Uniform representations of the incomplete beta function in terms of elementary functions

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Abstract

We consider the incomplete beta function $B_z(a,b)$ in the maximum domain of analyticity of its three variables: $a,b,z\in\mathbb{C}$, $-a\notin\mathbb{N}$, $z\notin[1,\infty)$. For $\Re b\le 1$ we derive a convergent expansion of $z^{-a}B_z(a,b)$ in terms of the function $(1-z)^b$ and of rational functions of z that is uniformly valid for z in any compact in $\mathbb{C}\setminus[1,\infty)$. When $-b\in\mathbb{N}\cup\{0\}$, the expansion also contains a logarithmic term of the form $\log(1-z)$. For $\Re b\ge 1$ we derive a convergent expansion of $z^{-a}(1-z)^bB_z(a,b)$ in terms of the function $(1-z)^b$ and of rational functions of z that is uniformly valid for z in any compact in the exterior of the circle $|z-1|=r$ for arbitrary $r>0$. The expansions are accompanied by realistic error bounds. Some numerical experiments show the accuracy of the approximations.

2010 AMS *Mathematics Subject Classification:* 33B20; 41A58; 41A80.

Keywords & Phrases: incomplete beta function; convergent expansions; uniform expansions.

1 Introduction

We may find in the literature a large variety of convergent or asymptotic expansions of the special functions of the mathematical physics that have the important property of being given in terms of elementary functions: direct or inverse powers of a certain complex variable z and, sometimes, other elementary functions of z. However, quite often, these expansions are not simultaneously valid for small and large values of $|z|$. Thus, it would be interesting to derive new convergent expansions of these functions in terms of elementary functions that hold uniformly in z in a large region of the complex plane that include small and large values of $|z|$.

In [1] and [6], the authors derived new uniform convergent expansions of the incomplete gamma function $\gamma(a,z)$ and the Bessel functions $J_\nu(z)$ and $Y_\nu(z)$ in terms of elementary functions of z that hold uniformly in unbounded regions of $\mathbb{C}$ that contain the point $z=0$. The starting point of the technique used in [1] and [6] is an appropriate integral representation of these functions. The key point is the use of the Taylor expansion, at an appropriate point of the integration interval, of a certain factor of the integrand that is independent of the variable z. This fact, the independence of this factor with respect to z, translates into a convergent uniform expansion in a large region of the complex $z-$plane. The expansions given in [1] and [6] are accompanied by error bounds and numerical experiments showing the accuracy of the approximations.

In this work, we continue that line of investigation considering the incomplete beta function $B_z(a,b)$. This function is used extensively in statistics as the probability integrals of the beta distribution and as special cases of the (negative) binomial distribution, Student's distribution, and the F (variance-ratio) distribution [3]. Among its physical applications, we mention its use in Monte Carlo simulations in statistical mechanics [4] and in cosmology [2]. We consider $B_z(a,b)$ as a function of the complex variable z, and derive new convergent expansions uniformly valid in an unbounded region of the complex $z-$plane that contains the point $z=0$. The starting point is the integral definition of the incomplete beta function [9, Sec. 8, Eq. 8.17.1],

$$z^{-a}\mathrm{B}_z(a,b) := \int_0^1 t^{a-1}(1-zt)^{b-1}dt, \tag{1}$$

valid for $\Re a>0$ and $z\in\mathbb{C}\setminus[1,\infty)$. The incomplete beta function $B_z(a,b)$ reduces to the ordinary beta function $B(a,b)$ when $z=1$ and, except for positive integer values of b, has a branch cut discontinuity in the complex $z-$plane running from 1 to ∞. When a or b are positive integers, the incomplete beta function is an elementary function of z.

For reasons that will become clear later, it is convenient to consider the integral (1) only for $\Re b\le 1$. When $\Re b\ge 1$, we consider instead the following integral representation of $\mathrm{B}_z(a,b)$ that may be obtained from (1) after the change of variable $t\to 1-t$:

$$z^{-a}\mathrm{B}_z(a,b) = (1-z)^{b-1}\int_0^1 (1-t)^{a-1}\left(1+\frac{z}{1-z}t\right)^{b-1}dt, \tag{2}$$

valid for $\Re a>0$ and $z\in\mathbb{C}\setminus[1,\infty)$.

By using the recurrence relation [9, Sec. 8, Sec. 8.17.20],

$$\mathrm{B}_z(a,b)=\frac{a+b}{a}\mathrm{B}_z(a+1,b)+\frac{z^a(1-z)^b}{a},$$

we find that the function $\mathrm{B}_z(a,b)$ may be analytically continued in the complex variable a to the negative half-plane $\Re a\le 0$ with poles at the negative integers $a=-1,-2,-3,\ldots$.

And reciprocally, by using repeatedly this formula we have that $\mathrm{B}_z(a,b)$, with $\Re a \leq 0$, may be written as a linear combination of elementary functions of its three variables and an incomplete beta function with $\Re a > 0$. Therefore, without loss of generality, in the remaining of the paper we restrict ourselves to $\Re a > 0$.

