Avoiding the order reduction when solving second-order in time PDEs with Fractional Step Runge-Kutta-Nyström methods

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Abstract

We study some of the main features of Fractional Step Runge-Kutta-Nyström methods when they are used to integrate Initial-Boundary Value Problems of second order in time, in combination with a suitable spatial discretization. We focus our attention in the order reduction phenomenon, which appears if classical boundary conditions are taken at the internal stages. This drawback is specially hard when time dependent boundary conditions are considered. In this paper we present an efficient technique, very simple and computationally cheap, which allows us to avoid the order reduction; such technique consists of modifying the boundary conditions for the internal stages of the method.
Keywords:
Fractional Step Runge-Kutta-Nyström methods, second-order partial differential equations, order reduction, stability, consistency

1. Introduction

In this paper we deal with the development of efficient numerical algorithms for solving Initial Boundary Value Problems (IBVP) of second order in time. As it is well-known, the numerical integration of this kind of evolutionary problems can be realized by means of the method of lines (see [1]). Such process consists of combining a numerical time integrator with a suitable spatial discretization technique; typically, if we choose to discretize firstly in space, using for example finite differences, finite elements or spectral methods, a family of stiff Initial Value Problems of second order in time is obtained, which must

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be suitably integrated in time afterwards. If the (elliptic) spatial differential operator is one-dimensional, there exist several methods which integrate adequately in time, for example, Runge-Kutta (RK) or Runge-Kutta-Nyström (RKN) methods. In this way, we obtain a totally discrete scheme which can be computationally interesting. But, if the elliptic operator is M-dimensional the computational cost can be very high, whether you use explicit or implicit methods for the time discretization. In order to avoid such drawback, in [2] there was introduced a new type of methods for the time discretization named Fractional Step Runge-Kutta-Nyström methods (FSRKN). These FSRKN methods have been designed by combining the ideas of Fractional Step Runge-Kutta methods (FSRK) for parabolic problems (see [3, 4, 5]), together with RKN methods (see [6, 7]). In fact, FSRKN methods can be viewed as a generalization of the alternating direction methods proposed in [3, 8] for solving the wave equation efficiently.

The main advantage of FSRKN methods is the obtaining of a numerical solution from unconditionally convergent schemes, which provide a low computational cost. To apply such methods in an efficient way we must firstly split the spatial operator as a sum of simpler operators in a certain sense; thus, only a piece of the splitting acts implicitly at each fractional step. Such decomposition is very important in order to obtain good results.

As it is well-known, one of the main drawbacks of many classical one-step time integrators is that they suffer an order reduction when they are used in this context; this phenomenon is specially hard in the case of considering time dependent boundary data. In the literature we can find an important number of papers about the order reduction phenomenon (see [9, 10, 11, 12] for RK methods, [13, 14] for RKN methods). In [11] the authors prove that for parabolic IBVP, RK methods present superconvergence in the interior; thus it is well known that, for RK or RKN methods, the order reduction is due to a non suitable election of the boundary conditions for the internal stages. This drawback also appears when FSRKN methods are used in the time discretization of second-order in time problems. In these methods the order reduction is related to the order of their internal stages, as in RK or RKN methods. When the FSRKN method has all its stages implicit, this relation is specially restrictive because the order reduction is very harsh.

We show a technique which permits us to recover the lost order in a extremely cheap way, from the point of view of the computational cost involved. The basis of this strategy is to obtain an improvement for the boundary conditions of the internal stages by following a simple recurrence process which involves only data of the given problem. Both the introduction of this technique and the subsequent analysis of the consistency of the method requires to consider the two discretization procedures in the inverse order, i.e., we will discretize firstly in time, using FSRKN methods, and afterwards we will solve the family of boundary value problems derived of this process.

This paper is structured as follows: in the following Section we describe the problem as well as the time discretization methods proposed and we study the local error; we prove that the order reduction is due to the boundary conditions and we show the technique to diminish it as far as reaching the classical order. In Section 3 the global error is studied; the theoretical results proven in this Section are corroborated by means of a numerical test shown in Section 4, where we have used spectral methods for the spatial discretization because they reach high orders of convergence. Finally, Section 5 presents some technical results and the proofs of the main theorems of this paper.

Henceforth we denote with C any constant independent of the size of the time step and the number of nodes of the spatial mesh.

