T-overlap Functions: a generalization of bivariate overlap functions by t-norms

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Abstract. This paper introduces a generalization of overlap functions by extending one of the boundary conditions of its definition. More specifically, instead of requiring that "the considered function is equal to zero if and only if some of the inputs is equal to zero", we allow the range in which some t-norm is zero. We call such generalization by a t-overlap function with respect to such t-norm. Then we analyze the main properties of t-overlap function and introduce some construction methods.

Keywords: aggregation function, overlap function, t-norm

1 Introduction

The notion of overlap function [1, 4, 7–10] has shown itself very useful to deal with situations in which it is necessary to determine up to what extent a given element belongs to one or several classes whose boundaries are not crisp. It has been used, e.g., in image processing [12], classification problems [13, 14] and decision making [11].

Our goal here is to generalize the notion of overlap function by relaxing one of the boundary condition. In particular, instead of demanding that "the considered function is equal to zero if and only if some of the inputs is equal to zero", we allow for some kind of threshold, defined in terms of a t-norm T. We call such generalization by a t-overlap function with respect to T.

We notice that, this simple generalization allows us to state several interesting properties, which may allow for application in fuzzy rule-based system in order to discard bad rules when computing the compatibility degree. Section 2 presents some preliminary concepts. In Sect. 3, besides studying the main properties, we also propose some construction methods. Section 4 is the Conclusion.

2 Preliminaries

This section aims at introducing the background necessary to understand the paper.

Definition 1. *A fuzzy negation is a function $N\colon [0,1] \to [0,1]$ satisfying: **(N1)** the boundary conditions: $N(0) = 1$ and $N(1) = 0$; **(N2)** N is decreasing: if $x \leq y$ then $N(y) \leq N(x)$.*

A fuzzy negation N is said to be strong if: $\forall x \in [0,1] : N(N(x)) = x$ (the involutive property). The standard negation or the Zadeh's negation is given by $N_Z(x) = 1 - x$.

Definition 2. *[3, 15] A function $A : [0,1]^n \to [0,1]$ is said to be an n-ary aggregation operator if the following conditions hold:*

(A1) *A is increasing* [6] *in each argument: for each $i \in \{1, \ldots, n\}$, if $x_i \leq y$, then $A(x_1, \ldots, x_n) \leq A(x_1, \ldots, x_{i-1}, y, x_{i+1}, \ldots, x_n)$;*
(A2) *A satisfies the Boundary conditions: $A(0, \ldots, 0) = 0$ and $A(1, \ldots, 1) = 1$.*

Definition 3. *A t-norm is a bivariate aggregation function $T : [0,1]^2 \to [0,1]$ satisfying the following properties, for all $x, y, z \in [0,1]$:*

(T1) *Commutativity: $T(x, y) = T(y, x)$;*
(T2) *Associativity: $T(x, T(y, z)) = T(T(x, y), z)$;*
(T3) *Boundary condition: $T(x, 1) = x$.*

Example of t-norms are the Łukasiewicz and Yager t-norms, defined, respectively, by $T_{Ł}(x, y) = \max\{0, x+y-1\}$ and $T_Y(x, y) = \max\{0, 1 - \sqrt{(1-x)^2 + (1-y)^2}$.

An element $x \in]0,1]$ is a non-trivial zero divisor of T if there exists $y \in]0,1]$ such that $T(x, y) = 0$. A t-norm is positive if and only if it has no non-trivial zero divisors, i.e., if $T(x, y) = 0$ then either $x = 0$ or $y = 0$. Examples of continuous and positive t-norms are the minimum and the product t-norms, defined, respectively, by $T_M(x, y) = \min\{x, y\}$ and $T_P(x, y) = xy$.

The main concern of this paper is the concept of overlap function [1, 4, 7–9, 12].

Definition 4. *[4] An overlap function is a bivariate function $O\colon [0,1]^2 \to [0,1]$ satisfying the following properties, for all $x, y \in [0,1]$:*

(O1) *O is commutative: $O(x, y) = O(y, x)$;*
(O2) *$O(x, y) = 0$ if and only if $x = 0$ or $y = 0$;*
(O3) *$O(x, y) = 1$ if and only if $x = y = 1$;*
(O4) *O is increasing;*
(O5) *O is continuous.*

3 Introducing t-Overlap Functions

This section generalizes the concept of overlap functions by changing the condition **(O2)** of Definition 4, namely, the property that requires that, for all $x, y \in [0,1]$ and overlap function $O : [0,1]^2 \to [0,1]$ it holds that $O(x, y) = 0 \Leftrightarrow xy = 0$. In our generalization, we replace the product operation by a t-norm $T : [0,1]^2 \to [0,1]$.