The power series expansion of the incomplete Beta function is given by [10]

$$z^{-a} B_z(a,b) = \sum_{n=0}^{\infty} \frac{(1-b)_n}{n!(a+n)} z^n. \tag{3}$$

This expansion may be derived from the integral representation (1) by replacing the factor $(1-zt)^{b-1}$ by its Taylor series at the origin and interchanging series and integral. This Taylor series expansion converges for $t \in [0,1]$, but the convergence is not uniform in $|z|$. Therefore, expansion (3) is convergent, but not uniformly in $|z|$ as the remainder is unbounded when $|z| \to \infty$.

From the hypergeometric function representation of $B_z(a,b)$ [9, Sec. 8.17, Eq. (8.17.7)],

$$B_z(a,b) = \frac{z^a}{a} {}_2F_1\left(a, 1-b; a+1; z\right),$$

and combining the formulas [8, Sec. 15.2, Eq. (15.2.2)] and [8, Sec. 15.8, Eqs. (15.8.2) and (15.8.8)], we obtain, for $1-a-b \notin \mathbb{N} \cup \{0\}$ and $|\mathrm{ph}(-z)| < \pi$, the asymptotic expansion

$$\begin{aligned} z^{-a} B_z(a,b) \sim & \frac{\pi \Gamma(a)}{\Gamma(a+b)\sin[\pi(1-a-b)]} \\ & \times \left[\frac{(-z)^{-a}}{\Gamma(1-b)} - \frac{(-z)^{b-1}}{\Gamma(a)\Gamma(1-a-b)} \sum_{k=0}^{\infty} \frac{(1-b)_k}{(1-a-b-k)k! z^k} \right]. \end{aligned} \tag{4}$$

On the other hand, if $1-a-b \in \mathbb{N} \cup \{0\}$, $|z| > 1$ and $|\mathrm{ph}(-z)| < \pi$, we have

$$\begin{aligned} z^{-a} B_z(a,b) \sim & \frac{\Gamma(a)(-z)^{-a}}{\Gamma(1-b)} \sum_{k=0}^{-a-b} \frac{(a)_k(-a-b-k)!}{k!\,\Gamma(1-k)\, z^k} \\ & + (-z)^{-a} \sum_{k=0}^{\infty} \frac{(-1)^k (1-b)_k}{k!(k+1-a-b)!\,\Gamma(a+b-k)\, z^{k+1-a-b}} \\ & \times \left[\log(-z) + \psi(k+1) + \psi(k+2-a-b) - \psi(1-b+k) - \psi(a+b-k) \right], \end{aligned} \tag{5}$$

where ψ denotes the digamma function. Expansions (4) and (5) are asymptotic expansions of the incomplete beta function for large $|z|$, but the remainders are unbounded when $|z| \to 0$ and then, these expansions are not uniform in $|z|$ either. Other large parameter asymptotic approximations with certain uniformity properties with respect to the parameters can be found in [7, 11].

Expansions (3), (4) and (5) have the good property of being given in terms of elementary functions of z, but they have the inconvenience of not being uniform in $|z|$ in unbounded regions of the complex plane that include the point $z = 0$. In this paper we show that it is possible to derive convergent expansions of $B_z(a, b)$ in terms of elementary functions that hold uniformly for z in an unbounded region of $\mathbb{C}$ that includes the point $z = 0$. As an illustration of the approximations that we are going to obtain (see Theorem 1 below), we derive, for example, the following one,

$$\frac{1}{z^{5/2}} B_z\left(\frac{5}{2}, \frac{1}{2}\right) = \frac{(32 + 40z - 5z^2) - (27z^2 + 56z + 32)\sqrt{1 - z}}{40\sqrt{2}\, z^3} + \epsilon(z), \tag{6}$$

with $|\epsilon(z)| < 0.0089$ in the negative half plane $\Re z \leq 0$. When $z = 0$, the right hand side of (6) must be understood in the limit sense.

In order to derive these kinds of approximations, we use in this paper the technique proposed in [1] and [6]: we consider a Taylor expansion of the factor t^{a-1} in (1) and of the factor $(1 - t)^{a-1}$ in (2). The factor t^{a-1} in (1) is not analytic at the origin unless $a \in \mathbb{N}$ (equivalently, the factor $(1 - t)^{a-1}$ in (2) is not analytic at $t = 1$). Following the arguments given in [6], we must consider the expansion of the factors t^{a-1} and $(1 - t)^{a-1}$ at the middle point $t = 1/2$ of the integration interval $(0, 1)$ in the respective integrals (1) and (2), in such a way that we assure that the integration interval is contained into the disk of convergence of the Taylor series. This Taylor expansion is convergent for any t in the integration interval of (1) or (2) and, obviously, it is independent of z. After the interchange of the series and the integral, the independence with respect to z, translates into a remainder that may be bounded independently of z in a large unbounded region of the complex $z-$plane that contains the point $z = 0$ and that we specify in Theorems 1 and 2 below. In the following section we consider the integral representation (1) for $\Re b \leq 1$. In Section 3 we consider the integral representation (2) for $\Re b \geq 1$. Throughout the paper we use the principal argument $\arg z \in (-\pi, \pi]$.

