Proceedings of the Royal Society of Edinburgh, **134A**, 537–555, *2004*

Convergent asymptotic expansions of Charlier, Laguerre and Jacobi polynomials

José L. López
Departamento de Matématica e Informática,
Universidad Pública de Navarra, 31006 Pamplona, Spain
(jl.lopez@unavarra.es)

Nico M. Temme
CWI, PO Box 94079, 1090 GB Amsterdam,
The Netherlands (nicot@cwi.nl)

(MS received 7 January 2003; accepted 27 November 2003)

Convergent expansions are derived for three types of orthogonal polynomials: Charlier, Laguerre and Jacobi. The expansions have asymptotic properties for large values of the degree. The expansions are given in terms of functions that are special cases of the given polynomials. The method is based on expanding integrals in one or two points of the complex plane, these points being saddle points of the phase functions of the integrands.

1. Introduction

In a previous paper [9], we have studied the expansion of an analytic function at two finite points in the complex plane. The domain of convergence is a Cassini oval around the two points. The main motivation for that paper was to obtain the coefficients of asymptotic expansions of certain integrals. In the present paper we give a few examples in which the expansion of an integral at two saddle points yields a convergent expansion that has an asymptotic property for large values of a parameter.

In the well-known methods for deriving asymptotic expansions of integrals, a basic step is transforming the integral into a standard form, and the transformation usually gives a new integral in which the integrand contains implicitly defined functions that are difficult to handle. In the method of this paper, we avoid a transformation and, in addition, we derive convergent expansions.

We start with a simple example in which only one saddle point occurs, and in which a function is expanded at that saddle point. This gives an expansion for the Charlier polynomials.

In two other examples (Laguerre and Jacobi polynomials) we take into account two saddle points, and again two convergent expansions can be constructed with the desired property. The approximants belong to the same class of polynomials as the original ones, but they are of a simpler type (Hermite and Chebyshev, respectively). The asymptotic property follows from recursion relations for functions appearing in the expansions. The convergence follows from the fact that an integral along a finite contour is expanded inside a domain of uniform convergence.

In the examples given in this paper, the contour integrals are based on Cauchy-type integrals obtained from generating functions. When the contour is finite, a proof of the convergence is usually rather easy. For more general finite contours and more general integrals we expect that the method can be applied as well. For example, we can apply the method to the Gauss hypergeometric function and the incomplete gamma function, with different integral representations.

Also, the methods of this paper can be generalized by considering Taylor expansions at more than two points. In [10], we give details on the theory of multi-point Taylor expansions, and in a future paper we will give details on applications to integrals with, for example, three saddle points.

We show a few graphs that indicate the nature of the approximations, and in a final section we mention a few examples in which other functions are considered.

2. A simplified version of the saddle-point method

Throughout this paper we are concerned with finding asymptotic expansions of integrals of the form

$$F(n) \equiv \int_{\Gamma} f(w)\mathrm{e}^{ng(w)}\frac{\mathrm{d}w}{w^{n+1}}, \tag{2.1}$$

where $f(w)$ and $g(w)$ are analytic in a domain Ω of the complex plane that contains the origin, Γ is a circle with centre at the origin and contained in Ω and n is a large positive integer. We assume, as it usually happens to be the case, that the asymptotic behaviour of the integral $F(n)$ for large n is determined by contributions from the saddle points of $\varphi(w) = g(w) - \ln w$ (see [15, ch. 2, § 4]).

The standard saddle-point method consists of

(i) deforming the contour of integration Γ into a new path that crosses one or some of the saddle points of $\varphi(w)$;

(ii) a suitable change of the variable of integration;

(iii) application of Watson's lemma or Laplace's method.

Instead of applying the standard saddle-point method, we will proceed in a simpler way: just substitute a power-series expansion at one or more saddle points of the function $f(w)$ in (2.1). If there is just one saddle point w_0, then that power series is its Taylor expansion at w_0,

$$f(w) = \sum_{k=0}^{\infty} \frac{f^{(k)}(w_0)}{k!}(w - w_0)^k, \tag{2.2}$$

which is uniformly convergent for w in a disk $D_r(w_0) \equiv \{w \in \Omega,\ |w - w_0| < r\}$ with centre at w_0 and radius $r = \inf_{w\in\mathbb{C}\setminus\Omega} |w - w_0|$. If there are two saddle points w_1 and w_2, then that power series is its two-point Taylor series at w_1 and w_2 [9],

$$f(w) = \sum_{n=0}^{\infty}[a_n(w - w_1) + a'_n(w - w_2)](w - w_1)^n(w - w_2)^n, \tag{2.3}$$

where

$$a_0 \equiv \frac{f(w_2)}{w_2 - w_1}, \qquad a'_0 \equiv \frac{f(w_1)}{w_1 - w_2} \tag{2.4}$$

and, for $n = 1, 2, 3, \dots,$

$$a_n \equiv \frac{1}{n!} \sum_{k=0}^{n} \frac{(n+k-1)!}{k!(n-k)!} \frac{(-1)^{n+1} n f^{(n-k)}(w_2) + (-1)^k k f^{(n-k)}(w_1)}{(w_1 - w_2)^{n+k+1}}, \tag{2.5}$$

$$a'_n \equiv \frac{1}{n!} \sum_{k=0}^{n} \frac{(n+k-1)!}{k!(n-k)!} \frac{(-1)^{n+1} n f^{(n-k)}(w_1) + (-1)^k k f^{(n-k)}(w_2)}{(w_2 - w_1)^{n+k+1}}. \tag{2.6}$$

The expansion (2.3) is uniformly convergent for w in a Cassini oval

$$O_r(w_1, w_2) \equiv \{w \in \Omega,\ |w - w_1||w - w_2| < r\}$$

with foci at w_1 and w_2 and 'radius' $r = \inf_{w \in \mathbb{C} \setminus \Omega} \{|w - w_1||w - w_2|\}$ (see [9]).

If we now substitute (2.2) or (2.3) in (2.1) and interchange summation and integration, we obtain an expansion of $F(n)$. This is proved in the following two propositions.

PROPOSITION 2.1. *Let the right-hand side of (2.2) converge uniformly to $f(w)$ for $w \in D_r(w_0)$ with $|w_0| < r$. Then*

$$F(n) = \sum_{k=0}^{\infty} \frac{f^{(k)}(w_0)}{k!} \int_{\Gamma} (w - w_0)^k \mathrm{e}^{n\varphi(w)} \frac{\mathrm{d}w}{w}. \tag{2.7}$$

Proof. If $|w_0| < r$, then $0 \in D_r(w_0)$. Then we can choose a small enough circle Γ in (2.1) such that $\Gamma \in D_r(w_0)$. Therefore, expansion (2.2) is uniformly convergent for $w \in \Gamma$. Introducing (2.2) in (2.1) and interchanging summation and integration we obtain (2.7). □

PROPOSITION 2.2. *Let the right-hand side of (2.3) converge uniformly to $f(w)$ for $w \in O_r(w_1, w_2)$ with $|w_1 w_2| < r$. Then*

$$\begin{aligned} F(n) = \sum_{k=0}^{\infty} a_k \int_{\Gamma} (w - w_1)^{k+1} (w - w_2)^k \mathrm{e}^{n\varphi(w)} \frac{\mathrm{d}w}{w} \\ + \sum_{k=0}^{\infty} a'_k \int_{\Gamma} (w - w_1)^k (w - w_2)^{k+1} \mathrm{e}^{n\varphi(w)} \frac{\mathrm{d}w}{w}. \end{aligned} \tag{2.8}$$

Proof. The proof is similar to that of proposition 2.1. □

In the remaining part of the paper we apply proposition 2.1 or proposition 2.2 to three specific examples of integrals $F(n)$ representing Charlier polynomials $C_n^a(nx)$, Laguerre polynomials $L_n^{\alpha}(nx)$ and Jacobi polynomials $P_n^{(\alpha,\beta)}(x)$. In this way, we obtain expansions of these polynomials for large values of n. In each example, we prove that the corresponding expansions (2.7) or (2.8) are convergent in a certain region of the variable x and that, in fact, they have an asymptotic nature for large n, uniformly with respect to x in certain domains of the region of convergence.