[6] In this paper, a increasing (decreasing) function does not need to be strictly increasing (decreasing).

Definition 5. *Let $T : [0,1]^2 \to [0,1]$ be a t-norm. A function $O_T : [0,1]^2 \to [0,1]$ is said to be a t-overlap function with respect to T if the following conditions hold:*

(O_T1) $O_T(x,y) = O_T(y,x)$,
(O_T2) $O_T(x,y) = 0 \Leftrightarrow T(x,y) = 0$,
(O_T3) $O_T(x,y) = 1 \Leftrightarrow x = y = 1$,
(O_T4) *O_T is increasing,*
(O_T5) *O_T is continuous.*

Remark 1. Observe that, considering a fuzzy rule-based system, this generalization allows to discard bad rules when computing the compatibility degree. This is due to the fact that the membership degrees of the input with the antecedents would be low for bad rules and, consequently, t-overlap functions may return 0 instead of a low value, which can mislead the final prediction. Accordingly, we have maintained the third condition, since, intuitively, it is not interesting to give the same value to all the rules whose membership degrees are high, since it may imply a decrease in the predictive power.

Remark 2. Notice that the proposed generalization enlarge the use of overlap function. For example, consider the overlap function $O = \frac{\sqrt{xy}}{\sqrt{xy}+(1-xy)}$, which only becomes zero in the case where $x = 0$ or $y = 0$, by condition **(O2)**, which means that overlap functions are t-overlap functions with respect to t-norms without zero divisors. Our definition overcomes this limitation by changing the condition **(O2)** by the condition(O_T2), where T is a t-norm that can have zero divisors. See, for example, the t-overlap function with respect to the Łukasiewicz t-norm $T_{Ł}$ given by:

$$O_{T_{Ł}}(x,y) = \frac{\max\{0,(1+\lambda)(x+y-1)-\lambda xy\}}{\max\{0,(1+\lambda)(x+y-1)-\lambda xy\}+\min\{1,1-(1+\lambda)(x+y-1)+\lambda xy\}}.$$

Example 1. Let $G : [0,1]^2 \to [0,1]$ be defined by $G(x,y) = (\min\{x,y\})^p$, with $p > 0$, and consider the Łukasiewicz and Yager t-norms, $T_{Ł}(x,y) = \max\{0, x+y-1\}$ and $T_Y(x,y) = \max\{0, 1-\sqrt{(1-x)^2+(1-y)^2}$. Then, the functions $O^G_{T_{Ł}}, O^2_{T_{Ł}}, O_{T_Y}$: $[0,1]^2 \to [0,1]$, defined by $O^G_{T_{Ł}}(x,y) = G(x,y)T_{Ł}(x,y)$, $O^2_{T_{Ł}}(x,y) = 2^{T_{Ł}} - 1$ and $O_{T_Y}(x,y) = 2^{T_Y} - 1$ are t-overlap functions whit respect to $T_{Ł}$ and T_Y.

The previous example may be generalized as the following results:

Remark 3. Let $T : [0,1]^2 \to [0,1]$ be a continuous t-norm. Then T is a t-overlap function with respect to itself.

Proposition 1. *Let $O : [0,1]^2 \to [0,1]$ be an overlap function and $T : [0,1]^2 \to [0,1]$ be a continuous t-norm. Then the function $O_T : [0,1]^2 \to [0,1]$, defined, for all $x, y \in [0,1]$, by $O_T(x,y) = O(x,y)T(x,y)$ is a t-overlap function with respect to T.*

Proof. (O_T1) It is immediate.
(O_T2) For all $x, y \in [0,1]$, it follows that:

$$\begin{aligned} O_T(x,y) = 0 &\Leftrightarrow O(x,y)T(x,y) = 0 \\ &\Leftrightarrow O(x,y) = 0 \ \vee \ T(x,y) = 0 \\ &\Leftrightarrow x = 0 \ \vee \ y = 0 \ \vee \ T(x,y) = 0 \ \text{ by } \textbf{(O2)} \\ &\Leftrightarrow T(x,y) = 0. \end{aligned}$$

(O_T3) For all $x, y \in [0,1]$, it follows that:

$$\begin{aligned} O_T(x,y) = 1 &\Leftrightarrow O(x,y)T(x,y) = 1 \\ &\Leftrightarrow O(x,y) = 1 \ \wedge \ T(x,y) = 1 \\ &\Leftrightarrow x = 1 \ \wedge \ y = 1 \ \wedge \ T(x,y) = 1 \ \text{ by } \textbf{(O3)} \\ &\Leftrightarrow T(x,y) = 1. \end{aligned}$$

(O_T4-5) Since both O and T are continuous and increasing, then the results are immediate.