2. The time discretization method

Second-order in time evolution IBVP governed by partial differential equations can be written in an abstract form as follows:

"Find $u : [0, T] \to \mathcal{H}$ solution of

$$\left\{ \begin{array}{rcl} u''(t) & = & A\,u(t) + f(t), \quad 0 \le t \le T < \infty, \\ \partial u(t) & = & g(t), \\ u(0) & = & u_0, \\ u'(0) & = & v_0, \text{''} \end{array} \right. \tag{1}$$

where, typically, $\mathcal{H}$ is a Hilbert space of functions defined in a certain bounded domain $\Omega \subseteq \mathbb{R}^M$, integer $M \ge 1$ with smooth boundary Γ and $A : \mathcal{D}(A) \subseteq \mathcal{H} \to \mathcal{H}$ is a linear differential operator of order d (integer $d \ge 1$) that contains the spatial derivatives and which is defined on a dense subset $\mathcal{D}(A) \subset \mathcal{H}$.

In order to ensure a well-posedness for problem (1) in the sense of Hadamard, we will assume:

(A1) The boundary operator $\partial : \mathcal{D}(A) \subset \mathcal{H} \to \mathcal{H}^b$ is onto, with $\ker(\partial)$ dense in $\mathcal{D}(A)$, where $\mathcal{H}^b$ is a Hilbert space of functions.

(A2) The restriction of A to $\ker(\partial)$, denoted by $A^0 \equiv A|_{\ker\partial}$ being $A^0 : \mathcal{D}(A^0) = \ker(\partial) \subseteq \mathcal{H} \to \mathcal{H}$, is self-adjoint and negative definite.

(A3) There exists $\widetilde{\omega} < 0 \in \mathbb{R}$ (see [15]) such that for each $\mu \in \mathbb{R}$ with $\mu > \widetilde{\omega}$, the problem [1]

$$\begin{cases} (\mu I - A)u &= 0, \\ \partial u &= v, \end{cases}$$

has, for every $v \in \mathcal{H}^b$, a unique solution $u = S(\mu)v$; which satisfies $\|S(\mu)v\| \leq L\|v\|$, for certain constant $L > 0$ independently of μ for $\mu > \omega^0 > \widetilde{\omega}$.

Also, in order to guarantee the convergence results we suppose the initial and boundary data to be sufficiently smooth.

From hypothesis (A2), we have that the operator A^0 is the infinitesimal generator of a cosine function, of type $\omega = 0$, on $\mathcal{H}$. This guarantees the well-posedness of problem (1) in the energy norm.

Many results of this article can be extended to the more general hipothesis

(A2') The operator A^0 is the infinitesimal generator of a C_0-semigroup of type $\tilde{\omega} \leq 0$.

When solving this type of problems with FSRKN methods, the elliptic operator A is assumed to be split as a sum of m linear differential operators of order less than or equal to d, each of them simpler in a certain sense, that is, $A = \sum_{\ell=1}^m A_\ell$, where $A_\ell : \mathcal{D}(A_\ell) \subseteq \mathcal{H} \to \mathcal{H}$ and $\cap_{\ell=1}^m \mathcal{D}(A_\ell) = \mathcal{D}(A)$. Besides, associated to every operator A_ℓ, $\ell = 1, \cdots, m$ we will define the boundary operators $\partial_\ell : \mathcal{D}(A_\ell) \subset \mathcal{H} \to \mathcal{H}_\ell^b$, $\ell = 1, \cdots, m$, and we will denote by $A_\ell^0 : \mathcal{D}(A_\ell^0) = \ker(\partial_\ell) \subset \mathcal{H} \to \mathcal{H}$ the restriction of A_ℓ to $\ker(\partial_\ell)$, with $\ker(\partial) = \cap_{\ell=1}^m \ker(\partial_\ell)$.

To simplify the exposition we also consider a decomposition of the source term in m smooth addends, $f(t) = \sum_{\ell=1}^m f_\ell(t)$. Then, problem (1) can be written as

$$\begin{cases} u''(t) &= \sum_{\ell=1}^m (A_\ell u(t) + f_\ell(t)), \qquad 0 \leq t \leq T < \infty, \\ \partial_\ell u(t) &= g_\ell(t), \qquad 0 \leq t \leq T < \infty, \qquad \ell = 1, \cdots, m, \\ u(0) &= u_0, \\ u'(0) &= v_0. \end{cases} \tag{2}$$

[1] $I : \mathcal{H} \to \mathcal{H}$ is the identity operator

To assure that problem (2) with vanishing boundary conditions is well posed we suppose that

(B1) The boundary operators $\partial_\ell : \mathcal{D}(A_\ell) \subset \mathcal{H} \to \mathcal{H}_\ell^b$ are onto, with $\ker(\partial_\ell)$ dense in $\mathcal{D}(A_\ell)$.

(B2) The operators A_ℓ^0 are self-adjoint and negative definite.

