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Marina DONNELLY LECEA

RELATIONAL UNDERSTANDING OF ADDITION AND SUBTRACTION TO IMPLEMENT PROBLEM SOLVING METHODS

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Lehen Hezkuntzako Irakasleen Gradua

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Preámbulo

El Real Decreto 1393/2007, de 29 de octubre, modificado por el Real Decreto 861/2010, establece en el Capítulo III, dedicado a las enseñanzas oficiales de Grado, que "estas enseñanzas concluirán con la elaboración y defensa de un Trabajo Fin de Grado […] El Trabajo Fin de Grado tendrá entre 6 y 30 créditos, deberá realizarse en la fase final del plan de estudios y estar orientado a la evaluación de competencias asociadas al título".

El Grado en Maestro en Educación Primaria por la Universidad Pública de Navarra tiene una extensión de 12 ECTS, según la memoria del título verificada por la ANECA. El título está regido por la *Orden ECI/3857/2007, de 27 de diciembre, por la que se establecen los requisitos para la verificación de los títulos universitarios oficiales que habiliten para el ejercicio de la profesión de Maestro en Educación Primaria*; con la aplicación, con carácter subsidiario, del reglamento de Trabajos Fin de Grado, aprobado por el Consejo de Gobierno de la Universidad el 12 de marzo de 2013.

Todos los planes de estudios de Maestro en Educación Primaria se estructuran, según la Orden ECI/3857/2007, en tres grandes módulos: uno, *de formación básica*, donde se desarrollan los contenidos socio-psico-pedagógicos; otro, *didáctico y disciplinar*, que recoge los contenidos de las disciplinares y su didáctica; y, por último, *Practicum*, donde se describen las competencias que tendrán que adquirir los estudiantes del Grado en las prácticas escolares. En este último módulo, se enmarca el Trabajo Fin de Grado, que debe reflejar la formación adquirida a lo largo de todas las enseñanzas. Finalmente, dado que la Orden ECI/3857/2007 no concreta la distribución de los 240 ECTS necesarios para la obtención del Grado, las universidades tienen la facultad de determinar un número de créditos, estableciendo, en general, asignaturas de carácter optativo.

Así, en cumplimiento de la Orden ECI/3857/2007, es requisito necesario que en el Trabajo Fin de Grado el estudiante demuestre competencias relativas a los módulos de formación básica, didáctico-disciplinar y practicum, exigidas para todos los títulos universitarios oficiales que habiliten para el ejercicio de la profesión de Maestro en Educación Primaria.

En este trabajo, el módulo *de formación básica nos* ha permitido contextualizar y enmarcar correctamente a nivel comunitario y cultural las prácticas que se llevan a cabo en los centros educativos. Asimismo, los conocimientos adquiridos a lo largo del grado en las asignaturas de Matemáticas y su didáctica I, Matemáticas y su Didáctica II Didáctica de las Matemáticas, Desarrollo Evolutivo y Aprendizaje y Sociedad, Familia y Escuela Inclusiva nos ha permitido el desarrollo de un contexto teórico y por ende de una propuesta didáctica que tiene en cuenta el desarrollo integral del alumnado.

El módulo *didáctico y disciplinar* se concreta en un proyecto en el que se han tenido en cuenta las distintas concepciones del proceso de enseñanza-aprendizaje que se pueden observar en las propuestas didácticas que hemos planteado. Además, a través de diversas materias del grado, hemos podido tomar en consideración algunas dimensiones del proceso de enseñanza-aprendizaje como la interdisciplinariedad, el aprendizaje significativo y la educación inclusiva. Por otro lado, nos ha proporcionado un bagaje de herramientas necesarias para llevar a cabo la implementación y el desarrollo de propuestas educativas que logren un mayor y mejor entendimiento de las matemáticas.

Asimismo, el módulo *practicum* nos ha permitido por un lado la adquisición de diversas competencias del grado, y por otro nos ha proporcionado un contexto real en el cual hemos podido diseñar, desarrollar y evaluar los procesos de enseñanza-aprendizaje. En este contexto de aula, hemos podido observar el desarrollo habitual de las clases de matemáticas, cómo se lleva a cabo este proceso de enseñanza-aprendizaje de los contenidos matemáticos y cómo ha cambiado la percepción de las matemáticas tanto en los niños y niñas como en los docentes después de implementar la propuesta didáctica.

Resumen

En el presente trabajo abordamos las matemáticas desde el entendimiento relacional de la suma y la resta para trabajar la resolución de problemas, en un aula de 1º de Educación de Primaria en un colegio público de Pamplona. En primer lugar incluimos un marco teórico en el que planteamos la importancia del aprendizaje de las matemáticas en la educación primaria, la importancia de un entendimiento relacional de la suma y la resta vinculado a situaciones o contextos como pueden ser los problemas matemáticos. En segundo lugar proponemos una secuencia de actividades detalladas para la resolución de problemas basadas en el método de Polya (1945) y analizamos los beneficios de esta propuesta en el proceso de enseñanza-aprendizaje. Por último, finalizamos nuestro trabajo con una conclusión y unas propuestas de mejora.

Palabras clave: matemáticas; resolución de problemas; entendimiento relacional; adición; sustracción.

Abstract

In this study, we approach mathematics from a relational understanding of addition and subtraction in relation to problem solving in a 1st year Primary class based in a public school in Pamplona. Firstly, we include a theoretical framework, in which we propose the importance of learning mathematics in primary education and the importance of relational understanding of addition and subtraction, linked to situations or contexts such as, mathematical word-problems. Secondly we propose a sequence of detailed activities for solving problems, based on Pólya's (1945) method and we analyse the outcome of the teaching-learning process. Finally, we present a series of topics which would further advance our findings.

Keywords: mathematics; problem solving; relational understanding; addition; subtraction.

Contenido

ITRODUCTION

Mathematics has always been a fundamental part of our learning, not only for academic success, but in developing our aptitudes as informed citizens. Many aspects of our daily lives involve different mathematical behaviours in one way or another, although we may not always notice them.

This study aims to explore the need to learn mathematics, enabling us to develop aptitudes and skills. As I mention in "1.1 Precedents: my own feelings" I have always had an innate interest in education. I am currently collaborating with an Erasmus + Project at the Public University of Pamplona called ANFOMAM. The aim of this project is to educate teachers in mathematical skills, through the observation of students. During this period, to date, I have also been volunteering as a teaching assistant at a mathematical workshop designed for children with Down syndrome, based in Pamplona and Zaragoza. The accumulation of my findings while immersed in these environments, has further confirmed in my mind, the need to teach relational understanding in mathematics to children.

The importance of teaching and learning mathematics in primary education, is covered in the second chapter of this Graduation Thesis, in the section *"2.1. The importance of learning mathematics in Primary Education*". We examine the past literature, on problem solving methodologies and we realized that there were several methods proposed in the past, two of which were developed by: Polya (1945) and Guzman (2006). These methods focused on developing a sequence of steps and suggestions, to help children have a relational understanding, defined by Skemp (1989), of word-problems as can be seen in the section "*2.2 Problem-Solving methods".* However, the approaches that Polya (1945) and Guzman (2006) took, are not geared towards young children, therefore we have modified the language to make it suitable for a younger age bracket.

We present word-problems, as a powerful tool for conducting the mathematical learning processes, and equate them into real life situations, where the numerical value of several things is known and others unknown. We also have an interest in discovering some numerical information that has not been previously provided. Although there are many other kinds of mathematical problems, we will only refer to these ones.

However not all word-problems are suitable for all age groups, as some require higher cognitive processes than others. Therefore we have developed a progressive sequence, based on the level of difficulty, found with regarding to the relationship associated with addition and subtraction. In addition, we have also tackled the relational understanding of addition and subtraction algorithms, based on Fuson et al. (1997) approach. “”

The third chapter presents a learning proposal, based on a sequence of four detailed activities, that will help teachers develop the strategies and skill needed in order to help their students have a relational understanding of word-problems. “*3.2.4 Lesson Plans”*. As an example of how detailed these activities are, the process to solve “*one”* mathematical word-problem, required a span of fifteen pages in order to develop sufficient questions and strategies to enable 1st year children, to solve a word-problem.

The purpose for conducting this research, was to establish whether there would be significant change in the problem solving methods currently used by teachers, as opposed to using our findings in conjunctions with Polya (1945) and De-Guzman’s (2006) methods. Although we have not been able to conduct this research to date, due to the current pandemic, we have observed a positive change, in both mainstream students and students with special needs attitudes towards maths’ classes and a deeper and more relational comprehension of addition and subtraction algorithms while problem-solving. This has led us to conclude, that the results are a great success, as referred in “*3.2.5 Evaluations and Assessment”.*

1. JUSTIFICATION FOR THE CHOICE OF MY THESIS' SUBJECT

1.1. Precedents: my own feelings.

The completion of this paper has required to review our personal, academic and professional life. Firstly we had to decide on the subject matter which the thesis would be based on. Secondly we decided which way we would incorporate all the additional skills and knowledge acquired while following the University curriculum. Finally we had to address where and how we would implement both the knowledge and practical experience gained from the work-in-practise placement.

I always had an innate interest in education. My mother is the Head Teacher of a state school in Pamplona and a fluent English speaker and she has always been my greatest inspiration. My aunt is also a Drama teacher in Dublin and being in contact with both disciplines, teaching and drama, from a very young age has allowed me to develop a passion for finding creative ways of teaching that will encourage children to love learning. On the other hand, my father and uncles are engineers and somehow they have conveyed to me a deep interest in numbers and mathematics. And, as it is said in mathematical terms, the intersection of both parameters made me consider the idea of being a mathematics teacher.

However, during my primary and secondary school years I was faced with a type of mathematics that I did not understand. Many of the concepts that I was supposed to learn involved the learning of various algorithms and formulas that were not explained to me. Which made my learning process very hard.

During the first three years of the Bachelor's Degree in Primary Education, I was faced again with Mathematics. Likewise during my early years, I had great difficulties in comprehending the mathematical concepts that they were teaching to me. I even had to memorize some of the formulas without any context in order to pass the exams. It was not until the third year, with the subject Didactics of Mathematics, that I had the great pleasure to meet Inmaculada Lizasoain, my teacher at that moment. She is convinced that mathematics can be understood and taught to anyone using the appropriate methodologies. She changed my perspective, my view of mathematics and she gave me the chance to learn how to teach mathematics. She also offered to me the possibility to participate in mathematical workshops that they were being done in Zaragoza with children with Down syndrome under the leadership of Elena Gil Clemente, and It was not until that moment, that I realized, as Gil Clemente and Cogolludo-Agustín (2019) say,

> "…the lack of progress in learning mathematics of children with Down syndrome has more to do with an inadequate choice of concepts and with a methodology that does not take advantages of their strengths, than with a real genetic impairment related to a poor conceptual understanding and innate difficulties with abstractions."

Which means that every one of us can understand and learn mathematics.

In the same period of time I met my mentor Raquel Garcia, who gave me a new perspective about what teaching mathematics really means and how children can achieve a rational understanding in this matter. Her distinctive way of teaching is mainly based on Polya and De-Guzmán's (1945; 2006) approach of problem solving and introducing arithmetical concepts through this particular method. Additionally, her first lessons were based on Hazekamp (2011), and the relational understanding by Skemp, (1989).

Consequently, I decided to improve my teaching abilities in this subject with the aim of helping my future pupils, regardless of their learning disabilities, to comprehend mathematics.

1.2. Justification for topic choice

This Graduation Thesis starts from an essential motivating element which was recognised while we were based on a teaching placement in a Primary School. The purpose of our devised work plan is to enable students to acquire and develop the foundation of mathematical skills through problem solving, by a group of diverse pupils enrolled in the first year of Primary Education in a mainstream classroom in a state school.

1.3. Objectives

The general objective proposed in this Graduation Thesis is to elaborate motivating educational responses that will improve the efficiency and the competence development of mathematical concepts through Problem Solving in a mainstream classroom. The specific objectives derived are:

- Elaborate a theoretical framework that will allow us to design a learning proposal to develop the understanding of addition and subtraction through problem solving in a first of Primary Education classroom.
- Design and develop a Learning Proposal which will allow the students a more meaningful way to understand and solve mathematical problems.
- Foster teamwork to enhance the motivation, improve sharing knowledge and strengthen meaningful and equitable learning outputs in a group of discouraged pupils because of some types of activities.
- Asses the teaching-learning development and the results obtained to improve the process.
- Identify the learning proposal’s main strengths and weaknesses so that later on, other teachers may develop similar proposals in other classrooms.

In order to achieve these objectives we have developed various tasks. The first task involves researching bibliographic resources about the Importance of Mathematics in Primary Education, Solving Problems methods, Addition and Subtraction teaching methods and Learning Centres methodology. This has enabled us to know their characteristics, objectives, methodologies and evaluation. The second task involves the design and development of a learning proposal. The third task entails the analysis of data collection and the comparison with the documents consulted. The final task involves the writing and synthesis of the present paper.

2. THEORETICAL FRAMEWORK

2.1. The importance of learning Mathematics in Primary Education

Mathematical learning and acquisition is not an abstract process of memorizing algorithms, formulas, definitions and the applications of these. On the contrary, a complex process involves an intricate understanding of the concepts origins, their basic properties and deep meaning, the interaction between the diverse personalities of the learners, the teachers and their intentions and actions, and the overall learning process. Mathematical learning process is also an interplay of teachers' and students' background, experience, perceptions and learning environment. In this case, solving problems methodologies play a crucial role in how mathematics is acquired, and whether the learners acquire adequate skills to understand effectively mathematical concepts.

Also, mathematical knowledge has always been parts of our history and our lives. There is evidence of prehistoric artefacts that can, perhaps, be interpreted as mathematical, found in Africa and date as back as 37.000 years. There are many written documents that show evidence for some level of mathematical knowledge in the Ancient Near East, in Ancient Greece and the medieval Islamic empire.

Therefore, mathematics are part of the human history and the study of its development along the centuries is also useful for understanding the steps that have been needed for defining some basic notions, some basic elements, and the difficulties of the process of learning them. We can also see that somehow we experience in our own learning the steps, failures, difficulties and needs of the development of mathematics along the time (Gasca, 2004, 2012).

The aim of teaching mathematics in primary educations is not only that children can learn the basic algorithms and contents of mathematics, but to understand better the world around them because of the mathematics they have learnt. They will need to be able to solve problems in future, but also right now. Their lives are full of experiences that can be understood better if we teach them the mathematics that explain and solve them. Because of this, mathematics has to be explained intimately related with the children's desires, experiences and needs. Perhaps we should break with the tradition of teaching mathematics that were only useful two centuries ago, and pay more attention to the mathematics that are interesting right now (Fuson, 1992; Hiebert, 1996).

The aim of teaching mathematics is also the development of mental skills, reasoning capacities, logic reasoning, etc. This is useful for everyone in every moment of their life. In words of Ronda Faragher (2013), "*we do not need to be smart for learning maths; the learning of maths makes everybody smarter*". This means that the teacher must understand the difficulties that their pupils may have in order to adapt the teaching strategies; and also grasp the strengths of any of their pupils and use them for getting a meaningful learning. This will open the child's mind, preparing them for a new challenge (Faragher and Clark, 2008; Faragher, Brady, Clark & Gervasoni, 2013; Diezmann, Lowrie, Bicknell, Faragher & Putt, 2004).

Moreover, teaching mathematics helps people to grow, develop the acquisition of self-esteem, the learning from errors and mistakes, the evaluation of risks, the planning of steps for solving general problems, the willing of researching and discovering and learning more (Siemon, Beswick, Brady, Clark, Faragher & Warren, 2011)

Mathematics has proved to be a daily challenge, a new exciting experience for every day. Then, why do children and teachers still seem to struggle with this subject? This is not a new question. Many researchers have asked themselves this same question, and have offered answers and solutions, like Richard Skemp (2002).

Teachers must improve their understanding of mathematics, they must enjoy them and have the willing of breaking with the tradition and the instrumentality of mathematics. It is an actually big effort, but we are disposed to do it.

2.2. Problem-Solving Methods

> "A great discovery solves a great problem but there is a grain of discovery in the solution of any problem. Your problem may be modest; but if it challenges your curiosity and brings into play your inventive faculties, and if you solve it by your own means, you may experience the tension and enjoy the triumph of discovery. Such experience at a susceptible age may create a taste for mental work and leave their imprint on mind and character for a lifetime" (Polya, 1945).

As Polya (1945) says, the key to solving problems is not based on providing students with out of context and boring problems, it is to create challenging circumstances, proportionate to their knowledge that will increase their curiosity and will help them solve problems more independently. That is way it is important to distinguish between activities and real word problems.

Nowadays, most of the text books for primary education are filled with activities based on the same mathematical algorithms, that is why, in most cases, if you have understood and followed their prior theory you are able, almost without thinking about it, to figure out which algorithm you have to use in order to solve the activity. For instance, if the objective of the unit is learn the addition algorithm and the children have been practicing day after day this algorithm trough different activities, mostly all of them will know that the word problem may be solved with the addition algorithm. Probably, they will not check if they have understood the problem, how they must solve it or if the solution is in line with the main question.

On the other hand a real problem is based on circumstances that children more or less understand the outcome but they do not know how to get to it or which steps they must follow in order to solve it (De-Guzman, 2006 p. 3) For example: The teacher tells one of their pupils to go to the canteen but they has never been in the school canteen before. In this case the pupil knows the outcome: the canteen is the place I have to go to, but does not know the step he has to follow to solve the problem, and claims: "but I do not know how to get to the canteen". A more mathematical approach would be: I have five sweets and my mother wants me to share them with my three brothers.