3. Asymptotic expansions of Charlier polynomials in terms of Gamma functions

The Charlier polynomials are defined by the generating function

$$\mathrm{e}^{-aw}(1+w)^x = \sum_{n=0}^{\infty} C_n^a(x)\frac{w^n}{n!}, \tag{3.1}$$

and have the explicit expression

$$C_n^a(x) = \sum_{k=0}^{n} \binom{n}{k}\binom{x}{k} k!(-a)^{n-k}. \tag{3.2}$$

The Charlier polynomials are orthogonal with respect to a discrete distribution on the positive real line. For an overview of properties, see [7]. Recent papers on asymptotics include [1,2,5,6]. In this section we give a simple convergent expansion of $C_n^a(xn)$, that has an asymptotic property for large n, uniformly for complex a in compact sets and for complex x bounded away from 1. A uniform expansion that holds for x in a compact neighbourhood of $x=1$ is given in [1], where J-Bessel functions are used in the approximations. Also, uniform expansions that hold for $-\infty < x < \infty$ are given in [2].

Theorem 3.1. *For $x \neq 1$, $a \in \mathbb{C}$ and $n \in \mathbb{N}$, the Charlier polynomials have the expansion*

$$C_n^a(xn) = \mathrm{e}^{a/(1-x)} \sum_{k=0}^{\infty} \frac{(-a)^k}{k!} \Phi_k(x,n), \tag{3.3}$$

where

$$\Phi_0(x,n) \equiv \frac{\Gamma(nx+1)}{\Gamma(nx+1-n)}, \qquad \Phi_1(x,n) \equiv \frac{\Phi_0(x,n)}{(1-x)(n(x-1)+1)} \tag{3.4}$$

and, for $k=0,1,2,\dots$,

$$\Phi_k(x,n) \equiv \frac{\Gamma(nx+1)}{\Gamma(nx-n+1)} \frac{1}{(1-x)^k}\, {}_2F_1(-k,-n,nx-n+1;1-x). \tag{3.5}$$

The sequence $\{\Phi_k(x,n),\ k=0,1,2,\dots\}$ satisfies the recurrence

$$\Phi_k(x,n) = \frac{1}{n(x-1)+k}\left[\frac{x(1-k)-k}{x-1}\Phi_{k-1}(x,n) + \frac{x(1-k)}{(x-1)^2}\Phi_{k-2}(x,n)\right], \tag{3.6}$$

where $k=2,3,\dots$, and is an asymptotic sequence for large n. For fixed k, we have

$$\Phi_k(x,n) = \mathcal{O}(n^{-\lfloor (k+1)/2 \rfloor}(n|x|)_n), \tag{3.7}$$

when $n\to\infty$, where $\lfloor \alpha \rfloor$ is the integer part of the real number α. The asymptotic property holds uniformly with respect to complex x, $|x-1| \geqslant \delta > 0$.

Observe that, in the notation of Pochhammer's symbol $(a)_n$, defined by

$$(a)_0 = 1, \quad (a)_n = \frac{\Gamma(a+n)}{\Gamma(a)} = a(a+1)\cdots(a+n-1), \quad n=0,1,2,\dots, \tag{3.8}$$

we have

$$\Phi_0(x) = (-1)^n \frac{\Gamma(n-nx)}{\Gamma(-nx)} = (-1)^n(-nx)_n = (nx)(nx-1)\cdots(nx-(n-1)). \quad (3.9)$$

Proof. From (3.1) we derive the integral representation

$$C_n^a(x) = \frac{n!}{2\pi\mathrm{i}} \int_\Gamma \mathrm{e}^{-aw}(1+w)^x \frac{\mathrm{d}w}{w^{n+1}},$$

where Γ is a circle with centre at the origin and radius less than 1. We write this in the form

$$C_n^a(nx) = \frac{n!}{2\pi\mathrm{i}} \int_\Gamma \mathrm{e}^{-aw}\mathrm{e}^{n\varphi(x,w)} \frac{\mathrm{d}w}{w}, \quad (3.10)$$

where

$$\varphi(x,w) \equiv x\log(1+w) - \log w.$$

The only saddle point of $\varphi(x,w)$ is $w_0 = (x-1)^{-1}$. The function e^{-aw} is an entire function of w. Hence the expansion

$$\mathrm{e}^{-aw} = \sum_{k=0}^{\infty} \frac{(-a)^k \mathrm{e}^{-aw_0}}{k!}(w-w_0)^k \quad (3.11)$$

is locally uniformly convergent for $w \in \mathbb{C}$ and $x \neq 1$. Therefore, after substituting this expansion in (3.10) and using proposition 2.1, we obtain (3.3) with

$$\Phi_k(x,n) = \frac{n!}{2\pi\mathrm{i}} \int_\Gamma (w-w_0)^k (1+w)^{xn} \frac{\mathrm{d}w}{w^{n+1}}. \quad (3.12)$$

To obtain the recurrence (3.6), we write

$$\Phi_k(x,n) = \frac{(n-1)!}{2\pi\mathrm{i}} \frac{1}{x-1} \int_\Gamma (w-w_0)^{k-1}(1+w) \frac{\partial \mathrm{e}^{n\varphi(x,w)}}{\partial w}\,\mathrm{d}w.$$

Integrating by parts and performing a few straightforward manipulations, we obtain equation (3.6). Equalities (3.4) and (3.5) follow after simple calculations. The asymptotic behaviour in (3.7) for large n follows from (3.4) and (3.6). From (3.4), we see that $\Phi_0 = \mathcal{O}((n|x|)_n)$ (see formula (3.9)) and that $\Phi_1 = \mathcal{O}((n|x|)_n/n)$. Therefore, equation (3.7) is true for $k = 0, 1$. From here, the proof follows by induction over k. If (3.7) holds up to k, then

$$\Phi_{k-1} = \mathcal{O}(n^{-[k/2]}(n|x|)_n) \quad \text{and} \quad \Phi_k = \mathcal{O}(n^{-[(k+1)/2]}(n|x|)_n).$$

Using this in (3.6), with k replaced by $k+1$, we have that

$$\Phi_{k+1} = \mathcal{O}(n^{-[(k+2)/2]}(n|x|)_n).$$

Property (3.7) holds uniformly for $|x-1| \geqslant \delta > 0$, and also for complex x. Detailed information on the asymptotic behaviour can easily be obtained from (3.12). □

From (3.9) it follows that $\Phi_0(x)$ has n zeros at $x = m/n$, $m = 0, 1, \ldots, n-1$ ($\Phi_1(x)$ has the same zeros, except for $x = (n-1)/n$). See also table 1, where we

Table 1. *The zeros of* $C_n^a(nx)$ *for* $n = 10$ *and* $a = 1$

0.000 000 090	0.534 449 998
0.100 006 223	0.680 932 968
0.200 157 621	0.855 641 877
0.301 812 498	1.068 772 397
0.410 358 953	1.347 867 376

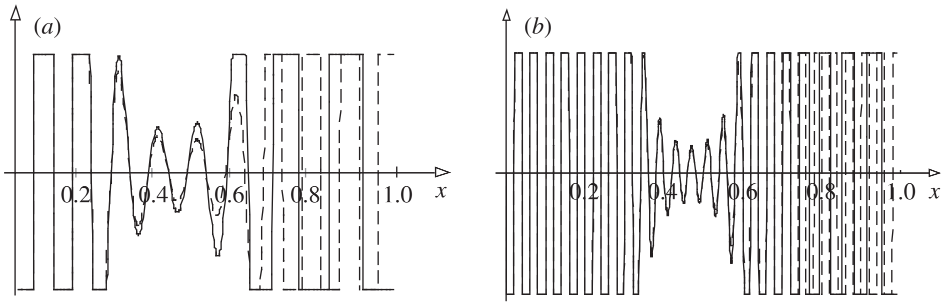

Figure 1. Numerical experiment on the approximation given in theorem 3.1 for large n and $x \in [0, 1)$. Continuous lines represent the Charlier polynomial $C_n^1(nx)$ for (a) $n = 20$ and (b) $n = 50$. Dashed lines represent the first-order approximation given by $\mathrm{e}^{a/(1-x)}\Phi_0(x, n)$. Both graphics are cut for extreme values of the polynomials.

give numerical values of the zeros of $C_n^a(nx)$ for $n = 10$ and $a = 1$. From the graphs in figure 1, we also see that the early zeros are approximated quite well.

