)| \leq e^{-\Re z} \frac{|(1+a-b)_n|}{n!\,|\Gamma(a)|} \int_0^1 t^{n+\Re a-1} {}_2F_1\left(\begin{array}{c} n+1+\Re a-\Re b,\ 1 \\ n+1 \end{array}\middle| -t\right) dt =$$
$$e^{-\Re z} \frac{|(1+a-b)_n|}{n!\,|\Gamma(a)|\,(n+\Re a)} {}_3F_2\left(\begin{array}{c} n+1+\Re a-\Re b,\ 1,\ n+\Re a \\ n+1,\ n+1+\Re a \end{array}\middle| -1\right). \tag{34}$$

3.2 A uniform convergent expansion of $U_2(a,b;z)$

The uniform expansion of $U_2(a,b;z)$ requires a different strategy as the use of the Taylor expansion of $f(t)=t^{a-1}(1+t)^{b-a-1}$ does not work. Then, we write $f(t)=t^{b-2}(1+t^{-1})^{b-a-1}$ and consider the Taylor expansion of $(1+t^{-1})^{b-a-1}$ at $t=\infty$:

$$(1+t^{-1})^{b-a-1}=\sum_{k=0}^{n-1}\frac{(1+a-b)_k}{k!}(-t)^{-k}+r_n^2(a,b,t), \qquad t\in(1,\infty), \tag{35}$$

where $r_n^2(a,b,t)$ is the Taylor remainder:

$$r_n^2(a,b,t):=\sum_{k=n}^{\infty}\frac{(1+a-b)_k}{k!}(-t)^{-k}, \qquad t\in(1,\infty).$$

After straightforward manipulations we obtain

$$r_n^2(a,b,t)=\frac{(1+a-b)_n}{n!(-t)^n}\,{}_2F_1\left(\begin{array}{c} n+1+a-b,\ 1 \\ n+1 \end{array}\middle| -\frac{1}{t}\right), \qquad t\in[1,\infty).$$

Replacing (35) into second integral of (30) and interchanging sum and integral we obtain

$$U_2(a,b;z)=\frac{z^{1-b}}{\Gamma(a)}\sum_{k=0}^{n-1}\frac{(1+a-b)_k}{k!}(-z)^k\Gamma(b-1-k,z)+R_n^2(a,b;z), \tag{36}$$

where $\Gamma(b,z)$ is an incomplete gamma function and

$$R_n^2(a,b;z):=\frac{1}{\Gamma(a)}\int_1^\infty e^{-zt}t^{b-2}r_n^2(a,b,t)dt.$$

The incomplete gamma functions in the right hand side of (36) may be computed recurrently from (12) and from [5, Sec. 8.8, eq. 8.8.2],

$$\Gamma(a+k+1,z)=a\Gamma(a+k,z)+z^{a+k}e^{-z}, \qquad k=0,1,2,...$$

As $t^{b-2}r_n^2(a,b,t)$ is an integrable function in $(1,\infty)$ we have that

$$|R_n^2(a,b;z)|\le\frac{e^{-\Re z}}{|\Gamma(a)|}\int_1^\infty t^{\Re b-2}|r_n^2(a,b,t)|dt.$$

This means that the expansion (36) is uniform in z for $\Re z\ge\Re z_0$ with fixed $z_0\in\mathbb{C}$. Moreover, we can give a bound for the remainder $R_n^2(a,b;z)$. Using the integral representation of the hypergeometric function [7, Sec. 15.6, eq. 15.6.1] we find that

$$|r_n^2(a,b,t)|\le\frac{|(1+a-b)_n|}{n!\,t^n}\,{}_2F_1\left(\begin{array}{c} n+1+\Re a-\Re b,\ 1 \\ n+1 \end{array}\middle| -\frac{1}{t}\right), \tag{37}$$

and then

$$|R_n^2(a,b;z)| \le e^{-\Re z}\frac{|(1+a-b)_n|}{n!\,|\Gamma(a)|}\int_0^1 t^{n-\Re b}{}_2F_1\left(\begin{array}{c} n+1+\Re a-\Re b,\ 1 \\ \\ n+1 \end{array}\middle| -t\right) dt =$$

$$e^{-\Re z}\frac{|(1+a-b)_n|}{|\Gamma(a)|\,n!\,(n+1-\Re b)}{}_3F_2\left(\begin{array}{c} n+1+\Re a-\Re b,\ 1,\ n+1-\Re b \\ \\ n+1,\ n+2-\Re b \end{array}\middle| -1\right). \tag{38}$$