2 A uniform convergent expansion of $B_z(a, b)$ for $\Re b \leq 1$

In this section we consider the integral representation (1). We define the extended sector (see Figure 1):

$$S_\theta := \{\theta \leq |\arg(z)| \leq \pi\} \cup \left\{z \in \mathbb{C}; \left|z - \frac{1}{2}\right| \leq \frac{1}{2} \text{ and } |z - 1| \geq \sin\theta\right\}, \tag{7}$$

with arbitrary $0 < \theta \leq \pi/2$. We have the following theorem.

Theorem 1. *For $\Re a > 0$, $\Re b \leq 1$, $z \in S_\theta$, with $0 < \theta \leq \pi/2$, and $n = 1, 2, 3, \ldots$,*

$$z^{-a} B_z(a, b) = 2^{1-a} \sum_{k=0}^{n-1} \frac{(1 - a)_k}{k!} \beta_k(z, b) + R_n(z, a, b), \tag{8}$$

where $\beta_k(z,b)$ are the elementary functions

$$\beta_k(z,b) := \frac{1}{z^{k+1}} \sum_{j=0}^{k} \binom{k}{j} 2^j \, (z-2)^{k-j} \left[\frac{1-(1-z)^{j+b}}{j+b} (1-\delta_{j,-b}) - \delta_{j,-b} \log(1-z) \right]. \quad (9)$$

For $k = 1, 2, 3, \ldots$ and $b \neq 0$, they satisfy the recurrence relation

$$\beta_k(z,b) = \frac{1}{zb} \left[1 - (-1)^k (1-z)^b \right] - \frac{2k}{zb} \beta_{k-1}(z, b+1), \quad \beta_0(z,b) = \frac{1}{zb} \left[1 - (1-z)^b \right]. \quad (10)$$

On the other hand, for $k = 1, 2, 3, \ldots$ and $b = 0$,

$$\beta_k(z,0) = \frac{1-(-1)^k}{kz} + \left(1 - \frac{2}{z} \right) \beta_{k-1}(z,0), \quad \beta_0(z,0) = -\frac{1}{z} \log(1-z). \quad (11)$$

When $z = 0$, the above expressions must be understood in the limit sense. In the extended sector S_θ the remainder is bounded in the form

$$|R_n(z,a,b)| \leq [\sin(\theta)]^{\Re b - 1} \frac{e^{\pi |\Im b|} |(1-a)_n|}{n! \, 2^{\Re a - 1} \Re a} \max\{2^{\Re a - n - 1}, 1\}. \quad (12)$$

For $n \geq \Re a - 1 > 0$, the remainder term may also be bounded in the form

$$|R_n(z,a,b)| \leq [\sin(\theta)]^{\Re b - 1} \frac{e^{\pi |\Im b|} \, 2^{1-\Re a} n |(1-a)_n|}{(n+1)! \, (\Re a - 1)}. \quad (13)$$

The remainder term behaves as $R_n(z,a,b) \sim n^{-\Re a}$ as $n \to \infty$ uniformly in $|z|$ in the extended sector S_θ.

Proof. Consider the truncated series Taylor expansion of the factor t^{a-1} in the integrand of the integral definition (1) of $B_z(a,b)$ at the middle point $t = 1/2$ of the integration interval,

$$t^{a-1} = \frac{1}{2^{a-1}} \sum_{k=0}^{n-1} \frac{(1-a)_k}{k!} (1-2t)^k + r_n(t,a), \quad t \in (0,1], \quad (14)$$

where $r_n(t,a)$ is the Taylor remainder

$$r_n(t,a) := \frac{1}{2^{a-1}} \sum_{k=n}^{\infty} \frac{(1-a)_k}{k!} (1-2t)^k, \quad t \in (0,1]. \quad (15)$$

After suitable manipulations we can write

$$r_n(t,a) = \frac{(1-a)_n}{2^{a-1} n!} (1-2t)^n {}_2F_1\left(n+1-a; n+1; 1-2t \right), \quad t \in (0,1]. \quad (16)$$

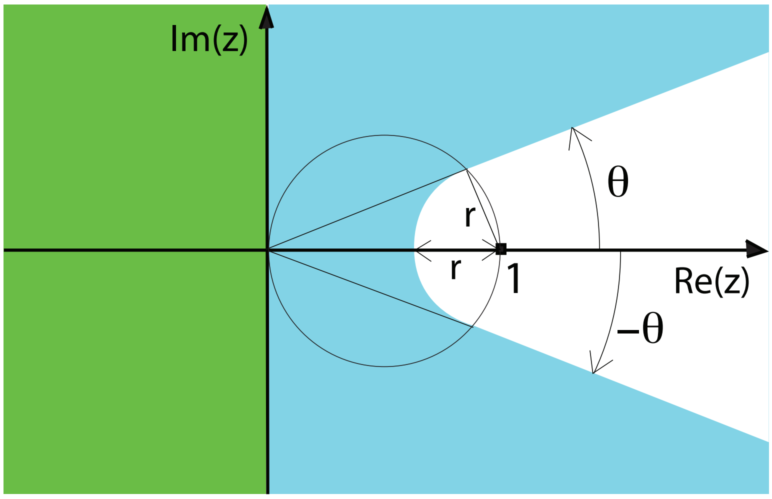

Figure 1: The blue and green regions comprise the extended sector S_θ defined in (7), with $r := \sin\theta$, $0 < \theta \le \pi/2$. In particular, $S_{\pi/2}$ is just the half plane $\Re z \le 0$ and $\lim_{\theta\to 0} S_\theta = \mathbb{C}\setminus[1,\infty)$. In the region S_θ, the remainder $R_n(z,a,b)$ is bounded independently of $|z|$ by the right hand side of (12).