(B3) There exists $\widetilde{\omega}_\ell \in \mathbb{R}$ such that for each $\mu \in \mathbb{R}$ with $\mu > \widetilde{\omega}_\ell$, the problem

$$\begin{cases} (\mu I - A_\ell)u &= 0, \\ \partial_\ell u &= v, \end{cases}$$

has, for every $v \in \mathcal{H}_\ell^b$, a unique solution $u = S_\ell(\mu)v$; which satisfies $\|S_\ell(\mu)v\| \leq L_\ell \|v\|$, for certain constant $L_\ell > 0$ independently of μ for $\mu > \omega_\ell^0 > \widetilde{\omega}_\ell$.

From hypothesis (B2), we have that the operator A_ℓ^0 is the infinitesimal generator of a cosine function, of type $\omega = 0$, on $\mathcal{H}$. Thus, we have that A_ℓ^0 is the infinitesimal generator of a C_0-semigroup of type $\tilde{\omega}_\ell < 0$. Then, $(\mu_\ell I - A_\ell^0)^{-1}$ exists and is bounded for every μ_ℓ with $Re(\mu_\ell) > \bar{\omega}_\ell$.

Furthermore, in what follows, we will assume that

$$\|A_{\ell_1} \cdots A_{\ell_k} u^{(j)}(t)\| \leq C \qquad \text{and} \qquad \|A_{\ell_1} \cdots A_{\ell_k} f_{\ell_{k+1}}^{(j)}(t)\| \leq C, \tag{3}$$

for certain integers j, k as big as needed, with $\ell_i \in \{1, \cdots, m\}$, for $i = 1, \cdots, k+1$.

When solving a linear problem like (2), FSRKN methods are defined by the following algorithm,

$$\begin{aligned} K_{n,i} &= U_n + c_i \tau V_n + \tau^2 \sum_{\ell=1}^{m} \sum_{j=1}^{i} a_{\ell,ij} \left(A_\ell K_{n,j} + f_\ell(t_{n,j}) \right), \quad i = 1, \cdots, s, \\ V_{n+1} &= V_n + \tau \sum_{\ell=1}^{m} \sum_{j=1}^{s} b_{\ell,j} \left(A_\ell K_{n,j} + f_\ell(t_{n,j}) \right), \\ U_{n+1} &= U_n + \tau V_n + \tau^2 \sum_{\ell=1}^{m} \sum_{j=1}^{s} \beta_{\ell,j} \left(A_\ell K_{n,j} + f_\ell(t_{n,j}) \right), \end{aligned} \tag{4}$$

where $t_{n,j} = t_n + c_j \tau$, for $j = 1, \ldots, s$ and $t_n = n\tau$, $n = 1, \cdots, N$, being $\tau = T/N$ the time step size and N the number of steps (see [2]). $K_{n,i}$ are the intermediate stages, which can be considered as numerical approximations to the exact solution at time $t_{n,i}$, $i = 1, \cdots, s$, and $(U_n, V_n)^T$ is the numerical approximation to the exact solution $(u(t_n), u'(t_n))^T$. Following the ideas of FSRK methods, we will assume that $a_{\ell_i,ii} > 0$,

$i = 1, \cdots, s$, $\ell = 1, \cdots, m$ and we will group the coefficients $a_{\ell,ij}$, $b_{\ell,j}$, $\beta_{\ell,j}$ and c_i which appear in (4) in the following tableau

$$
\begin{array}{c|c|c|c}
c & \mathcal{A}_1 & \cdots & \mathcal{A}_m \\
\hline
 & \beta_1^T & \cdots & \beta_m^T \\
\hline
 & b_1^T & \cdots & b_m^T
\end{array}
=
\begin{array}{c|ccc|c|ccc}
c_1 & a_{1,11} & & & & a_{m,11} & & \\
\vdots & \vdots & \ddots & & & \vdots & \ddots & \\
c_s & a_{1,s1} & \cdots & a_{1,ss} & & a_{m,s1} & \cdots & a_{m,ss} \\
\hline
 & \beta_{1,1} & \cdots & \beta_{1,s} & \cdots & \beta_{m,1} & \cdots & \beta_{m,s} \\
\hline
 & b_{1,1} & \cdots & b_{1,s} & \cdots & b_{m,1} & \cdots & b_{m,s}
\end{array}
$$