3.3 A uniform convergent expansion of $U(a,b;z)$

From formulas (32) and (36) we obtain the following expansion of the confluent hypergeometric function $U(a,b;z)$:

$$U(a,b;z) = \sum_{k=0}^{n-1}\frac{(1+a-b)_k}{\Gamma(a)\,k!\,(-1)^k}\left(z^{-a-k}\gamma(a+k,z) + z^{1-b+k}\Gamma(b-1-k,z)\right) + R_n(a,b;z), \tag{39}$$

with $R_n(a,b;z) = R_n^1(a,b;z) + R_n^2(a,b;z)$. From the integral representation of the hypergeometric function ${}_3F_2$ and assuming that $n+1 > \Re b$, it is straightforward to see that the absolute value of the right sides of (34) and (38) are both bounded by 1. Then, by using [7, Sec. 15.6, eq. 15.6.1] and [8, Sec. 16.5, eq. 16.5.2] we find

$$|R_n(z,a,b)| \le \frac{e^{-\Re z}|(1+a-b)_n|}{n!\,|\Gamma(a)|}\left(\frac{1}{(n+\Re a)} + \frac{1}{(n-\Re b+1)}\right). \tag{40}$$

From the right side of (40) and the Stirling formula for the gamma function we find that $R_n(a,b;z) \sim n^{-(\Re b-\Re a+1)}$ when $n\to\infty$ and the series (39) is convergent for $\Re b - \Re a > -1$. Moreover, the expansion (39) is uniformly convergent in $|z|$ for $\Re z > 0$.

4 Final remarks and numerical experiments

The incomplete gamma function and the Bessel functions are particular cases of the confluent hypergeometric function $M(a,b;z)$ [6, Sec. 13.6, eqs. 13.6.5 and 13.6.9]:

$$a\gamma(a,z) = z^a M(a,a+1,-z),$$

and

$$\Gamma(\nu+1)\,I_\nu(z) = (z/2)^\nu\,e^{-z}\,M(\nu+1/2,2\nu+1,2z).$$

The expansions [2, eq.(8)] and [3, eq.(5)] may be derived from these relations and (21).

We observe from the bounds given in (16), (28) and (40) that the approximations given in formulas (10), (17), (21), (29) and (39) may be not very accurate for small values of $\Re a$

or of $\Re b - \Re a$. Then, the smaller $\Re a$ and/or $\Re b - \Re a$ are, the more terms we need in the expansions (10), (17), (21), (29) and (39) to obtain a given accuracy. For small values of $\Re a$ or of $\Re b - \Re a$, it is preferable to combine these expansions with the recurrence relations for the parameters a and b given in [6, Sec. 13.3(i)].

In Tables 1 and 2 we compute, for several values of a, b and z, the number of terms required by the Taylor expansions, the asymptotic expansions, and the uniform expansions, to get a relative error of the order 10^{-6} in the approximation of the functions $M(a,b;z)$ and $U(a,b;z)$. Blank spaces in the column corresponding to the asymptotic expansions mean that the accuracy 10^{-6} is not attained at any order of the approximation.