Table 2 gives approximate values of $C_n^a(x, a)$ for $x = 0.25$, $a = 1$, and several values of n.

Remark 3.2. When in expansion (3.11), the expansion point w_0 is not equal to the saddle point $(x-1)^{-1}$, we are not able to prove the asymptotic nature of expansion (3.3). This follows from the integration by parts procedure mentioned in the proof of theorem 3.1. On the other hand, we can show (3.7) directly from the definition (3.12) with a change of variable like in the standard saddle-point method.

3.1. Details on the convergence

It is of interest to verify the speed of convergence of the expansion in (3.3). We consider equation (3.12), with $k = \kappa n$, and determine the saddle point of $(w - w_0)^\kappa (1 + w)^x w^{-1}$, where κ is large. We consider x fixed, and for $x < 1$ we verify that a positive saddle point w_+ occurs with $w_+ \sim 1/[(1 - x)\kappa]$. There is a negative saddle point, which is not relevant. We have

$$[(w_+ - w_0)^\kappa (1 + w_+)^x w_+^{-1}]^n \sim \frac{\kappa^n \mathrm{e}^n (1 - x)^n}{(1 - x)^k}.$$

Multiplying this with $a^k n!/k!$ (see (3.3) and (3.12)) and using in Stirling's approximation of the factorials only the dominant parts, that is, $k! \sim (k/\mathrm{e})^k$, we see that

Table 2. *Numerical experiment on the convergence rate of expansion (3.3) for $x = 0.25$ and $a = 1$*

(Here,

$$\boldsymbol{C}_n^a(x, N) \equiv \mathrm{e}^{a/(1-x)} \sum_{k=0}^{N} \frac{(-a)^k}{k!} \Phi_k(x, n)$$

represents the truncated series in (3.3). All the rows are multiplied by an appropriate constant in order to keep the numbers small.)

n	$C_n^1(0.25n)$	$\boldsymbol{C}_n^a(x,0)$	$\boldsymbol{C}_n^a(x,1)$	$\boldsymbol{C}_n^a(x,2)$	$\boldsymbol{C}_n^a(x,3)$	$\boldsymbol{C}_n^a(x,4)$	$\boldsymbol{C}_n^a(x,5)$
10	−1.036 30	−0.977 36	−1.023 35	−1.047 47	−1.004 38	−1.036 33	−1.036 3
30	4.358 72	4.038 23	4.288 67	4.350 77	4.357 62	4.358 58	4.358 70
50	−4.867 27	−4.658 13	−4.828 29	−4.864 64	−4.867 01	−4.867 25	−4.867 26
90	−2.948 51	−2.879 26	−2.936 99	−2.948 08	−2.948 48	−2.948 51	−2.948 51

the main information on $a^k n! \Phi_k(k, n)/k!$ is given by

$$\frac{\mathrm{e}^k k^n a^k (1-x)^n}{(1-x)^k k^k},$$

where k is large compared with n and x, $x < 1$. We see that the ratio of successive terms is about $a/[(1-x)k]$.

For other values of x, also complex, a similar analysis can be given, with some care in choosing the saddle points and defining the branches in the complex plane.

4. Asymptotic expansions of Laguerre polynomials in terms of Hermite polynomials

The Laguerre polynomials can be defined by the generating function

$$(1-t)^{-\alpha-1} \mathrm{e}^{-tx/(1-t)} = \sum_{n=0}^{\infty} L_n^{\alpha}(x) t^n, \quad \alpha, x \in \mathbb{C}, \quad |t| < 1, \tag{4.1}$$

and have the representation

$$L_n^{\alpha}(x) = \sum_{k=0}^{n} (-1)^k \binom{n+\alpha}{n-k} \frac{x^k}{k!}. \tag{4.2}$$

To derive the asymptotic expansion, we use the Cauchy integral that follows from function (4.1),

$$L_n^{(\alpha)}(x) = \frac{1}{2\pi \mathrm{i}} \int_{\Gamma} \mathrm{e}^{xw/(w-1)} (1-w)^{-\alpha-1} \frac{\mathrm{d}w}{w^{n+1}}, \tag{4.3}$$

where Γ is a circle around the origin with radius less than 1. The many-valued functions $(1-w)^{\mu}$ appearing here and in the theorem assume the principal branch that is equal to 1 at $w = 0$.

The asymptotics for large n, fixed α, is considered in [4]. For real x, two uniform expansions are given, one involving the J-Bessel function for x in an interval that contains the origin, and one in terms of the Airy function for x in an interval containing the transition near the largest zero of $L_n^\alpha(x)$. In this section we give an asymptotic expansion of $L_n^\alpha(nx)$ in terms of $L_n^{1/2}(nx)$, which, in fact, is an Hermite polynomial. We consider $x>1$ and for these values the expansion is convergent and is in particular of interest because this is the interval that contains the large zeros and the transition point at $x=4$.

When the parameter α of the Laguerre polynomial is large, the asymptotic behaviour can be described in terms of Hermite polynomials (see [4]). For example, we have the limit

$$\lim_{\alpha\to\infty}\alpha^{-n/2}L_n^\alpha(x\sqrt{\alpha}+\alpha)=\frac{(-1)^n2^{-n/2}}{n!}H_n\left(\frac{x}{\sqrt{2}}\right). \tag{4.4}$$

In [8] we have extended this limit by giving an asymptotic representation for large α and n fixed in terms of Hermite polynomials. For more details on large α asymptotics, we refer to [12]. The approach in this section is quite different because we take α fixed and n large.

From (4.3), we obtain

$$L_n^{(\alpha)}(nx)=\frac{1}{2\pi \mathrm{i}}\int_\varGamma f(w)\frac{\mathrm{e}^{n\varphi(x,w)}}{(1-w)^{3/2}}\frac{\mathrm{d}w}{w}, \tag{4.5}$$

where

$$\varphi(x,w)\equiv\frac{xw}{w-1}-\log w,\qquad f(w)\equiv(1-w)^{1/2-\alpha}. \tag{4.6}$$

The function $\varphi(x,w)$ has two conjugate saddle points,

$$w^\pm=1-\tfrac{1}{2}x\pm\tfrac{1}{2}\mathrm{i}\xi,\qquad \xi=\sqrt{x(4-x)}. \tag{4.7}$$

The square root defining ξ is positive for $0<x<4$; for $x\geqslant 4$, we define $\xi=\mathrm{i}\sqrt{x(x-4)}$, again with positive square root. In the expansion of the Laguerre polynomials, we allow that the saddle points coalesce.