□

The previous theorem may be generalized using a special t-norm T' instead of the product between the overlap function O and the t-norm T with which the function O_T is a t-overlap with respect to T.

Proposition 2. *Let $O : [0,1]^2 \to [0,1]$ be an overlap function and $T : [0,1]^2 \to [0,1]$ be a continuous t-norm. For any continuous and positive t-norm $T' : [0,1]^2 \to [0,1]$, one has that the function $O_T : [0,1]^2 \to [0,1]$, defined, for all $x, y \in [0,1]$, by $O_T(x,y) = T'(O(x,y), T(x,y))$, is a t-overlap function with respect to T.*

Proof. (O_T1) It is immediate.
(O_T2) For all $x, y \in [0,1]$, it follows that:

$$\begin{aligned} O_T(x,y) = 0 &\Leftrightarrow T'(O(x,y), T(x,y)) = 0 \\ &\Leftrightarrow O(x,y) = 0 \ \ \vee \ \ T(x,y) = 0 \ \text{ Since } T' \text{ is positive} \\ &\Leftrightarrow x = 0 \ \ \vee \ \ y = 0 \ \ \vee \ \ T(x,y) = 0 \ \text{ by } \textbf{(O2)} \\ &\Leftrightarrow T(x,y) = 0. \end{aligned}$$

(O_T3) For all $x, y \in [0,1]$, it follows that:

$$\begin{aligned} O_T(x,y)(x,y) = 1 &\Leftrightarrow T'(O(x,y), T(x,y)) = 1 \\ &\Leftrightarrow O(x,y) = 1 \ \ \vee \ \ T(x,y) = 1 \\ &\Leftrightarrow x = y = 1 \ \ \vee \ \ T(x,y) = 1 \\ &\Leftrightarrow T(x,y) = 1. \end{aligned}$$

(O_T4-5) It is immediate.

□

Note that if a t-norm T is positive, then O_T is an overlap function.

Theorem 1. *Let $O_T^1, \ldots, O_T^n : [0,1]^2 \to [0,1]$ be t-overlap functions with respect to a t-norm $T : [0,1]^2 \to [0,1]$ and $\omega_1, \ldots, \omega_n \in [0,1]$ be weights with $\sum_{i=1}^n \omega_i = 1$. Then the function $O_T : [0,1]^2 \to [0,1]$, defined, for all $x, y \in [0,1]$, by $O_T(x,y) = \sum_{i=1}^n \omega_i O_T^i(x,y)$ is also a t-overlap function with respect to T.*

Proof. (O_T1) It is immediate.

(O_T2) For all $x, y \in [0,1]$, it follows that:

$$\mathrm{O}_T(x,y) = 0 \Leftrightarrow \sum_{i=1}^{n} \omega_i O_T^i(x,y) = 0$$
$$\Leftrightarrow \omega_i O_T^i(x,y) = 0, \forall i = 1, \ldots, n.$$

Since $\sum_{i=1}^{n} \omega_i = 1$, then there exists $k \in \{0, \ldots, n\}$ such that $\omega_k \neq 0$, and, thus $O_T^k(x,y) = 0$. By (O_T2), it holds that $T(x,y) = 0$. The reciprocal is analogous.

(O_T3) For all $x, y \in [0,1]$, it follows that:

$$\mathrm{O}_T(x,y) = 1 \Leftrightarrow \sum_{i=1}^{n} \omega_i O_T^i(x,y) = 1 = \sum_{i=1}^{n} \omega_i.$$

One has that $\sum_{i=1}^{n} \omega_i O_T^i(x,y) - \sum_{i=1}^{n} \omega_i = 0$, i.e., $\sum_{i=1}^{n} \omega_i(O_T^i(x,y) - 1) = 0$. This means that, for all $i = 1, \ldots, n$, it holds that $\omega_i O_T^i(x,y) - \sum_{i=1}^{n} \omega_i = 0$. However, since $\sum_{i=1}^{n} \omega_i \neq 0$, there exist $k \in \{1, \ldots, n\}$ such that $\omega_k \neq 0$. Thus, one has that $O_T^k(x,y) = 1$, and, by (O_T3), it follows that $x = y = 1$. The reciprocal is analogous.