	$a = 2.1+i,\ \ b = 4.2+1.2i$			
z	Taylor	Asympt.	(10)	(21)
-5	19	-	44	> 50
-10	30	-	10	> 50
-15	45	4	5	> 50
-20	> 50	3	4	> 50
$5e^{3i\pi/4}$	19	-	> 50	> 50
$10e^{3i\pi/4}$	33	-	> 50	> 50
$15e^{3i\pi/4}$	47	5	16	> 50
$20e^{3i\pi/4}$	> 50	4	5	> 50
5i	18	-	> 50	> 50
10i	32	-	> 50	> 50
15i	46	7	> 50	> 50
20i	> 50	5	> 50	> 50
5	15	-	> 50	> 50
10	24	-	29	> 50
15	32	9	9	> 50
20	40	6	7	> 50

	$a = 12.1+i,\ \ b = 24.2+1.2i$			
z	Taylor	Asympt.	(10)	(21)
-5	16	-	> 50	> 50
-10	28	-	27	> 50
-15	40	-	23	46
-20	> 50	-	19	49
$5e^{3i\pi/4}$	16	-	> 50	> 50
$10e^{3i\pi/4}$	27	-	30	> 50
$15e^{3i\pi/4}$	40	-	27	> 50
$20e^{3i\pi/4}$	> 50	-	25	47
5i	15	-	> 50	> 50
10i	25	-	40	34
15i	36	-	41	35
20i	48	20	45	37
5	13	-	> 50	> 50
10	20	-	32	44
15	27	-	26	41
20	34	-	22	45

Table 1: Number of terms n required to get a relative error of the order 10^{-6} in the approximation of $M(a,b;z)$ for several values of a, b and z; using the Taylor expansion (1), the asymptotic expansion (3)-(4) and the uniform expansions (10) and (21) (these last ones combined with Kummer's transformation [6, Chap. 13, eq. 13.2.39], $M(a,b;z) = e^z M(b-a,b;-z)$, when $\Re z > 0$).

	$a = 1.8 + 0.7i, \quad b = 4.2 + 2.8i$		
z	Taylor	Asymptotic	(39)
5	19	-	> 50
10	32	-	20
15	46	9	10
20	> 50	6	7
$5\ e^{i\pi/6}$	19	-	35
$10\ e^{i\pi/6}$	33	-	24
$15\ e^{i\pi/6}$	46	9	10
$20\ e^{i\pi/6}$	> 50	6	7
$5\ e^{-i\pi/4}$	18	-	> 50
$10\ e^{-i\pi/4}$	31	-	49
$15\ e^{-i\pi/4}$	44	9	23
$20\ e^{-i\pi/4}$	> 50	6	8

	$a = 1.8 + 5.2i, \quad b = 12.3 + 0.8i$		
z	Taylor	Asymptotic	(39)
5	25	-	18
10	38	-	19
15	> 50	15	16
20	> 50	12	13
$5\ e^{i\pi/6}$	24	-	17
$10\ e^{i\pi/6}$	37	-	17
$15\ e^{i\pi/6}$	49	15	15
$20\ e^{i\pi/6}$	> 50	12	13
$5\ e^{-i\pi/4}$	26	-	25
$10\ e^{-i\pi/4}$	41	-	32
$15\ e^{-i\pi/4}$	> 50	19	31
$20\ e^{-i\pi/4}$	> 50	14	27

Table 2: Number of terms n required to get a relative error of the order 10^{-6} in the approximation of $U(a, b; z)$ for several values of a, b and z; using the Taylor expansion (2), the asymptotic expansion (3) and the uniform expansion (39).

The presence of the exponential factor in the function $H(z)$ for $\Re z > 0$ suggests that it is more appropriate to use the expansions (17) and (29) for positive values of $\Re z$ and (10) and (21) for negative values of $\Re z$.

Expansions (10), (17), (21), (29) and (39) have the three important properties mentioned in the introduction: (i) they are convergent (in the prescribed regions of the parameters a and b), (ii) they are given in terms of elementary functions (or at least more elementary than the functions M and U) and (iii) they are uniformly valid in z (with $\Re z$ bounded from below or from above). As far as we know, other expansions of the confluent hypergeometric functions given in the literature, in particular the ones mentioned in the introduction, do not satisfy these three properties simultaneously.

In Figures 1-3 we show the accuracy of the approximations (10), (29) and (39) for certain values of the parameters a and b, certain regions of the variable z and two different orders of the approximation $n = 3$ and $n = 5$.