The boy knows the outcome which is giving some of his sweets away but he does not know how he could do it so that all of them have the same amount, he does not know the steps to solve this problem. These are the types of situations that we encounter in our daily lives and in Guzmán's (2006) words, these are the circumstances an investigator has to face.

But why is it so difficult for children to understand mathematics? As Skemp (1989) says, *"Mathematics is much more abstract than any of the other subjects which children are taught, at the same age, and this leads to special difficulties of communication"* This means that teachers must not only have an intuitive level of knowledge but they must be able to put this knowledge into words, for that teachers must have an explicit understanding of the main ideas before being able to teach them and consequently making it easier for learners to understand.

As teachers, we think that most of our students understand how to do the addition algorithm if they are able to reproduce the algorithm by themselves after a huge number of repetitive examples. Through practise children remember that they must start adding up from the right column and if the number of ones is bigger than ten they must "forget" the first digit of the sum and add up one to the numbers in next column on the left, and so on… At this point we are only looking for a procedure understanding: children learn what to do, recalling the algorithm as a sequence of steps that mean nothing. With the definition that Skemp (1989) establishes about relational or instrumental understanding, children do not even acquire an instrumental understanding, thereby the effort is completely useless:

> "*We say an individual has an instrumental understanding of a concept if he or she can state the definition of the concept, is aware of the important theorems associated with that concept, and can apply those theorems in specific instances. We say an individual has a relational understanding of a concept if he or she understands the informal notion this concept was created to exhibit, why the definition is a rigorous demonstration of this intuitive notion, and why the theorems associated with this concept are true"(Weber, 2002 p.2).*

As Skimp (1989) says, instrumental understanding is usually easier to grasp for children as less knowledge is involved, they do not need to think why they are doing certain things, they only need to recall the steps or algorithm thereby the reward is immediate because they get the right answer. However, children will not be able to apply this algorithm in new task, probably if they stop practising the algorithm they will forget the steps and in many cases they will swap steps with other algorithms later on, which will lead them to frustration.

As another example, in order to teach the standard subtraction algorithm a teacher draws on the board this table and relates the steps of the algorithm that children must follow:

$$\begin{array}{r} 28 \\ - \quad \underline{\ \ 9} \end{array}$$

1. Let's start with the ones: Because we can't subtract 9 form 8 we go to the next ten which is 18.

2. Now subtract 9 from 18: *It's 9*

3. Very good! Now we mustn't forget the 1 from 18. Take that one and write it under the number 2.

4. Finally, subtract 1 from 2: *It's 1*

5. Great! So the answer is 19.

After the question "have you understood?" the answer is simple: "yes, I understand what I have to do, just need to remember the steps". Some students will defiantly be able to reproduce these steps and apply this subtraction algorithm to many examples. But have they really understood what they are doing? Why do those steps work and calculate the rest? What is the purpose of the subtraction algorithm and what does every step leads us to?

A rational understanding approach in this example would imply a groundwork that the teacher should give to the students in preparation for following learning. Explaining our pupils what subtracting really means, giving students a context which they can relate to and make learning more meaningful for them. As Donaldson (1978) states, children acquire linguistic skills before they become aware of them, they do not interpret word isolation, but interpret situations. Thus, we should reproduce a manipulative procedure that explains a longer and different algorithm (lending) which will lead us to the long borrowing algorithm so that, and after practice, they will be able to understand the short standard algorithm presented upwards.

As we can see there are many things to do before we present algorithms to children, and the first and possible the most important one is to present the operation as a procedure that they already do in different situations. These situations can be related as word-problems. As children are able to interpret situations why not teaching mathematical concepts through word-problems. *"The main goal in teaching mathematical problem-solving is for the students to develop a generic ability in solving real-life problems and to apply mathematics in real life situations….Learners need to develop many skills to solve problems effectively and this makes problem-solving a hard topic to teach"* (Guzman Gurat, 2018 p.53).

As for this, this topic has been widely studies, but most of the studies have tackled the question from a theoretical point of view and for general mathematical problems. As a result, very few examples of a right procedure for elemental arithmetic problems can be found in literacy.

2.2.1 Pólya's Problem-Solving Method

"Having tasted the pleasure in mathematics (students) will not forget it easily and then there is a good chance that mathematics will become something (for them)" (Polya, 1945 p.vi).

For Polya (1945), the acquisition of independent work is as important as making word-problems comprehensible for learners thereby, teachers have a very important role in the process. Indeed they need to help their pupils to understand word-problems because if they are left alone they may not progress, but if they are helped too much, pupils will not feel the joy of triumph. This type of help is what the author states as unobtrusively and natural help. In order to acquire this, teachers must put themselves in their students' point of view, trying to understand what and how students are thinking.

Another important aspect for the author is the use of general and common sense questions that will lead the learners not only to solve one specific word problem but to be able to solve many different problems. General questions that can be applied to many other situations, and common sense questions, which come naturally to us when we are concerned about the problem, are ideal, because children can make themselves these same questions no matter the problem they are trying to solve.

Although in many occasion the intention of the teacher is to help their students by making questions and suggestions, it is important to remember that not all the questions are right nor asked in the fitting moment. We do not want to make suggestions when the pupils have not seen all the aspects at which it is driving them yet, we also do not want to giving them too much information and leaving them with very little to do. On the contrary we want our questions to lead them to learning for future problems and to an understanding of how the teacher came to the idea of asking that specific question. As Fuson et al. (1997) address

> *"…the teacher plays an active role in the classroom by posing the problems, coordinating the discussion of strategies, and joining the students in asking questions about strategies. The intent is to create an environment in which teachers support students' efforts to construct their own solutions methods".*

In order to help students unobtrusively the author has developed four steps and many questions or suggestions that will guide the students in problems solving. Students will be able to understand the problem, devise a plan, carry out the plan and learn from what has been done.

1st Phase: Understanding a problem

First of children must understand and be willing to solve a problem. For this reason teachers should not only create the appropriate atmosphere to motivate their pupils, but also encourage the students to have a relational understanding of the word problem, which means more than comprehending the words or finding out what the problem requires.

Understanding a problem means that we are able to know what the problem is talking about, we are capable of identifying the categories and the relations among them and to make graphic representations without adding numbers. For this, we must keep in mind that problems should not be too difficult or too easy but natural and interesting for the students.

Firstly, teachers have to make sure that their pupils understand the meaning of the word problem. They can do so by asking the students to express the problem without numbers with their own words. The first questions to be answered is “What does the problem talk about?”

Secondly, learners should be able to point out the principal parts of the problem, which are the magnitudes that play a role, and the relations between them. Some examples of questions that the teacher can ask are: What plays a role in the problem that could be given a numerical value? What is counted, or measured? What can be counted or measured? Are they independent? If we changed some values, what would also change? Which are the relations between the related magnitudes?

In the third step, pupils should try to make a pictorial representation of the information they have extracted before from the problem. It is important to let children come up with their own representations, the one that suits them better. However, we can also provide them some useful representations for the relations. For instance, in a part-whole relation the Pie Charts, Ven Diagrams, Number Bond Diagrams and Bar Model are useful. (Figure x)

On this pictorial representation we show the unknowns of the word-problem and the question we are asked to answer. Then we ask children to tell the problem just looking at the representations and check out if all the information (except the specific numerical values) of the word-problem are present in the representation. When the pupils are able to represent the unknowns, it means that they have understood the problem. Teachers should help them in this procedure, with questions like: Where do you show this information? How have you shown this relation? Which kind of relation does this picture stand for? What does every picture mean? How do I know whether this is known or unknown? Which was the question I had to answer? Finally, we can write the numbers of the known magnitudes.

2nd Phase: Devising a plan

We say we have a plan when we have a list of steps, calculations or constructions that we must perform in order to access the unknown. The main goal in solving problems is to conceive the idea of a plan and being able to develop it. Among children this process is not easy due to the lack of knowledge in the subjects and because nowadays we tend to solve their real-life problems letting them a very limited margin to develop their own ideas.

The creative process is one of the highest and more complex potentialities in human beings. It implies thinking skills and the integration of less complicated cognitive processes and new or higher ones in order to develop an idea. Creativity has been among humans since our very early stages, millions of years ago and therefore it is in our own nature to be creative. However, good ideas are based on our own past experiences and the knowledge acquired through them; so it is very important to give such opportunity to children as otherwise mere remembering facts is not enough to create new ideas.

Among our learners, some ideas may emerge suddenly but most of these ideas will arise gradually and after many, *apparently*, unsuccessful trials. And we say *apparently*, because with every attempt, our pupils acquire more knowledge. Thereby, teachers must unobtrusively help their students to conceive these ideas and a way to achieve this is making some of the questions and suggestions that Pólya (1945) designed.

For instance, we can ask our pupils if they remember or know any problem that is related to the one we are trying to solve at the moment. As there might be many problems somewhat related, we can help our pupils by asking them: What should we look at to choose one or the other? The relations in the problem. Can you think of any problem that has the same relations? Once they do it, it is time to decide whether we can use the problem or not. Through this process, pupils are connecting their past experiences, facts and ideas to the new ones.

However, some word problems may still be too challenging for some learners and one way to address this situations is transforming or slightly modifying the problem. As De-Guzmán (2006) says, if the problem is too difficult to solve because there are too many elements, create an easier, less complex but similar one.

At this point students should be able to devise their plan, to write down the steps and to ask themselves if they have used all the data that they already know, even though may be not all of them are necessary.

3rd Phase: Carrying out the plan

We have already mentioned that conceiving the idea of the solution or devising a plan is not easy as formerly acquired knowledge, concentration and good mental habits are needed in order to success. On the other hand, carrying out a plan is much easier, if everything works out; what is mainly needed is understanding of the operations. Nevertheless, we still need to develop those strategies and make sure that every step is followed.

The plan is a general outline of what has to be done, we need to make sure that the details fit, examining all of them one after the other till it is all perfectly clear and nothing is left out in which an error could be hidden. However, the main danger lies in forgetting the plan. As the Polya (1945) says, forgetting a plan may occur if the learners do not develop their plan and it is given form the outside, i.e. the teacher. Although if the pupils work for themselves with unobtrusively help and conceive their own idea, they will not lose it so easily.

4th Phase: Looking back

Re-examining and reconsidering the result is at least as important as any other of the previous steps, as leads learners to consolidate their knowledge and develop the ability to solve problems, that is part of a teachers' duty.

By looking back at the solution, plan, argument … we are giving pupils the opportunity not only to check the results but to investigate other ways of solving the problem or improving the solution and, thereby, improve their knowledge in problem-solving. As the Polya (1945) states, we prefer to have two different proofs to convince ourselves.

We can also help children to establish connections between problems by encoring our learners to imagine different scenarios where they could use the same procedure or apply the result obtained. *Do you think we could use the same result in other problems? Could we use the same method or plan in another situation?*

2.3. Addition and Subtraction of Natural Numbers

In order to explain to children what adding and subtracting mean and for them to have a relational understanding, it is important to understand how children acquire natural numbers and their representations and how we must teach them.

2.3.1 Natural Numbers

A natural number is the only abstract property that all the equipollent sets share. Two sets are said to be equipollent if a bijection (one-to-one and onto map) can be defined between them. The set of Natural Numbers is precisely defined by Peano Axioms, as the only set verifying:

1. Number 1 belongs to it.

2. Any natural number has a following.

3. There is no natural number which following is number 1.

4. Two different natural numbers cannot have the same following.

5. If A is a subset of natural numbers, such that 1 belongs to A, and for any natural number in A, its following also belongs to A, then there are not natural numbers out of A.

But the common understanding of natural numbers is that they represent the quantities of living beings that can be found in nature. So they were born for counting objects, although we also use them for many different purposes. Children learn them for the first time through a short verbal chain of natural numbers as a simple measure of time in their games or though songs. Later they are used for counting small sets of things.

Counting and thereby calculating the cardinality of a set of elements in a given finite set *A* means establishing a bijection between this set of elements and a finite chain of natural numbers beginning with the number 1. In order to count, learners need to verbalize the numerical chain as they are relating the words in the chain to a particular object in the set.

The teacher must understand the details of the definition of the bijection, because there lie the details of a right counting:

1. *A bijection is a map, a function*. This means that any object of the set has an image and only one image. So, every object has to be related to a number, and only one number must be related to an object. Explained to children it means that we cannot leave any object without being counted, and that we cannot count an object twice.

2. *A bijection is injective*. This means that two different objects cannot have the same image, thereby we cannot give the same number to two different objects. Explained to children it means that we cannot repeat any number when we recite the chain of natural numbers.

3. *A bijection is surjective*. This means that any element in the image set has to be the image by the function of some element in the origin set. So, in our verbal chain of numbers, all numbers recited have to be related to some object in the set that we are counting. Explained to children it means that we cannot make gaps in saying the numbers and that we cannot recite a number without relating it to an object.

4. *Any bijection works*. This means that counting does not depend on the particular bijection that we establish between the sets. Any bijection is valid and thereby the correspondence between an element in the origin set and its image can be modified. Explained to children it means that it does not matter the order chosen when we establish the relation between objects and numbers. We can begin with different object and we can select them in a different order. Any person or any time that we count, may establish a different concrete relation between the objects and the words representing the numbers.

All these aspects that the establishment of a bijection implies must be acquired by children when counting.

However, counting is not the only strategy that children need to practise in order to master understanding the cardinality of a set of elements. There are other possible strategies as calculating the number of elements at first sight, through multiplications and addition or through an estimation.

2.3.2 Representations of Natural Numbers

Nowadays we use a decimal positional numeral system but this is not the only system of number representation that has existed, there have been many others before. For example the Unary Numeral System also called Tally System, the Simple Grouping System, or the Multiplicative Grouping. We summarize them here in order to show how these systems are useful nowadays for explaining children some aspects of the operations with algorithms.

1. Unary Numeral System, Tally System or Simple Representation.

Primitive societies had little need for large numbers and a practical method of keeping accounts was the unary numeral system or tally system. This system represents every natural number with a corresponding quantity of tally sticks |. For instance, when people wanted to keep track of the number of sheep in a field they would make a notch or tally on a stick for every sheep. So a drawing (that will be called symbol) is designated to the unit and the only rule for representing a natural number *n* is drawing the symbol repeated *n* times.

For example, if we take the symbol *, the number six would be represented as ******, or for instance, if we use the symbol | (a tally mark), the number six is represented as ||||||.

However, counting tally sticks can be very tiring and complicated when using large numbers, as large numbers would require many strokes and it is somewhat difficult to read such big numbers. Think about number 54, for example: ||||||||||||||||||||||||||||||||||||||||||||||||||||||.

A more advanced method uses an additive approach as the Simple Grouping system. For instance, the same number 54 before could be represented with tallies spaced as |||||||||| |||||||||| |||||||||| |||||||||| |||||||||| ||||. Now the reading is easier, but again we would fail when reading 546, for instance. However, the simple grouping tally system is efficient for small numbers and very helpful for explaining the addition and subtraction algorithms. The reason is that children have the intuitive believe that the addition means putting things together, as the tally system is very graphic showing the idea that adding is actually putting the tally sticks altogether and subtracting is splitting a group of tally sticks into two groups, take away one of them, and look at what remains.

2. Additive System

The early tally system led to the essentials of more advanced numeral systems like that of grouping. In this system groups of *b* elements are replaced by a new symbol, and this rule is followed for any symbol. In this way a symbol can be repeated *(b-1)* times at last. The quantity b is called *base* of the number system.

For example with *b* = 5, if the symbol | represents a unit, the symbol ◊ means 5 symbols | (and so 5 units), and the symbol * means 5 symbols ◊ (and so 25 units), then the number six is represented as ◊ |; the number eighteen as ◊◊◊ |||; and the number 54 as **||||. The representation of any number bigger that 125 needs the election of a new symbol representing 125, because in this system every power of *b* needs a different symbol.

The ancient Egyptians system (Figure 1) is an example of this system. They used *b* = 10 and different symbols to represent the powers of 10. In this way, for representing numbers from 1 to 9 they used tally strokes, then a new symbol to represent 10 which is repeated twice for 20, three times for 30, four times for 40, and so on, up to 90, and again a different symbol for 100, 1.000, 10.000, 100.000 and 1.000000. With this system Egyptians could represent numbers up to 9.999.999 with just seven symbols as shown in the following image. (Boyer, 1944)

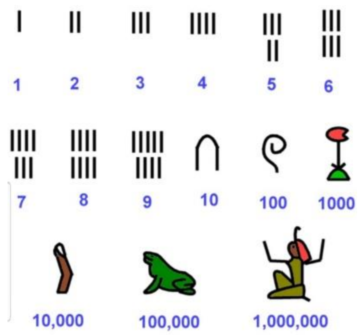

Figure 1: Ancient Egyptian Number System

Other systems where most likely inspired by the Egyptians Number System as "The Greek Number System" and "The Roman Number System". Both of which developed a number of symbols of their alphabet to represent different numerals.

Although an additive system is a better system than the tally one, none of the above systems are relatively good for representing large numbers as they imply the existence of an infinite number of different symbols.

3. Multiplicative System

In a multiplicative system there is again a base *b*. The difference with the additive system is that there are two different sets of symbols. One of them (let's call it A) is, just like before, related to the different powers of *b*. The other set (let's call it B) is finite, and has exactly (*b*-1) symbols, standing for quantities from 1 to (*b*-1). The rule for representing a number is that the repetition of a symbol from group A is replaced by a symbol from set B (showing the number of repetitions) and just once the symbol from A.