The coefficients satisfy the additional hypotheses $a_{\ell,ij} = 0$, $\beta_{\ell,j} = 0$, $b_{\ell,j} = 0$ for $\ell \neq \ell_j \in \{1, \cdots, m\}$, $1 \leq j \leq s$; these hypotheses allow us to compact the notation in the following way

$$
\begin{aligned}
K_{n,i} &= U_n + c_i \tau V_n + \tau^2 \sum_{j=1}^{i} a_{\ell_j,ij} \left(A_{\ell_j} K_{n,j} + f_{\ell_j}(t_{n,j}) \right), \quad i = 1, \cdots, s, \\
V_{n+1} &= V_n + \tau \sum_{j=1}^{s} b_{\ell_j,j} \left(A_{\ell_j} K_{n,j} + f_{\ell_j}(t_{n,j}) \right), \\
U_{n+1} &= U_n + \tau V_n + \tau^2 \sum_{j=1}^{s} \beta_{\ell_j,j} \left(A_{\ell_j} K_{n,j} + f_{\ell_j}(t_{n,j}) \right).
\end{aligned}
\tag{5}
$$

Note that the structure of the coefficients of FSRKN methods implies that in every stage only one elliptic operator A_ℓ acts implicitly and, in this way, when a multidimensional problem is solved with an FSRKN method, at each intermediate stage we must solve a problem which can be much simpler than the first one; thus, by choosing adequately the split of the operator A, we can obtain important reductions in the computational cost of these methods, compared to the computational costs associated to the use of other time integrators like, for example, implicit RKN methods.

In order to have a unique solution from (4) we must assure that the intermediate stages are well defined and that they have a unique solution. To obtain this solution we must determine the values of the boundary conditions of such intermediate stages, thus we must solve, for $i = 1, \cdots, s$,

$$
\begin{aligned}
(I - \tau^2 a_{\ell_i,ii} A_{\ell_i}) K_{n,i} &= U_n + \tau c_i V_n + \tau^2 \sum_{j=1}^{i-1} a_{\ell_j,ij} A_{\ell_j} K_{n,j} + \tau^2 \sum_{j=1}^{i} a_{\ell_j,ij} f_{\ell_j}(t_{n,j}), \\
\partial_{\ell_i} K_{n,i} &= G_{\ell_i,n,i}.
\end{aligned}
\tag{6}
$$

By denoting $G_{\ell,n} = [\partial_\ell K_{n,1}, \cdots, \partial_\ell K_{n,s}]^T$, $\ell = 1, \cdots, m$, $e = [1, \cdots, 1]^T$, $K_n = [K_{n,1}, \cdots, K_{n,s}]^T$

and $f_{\ell,n} = [f_\ell(t_{n,1}), \cdots, f_\ell(t_{n,s})]^T$ the internal stages, in tensorial form, are given by[2]

$$\begin{aligned} K_n &= (e \otimes I) U_n + \tau (c \otimes I) V_n + \tau^2 \sum_{\ell=1}^{m} (\mathcal{A}_\ell \otimes I)((\mathcal{I} \otimes A_\ell) K_n + f_{\ell,n}), \quad (7) \\ (\partial_1, \cdots, \partial_m) K_n &= (G_{1,n}, \cdots, G_{m,n}). \end{aligned}$$

Once K_n has been obtained,

$$V_{n+1} = V_n + \tau \sum_{\ell=1}^{m} (b_\ell^T \otimes I)((\mathcal{I} \otimes A_\ell) K_n + f_{\ell,n}), \tag{8}$$

$$U_{n+1} = U_n + \tau V_n + \tau^2 \sum_{\ell=1}^{m} (\beta_\ell^T \otimes I)((\mathcal{I} \otimes A_\ell) K_n + f_{\ell,n}). \tag{9}$$

To prove that (6) possess a unique solution, it is enough to consider problems

$$\begin{aligned} (I - \tau^2 a_{\ell_i,ii} A_{\ell_i}) K_{n,i}^b &= 0, \quad (10) \\ \partial_{\ell_i} K_{n,i}^b &= G_{\ell_i,n,i}, \end{aligned}$$

for $i = 1, \ldots, s$ and once that $K_n^b = [K_{n,1}^b, \cdots, K_{n,s}^b]^T$ has been obtained, we must solve

$$\begin{aligned} (\mathcal{I} \otimes I - \tau^2 \sum_{\ell=1}^{m} \mathcal{A}_\ell \otimes A_\ell) K_n^0 &= (e \otimes I) U_n + \tau (c \otimes I) V_n + \tau^2 \sum_{\ell=1}^{m} (\mathcal{A}_\ell \otimes I) \Big((\mathcal{I} \otimes A_\ell) \tilde{K}_n^b + f_{\ell,n} \Big), \\ (\partial_1, \cdots, \partial_m) K_n^0 &= (0, \cdots, 0), \quad (11) \end{aligned}$$

with $\tilde{K}_n = [0, K_{n,1}^b, \cdots, K_{n,s-1}^b]^T$.

With this decomposition it is immediately observed that the solution of (6) can be expressed as $K_n = K_n^0 + K_n^b$.

