Weak and directional monotonicity of functions on Riesz spaces to fuse uncertain data

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Abstract

In the theory of aggregation, there is a trend towards the relaxation of the axiom of monotonicity and also towards the extension of the definition to other domains besides real numbers. In this work, we join both approaches by defining the concept of directional monotonicity for functions that take values in Riesz spaces. Additionally, we adapt this notion in order to work in certain convex sublattices of a Riesz space, which makes it possible to define the concept of directional monotonicity for functions whose purpose is to fuse uncertain data coming from type-2 fuzzy sets, fuzzy multisets, n-dimensional fuzzy sets, Atanassov intuitionistic fuzzy sets and interval-valued fuzzy sets, among others. Focusing on the latter, we characterize directional monotonicity of interval-valued representable functions in terms of standard directional monotonicity.

Keywords: Aggregation function; Directional monotonicity; Interval-valued function; Riesz space; Type-2 fuzzy set

1. Introduction

The theory of aggregation functions addresses the problem of obtaining a single number that is representative for a collection of values. This issue is prevalent in any process that involves working with real data. Before turning into an independent theory, there had been various works in the literature on the topic of aggregation [20, 40, 49]. The first monograph on aggregation functions was published [17] in 2001. Classically, an aggregation function A is a function defined on the unit hypercube with values in the unit interval that satisfies

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certain boundary conditions and is increasing with respect to every argument. This family of functions has been extensively used in different theoretical and applied fields [29, 31, 45].

Various works about the state-of-the-art of the theory of aggregation functions [30, 41] declare that the following are two of the major trends in the aggregation theory:

1. To relax the monotonicity constraints in the definition of aggregation functions;

2. To extend the concept of aggregation functions to be capable of handling more scales apart from numbers.

The need of relaxing the monotonicity condition was originated due to the existence of functions that are valid to fuse information but do not qualify as aggregation functions because they violate the monotonicity condition. This is the case, for example of the Lehmer mean [11]. Consequently, and pursuing the creation of a framework of functions for fusing data, the notion of weak monotonicity was introduced [52]. This concept was then generalized by directional monotonicity [15], which studies the monotonicity of a function along a real vector, or a ray, in $\mathbb{R}^n$. Directional monotonicity was established as the new axiom replacing standard monotonicity and this lead to the introduction of a class of functions resembling aggregation functions but with relaxed monotonicity constrictions [39]. Subsequently, new notions of monotonicity have arised [7, 14, 48] and have been applied with success to problems of edge detection in computer vision [14, 47] and fuzzy rule-based classification [37, 38].

Regarding the second item, the theory of aggregation has been extended to work with posets [25, 35], with graphs [51], with infinite sequences [42, 46] and with intervals [16, 22], among others. Furthermore, it is not unusual that there exists a degree of uncertainty around the data to aggregate (missing inputs, measurement errors, etc.) and therefore aggregation functions have been extended to work with values coming from structures that model uncertainty. This is the case of the different extensions of fuzzy sets, there are works in the literature describing how this type of uncertainty modeling techniques have been succesfully applied to real problems, e.g., type-2 fuzzy sets [33, 36], n-dimensional fuzzy sets [24], Atanassov intuitionistic fuzzy sets [21] and interval-valued fuzzy sets [10, 12, 16], among others.

In this work we combine both trends in the theory of aggregation, and based on the structure of Riesz spaces, we provide a framework to define directional monotonicity for functions that handle various types of uncertain data coming from different extensions of fuzzy sets. In particular, we define directional monotonicity for fusing type-2 fuzzy values, fuzzy multiset and n-dimensional fuzzy values, Atanassov intuitionistic fuzzy values and interval-valued values. We also study the properites of this class of functions and, focusing on the interval-valued setting, we show the relation between directional monotonicity for interval-valued representable functions and standard directional monotonicity presented in [15]. This relation permits to construct examples belonging to

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this class of functions on the basis of two functions defined in $[0,1]$ and with values in $[0,1]$. Additionally, we study the particular case when the directions of increasingness are formed by interval values. We refer to this concept as interval directional monotonicity (IDM).

This work is organized as follows. In Section 2 we present some preliminary concepts and results in order to make the work self-contained. In Section 3 we recall the notion of a Riesz space and expound some of the specific instances of Riesz spaces that we use later in this work. In Section 4 we introduce the concept of directional monotonicity for functions that take values in a Riesz space, as well as some properties and how this notion can be modified in order to work in certain sublattices of a Riesz space. In Section 5 we make use of the mentioned sublattices to show how we can recover the concept of directional monotonicity in order to fuse data that comes from different extensions of fuzzy sets. In Section 6 we explicitly present the notion of directional monotonicity for interval-valued functions and give the relation between this notion for representable interval-valued functions and standard directional monotonicity. We also propose the concept of interval directional monotonicity, IDM, which results from restricting the directions of increasingnes to vectors that are formed with intervals. We finalize this work with some conclusions in Section 7.

2. Preliminaries

2.1. Aggregation functions and directional monotonicity

We recall the definition of aggregation functions [5, 18, 32].

Definition 2.1. An aggregation function is a function $A \colon [0,1]^n \to [0,1]$ such that

(i) $A(0,\ldots,0) = 0$;

(ii) $A(1,\ldots,1) = 1$;

(iii) A is increasing, i.e., if $(x_1,\ldots,x_n), (y_1,\ldots,y_n) \in [0,1]^n$ such that $x_i \leq y_i$ for all $1 \leq i \leq n$, then $A(x_1,\ldots,x_n) \leq A(y_1,\ldots,y_n)$.

Seeking the relaxation of the monotonicity condition, in [52] the notion of weak monotonicity was introduced.

Definition 2.2 ([52]). Let $F : [0,1]^n \to [0,1]$, we say that F is weakly increasing (weakly decreasing), if for all $c > 0$ and $\mathbf{x} = (x_1,\ldots,x_n) \in [0,1]^n$ such that $0 \leq x_i+c \leq 1$ for all $1 \leq i \leq n$, it holds that $F(x_1+c,\ldots,x_n+c) \geq F(x_1,\ldots,x_n)$ $(F(x_1 + c,\ldots,x_n + c) \leq F(x_1,\ldots,x_n))$.

Weak monotonicity can be seen as monotonicity along the ray $\vec{1} = (1,\ldots,1)$ and this concept is generalized by directional monotonicity in [15].

Definition 2.3 ([15]). Let $F : [0,1]^n \to [0,1]$ and $\vec{r} \in \mathbb{R}^n$, we say that F is $\vec{r}$-increasing (decreasing), if for all $c > 0$ and $\mathbf{x} \in [0,1]^n$ such that $\mathbf{x} + c\vec{r} \in [0,1]^n$, it holds that $F(\mathbf{x} + c\vec{r}) \geq F(\mathbf{x})$ $(F(\mathbf{x} + c\vec{r}) \leq F(\mathbf{x}))$.

The relaxation of monotonicity in the definition of aggregation functions by directional monotonicity has led to good results in fuzzy rule-based classification systems [38, 39].

Let us now present two results about directionally monotone functions. The first deals with increasingness along the convex combination of two different directions.

Theorem 2.4 ([15]). *Let $\vec{r}, \vec{s} \in \mathbb{R}^n$ and $a, b \geq 0$ such that $a + b > 0$ and let us set $\vec{u} = a\vec{r} + b\vec{s}$. Let $\mathbf{x} \in [0,1]^n$ and $c > 0$ such that $\mathbf{x} + c\vec{u} \in [0,1]^n$ and either $\mathbf{x} + ca\vec{r}$ or $\mathbf{x} + cb\vec{s} \in [0,1]^n$. Then, if a function $F : [0,1]^n \to [0,1]$ is both $\vec{r}$- and $\vec{s}$-increasing, it is also $\vec{u}$-increasing.*

The second is a characterization of standard monotonicity in terms of directional monotonicity.