4.1. Construction of the expansion

The function $f(w)$ of (4.5) is analytic in $\varOmega=\mathbb{C}\setminus[1,\infty)$ and we can expand $f(w)$ in in a two-point Taylor expansion at the two saddle points $w^\pm$, using a slightly different form of (2.3),

$$f(w)=\sum_{k=0}^\infty[A_k+B_kw](w-w^+)^k(w-w^-)^k. \tag{4.8}$$

After substituting expansion (4.8) in (4.5) and interchanging summation and integration, we obtain

$$L_n^{(\alpha)}(xn)=\sum_{k=0}^\infty[A_k\varPhi_k(x,n)+B_k\varPsi_k(x,n)], \tag{4.9}$$

where

$$\varPhi_k(x,n)=\frac{1}{2\pi\mathrm{i}}\int_\varGamma(w-w^+)^k(w-w^-)^k\frac{\mathrm{e}^{n\varphi(x,w)}}{(1-w)^{3/2}}\frac{\mathrm{d}w}{w} \tag{4.10}$$

and

$$\Psi_k(x,n)=\frac{1}{2\pi\mathrm{i}}\int_\Gamma (w-w^+)^k(w-w^-)^k\frac{\mathrm{e}^{n\varphi(x,w)}}{(1-w)^{3/2}}\,\mathrm{d}w. \tag{4.11}$$

We have

$$\left.\begin{aligned}\Phi_0(x,n)&\equiv L_n^{(1/2)}(nx)=\frac{(-1)^n}{n!2^{2n+1}\sqrt{nx}}H_{2n+1}(\sqrt{nx}),\\ \Psi_0(x,n)&\equiv L_{n-1}^{(1/2)}(nx)=\frac{(-1)^{n-1}}{(n-1)!2^{2n-1}\sqrt{nx}}H_{2n-1}(\sqrt{nx}),\end{aligned}\right\} \tag{4.12}$$

and, for $k=1,2,3,\dots,$

$$\Phi_k(x,n)\equiv\sum_{j=0}^{k}\binom{k}{j}x^{k-j}L_{n-k+j}^{(1/2-2j)}(nx),\qquad \Psi_k(x,n)\equiv\sum_{j=0}^{k}\binom{k}{j}x^{k-j}L_{n-k+j-1}^{(1/2-2j)}(nx). \tag{4.13}$$

The sequences $\{\Phi_k(x,n)\}$ and $\{\Psi_k(x,n)\}$, $k=0,1,2,\dots,$ satisfy the recurrences

$$\Phi_k=\frac{1}{n-2k+\frac{3}{2}}\{a_1\Phi_{k-1}+a_2\Phi_{k-2}+b_1\Psi_{k-1}+b_2\Psi_{k-2}\}, \tag{4.14}$$

with

$$\begin{aligned}a_1&=(k-1)(x^2-2x-2)-\tfrac{1}{2},\\ a_2&=(k-1)x(2-x),\\ b_1&=(k-1)(2-3x)+\tfrac{1}{2}(1-x),\\ b_2&=(k-1)x(4x-x^2-2),\end{aligned}$$

and

$$\Psi_k=\frac{1}{n-2k+\frac{1}{2}}\{c_0\Phi_k+c_1\Phi_{k-1}+c_2\Phi_{k-2}+d_1\Psi_{k-1}+d_2\Psi_{k-2}\}, \tag{4.15}$$

with

$$\begin{aligned}c_0&=(2-3k)x+2(k-1)+\tfrac{1}{2}(1-x),\\ c_1&=(1-k)x^3+4(k-1)x^2+kx+2(1-k)+\tfrac{1}{2}(x-1),\\ c_2&=-b_2,\\ d_1&=(4k-3)x^2+2(4-5k)x+2(k-1)+\tfrac{1}{2}(x^2-3x+1),\\ d_2&=(k-1)x(x^3-6x^2+9x-2).\end{aligned}$$

To verify the recursions (4.14) and (4.15), we write

$$\Phi_k(x,n)=-\frac{1}{2\pi\mathrm{i}n}\int_C (w-w^+)^{k-1}(w-w^-)^{k-1}\sqrt{1-w}\frac{\partial\mathrm{e}^{n\varphi(x,w)}}{\partial w}\,\mathrm{d}w, \tag{4.16}$$

$$\Psi_k(x,n)=-\frac{1}{2\pi\mathrm{i}n}\int_C (w-w^+)^{k-1}(w-w^-)^{k-1}w\sqrt{1-w}\frac{\partial\mathrm{e}^{n\varphi(x,w)}}{\partial w}\,\mathrm{d}w, \tag{4.17}$$

$$\Psi_{k-1}(x,n) - w^+\Phi_{k-1}(x,n) = \frac{1}{2\pi\mathrm{i}}\int_C (w-w^+)^k(w-w^-)^{k-1}\frac{\mathrm{e}^{n\varphi(x,w)}}{(1-w)^{3/2}}\frac{\mathrm{d}w}{w}, \tag{4.18}$$

$$\Psi_{k-1}(x,n) - w^-\Phi_{k-1}(x,n) = \frac{1}{2\pi\mathrm{i}}\int_C (w-w^+)^{k-1}(w-w^-)^{k}\frac{\mathrm{e}^{n\varphi(x,w)}}{(1-w)^{3/2}}\frac{\mathrm{d}w}{w}. \tag{4.19}$$

Integrating by parts in (4.16) and (4.17), using (4.18) and (4.19) and after straightforward manipulations we obtain (4.14) and (4.15). Formulae (4.12) and (4.13) follow from (4.10) and (4.11) after simple calculations.

Theorem 4.1. *Expansion (4.9) is convergent, uniformly for* $\alpha \in \mathbb{C}$ *in compact sets, and* $x \geqslant 1+\delta > 1$. *Moreover,* $\{\Phi_k(x,n)\}$ *and* $\{\Psi_k(x,n)\}$ *are asymptotic sequences for large* n,

$$\left.\begin{aligned} \Phi_k(x,n) &= \mathcal{O}(n^{-\lfloor (k+1)/2\rfloor})[|\Phi_0(x,n)| + |\Psi_0(x,n)|],\\ \Psi_k(x,n) &= \mathcal{O}(n^{-\lfloor (k+1)/2\rfloor})[|\Phi_0(x,n)| + |\Psi_0(x,n)|],\end{aligned}\right\} \tag{4.20}$$

as $n \to \infty$ *and* $k = 0, 1, 2, \dots$.

Proof. We apply proposition 2.2. Expansion (4.8) is uniformly convergent for w inside the Cassini oval with foci w^+ and w^- and 'radius' $r = |w_1 - w^+||w_1 - w^-|$, where $w_1 = 1$ is the singular point of f. Using (4.7), it follows that $r = x$. The points w are inside the Cassini oval if they satisfy $|w-w^+||w-w^-| < r = x$. Because $w^+w^- = 1$, the origin $w = 0$ is inside the oval only if $x > 1$. Hence the contour Γ of (4.5) can be taken completely inside the oval only if $x > 1$ (see also figure 2). This proves the convergence of (4.9) for $x > 1$. The asymptotic behaviour in (4.20) follows from (4.12) and the recursions (4.14) and (4.15). More detailed asymptotic information can be obtained from the integrals in (4.10) and (4.11). □

Figure 3 gives graphs of $L_n^{(4)}(nx)$ and its approximations based on the first two terms in expansion (4.9). In table 3 we give approximate values of $L_n^{(\alpha)}(xn)$ for $\alpha = 1$, $x = 3.5$, and several values of n.

Remark 4.2. The expansion in (4.9) has a meaning for all complex x, and has for all fixed x an asymptotic meaning. The expansion is uniformly convergent for $|x| \geqslant 1+\delta > 1$.