In Figures 4-7 we show some numerical experiments that compare the expansions (10) and (39) with the Taylor expansions (1) and (1)-(2) and the asymptotic expansions (3) and (3)-(4). The Taylor expansions are more competitive for small $|z|$, the asymptotic expansions are more competitive for large $|z|$, whereas expansions (10), (17), (21), (29) and (39) are more competitive for intermediate values of $|z|$. And more importantly, these later expansions are uniform in $|z|$ (with $\Re z$ bounded from below or from above).

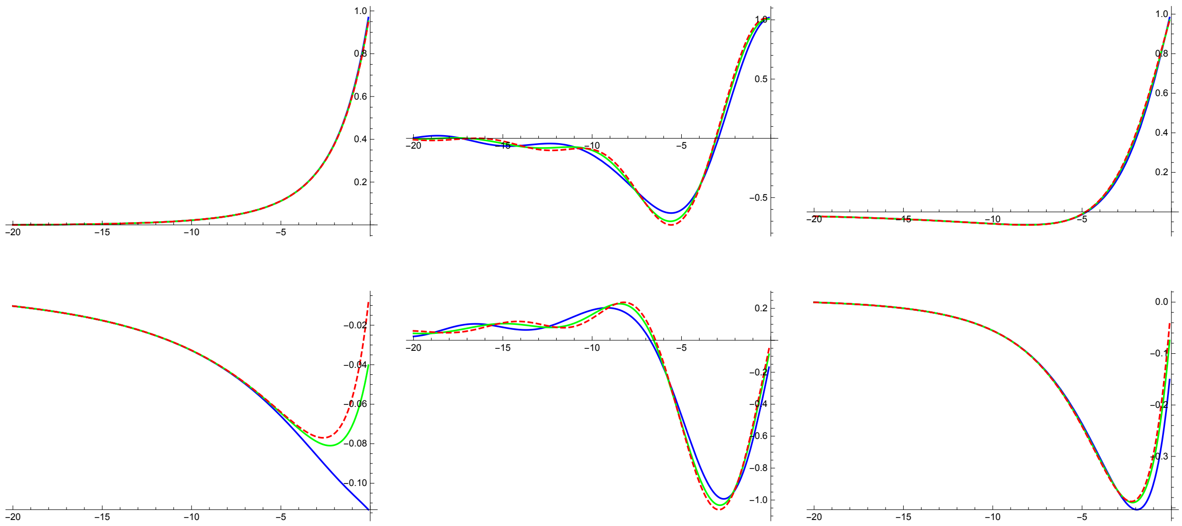

Figure 1: Graphics of $M(2.1+i, 4.2+1.2i; z)$ (red dashed) and the approximations given in (10) for $n = 3$ (blue), $n = 5$ (green) in several intervals: $[-20, 0]$ (left), $[-20i, 0]$ (middle) and $[-20e^{i\pi/4}, 0]$ (right). The top graphics represent the real part and the bottom pictures the imaginary part.