Replacing (14) into the integral representation of $B_z(a,b)$ given in (1) and interchanging sum and integral we obtain (8) with

$$R_n(z,a,b) := \int_0^1 r_n(t,a)(1-zt)^{b-1}\,dt \tag{17}$$

and

$$\beta_k(z,b) := \int_0^1 (1-2t)^k(1-zt)^{b-1}dt = \frac{1}{z}\int_{1-z}^1 \left(1-\frac{2}{z}+2\frac{u}{z}\right)^k u^{b-1}du. \tag{18}$$

Expanding the first factor of the integrand in the second integral in powers of u and integrating term-wise we obtain (9).

Integrating by parts in any of the integrals in (18), it is straightforward to see that, for $k = 1,2,3,\ldots$, the functions $\beta_k(z,b)$ satisfy the recurrence relations (10) and (11).

In order to derive the bound (12), we need a bound for the factor $(1-zt)^{b-1}$ uniformly valid for $t\in[0,1]$. It is straightforward to check that, for $t\in[0,1]$ we have that $|(1-zt)^{b-1}| \le e^{\pi|\Im b|}M(z,b)$, with

$$M(z,b) := \begin{cases} 1, & \text{if } \Re(z)\le 0,\\ |1-z|^{\Re b-1}, & \text{if } \Re(1/z)\ge 1,\\ |\sin(\arg(z))|^{\Re b-1}, & \text{if } 0<\Re(1/z)<1. \end{cases} \tag{19}$$

The regions of the complex $z-$plane considered in this formula are depicted in Figure 2. For $z\in S_\theta$, with $0<\theta\le\pi/2$, we have that $M(z,b)\le[\sin(\theta)]^{\Re b-1}$. This inequality may be proved by using the following geometrical arguments: (i) at the points of the circle $|z-1/2| = 1/2$ we have that $|1-z| = |\sin(\arg(z))|$; (ii) the closest points of the sector

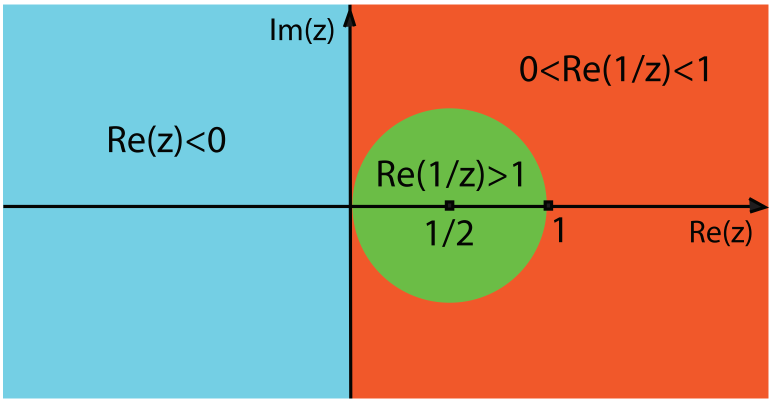

Figure 2: Different regions considered in formula (19). The green region $\Re(1/z) > 1$ is the open disk of radius $1/2$ with center at $z = 1/2$. The red region $0 < \Re(1/z) < 1$ is the intersection of the half plane $\Re z > 0$ with the exterior to this disk.

$\theta \le |\arg(z)| < \pi/2$ to the point $z = 1$ are just the two points obtained from the intersection of the rays $\arg z = \pm\theta$ with the circle $|z - 1/2| = 1/2$; (iii) the closest points of the region $\left\{z \in \mathbb{C}; \left|z - \frac{1}{2}\right| \le \frac{1}{2} \text{ and } |z - 1| \ge \sin\theta\right\}$ to the point $z = 1$ are those of the portion of circle $|z - 1| = \sin\theta$ contained inside this region.