Theorem 2.5 ([15]). *Let $n \in \mathbb{N}$, $F : [0,1]^n \to [0,1]$ and $\{\vec{e}_i\}_{i=1}^n$ be the canonical basis of $\mathbb{R}^n$. Then, the following are equivalent*

(i) *F is increasing;*

(ii) *F is $\vec{e}_i$-increasing for all $1 \leq i \leq n$.*

2.2. Interval-valued aggregation functions

We call $L(\mathbb{R})$ the set of closed intervals of the real numbers, i.e., $L(\mathbb{R}) = \{[x,y] \mid x, y \in \mathbb{R}, x \leq y\}$. The restriction to the intervals in the unit interval is denoted by $L([0,1])$.

The set of closed intervals $L(\mathbb{R}) = \{[x,y] \mid x, y \in \mathbb{R}, x \leq y\}$ and the half-space $K(\mathbb{R}) = \{(x,y) \in \mathbb{R}^2 \mid x \leq y\}$ of $\mathbb{R}^2$ are isomorphic lattices with respect to the standard partial order of intervals $\leq_L$ defined by

$$[a,b] \leq_L [c,d] \text{ if and only if } a \leq c \text{ and } b \leq d.$$

The top and bottom elements of $(L([0,1]), \leq_L)$ are $1_L = [1,1]$ and $0_L = [0,0]$, respectively.

Thus, we can define the concept of an interval-valued (IV) aggregation function. We denote the product space as $L([0,1])^n = L([0,1]) \times \ldots \times L([0,1])$ and the component-wise order in $L([0,1])^n$ by $\leq_{L^n}$.

Definition 2.6. Let $A : L([0,1])^n \to L([0,1])$. We say that A is an IV aggregation function if it satisfies the following conditions.

(i) $A(0_L, \ldots, 0_L) = 0_L$;

(ii) $A(1_L, \ldots, 1_L) = 1_L$;

(iii) A is increasing with respect to $\leq_L$.

Interval-valued aggregation operators have been, and continue to be, extensively studied both from the theoretic and applied points of view [6, 26, 44].

2.3. Fuzzy sets and generalizations

We end the preliminaries section with this subsection about different generalizations of fuzzy sets. We present the definition of the concepts that are mentioned in this work and some remarks about their relation. The history and main properties of the different generalizations of fuzzy sets can be found in [13].

Let us start by recalling the concept of a fuzzy set [55].

Definition 2.7. Given a non-empty universe X, a fuzzy set A on X is a function $A : X \to [0,1]$. Given $x \in X$, the membership degree of x to the fuzzy set is $A(x)$.

We denote the set of all fuzzy sets over the universe X as $FS(X)$. Fuzzy sets are also known as type-1 fuzzy sets, due to the ideas presented in [56], where some extensions of fuzzy sets were presented, the so-called type-n fuzzy sets $(T_n FS)$.

Definition 2.8. Given a non-empty universe X and $n > 1$, a type-n fuzzy set A on X is a function $A : X \to T_{n-1}FS([0,1])$.

In other words, a type-n fuzzy set is a fuzzy set in which the membership of the elements are described by a type-$(n-1)$ fuzzy set.

Among these extensions, type-2 fuzzy sets have been shown to hold a prominent position as they have been successfully applied in diverse fields [19, 23]. The membership of an element $x \in X$ to a type-2 fuzzy set A is given by a type one fuzzy set on the universe $[0,1]$, therefore a type-2 fuzzy set can be identified with an operator $A : X \to [0,1]^{[0,1]}$, where $[0,1]^{[0,1]}$ is the set of functions whose domain and codomain is $[0,1]$.

An additional generalization of fuzzy sets that we discuss in this work is the so-called fuzzy multisets, which were introduced by Yager in [53].

Definition 2.9. Let $n \geq 1$ and $X \neq \emptyset$. A fuzzy multiset A on X is a function $A : X \to [0,1]^n$.

If in the preceding definition (Definition 2.9), if we refer to the membership of an element $x \in X$ by $A(x) = (A_1(x), \ldots, A_n(x))$ and it holds that $A_1(x) \leq \ldots \leq A_n(x)$ for all $x \in X$, then we say that A is a n-dimensional fuzzy set [4]. In the literature one can find an extensive list of works on fuzzy multisets and n-dimensional fuzzy sets and their application [43].

We address two more extensions of fuzzy sets in this work: intuitionistic Atanassov fuzzy sets (AIFS) [1] and interval-valued fuzzy sets (IVFS) [56]. To define the concept of an AIFS let us first set: $D([0,1]) = \{(x,y) \in [0,1]^2 \mid x + y \leq 1\}$.

Definition 2.10. Given a non-empty universe X, an Atanassov intuitionistic fuzzy set A on X is a function given by $A : X \to D([0,1])$. For $x \in X$, we have $A(x) = (\mu_A(x), \nu_A(x))$, where $\mu_A(x)$ denotes the membership of x to the AIFS and $\nu_A(X)$ its non-membership.

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Finally, the concept of an interval-valued fuzzy set is defined as follows.

Definition 2.11. Given a non-empty universe X, an interval-valued fuzzy set A on X is a function given by $A : X \to L([0,1])$.

From a formal point of view, the last two concepts are equivalent, as there exists a one-to-one mapping between the set of all AIFSs on X, $AIFS(X)$, and the set of all IVFSs on X, $IVFS(X)$ (see [2, 28]):

$$\psi : \begin{array}{ccc} IVFS(X) & \to & AIFS(X) \\ [\underline{A}, \overline{A}] & \mapsto & (\underline{A}, 1 - \overline{A}). \end{array}$$

Moreover, IVFSs on X are n-dimensional fuzzy sets for $n = 2$. Hence, all the results for n-dimensional fuzzy values are valid for intervals in $L([0,1])$.

3. Riesz spaces

Although we ultimately aim at interval-valued functions, the theoretical results in this work are developed in a more general framework of which the set of intervals is a relevant example. In particular, we deal with vector spaces and we consider vector spaces over $\mathbb{R}$ instead of general fields F.

A vector space V endowed with a partial order relation $\leq_V$ is said to be a partially ordered vector space if the order structure and the vector space structure are compatible, that is, if the following conditions hold for any $u, v \in V$:

(i) If $u \leq_V v$, then $u + w \leq_V v + w$ for every $w \in V$;

(ii) If $u \leq_V v$, then $\alpha u \leq_V \alpha v$ for every real $\alpha \geq 0$.

Condition (ii) can be equivalently formulated as follows: if $u \geq_V 0_V$, then $\alpha u \geq_V 0_V$ for every real $\alpha \geq 0$.

We denote the Cartesian product of such spaces as $V^n = V \times \ldots \times V$, which is a partially ordered vector space with respect to the product order $\leq_{V^n}$, which results from considering $\leq_V$ component-wise. Namely, if $\mathbf{v} = (v_1, \ldots, v_n), \mathbf{u} = (u_1, \ldots, u_n) \in V^n$, we say that $\mathbf{v} \leq_{V^n} \mathbf{u}$ if $v_i \leq_V u_i$ for $1 \leq i \leq n$.

Furthermore, if V with the order relation forms a lattice, then V is said to be a Riesz space (also known as vector-lattice). Note that in this case also V^n forms a Riesz space.

All instances of ordered vector spaces that we mention in this work are in fact Riesz spaces. We provide a brief description of each in the next example.

Example 3.1. The following are various instances of Riesz spaces.

(1) The real line $\mathbb{R}$ with the standard linear order structure and operations is a Riesz space.

(2) The space $\mathbb{R}^n$ with $n \geq 2$ with the component-wise order is also a Riesz space.

(3) The space $\mathbb{R}^2$ with either the first or the second lexicographical order is a Riesz space.

(4) Let V be the space formed from all real functions defined on a non-empty set X with point-wise addition, scalar multiplication and order, respectively:

- $(f + g)(x) = f(x) + g(x)$,
- $(\alpha f)(x) = \alpha f(x)$,
- $f \leq_V g$ if $f(x) \leq g(x)$,

for all $x \in X$ and $\alpha \geq 0$. Thus, V is a Riesz space.

(5) The set $C(X)$ of continuous real functions on a topological space X with the point-wise order and linear operations is also a Riesz space.

(6) For every $0 < p \leq \infty$, the spaces L_p, spaces of functions whose absolute value to the p-th power is Lebesgue integrable for $0 < p < \infty$ and the set of all measurable bounded functions when $p = \infty$, are Riesz spaces with the almost everywhere (a.e.) point-wise order for functions. If $p \geq 1$, L_p is also a Banach space.