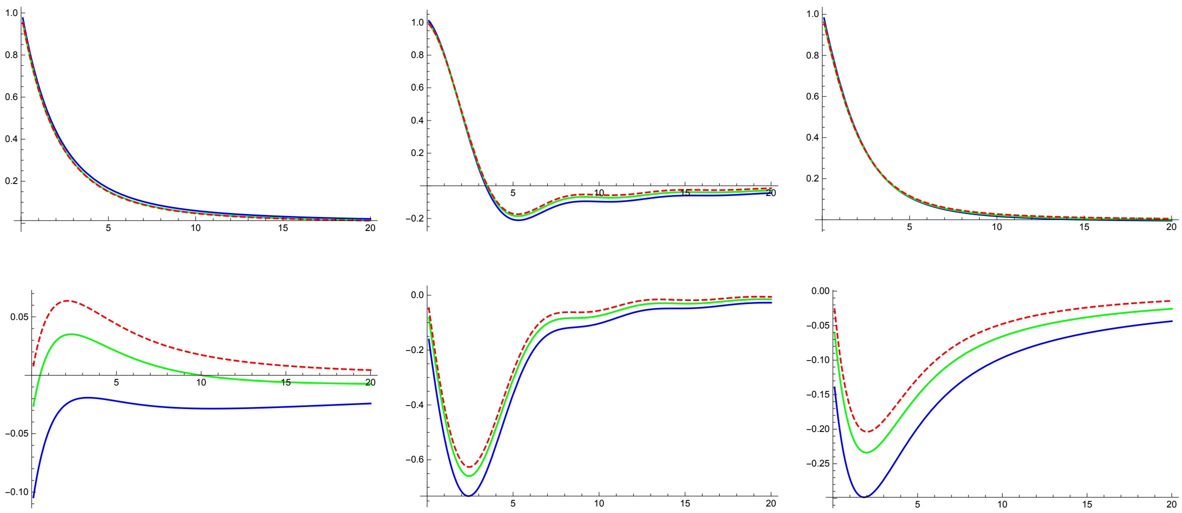

Figure 2: Graphics of $e^{-z}M(2.1+i, 4.2+1.2i; z)$ (red dashed) and the approximations given in (29) for $n = 3$ (blue), $n = 5$ (green) in several intervals: $[0, 20]$ (left), $[0, 20i]$ (middle) and $[0, 20e^{i\pi/4}]$ (right). The top graphics represent the real part and the bottom pictures the imaginary part.

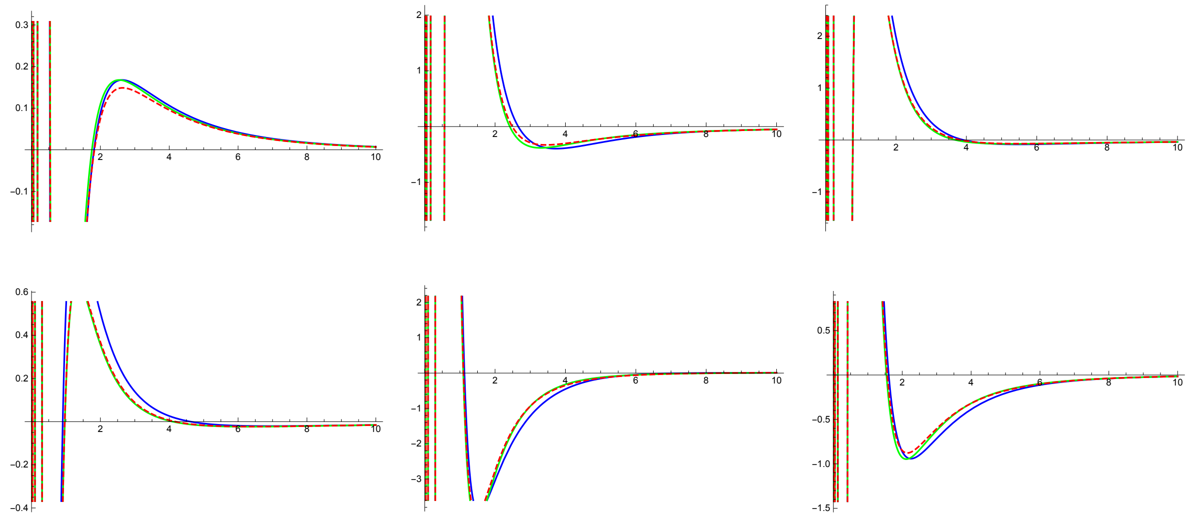

Figure 3: Graphics of $U(1.8+0.7i, 4.2+2.8i; z)$ (red dashed) and the approximations given in (39) for $n=3$ (blue), $n=5$ (green) in several intervals: $[0,10]$ (left), $[0, 10e^{i\pi/3}]$ (middle) and $[0, 10e^{i\pi/4}]$ (right). The top graphics represent the real part and the bottom pictures the imaginary part.