For instance, if again we take *b* = 5, the set B={a,c,d,e}, and in the set A we have the symbol | representing a unit, the symbol ◊ meaning 5 symbols | (and so 5 units), and the symbol * meaning 5 symbols ◊ (and so 25 units), then, the number eighteen is represented by d◊ d|; and the number 54 as c*e|. The representation of any number bigger that 125 needs the election of a new symbol representing 125, because again in this system every power of *b* needs a different symbol.

The traditional Chinese Numeration (Figure 2) uses a multiplicative representation system. This system has different symbols for the units, a different one for the tens, another one for the hundreds and so on for the thousands and the tens of thousands. But they do not have more different symbols for a bigger power of ten. For instance one hundred thousand is written as ten times ten thousand, or one million is written as one hundred times ten thousand. They also use different symbols for quantities from 1 to 9.

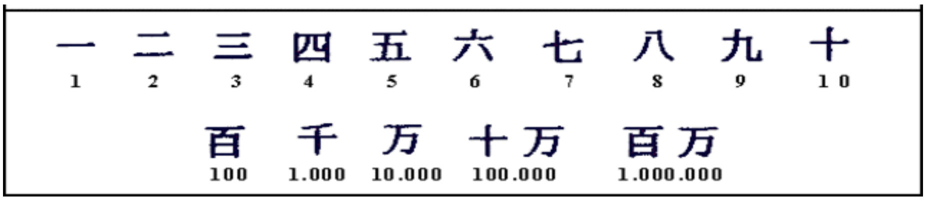

Figure 2: Traditional Chinese Number System

In fact the Chinese write their numbers in a similar way to how we speak our numbers, so the number 358 (three hundred and fifty-eight, being fifty a conjunction for five tens) written in Chinese symbols would be: symbol for three and symbol for hundreds, symbol for five and symbol for tens and finally symbol for eight (Cycleback, 2014).

The main problem again with this system is that we need an infinite set A of symbols that represent the powers of the base. As there cannot be infinite symbols only a finite subset of natural numbers can be represented this way.

In teaching the addition and subtraction to young children, the Open Calculation Based on Numbers (ABN) method uses this Multiplicative System, as will be seen further on.

4. Positional Systems

The need of an infinite quantity of symbols representing the powers of the base can be overcome by the introduction of a position. An agreement of value as power of the base for every possible position of a symbol from the set B in the writing of a number, avoids the need of symbols in set A, since the power related to each multiplier (symbol from B) is understood by the position that the multiplier occupies (Pascal, 1908).

Let's suppose again that *b*=5 is chosen, and that B = {a, c, d, e}, where a stands for one, c for two, d for three and e for four. The beginning and ending of the number will be shown with parenthesis and we will show the positions separating them with the symbol |. Then, the number six will be written as (a|a); the number eighteen will be written (d|d) and number fifty four will be written (c| |e).

The Sumerian and Babylonian Number System or the Hindu-Arabic Number System, that actually is the Decimal System that we use nowadays, are examples of positional systems.

The Babylonian are most likely to be considered the precursors of the Positional System sometime in the second millennium BC. This civilization used 60 as its number base and only two cuneiform symbols which were arranged into fifty-nine base units; a symbol for one and a symbol for ten. As they used a positional number system, they arranged the numbers into columns. The first column contained the fifty-nine base unit symbols, the next column used multiples of sixty, the third column represented 60^2 and so on (Boyer, 1944).

Figure 3: Hindu-Arabic Number System

The Hindu-Arabic system is another example of the Positional System. This system was developed by Aryabhata of Kusumapura and Brahmagupta between the 5th and 6th century and they are credited for developing the Decimal Numeral System that we use nowadays. This was also a multiplicative grouping system in its origin, creating symbols for the quantity of units (from 1 to 9) and also for the different orders until they began to represent the numbers following a place value order for making various calculus with them.

This system also introduces a symbol for zero that would slowly spread to other countries (Romero, 2019). When Brahmagupta invented this zero symbol, meaning that there was no units, tens, hundreds… in that specific position, the vertical lines were not needed any more and disappeared.

Let's suppose again that *b*=5 is chosen, and that now B={0,a,c,d,e}, where a stands for one, c for two, d for three and e for four, just like before, and 0 means that there is nothing there. Then, number six will be written as aa; the number eighteen will be written dd and number fifty four will be written c0e.

Also, our decimal numeral system is a place-value system and although we have been using it for hundreds of years, it is understandable that children find it hard to grasp, as it is the final step in a very long evolution of number representations. Since the comprehension of this system is essential for understanding the addition and the subtraction algorithms, whichever it is used, we will need to explain very patiently its definition, generation and writing rules, probably following the historical evolution described before.

5. Numeral Systems

Several systems for representing natural numbers have been seen. But not all of them can be considered numeral systems. A numeral system is a finite set of symbols and a finite set of rules and conventions that provide a way to represent and identify any natural number. Most of the societies have built their own numeral system, some of them differ from one another but they all share two common properties that must be verified:

a. They must use symbols to represent some units and other symbols or rules to represent groups of units.
b. Any number can be represented and identified by those symbols regardless of how big it is.

Anyway, we must understand that any writing of a number is just a way of representing it, not the number itself. This means that it does not matter how we represent the number twenty-three; 23 or $10111_{2)}$, it is still the same number, just another word in another language. This is easily understood with common language: a table is a table for every person in the world, it does not matter if we use the word *mesa* or the word *Tisch* for naming it.

2.3.3 The use of different numeral systems in addition and subtraction teaching

In teaching the addition and subtraction to young children, the Open Calculation Based on Numbers (ABN) method uses a Multiplicative System. The ABN method was created by Jaime Montero in 2000 similar to one developed by Fuson et al., (1997), and its main objective is to change the teaching-learning process of problem solving and calculation in primary education, using a model based on semantic categories as formal support for both approaches.

The arithmetical algorithms in this method are based on numbers, not on the specific writing of the numbers, thereby it facilitates children's natural intuitive process and helps them to develop a dynamic approach of the number sense. Additionally, these algorithms are adjustable, as they allow each learner to use their own processing system when doing calculations; and are transparent, because, as with the tally system, children can easily understand the bijection between a set of elements and the number-words in the numerical chain (Canto s.f.).

In order to explain how to use this method and why it has so many advantages in comparison with the more traditional methods we will discuss both of them. We will begin explaining how the traditional addition and subtraction algorithms are used and furthermore we will explain how the ABN method is used.

1. Traditional Addition Algorithm:

The procedure can only be done following these steps:

a. Decomposing the addends into units, tens, hundreds, etc.;

b. Lining them up vertically by matching the place values: the symbol that represents quantity of units under the units, the symbol representing quantity of tens under the tens, etc.;

c. Combining them by adding the numbers that share the same value place, starting with the units and moving left wise;

d. If any of the categories are bigger than ten, we take the figure in the tens' place value and add up to the next column.

In this traditional way, no modifications of the rules are allowed and students are required to memorize in order these four steps with no understanding involved.

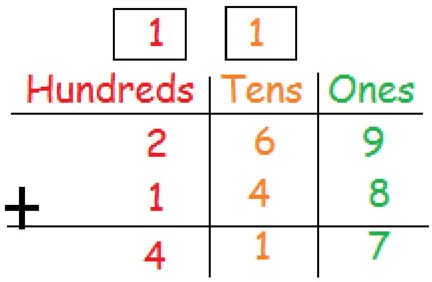

Figure 4: Traditional Addition Algorithm (Own development)

This procedure may lead to many difficulties among children as they may not understand the semantics of numeral, the cardinality of the numbers and the implicit meaning of the digits. Some of the difficulties that children come across stated by Lengnink & Schlimm (2010), are regarding the columns in general.

If a learner does not understand the place-value when doing an algorithm in this way, they may add the numerals regardless of their place-value. For instance; 24+58, as 2+4+5+8. Another difficulty that they could come across is with the direction in which the columns are dealt with as for example; 24+58, adding 2+5 first and then 4+8, giving 712 as result. Additionally, according to Ashlock (1998), the most frequent error made by children is when handling the carries. If children do not understand what adding the carries really means it will be easily forgotten, as for instance; 98+3 = 911 or 91.

2. Traditional Subtraction Algorithm

The procedure followed when doing this algorithm is:

a. Decomposing the minuend and the subtrahend into units, tens, hundreds…;
b. Placing them properly which means lining them up vertically by matching the place values;
c. Subtract the subtrahend units form the minuend units, the tens from the tens, hundred from hundreds and so on, moving from right to left.
d. If in a category the digit in the minuend is smaller than the digit in the subtrahend we suppose that in the minuend there are ten more units, make the subtraction and add a unit in the next column in the subtrahend.

Unlike the addition, the order of the two numbers involved in the subtraction is crucial. The reason is the nature of the sets they stand for. In an addition both numbers stand for the parts of a total set that is being formed and could be seen as parallel numbers. In a subtraction, the minuend stands for that total set and the subtrahend represents the quantity of those elements in the total set that are going to be isolated. In subtraction they are not at all parallel numbers.

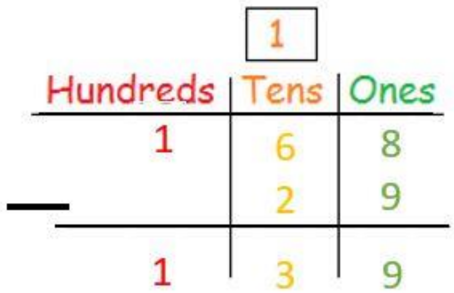

Figure 5: Traditional Subtraction Algorithm (Own development)

Again, following this procedure students are required to memorize all the steps and no modifications of the rules are allowed.

Following this procedure or other similar ones may lead to many difficulties among children as they may not understand the cardinality of the numbers, the semantics of numeral and the implicit meaning of the digits. Some of the difficulties that children come across according to Lengnink & Schlimm (2010), are regarding what to subtract from what.

Most students are taught that “you cannot subtract from a smaller number a larger one” without the order of natural numbers is explained and without understanding how this can be deduced from the writing of the numbers, which leads them to find the smallest digit and subtract it from the bigger one in every column, doing, for instance, 54-28 = 34.

As in the addition algorithm, another difficulty relies on which direction the columns are to be processed. Learners always proceed form left to right when reading books and writing the numbers, and so they follow this direction when performing the algorithm, and could do, for instance 254-163 = 190, where the result is wrong even though the formal operation of the borrowing algorithm is performed correctly.

With both operations, the most frequent error made by children when they follow these algorithms is when handling the carries or borrows. If children do not understand what carries really mean it will be easily forgotten, or added up to a wrong digit in the subtraction (because it does not matter in the addition). For instance 54+28=72, or 54-28=36, or 54-28 = 46. Additionally if it is not explained properly it can also lead children to apply it even if it is not necessary.

However all these mistakes have the same origin: children do not know why they are doing what they are doing.

3. Relational understanding methods

The aim of ABN method is to make implicit meanings of the explicit symbols. It uses different physical resources to represent specific numbers, according to what Bruner (1966) called the concrete, pictorial and abstract teaching approach. By doing so, the learners may develop a deeper and more sustainable understanding in mathematics. *“The principle of the ABN method is to maintain the numerosity of quantities all the time, in terms of knowledge, composition and decomposition, as well as taking into consideration how they operate in relation to other quantities* (Martinez-Montero, 2011).

This method avoids teaching in a unidirectional and sequential way allowing the pupils to learn mathematics in a more realistic and flexible way, far from the mere acquisition of strategies and symbols, that does not provide at all a relational understanding.

In order to teach children how to do calculate additions and subtractions we will use the principles shown by Karen Fuson (1997), concretely applying the simple grouping tally system in a manipulative way, then passing to a multiplicative system pictorial representation of the quantities involved, and finally arriving to explain the standard algorithms in an abstract way providing a relational understanding of what is done and why it works.

4. Relational Addition and Subtraction algorithms using the Tally System

The nature of addition is to join disjoint sets. So, the addends are the cardinalities of the disjoint sets and the sum is the cardinal of the union set. The nature of subtraction is splitting a set into two disjoint subsets, take away one of them and maintain the other one. Here the minuend is the cardinal of the total set, the subtrahend is the cardinal of the separated subset and the rest is the cardinal of the subset that remains.

Initially, children learn how to solve problems involving multidigit numbers by representing the problems with single-unit counter. As counter we mean, sticks, blocks or any other set of counters that children can manipulate easily. However, when addressing single-unit problems, children are not required to understand the conceptions of place value, they only need the ability to count. Children learn to understand place-value concepts when they manipulate any base-ten materials (Fuson et al., 1997).

Firstly, teachers should allow their students to explore the counters (from now on, we will call them sticks), counting them, making groups with an elastic rubber-band, estimating how many they have got… We can help our pupils by asking: “How many sticks have you got? How do you know that? How many sticks do you think you have got in those groups?”

Then, we introduce the only rule that is required in order to represent the place-value: whenever we have ten elements we have to make a group with those ten elements and put an elastic rubber-band to hold them. If we only work with numbers smaller tan 99, we just make groups of ten sticks, and we ask children: "How many sticks have you got in each group? I have ten sticks". After making several groups and being sure that our pupils know that each group has ten and only ten sticks, we can ask them to represent different multi-digit numbers: "How would you represent 11 sticks? And 15?" Whenever we use numbers bigger than 100 and smaller than 999, we will make new bigger groups with ten groups of ten sticks (so 100 sticks). With teachers' encouragement, pupils come to recognise that they do not need to count all the individual sticks every time they want to construct two-digit quantities, they only need to make collections of tens and ones (Fuson et al., 1997). Other activities can be used to give children more practice in manipulating ones and tens, as for example giving children a collection of tens and ones and they have to recognise the number that the sticks represent.

Once children have a comprehensive grasp of tens and ones, we can move onto adding them or subtracting them. At this point teachers could begin asking their pupils to add two sets of ones, for instance ||||| + |||||| and remind them the only rule that must be followed: when you get ten sticks you must put them together with an elastic rubber-band. Then teachers can ask pupils how many sticks they have got. If they have been practising counting tens and ones, they should be able to tell, almost immediately, that they have 11 sticks and when asked: "how do you know that you have eleven?" Children should be able to explain that they have a group of tens and one unit. Therefore, the carries in addition are very natural.

The practise of the subtraction could be conducted by asking children to get a certain number of sticks and then taking away part of the amount, for instance |||||| and take away |||, now look at the remaining ones, "how many are there left?" Through practice children learn that they cannot take away an amount of objects from a set with a smaller quantity of elements.

When approaching the carries in the subtraction we just must let children to discover what to do when they have to take away for instance 8 sticks from a set with two tens and two ones. Many children could think that after have taken away the two sticks they do not have enough for taking away the other six ones. But just asking "don't you really have more sticks?" they soon realise that they can open one of the groups having new ten sticks for going on with the process. This is just the principle of the *lending* algorithm, much easier and intuitive for children than the standard borrowing algorithm presented upwards. With this simple manipulative method it is easy to see that whenever the number of elements in a position in the minuend is smaller than the number of units elements in the same position in the subtrahend, we just have to take a group of the next position from the minuend and split them into ten elements in the current position. It is also very important that pupils are able to discover this procedure by themselves, and so they get a relational understanding of what is done and it will be harder to forget.

In further activities, this strategies should be paired with the expanded algorithm for additions. Many authors as Fuson (1997), Skemp (1989), Hazekamp (2011), Faragher (2014), Millan-Gasca (2014) and many others, advise to work both algorithms together.

After this manipulative work, children may afford the pictorial representation and simultaneously the abstract writing of what is done. But we will always immerse the operation in a story or word-problem, which will provide meaning to the operation. For instance in a simple (just one part-whole relation) word-problem like: "In a party everybody is holding balloons. 16 of them are holding red balloons and 15 are holding blue balloons. How many balloons are there at the party?

Children interpretation with sticks:

Teachers' interpretation with the expanded algorithm:

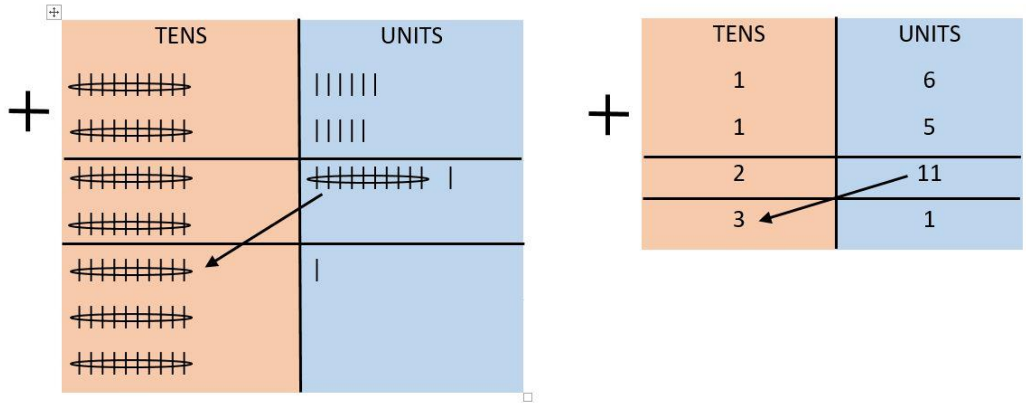

Figure 6: Addition Algorithm using Tally System (Own development)

The simple word-problem giving context to the subtraction could be "in a party there are red and blue balloons. How many red balloons are there if there are 24 balloons altogether, and only 8 of them are red?"