4.2. Details on the coefficients

The expressions (2.5) and (2.6) for the coefficients in (2.3) can be used in the present case also. We first write

$$f(w) = \sum_{k=0}^{\infty}[a_k(w-w^-) + a_k'(w-w^+)][(w-w^-)(w-w^+)]^k, \tag{4.21}$$

and compare this with (4.8). By comparing coefficients of equal powers, it follows that A_k and B_k can be expressed in terms of a_k and a_k'. We have, for $k = 0, 1, 2, \dots$,

$$A_k = -a_k w^+ - a_k' w^-, \qquad B_k = a_k + a_k'.$$

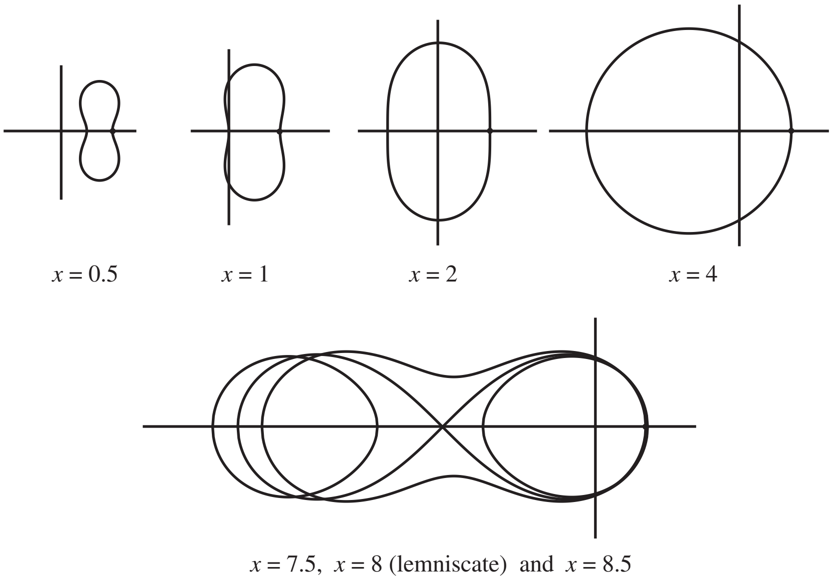

Figure 2. Cassini ovals for the expansion (4.8) for several values of x. For w inside the ovals, the expansion is convergent. For $0 < x < 1$, the origin is outside the oval; for $x = 4$, it is a circle; for $x = 8$, a lemniscate. For $x > 8$, the oval splits up into two parts. All ovals go through the point $w = 1$, a singular point of $f(w)$.

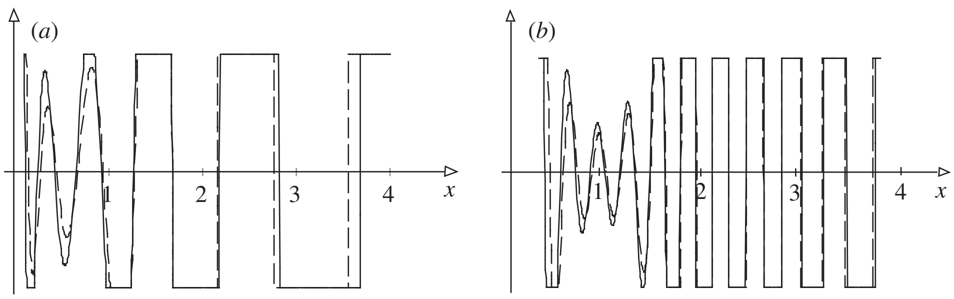

Figure 3. Numerical experiments on the approximation of theorem 4.1 for large n and $x \in (0, 4]$. Continuous lines represent the Laguerre polynomial $L_n^{(4)}(nx)$ for (a) $n = 10$ and (b) $n = 20$. Dashed lines represent the first-order approximation given by $A_0\Phi_0(x, n) + B_0\Psi_0(x, n)$. Both graphics are cut for extreme values of the polynomials.

We have

$$A_k \equiv -2\,\mathrm{Re}(w^+ a_k), \qquad B_k \equiv 2\,\mathrm{Re}(a_k), \qquad a_0 = \mathrm{i}(1 - w^-)^{1/2-\alpha}\xi^{-1}, \tag{4.22}$$

and, for $k = 1, 2, 3, \dots,$

$$a_k = \sum_{j=0}^{k} \frac{(k+j-1)!(\alpha - \frac{1}{2})_{k-j}}{k!j!(k-j)!(\mathrm{i}\xi)^{k+j+1}} \left\{ \frac{(-1)^j j}{(1-w^+)^{\alpha+k-j-1/2}} - \frac{(-1)^k k}{(1-w^-)^{\alpha+k-j-1/2}} \right\}. \tag{4.23}$$

Table 3. *Numerical experiment on the convergence rate of expansion (4.9) for $x = 3.5$ and $\alpha = 1$*

(Here,

$$\boldsymbol{L}_n^{(\alpha)}(x, N) \equiv \sum_{k=0}^{N}[A_k \Phi_k(x, n) + B_k \Psi_k(x, n)]$$

represents the truncated series in (4.9). All the rows are multiplied by an appropriate constant in order to keep the numbers small.)

n	$L_n^{(1)}(3.5n)$	$\boldsymbol{L}_n^{(1)}(3.5, 0)$	$\boldsymbol{L}_n^{(1)}(3.5, 1)$	$\boldsymbol{L}_n^{(1)}(3.5, 2)$
10	0.340 506	0.343 249	0.341 724	0.340 495
30	−8.940 39	−8.865 31	−9.035 30	−8.957 98
50	−5.056 78	−5.059 41	−5.067 64	−5.058 01
90	6.565 56	6.563 28	6.575 72	6.566 01

n	$\boldsymbol{L}_n^{(1)}(3.5, 3)$	$\boldsymbol{L}_n^{(1)}(3.5, 4)$	$\boldsymbol{L}_n^{(1)}(3.5, 5)$
10	0.340 449	0.340 490	0.340 504
30	−8.942 13	−8.940 45	−8.940 38
50	−5.056 89	−5.056 80	−5.056 78
90	6.565 47	6.565 53	6.565 56

The coefficients can also be computed from the recursion relations

$$\left.\begin{aligned} x(k+1)A_{k+1} - x(k+1)B_{k+1} &= (\alpha - \tfrac{1}{2} + 2k)A_k - (xk+1)B_k, \\ x(k+1)A_{k+1} - x(x-3)(k+1)B_{k+1} &= (\alpha + \tfrac{1}{2} + 2k)B_k, \end{aligned}\right\} \tag{4.24}$$

where $k = 0, 1, 2, \dots$. Let, for $1 < x \leqslant 4$, $x = 4\sin^2(\frac{1}{2}\theta)$. Then $w^{\pm} = \mathrm{e}^{\pm \mathrm{i}\theta}$, and

$$A_0 = -2^{\beta+1}\sin^{\beta}(\tfrac{1}{2}\theta)\cos[\tfrac{1}{2}(\theta-\pi)\beta - \theta], \qquad B_0 = 2^{\beta+1}\sin^{\beta}(\tfrac{1}{2}\theta)\cos[\tfrac{1}{2}(\theta-\pi)\beta].$$

where $\beta = \frac{1}{2} - \alpha$. This gives real expressions for the first coefficients to start the recursion relations in (4.24). For $x \geqslant 4$, we can obtain expressions in terms of hyperbolic functions by writing $x = 4\cosh^2(\frac{1}{2}\theta)$, which gives $w^{\pm} = -\mathrm{e}^{\pm\theta}$ and

$$A_0 = -2^{\beta+1}\cosh^{\beta}(\tfrac{1}{2}\theta)\cosh[(\theta(\tfrac{1}{2}\beta - 1)], \qquad B_0 = 2^{\beta+1}\cosh^{\beta}(\tfrac{1}{2}\theta)\cosh(\tfrac{1}{2}\theta\beta).$$

4.3. An alternative form of the expansion

By using in (4.5) the substitution

$$f(w) = \alpha_0 + \beta_0 w + (w - w^-)(w - w^+)g_0(w), \tag{4.25}$$

where α_0 and β_0 follow from substituting $w = w^{\pm}$, we obtain, by integrating by parts,

$$L_n^{(\alpha)}(nx) = \alpha_0 \Phi_0(x, n) + \beta_0 \Psi_0(x, n) + \frac{1}{2\pi \mathrm{i} n}\int_{\Gamma} f_1(w)\frac{\mathrm{e}^{n\varphi(x,w)}}{(1-w)^{3/2}}\frac{\mathrm{d}w}{w}, \tag{4.26}$$

where

$$f_1(w) = w(1-w)^{3/2}\frac{\mathrm{d}}{\mathrm{d}w}[(\sqrt{1-w}g_0(w)]. \tag{4.27}$$