(7) For every $0 < p \leq \infty$, the sequence spaces ℓ_p are Riesz spaces with component-wise order. If $p \geq 1$, ℓ_p is also a Banach space.

For more insight about this examples and about partially ordered vector spaces and Riesz spaces, see [54].

4. Weak and directional monotonicity in Riesz spaces

Let $(V, \leq_V)$ be a partially ordered vector space over $\mathbb{R}$, and let us denote by $\vec{0} \in V$ the identity element for addition. We can define monotonicity for functions whose inputs come from V^n and have values in V as follows.

Definition 4.1. Let $n \in \mathbb{N}$, $(V, \leq_V)$ be a partially ordered vector space over $\mathbb{R}$. We say that a function $F : V^n \to V$ is increasing (decreasing) if for all $\mathbf{x}, \mathbf{y} \in V^n$ such that $\mathbf{x} \leq_{V^n} \mathbf{y}$ it holds that $F(\mathbf{x}) \leq_V F(\mathbf{y})$ $(F(\mathbf{x}) \geq_V F(\mathbf{y}))$.

Moreover, we can define directional monotonicity for this class of functions, understanding that the directions are non-zero vectors from V^n.

Definition 4.2. Let $n \in \mathbb{N}$, $(V, \leq_V)$ be a partially ordered vector space over $\mathbb{R}$ and let $\mathbf{v} = (v_1, \ldots, v_n) \in V^n$ such that $v_i \neq \vec{0}$ for some $1 \leq i \leq n$. We say that a function $F : V^n \to V$ is $\mathbf{v}$-increasing ($\mathbf{v}$-decreasing) if for all $\mathbf{x} \in V^n$ and $c > 0$, it holds that $F(\mathbf{x} + c\mathbf{v}) \geq_V F(\mathbf{x})$ $(F(\mathbf{x} + c\mathbf{v}) \leq_V F(\mathbf{x}))$. If F is both $\mathbf{v}$-increasing and $\mathbf{v}$-decreasing, we say that F is $\mathbf{v}$-constant.

Analogously, driven by the concept of weak monotonicity for real valued functions, we propose the concept of w-weak monotonicity, focusing on a fixed $\vec{0} \neq w \in V$.

Definition 4.3. Let $n \in \mathbb{N}$, $(V, \leq_V)$ be a partially ordered vector space over $\mathbb{R}$ and let $\vec{0} \neq w \in V$. We say that a function $F : V^n \to V$ is w-weakly increasing (decreasing) if for all $\mathbf{x} \in V^n$ and $c > 0$, it holds that $F(\mathbf{x} + c(w, \ldots, w)) \geq_V F(\mathbf{x})$ $(F(\mathbf{x} + c(w, \ldots, w)) \leq_V F(\mathbf{x}))$.

Remark 4.4. Note that, in the conditions of Definition 4.3, if $u = cw$ for some real number $c > 0$, then u-weak increasingness (u-weak decreasingness) and w-weak increasingness (w-weak decreasingness) coincide. On the other hand, if $c < 0$, then u-weak increasingness (u-weak decreasingness) coincides with w-weak decreasingness (w-weak increasingness). Observe the generalization of the concept of weak monotonicity of real functions introduced in [52] (see also Definition 2.2). In fact, weak increasingness is just 1-weak increasingness.

4.1. Properties

In this section we discuss some properties of directionally monotone functions in this general setting. These properties serve as baseline and in the subsequent sections, where we focus on less general domains, we study which ones still hold true and which do not.

We start with a remark about the directions of increasingness for a function F when the ordered vector space $(V, \leq_V)$ we consider is in fact a normed space.

Remark 4.5. Given $\mathbf{v} \in V^n$, for a function $F : V^n \to V$ it is equivalent to be $\mathbf{v}$-increasing and to be $k\mathbf{v}$-increasing for any positive constant k. Consequently, in the cases when the space V^n can be equipped with a norm $\| \cdot \|$, without loss of generality we can characterize each direction $\mathbf{v} \in V^n$ with the one that satisfies $\|\mathbf{v}\| = 1$.

Similarly, the next result shows that it is equivalent or a function F to increase along one direction and to decrease along the opposite one. Thus, without loss of generality, we can develop our results focusing on increasingness.

Proposition 4.6. *Let $n \in \mathbb{N}$, $(V, \leq_V)$ be a partially ordered vector space over $\mathbb{R}$ and let $\mathbf{v} \in V^n$ such that $v_i \neq \vec{0}$ for some $1 \leq i \leq n$. A function $F : V^n \to V$ is $\mathbf{v}$-increasing if and only if F is $(-\mathbf{v})$-decreasing.*

Proof. Let F be $\mathbf{v}$-increasing. Let $\mathbf{x} \in V^n$ and $c > 0$, then

$$F(\mathbf{x} + c(-\mathbf{v})) \leq_V F(\mathbf{x} + c(-\mathbf{v}) + c\mathbf{v}) = F(\mathbf{x}),$$

and therefore F is $(-\mathbf{v})$-decreasing. The converse statement follows similarly. $\square$

Now, as in the real case with Theorem 2.4, we study whether a function $F : V^n \to V$ that is $\mathbf{v}$-increasing and $\mathbf{u}$-increasing is also increasing along a linear combination of $\mathbf{v}$ and $\mathbf{u}$.

Theorem 4.7. *Let $n \in \mathbb{N}$, $(V, \leq_V)$ be a partially ordered vector space over $\mathbb{R}$ and let $\mathbf{v}, \mathbf{u} \in V^n$ be such that $v_i \neq \vec{0}$ for some $1 \leq i \leq n$ and $u_j \neq \vec{0}$ for some $1 \leq j \leq n$. If a function $F : V^n \to V$ is $\mathbf{v}$-increasing and $\mathbf{u}$-increasing, then F is $(a\mathbf{v} + b\mathbf{u})$-increasing for any $a, b > 0$.*

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Proof. Let F be **v**- and **u**-increasing, $a, b > 0$ and $\mathbf{x} \in V^n$. Then, if $c > 0$,

$$F(\mathbf{x} + c(a\mathbf{v} + b\mathbf{u})) \geq_V F(\mathbf{x} + cb\mathbf{u}) \geq_V F(\mathbf{x}),$$

where the inequalities hold due to the **v**- and **u**-increasingness of F, respectively.
$\square$

The next two results deal with the directional increasingness of functions that are **v**-increasing for some $\mathbf{v} \in V^n$.

Proposition 4.8. *Let $n \in \mathbb{N}$, $(V, \leq_V)$ be a partially ordered vector space over $\mathbb{R}$, $\mathbf{v} \in V^n$ such that $v_i \neq \vec{0}$ for some $1 \leq i \leq n$ and $F : V^n \to V$ be a **v**-increasing function. Let $\varphi : V \to V$ be an increasing (decreasing) function. Then, the function $\varphi \circ F$ is **v**-increasing (decreasing).*

Proof. Let F be **v**-increasing and φ be increasing. Let $\mathbf{x} \in V^n$ and $c > 0$, then

$$(\varphi \circ F)(\mathbf{x} + c\mathbf{v}) = \varphi(F(\mathbf{x} + c\mathbf{v})) \geq_V \varphi(F(\mathbf{x})) = (\varphi \circ F)(\mathbf{x}).$$

The case in which φ is decreasing is analogous.
$\square$

Proposition 4.9. *Let $n \in \mathbb{N}$, $(V, \leq_V)$ be a partially ordered vector space over $\mathbb{R}$, $\mathbf{v} \in V^n$ such that $v_i \neq \vec{0}$ for some $1 \leq i \leq n$ and $F_1, \ldots, F_k : V^n \to V$ be **v**-increasing functions. Let $A : V^k \to V$ be an increasing (decreasing) function. Then, the function $A(F_1, \ldots, F_k)$ is **v**-increasing (decreasing).*

Proof. Let $F_1, \ldots, F_k$ be **v**-increasing functions and A be increasing. Let $\mathbf{x} \in V^n$ and $c > 0$, then

$$\begin{aligned}
A(F_1, \ldots, F_k)(\mathbf{x} + c\mathbf{v}) &= A(F_1(\mathbf{x} + c\mathbf{v}), \ldots, F_k(\mathbf{x} + c\mathbf{v})) \\
&\geq_V A(F_1(\mathbf{x}), \ldots, F_k(\mathbf{x})) = A(F_1, \ldots, F_k)(\mathbf{x}).
\end{aligned}$$

The case in which A is decreasing is analogous.
$\square$

Finally, as in the case of real functions (Theorem 2.5), we can characterize monotonicity in Riesz spaces in terms of directional monotonicity. To that end, let us define the set $V^+ = \left\{ v \in V \mid v \geq_V \vec{0} \right\}$.