(O_T4-5) It is immediate.

□

Let $T : [0,1]^2 \to [0,1]$ be a t-norm and denote $\mathrm{K}_T = \{(x,y) \in [0,1]^2 \mid T(x,y) = 0\}$. Obviously, any t-overlap function with respect to a t-norm T coincides with an overlap function if and only if $\mathrm{K}_T = \{(x,y) \in [0,1]^2 \mid x = 0 \vee y = 0\}$.

Denote by Θ the set of all t-overlap functions with respect of any t-norm T. The following result is immediate.

Theorem 2. *The ordered set $\mathfrak{S} = (\Theta, \leq_\Theta)$ is a lattice, where $\leq_\Theta$ is defined, for all $O_{T_1}, O_{T_2} \in \Theta$, by $O_{T_1} \leq_\Theta O_{T_2}$ if and only if $O_{T_1}(x,y) \leq O_{T_2}(x,y)$, for all $(x,y) \in [0,1]^2$.*

Theorem 3. *Let O_{T_i} be a t-overlap function with respect to the t-norms $T_1, \ldots, T_n : [0,1]^2 \to [0,1]$ and let $\omega_1, \ldots, \omega_n \in [0,1]$ be weights such that $\sum_{i=1}^{n} \omega_i = 1$. If $T = \sum_{i=1}^{n} \omega_i T_i : [0,1]^2 \to [0,1]$ is a t-norm, then O_{T_i} is a t-overlap function with respect to T.*

Proof. (O_T1) It is immediate.

(O_T2) $(\Rightarrow)$ Since O_{T_i} is a t-overlap function with respect to the t-norms $T_1, \ldots, T_n$, then, by (O_T2), for all $i = 1, \ldots, n$, it holds that whenever $O_{T_i}(x,y) = 0$ then $T_i(x,y) = 0$, for all $x, y \in [0,1]$. Then, it follows that $\sum_{i=1}^{n} \omega_i T_i(x,y) = 0$. $(\Leftarrow)$ If $\sum_{i=1}^{n} \omega_i T_i(x,y) = 0$, then, since $\sum_{i=1}^{n} \omega_i \neq 0$, there exists $k = 1, \ldots, n$ such that $\omega_k \neq 0$. It follows that $T_k(x,y) = 0$. Since O_{T_i} is a t-overlap function with respect to the t-norm T_k, one has that $O_{T_i}(x,y) = 0$. It follows that O_{T_i} is a t-overlap function with respect to the t-norm T.

(O_T3-5) It is immediate.

□

Theorem 4. *Let $O_1, O_2 : [0,1]^2 \to [0,1]$ be t-overlap functions with respect to the t-norms $T_1, T_2 : [0,1]^2 \to [0,1]$, respectively. Consider $\omega_1, \omega_2 \in [0,1]$ such that $\omega_1 + \omega_2 = 1$. If $T'[0,1]^2 \to [0,1]$ is a positive t-norm then*

$$O_T(x,y) = \omega_1 O_1(x,y) + \omega_2 O_2(x,y)$$

is a t-overlap function with respect to the t-norm $T : [0,1]^2 \to [0,1]$, defined, for all $x, y \in [0,1]$, by $T(x,y) = T'(T_1(x,y), T_2(x,y))$.

Proof. (O_T1) It is immediate.
(O_T2) For all $x, y \in [0,1]$, it follows that:

$$\begin{aligned} O_T(x,y) = 0 &\Leftrightarrow \omega_1 O_1(x,y) + \omega_2 O_2(x,y) = 0 \Leftrightarrow \omega_1 O_1(x,y) = \omega_2 O_2(x,y) = 0 \\ &\Leftrightarrow \omega_1 = 0 \vee O_1(x,y) = 0 \text{ and } \omega_2 = 0 \vee O_2(x,y) = 0. \end{aligned}$$