Continuing this procedure, we obtain the expansion in negative powers of n,

$$L_n^{(\alpha)}(nx) = \Phi_0(x,n)\sum_{k=0}^{\infty}\frac{\alpha_k}{n^k} + \Psi_0(x,n)\sum_{k=0}^{\infty}\frac{\beta_k}{n^k}, \tag{4.28}$$

where α_k and β_k follow from

$$f_k(w^-) = \alpha_k + \beta_k w^-, \qquad f_k(w^+) = \alpha_k + \beta_k w^+, \tag{4.29}$$

where, with $f_0(w) = f(w)$ and for $k = 0, 1, 2, \dots,$

$$\left.\begin{aligned} f_k(w) &= \alpha_k + \beta_k w + (w - w^-)(w - w^+)g_k(w),\\ f_{k+1}(w) &= w(1-w)^{3/2}\frac{\mathrm{d}}{\mathrm{d}w}[(\sqrt{1-w}g_k(w)]. \end{aligned}\right\} \tag{4.30}$$

The expansion in (4.28) also follows from rearranging expansion (4.9) by using the recursion relations for $\Phi_k(x,n)$ and $\Psi_k(x,n)$ in (4.14) and (4.15).

5. Asymptotic expansions of Jacobi polynomials in terms of Chebyshev polynomials

The large-n asymptotics for the Jacobi polynomials is discussed in [3, vol. II, § 10.14], in particular, for $x \in (-1,1)$. For x bounded away from the points ± 1, elementary functions (sine and cosine functions) can be used for describing the asymptotics. For x close to ± 1, Bessel functions can be used (Hilb-type formulae).

In this section we develop a convergent expansion that is valid for $x \in (-1,1)$ and the terms of the expansion constitute asymptotic scales for large n. It is possible to extend the results to complex values of x, but this will not be considered here. The first approximants are Chebyshev polynomials, which, in fact, are elementary functions, and the other terms can be obtained from recursions that show the asymptotic property.

5.1. Construction of the expansion

The starting point is the integral representation that follows from [3, vol. II, p. 172],

$$P_n^{(\alpha,\beta)}(x) = \frac{1}{2\pi \mathrm{i}}\frac{(-1)^n}{2^n}\int_{\Gamma}\frac{(1-w-x)^{\alpha}(1+w+x)^{\beta}}{(1-x)^{\alpha}(1+x)^{\beta}}\mathrm{e}^{n\varphi(x,w)}\frac{\mathrm{d}w}{w}, \tag{5.1}$$

where we consider $x \in (-1,1)$; the function $\varphi(x,w)$ is defined by

$$\varphi(x,w) \equiv \log(1+w+x) + \log(1-w-x) - \log w$$

and Γ is a simple closed contour, in the positive sense, around $w = 0$. The points $w = -x \pm 1$ are outside the contour, and $(1-w-x)^{\alpha}/(1-x)^{\alpha}$ and $(1+w+x)^{\beta}/(1+x)^{\beta}$ are to be taken as unity when $w = 0$.

The function $\varphi(x,w)$ has two conjugate saddle points:

$$w^{\pm} = \pm w_0, \qquad w_0 = \mathrm{i}\sqrt{1-x^2}. \tag{5.2}$$

We expand the integral by using the function

$$f(w) \equiv (1-x)^{-\alpha-1/2}(1+x)^{-\beta-1/2}(1-w-x)^{\alpha+1/2}(1+w+x)^{\beta+1/2},$$

which is analytic in

$$\mathbb{C} \setminus \{(-\infty, -1-x] \cup [1-x, \infty)\}.$$

We expand, using a slightly different form of (2.3),

$$f(w) = \sum_{k=0}^{\infty} [A_k + B_k w](w^2 - w_0^2)^k, \tag{5.3}$$

where the coefficients A_k and B_k can be expressed in terms of the derivatives of $f(w)$ at $w = \pm w_0$ (see the next subsection for more details).

After substituting expansion (5.3) in (5.1) and interchanging summation and integration, we obtain

$$P_n^{(\alpha,\beta)}(x) = \sum_{k=0}^{\infty} [A_k \Phi_k(x,n) + B_k \Psi_k(x,n)], \tag{5.4}$$

where

$$\Phi_k(x,n) = \frac{(-1)^n}{2\pi \mathrm{i}} \frac{\sqrt{1-x^2}}{2^n} \int_\Gamma \frac{(w^2+1-x^2)^k}{W(x,w)} \mathrm{e}^{n\varphi(x,w)} \frac{\mathrm{d}w}{w}, \tag{5.5}$$

$$\Psi_k(x,n) = \frac{(-1)^n}{2\pi \mathrm{i}} \frac{1}{2^n\sqrt{1-x^2}} \int_\Gamma (w^2+1-x^2)^k W(x,w) \mathrm{e}^{n\varphi(x,w)} \frac{\mathrm{d}w}{w} \tag{5.6}$$

and

$$W(x,w) \equiv \sqrt{(1-w-x)(1+w+x)}. \tag{5.7}$$

We have

$$\Phi_0(x,n) \equiv P_n^{(-1/2,-1/2)}(x), \qquad \Psi_0(x,n) \equiv -\tfrac{1}{2}(1-x^2)P_{n-1}^{(1/2,1/2)}(x). \tag{5.8}$$

These Jacobi polynomials are Chebyshev polynomials,

$$P_n^{(-1/2,-1/2)}(x) = \frac{2^{-2n}(2n)!}{(n!)^2} T_n(x), \qquad P_n^{(1/2,1/2)}(x) = \frac{2^{-2n}(2n+1)!}{n!(n+1)!} U_n(x). \tag{5.9}$$

In terms of elementary functions,

$$T_n(\cos\theta) = \cos n\theta, \qquad U_n(\cos\theta) = \frac{\sin(n+1)\theta}{\sin\theta}. \tag{5.10}$$

Furthermore,

$$\Phi_1(x,n) \equiv \frac{1-x^2}{4(n+1)} [4(n+1)\Phi_0(x,n) - 4x(n-1)\Psi_0(x,n) + (2n-1)\Psi_0(x,n-1)], \tag{5.11}$$

$$\Psi_1(x,n) \equiv \frac{-3(1-x^2)}{4(n+1)(n+2)} \{2[n+1-2x^2n]\Psi_0(x,n) + x(2n-1)\Psi_0(x,n-1)\} \tag{5.12}$$

and, for $k = 2, 3, 4, \ldots,$

$$\Phi_k(x,n) \equiv \sum_{j=0}^{k} \binom{k}{j} \frac{(1-x^2)^{k+j}}{4^j} P_{n-2j}^{(2j-1/2,2j-1/2)}(x), \tag{5.13}$$

$$\Psi_k(x,n) \equiv -\tfrac{1}{2}(1-x^2) \sum_{j=0}^{k} \binom{k}{j} \frac{(1-x^2)^{k+j}}{4^j} P_{n-1-2j}^{(2j+1/2,2j+1/2)}(x). \tag{5.14}$$

The sequences $\{\Phi_k(x,n)\}$ and $\{\Psi_k(x,n)\}$ satisfy the recursion relations

$$\Phi_k = \frac{1}{n+2k-1}[a_1\Phi_{k-1} + a_2\Phi_{k-2} + b_1\Psi_{k-1} + b_2\Psi_{k-2}], \tag{5.15}$$

$$\Psi_k = \frac{1}{n+2k}[c_0\Phi_k + c_1\Phi_{k-1} + c_2\Phi_{k-2} + d_1\Psi_{k-1} + d_2\Psi_{k-2}], \tag{5.16}$$

where

$$\begin{aligned}
a_1 &= (1-x^2)(6k-5),\\
a_2 &= -4(k-1)(1-x^2)^2,\\
b_1 &= -x(4k-3),\\
b_2 &= 4x(k-1)(1-x^2),\\
c_0 &= -x(4k-1),\\
c_1 &= x(1-x^2)(8k-5),\\
c_2 &= -4x(1-x^2)^2(k-1),\\
d_1 &= 3(2k-1)(1-x^2),\\
d_2 &= -4(1-x^2)^2(k-1).
\end{aligned}$$

For the relations between Jacobi and Chebyshev polynomials, we refer to [13, pp. 152, 153]. The expressions in (5.11) and (5.12) follow from contiguous relations of the Jacobi polynomials (see [13, p. 166]).