The solvability of (10) is a direct consequence of hypothesis (B3) because, as we are assuming $a_{\ell_i,ii} > 0$, we have that for $(\tau^2 a_{\ell_i,ii})^{-1} > \omega_{\ell_i}^0 > \tilde{\omega}_{\ell_i}$, expression $S_{\ell_i}((\tau^2 a_{\ell_i,ii})^{-1}) G_{\ell_i,n,i}$ is solution of (10), with

$$\| S_{\ell_i}((\tau^2 a_{\ell_i,ii})^{-1}) G_{\ell_i,n,i} \| \leq L_{\ell_i} \| G_{\ell_i,n,i} \|,$$

where $L_{\ell_i} > 0$ is a constant independent of $\tau^2 a_{\ell_i,ii}$.

The solvability of (11) was proven in [2].

2.1. Local error

Now, we study the local error that is made when problem (2) is solved in time by using an FSRKN method. The boundary values of the internal stages appear as data to

[2] Note that, $(\mathcal{A}_\ell \otimes I)(\mathcal{I} \otimes A_\ell) = (\mathcal{A}_\ell \otimes A_\ell)$; $(b_\ell^T \otimes I)(\mathcal{I} \otimes A_\ell) = (b_\ell^T \otimes A_\ell)$ and $(\beta_\ell^T \otimes I)(\mathcal{I} \otimes A_\ell) = (\beta_\ell^T \otimes A_\ell)$

introduce and by following classical ideas for RK, RKN or FSRK methods, the first option is to take these boundary values as $\partial_\ell K_{n,i} = \partial_\ell u(t_{n,i}) = g_\ell(t_{n,i})$, assuming that the internal stages can be considered as approximations of the solution at the intermediate times $t_{n,i}$. As we will see later, this is not the best choice, since this implies a reduction in the order of the error observed. This order is related to the stage order, and this stage order is only 1 when the FSRKN method has not got any explicit stage, as it will be proven in Lemma 2.3. We will show that by choosing these boundary values in an adequate way, the order reduction can be avoided.

We introduce for FSRKN methods the concepts of classical and stage order which are defined in a similar way as to RKN methods:

Definition 2.1. *An FSRKN method given by (4) (or (5)) is said to have classical order p when it is applied to numerically solve problem (2) if*

$$\|\xi_{n+1}\| \equiv \|u'(t_{n+1}) - \bar{V}_{n+1}\| = \mathcal{O}(\tau^{p+1}) \qquad \text{and} \qquad \|\rho_{n+1}\| \equiv \|u(t_{n+1}) - \bar{U}_{n+1}\| = \mathcal{O}(\tau^{p+1})$$

with $\left(\bar{U}_{n+1}, \bar{V}_{n+1}\right)^T$ the numerical solution obtained from the exact solution $\left(u(t_n), u'(t_n)\right)^T$ by taking a time step-size τ.

Definition 2.2. *The stage order of an FSRKN method is defined as $q = \min\{\tilde{q}, p\}$, being p the classical order of the method and $\tilde{q}$ the maximum value such that, for $\ell = 1, \cdots, m$,*

$$c^k = k(k-1)\mathcal{A}_\ell c^{k-2}, \quad k = 2, \cdots, \tilde{q},$$

where $c^k = [c_1^k, \cdots, c_s^k]^T$ and $c^0 = e$. Similarly, it can be expressed as

$$c_i^k = k(k-1)\sum_{j=1}^{i} a_{\ell,ij} c_j^{k-2}, \quad i = 1, \cdots, s, \quad k = 2, \cdots, \tilde{q}. \tag{12}$$

When the above conditions are not satisfied by any $\tilde{q} \geq 2$, then $\tilde{q}$ is taken equal to 1.

Notice that the minimum stage order that is obtained is 1.

Lemma 2.3. *Let an FSRKN method be given by (4) (or (5)) whose coefficients satisfy $a_{\ell_i,ii} \neq 0$, $\forall\, i = 1, \cdots, s$, (that is, all its stages are implicit). Then the maximum stage order achieved is* 1.