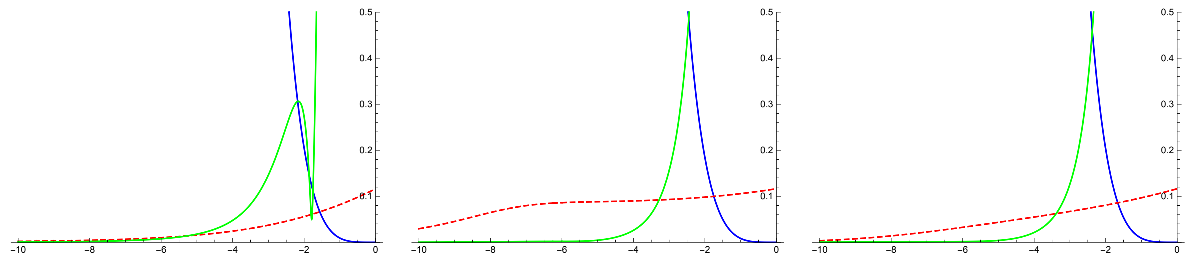

Figure 4: Absolute value of the relative errors in the third order approximation ($n=3$) of $M(2.1+i, 4.2+1.2i, z)$ by using the uniform expansion (10) (green), the Taylor expansion (1) (blue) and the asymptotic expansion (3)-(4) (red dashed) in the intervals $z\in[-10,0]$ (left), $z\in[-10e^{-i\pi/3},0]$ (middle) and $z\in[-10e^{-i\pi/4},0]$ (right).

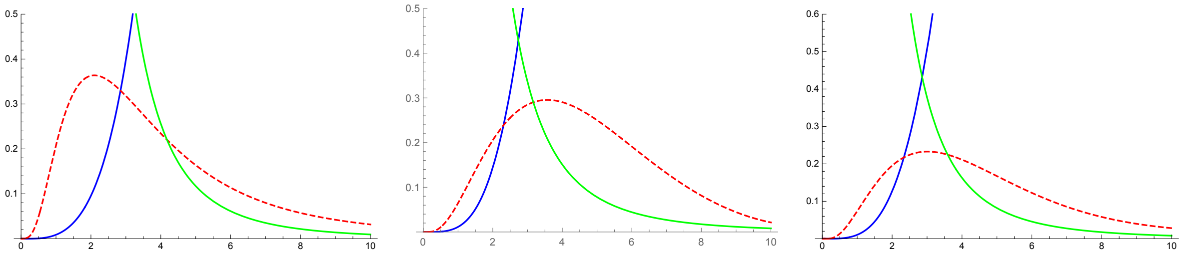

Figure 5: Absolute value of the relative errors in the third order approximation ($n = 3$) of $U(1.8 + 0.7i, 4.2 + 2.8i; z)$ by using the uniform expansion (39) (red dashed), the Taylor expansion (1)-(2) (blue) and the asymptotic expansion (3) (green) in the intervals $z \in [0, 10]$ (left), $z \in [0, 10e^{i\pi/3}]$ (middle) and $z \in [0, 10e^{i\pi/4}]$ (right).