Theorem 4.10. *Let $n \in \mathbb{N}$, $(V, \leq_V)$ be a partially ordered vector space over $\mathbb{R}$. A function $F : V^n \to V$ is increasing (decreasing) if and only if F is **v**-increasing (**v**-decreasing) for every $\mathbf{v} \in (V^+)^n$ such that $v_i \neq \vec{0}$ for some $1 \leq i \leq n$.*

Proof. Let F be increasing and let $\mathbf{x} \in V^n$. Now, given $c > 0$ and $\mathbf{0} \neq \mathbf{v} \in (V^+)^n$, it holds that $\mathbf{x} <_{V^n} \mathbf{x} + c\mathbf{v}$. Hence, since F is increasing, it also is **v**-increasing.

Conversely, let F be **v**-increasing for every $\mathbf{0} \neq \mathbf{v} \in (V^+)^n$. Let $\mathbf{x}, \mathbf{y} \in V^n$ such that $\mathbf{x} \leq_{V^n} \mathbf{y}$. Since the case $\mathbf{x} = \mathbf{y}$ is straight, we can assume $\mathbf{x} <_{V^n} \mathbf{y}$. Thus, we set $\mathbf{v} = \mathbf{y} - \mathbf{x}$. Clearly, $\mathbf{v} \in (V^+)^n$ and $\mathbf{v} \neq \mathbf{0}$. Therefore, since F is **v**-increasing, it holds that $F(\mathbf{x}) \leq_V F(\mathbf{y})$.
$\square$

Theorem 4.7 and Theorem 4.10 derive the following result.

Corollary 4.11. *Let $n \in \mathbb{N}$, $(V, \leq_V)$ be a partially ordered vector space over $\mathbb{R}$. Let B be the set of vectors $\mathbf{v} \in (V^+)^n$ that span $(V^+)^n$. Then, a function $F : V^n \to V$ is increasing (decreasing) if and only if F is $\mathbf{v}$-increasing ($\mathbf{v}$-decreasing) for every $\mathbf{v} \in B$ such that $v_i \neq \vec{0}$ for some $1 \leq i \leq n$.*

4.2. Restriction of V to an interval sublattice

In this section we study directional monotononicity of functions as in Section 4 but whose domains and codomains are restricted. Let $(V, \leq_V)$ be a partially ordered vector space over $\mathbb{R}$ and let $r, s \in V$ such that $r \leq_V s$, then we set the following subset of V:

$$V_r^s = \{v \in V \mid r \leq_V v \leq_V s\}. \tag{1}$$

V_r^s is a sublattice with top and bottom elements s and r, respectively. However, V_r^s is not a vector space and, hence, the definitions of directional and weak monotonicity for functions $F : (V_r^s)^n \to V_r^s$ need some adaptations.

Definition 4.12. Let $n \in \mathbb{N}$, $(V, \leq_V)$ be a partially ordered vector space over $\mathbb{R}$ and set V_r^s as in eq. (1). Let $\mathbf{v} = (v_1, \ldots, v_n) \in V^n$ such that $v_i \neq \vec{0}$ for some $1 \leq i \leq n$. We say that a function $F : (V_r^s)^n \to V_r^s$ is $\mathbf{v}$-increasing ($\mathbf{v}$-decreasing) if for all $\mathbf{x} \in (V_r^s)^n$ and $c > 0$ such that $\mathbf{x} + c\mathbf{v} \in (V_r^s)^n$, it holds that $F(\mathbf{x} + c\mathbf{v}) \geq_V F(\mathbf{x})$ ($F(\mathbf{x} + c\mathbf{v}) \leq_V F(\mathbf{x})$). If F is both $\mathbf{v}$-increasing and $\mathbf{v}$-decreasing, we say that F is $\mathbf{v}$-constant.

Note that the direction $\mathbf{v}$ does not necessarily belong to the restricted set $(V_r^s)^n$, but to the original vector space V^n. We allow the directions to exit the restricted space and we require the condition of monotonicity to the points $\mathbf{x} \in (V_r^s)^n$ such that $\mathbf{x} + c\mathbf{v} \in (V_r^s)^n$. This resembles the original case ([15]), where directional monotonicity is studied for functions $F : [0, 1]^n \to [0, 1]$ and the directions of increasingness belong to the more general $\mathbb{R}^n$.

Definition 4.13. Let $n \in \mathbb{N}$, $(V, \leq_V)$ be a partially ordered vector space over $\mathbb{R}$, V_r^s as in eq. (1) and let $\vec{0} \neq w \in V$. We say that a function $F : (V_r^s)^n \to V_r^s$ is w-weakly increasing (decreasing) if for all $\mathbf{x} \in (V_r^s)^n$ and $c > 0$ such that $\mathbf{x} + c(w, \ldots, w) \in (V_r^s)^n$, it holds that $F(\mathbf{x} + c(w, \ldots, w)) \geq_V F(\mathbf{x})$ ($F(\mathbf{x} + c(w, \ldots, w)) \leq_V F(\mathbf{x})$).

Real functions defined on a Cartesian product of a closed real interval with itself and taking values in the same interval are an example of the functions described in this section. In particular, aggregation functions can be seen as a particular case. Recall that an aggregation function is defined as $f : [0, 1]^n \to [0, 1]$ such that $f(0, \ldots, 0) = 0$, $f(1, \ldots, 1) = 1$ and f is increasing with respect to each component. Therefore, it suffices to set $V_r^s = [0, 1]$ as a subset of $V = \mathbb{R}$ and we recover the notion of directional monotonicity introduced in [15].

In relation to the properties that this class of functions satisfy, all the properties studied in Section 4.1 hold similarly for functions defined in V_r^s taking

Definition 4.12 into account. The only result that needs an additional assumption is Theorem 4.7 and the new formulation is as follows.

Theorem 4.14. *Let $n \in \mathbb{N}$, $(V, \leq_V)$ be a partially ordered vector space over $\mathbb{R}$, let V_r^s be as in eq. (1), let $a, b > 0$ and $\mathbf{v}, \mathbf{u} \in V^n$ such that $v_i \neq \vec{0}$ for some $1 \leq i \leq n$ and $u_j \neq \vec{0}$ for some $1 \leq j \leq n$. Assume that for all $\mathbf{x} \in (V_r^s)^n$ and $c > 0$ such that $\mathbf{x} + c(a\mathbf{v} + b\mathbf{u}) \in (V_r^s)^n$, then $\mathbf{x} + ca\mathbf{v} \in (V_r^s)^n$ or $\mathbf{x} + cb\mathbf{u} \in (V_r^s)^n$. Thus, if a function $F : (V_r^s)^n \to V_r^s$ is $\mathbf{v}$-increasing and $\mathbf{u}$-increasing, then F is $(a\mathbf{v} + b\mathbf{u})$-increasing.*

4.3. Restriction of V to a convex cone

In this section we introduce the concepts of weak and directional monotonicity for functions that take values on a convex cone C of a Riesz space V. Note that the notions presented in this section are not a particular case of the developments in Section 4 since a convex cone C is not a vector space due to the non existence of an inverse for the addition in general.

Definition 4.15. Let V be a vector space over $\mathbb{R}$. We say that a subset $C \subset V$ is a cone if for every $x \in C$ and $a \geq 0$ it holds that $ax \in C$. A cone C is a convex cone if for all $a, b > 0$ and $x, y \in C$, it holds that $ax + by \in C$.

Let us point out that the set of closed real intervals $L(\mathbb{R})$ can be seen as a convex cone of the vector space $\mathbb{R}^2$ and that, indeed, there does not exist an inverse for the addition in general: $[2, 3] - [2, 4] = [0, -1] \notin L(\mathbb{R})$.