Now suppose that $\omega_1 \neq 0$. Then one has that $O_1(x,y) = 0$ and, by (O_T2), it holds that $T_1(x,y) = 0$. It follows that $T(x,y) = T'(T_1(x,y), T_2(x,y)) = 0$. The reciprocal is analogous, taking into account that T' is a positive t-norm.
(O_T3) For all $x, y \in [0,1]$, it follows that:

$$\begin{aligned} O_T(x,y) = 1 &\Leftrightarrow \omega_1 O_1(x,y) + \omega_2 O_2(x,y) = 1 \\ &\Leftrightarrow \omega_1 O_1(x,y) + \omega_2 O_2(x,y) = \omega_1 + \omega_2 \\ &\Leftrightarrow \omega_1(1 - O_1(x,y)) + \omega_2(1 - O_2(x,y)) = 0 \\ &\Leftrightarrow \omega_1 = 0 \vee 1 - O_1(x,y) = 0 \text{ and } O_2 = 0 \vee 1 - O_2(x,y) = 0 \\ &\Leftrightarrow \omega_1 = 0 \vee O_1(x,y) = 1 \text{ and } O_2 = 0 \vee O_2(x,y) = 1 \\ &\Leftrightarrow \omega_1 = 0 \vee x = y = 1 \text{ and } \omega_2 = 0 \vee x = y = 1. \end{aligned}$$

Now, since $\omega_1 + \omega_2 = 1$ it holds that $x = y = 1$. The reciprocal is immediate.
$(O_T4 - 5)$ It is immediate.

□

Theorem 5. *The function $O_T : [0,1]^2 \to [0,1]$ is a t-overlap function with respect to a t-norm $T : [0,1]^2 \to [0,1]$ if and only if*

$$O_T(x,y) = \frac{f(x,y)}{f(x,y) + h(x,y)},$$

for all $x, y \in [0,1]$ and some functions $f, h : [0,1]^2 \to [0,1]$ such that

(i) *f and h are commutative.*
(ii) *f is increasing and h is decreasing.*
(iii) *$f(x,y) = 0$ if and only if $T(x,y) = 0$.*
(iv) *$h(x,y) = 0$ if and only if $x = y = 1$.*
(v) *f and h are continuous.*

Proof. ($\Rightarrow$) Suppose that O_T is a T-overlap function with respect to a t-norm T. Consider that $O_T(x,y) = f(x,y)$ and $h(x,y) = 1 - f(x,y)$. It is immediate that (i) f and h are symmetric, (ii) f is increasing and h is decreasing and (v) f and h are continuous. (iii) Now, by (O_T2), $f(x,y) = 0$ if and only if $T(x,y) = 0$. (iv) Similarly, by (O_T3), $h(x,y) = 0$ if and only if $f(x,y) = 1$ if and only if $x = y = 1$. Since $f(x,y) + h(x,y) = 1$ then

$$O_T(x,y) = f(x,y) = \frac{f(x,y)}{1} = \frac{f(x,y)}{f(x,y)+h(x,y)}.$$

($\Leftarrow$) Consider two functions $f, g : [0,1]^2 \to [0,1]$ satisfying the conditions (i)-(v), and the function $O_T : [0,1]^2 \to [0,1]$, defined, for all $x, y \in [0,1]$, by

$$O_T(x,y) = \frac{f(x,y)}{f(x,y)+h(x,y)}.$$

(O_T1) It is immediate.
(O_T2) For all $x, y \in [0,1]$, it follows that:

$$O_T(x,y) = 0 \Leftrightarrow \frac{f(x,y)}{f(x,y)+h(x,y)} = 0 \Leftrightarrow f(x,y) = 0 \Leftrightarrow T(x,y) = 0.$$

(O_T3) For all $x, y \in [0,1]$, it follows that:

$$\begin{aligned} O(x,y) = 1 \Leftrightarrow \frac{f(x,y)}{f(x,y)+h(x,y)} &= 1 \Leftrightarrow f(x,y) = f(x,y) + h(x,y) \\ \Leftrightarrow h(x,y) &= 0 \Leftrightarrow x = y = 1. \end{aligned}$$

(O_T4) Let $x, y, z \in [0,1]$ be such that $x \le y$, then $f(x,z) \le f(y,z)$ and $h(y,z) \le h(x,z)$. It follows that

$$\begin{gathered} f(x,z)h(y,z) \le f(y,z)h(x,z) \Rightarrow \\ f(x,z)h(y,z) + f(x,z)f(y,z) \le f(y,z)h(x,z) + f(x,z)f(y,z) \Rightarrow \\ f(x,z)(h(y,z)+f(y,z)) \le f(y,z)(h(x,z)+f(x,z)) \Rightarrow \\ \frac{f(x,z)}{h(x,z)+f(x,z)} \le \frac{f(y,z)}{h(y,z)+f(y,z)} \Rightarrow \\ O(x,z) \le O(y,z). \end{gathered}$$

(O_T5) It is immediate.