Children interpretation with sticks:

Teachers' interpretation with the expanded algorithm:

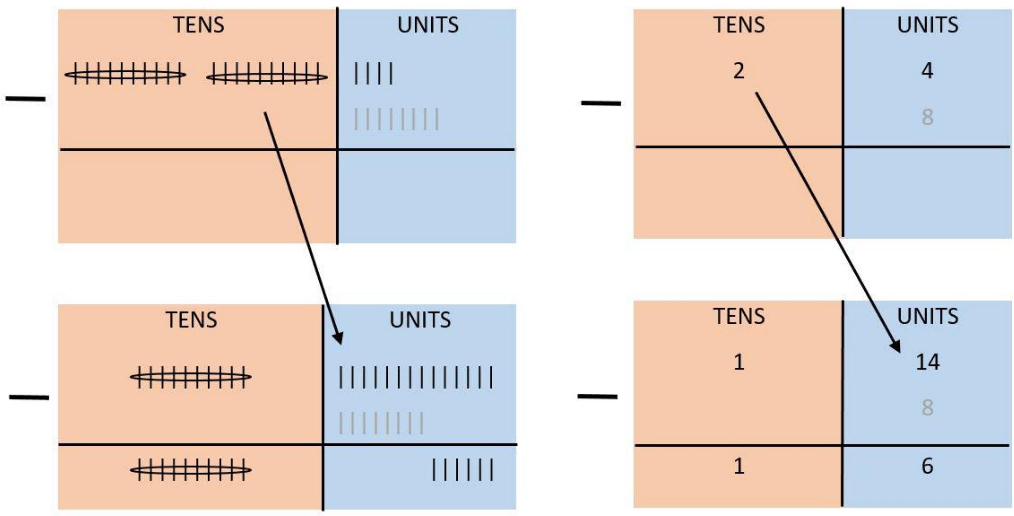

Figure 7: Subtraction Algorithm using Tally System (Own development)

2.4. Addition and Subtraction in Word-problems

In order to be able to teach addition and subtraction to children, and to create word-problems for children that provide meaning to the operations, teachers must understand some fundamental principles form a constructivist point of view.

First of all, teachers must understand that for having a proper addition there have to be two sets of things, so that there must be at least a characteristic that the elements in one set have and the elements in the other set do not have; but also all the elements in both sets must share a common property. The difference between elements in each set will provide disjoint sets, and the common characteristic will allow us to make the union meaningful. For instance, the addition cannot be made between 4 pies and 3 cats (which is the result? 7 what?), or between 6 children with curled hair and 5 ones with blond hair (how many children have curly blonde hair?).

Teachers should also take into consideration that in order to create word-problems that will equip the subtraction with meaning, the set that is subtracted has to be part of the total set (a proper subset). For instance we cannot subtract 6 sheep from a set of 10 wolves. But this example is too clear. Let's give another one: "This morning I bought a box with 8 ice-creams, I've eaten 3 ice-creams. How many ice-creams do I have now?" In this situation, some students could answer 8 ice-creams and others 5 ice-creams, but the truth is that any answer is possible, because the problem is not specific: maybe I eat ice creams form the box, or other ones; maybe I had other ice creams at home that I did not buy this morning, and may be the box fell down and lost all of them. I we want to work open-answer problems, this is a beautiful one, but it is not a good word-problem for illustrating a subtraction.

As we have mentioned before, when we want to use word-problems as the driven shaft of any mathematical content, because we believe that it is easier for children to understand this concepts if they can be related to real situations, the word-problems have to be well structured.

We should also know that not all word-problems are suitable for all age groups as some require higher cognitive processes than others. The following list shows the relations that may appear in arithmetic problems. It is important to say that this list is not the order in which word-problems should be taught.

1. Part-whole relation:
    a. Discrete part-whole relation.
    b. Continuous part-whole relation.
2. Comparison problems:
    a. Additive comparison relation.
    b. Multiplicative comparison relation.
    c. Additive-Multiplicative comparison relation.
    d. Comparison relation related with fractions, ratio and percentage.
3. Before-after relation:
    a. Basic before-after relation.
    b. Fractions in before-after relation.
    c. Percentage in before-after relation.
    d. Before-after relation involving more than one quantity.

The difficulty of the problems raises as long as the number of relations that appear in the problem. The first ones involve just one simple relation.

As the placement was based on the 1st year of Primary Education we will work only addressing discrete part-whole, additive comparison and basic before-after relations. These relations should also be provided for discrete (countable) and continuous (measurable) magnitudes. The progressive difficulty (for most children) of relations involving addition and subtraction operations is given by this order

- part-whole problems with total as unknown;
- part-whole problems with a part as unknown;
- before-after problems, with after as unknown;
- additive comparison problems with referred as unknown;
- additive comparison problems with comparison as unknown;
- before-after problems, with process as unknown;

- additive comparison problems with referent as unknown;
- before-after problems, with before as unknown;

A Part-Whole relation involves three magnitudes: two disjoint independent parts and the total set of elements. These type of relation can be tacked from two different points of view: in the first one we know the cardinality of both subsets (it can be extended to more than just two parts, always that we have pairwise disjoint sets, because of the associative property of addition) and the question asks for the cardinality of the whole; in the second one we know the cardinality of the whole and of one of the subsets and the question asks for the cardinality of the other subset. It cannot be immediately extended to more parts.

An Additive Comparison relation involves two magnitudes and the comparison between them. One magnitude is called referenced or referred, and the other one is called referent. The referenced magnitude value is always related in terms of the referent one and the comparison.

A Basic Before-After relation uses just one magnitude, which value changes in time.

3. EMPIRICAL STUDY

The aim of this section is to present innovative and motivating lesson plans for the mathematics classes. While elaborating the learning proposal we have taken into consideration the real context that provides a 1st year of Primary Education.

Some of the lesson plans have been carried out during the months of February and March, 2020. However, due to the global pandemic caused by the virus SARS-CoV-2, the educational institutions have been closed by the health authorities, thereby it has been impossible to continue with the implementation and the evaluation of the Learning Proposal.

3.1 Procedure and Phases of the Study

The following table describes the steps taken and the periods in which the present Graduation Thesis has been developed.

Table 1: Phases of the Study (Own development)

	November	December	January	February	March	April	May
Bibliographic research	■	■	■	■	■		
Elaboration of the theoretical framework			■	■	■	■	■
Development of the proposal			■	■	■	■	■
Implementation of the proposal				■	■		
Conclusions							■
Editing of the TFG							■

3.2 Learning Proposal

3.2.1 Justification for the choice of the Learning Proposal

This learning proposal has been developed for 5 and 6 year old children in 1st year of Primary Education. The aim of the proposal is to implement four mathematic lesson plans during the second trimester with an approximate duration of eight weeks – 16 sessions – which will be carried out in two of the sessions per week that they dedicate to mathematics lessons. Each lesson will take approximately four session, one for each of Pólya's (1945) phases. Through this proposal we aim to help children develop an actual relational understanding of problem solving, of the decimal positional system and the addition and subtraction operations.

This Learning Proposal imbibes from three of the four perspectives of the syllabus (MEC, 1989): the sociological, the psychological and the pedagogical.

From a sociological point of view because through the activities proposed it fulfils some of the functions attributed to schools as Lorenzo (1994) states, like developing educational activities with social purposes, train responsible citizens or transmit social values and not just productive ones.

From a psychological point of view because all the activities that we have developed take into consideration the age and level of the children and their evolutionary development. Likewise, according to Ausubel (1963) we considered the learners' previous knowledge as the base of their learning development, and in every step we provide mathematics of real meaning.

And finally, from a pedagogical perspective since taking into account a more inclusive and equitable teaching.

3.2.2 Context and Population

We have developed the empirical study in a Primary Education state school in Pamplona called CPEIP Ermitagaña.

This school has adequate endowment teaching materials and sufficient human resources for the educational work. The professional staff of the school consists of: one counsellor, two speech therapists, four special needs teachers, twenty classrooms, twenty form teachers, five teaching assistants, four specialists and four janitors. It also has several facilities such as a gym, a library, an ICT room, a music classroom, a staff room, an administrative office, a speech therapist classroom and a special needs teacher classroom.

The school can be defined as an institution that supports several values as participation, communication, inclusion, cooperation and effectively engaging the community, where the people that participate in it fell safe and welcomed. It is also very committed with the environment, the multiculturalism and the coexistence among all the individuals.

The students in the school come from different backgrounds and nationalities where multiculturalism is an important feature. It should also be noted that many of the students either do not know the language or they are fluent in it and some come from either social, economic or cultural disadvantaged backgrounds and/or at risk of social exclusion. Nevertheless, the uniqueness of each student is recognized, accepted and respected and each individual is encouraged to reach his or her fullest potential.

The target population for this study was all 1st of Primary Education learners in a public school. The selected sample involves two diverse classes of mix ability students, 1º A and 1ºB. As is can be seen in Appendix 2 between both groups of students there are three children with curricular adaptations due to their intellectual disability, four children with a higher learning capacity, six children with learning disabilities specially in mathematics and four ones that are repeating the 1st year. Due to all this diversity is has been challenging to create the appropriate environment that would motivate all the children and at the same time to adapt the lessons and make them accessible to all.

The pupils take Mathematics classes four sessions per week plus one session for problem solving. They also take one session in English. During the ongoing course, the students received instrumental understanding in mathematical concepts in four of the sessions, three in Spanish and one in English, and relational understanding through problem solving in the two remaining sessions.

These approaches, i.e. (instrumental and relational understanding) were given to the same group of students means than this study does not have a control group. Therefore is has been a convoluted task to draw inferences as to what will be the cause of improvement or poor performance. Improvement could be constructed to mean that:

- Combined approaches had better outcomes
- The introduction of relational understanding has been responsible for the improvement.

Either way, the above inferences are not conclusive enough and more research needs to be done on the topic, using a controlled group in order to establish the specific impact of one of the particular methods. However, this thesis does provide an insight on how to make mathematics classes more meaningful and how to make children understand word problems. Thus, we would like to develop a proper research on this matter, with a wide population and selected groups of children.

3.2.3 Contents, evaluation criteria and evaluable knowledge

The following tables show the contents, evaluation criteria and learning standards chosen form the Decreto Foral 60/2014, which we have used in order to design and develop the learning proposal.

Table 2: Contents, Evaluation Criteria and Learning Standards. Matemáticas 1º E.P.

	Matemáticas 1º Educación Primaria		
	Contenidos	Criterios de Evaluación	Estándares de Aprendizaje Evaluables
Bloque 1: Planificación del proceso de resolución de problemas	Análisis y comprensión de enunciados.	1. Expresar verbalmente de forma razonada el proceso seguido en la resolución de un problema.	1.1. Se inicia en la comunicación verbal de forma razonada del proceso seguido en la resolución de un problema de matemáticas o en contextos de realidad.
	Estrategias y procedimientos puestos en práctica: hacer un dibujo, una tabla, un esquema de la situación, ensayo y error razonado, operaciones matemáticas razonadas, etc.	2. Utilizar procesos de razonamiento y estrategias de resolución de problemas, realizando los cálculos necesarios y comprobando las soluciones obtenidas.	2.1. Se inicia en el análisis y comprensión del enunciado de los problemas (datos, relaciones entre datos, contexto del problema). 2.2. Se inicia en la utilización de estrategias heurísticas y procesos de razonamiento den la resolución de problemas. 2.3. Se inicia en la reflexión sobre el proceso de resolución de problemas: revisa las operaciones utilizadas, las unidades de los resultados, comprueba e interpreta las soluciones en el contexto de la situación, busca otras formas de resolución, etc.
		3. Describir y analizar las situaciones de cambio, para encontrar patrones, regularidades y leyes matemáticas, en contextos numéricos, geométricos y funcionales, valorando su utilidad para hacer predicciones.	3.1. Se inicia en la identificación de patrones, regularidades y leyes matemáticas en situaciones de cambio, en contextos numéricos.
		4. Profundizar en problemas resueltos, planteando pequeñas variaciones en los datos, otras preguntas, etc.	4.1. Se inicia en la profundización en problemas una vez resueltos, analizando la coherencia de la solución y

			buscando otras formas de resolverlo. 4.2. Se inicia en el planteamiento de nuevos problemas, a partir de uno resuelto: variando los datos, proponiendo nuevas preguntas, etc.
		11. Reflexionar sobre las decisiones tomadas, aprendiendo para situaciones similares.	11.1 Se inicia en la reflexión sobre problemas resueltos y procesos desarrollados, valorando las ideas claves, aprendiendo para situaciones futuras similares, etc.
Bloque 2: Números Naturales	Estimación de resultados Ordenación de los números de la primera centena. Operaciones con números naturales: adición, substracción Propiedad conmutativa de la suma utilizando números naturales.	1. Leer, escribir y ordenar los números naturales hasta el 99, utilizándolos en la interpretación y la resolución de problemas en contextos reales.	1.1. Cuenta objetos reales o dibujados. 1.4 Conoce la unidad y la decena y su valor. 1.5 Identifica el valor posicional de las cifras. 1.6 Compone y descompone números de forma aditiva atendiendo al valor posicional de las cifras. 1.11 Establece equivalencias entre D,U.
		2. Realizar cálculos numéricos básicos con las operaciones de suma y resta, utilizando diferentes estrategias y procedimientos.	

In comparison with all the contents in the curriculum through the problem solving lesson we are able to cover many of these without having to dedicate separated lessons to teach children the various mathematical contents.

3.2.4 Lesson Plans

This section shows a collection of four activities and practical ideas for the classroom based on the priorities identified in the preceding chapter. They are all intended to encourage real comprehension of mathematical problems and to develop confidence and willingness for future activities. They are also designed to build upon the capacities, addressed in the first chapter of this report, which children bring to the classroom. The activities develop the capacity for understanding a problem, the ability to plan strategies and devise a plan, the capability for carrying out a plan or strategy and the skill to look back and learn from what has been done.

All the lesson plans follow the same structure developed by Polya (1945): procedure, independent problem solving to verify comprehension and similar sample problems. At the same time the procedure is sub divided into: creating a positive classroom atmosphere, the proposal of the problem and the solutions, which are comprised of four phases, totalling 14 steps. Accordingly, the questions asked by the teacher will appear in black and the answers that children might give, in grey.

By creating a positive classroom atmosphere, we increase children's natural motivation to participate in the learning process and develop their desire to learn and their self-esteem. Students who are motivated, undertake the activity for pure enjoyment and are rewarded with a sense of accomplishment. (Lumsden, 1994; Scaramuzzo, G. 2010, 2016). In the following activities we take these aspects into consideration and provide the students with an atmosphere that will evoke the desire to learn, participate and solve word-problems.

The materials needed for all the lesson plans include a whiteboard, small boxes (if possible) and a selections of resource materials e.g. snap-cubes, lego blocks, beans, beads...all of which aid multiple equation formations. This resources will be available for all students at any point throughout the lesson.

In order to meet students' demands we have developed some activities that they can do as a follow-up as for example: draw a picture that represents the steps that we have followed to solve the problem, write down the algorithm we have used to solve the problem, create a new problem using the same magnitudes, etc.

The sequence of these activities are grouped according to the demand and response from the children in our classroom. However, if the underlying principles of the activities are clear, then each of us can give it a format which suits every specific circumstance.

The activity 0 shows the sequence of steps and questions that usually teachers follow when they want to teach their pupils how to face a mathematical problem. Nevertheless, these type of questions, focused on data, numbers and algorithms lead to a lack of comprehension. As Pólya (1945) says, questions must be general, natural and applicable in many cases in order to not only help students solve the specific problem but to develop their ability to solve future problems by themselves.

Activity 0 "My grandfather the shoemaker"

We begin the lesson by telling the children a story about the teacher's grandfather who was a shoemaker:

"My grandfather was a shoemaker and loved shoes. He use to make all kind of footwear like boots, slippers, runners, school shoes, etc. As you can see, he was obsessed with footwear and even counted how many shoes and boots were in every shop or house he entered". As a result, I have the same obsession so this morning when I came into the classroom I counted how many children were wearing shoes and how many children were wearing boots. Have you ever asked yourselves so? Do you want to know the answer? Let's find out!"

We continue the lesson by writing down the problem on one side of the board.

"This morning when I came into the classroom I counted 11 children wearing boots and 10 children wearing runners. How many children are there in the classroom? "

After we all make sure that none of the children are barefooted we ask the pupils to stand up and think how they could perform the story. We continue asking questions until one of them says that they could make two groups of children on each side of the classroom: the ones wearing boots and the ones that don't. All the children then move to the corresponding side of the classroom. As the teacher writes on the other side of the board "children wearing boots" and "children wearing shoes"

We ask one pupil form each group to count how many children (including themselves) are in their groups. As they tell out loud the results, the teacher draws on the board red shoeboxes representing the children wearing boots and blue shoeboxes which will represent children wearing shoes, under the corresponding column.

Once everything is drawn on the board we ask a number of questions to ensure the comprehension of the problem:

- "How many children are wearing boots? How many are wearing shoes?"

In order to find out the answer to the question "how many children are there in the classroom?" they must discuss about which action they need to perform.

Children should start by having their hands wide open and then join them together. This action represents two separate sets of things that are joined which is the equivalence of addition.

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With this procedure we would have not made children to understand the mathematical relations that live under the problem, we would have guided them towards the data and we would have not shown them neither the strategy followed nor the applications of what has been done to other problems. We will have failed even with the questions asked to children, because they cannot be applied to any other, even slightly, different problem.