Proof. As $a_{\ell_i,ii} \neq 0$, $\forall\, i = 1, \cdots, s$, in particular, $a_{\ell_1,11} \neq 0$ and $a_{\ell,11} = 0$, $\ell = 1, \cdots, m$, $\ell \neq \ell_1$ for certain $\ell_1 \in \{1, \cdots, m\}$. When (12) is imposed to obtain order 2, for $i = 1$,

$$\frac{c_1^2}{2} = a_{\ell,11}, \quad \ell = 1, \cdots, m,$$

which leads, for $\ell = \ell_1$ to $c_1 \neq 0$. On the other hand, if $\ell \neq \ell_1$, then $a_{\ell,11} = 0$ is deduced, which implies $c_1 = 0$, that is against to the fact that $c_1 \neq 0$. □

Remark 2.4. *For these methods, (except for those ones with classical order 1, which do not have much interest in practice), it is always observed an order reduction when solving problems like (1), so it is important to have techniques at our disposal to recover the lost order.*

As it was explained before, certain boundary values for the intermediate stages must be chosen. These boundary values determine the local order observed, as it will be shown. To avoid this order reduction, the procedure is to calculate, in a recursive way, the value of these intermediate stages at the boundary. Thus, by taking as first choice the natural boundary conditions, we define

$$\begin{aligned} K_n^{[0]} &= [K_{n,1}^{[0]}, \ldots, K_{n,s}^{[0]}]^T = [u(t_{n,1}), \ldots, u(t_{n,s})]^T, \\ G_{\ell,n}^{[0]} &= [G_{\ell,n,1}^{[0]}, \cdots, G_{\ell,n,s}^{[0]}]^T = [g_\ell(t_{n,1}), \ldots, g_\ell(t_{n,s})]^T = \partial_\ell K_n^{[0]}, \quad \ell = 1, \cdots, m, \end{aligned} \tag{13}$$

and from this definition, for integer $r \geq 1$ we obtain

$$\begin{aligned} K_n^{[r]} &= (e \otimes I)u(t_n) + \tau(c \otimes I)u'(t_n) + \tau^2 \sum_{\ell=1}^{m} (\mathcal{A}_\ell \otimes I)((\mathcal{I} \otimes A_\ell)K_n^{[r-1]} + f_{\ell,n}), \\ G_{\ell,n}^{[r]} &= \partial_\ell K_n^{[r]}, \quad \ell = 1, \cdots, m. \end{aligned} \tag{14}$$

Then, $\bar{K}_n^{[r]}$, integer $r \geq 0$, is defined as the vector that satisfies

$$\begin{aligned} \bar{K}_n^{[r]} &= (e \otimes I)u(t_n) + \tau(c \otimes I)u'(t_n) + \tau^2 \sum_{\ell=1}^{m} (\mathcal{A}_\ell \otimes I)((\mathcal{I} \otimes A_\ell)\bar{K}_n^{[r]} + f_{\ell,n}), \\ (\partial_1, \cdots, \partial_m)\bar{K}_n^{[r]} &= (G_{1,n}^{[r]}, \cdots, G_{m,n}^{[r]}). \end{aligned} \tag{15}$$

From $\bar{K}_n^{[r]}$ the approximations $\bar{V}_{n+1}^{[r]}$ and $\bar{U}_{n+1}^{[r]}$ are given by

$$\bar{V}_{n+1}^{[r]} = u'(t_n) + \tau \sum_{\ell=1}^{m} (b_\ell^T \otimes I)((\mathcal{I} \otimes A_\ell)\bar{K}_n^{[r]} + f_{\ell,n}), \tag{16}$$

$$\bar{U}_{n+1}^{[r]} = u(t_n) + \tau\, u'(t_n) + \tau^2 \sum_{\ell=1}^{m} (\beta_\ell^T \otimes I)((\mathcal{I} \otimes A_\ell)\bar{K}_n^{[r]} + f_{\ell,n}). \tag{17}$$

Thus, the local errors in the derivative and in the solution are defined now as

$$\xi_{n+1}^{[r]} = u'(t_{n+1}) - \bar{V}_{n+1}^{[r]} \qquad \text{and} \qquad \rho_{n+1}^{[r]} = u(t_{n+1}) - \bar{U}_{n+1}^{[r]}, \tag{18}$$

for integer $n \geq 0$. Then, the following Theorem can be proven

Theorem 2.5. *Let (4) (or 5) be the time semidiscretization obtained by using an FSRKN with stage order* $q = \min\{\tilde{q}, p\}$ *and classical order* p*, for problem (2), with* A *satisfying hypotheses (A1–A3) and* $\{A_\ell\}_{\ell=1}^m$ *satisfying hypotheses (B1–B3).*

If the exact solution and the split of the source term of such problem satisfy bound (3) and the boundary values are taken as $G_{\ell,n}^{[r]} = \partial_\ell K_n^{[r]}$ *with* $K_n^{[r]}$ *given by (14), integer* $r \geq 0$*, then the local errors satisfy*

$$\|\xi_{n+1}^{[r]}\| = \mathcal{O}(\tau^{\min\{\tilde{q}+2r, p+1\}}) \qquad \text{and} \qquad \|\rho_{n+1}^{[r]}\| = \mathcal{O}(\tau^{\min\{\tilde{q}+2r+1, p+1\}}).$$

Notice that for $r = 0$, that is, when the boundary values for the intermediate stages are chosen as $G_{\ell,n,i}^{[0]} = \partial_\ell u(t_{n,i})$, $\ell = 1, \cdots, m$, $i = 1, \cdots, s$, the local error is referred to the stage order. The order reduction can be completely avoided when the solution is regular enough and the sufficient number of iterations is made.