We now present the concepts of usual, directional and weak monotonicity for functions that take values on a convex cone C. These notions are also valid for interval-valued functions setting $C = L(\mathbb{R})$.

Definition 4.16. Let $n \in \mathbb{N}$, $(V, \leq_V)$ be a partially ordered vector space over $\mathbb{R}$ and let $C \subset V$ be a convex cone. We say that a function $F : C^n \to C$ is increasing (decreasing) if for all $\mathbf{x}, \mathbf{y} \in C^n$ such that $\mathbf{x} \leq_{V^n} \mathbf{y}$ it holds that $F(\mathbf{x}) \leq_V F(\mathbf{y})$ $(F(\mathbf{x}) \geq_V F(\mathbf{y}))$.

Definition 4.17. Let $n \in \mathbb{N}$, $(V, \leq_V)$ be a partially ordered vector space over $\mathbb{R}$ and let $C \subset V$ be a convex cone and let $\mathbf{v} = (v_1, \ldots, v_n) \in V^n$ such that $v_i \neq \vec{0}$ for some $1 \leq i \leq n$. We say that a function $F : C^n \to C$ is $\mathbf{v}$-increasing ($\mathbf{v}$-decreasing) if for all $\mathbf{x} \in C^n$ and $c > 0$ such that $\mathbf{x} + c\mathbf{v} \in C^n$, it holds that $F(\mathbf{x} + c\mathbf{v}) \geq_V F(\mathbf{x})$ $(F(\mathbf{x} + c\mathbf{v}) \leq_V F(\mathbf{x}))$. If F is both $\mathbf{v}$-increasing and $\mathbf{v}$-decreasing, we say that F is $\mathbf{v}$-constant.

Definition 4.18. Let $n \in \mathbb{N}$, $(V, \leq_V)$ be a partially ordered vector space over $\mathbb{R}$ and let $C \subset V$ be a convex cone and let $\vec{0} \neq v \in V$. We say that a function $F : C^n \to C$ is v-weakly increasing (decreasing) if for all $\mathbf{x} \in C^n$ and $c > 0$ such that $\mathbf{x} + c(v, \ldots, v) \in C^n$, it holds that $F(\mathbf{x} + c(v, \ldots, v)) \geq_V F(\mathbf{x})$ $(F(\mathbf{x} + c(v, \ldots, v)) \leq_V F(\mathbf{x}))$.

The restriction of V to a cone C has not great impact in the properties studied in Section 4.1, all the properties hold for functions $F : C^n \to C$ with minor adjustments. Note that although a convex cone C loses the vector space structure, it still is closed under convex combinations, and hence the adaptation of Theorem 4.7 for this framework is straightforward. This is relevant because the mentioned property is meaningful in the setting of directional monotonicity. In particular, for the interval-valued case with $C = L(\mathbb{R})$ and $V = \mathbb{R}^2$.

5. Directional monotonicity of functions to fuse data from different fuzzy settings

In this section we present some prominent particular cases of the theoretical developments of Section 4. We show that functions to fuse data from different fuzzy settings can be seen as either an interval sublattice or a convex cone of some of the Riesz spaces V presented in Example 3.1. Concretely, we study the cases of type-2 fuzzy sets, fuzzy multisets, n-dimensional fuzzy sets, interval-valued fuzzy sets and Atanassov intuitionistic fuzzy sets.

5.1. Type-2 fuzzy values

Let us set V as in Example 3.1 (4), the set of all real functions defined in a set X, and let $X = [0, 1]$. Thus, V with the point-wise order is a Riesz space.

Now, let $f_0, f_1 : [0, 1] \to \mathbb{R}$ be the functions given by $f_0(x) = 0$ and $f_1(x) = 1$ for all $x \in [0, 1]$, respectively. If we consider the interval sublattice $V_{f_0}^{f_1}$, we obtain that all the functions defined in $[0, 1]$ with values in $[0, 1]$ belong to $V_{f_0}^{f_1}$, and due to the definition of the point-wise order, no other function belongs to that subset. Therefore, we can see $V_{f_0}^{f_1}$ as $[0, 1]^{[0,1]}$ and, hence, from Section 4.2 we can retrieve a definition and properties of directional monotonicity for type-2 fuzzy valued functions.

5.2. Fuzzy multiset values and n-dimensional fuzzy values

Since $[0, 1]^n$ can be seen as an interval sublattice of the Riesz space $\mathbb{R}^n$, the developments in Section 4.2 are applicable to functions that are intended to fuse information coming from fuzzy multisets.

For the case of n-dimensional fuzzy sets, the set of n-dimensional fuzzy values is given by

$$L_n = \{(x_1, \ldots, x_n) \in \mathbb{R}^n \mid 0 \le x_1 \le \ldots \le x_n \le 1\}. \tag{2}$$

Clearly, L_n is the intersection of the interval sublattice $[0, 1]^n$ of the Riesz space $\mathbb{R}^n$ and the convex cone $\{(x_1, \ldots, x_n) \mid x_1 \le \ldots \le x_n\}$ of the same Riesz space $\mathbb{R}^n$. Therefore, we can take into account the adaptations in the definitions and properties of directional monotonicity in Sections 4.2 and 4.3, to define directional monotonicity for n-dimensional fuzzy valued functions.

5.3. Interval-valued fuzzy values and Atanassov intuitionistic fuzzy values

Since interval-valued fuzzy sets are a particular instance of n-dimensional fuzzy sets for $n = 2$, and since IVFSs and AIFSs are formally equivalent, the points made in the preceding subsection are also valid for interval-valued and Atanassov intuitionistic fuzzy sets.

However, in the following section of this work we study further the particular case of interval-valued functions, for intervals in $L([0, 1])$ (as is the case of interval-valued fuzzy values), giving explicit definitions, examples and properties.

Note that although we focus on functions defined and with values in $L([0, 1])$, the developments in Section 4.3 generalize the concept of directional monotonicity for functions defined in $L(\mathbb{R})$ as well, since $L(\mathbb{R})$ can be seen as a convex cone of the vector space $\mathbb{R}^2$. Indeed, it is isomorphic to the set $K(\mathbb{R}) \subset \mathbb{R}^2$, which is a half-space as $K(\mathbb{R}) = \{(x, y) \in \mathbb{R}^2 \mid y - x \geq 0\}$, and therefore $K(\mathbb{R})$ is a convex cone of $\mathbb{R}^2$.

6. Weak and directional monotonicity on the interval-valued setting

6.1. Restriction to intervals in $L([0, 1])$

In this subsection we present explicit definitions for interval-valued functions that are defined over $L([0, 1])$, as they are recurrent in both theoretic and applied works in the literature [10, 12, 16]. This type of functions can be seen as the result of restricting the former space V to be the intersection of an interval sublattice and a convex cone, as in Sections 4.2 and 4.3, respectively. Note that these developments are equivalent for the case of any other closed interval, i.e., they are equivalent for $L([a, b])$.

We now present the explicit definitions for standard, directional and weak monotonicity for functions that take values on $L([0, 1])$.

Definition 6.1. Let $n \in \mathbb{N}$. We say that a function $F : L([0, 1])^n \to L([0, 1])$ is increasing (decreasing) if for all $\mathbf{x}, \mathbf{y} \in L([0, 1])^n$ such that $\mathbf{x} \leq_{L^n} \mathbf{y}$ it holds that $F(\mathbf{x}) \leq_L F(\mathbf{y})$ ($F(\mathbf{x}) \geq_L F(\mathbf{y})$).

Definition 6.2. Let $n \in \mathbb{N}$ and let $\mathbf{v} = (a_1, b_1, \ldots, a_n, b_n) \in (\mathbb{R}^2)^n$ such that $(a_i, b_i) \neq \vec{0}$ for some $1 \leq i \leq n$. We say that a function $F : L([0, 1])^n \to L([0, 1])$ is $\mathbf{v}$-increasing ($\mathbf{v}$-decreasing) if for all $\mathbf{x} \in L([0, 1])^n$ and $c > 0$ such that $\mathbf{x} + c\mathbf{v} \in L([0, 1])^n$, it holds that $F(\mathbf{x} + c\mathbf{v}) \geq_L F(\mathbf{x})$ ($F(\mathbf{x} + c\mathbf{v}) \leq_L F(\mathbf{x})$). If F is both $\mathbf{v}$-increasing and $\mathbf{v}$-decreasing, we say that F is $\mathbf{v}$-constant.