As shown in the previous chapter, in order to actually help pupils to develop the ability to comprehend and do mathematical problems, Pólya developed a variety of questions and suggestions as well as steps that must be followed every time discussing problems with children. Thus, pupils will eventually use the right questions and suggestions to solve not only algebraic problems but also geometrical, nonmathematical, theoretical or practical problems. [Pólya, (1945)]

In this way, we have developed a set of four different activities following Pólya's steps and guidance in Problem Solving. This first activity is an improvement of the activity 0.

Activity 1 “My grandfather, the shoemaker”

Objectives and Outcomes:

The particular intention behind this activity is to strengthen the understanding of addition and subtraction, to recognise the part-whole relation with just two parts and to comprehend the concept of any natural number by its additive decompositions. As a result, children will begin to practice a procedure useful for understanding a Word-problem and the process of developing a suitable pictorial representation of a Word-problem. From here they will be able to develop strategies for arithmetical comparison and before-after relations.

Procedure:

a) Create the appropriate atmosphere:

“Do you know anyone that has a compulsive obsession? People who cannot think about nothing else but a specific issue? For example, there are people that spend most of their days thinking about music, animals or cars … My grandfather was a shoemaker and he loved shoes. He used to make all kind of footwear like boots, slippers, runners, school shoes, etc. But this was not only a job for him, he was actually obsessed with footwear. The first thing he stared at when he met people were their shoes; in a party he did never look at the guests faces, but their feet; and even counted how many shoes and boots were in every shop or house he entered. Have you ever done this? No? Never? Ever? Well, I have to confess that I share the same obsession.”

At this point, pupils will probably start staring at their partner’s shoes and talking about the topic. After 3 minutes, they are in the right atmosphere. Then it’s time to propose the problem.

b) Proposal of the problem: (5 minutes)

"So this morning when I came into the classroom I began counting how many children were wearing shoes and how many children were wearing boots. And now I need to know how many of you are here in the classroom, because then I will know if someone has not come to class today. Could you help me, please? But I refuse to count all of you. Let's find out with the pieces of information that we already have! Are any of you barefooted right now?"

We continue the lesson by writing down the problem on the board.

"This morning, when I came into the classroom I counted 11 children wearing boots and 10 children wearing shoes, and no one wearing any other type of footwear. How many children are there in the classroom?"

c) Solution Process:

As Pólya says, we need to distinguish four phases in the process of solving a problem:

- In the first phase we will help our pupils to understand the problem, that is much more than understanding the words or to find out what is required. Children should be able to state what the problem is about and the principal parts of it. They need to discover the items of the problem, as well as the connections or relations among those items, some of them with known value whereas others will be unknown. One of them is the answer that we look for.

- In the second phase they will have to devise a plan which may lead them to the solution, and learn to modify it in case it does not work well enough.

- In the third phase children will carry out the plan and will realise whether it leads to the solution or not, and discover the lacks it has for not working.

- On the final and fourth phase we get used to learn from what has been done. It is time for looking back at the solution and discuss it widely.

1ST PHASE: UNDERSTAND THE PROBLEM.

Step 1: Finding the problem's topic

First of all we scratch the numbers of the problem on the board, so that we know that there are numbers there, but we do not see it.

After we all make sure that none of the children are barefooted we ask the pupils to think how they could perform the story. We continue asking questions until one of them says that the problem talks about two groups of children: the ones wearing boots and the ones that do not.

These questions could be like this:

- What does the problem talk about? Children wearing boots and shoes
- Ok, and in general? Children wearing footwear
- And what would happen if we swapped children for adults? Is there any information that would not change? Yes, footwear.
- Great, and in particular? Types of footwear
- Awesome, and which categories (in this example "types of footwear") does the problem talk about? Boots and shoes.
- So we count boots and we count shoes…. No, we count children!
- So which "kinds" of children does the problem talk about? Children wearing boots, and children wearing shoes.
- No more groups? Let's read again: Children in the classroom.

Step 2: Discover the Relations

- Is there any relation among the categories? What does that mean?

- The reason why you wear today one kind of footwear is that your partner wears the same? No, it's just on me.

- So, being today in one of the groups is just random, Does the number of children in one group does depend on the number in the other group? Not at all; we could all wear boots today or no one could be wearing boots.

- Can those numbers be any? Yes, they can be zero.

- Can those numbers be as big as we want? The numbers cannot be bigger than 25!

- Why? Because we are 25.

- And? The children wearing boots are in the class, and the children not wearing boots too.

- Could they two both be 20? No, together we make the class, and together we are 25!

- Great. So, we have a class split into two groups. Any group is a part of the total and with both parts we make the total, right? (Miming with our hands is essential while we pronounce these words) Yes

Step 3: Representation

At this point we hand out to the students a piece of paper and crayons or pencils.

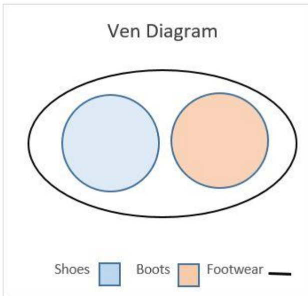
- Does anyone know how a very simple picture of the problem could be done?

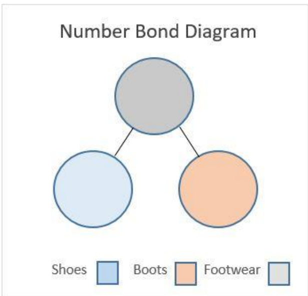
For any of their representations we discuss whether all the information of the problem is or not represented:

- Are the two groups independent?
- Do they make together the class?
- What is known? What is unknown?
- What have we been asked to find?

Likewise, we must give them some different examples if they are not acquainted with the different types of graphic representations.

Some examples are:





- Using the Pie Chart: Footwear is represented as the whole and each part of the pie represents one of the parts; the pink side shows the boots' category and the blue side represents the shoes' category. We can clearly see that both parts (children wearing boots and children wearing shoes) represent the total amount of children in the classroom.

- Using the Ven Diagram: In this case we can clearly see that both categories are independent and the line surrounding both circles represent the total amount of footwear. We must take into account that the gap between both circles and the surrounding line does represent any other type of footwear.

- Using the Number Bond Diagram: By using this diagram we are able to distinguish between the total (in the upper circle) and the parts as two separated circles. Keeping in mind that the total is split into two separated parts.

- Using the Bar Model: Both parts (children wearing boots and children wearing shoes) are represented by two different bars that when put together, one after the other, they represent the total of footwear as shown in the longest bar.

When we draw these examples we need to make sure that our pupils understand the relations that each bar or circle represents. We can do so just by taking a piece of paper, representing the total and tearing it in two parts.

- Now, if we only had the graphic representation and categories without the numbers written down, could we still tell the problem? Yes, we have two different types of footwear which are shoes and boots, we don't know how many pairs of footwear we have but we know how many children are wearing boots and how many children are wearing shoes in this classroom.

It is also helpful if the children use different colours for each category. We will do so on the board, as shown in the previous figure.

Step 4: Add the information to the graphic representation.

After each pupil has decided and drawn their desired diagram, then it is time to add the information that we have. We ask the following questions to guide the pupils in their procedure.

- So, how many magnitudes do we have? Three: number of children wearing boots, number of children wearing shoes and number of children in total.

- Which ones do we know? None, you have crossed them out!

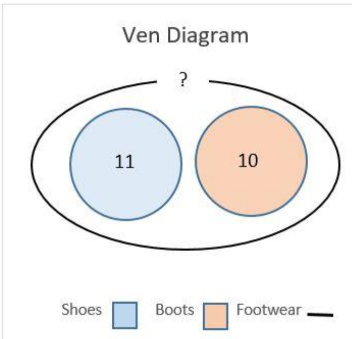
- Which ones do we know, although we do not see them? The number of children wearing boots, number of children wearing shoes.

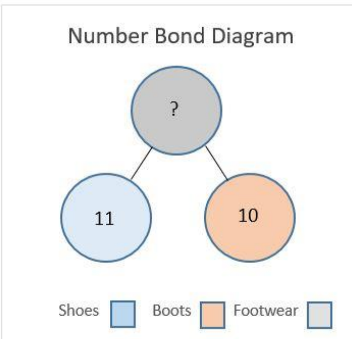
Let's retrieve those numbers. Ok, so we write this information in one of the circles, in one part of the circle or in one of the bars.

- What don't we know? The number of children in total.

- Let's write an interrogation sign there.

- What are we asked to find? Just that!





Step 6: Let's Recall

This final step is very important to ensure that all of our pupils have comprehended the problem. And it should be done among all the children.

- So, there is nothing else to add, isn't there? let's summarize looking at our representation:
- What does the problem talk about? The children wear different footwear: boots and shoes.
- Do we count every single shoe or boot? No, we count people wearing a particular type of footwear.
- Which means that we have two types of people: the ones wearing boots and the ones wearing shoes.
- Is anyone wearing boots and shoes at the same time? No. we have drawn them separately.
- Ok, so we have two separate groups.
- Is anyone barefooted? No.
- And that means? That we are all in one group or the other.
- So, joining both groups we have the total.
- Do we know how many are there in each group? Yes
- Do we know how many are there in total? That's the question.

2nd PHASE: PLAN A STRATEGY.

It is not easy to devise a plan or to conceive the idea of a plan for a difficult problem. It will probably not emerge suddenly and it will take many trials and periods of hesitation until the correct one is developed. We have to prepare children to be able to plan by themselves, starting from the simplest problems. That is why, as Pólya (1945) says, it is very important that teachers procure for their pupils unobtrusive help. Later the child will face the problem alone by putting into practice the skills they have learned. It is also important to keep in mind that they might not know what they have to do considering that they have little knowledge of the subject. As De-Guzmán says (2006), if the problem is too challenging we can create a similar one which is easier to solve and it is based on past experience. This particular problem will be their base, their experience, for all the others that will come. We are indeed planting the seed with care, patience and details.

By asking some of the following questions, students will be able to come up with a plan that will allow them to solve the problem.

Step 7: Devise a plan

It is convenient that in this phase children use their graphic representation, this will help to check each step while they develop their plan. They may also use snap-cubes, lego blocks, beans, beads... and small boxes to place on the diagram.

- Do you know what to do? I am not asking about maths! I mean what to do with us, here, in the class? No
- Let's try it on our desks. Let's suppose that we have just 4 children wearing boots and just 3 children wearing shoes. Draw them on the diagrams that you had.
- Now, can you answer the question of the problem? How many children are there? Yes, there are 7 children.
- Great. What have you done? I have counted them all.
- Very well, but for doing so, what have you done first? I have joined them together.

- Excellent. So, the solution has two steps: put together, and count? Yes.

- Let's do it with the original problem: put together and count. But you said you didn't want to count us all!

- Yes, I did. We will cross that bridge when we arrive.

- Now let's write down our plan "a) put together the sets of children with boots and children with shoes; b) count this total set."

3rd PHASE: CARRYING OUT THE PLAN.

Being able to carry out a plan, mean we must follow all the steps and then swap them or translate them into mathematical language. Associate the verb (join, put together…) to the arithmetic operation.

Step 8: Which mathematical content do we need?

At this step children will have to "translate into maths" the manipulations described in the previous section, exactly those manipulations that they have sequenced before, in that order, and realise if they are able to get a sequence of partial results that leads to the final answer.

For doing so, we should provide them before with the relational comprehension Skemp, (1989) of the arithmetic operations. Then we will be able to stablish this conversation:

- Which mathematical operation means "put or join things together and count"? Addition

- So, let's translate into maths our solution: we have to put together and count two groups means: that we have to add up two quantities, it is, the solution is 11+10? Yes, "11 + 10 children are in the classroom" is the solution.

- Would you be able to get that number without an algorithm? Yes/No

- If not, which algorithm should we use? The addition algorithm.

Step 9: Answer the problem's question.

- Ok, have you joined the objects together? Yes

- Which is the answer? 21

- 21 of what exactly? 21 footwear

- Are you sure? Let's recall, what were we counting? People wearing footwear

- Which means? 21 people wearing footwear.

- Does "21 people wearing footwear" answer our question? The question was? How many children are there in the classroom?

- So the correct answer is? There are 21 children in the classroom.

4th PHASE: LOOKING BACK.

This final phase is key to assure that children have acquired sufficient knowledge so that they can start making relations between problems and different ways of solving them. By looking back and re-examining the result and the steps that they have followed, children will be able to consolidate their knowledge and develop the ability to solve future problems. It is also a very useful resource for the teachers as they can assess and evaluate what children have learnt.

Step 10: Does my answer fit the problem?

- Have we given an accurate answer to our problem? Yes, we said how many people are in the classroom.

- How many children are there usually in the classroom? 23 children

- So, is 21 a possible answer? Yes, because if we are 23 in class, 21 is close.

- Could the answer be 24? No, because we are 23, so the answer could be 23 or any number below that.

- Can you verify the result? Yes, because there are two pupils missing today.

Step 11: Could we improve the strategy?

- Could we have obtained the answer through an alternative path? Yes, taking into account the two children that are absent today.

- And if we would have been in the playground or the canteen? No, it would have been very difficult because some children could be in the classroom doing other subjects.

- So, what we have done seems to be the best way to solve this problem.

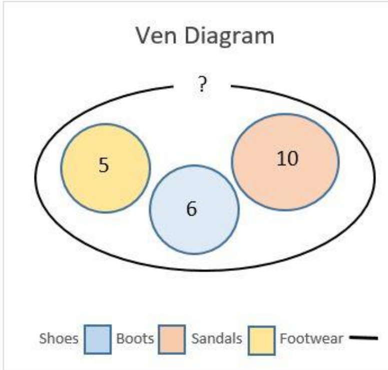
Step 12: Could I create a similar problem?

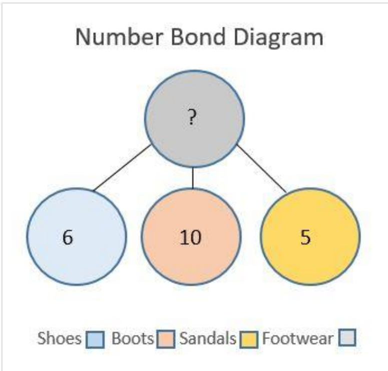
- Can you construct a similar problem using the same structure?

At this point children can think and propose similar problems and tell them to their partners.

- And what would happen if some children were wearing sandals today? Could we use the same graphic representation?

We allow children to discuss about the new situation and on how they could incorporate a new category to their plan. At this point we could also present the same problem but using three categories instead of two for instance, 10 children wearing boots, 6 children wearing shoes and 5 children wearing sandals.





Step 13: What if the numbers are much bigger?

- Could we apply the same strategy if the numbers were much bigger, for instance all the children in the school? Yes/ No/ It would be very difficult to count……

- Would we have to change many of the steps that we have followed? Yes/No

- Would we still have two categories one for people wearing boots and another one for people wearing shoes and a one for all footwear? Yes

- So, would we have to change our graphic representation? No

- If the amount of people wearing shoes was higher would that have an effect on the people wearing boots? No

- So, are the relations among the categories different? No

- So, what are the differences? That the numbers are much higher

- In order to find out how many students are there in the school, could we apply the same strategy? Yes, join things together. But we will not count, it's boring!

- The translation into maths of our strategy would be different? No, we will have to make an addition. But we do not know how to add up 200 and 500!!!

- Ok, and what could we do? An algorithm/ use a calculator…

Step 14: What if the unknown is somewhere else?

We take some of the problems that the children have proposed and vary the knowns and the unknown, kind of "I have bought chocolate and strawberry ice creams, 12 in total. Only 4 of them have chocolate flavour. How many strawberry ice creams have I bought?" We must discuss with them all the steps that we have followed and discover whether they can be applied exactly the same or nor. We find a change in step 4, because the question is now on one of the groups. This changes the strategy, and it will be discussed later, in a following problem.

Step 15: What have I learned?

It's time for making the children to use the problem as a tool of knowledge. All through the previous steps followed we have provided them some skills for facing and solving problems. But the problem itself is a piece of abstract knowledge that we have acquired. We have to make the child aware of what they have learned. This is what learnability means.

- I have learnt to design and carry out a plan for each time I have a group of things that can be separated into two different groups.

- I have learnt how calculate the total amount of something if I know the amount of things in each of its parts.

- In order to begin, I only need to figure out what item appears in the problem in general and the item’s types.

- It is important to see how the items are related to each other.

- I do not need to know the answer from the very beginning. There are many things to do before and I will always be able to do those things.

- For making a plan I can take small numbers and paint the problem.

- I can always know if the result is acceptable.

<u>Similar problems</u>

Below are some similar problems related to “My Grandfather the Shoemaker” that could be done as a follow up in this activity.

A) “So this morning when I came into the classroom I began counting how many children were wearing shoes and how many children were wearing boots, but I got distracted from counting because of a sudden noise and could not count all of them. And as you know, I really need to figure this out. Could you help me, please? But do not stand up. Let’s find out with the pieces of information that we already have!”

B) “So this morning when I came I began counting how many children were wearing shoes and how many children were wearing boots in the other classroom. And now I do need to know if all of them have come to class today. There should be 25 children. Could you help me, please? As we cannot interrupt them now, let’s find out with the pieces of information that I already have!”

Activity 2 “My favourite Ice-cream”

Objectives and Outcomes:

The particular intention behind this activity is to continue strengthening the understanding of addition and subtraction and to recognise the part-whole relation of a word problem. However, in this problem the unknown is in one of the parts therefore children will learn to develop a strategy that will lead them solve problems in which the whole and one of the parts are given and the other part is unknown. We will also continue practicing the procedure for understanding Word-problem and pictorial representations.