3. Space Discretization

In this part we deal with the complete discretization of problem (2). Now, we describe a general context which permits us to include spectral discretizations as well as some finite element and finite difference methods.

We should to take into account that, although at each stage of the time discretization we are obtaining several simpler problems, they are related in a way that all of them belong to the same space. For every $\ell = 1, \cdots, m$ we want to solve

"Find $u : \Omega \to \mathcal{H}$ solution of

$$\begin{cases} A_\ell u &= F_\ell, \quad \text{in } \Omega, \\ \partial_\ell u &= G_\ell, \quad \text{in } \Gamma_\ell = \partial_\ell \Omega, \text{"} \end{cases}$$

where $F_\ell \in \mathcal{H}$, $G_\ell \in \mathcal{H}_\ell^b$ and $u \in \mathcal{D}(A_\ell)$. Let us assume that operators A_ℓ and ∂_ℓ satisfy the hypotheses pointed in the previous Section.

For the space discretization of this problem, we consider in $\Omega \cup \partial\Omega$ a grid Ω_J (not necessarily uniform) associated to a natural parameter J related to the number of nodes on it. In this grid, we denote the interior nodes as Ω_J^I and the boundary ones as Ω_J^b, with $\Omega_J = \Omega_J^I \cup \Omega_J^b$ and $\Omega_J^I \cap \Omega_J^b = \emptyset$. After that, we take $\mathcal{H}_J \subseteq \mathcal{D}(A)$, a finite-dimensional space associated to this parameter, considering the subspace (or space of less dimension than $\mathcal{H}_J$) $\mathcal{H}_J^0$ that contains the elements of $\mathcal{H}_J$ which vanish in some way on the boundary $\partial\Omega$. Then, the collocation problem is as follows

"Find $u_J : \Omega_J \to \mathcal{H}_J$ solution of

$$\left\{ \begin{array}{rcll} A_\ell u_J &=& F_\ell, & \text{in } \Omega_J \setminus \Omega_J^b, \\ \partial_\ell u_J &=& G_\ell, & \text{in } \Omega_{\ell,J}^b," \end{array} \right.$$

being $\Omega_{\ell,J}^b = \Gamma_\ell \cap \Omega_J^b$.

In order to obtain the numerical solution, we take operators $P_J : \mathcal{H} \to \mathcal{H}_J^0$, the projection operator; $A_{\ell,J}^0 \equiv P_J A_\ell|_{\mathcal{H}_J^0} : \mathcal{H}_J^0 \to \mathcal{H}_J^0$, symmetric and negative definite; $\kappa_{\ell,J}^b$ operator such that $\kappa_{\ell,J}^b G_\ell$ interpolates G_ℓ in $\Omega_{\ell,J}^b$, and vanishes in $\Omega_J \setminus \Omega_{\ell,J}^b$; $S_{\ell,J} \equiv P_J A_\ell \kappa_{\ell,J}^b : \mathcal{H}_\ell^b \to \mathcal{H}_J^0$ and $R_J : \mathcal{D}(A) \to \mathcal{H}_J^0$ the operator such that $R_J u$ is the numerical approximation to u in $\mathcal{H}_J^0$. Because of this, it coincides with u_J in Ω_J^I and it vanishes in Ω_J^b.

Then, we should solve problem

$$A_{\ell,J}^0 R_J u + S_{\ell,J} \partial_\ell u = P_J A_\ell u \qquad \text{or similarly} \qquad A_{\ell,J}^0 R_J u + S_{\ell,J} G_\ell = P_J F_\ell. \tag{19}$$

Notice that because of their definition, $P_J u - R_J u$ vanishes in Ω_J^b. Besides, we consider operators $A_J^0 = P_J A|_{\mathcal{H}_J^0}$ and $S_J = P_J A \kappa_J^b$, with $\kappa_J^b g$ interpolating g in Ω_J^b and vanishing in Ω_J^I, satisfying that $A_J^0 R_J u + S_J g = P_J f$.

Apart from that, we denote by $\| \cdot \|_J$ an approximation to the norm in $\mathcal{H}$, assuming that it defines a discrete norm in $\mathcal{H}_J^0$ associated to a scalar product, such that for smooth enough $u \in C(\Omega) \subset \mathcal{H}$ and big enough J, the following compatibility relation between norms is satisfied: $\|P_J u\|_J = \mathcal{O}(\|u\|)$.