Definition 6.3. Let $n \in \mathbb{N}$ and let $\vec{0} \neq (a, b) \in \mathbb{R}^2$. We say that a function $F : L([0, 1])^n \to L([0, 1])$ is (a, b)-weakly increasing (decreasing) if for all $\mathbf{x} \in L([0, 1])^n$ and $c > 0$ such that $\mathbf{x} + c(a, b, \ldots, a, b) \in (\mathbb{R}^2)^n$, it holds that $F(\mathbf{x} + c(a, b, \ldots, a, b)) \geq_L F(\mathbf{x})$ ($F(\mathbf{x} + c(a, b, \ldots, a, b)) \leq_L F(\mathbf{x})$).

Example 6.4. The following are two examples of interval-valued functions and their directions of increasingness in terms of some parameters.

(1) Let $F : L([0,1])^2 \to L([0,1])$ be a function given by

$$F([\underline{x_1}, \overline{x_1}], [\underline{x_2}, \overline{x_2}]) = \left[\frac{\underline{x_1} + \underline{x_2}}{2}, \min\left(\frac{\underline{x_1} + \overline{x_2}}{2}, \frac{\overline{x_1} + \underline{x_2}}{2} \right) \right].$$

Then, given $a, b \in \mathbb{R}$, F is (a, b)-weakly increasing if and only if $a > 0$ and $a + b \geq 0$, or $a = 0$ and $b > 0$. Indeed, it follows the fact that, given $c > 0$,

$$F(([\underline{x_1}, \overline{x_1}], [\underline{x_2}, \overline{x_2}]) + c(a, b, a, b)) = F([\underline{x_1}, \overline{x_1}], [\underline{x_2}, \overline{x_2}]) + c\left[a, \frac{a + b}{2} \right].$$

(2) Let $\lambda \in]0, 1[$ and let $L_\lambda : [0,1]^2 \to [0,1]$ be the weighted Lehmer mean [15], which is given (with the convention $\frac{0}{0} = 0$) by

$$L_\lambda(x, y) = \frac{\lambda x^2 + (1 - \lambda)y^2}{\lambda x + (1 - \lambda)y}.$$

Let $F : L([0,1])^2 \to L([0,1])$ be a function given by

$$F([\underline{x_1}, \overline{x_1}], [\underline{x_2}, \overline{x_2}]) = \left[\frac{1}{2} L_\lambda(\underline{x_1}, \underline{x_2}), L_\lambda(\overline{x_1}, \overline{x_2}) \right],$$

(with the convention $\frac{0}{0} = 0$). F is a well-defined because $\frac{1}{2} L_\lambda(x, y) \leq L_\lambda(z, t)$ for all $x, y, z, t \in [0,1]$ such that $x \leq z$ and $y \leq t$.

We show in the following subsection that the function F is $((1 - \lambda, 1 - \lambda), (\lambda, \lambda))$-increasing. In fact, it only increases along that particular direction (up to positive scalar multiplication).

The next result is an adaptation of Theorem 2.5 for the interval-valued case and succeeds to characterize regular monotonicity for interval-valued functions with respect to the partial order $\leq_L$. In it, we make use of the canonical basis of $(\mathbb{R}^2)^n$, i.e., the set of vectors $\{\mathbf{e}_i\}_{i=1}^{2n}$. In the case of $n = 2$, the vectors of the canonical basis of $(\mathbb{R}^2)^2$ are the following:

$$\mathbf{e}_1 = ((1, 0), (0, 0)); \qquad \mathbf{e}_2 = ((0, 1), (0, 0));$$
$$\mathbf{e}_3 = ((0, 0), (1, 0)); \qquad \mathbf{e}_4 = ((0, 0), (0, 1)).$$

Theorem 6.5. *Let $n \in \mathbb{N}$, let $\leq_L$ be the partial order on $L([0,1])$, let $F : L([0,1])^n \to L([0,1])$ and let $\{\mathbf{e}_i\}_{i=1}^{2n}$ be the canonical basis of $(\mathbb{R}^2)^n$. Then, F is increasing if and only if F is $\mathbf{e}_i$-increasing for all $1 \leq i \leq 2n$.*

Proof. Let F be increasing with respect to $\leq_L$ and let $\mathbf{x} = ([\underline{x_1}, \overline{x_1}], \ldots, [\underline{x_n}, \overline{x_n}]) \in L([0,1])^n$. Now, given $c > 0$ such that $\mathbf{x} + c\mathbf{e}_i \in L([0,1])^n$, it is straight to check that $\mathbf{x} \leq_L \mathbf{x} + c\mathbf{e}_i$. Hence, the increasingness of F ensures $\mathbf{e}_i$-increasingness.

Conversely, let F be $\mathbf{e}_i$-increasing for all $1 \leq i \leq 2n$. Let $\mathbf{x}, \mathbf{y} \in L([0,1])^n$ such that $[\underline{x_i}, \overline{x_i}] \leq_L [\underline{y_i}, \overline{y_i}]$ for all $1 \leq i \leq n$. From the definition of $\leq_L$, it follows that $\underline{x_i} \leq \underline{y_i}$ and $\overline{x_i} \leq \overline{y_i}$ for all $1 \leq i \leq n$ and, hence, for each i there

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exist $a_i, b_i \geq 0$ with $a_i + b_i > 0$ such that $[\underline{y_i}, \overline{y_i}] = [\underline{x_i}, \overline{x_i}] + a_i(1, 0) + b_i(0, 1)$. Consequently,

$$\mathbf{y} = \mathbf{x} + \sum_{i=1}^{n} a_i \mathbf{e}_{2i-1} + \sum_{i=1}^{n} b_i \mathbf{e}_{2i},$$

and by the straight adaptation of Theorem 4.14 for $L([0, 1])$, it holds that F is $\mathbf{v}$-increasing for $\mathbf{v} = \sum_{i=1}^{n} a_i \mathbf{e}_{2i-1} + \sum_{i=1}^{n} b_i \mathbf{e}_{2i}$. Hence, $F(\mathbf{x}) \leq_L F(\mathbf{y})$. $\qquad \square$

6.2. Representable interval-valued functions

In this subsection we focus on the special class of IV functions $F : L([0, 1])^n \to L([0, 1])$ that verifies

$$F(\mathbf{x}) = F([\underline{x_1}, \overline{x_1}], \ldots, [\underline{x_n}, \overline{x_n}]) = [f(\underline{x_1}, \ldots, \underline{x_n}), g(\overline{x_1}, \ldots, \overline{x_n})], \qquad (3)$$

for some functions $f, g : [0, 1]^n \to [0, 1]$ such that $f(x_1, \ldots, x_n) \leq g(y_1, \ldots, y_n)$ whenever $x_i \leq y_i$ for all $1 \leq i \leq n$. This type of interval-valued functions are said to be representable [27].

Note that Example 6.4 brings an example of a non-representable function (see item (1)) and an example of a representable function (see item (2)).