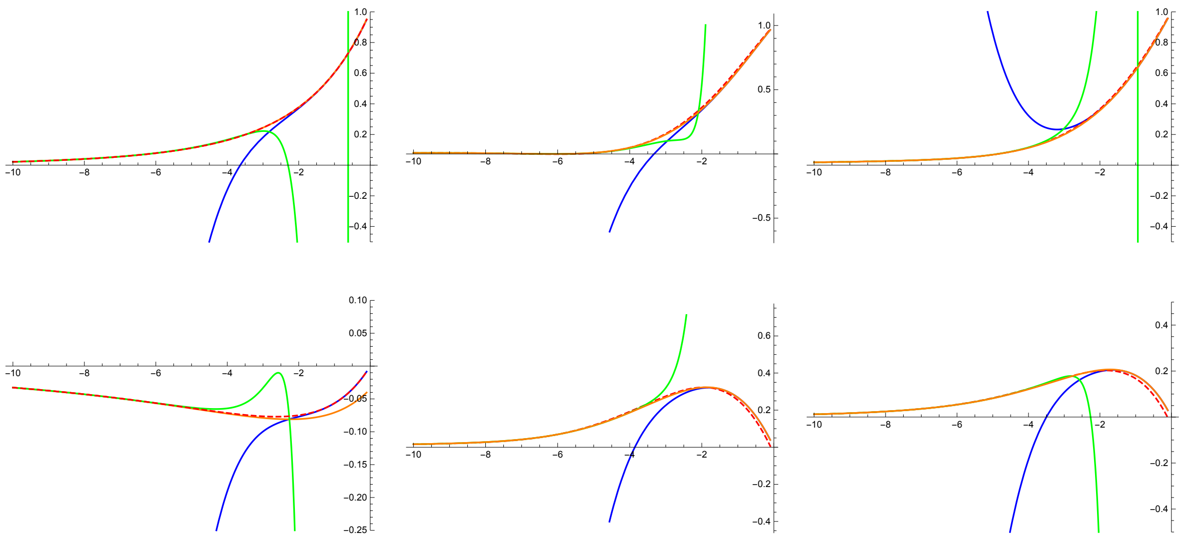

Figure 6: Graphics of $M(2.1 + i, 4.2 + 1.2i; z)$ (red dashed) and the fifth order approximations ($n = 5$) given in (10) (orange), the Taylor expansion (1) (blue) and the asymptotic expansion (3)-(4) (green) in several intervals: $[-10, 0]$ (left), $[-10e^{-i\pi/3}, 0]$ (middle) and $[-10e^{-i\pi/4}, 0]$ (right). The top graphics represent the real part and the bottom pictures the imaginary part.

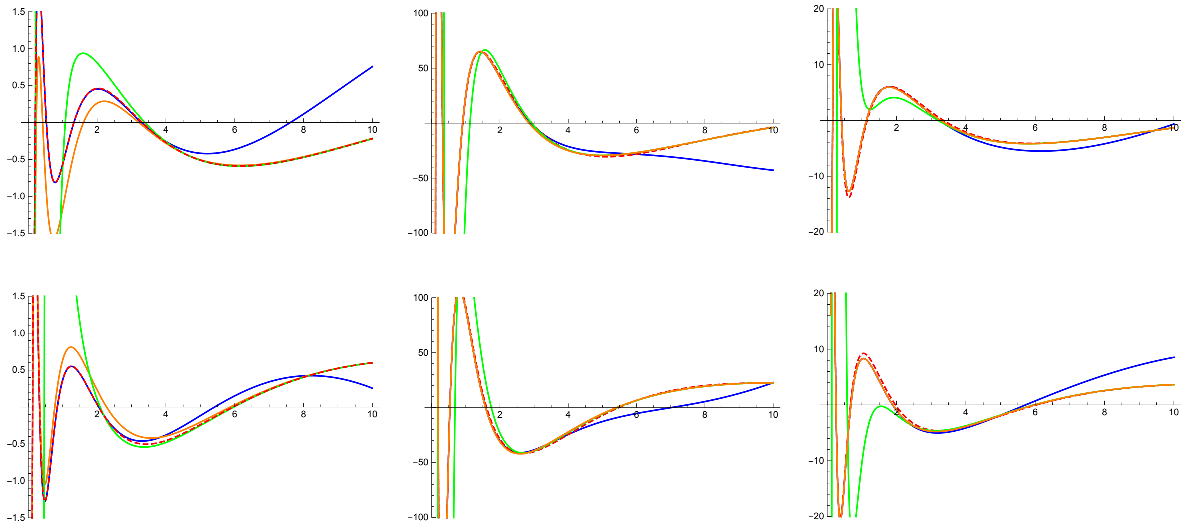

Figure 7: Graphics of $U(0.1 + 2i, 2.2 + 3.2i; z)$ (red dashed) and the third order approximations ($n = 3$) given in (39) (orange), the Taylor expansion (1)-(2) (blue) and the asymptotic expansion (3) (green) in several intervals: $[0, 10]$ (left), $[0, 10i]$ (middle) and $[0, 10e^{i\pi/4}]$ (right). The top graphics represent the real part and the bottom pictures the imaginary part.

5 Acknowledgments

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