Procedure:

a) Creating a positive classroom atmosphere

“What’s your favourite thing to eat during the summer? Can you guess which one is mine? Yes/No. Well let me tell you, it’s ice-cream. Who doesn’t like ice-cream? And what’s your favourite flavour? Mine is chocolate and my brother’s is strawberry, and we don’t like any other flavour. I only like chocolate and my brother only likes strawberry. So my mum always has to buy two boxes of ice-crams”.

It is at this moment when pupils will probably start talking about their favourite ice-creams. After 3 minutes, they are in the right atmosphere. Then it’s time to propose the problem.

b) Proposal of the problem:

“So let me tell you what happened to me the other day. My mum came home with a box of 12 ice-creams for both of us, she said that all the ice-creams in the box were either chocolate or strawberry. We were both so excited, we both had our favourite ice-creams in the same box. Although it was before lunch, my brother couldn’t resist it and opened the box. He took all of the strawberry flavoured ice-creams out of the box and he said that there were 7, suddenly my mum came into the kitchen and took all the ice-creams, put them back in the box and into the freezer. She got very cross!!! But my problem was that I didn't get to count how many chocolate ice-creams were in the box, so I don't know how many are for me. Could you help me please? I really want to find out."

We continue the lesson by writing down the problem on the board.

"My mum has bought a box of 12 ice-creams that are either chocolate or strawberry ice-creams. 7 of the ice-creams are strawberry. How many chocolate ice-creams are there in the box?"

c) Solution Process: Following Pólya's phases and steps.

1st PHASE: UNDERSTAND THE PROBLEM.

Step 1: Finding the problem's topic

First of all, we scratch the numbers of the problem on the board, so that we know that there are numbers there, but we do not see them.

We ask the pupils to think how they could perform the story. We continue asking questions until one of them says that the problem talks about two different types of ice-creams: chocolate and strawberry.

The questions for this problem could be like this:

- What's the problem about? You and your brother's favourite ice-creams.
- And what would happen if we swapped my brother and I for you and your brother? Is there any information that would not change? Yes, the strawberry ice-creams the chocolate ice-creams and the box of ice-creams.
- Great, so in particular? Types of ice-creams.

- Awesome, and which categories (in this example “types of ice-creams”) does the problem state? Strawberry and chocolate.

- So we count strawberries and bars of chocolates? No, we count strawberry ice-creams and chocolate ice-creams.

- Are there any more groups? Yes, the box with all the ice-creams.

Step 2: Discover the Relations

Is there any relation among the categories? What does that mean?

- Is the reason that I like chocolate ice-cream, because my brother likes strawberry ice-creams? No, you like chocolate and your brother could like any other flavour.

- Does the number of chocolate ice-creams depend on the number of strawberry ice-creams? What does that mean?

- Imagine that the box didn’t specify the flavours, would the amount of one flavour depend on the amount of the other? Not at all; they could all be chocolate or they could all be strawberry or any other flavour.

- And could the number in each category be any number? Yes it could be 0.

- And could they be as big as we want? No, they cannot be bigger that the number of ice-creams in the box.

- So, between both of them they cannot be bigger than the amount of ice-creams in the box, right? Yes.

- Great. So, we have a box of ice-creams with two groups. One of the groups is the amount of strawberry ice-creams and the other group is the amount of chocolate ice-creams. We know that the numbers could be 12 or smaller. And we know that in the box we have all the ice-creams, right? Yes.

Step 3: Representation

At this point we hand out to the students a piece of paper and crayons or pencils.

- Does anyone recall a similar problem were we had a bigger group and two independent groups? Yes, the one about your grandfather the shoemaker.
- Yes! You are right. Could we use that representation again for this problem? Yes /no
- Let’s check!.
- Are the two groups (chocolate ice-crams and strawberry ice-creams) independent? Yes.
- Do they both make the total amount (whole)? Yes.
- Is the known represented? Yes.
- Is the unknown represented? Yes.

For some examples of different types of representations refer to previous activity.

Step 4: Add the information to the graphic representation.

After each pupil has decided and drawn their desired diagram, then it is time to add the information that we have. We ask the following questions to guide the pupils in their procedure.

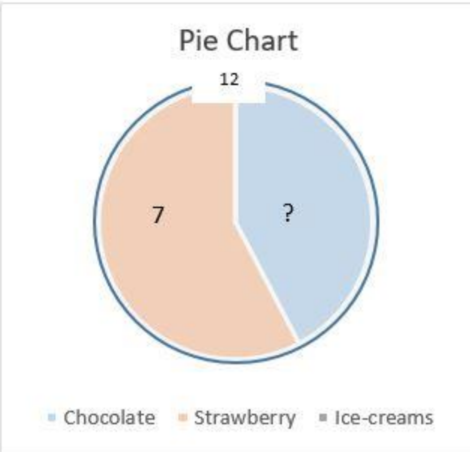
- So, how many magnitudes do we have? Three: the total amount of ice-creams in the box, the chocolate ice-creams and the strawberry ice-creams.
- Which ones do we know? None, you have crossed them out!

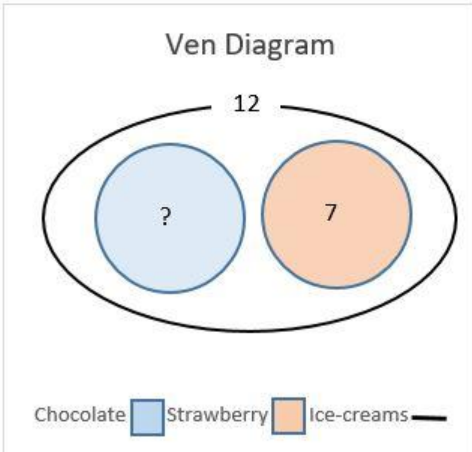
- Which ones do we know, although we do not see them? The total amount of ice-creams and the amount of strawberry ice-creams.

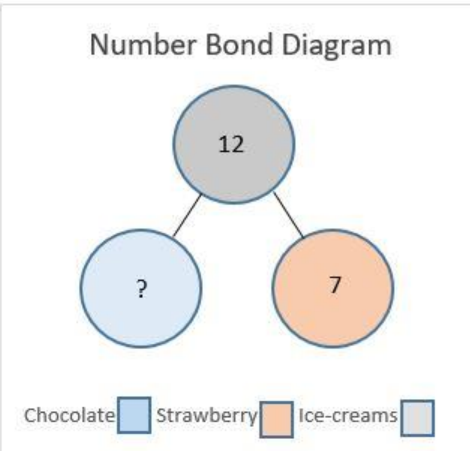
Let's retrieve those numbers.

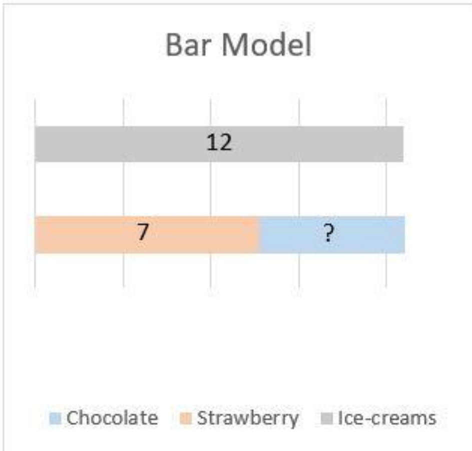
- Ok, so we write this information down.
- What don't we know? The amount of chocolate ice-creams that are in the box.
- Let's write an interrogation sign there.
- What are we asked to find? Just that!

For example:









Step 5: Let's Recall

This final step is very important to ensure that all of our pupils have comprehended the problem. And it should be done among all the children.

- So, there is nothing else to add, isn't there? Yes/No.
- let's summarize looking at our representation:
- What does the problem say? Different types of ice-creams in a box: chocolate ones and strawberry ones.
- Are there any ice-creams in the box that can be chocolate and strawberry at the same time? No, we have drawn them separately, they can either be chocolate or strawberry.
- Ok, so we have two separate groups.
- And, joining both groups we have the total.
- Do we know how many are there in each group? No, we only know one of them.
- Do we know how many are in there other group? No, that's the question.
- Do we know how many are altogether? Yes, the box has 12 ice creams altogether.

2nd PHASE: PLAN A STRATEGY.

Step 6: Devise a plan

It is convenient that in this phase children use their graphic representation, this will help them to check each step while they develop their plan. They could also use snap-cubes, lego blocks, beans, beads... and small boxes to place on the diagram.

- Do you know what to do? I am not asking about maths! I mean what to do with the ice-creams? No.

- Let's try it on our desks. Let's suppose that we have just 4 ice-creams in total and just 3 of them are strawberry. Take some snap-cubes, let's pretend they are ice-creams.

- Now, can you answer the question of the problem? Yes.

- How many are chocolate ones? 1!

- Great. What have you done? I have looked at the ones that are left.

- Great, but what have you done before that? I have separated 3 from the four that I have.

- Why? Because if I separate 3 from the total amount I know how many are chocolate.

Now let's write down our plan "a) separate from the total amount of ice-creams the ones that are strawberry; b) count what is left"

3rd PHASE: CARRYING OUT THE PLAN.

Being able to carry out a plan, means we must follow all the steps and then swap them or translate them into mathematical language. Associate the verb (join, put together, split …) to the arithmetic operation.

Step 7: Which mathematical content do we need?

In this step, children will have to “translate into maths” the manipulations described in the previous section, exactly those manipulations that they have sequenced before, in that order, and realise if they are able to get a sequence of partial results that leads to the final answer.

- Which mathematical operation means “separate an amount of objects from a bigger amount of objects, and look at what is left”? Subtraction.

- So, let’s translate into maths our solution: Separating one set of things form a bigger set of things means, subtracting, and then we look at what is left right? Yes.

- What do we take away? The strawberry ice-creams.

- From where? From the box that has the ice-creams altogether.

- How many ice creams are there altogether? 12.

- And how many strawberry ice creams? 7.

- So, who can tell me how to do it? Me, “12 - 7 chocolate ice-creams” is the solution.

- Would you be able to get that number without an algorithm? Yes/No.

- If not, which algorithm should we use? The subtraction algorithm.

Step 8: Answer the problem’s question.

- Did we take away a part from the total? Yes.

- Which operation have we done? 12-7.

- How has each one of you done the operation? Counting down; subtracting first 2 and then 5; separating 12 as 10+2, subtracting 5 from 10 and 2 from 2; separating 12 as 10+2, subtracting 7 from 10 and adding the other 2; from 7 to 10 I need 3 so for arriving to 12 I need 5...

For a deeper understanding on how to subtract with carries refer to *2.3.4 Addition and Subtractions of Natural Numbers in the Theoretical Framework.*

- Great, and have you looked at what it is left? Yes.

- What's the answer? 5.

- 5 what? 5 chocolate ice-creams.

- Does "5 chocolate ice-creams" answer our question? The question was? How many chocolate ice-creams are there in the box?

- So the answer is? There are 5 chocolate ice-creams in the box.

4th PHASE: LOOKING BACK.

This final phase is key to assure that children have acquired sufficient knowledge so that they can start making relations between problems and different ways of solving them. By looking back and re-examining the result and the steps that they have followed, children will be able to consolidate their knowledge and develop the ability to solve future problems. It is also a very useful resource for the teachers as they can assess and evaluate what children have learnt.

Step 9: Does my answer fit the problem?

- Have we given an accurate answer to our problem? Yes, we said how many chocolate ice-creams were in the box.

- How many ice-creams are there in the box? 12.

- So, is 5 a possible answer? Yes, because it is smaller than 12.

- Could the answer be 24? No, because there are 12 ice-creams in the box.

- Can you verify the result? Yes, because if we add 5 and 7, it makes 12.

Step 10: Could we improve the strategy?

- Could we have obtained the answer through an alternative path? Yes, asking your mother.

- But if my mother was working, could I have guessed the amount? No.

- So, what we have done seems to be the best way to solve this problem.

Step 11: Could I create a similar problem?

- Can you construct a similar problem using the same structure?

At this point children can think and propose similar problems and tell them to their partners.

Step 12: What if the numbers were much bigger?

- Could we apply the same strategy if the numbers were much bigger, for instance a box with 50 ice-creams? Yes/ No/ It would be very difficult to count……

- Would we have to change many of the steps that we have followed? Yes/No.

- Would we still have two categories, one for strawberry ice-creams and another one for chocolate ice-creams, and then another one for the total amount of ice-creams? Yes.

- So, would we have to change our graphic representation? No.

- If the amount of strawberry ice-creams was higher would that have an effect on the amount of chocolate ice-creams? No.

- So, are the relations among the categories different from each other? No.

- So, what are the differences? That the numbers are much higher.

- In order to find out how many ice-creams were chocolate, could we apply the same strategy? Yes, separate the amount of strawberry ice-creams from the total amount of ice-creams and look at what is left.

- Would the translation into maths of our strategy be different? No, we will have to make a subtraction. But if the numbers were much higher like "187 – 85 "we wouldn't know how to do it!!!

- Ok, and what could we do? An algorithm/ use a calculator…

The most important thing is that we know what to do, even though we cannon calculate it right now without a calculator. So, we know how to solve this problem, and many other similar ones.

Step 13: What if the unknown is somewhere else?

We take some of the problems that the children have proposed and vary the knowns and the unknown, kind of "I have bought a bag of fruits. 14 of the fruits are bananas and the other 18, are apples. How many fruits have I bought?" We must discuss with them all the steps that we have followed and discover whether they can be applied exactly the same or not. We find a change in step 4, because the question is now on the whole. This changes the strategy, and was discussed in the previous activity.

Step 14: What have I learned?

It's time to make the children use the problem as a tool of knowledge. All through the followed previous steps, we have provided them with some skills for facing and solving a new simple problem. But the problem itself is a piece of abstract knowledge that we have acquired. We have to make the child aware of what they have learned. This is what learnability means.

- In the previous activities I learnt, how to design and carry out a plan for each time I have a group of things that can be separated into two different groups. And I can apply it here even though the question is different.

- These two problems are very similar, although the operation used is a subtraction and not an addition.

- The only difference was that the unknown was not the total, but one of the two parts.

- The procedure that I learnt solving the Shoemaker problem has been very useful in this new problem.

- I can summarize that if I know the parts, the total amount is the addition of the parts and if I only know one part, but I know the total, the other part can be obtained with a subtraction.

- It’s clear that addition and subtraction are operations that appear in problems that can have the same design. The difference lies is in the unknown.

Similar problems:

Below there are some similar problems related to “My favourite Ice-cream” that could be done as a follow up in this activity.

A) “So yesterday my mum bought another box of chocolate and strawberry ice-creams for my brother and for me. As always, my brother took out all the ice-creams out of the box, but this time my mother didn’t catch us! So, my brother counted 3 strawberry ice-creams and I counted 9 chocolate ice-creams. I was the lucky one this time! But we forgot to count how many ice-creams where in total. Could you help me figure this out, please?

B) "So yesterday my mum bought another box of chocolate and strawberry ice-creams for my brother and for me. As always, my brother took out all the ice-creams out of the box. So, my brother counted that there were at least 3 strawberry ice-creams and I counted at least 9 chocolate ice-creams. I think that I was the lucky one this time!

Activity 3 "What happened to my favourite stones?"

Objectives and Outcomes:

The particular intention behind this activity is to practise before-after word-problems. Children will begin to understand that in this type of problems there is a before situation represented by a number of units, a middle situation that represents that something has changed (or either you win or lose some unites) and a new situation afterwards. They will also begin establish relations between before-after problems and part-whole problems.

We will continue to practise the concept of any natural number by its additive decompositions and the process of developing a suitable pictorial representation of a Word-problem.

Procedure:

a) Creating a positive classroom atmosphere

"Do you remember the problem we did a few weeks ago about my obsession with counting shoes? Well, I must confess that I have another obsession which is collecting rare and unique stones. For example, I have a stone with the shape of a heart, another one which looks like a diamond, one that looks like a house and many more. I keep them in a treasure box in my room so none of them gets lost because is very hard to find them and whenever I have lost one I get really sad. Do you have a collection of stones? No? And of any other object? Like marbles or toy figures...? Yes. And have you ever lost any?"

At this point, pupils will probably start talking to each other about their collections and how they feel when they lose any of them. After 3 minutes, they are in the right atmosphere. Then it's time to propose the problem.

b) Proposal of the problem:

"Well, let me tell you what happened to me yesterday. I told my best friend that I had a big collection of rare stones and that they were very special to me. She told me that she wanted to see them, so I decided to take seven of my favourite ones to her house. One looked like a heart, another one looked like a car, another one like a cat, there is another one that looks like a dinosaur, another one that looks like a diamond, another one like a house and the last one was green and orange and looked like a hill covered with flowers. So I placed all the seven stones in my pocket and I walked to her house but when I got there and I put my hand inside my pocket, I could only find two of them, I took them out, I took my jacket off, I pulled the pocket inside-out, but nothing I only had two stones. I felt so sad…. However my problem is that I don't know what happened. I don't know how many I lost? Could you help me, please?"

We continue the lesson by writing down the problem on the board.

We read: "Yesterday, I took 7 stones from my house to show my best friend, so I put them in my pocket but when I got to my friend's house I had only 3 stones in my pocket. How many stones did I lose?", and we write "I had 7 stones at my home. I had 3 stones at my friend's house".

c) Solution Process:

1st PHASE: UNDERSTAND THE PROBLEM.

Step 1: Finding the problem’s topic

First of all we scratch the numbers of the problem on the board, so that we keep in mind that we know the number of stones I had at my home, and the number of stones that I had when I arrived to my friend’s house. We also keep in mind that I had more before than later on.