Theorem 6.6. *Let $F : L([0, 1])^n \to L([0, 1])$ be an interval-valued function satisfying eq. (3) for some functions $f, g : [0, 1]^n \to [0, 1]$ such that $f(x_1, \ldots, x_n) \leq g(y_1, \ldots, y_n)$ if $x_i \leq y_i$ for $1 \leq i \leq n$. Let $\vec{a} = (a_1, \ldots, a_n)$ and $\vec{b} = (b_1, \ldots, b_n) \in \mathbb{R}^n$ be vectors such that $\vec{a}, \vec{b} \neq (0, \ldots, 0)$. Then, F is $((a_1, b_1), \ldots, (a_n, b_n))$-increasing if and only if f is $\vec{a}$-increasing and g is $\vec{b}$-increasing.*

Proof. Let F be satisfy eq. (3) for $f, g : [0, 1]^n \to [0, 1]$ such that they satisfy $f(x_1, \ldots, x_n) \leq g(y_1, \ldots, y_n)$ if $x_i \leq y_i$ for $1 \leq i \leq n$ and assume that F is $((a_1, b_1), \ldots, (a_n, b_n))$-increasing. Let us now show that f is $\vec{a}$-increasing. Let $(x_1, \ldots, x_n) \in [0, 1]^n$ and $c > 0$ such that $(x_1, \ldots, x_n) + c\vec{a} \in [0, 1]^n$. We can find $y_1, \ldots, y_n \in [0, 1]$ such that $x_i \leq y_i$ for all $1 \leq i \leq n$ and such that

$$([x_1, y_1], \ldots, [x_n, y_n]) + c\left((a_1, b_1), \ldots, (a_n, b_n)\right) \in L([0, 1])^n.$$

Now, since F is $((a_1, b_1), \ldots, (a_n, b_n))$-increasing, it follows that $f((x_1, \ldots, x_n) + c\vec{a}) \geq f(x_1, \ldots, x_n)$ and, hence, f is $\vec{a}$-increasing. Similarly, it can be shown that g is $\vec{b}$-increasing.

For the converse, let f and g be $\vec{a}$- and $\vec{b}$-increasing, respectively. Let $\mathbf{x} \in L([0, 1])^n$ and $c > 0$ such that $\mathbf{x} + c\left((a_1, b_1), \ldots, (a_n, b_n)\right) \in L([0, 1])^n$. Then,

$$
\begin{aligned}
F(\mathbf{x} + c\left((a_1, b_1), \ldots, (a_n, b_n)\right)) &= F([\underline{x_1}, \overline{x_1}] + c(a_1, b_1), \ldots, [\underline{x_n}, \overline{x_n}] + c(a_n, b_n)) \\
&= [f(\underline{x_1} + ca_1, \ldots, \underline{x_n} + ca_n), g(\overline{x_1} + cb_1, \ldots, \overline{x_n} + cb_n)] \\
&= [f((\underline{x_1}, \ldots, \underline{x_n}) + c\vec{a}), g((\overline{x_1}, \ldots, \overline{x_n}) + c\vec{b})] \\
&\geq_L [f((\underline{x_1}, \ldots, \underline{x_n})), g((\overline{x_1}, \ldots, \overline{x_n}))] \\
&= F(\mathbf{x}),
\end{aligned}
$$

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and, hence, F is $((a_1, b_1), \ldots, (a_n, b_n))$-increasing.

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ $\square$

As a direct consequence, we have the following corollary. Let us fix the notation $\mathcal{D}^\uparrow(F)$ to denote the set of vectors along which the function F is directionally increasing. Of course, $\mathcal{D}^\uparrow$ is a subset of the Riesz space where the directions of F are defined. In the case of an interval-valued function $F : L([0,1])^n \to L([0,1])$, it holds that $\mathcal{D}^\uparrow(F) \subset (\mathbb{R}^2)^n$.

Corollary 6.7. *Let $F : L([0,1])^n \to L([0,1])$ be an interval-valued function satisfying eq. (3) for some functions $f, g : [0,1]^n \to [0,1]$ such that $f(x_1, \ldots, x_n) \leq g(y_1, \ldots, y_n)$ if $x_i \leq y_i$ for $1 \leq i \leq n$. Then, it holds that*

$$\mathcal{D}^\uparrow(F) = \left\{ ((a_1, b_1), \ldots, (a_n, b_n)) \in (\mathbb{R}^2)^n \mid \vec{a} \in \mathcal{D}^\uparrow(f) \text{ and } \vec{b} \in \mathcal{D}^\uparrow(g) \right\}.$$

Consequently, one can define many instances of functions F that increase along some direction. Indeed, Theorem 6.6 shows that in the case of representable functions, the study of directions in which a function F is increasing (decreasing) is reduced to the study of directional monotonicity of component functions f and g. In particular, the function based on the weighted Lehmer mean in Example 6.4 item (2) increases only along the direction $((1 - \lambda, 1 - \lambda), (\lambda, \lambda))$ because the weighted Lehmer mean L_λ is only directionally increasing with respect to the vector $(1 - \lambda, \lambda)$, up to positive scalar multiplication.

In a similar manner, we can construct other instances of functions F and characterize the set of vectors along which it increases.

Example 6.8. Let $F : L([0,1])^n \to L([0,1])$ be a function given by

$$F([\underline{x_1}, \overline{x_1}], \ldots, [\underline{x_n}, \overline{x_n}]) = \left[\min(\underline{x_1}, \ldots, \underline{x_n}), \frac{1}{n} \sum_{i=1}^n \overline{x_i} \right].$$

Clearly, F is a well-defined representable interval-valued function and following the notation in eq. (3), the function f in this case is the minimum ($f = \min$) and the function g is the arithmetic mean ($g = AM$).

Now, these are the set of directions for which the minimum and the arithmetic mean are increasing:

$$\mathcal{D}^\uparrow(\min) = \left\{ \vec{r} = (r_1, \ldots, r_n) \mid r_i \geq 0 \text{ for all } 1 \leq i \leq n \right\},$$

$$\mathcal{D}^\uparrow(AM) = \left\{ \vec{r} = (r_1, \ldots, r_n) \mid \sum_{i=1}^n r_i \geq 0 \right\}.$$

Therefore, by Corollary 6.7, the set of directions along which the function F is the following.

$$\mathcal{D}^\uparrow(F) = \left\{ ((a_1, b_1), \ldots, (a_n, b_n)) \in (\mathbb{R}^2)^n \mid a_i \geq 0 \text{ for all } 1 \leq i \leq n, \text{ and } \sum_{i=1}^n b_i \geq 0 \right\}.$$

For example, one of the directions of increasingness for $n = 2$ is $((1,1),(0,-1))$.

Interested readers can find numerous examples of directionally monotone functions in [7–9, 15, 52], which enable to construct directionally monotone representable IV functions.

A remarkable example of such an IV function is the interval-valued Choquet integral, as it has been proved to be useful in diverse applications [34].

Example 6.9. Let $X = \{1, \ldots, n\}$ and $m : 2^X \to [0,1]$ be a fuzzy measure (see [50]). The discrete Choquet integral $C_m : [0,1]^n \to [0,1]$ is defined as:

$$C_m(x_1, \ldots, x_n) = \sum_{i=1}^{n} x_{\sigma(i)} \left(m(\{\sigma(i), \ldots, \sigma(n)\}) - m(\{\sigma(i+1), \ldots, \sigma(n)\}) \right),$$

where $\sigma : \{1, \ldots, n\} \to \{1, \ldots, n\}$ is a permutation such that $x_{\sigma(1)} \leq \ldots \leq x_{\sigma(n)}$ and, by convention, $\{x_{\sigma(n+1)}, x_{\sigma(n)}\} = \emptyset$.

The set of vectors for which C_m directionally increases was characterized in [15]:

$$\mathcal{D}^{\uparrow}(C_m) = \left\{ (r_1, \ldots, r_n) \in \mathbb{R}^n \mid \sum_{i=1}^{n} r_i(m(\{\sigma(i), \ldots, \sigma(n)\}) \right.$$
$$\left. - m(\{\sigma(i+1), \ldots, \sigma(n)\})) \geq 0 \text{ for all } \sigma \in S_n \right\},$$

where S_n denotes the set of all permutations of n elements.