Now we use that $r_n(t, a)$ is integrable in $(0, 1)$, the bound $|(1 - zt)^{b-1}| \le e^{\pi|\Im b|}[\sin(\theta)]^{\Re b - 1}$ for $t \in [0, 1]$ and introduce (16) in (17). We obtain

$$|R_n(z, a, b)| \le e^{\pi|\Im b|}[\sin(\theta)]^{\Re b-1} \frac{|(1-a)_n|}{n!\, 2^{\Re a - 1}} \int_0^1 |1 - 2t|^n \left| {}_2F_1(n + 1 - a, 1; n + 1; 1 - 2t)\right| dt.$$

From the integral representation of the hypergeometric function [8, Sec. 15.6, Eq. (15.6.1)] we find that, for $t \in (0, 1)$,

$$\left|{}_2F_1(n + 1 - a, 1; n + 1; 1 - 2t)\right| \le {}_2F_1(n + 1 - \Re a, 1; n + 1; 1 - 2t).$$

Then,

$$\begin{aligned} |R_n(a, z, b)| &\le e^{\pi|\Im b|}[\sin(\theta)]^{\Re b-1} \frac{|(1-a)_n|}{n!\, 2^{\Re a-1}} \int_0^1 |1-2t|^n \, {}_2F_1(n+1-\Re a, 1; n+1; 1-2t)\, dt \\ &= e^{\pi|\Im b|}[\sin(\theta)]^{\Re b-1} \frac{|(1-a)_n|}{n!\, 2^{\Re a}} \left[\frac{1}{\Re a} + \frac{1}{n+1} {}_2F_1(n+1-\Re a, 1; n+2; -1)\right]. \end{aligned} \tag{20}$$

Using now the contiguous function [8, Sec. 15.5, Eq. (15.5.14)] with $a = 1$, $b = n+1-\Re a$, $c = n + 1$ and $z = -1$ we find that

$$\begin{aligned} \frac{1}{n+1} {}_2F_1(n+1-\Re a, 1; n+2; -1) =& \frac{\Re a + 1}{n\Re a} {}_2F_1(n+1-\Re a, 1; n+1; -1) \\ & - \frac{2}{n\Re a} {}_2F_1(n+1-\Re a, 2; n+1; -1), \end{aligned}$$

and applying [8, Sec. 15.5, Eq. (15.5.11)] in the second hypergeometric function, we can write

$$2\,{}_2F_1\left(n+1-\Re a,2;n+1;-1\right)=n+(\Re a+1-2n){}_2F_1\left(n+1-\Re a,1;n+1;-1\right).$$

Thus, introducing these formulas into (20), we get

$$|R_n(z,a,b)|\le e^{\pi|\Im b|}[\sin(\theta)]^{\Re b-1}\frac{|(1-a)_n|}{n!\,2^{\Re a-1}\,\Re a}\,{}_2F_1\left(n+1-\Re a,1;n+1;-1\right). \tag{21}$$

From the integral representation of the hypergeometric function [8, Sec. 6, eq. (15.6.1)] we have that

$${}_2F_1\left(n+1-\Re a,1;n+1;-1\right)=n\int_0^1(1-t)^{n-1}(1+t)^{\Re a-n-1}dt\le\max\{2^{\Re a-n-1},1\}.$$

Bound (12) follows from (21) and this last inequality.
When $n\ge\Re a-1>0$, we consider again the integral representation of the hypergeometric function [8, Sec. 15.6, Eq. (15.6.1)]:

$$|{}_2F_1\left(n+1-a,1;n+1;1-2t\right)|\le n\int_0^1(1-s)^{n-1}[1-(1-2t)s]^{\Re a-n-1}ds.$$

When $t\in(0,1)$, $[1-(1-2t)s]^{\Re a-n-1}\le(1-s)^{\Re a-n-1}$ and then

$$|{}_2F_1\left(n+1-a,1;n+1;1-2t\right)|\le\frac{n}{\Re a-1}.$$

Therefore, from (16) we have that

$$|r_n\left(t,a\right)|\le\frac{|(1-a)_n|\,|1-2t|^n}{2^{\Re a-1}(n-1)!(\Re a-1)}.$$

Formula (13) follows straightforward introducing this bound in (17).

Finally, using the Stirling formula and [5, Eq. (30)] in (12) or (13) we obtain that $R_n(a,z)\sim n^{-\Re a}$ as $n\to\infty$. Then, any of the bounds (12) or (13) show the uniform character of the expansion (8) in the extended sector S_θ. □

Formula (6) follows from Theorem 1 with $a=5/2$, $b=1/2$ and $n=3$.

An error bound simpler than the bounds given in (12) and (13) can be found when a is real. It is given in the following proposition.

Proposition 1. *For $a>0$, $\Re b\le1$, $z\in S_\theta$ and $n=1,2,3,\ldots$, the error term $R_n(z,a,b)$ in Theorem 1 may be bounded in the form*

$$|R_n(z,a,b)|\le[\sin(\theta)]^{\Re b-1}\frac{e^{\pi|\Im b|}|(1-a)_n|}{2^{a-1}\,a\,n!}. \tag{22}$$

Proof. Take $p := \lfloor a \rfloor$ and define $\alpha := a - p$. Then we have that, for $k \geq p$,

$$(1-a)_k = (-1)^p (\alpha)_p (1-\alpha)_{k-p}. \tag{23}$$

Using this equality in (15) we find that

$$|r_n(t,a)| \leq \frac{(\alpha)_p}{2^{a-1}} \sum_{k=n}^{\infty} \frac{(1-\alpha)_{k-p}}{k!} |1-2t|^k.$$

We introduce this bound in (17) and, using that $|(1-zt)^{b-1}| \leq e^{\pi|\Im b|}[\sin(\theta)]^{\Re b-1}$ for $t \in [0,1]$ and (23), we find (22). □