□

From the previous theorem, one may consider the particular case where the function f is the t-norm T (with respect to the function O_T is a t-overlap function), and the function h is $N(T)$, where $N : [0,1] \to [0,1]$ is a strong negation. It is immediate that:

Corollary 1. *Let T be a continuous t-norm and N a strong negation. Then the function $O_T : [0,1]^2 \to [0,1]$, defined, for all $x, y \in [0,1]$, by*

$$O_T(x,y) = \frac{T(x,y)}{T(x,y)+N(T(x,y))}$$

is a t-overlap function.

Supported by the previous corollary, we give some examples of t-overlap functions that are not overlap functions, considering continuous and positive t-norms.

Example 2. The following functions are some examples of associative t-overlap functions that are not overlap functions, since the property **(O2)** does not hold:

(i) Consider the standard negation $N_Z(x) = 1 - x$ and the family of Lukasiewicz t-norms $\mathrm{T}_{\text{Ł}}(x,y) = \max\{0, (1+\lambda)(x+y-1) - \lambda xy\}$, where $\lambda \ge -1$. The function $O_1 : [0,1]^2 \to [0,1]$, defined, for $x, y \in [0,1]$, by

$$O_1(x,y) = \frac{\max\{0,(1+\lambda)(x+y-1)-\lambda xy\}}{\max\{0,(1+\lambda)(x+y-1)-\lambda xy\}+\min\{1,1-(1+\lambda)(x+y-1)+\lambda xy\}}$$

is a t-overlap function with respect to $T_Ł$.

(ii) Consider the Lukasiewicz t-norm $T_Ł(x,y) = \max\{0, x+y-1\}$ and the strong negation $N(x) = \sqrt{1-x^2}$. The function $O_2 : [0,1]^2 \to [0,1]$, defined, for all $x, y \in [0,1]$, by

$$O_2(x,y) = \frac{\max\{0,(x+y-1)\}}{\max\{0,x+y-1\}+\min\{1,\sqrt{1-(x+y-1)^2}\}},$$

is a t-overlap function with respect to $T_Ł$. Now, if one takes the strong negation $N(x) = \frac{2}{\pi}\arcsin[1-\sin(\frac{\pi}{2}x)]$, then the function $O_3 : [0,1]^2 \to [0,1]$, defined, for all $x, y \in [0,1]$, by

$$O_3(x,y) = \frac{\max\{0,x+y-1)\}}{\max\{0,x+y-1\}+\frac{2}{\pi}\arcsin[\frac{\pi}{2}\max\{o,x+y-1\}]}$$

is a t-overlap function with respect to $T_Ł$.

(iii) Consider the Yager t-norm $T_Y(x,y) = \max\{0, 1-\sqrt{(1-x)^2+(1-y)^2}$ and the strong negation $N(x) = \sqrt{1-x^2}$. The function $O_4 : [0,1]^2 \to [0,1]$, defined, for all $x, y \in [0,1]$, by

$$O_4(x,y) = \frac{\max\{0,1-\sqrt{(1-x)^2+(1-y)^2}\}}{\max\{0,1-\sqrt{(1-x)^2+(1-y)^2}\}+\sqrt{1-\max^2\{0,1-\sqrt{(1-x)^2+(1-y)^2}\}}}$$

is a t-overlap function with respect to T_Y. Now, if one takes the strong negation $N(x) = \frac{2}{\pi}\arcsin(1-\sin(x\frac{\pi}{2}))$, then the function $O_5 : [0,1]^2 \to [0,1]$, defined, for all $x, y \in [0,1]$, by

$$O_5(x,y) =$$

$$\frac{\max\{0, 1-\sqrt{(1-x)^2+(1-y)^2}\}}{\max\{0, 1-\sqrt{(1-x)^2+(1-y)^2}\}+\frac{2}{\pi}\arcsin(1-\sin(\frac{\pi}{2}\max\{0, 1-\sqrt{(1-x)^2+(1-y)^2}\}))}$$

is a t-overlap function with respect to T_Y.