To verify the recursions in (5.15) and (5.16), we write

$$\Phi_k(x,n) = \frac{-1}{2\pi \mathrm{i}} \frac{(-1)^n}{n2^n\sqrt{1-x^2}} \int_\Gamma (w^2+1-x^2)^{k-1} W(x,w) \frac{\partial \mathrm{e}^{n\varphi(x,w)}}{\partial w}\,\mathrm{d}w, \tag{5.17}$$

$$\Psi_k(x,n) = \frac{-1}{2\pi \mathrm{i}} \frac{(-1)^n}{n2^n\sqrt{1-x^2}} \int_\Gamma w(w^2+1-x^2)^{k-1} W(x,w) \frac{\partial \mathrm{e}^{n\varphi(x,w)}}{\partial w}\,\mathrm{d}w. \tag{5.18}$$

Integrating by parts in (5.17) and (5.18) and after straightforward manipulations we obtain (5.15) and (5.16). The asymptotic behaviour pointed out above follows from the definition of the Jacobi polynomials and the recurrences (5.15) and (5.16).

THEOREM 5.1. *Expansion (5.4) is convergent for $x \in (-1,1)$. Moreover, $\{\Phi_k(x,n)\}$ and $\{\Psi_k(x,n)\}$ are asymptotic sequences for large n,*

$$\Phi_k(x,n) = \mathcal{O}(n^{-\lfloor (k+1)/2 \rfloor}), \qquad \Psi_k(x,n) = \mathcal{O}(n^{-\lfloor (k+1)/2 \rfloor}),$$

as $n \to \infty$ and $k = 0, 1, 2, \ldots$.

Proof. Expansion (5.3) is uniformly convergent for w inside the Cassini oval of focus $w^{\pm}$ and 'radius'

$$r = \min\{|1+x+w^{+}||1+x+w^{-}|, |1-x+w^{+}||1-x+w^{-}|\} = 2(1-|x|).$$

Hence (5.3) is convergent for $w \in \mathbb{C}$ such that $|w^2+1-x^2| < 2(1-|x|)$. From proposition 2.2, it follows that (5.4) is convergent for $1-x^2 = |w^{+}w^{-}| < 2(1-|x|)$, that is, for all $x \in (-1,1)$. The asymptotic property follows from the recursion relations in (5.15) and (5.16). □

5.2. Details on the coefficients

The expressions for (2.5) and (2.6) for the coefficients in (2.3) can also be used in the present case. We first write

$$f(w) = \sum_{k=0}^{\infty}[a_k(w-w_0) + a'_k(w+w_0)](w^2-w_0^2)^k, \tag{5.19}$$

and compare this with (5.3). By comparing coefficients of equal powers, it follows that A_k and B_k can be expressed in terms of a_k and a'_k. We have, for $k = 0, 1, 2, \ldots,$

$$A_k = (a'_k - a_k)w_0, \qquad B_k = a_k + a'_k.$$

After straightforward manipulations, we find

$$A_k \equiv \frac{2\,\mathrm{Im}[a_k\sqrt{1-x^2}]}{(1-x)^{\alpha+1/2}(1+x)^{\beta+1/2}} \quad \text{and} \quad B_k \equiv \frac{2\,\mathrm{Re}\,a_k}{(1-x)^{\alpha+1/2}(1+x)^{\beta+1/2}}, \tag{5.20}$$

where

$$a_0 = \frac{(1-x+w_0)^{\alpha+1/2}(1+x-w_0)^{\beta+1/2}}{2w_0} \tag{5.21}$$

and, for $m = 1, 2, 3, \ldots,$

$$\begin{aligned} a_m =& \frac{1}{m!}\sum_{k=0}^{m}\frac{(m+k-1)!}{k!(m-k)!(2w_0)^{m+k+1}} \\ &\times \sum_{j=0}^{m-k}\binom{m-k}{j}\bigg\{\frac{k(-1)^{m+j}(-\alpha-\frac{1}{2})_j(-\beta-\frac{1}{2})_{m-k-j}}{(1-x-w_0)^{j-\alpha-1/2}(1+x+w_0)^{m-k-j-\beta-1/2}} \\ &\qquad - \frac{m(-1)^{k+j}(-\alpha-\frac{1}{2})_j(-\beta-\frac{1}{2})_{m-k-j}}{(1-x+w_0)^{j-\alpha-1/2}(1+x-w_0)^{m-k-j-\beta-1/2}}\bigg\}. \end{aligned} \tag{5.22}$$

Figure 4 gives graphs of $P_n^{(\alpha,\beta)}(x)$ and its approximations based on the first two terms in expansion (5.4). In table 4 we give approximate values of $P_n^{(\alpha,\beta)}(x)$ for $\alpha = \frac{3}{2}$, $\beta = \frac{1}{2}$, $x = 0$, and several values of n.

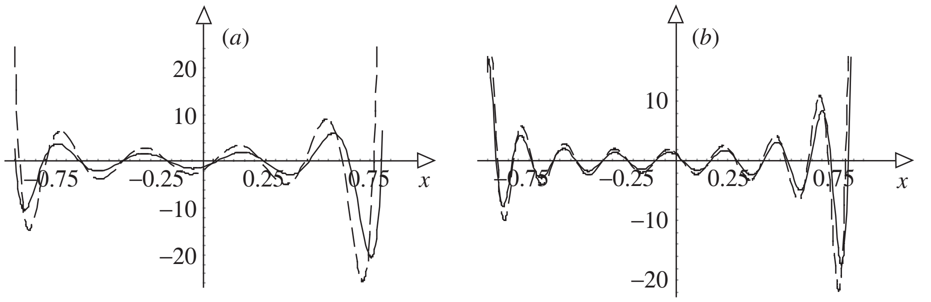

Figure 4. Numerical experiments about the approximation of theorem 5.1 for large n and $-1 < x < 1$. Continuous lines represent the Jacobi polynomial $P_n^{(3,4)}(x)$ for (a) $n = 10$ and (b) $n = 20$. Dashed lines represent the first-order approximation given by $A_0\Phi_0 + B_0\Psi_0$.