Table 1 shows the first few terms of the approximation of $z^{-a}B_z(a,b)$ given by the expansion (8) for $\Re b \leq 1$ and $-b \notin \mathbb{N} \cup \{0\}$. These terms are rational functions of z and functions of $(1-z)^b$. When $-b \in \mathbb{N} \cup \{0\}$, the terms of the expansion (8) also contain the term $\log(1-z)$.

n	$(1-a)_n \beta_n(z,b)/n!$
0	$\dfrac{\left(1-(1-z)^b\right)}{bz}$
1	$\dfrac{\left((a-2)(1+b)+(a-2-ab)(1-z)^b\right)}{b(b+1)z} + \dfrac{2(a-1)\left(1-(1-z)^b\right)}{b(b+1)z^2}$
2	$\dfrac{\left((-3+a)(-2+a)(1+b)(2+b)+\left(a(10+(-5+b)b)+a^2\left(-2+b-b^2\right)-2\left(6+b+b^2\right)\right)(1-z)^b\right)}{2b(b+1)(b+2)z}$ $+\dfrac{2(-1+a)\left(-(-3+a)(2+b)+(-6-a(-2+b)+b)(1-z)^b\right)}{b(b+1)(b+2)z^2} + \dfrac{4(-2+a)(-1+a)\left(1-(1-z)^b\right)}{b(b+1)(b+2)z^3}$

Table 1: First few terms in the expansion (8) of $z^{-a}B_z(a,b)$ when $-b \notin \mathbb{N} \cup \{0\}$.

In Figure 3 we plot $z^{-1.5}B_z(1.5, 0.5)$ and the approximations given in Theorem 1 for $n = 1, 2, 3, 4, 5$. This is a numerical experiment about the rate of convergence provided by (8). We also observe the uniform character of the approximation in the region S_θ.

3 A uniform convergent expansion of $B_z(a,b)$ for $\Re b \geq 1$

In this section we consider the integral representation (2). For any $0 < r \leq 1$, consider the punctured complex plane at $z = 1$ with the interval $[1, \infty)$ removed:

$$C_r := \{z \in \mathbb{C}; |z-1| \geq r, \ |\arg(1-z)| < \pi\}. \tag{24}$$

We have the following theorem.

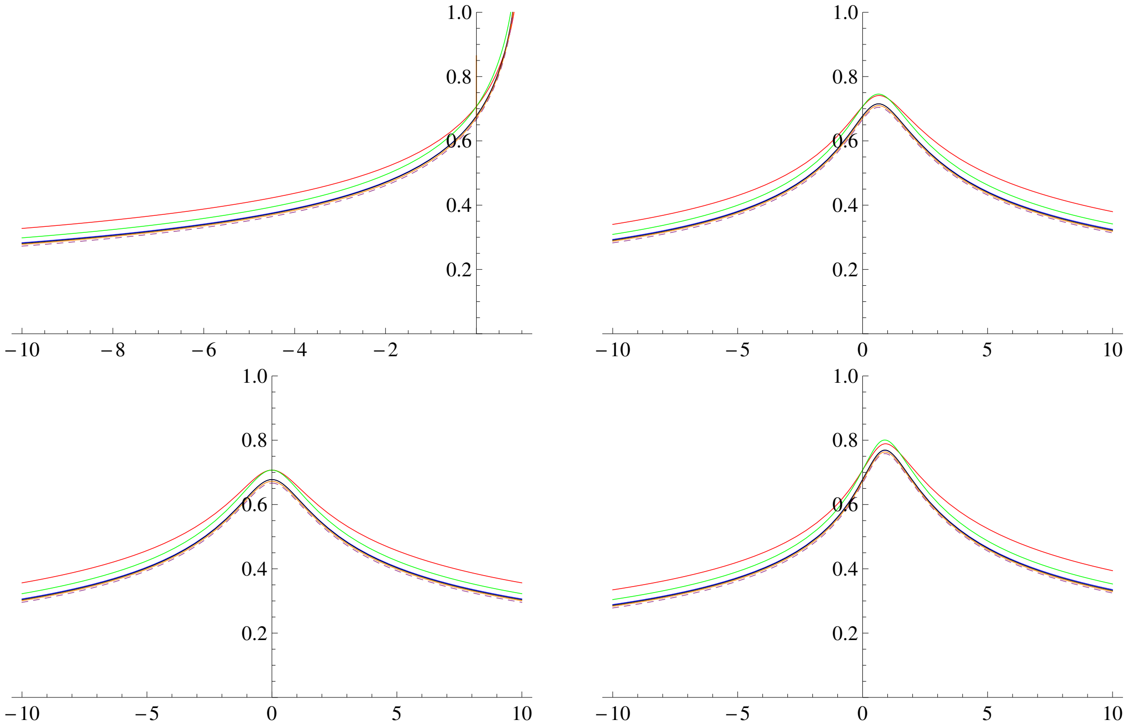

Figure 3: Plots of the absolute value of $z^{-1.5}B_z(1.5, 0.5)$ (dashed) and the approximations given in Theorem 1 for $n = 1$ (red), $n = 2$ (green), $n = 3$ (blue), $n = 4$ (black) and $n = 5$ (orange) in several intervals: $[-10, 1]$ (top left), $[-10e^{i\pi/4}, 10e^{i\pi/4}]$ (top right), $[-10e^{i\pi/2}, 10e^{i\pi/2}]$ (bottom left) and $[-10e^{-i\pi/3}, 10e^{-i\pi/3}]$ (bottom right).