These questions could be like this:

- What does the problem say? That you went to your friend’s house to show her your stones.
- Awesome, and which categories does the problem mention? Just the stones that you have in your pocket.
- Does that magnitude change in time? Yes, before it was bigger. Later smaller.
- So, what do you think happened? That on your way to your friend’s house you lost some stones.
- So, I had some stones, lost some of them and some others remained in my pocket. Do you think I can draw it? What about in comic format? Yes, we could make three pictures.
- Why three? At my home, along the way and at my friend’s.
- Ok, we will make a comic with three captures.

Now we encourage our pupils to draw such a comic.

Step 2: Discover the Relations

- Did I have at home the stones that I lost? Yes.

- So, is there any relation between the stones that I had at my home and the ones that I lost? They are part of them.

- Did I have at home the stones that I finally showed to my friend? Yes.

- Is there any relation between the stones that I had at my home and the ones that made it to my friend's house? They are also part of them.

- So I had some stones. A part of them made it to my friend's house, and the rest got lost. Right? Yes.

- Great. So, I had a set of stones. A part of them is the stones that I lost and the other part is the set of stones that I had when I arrived to my friend's house.

- And we know that when I left the house I had all the stones from both groups, and no other stone, right? Yes.

- Is it possible to lose a stone but then show it to my friend? Only if you find it again.

- Ok, but I didn't look for it before seeing my friend, so? No, the lost ones were lost.

Step 3: Representation

- Is some of this similar to any other problem? Yes, the problem with the ice-creams.

- Why? Because you had a lot of ice creams in a box and your brother took the strawberry ones, and only the chocolate ones remained. Yes, the box is like your pocket!

- Yes, right! So, could we rescue the picture we made there for this new situation?

- But with the ice creams we talked about the total and the parts. Which is the set now that is split into two subsets? The stones that you had at the beginning.

- Ok, we have the whole. Which are the parts? The lost stones, and the stones you did not lose.

- Right. Try a picture of that.

For any of their representations we discuss whether all the information of the problem is or not represented:

- Is the whole (total amount of stones) represented?

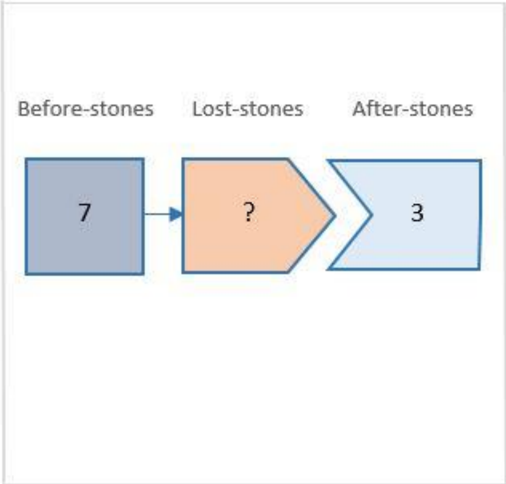
- Are the two groups (situations) independent?

- Is the unknown represented?

- Is the known represented?

- What have we been asked to find?

Likewise, we must give them some different examples if they are not acquainted with the different types of graphic representations.



Step 4: Add the information to the graphic representation.

After each pupil has decided and drawn their desired diagram, then it is time to add the information that we have. We ask the following questions to guide the pupils in their procedure.

- So, how many magnitudes do we have? Three: Total amount of stones, the stones that you lost and the stones that you had when you arrived to your friend's house.

- Which ones do we know? The total amount of stones and the stones that you had when you arrived to your friend's house.

- Can anyone think of a different type of representation that has two independent groups and the total amount? Yes, the Number-bonds.

- And can you recall any problem in which we know the total amount and only one of the parts? Yes, the one about your favourite ice-cream.

- Fantastic!

Let's retrieve those numbers. We write the numbers on the board.

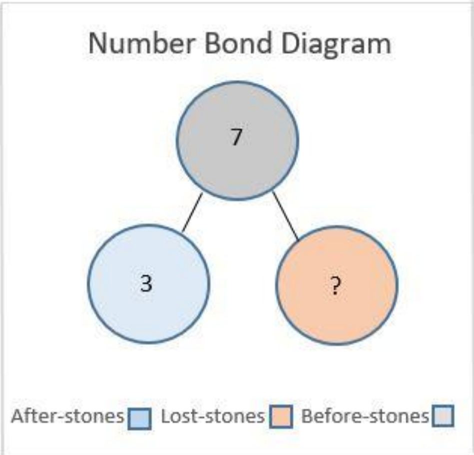
- Ok, so we write this information down in our diagrams.

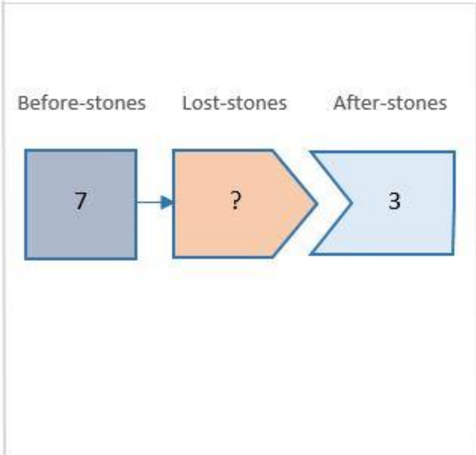
- What don't we know? The amount of stones that you lost.

- Let's write an interrogation sign there.

- What are we asked to find? Just that!

For instance,





As you can see by the Number Bond Diagram, Basic Before-After problems can also be transformed into Part-Whole problems. One of the parts being unknown and the whole being known.

Step 5: Let's Recall

This final step is very important to ensure that all of our pupils have comprehended the problem. And it should be done among all the children.

- So, there is nothing else to add, isn't there? let's summarize looking at our representation:

- What does the problem say? Your Stones.

- And what else? The stones that you lost and the stones that you had left.

- Which means? That there are two different groups, that's why we have drawn them separately.

- Ok, so we have two separate groups.

- Could I have some of the stones I lost in my pocket when I got to my friend's house? No, that's impossible.

- And that means? That all stones are in one group or the other.

- Do we know how many are there in total? Yes.

- Do we know how many are there in one group? Yes.

- Do we know how many are there in the other group? No, that's the question.

2nd PHASE: PLAN A STRATEGY.

By asking some of the following questions, students will be able to come up with a plan that will allow them to solve the problem.

Step 6: Devise a plan

It is convenient that in this phase children use their graphic representation, this will help them to check each step while they develop their plan. They could also use snap-cubes, lego blocks, beans, beads... and small boxes to place on the diagram.

- Do you know what to do? I am not asking about maths! I mean what to do with us, here, in the class? No.

- Can anyone recall a similar problem where we had to separate an amount of things from a bigger amount of things and look at what was left? Yes, when you didn't know how many chocolate ice-creams were in the box.

- Great! And can anyone remember what steps did we follow? Yes.

- Tell me. It had two steps: separate and look at the ones that are left.

- Fantastic!!!

Now let's write down our plan "a) separate from the total amount of stones the stones that I had when I got to my friend's house; b) count what is left"

3rd PHASE: CARRYING OUT THE PLAN.

Step 7: Which mathematical content do we need?

At this step children will have to "translate into maths" the manipulations described in the previous section, exactly those manipulations that they have sequenced before, in that order, and realise if they are able to get a sequence of partial results that leads to the final answer.

- Who remembers which mathematical operation means "separate one thing form another and look at what's left"? Subtraction.

- So, let's translate into maths our solution: to separate one thing from another and look at what is left, means subtract from the total part.

- Yes, so? "7-3 are the stones that you lost" That's the solution.

- Would you be able to get that rest? Yes/No.

- How? Counting back; separate 7 into 5+2 and take away those 2 and 1 from 5; we know the result; we know that 7 is 3+4...

Step 8: Answer the problem's question.

- Ok, have you separated a part from a total set? Yes.

- Have you looked at what was left? Yes, the other part.
- What's the answer? 4.
- 4 what? 4 stones are lost.
- Does "4 lost stones" answer our question? The question was? How many stones have I lost?
- So the correct answer is? You have lost 4 stones.

4th PHASE: LOOKING BACK.

Step 9: Does my answer fit the problem?

- Have we given an accurate answer to our problem? Yes, we said how many stones you had lost.
- How many stones did I have when I left the house? 7 stones.
- How many stones did I have when I got to my friend's house? 3.
- Could the answer be 8? No, you had only 7 stones, so the answer has to be smaller than 7.
- So, is 4 a possible answer? Yes, because it is smaller than 7, and you cannot loose stones that you did not have.

Step 10: Could we improve the strategy?

- Could we have obtained the answer through an alternative path? Yes, by trying different numbers from 1 to 6, adding 2 and see which would have given us 7.
- Would have you got the answer faster? No.

- So it seems that we chose the best strategy.

Step 11: Could I create a similar problem?

- Can you create a similar problem and give it to me? Yes

Once everyone has written a similar problem the teacher hands them out randomly and they will have to solve them following the same steps that we have followed before. In this activity we will ask our students to correct the problems that they have created.

Step 12: What if the numbers were much bigger?

- Could we apply the same strategy if the numbers were much bigger, for instance all my stones? Yes/ No/ It would be very difficult to count……
- Would we have to change many of the steps that we have followed? Yes/No.
- Would we still have two categories: the stones that I had when I left the house and the stones that I had when I got to my friend's house? Yes.
- So, would we have to change our graphic representation? No.
- If the amount of stones that I had when I got to my friend's house was much bigger would that have an effect on the stones that I had when I left the house? No.
- So, are the relations among the categories different? No.
- So, what are the differences? That the numbers are much bigger.
- In order to find out how many stones I had lost, could we apply the same strategy? Yes, separate and look at what is left.

- The translation into maths of our strategy would be different? No, we will have to make a subtraction. But we do not know how to subtract 24 from 78!!!

- Ok, and what could we do? An algorithm/ use a calculator…

Step 13: What if the unknown is somewhere else?

We take some of the problems that the children have proposed and vary the knowns and the unknown, kind of "I left the house with an amount of stones, on my way to my friend's house I lost 5 and when I got to her house I had 7 stones. How many stones did I had when I left the house?" We must discuss with them all the steps that we have followed and discover whether they can be applied exactly the same or not. We find a change in step 4, because the question is now about the total amount and It can be related to Activity 1.

Step 14: What have I learned?

- When I loose things, the set of things that I had before is split into two subsets: the lost objects and the elements that remain. The things that I had at the beginning is the total, the others are its parts.

- I do not need to know the answer from the very beginning. There are many things to do before and I will always be able to do those things again. For instance I can try to adequate the scheme of other problems that looked different, but had the same representation.

- A picture of the problem or a role play telling what has happened can help me to figure out a plan.

- I can always know if the result is acceptable: one part is always smaller than the total.

Similar problems:

Below are Similar problems of “My Grandfather the Shoemaker” that could be done as a follow up in this activity.

A) “Yesterday I left the house with 25 stones to show my best friend, on the way to her house I gave as a present 7 of the stones to a really nice boy. Could you help me find out how many stones I had when I arrived at my friend’s house?”

B) “I left the house with an amount of stones, but I do not remember how many there where. On the way to my friend’s house I realized I had left 8 stones at the bus stop and when I got to my friend’s house I had 16 stones. And now I do need to know how many I had when I left the house. Could you help me, please? Let’s find out.”

C) “ I left my house with an amount of stones, and as I was walking to my Friends house I found three beautiful stones so I picked them up and put them in my pocket. When I got to my friend’s house I had 15 stones. I can’t remember how many stones I had when I left the house, can you help me, please?

D) “This morning I left the house with 17 of my favourite stones and I put them in my pocket. Has I was on my way to school I found 5 stones that I really liked so I decided to pick them up and put them in my pocket. Now I need to know how many stones did I have when I got to school, can you help me, please?

Activity 4 “Who is faster?”

Objectives and Outcomes:

With this activity children will learn the relations established in an Additive Comparison Problem. These type of problems are different from the previous ones as there are only two magnitudes. Children will begin to understand that in order to solve these problems they need to compare both magnitudes and eventually discover that there is a third quantity that is given when compared the two magnitudes.

Procedure:

a) Creating a positive classroom atmosphere

"Do you like running? Yes, and do you like racing with your friend? Yes? I like running too, but what I really love is racing my friends and see who can run longer. I love running."

At this point, pupils will probably start talking to each other about running and racing. After 3 minutes, they are in the right atmosphere. Then it's time to propose the problem.

a. Proposal of the problem:

"Well let me tell, because I have a big problem. The other day I went with my best friend to our village and we decided to do a competition. The competition was about who could run more kilometres in half an hour. So we set our watches on and we started running, she is usually faster than me but I can run for longer distances so I wasn't worried. However, when we finished I had run 7 kilometres and my friend said that she had run 12 kilometres. Who won? How further did one of us arrive?"

We continue the lesson by writing down the problem on the board.

"I ran 7 kilometres and my friend ran 12 kilometres. Who run further? How many kilometres did my friend run more than me? "

b. Solution Process:

1st PHASE: UNDERSTAND THE PROBLEM.

Step 1: Finding the problem's topic

First of all, as usual, we scratch the numbers of the problem on the board, so that we know that there are numbers there, but we do not distinguish them. We continue asking questions until a child says that the problem talks about comparing the kilometres that I ran and the kilometres my friend ran.

These questions could be like this:

- What does the problem say? That you and your friend were racing and that you ran some kilometres and your friend ran some more.

- Why did she ran more? Because 12 is bigger than 7.

- Why? Count: 7,8,9, 10, 12.

- So, a number is bigger if we say it at the end? Ok, let's say 12, 11, 10, 9, 8, 7. Then 7 is bigger than 12? NO, paint them, you will see that you need more for painting 12.

- I believe you, but I would like to see that. Please, show it to me.

They can paint, take cubes, pencils, whatever they want. But they have to prove that actually 12 is bigger than 7.

- Ok, now I see that in fact 12 is bigger than 7 and so she ran more than me. So she won. But, was it a tight or a major victory? What does that mean?

- Was there a big or a small difference between us? Very big, 12 is much bigger than 7!

- How big? That's what we have to discover.

- Great! Now imagine that we swap the kilometres one person runs for the kilometres the other person runs, what information still does not change? The fact that one person ran more kilometres than the other in the same time.

- Fantastic! So which categories does the problem talk about? The kilometres that two people run in one hour.

Step 2: Discover the Relations

- Is there any relation among the categories? No, they run as much as they can.

- Can those numbers be as big as we want? Yes, if you run faster and faster, you can run many more kilometres, as many as you want.

- But humans run at 35 km/h in average. And the fastest man in the world, Usain Bolt, can run at a maximal speed of 44'7 km/h but just for 10 seconds. I've read that the marathon record speed is 21 km/h. So, I think that none of us can run more than 15 or 17 kilometres in half an hour. Wow, now I realise that my friend is really very fast...

Step 3: Representation

At this point we hand out to the students a piece of paper and crayons or pencils.

- Could we draw somehow the length that I ran and the length that my friend ran? We can paint on a map the race.

- Yes, it's a good idea. But we ran along a road that goes out from the city. It was a straight line all the time.

- Ok, let's see. How could we represent the length that both of us ran? Maybe drawing lines in two different colours.

- Which one will be longer? Your friend's one.

- Which part of the way did we both ran? All the kilometres you made.

- Which part of the way was run just by my friend? From where you arrived to the point where she arrived.

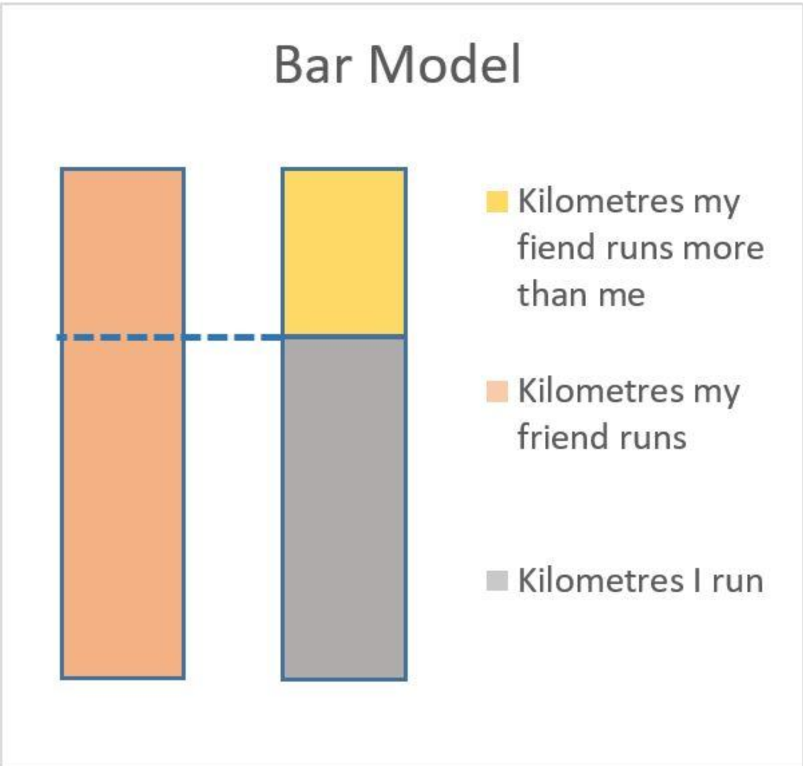
- So which is the difference between her race and mine? These kilometres that you did not ran.