Table 4. *Numerical experiment on the convergence rate of expansion (5.4) for $x = 0$, $\alpha = \frac{3}{2}$ and $\beta = \frac{1}{2}$*

(Here,

$$\boldsymbol{P}_n(x, N) \equiv \sum_{k=0}^{N}[A_k\Phi_k(x,n) + B_k\Psi_k(x,n)]$$

represents the truncated series in (5.4).)

n	$P_n^{(3/2,1/2)}(0)$	$\boldsymbol{P}_n(0,0)$	$\boldsymbol{P}_n(0,1)$	$\boldsymbol{P}_n(0,2)$
10	−0.336 376	−0.348 029	−0.348 029	−0.337 153
30	−0.201 847	−0.204 304	−0.204 304	−0.201 909
50	−0.157 618	−0.158 781	−0.158 781	−0.157 636
90	−0.118 124	−0.118 612	−0.118 612	−0.118 128

n	$\boldsymbol{P}_n(0,3)$	$\boldsymbol{P}_n(0,4)$	$\boldsymbol{P}_n(0,5)$
10	−0.336 376	−0.336 340	−0.336 360
30	−0.201 839	−0.201 845	−0.201 847
50	−0.157 615	−0.157 617	−0.157 618
90	−0.118 123	−0.118 124	−0.118 124

6. A few other examples of convergent asymptotic expansions

Consider the integral for the modified Bessel function (for properties of the special functions in this section, we refer to [3, 13])

$$I_\nu(z) = \frac{(2z)^\nu e^z}{\sqrt{\pi}\Gamma(\nu + \frac{1}{2})}\int_0^1 e^{-2zt}[t(1-t)]^{\nu-1/2}\,dt. \tag{6.1}$$

The usual method for obtaining the asymptotic expansion for large z consists of substituting the expansion

$$(1-t)^{\nu-1/2} = \sum \binom{\nu - \frac{1}{2}}{k}(-t)^k$$

and by interchanging the order of summation and integration. The resulting integrals are not evaluated over $[0,1]$ but over $[0,\infty)$. This latter step gives the divergent asymptotic expansions. If we integrate over $[0,1]$, we obtain an expansion in terms of incomplete gamma functions,

$$I_\nu(z) = \frac{\mathrm{e}^z}{\sqrt{2z\pi}\,\Gamma(\nu+\frac{1}{2})} \sum_{k=0}^{\infty} (-1)^k \binom{\nu-\frac{1}{2}}{k} \frac{\gamma(\nu+k+\frac{1}{2},2z)}{(2z)^k}. \tag{6.2}$$

For fixed values of z, the incomplete gamma functions have the asymptotic behaviour

$$\gamma(\nu+k+\tfrac{1}{2},2z) = \frac{\mathrm{e}^{-2z}(2z)^{\nu+k+1/2}}{\nu+k+\frac{1}{2}}[1+\mathcal{O}(k^{-1})], \quad k\to\infty, \tag{6.3}$$

and we see that the terms behave like $\mathcal{O}(k^{-\nu-3/2})$. So convergence is guaranteed if $\operatorname{Re}\nu > -\frac{1}{2}$. A further examination of the terms of the expansion shows that, for large z, it is better to use the asymptotic property of the expansion (the ratio of successive terms is of order $\mathcal{O}(1/z)$). The incomplete gamma functions can be computed by using a backward recursion scheme.

Another example is the K-Bessel function given by

$$K_\nu(z) = \frac{\sqrt{\pi}(2z)^\nu \mathrm{e}^{-z}}{\Gamma(\nu+\frac{1}{2})} \int_0^\infty \mathrm{e}^{-2zt}[t(1+t)]^{\nu-1/2}\,\mathrm{d}t. \tag{6.4}$$

Again, expanding

$$(t+1)^{\nu-1/2} = \sum \binom{\nu-\frac{1}{2}}{k} (t)^k$$

gives the standard expansion. A convergent expansion can be obtained by using

$$(t+1)^{\nu-1/2} = \sum c_k \left(\frac{t}{t+1}\right)^k, \qquad c_k = (-1)^k \binom{\frac{1}{2}-\nu}{k}. \tag{6.5}$$

This gives a convergent expansion in terms of confluent hypergeometric functions

$$K_\nu(z) = \sqrt{\frac{\pi}{2z}}\,\mathrm{e}^{-z} \sum_{k=0}^{\infty} (-1)^k \frac{(\nu+\frac{1}{2})_k(\nu-\frac{1}{2})_k}{k!} U(k,\tfrac{1}{2}-\nu,2z). \tag{6.6}$$

We have

$$U(0,\tfrac{1}{2}-\nu,2z) = 1, \qquad U(1,\tfrac{1}{2}-\nu,2z) = (2z)^{\nu+1/2}\mathrm{e}^{2z}\Gamma(-\tfrac{1}{2}-\nu,2z), \tag{6.7}$$

again an incomplete gamma function. Other U-functions can be obtained by recursion. For large k, the U-function behaves like [11],

$$k!\,U(k,\tfrac{1}{2}-\nu,2z) = \mathcal{O}(k^\alpha \mathrm{e}^{-2\sqrt{2kz}}),$$

where α is some constant. It follows that the convergence is better than in the previous example.

As a final example we consider an expansion of the Kummer function. Tricomi [14] has derived several convergent expansions of the ${}_1F_1$-function in terms of Bessel functions that are useful for evaluating the function when the parameters are large. For example, we have (see [14])

$$
{}_1F_1(a,c;z) = \mathrm{e}^{z/2}\Gamma(c)(\kappa z)^{(1-c)/2}\sum_{n=0}^{\infty} A_n(\kappa,\tfrac{1}{2}c)\left(\frac{z}{4\kappa}\right)^{n/2} J_{c-1+n}(2\sqrt{\kappa z}), \tag{6.8}
$$

where $\kappa = \frac{1}{2}c - a$ and the $A_n(\kappa,\lambda)$ are coefficients in the generating function

$$
\mathrm{e}^{2\kappa z}(1-z)^{\kappa-\lambda}(1+z)^{-\kappa-\lambda} = \sum_{n=0}^{\infty} A_n(\kappa,\lambda) z^n. \tag{6.9}
$$

The series in (6.8) is convergent in the entire z-plane. Moreover, it can be used for the evaluation of ${}_1F_1(a,c;z)$ for large κ, because the series has an asymptotic property. For further details on these expansions, we refer to [14].

Acknowledgments

The authors thank the referee for the comments on the first version of the paper.

J.L.L. thanks the CWI of Amsterdam for its scientific and financial support during the realization of this work. The financial support of the savings bank, Caja Rural de Navarra, is also acknowledged.

References

1 R. Bo and R. Wong. Uniform asymptotic expansion of Charlier polynomials. *Meth. Applic. Analysis* **1** (1994), 294–313.

2 T. M. Dunster. Uniform asymptotic expansion for Charlier polynomials. *J. Approx. Theory* **112** (1994), 93–133.

3 A. Erdélyi. *Higher transcendental functions* (ed. A. Erdélyi), vols 1–3 (McGraw-Hill, 1953).

4 C. L. Frenzen and R. Wong. Uniform asymptotic expansions of Laguerre polynomials. *SIAM J. Math. Analysis* **19** (1988), 1232–1248.

5 W. M. Y. Goh. Plancherel–Rotach asymptotics for the Charlier polynomials. *Constr. Approx.* **14** (1998), 151–168.

6 L. C. Hsu. Certain asymptotic expansions for Laguerre polynomials and Charlier polynomials. *Approx. Theory Applicat.* **11** (1995), 94–104.

7 R. Koekoek and R. F. Swarttouw. *The Askey scheme of hypergeometric orthogonal polynomials and its q-analogue* (1998). Delft University of Technology, Faculty of Technical Mathematics and Informatics, Report No. 94-05. (Available at http://aw.twi.tudelft.nl/~koekoek/askey.html.)

8 J. L. López and N. M. Temme. Approximations of orthogonal polynomials in terms of Hermite polynomials. *Meth. Applic. Analysis* **6** (1999), 131–146.

9 J. L. López and N. M. Temme. Two-point Taylor expansions of analytic functions. *Stud. Appl. Math.* **109** (2002), 297–311.

10 J. L. López and N. M. Temme. Multi-point Taylor expansions of analytic functions (2004). Accepted for publication in *Trans. Amer. Math. Soc.*

11 Y. L. Luke. *The special functions and their applications* (Academic, 1969).

12 N. M. Temme. Asymptotic estimates for Laguerre polynomials. *Z. Angew. Math. Phys.* **41** (1990), 114–126.

13 N. M. Temme. *Special functions. An introduction to the classical functions of mathematical physics* (Wiley, 1996).

14 F. G. Tricomi. *Funzioni ipergeometriche confluenti* (Roma: Edizione Cremonese, 1954).

15 R. Wong. *Asymptotic approximations of integrals* (Academic, 1989).

(*Issued 29 June 2004*)