Corollary 2. *Let $O_T : [0,1]^2 \to [0,1]$ be a t-overlap function with respect to a t-norm $T : [0,1]^2 \to [0,1]$ and $h : [0,1]^2 \to [0,1]$ satisfying the conditions (1), (ii), (iv) and (v) of Theorem 5. Then it holds that $O_T(x,x) = x$, for some $x \in [0,1[$ if and only if*

$$f(x,x) = \frac{x}{1-x}h(x,x).$$

Proof. For $x \in [0,1[$, it follows that:

$$O_T(x,x) = x \Leftrightarrow x = \frac{f(x,x)}{f(x,x)+h(x,x)} \text{ by Theorem 5}$$

$$\Leftrightarrow xf(x,x) + xh(x,x) = f(x,x) \Leftrightarrow f(x,x) = \frac{x}{1-x}h(x,x).$$

□

Given two t-norms $T_1, T_2 : [0,1]^2 \to [0,1]$, define $T_1T_2 : [0,1]^2 \to [0,1]$ by $T_1T_2(x,y) = T_1(x,y)T_2(x,y)$, for all $[x,y] \in [0,1]$.

Theorem 6. *Let $O : [0,1]^2 \to [0,1]$ be a overlap function, $T_1, T_2 : [0,1]^2 \to [0,1]$ be continuous t-norms such that $T_1, T_2 : [0,1]^2 \to [0,1]$ is a t-norm. Then the function $O_T : [0,1]^2 \to [0,1]$ defined, for all $[x,y] \in [0,1]$, by $O_T(x,y) = O(T_1(x,y), T_2(x,y))$, is a t-overlap function with respect to T_1T_2.*

Proof. (O_T1) It is immediate.
(O_T2) For all $[x,y] \in [0,1]$, it follows that:

$$\begin{aligned} O_T(x,y) = 0 &\Leftrightarrow O(T_1(x,y), T_2(x,y)) = 0 \Leftrightarrow T_1(x,y)T_2(x,y) = 0 \\ &\Leftrightarrow (T_1T_2)(x,y) = 0. \end{aligned}$$

$(O_T3$ For all $[x,y] \in [0,1]$, it follows that:

$$\begin{aligned} O_T(x,y) = 1 &\Leftrightarrow O(T_1(x,y), T_2(x,y)) = 1 \Leftrightarrow T_1(x,y) = T_2(x,y) = 1 \\ &\Leftrightarrow x = y = 1. \end{aligned}$$

$(O_T4 - 5)$ It is immediate.

□

Theorem 7. *Let $O_1, O_2 : [0,1]^2 \to [0,1]$ be t-overlap functions with respect to the t-norms $T_1, T_2 : [0,1]^2 \to [0,1]$, respectively, and $M : [0,1]^2 \to [0,1]$ be a continuous and positive function such that $M(x,y) = 1 \Leftrightarrow x = y = 1$. Then the function $O_T : [0,1]^2 \to [0,1]$, defined, for all $x, y \in [0,1]$, by $O_T(x,y) = M(O_1(x,y), O_2(x,y))$, is a t-overlap function with respect to T_1 or T_2.*

Proof. (O_T1) It is immediate.
(O_T2) For all $[x,y] \in [0,1]$, it follows that:

$$\begin{aligned} O_T(x,y) = 0 &\Leftrightarrow M(O_1(x,y), O_2(x,y)) = 0 \Leftrightarrow O_1(x,y) = 0 \vee O_2(x,y) = 0 \\ &\Leftrightarrow T_1(x,y) = 0 \vee T_2(x,y) = 0. \end{aligned}$$

(O_T3) For all $[x,y] \in [0,1]$, it follows that:

$$\begin{aligned} O_T(x,y) = 1 &\Leftrightarrow M(O_1(x,y), O_2(x,y)) = 1 \Leftrightarrow O_1(x,y) = O_2(x,y) = 1 \\ &\Leftrightarrow x = y = 1. \end{aligned}$$

$(O_T4 - 5)$ It is immediate.

□

4 Conclusion

In this work, we generalized the concept of overlap functions, by relaxing the requirement that "one of its inputs must be zero so that the overlap function is zero". For that, we considered overlap functions associated to positive t-norms, as the Luckasiewicz t-norm. Likewise, a method for constructing t-overlap functions based on certain simple conditions has been presented. Future work is concerned this generalization under an interval-valued approach, as in [2, 5, 6].

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