Theorem 2. *For $\Re a > 0$, $\Re b \geq 1$, $z \in C_r$ with $0 < r \leq 1$, and $n = 1, 2, 3, \ldots$,*

$$z^{-a}(1-z)^{1-b}B_z(a, b) = 2^{1-a}\sum_{k=0}^{n-1}\frac{(-1)^k(1-a)_k}{k!}\beta_k(z, b) + R_n(z, a, b), \tag{25}$$

where the functions $\beta_k(z, b)$ are the elementary functions

$$\beta_k(z, b) := \frac{1}{z^{k+1}}\sum_{j=0}^{k}\binom{k}{j}(-2)^j(2-z)^{k-j}\frac{(1-z)^{1-b}-(1-z)^{j+1}}{j+b}. \tag{26}$$

For $k = 1, 2, 3, \ldots$, they satisfy the recurrence relation

$$\beta_k(z, b) = \frac{1-z}{zb}\left[\frac{(-1)^k}{(1-z)^b} - 1\right] + \frac{2k(1-z)}{zb}\beta_{k-1}(z, b+1), \quad \beta_0(z, b) = \frac{1-z}{zb}\left[\frac{1}{(1-z)^b} - 1\right].$$

When $z = 0$, the above expressions must be understood in the limit sense. The remainder is bounded in the form

$$|R_n(z, a, b)| \leq \frac{e^{\pi|\Im b|}|(1-a)_n|}{n!\,2^{\Re a-1}\,\Re a\, r^{\Re b-1}}\max\{2^{\Re a-n-1}, 1\}. \tag{27}$$

For $n \geq \Re a - 1 > 0$, the remainder term may also be bounded in the form

$$|R_n(z,a,b)| \leq \frac{e^{\pi|\Im b|}2^{1-\Re a}|(1-a)_n|}{(n-1)!(n+1)\,(\Re a-1)r^{\Re b-1}}. \tag{28}$$

The remainder term behaves as $R_n(z,a,b) \sim n^{-\Re a}$ as $n \to \infty$ uniformly for $z \in C_r$.

Proof. It is similar to the proof of Theorem 1 but considering the integral representation (2) instead of (1). That is, we must consider the Taylor expansion of the factor $(1-t)^{a-1}$ at $t = 1/2$ instead of the expansion of the factor t^{a-1}. And we must replace z by $z/(z-1)$ in the factor $(1-zt)^{b-1}$. Then, we only give here a few significant details.

Replacing the truncated Taylor series expansion of $(1-t)^{a-1}$ at $t = 1/2$ on the right hand side of (2) we obtain (25) with

$$R_n(z,a,b) := \int_0^1 r_n(t,a)\left(1+\frac{z}{1-z}t\right)^{b-1}dt \tag{29}$$

and

$$\beta_k(z,b) := \int_0^1 (1-2t)^k\left(1+\frac{z}{1-z}t\right)^{b-1}dt = \frac{1-z}{z}\int_1^{(1-z)^{-1}}\left(\frac{2-z}{z}-\frac{2(1-z)}{z}u\right)^k u^{b-1}du.$$

Expanding the first factor of the integrand in the second integral in powers of u and integrating term-wise we obtain (26). Then, we obtain (25) with $R_n(z,a,b)$ given in (29). Now, in order to derive the bounds (27) and (28), instead of a bound for the factor $(1-zt)^{b-1}$ valid for every $t \in [0,1]$, we need a bound for the factor $(1-z(z-1)^{-1}t)^{b-1}$ valid for every $t \in [0,1]$. It is given by $|(1-z(z-1)^{-1}t)^{b-1}| \leq e^{\pi|\Im b|}\overline{M}(z,b)$, with

$$\overline{M}(z,b) := \max\{1, |1-z|^{1-\Re b}\}.$$

It is clear that $\overline{M}(z,b) \leq r^{1-\Re b}$ for $z \in C_r$ and then, instead of (12) and (13) we obtain (27) and (28).

□

A simpler error bound than the bounds (27) and (28) can be found when a is real. The proof is similar to the proof of Proposition 1 and we omit it.

Proposition 2. *For $a > 0$, $\Re b \geq 1$, $z \in C_r$, with C_r defined in (24) for $0 < r \leq 1$, and $n = 1, 2, 3, \ldots$, the error term $R_n(z,a,b)$ defined by (29) in Theorem 2 may be bounded in the form*

$$|R_n(z,a,b)| \leq \frac{e^{\pi|\Im b|}|(1-a)_n|}{a\,r^{\Re b-1}\,2^{a-1}\,n!}.$$

Table 2 shows the first few terms of the approximation of $z^{-a}(1-z)^{1-b}B_z(a,b)$ given by the expansion (25). These terms are rational functions of z and functions of $(1-z)^b$.