- Great! Let's represent it now.

For any of their representations we discuss whether all the information of the problem is or not represented:

- Is the whole (total amount of kilometres) represented?
- Are the two groups independent?
- Is the unknown represented?
- Is the known represented?
- What have we been asked to find?

Likewise, we must give them some different examples if they are not acquainted with the different types of graphic representations.



Step 4: Add the information to the graphic representation.

After each pupil has decided and drawn their desired diagram, then it is time to add the information that we have. We ask the following questions to guide the pupils in their procedure.

- So, how many magnitudes do we have? We had two, but we have painted now three: The total amount of kilometres your friend ran, the kilometres you ran and the kilometres your friend ran more than you.

- Why? Because in this way we can use the models that we already know.

- Ok, great idea. We have three magnitudes now.

- Which ones do we know? The total amount of kilometres your friend ran and the amount of kilometres you ran.

- Can anyone think of a different type of representation that has two independent groups and the total amount? Yes, the Number-bonds.

- And can you recall any problem in which we know the total amount and only one of the parts? Yes, the one about your favourite ice-cream.

- Fantastic!

Let's retrieve those numbers. We write the numbers on the board.

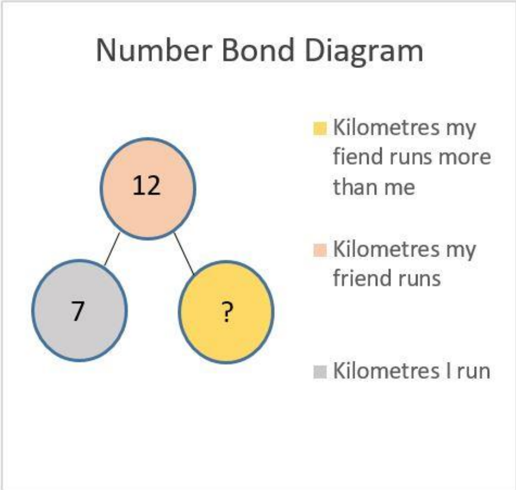
- Ok, so we write this information down in our diagrams.

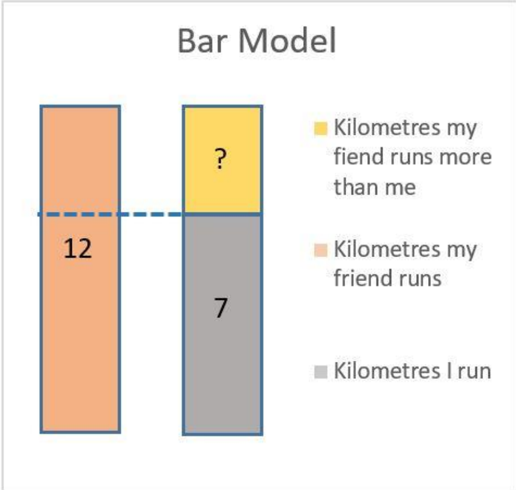
- What don't we know? The amount of kilometres that you friend ran more than you.

- Let's write an interrogation sign there.

- What are we asked to find? Just that!

For instance,





As you can see by the Number Bond Diagram, Additive Comparison Problems can also be transformed into Part-Whole problems. One of the parts being unknown and the whole being known.

Step 5: Let's Recall

This final step is very important to ensure that all of our pupils have comprehended the problem. And it should be done among all the children.

- So, there is nothing else to add, isn't there? let's summarize looking at our representation:
- What does the problem talk about? The kilometres you ran and the kilometres your friend ran.
- And what else did we discovered that was hidden? The difference between the kilometres you ran and the kilometres your friend ran.
- Which can be seen as the total now? The kilometres your friend ran.

- And the parts? The kilometres she ran like you and the kilometres she ran alone.

2nd PHASE: PLAN A STRATEGY.

By asking some of the following questions, students will be able to come up with a plan that will allow them to solve the problem.

Step 6: Devise a plan

It is convenient that in this phase children use their graphic representation, this will help them to check each step while they develop their plan. They could also use snap-cubes, lego blocks, beans, beads... and small boxes to place on the diagram.

- Do you know what to do? I am not asking about maths! I mean what to do in general? No

- Can anyone recall a similar problem where we had to separate an amount of things from a bigger amount of things and look at what was left? Yes, when you didn't know how many chocolate ice-creams were in the box.

- Great! And can anyone remember which steps we followed? Yes.

- Tell me. It had two steps: separate and look at the ones that are left.

- Fantastic!!!

Now let's write down our plan "a) separate from the total amount of stones the stones that that I had when I got to my friend's house; b) count what is left"

3rd PHASE: CARRYING OUT THE PLAN.

Step 7: Which mathematical content do we need?

At this step children will have to “translate into maths” the manipulations described in the previous section, exactly those manipulations that they have sequenced before, in that order, and realise if they are able to get a sequence of partial results that leads to the final answer.

- Who remembers which mathematical operation means “separate one thing form another and look at what’s left”? Subtraction.

- So, let’s translate into maths our solution: to separate one thing from another and look at what is left, means subtract from the total a part Is it 12-7? Yes, “12-7 are the kilometres your friend ran more than you” That’s the solution.

- Would you be able to get that value 12-7? Yes/No.

Step 8: Answer the problem’s question.

Ok, have you separated a smaller quantity from a bigger one? Yes.

- Have you looked at what was left? Yes.

- What’s the answer? 5.

- 5 what? 5 kilometres.

- Does “5 kilometres” answer our question? The question was? How many kilometres did my friend run more than me?

- So the correct answer is? Your friend ran 5 kilometres more than you.

4th PHASE: LOOKING BACK.

Step 9: Does my answer fit the problem?

- Have we given an accurate answer to our problem? Yes, we said how many kilometres your friend ran more than you.

- How many kilometres did she run? 12 kilometres.

- How many kilometres did I run? 7 kilometres.

- Then, could the answer be 14? No, because that would mean that your friend ran more kilometres and she only ran 12.

- So, is 5 a possible answer? Yes, because it is smaller than 12.

Step 10: Could we improve the strategy?

- Could we have obtained the answer through an alternative path? Yes, by trying different numbers from 0 to 11, adding 7 and see which would have given us 12.

- Would have you got the answer faster? No.

- So it seems that we chose the best strategy.

Step 11: Could I create a similar problem?

Each student creates a similar problem and then tells it to the rest of the class. The teacher writes these problems on the board for the students to choose their four favourite ones and then take them home as homework.

Step 12: What if the numbers were much bigger?

- Could we apply the same strategy if the numbers were much bigger, for instance in a marathon? Yes/ No/ It would be very difficult to count……

- Would we have to change many of the steps that we have followed? Yes/No.

- Would we still have two categories: the kilometres one person runs and the amount of kilometres another person runs more or less than the first one? Yes.

- So, would we have to change our graphic representation? No.

- If the amount of kilometres that I had run was much bigger would that have an effect on the amount of kilometres that the other person ran more than me? No.

- So, are the relations among the categories different? No.

- ▪ So, what are the differences? That the numbers are much higher.

- In order to find out how many kilometres my friend ran more than me, could we apply the same strategy? Yes, separate and look at what is left.

- The translation into maths of our strategy would be different? No, we will have to make a subtraction. But we do not know how to subtract such big numbers!!!

- Ok, and what could we do? An algorithm/ use a calculator…

Step 13: What if the unknown is somewhere else?

We take some of the problems that the children have proposed and vary the knowns and the unknown, kind of “My friend has 17 marbles and I have 5 marbles more than him. How many marbles have I got?” We must discuss with them all the steps that we have followed and discover whether they can be applied exactly the same or nor. We find a change in step 4, because the question is now about the total amount and it can be related to Activity 1

Step 14: What have I learned?

- When we compare two magnitudes the bigger one can be split into two parts: the part that equals the smaller one and the part that shows the difference between them. So the representation that we have worked is also valid here.

- It is again very, very important to see how the items are related to each other.

- I do not need to know the answer from the very beginning. There are many things to do before and I will always be able to do those things. Always painting the problem helps me to understand it. Try to solve it with small numbers makes me see the plan for solving it.

- I can always know if the result is acceptable.

Similar problems

Below is a variation of “My friend won the race” that could be done as a follow up in this activity.

A) “When we finished the race my friend didn’t know how many kilometres she had run, but I know I ran 5 kilometres less than her and I ran 8 kilometres. How many kilometres did my friend run?”

B) “When we finished the race my friend told me that she had run 8 kilometres, I know I ran 3 kilometres less than her. How many kilometres did I run?”

***

3.2.5 Evaluation and Assessment

We use the evaluation as a tool to improve the lesson plans that we have developed and to observe and analyse which strategies worked and which ones did not in the teaching-learning process.

The evaluation of this project will be formative and confirmed. Formative as we are taking into consideration the learning process of the students, their behaviour and attitudes; and confirmed as we established some initial objectives in order to evaluate our students in their learning process.

The tool that we have designed to evaluate is an observation rubric in order to establish different aspects that we consider important when learning mathematical contents.

Due to the pandemic we have not had enough time to evaluate the students' progress in understanding word-problems and developing strategies to solve them. However, during the time that we were able to attend the children we had the opportunity to observe and evaluate different aspects that are also important in the mathematical classes.

The results show that most children were curious, excited and willing to explore the new manipulative counters. They also seemed very enthusiastic and attentive when listening to the stories that we designed for the problem-solving lessons.

Regarding the understanding of the process in addition and subtraction problems, children have acquired a meaningful comprehension of the place-value concept in two-digit numbers. Even children that were not able to count, as their former teacher suggested, were able to do part-whole mathematical problems using counters.

In the following image it can be observed how this child was able to understand what this basic before-after word problem was about, to graphically express the before and after situation and answer the question representing what happened in the course between the situations. The word problem was: "This morning when I left the house I put 3 stones in my pocket but when I got to school I only had 2 stones. How many stones did I lose?"

Figure 8: Representation of a word-problem (1º E.P.)

The next picture shows how a student with intellectual disability represented the same basic before-after word-problem. Although, it does not show the relation among the magnitudes it can clearly be seen how the situation has changed when he represents the three stones and one of them is crossed out.

Figure 9: Representation of a word-problem (1º E.P.

FINAL CONCLUSIONS AND OPEN QUESTIONS

One of the driving ideas behind this Graduation Thesis, was to analyse and upgrade the teaching-learning method of mathematical lessons, in order to improve children's mathematical knowledge and skills. Using actual relational understanding of the maths which is part of their world and to enhance their attitudes towards this core subject.

These students are diverse; due to different learning capacities, abilities, styles, rhythms, culture and varying levels of interest in the subject, due to their sociocultural backgrounds. All these aspects require a need for change in current working methods. Traditional or individualistic learning means that, our students are doomed to remain on the perimeter of an educational system, which is based on the archaic views, that learning by rote and achieving results is the main goal. This is a learning process that leaves aside the education and integral growth of individuals, as well as the development of necessary skills to solve any current life problems or the ones they may face in the future.

The theoretical framework and the approach of different methodologies in the classroom, has meant a change in the mind-set of our students along with a change in the development of teaching.

In this Graduation Thesis, we have proposed the teaching of mathematical skills though word-problem solving, as the main objective. We consider the development of these skills to be essential for individuals, therefore for our learners. These skills are essential to having a better understanding of the world around them, for developing mental skills and for being able to excel socially. However, to achieve inclusivity, we need innovation, new proposals and new methodologies in schools. It has also made us reconsider our teaching practice and the need for human and professional training, to educate students who are part of a new society. A training in which interculturalism, individualised teaching and the appreciation to embrace the diversity of people, which is both positive and enriching.

Having said that, we need to take into consideration, the difficulties that these learners will continue to have because this new methodology is not common practise in schools. Methodologies based on activities where the repetition of steps and algorithms are the main point, will lead students to a meaningless sense of learning towards methodologies, where all students can learn, regardless of their additional needs to develop to their full potential.

Having been immersed in the process of accumulating all my new found knowledge for over two years, my attitude towards mathematics has dramatically changed. I can now confirm that this subject is accessible to everyone. I have also realised how time consuming it is to create motivating lesson plans, that will enhance the students desire to have a relational understanding of mathematics. If applied, these existing methods of Polya (1945) and De-Guzmán (2006) could be used as the foundations of a fundamental process of learning. My finding suggest that as these methods are not been widely put into practice, children have nothing to build upon.

This Graduation Thesis has allowed us to update and revise the conceptual and terminological knowledge of mathematics, as well as establishing connections between the theory and the *praxis,* along with the possibility to involve ourselves in a teaching environment and realising the importance of continuous training, that will lead us to improve the teaching-learning process of the individuals in our society.

Now that we have come to the end of our research process, we have reached a positive conclusion: by developing mathematical skills among our students, positive attitudes towards mathematics, attitudes of respect, collaboration and mutual help, motivating lesson proposals and activities along with knowledge that all students have developed by adopting our teaching methods.

With regard to further research, I would like to implement these methodologies to a cross section of students, over a longer period of time. This would allow me to proceed with a more in-depth analysis, regarding the benefits of these methods.

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Images

Figure 1: Zúñiga, Á. R. (2003). Historia y filosofia de las matemáticas. Euned.

Figure 2: Feito, M., & Sandoval, C. (2014). Matemáticas y competencias básicas a partir de la tablilla Plimpton 322 (1). Suma , 77(1), 31–40. doi: 10.1515/tmmp-2016-0027

Figure 3: CASADO SANTIAGO.- LOS SISTEMAS DE NUMERACIÓN A LO LARGO DE LA HISTORIA, www.thales.cica.es/rd/Recursos/rd/97

Normativa

Currículo de las enseñanzas de Educación Primaria en la Comunidad Foral de Navarra. En BON número 174, de 5 de septiembre de 2014

APPENDICES

Appendix l

LESSONS’

OBSERVATION GUIDE

Activity ____

1. What have been the initial attitudes of the children when told that they were going to listen to a story?

   DISAPPOINTED HAPPY EXCITED

2. What has been the attitude of the child though out the lesson?

   BORED PESSIMISTIC WILLING ATTENTIVE

3. Have they been able to work independently with unobtrusive help?

   NO OCCASIONALLY FREQUENTLY TOTALLY

4. Have they been able to identify what the problem was talking about?

   PARCIALLY TOTALLY

5. Have they been able to establish the relation among the magnitudes?

   PARCIALLY TOTALLY

6. Have they developed their own pictorial representations?

   PARCIALLY TOTALLY

7. Have they been able to add the information to the representations?

   PARCIALLY TOTALLY

8. Have they been able to come up with a plan or strategy?

   PARCIALLY TOTALLY

9. Have they been able to carry out the strategy?

   PARCIALLY TOTALLY

10. Have they been able to improve the strategy?

    PARCIALLY TOTALLY

11. Have they been able to create a similar problem?

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Appendix ll

	AGE	AGE GROUP	LEVEL	CURRICULAR ADAPTATIONS	LEARNING DISABILITIES	VEHICULAR LANGUAGE
STUDENT 1	7	January –April 2013	1ºEP	X	X	YES
STUDENT 2	7	January –April 2013	1ºEP			YES
STUDENT 3	7	January –April 2013	1ºEP			YES
STUDENT 4	7	January –April 2013	1ºEP	X		YES
STUDENT 5	7	January –April 2013	1ºEP			YES
STUDENT 6	7	January –April 2013	1ºEP		X	YES
STUDENT 7	7	January –April 2013	1ºEP			YES
STUDENT 8	7	January –April 2013	1ºEP			YES
STUDENT 9	7	May – August 2013	1ºEP			YES
STUDENT 10	8	May – August 2012	1ºEP			YES
STUDENT 11	6	May – August 2013	1ºEP	X		YES
STUDENT 12	7	May – August 2012	1ºEP			YES
STUDENT 13	7	May – August 2012	1ºEP			YES
STUDENT 14	6	May – August 2013	1ºEP	X		YES
STUDENT 15	6	May – August 2013	1ºEP			YES
STUDENT 16	7	May – August 2012	1ºEP		X	YES
STUDENT 17	6	May – August 2013	1ºEP			YES
STUDENT 18	6	May – August 2013	1ºEP			YES
STUDENT 19	6	May – August 2013	1ºEP			YES
STUDENT 20	6	May – August 2013	1ºEP		X	NO
STUDENT 21	6	May – August 2013	1ºEP			YES
STUDENT 22	6	May – August 2013	1ºEP			YES
STUDENT 23	6	May – August 2013	1ºEP			YES
STUDENT 24	6	May – August 2013	1ºEP			YES
STUDENT 25	6	September- December 2013	1ºEP		X	YES
STUDENT 26	6	September- December 2013	1ºEP	X		YES
STUDENT 27	6	September- December 2013	1ºEP	X		YES
STUDENT 28	6	September- December 2013	1ºEP	X		YES
STUDENT 29	6	September- December 2013	1ºEP			YES
STUDENT 31	6	September- December 2013	1ºEP			YES
STUDENT 32	6	September- December 2013	1ºEP		X	YES
STUDENT 33	6	September- December 2013	1ºEP			YES
STUDENT 34	6	September- December 2013	1ºEP			YES
STUDENT 35	6	September- December 2013	1ºEP			YES
STUDENT 36	6	September- December 2013	1ºEP			YES
STUDENT 37	6	September- December 2013	1ºEP			YES
STUDENT 38	6	September- December 2013	1ºEP			YES
STUDENT 39	6	September- December 2013	1ºEP			YES
STUDENT 40	6	September- December 2013	1ºEP			YES
STUDENT 41	6	September- December 2013	1ºEP			YES