The definition of the discrete IV Choquet integral follows Auman's approach to define integrals for set-valued functions [3]. The discrete IV Choquet integral $\mathbf{C}_m : L([0,1])^n \to L([0,1])$ is given by

$$\mathbf{C}_m([\underline{x_1}, \overline{x_1}], \ldots, [\underline{x_n}, \overline{x_n}]) = \left[C_m(\underline{x_1}, \ldots, \underline{x_n}), C_m(\overline{x_1}, \ldots, \overline{x_n}) \right].$$

Therefore, by Corollary 6.7, it holds that

$$\mathcal{D}^{\uparrow}(\mathbf{C}_m) = \left\{ ((a_1, b_1), \ldots, (a_n, b_n)) \in (\mathbb{R}^2)^n \mid \right.$$
$$\sum_{i=1}^{n} a_i(m(\{\sigma(i), \ldots, \sigma(n)\}) - m(\{\sigma(i+1), \ldots, \sigma(n)\})) \geq 0 \text{ and}$$
$$\left. \sum_{i=1}^{n} b_i(m(\{\sigma(i), \ldots, \sigma(n)\}) - m(\{\sigma(i+1), \ldots, \sigma(n)\})) \geq 0, \text{ for all } \sigma \in S_n \right\}.$$

6.3. Particular case: Interval Directions

In this section we study the particular case of directional monotonicity for functions $F : L([0,1])^n \to L([0,1])$ that increase along a direction formed

by intervals, i.e., the cases in which such function F is **v**-increasing for $\mathbf{v} = ([a_1, b_1], \ldots, [a_n, b_n]) \in L(\mathbb{R})^n$ (as opposed to $\mathbf{v} \in (\mathbb{R}^2)^n$). We refer to this notion as interval directional monotonicity (IDM).

Definition 6.10. Let $n \in \mathbb{N}$ and let $\mathbf{v} = ([a_1, b_1], \ldots, [a_n, b_n]) \in L(\mathbb{R})^n$ such that $[a_i, b_i] \neq 0_L$ for all $1 \leq i \leq n$. We say that a function $F : L([0, 1])^n \to L([0, 1])$ is IDM **v**-increasing (IDM **v**-decreasing) if for all $\mathbf{x} \in L([0, 1])^n$ and $c > 0$ such that $\mathbf{x} + c\mathbf{v} \in L([0, 1])^n$, it holds that $F(\mathbf{x} + c\mathbf{v}) \geq_L F(\mathbf{x})$ $(F(\mathbf{x} + c\mathbf{v}) \leq_L F(\mathbf{x}))$. If F is both IDM **v**-increasing and IDM **v**-decreasing, we say that F is IDM **v**-constant.

The restriction of the possible directions of increasingness from $(\mathbb{R}^2)^n$ to $L(\mathbb{R})^n$ has an impact in the properties studied in Section 4.1. However, all properties hold for functions $F : L([0, 1])^n \to L([0, 1])$ for which the vectors of increasingness belong in $L([0, 1])^n$, with the exception of Proposition 4.6, which deals with the inverse of addition. Note that there is no inverse of addition defined on $L([0, 1])^n$. The remaining properties of Section 4.1 hold similarly taking into account this new restriction.

Theorem 6.6 in Section 6.2 on representable interval-valued functions is also valid for IDM with some minor modifications. Theorem 6.6 is adapted as follows.

Theorem 6.11. Let $F : L(\mathbb{R})^n \to L(\mathbb{R})$ be an interval-valued function satisfying eq. (3) for some functions $f, g : \mathbb{R}^n \to \mathbb{R}$ such that $f(x_1, \ldots, x_n) \leq g(y_1, \ldots, y_n)$ if $x_i \leq y_i$ for $1 \leq i \leq n$. Let $\vec{a} = (a_1, \ldots, a_n)$ and $\vec{b} = (b_1, \ldots, b_n) \in \mathbb{R}^n$ be vectors such that $\vec{a}, \vec{b} \neq (0, \ldots, 0)$ and $a_i \leq b_i$ for $1 \leq i \leq n$. Then, F is $([a_1, b_1], \ldots, [a_n, b_n])$-increasing if and only if f is $\vec{a}$-increasing and g is $\vec{b}$-increasing.

With respect to finding a characterization of regular monotonicity in terms of IDM increasing functions, as in Theorem 6.5, note that in the conditions of Theorem 6.5 vectors $\mathbf{e}_{2i-1} \notin L(\mathbb{R})^n$ and hence it is not possible to find a characterization because the composition of vectors in $L(\mathbb{R})^n$ is not sufficient to reach every point $\mathbf{y} \in L([0, 1])^n$ from a given point $\mathbf{x} \in L([0, 1])^n$. Consequently, even if a function is IDM increasing for every possible direction, it need not be increasing. This fact is stated in the following remark.

Remark 6.12. A function $F : L([0, 1])^n \to L([0, 1])$ is not necessarily increasing even though F is IDM **v**-increasing for all $\mathbf{v} \in L(\mathbb{R})^n$.

Indeed, let

$$\mathbf{x}_0 = ([0.2, 0.5], [0, 0], \ldots, [0, 0]) \in L([0, 1])^n,$$
$$\mathbf{y}_0 = ([0.4, 0.5], [0, 0], \ldots, [0, 0]) \in L([0, 1])^n.$$

Clearly, $[0.2, 0.5] \leq_L [0.4, 0.5]$. However, it does not necessarily hold that $F(\mathbf{x}_0) \leq_L F(\mathbf{y}_0)$ because there do not exist a constant $c > 0$ and a vector $\mathbf{v} \in L(R)^n$ such that $\mathbf{y}_0 = \mathbf{x}_0 + c\mathbf{v}$.

For example, let us define a function $F : L([0,1])^n \to L([0,1])$ in the following way

$$F([\underline{x_1}, \overline{x_1}], \ldots, [\underline{x_n}, \overline{x_n}]) = [\overline{x_1} - \underline{x_1}, \overline{x_1} - \underline{x_1}].$$

It is straight to check that F is well-defined. Let us now consider an arbitrary direction $\mathbf{v} = ([a_1, b_1], \ldots, [a_n, b_n]) \in L(\mathbb{R})^n$. Now, for $c > 0$ it holds that

$$\begin{aligned}
F(([\underline{x_1}, \overline{x_1}], \ldots, [\underline{x_n}, \overline{x_n}]) + c\mathbf{v}) &= [\overline{x_1} + b_1 - \underline{x_1} - a_1, \overline{x_1} + b_1 - \underline{x_1} - a_1] \\
&= [\overline{x_1} - \underline{x_1} + b_1 - a_1, \overline{x_1} - \underline{x_1} + b_1 - a_1] \\
&\geq_L F([\underline{x_1}, \overline{x_1}], \ldots, [\underline{x_n}, \overline{x_n}]),
\end{aligned}$$

because the fact that $\mathbf{v} \in L(\mathbb{R})^n$ implies that $b_1 - a_1 \geq 0$. Therefore, F is IDM $\mathbf{v}$-increasing for all $\mathbf{v} \in L(\mathbb{R})^n$.

However,

$$F(\mathbf{x}_0) = [0.3, 0.3] >_L [0.1, 0.1] = F(\mathbf{y}_0),$$

and hence, F is not increasing.

Nevertheless, although it is not possible to find a characterization of regular monotonicity, we are able to find a partial result.

Proposition 6.13. *Let $n \in \mathbb{N}$, let $\leq_L$ be the partial order. If F is increasing, then F is IDM $\mathbf{v}$-increasing for all $\mathbf{v} \in L([0,1])^n$.*

Proof. Let $\mathbf{x} = ([\underline{x_1}, \overline{x_1}], \ldots, [\underline{x_n}, \overline{x_n}]) \in L([0,1])^n$. Now, given $c > 0$ such that $\mathbf{x} + c\mathbf{v} \in L([0,1])^n$, it is straight to check that $\mathbf{x} \leq_L \mathbf{x} + c\mathbf{v}$ because the fact that $\mathbf{v} \in L([0,1])^n$ implies that we are adding positive valued to every component. Hence, the increasingness of F ensures $\mathbf{v}$-increasingness. $\square$

7. Conclusions

Based on the concept of Riesz spaces, we have proposed a framework to handle uncertain data originating from different extensions of fuzzy sets, such as type-2 fuzzy sets, fuzzy multisets, n-dimensional fuzzy sets, Atanassov intuitionistic fuzzy sets and interval-valued fuzzy sets. We have introduced the concept of directional monotonicity for functions that handle this sort of uncertainty, combining two of the tendencies in the research on aggregation theory, the relaxation of the monotonicity condition and the extension of the domain. Moreover, we have studied in depth this concept for the particular case of interval-valued functions and we have characterized it in terms of standard directional monotonicity for functions that take values in the unit interval. Thus, we have provided a tool to construct such functions.

As a goal for future research, we intend to study the class of interval-valued directionally monotone functions to see whether they produce as good results as standard directionally monotone functions in classification problems.

Acknowledgments

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