In Figure 4 we plot $z^{-1.5}(1-z)^{-2}B_z(1.5,3)$ and the approximations given in Theorem 2 for $n = 1, 2$ and 3. This is a numerical experiment about the rate of convergence provided by (25). We also observe the uniform character of the approximation in the region C_r.

n	$(-1)^n(1-a)_n\beta_n(z,b)/n!$
0	$\frac{\left(-1+(1-z)^{-b}\right)(1-z)}{bz}$
1	$\frac{(1-z)^{-b}(-1+z)\left(2-2(1-z)^{1+b}-2(1+b)z+a\left(-2+z+bz+(1-z)^b(2+(-1+b)z)\right)\right)}{b(b+1)z^2}$
2	$\frac{\left(-1+(1-z)^{-b}\right)(1-z)}{bz}-\frac{(-1+a)(1-z)^{1-b}\left(2\left(-1+(1-z)^b\right)+\left(1+b-(1-z)^b+b(1-z)^b\right)z\right)}{b(b+1)z^2}$ $+\frac{\left((-2+a)(-1+a)(1-z)^{-b}(-1+z)\left(8\left(-1+(1-z)^b\right)+4\left(2+b-2(1-z)^b+b(1-z)^b\right)z+\left(2\left(-1+(1-z)^b\right)+b^2\left(-1+(1-z)^b\right)-b\left(3+(1-z)^b\right)\right)z^2\right)\right)}{2b(b+1)(b+2)z^3}$

Table 2: First few terms in the expansion (25) of $z^{-a}(1-z)^{1-b}B_z(a,b)$.

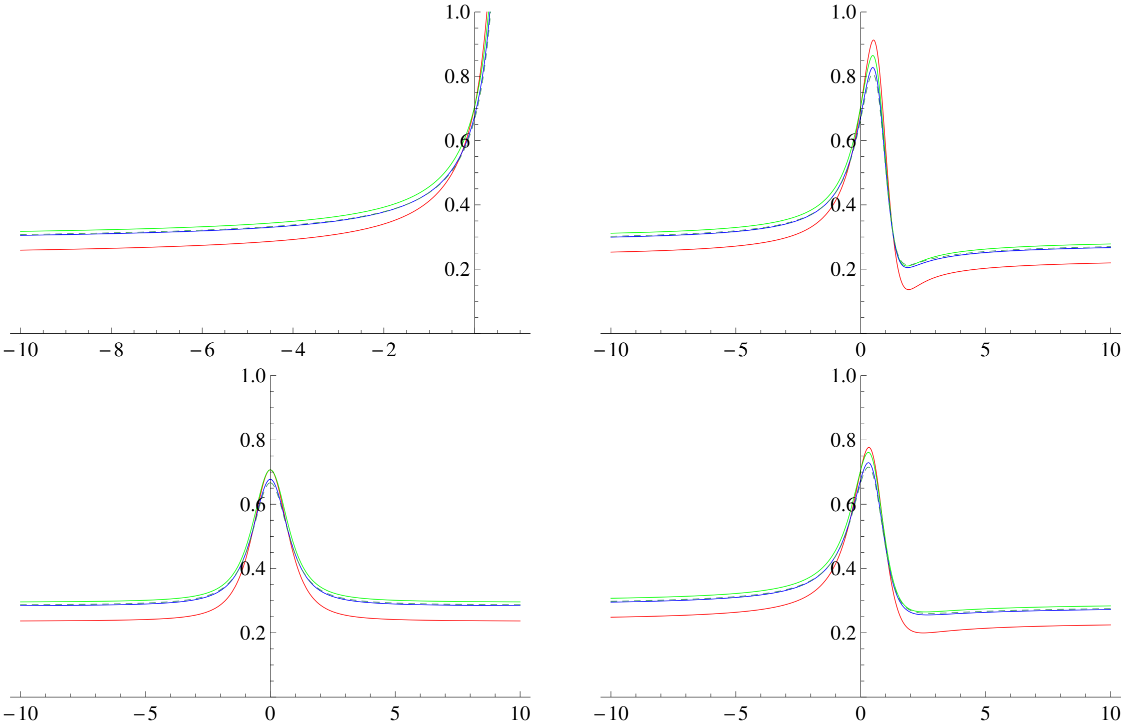

Figure 4: Plots of the absolute value of $z^{-1.5}(1-z)^{-2}B_z(1.5,3)$ (dashed) and the approximations given in Theorem 2 for $n=1$ (red), $n=2$ (green) and $n=3$ (blue) in several intervals: $[-10,1]$ (top left), $[-10e^{i\pi/4},10e^{i\pi/4}]$ (top right), $[-10e^{i\pi/2},10e^{i\pi/2}]$ (bottom left) and $[-10e^{-i\pi/3},10e^{-i\pi/3}]$ (bottom right).

4 Acknowledgments

This research was supported by the Spanish *Ministry of Economía y Competitividad*, project MTM2017-83490-P.

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