In what follows, we will assume that the following hypotheses are satisfied

(H1) There exists $\tilde{\alpha} > 0$ and a non-increasing function $\tilde{h} : (\tilde{\alpha}, \infty) \to (-\infty, 0)$ such that, for $\ell = 1, \cdots, m$, if $u \in H^\alpha(\Omega) \subset \mathcal{D}(A_\ell)$, with $\alpha > \tilde{\alpha}$ and J is big enough[3],

$$\|(R_J - P_J)u(t)\|_J = \mathcal{O}(J^{\tilde{h}(\alpha)} \|u(t)\|_{H^\alpha(\Omega)}). \tag{20}$$

(H2) For $u_J \in \mathcal{H}_J^0$, there exist constants $\widetilde{d} \geq 0$ and $\widetilde{\widetilde{d}} \geq 0$ such that

$$\begin{array}{rcll} \|A_J^0 u_J\|_J &=& \mathcal{O}(J^{2\widetilde{d}} \|u_J\|_J), & \\ \|B_J^0 u_J\|_J &=& \mathcal{O}(J^{\widetilde{d}} \|u_J\|_J), & \\ \|A_{\ell,J}^0 u_J\|_J &=& \mathcal{O}(J^{\widetilde{\widetilde{d}}} \|u_J\|_J), & \forall\, \ell = 1, \cdots, m, \end{array}$$

with B_J^0 the operator such that $(B_J^0)^2 = -A_J^0$.

[3]For spectral methods, the function $\tilde{h}(\alpha)$ strictly decreases when α increases. For finite-differences and finite-element methods $\tilde{h}(\alpha)$ is usually constant for $\widetilde{\alpha}$ sufficiently long.

(H3) The operators B_J^0 and $(\mathcal{I}\otimes I_J-\tau^2\sum_{\ell=1}^m \mathcal{A}_\ell\otimes A_{\ell,J}^0)$ are invertible[4], and their inverse is bounded independently of $\tau\in(0,\tau_0]$ and J.

(H4) The operators $\tau^2(b_\ell^T\otimes A_{\ell,J}^0)(\mathcal{I}\otimes I_J-\tau^2\sum_{\ell=1}^m \mathcal{A}_\ell\otimes A_{\ell,J}^0)^{-1}$ and $\tau^2(\beta_\ell^T\otimes A_{\ell,J}^0)(\mathcal{I}\otimes I_J-\tau^2\sum_{\ell=1}^m \mathcal{A}_\ell\otimes A_{\ell,J}^0)^{-1}$ are bounded independently of $\tau\in(0,\tau_0]$ and J, $\ell=1,\cdots,m$.

3.1. Global error

In order to obtain the totally discrete scheme, we must realize the spatial discretization of the scheme given by (7-9). Thus, for the spatial discretization of (7), we have, in tensorial form,

$$K_{n,J}^0=(e\otimes I_J)U_{n,J}^0+\tau(c\otimes I_J)V_{n,J}^0+\tau^2\sum_{\ell=1}^m(\mathcal{A}_\ell\otimes I_J)((\mathcal{I}\otimes A_{\ell,J}^0)K_{n,J}^0+(\mathcal{I}\otimes S_{\ell,J})\,G_{\ell,n}+(\mathcal{I}\otimes P_J)\,f_{\ell,n}),$$

where, by using hypothesis (H3), $K_{n,J}^0$ can be obtained. The numerical approximations to the function $u(t)$ and its derivative are given by

$$\begin{aligned}
V_{n+1,J}^0 &= V_{n,J}^0+\tau\sum_{\ell=1}^m(b_\ell^T\otimes I_J)((\mathcal{I}\otimes A_{\ell,J}^0)K_{n,J}^0+(\mathcal{I}\otimes S_{\ell,J})\,G_{\ell,n}+(\mathcal{I}\otimes P_J)\,f_{\ell,n}),\\
U_{n+1,J}^0 &= U_{n,J}^0+\tau V_{n,J}^0+\tau^2\sum_{\ell=1}^m(\beta_\ell^T\otimes I_J)((\mathcal{I}\otimes A_{\ell,J}^0)K_{n,J}^0+(\mathcal{I}\otimes S_{\ell,J})\,G_{\ell,n}+(\mathcal{I}\otimes P_J)\,f_{\ell,n}).
\end{aligned}$$

In Section 2 it is proven the relevance of making a good choice of the boundary conditions for the time discretization scheme. Now we prove the influence of such conditions for the final scheme too.

To obtain the totally discrete scheme, with less or without order reduction, we must also consider the new boundary conditions $G_{\ell,n}^{[r]}$ given by (13) and (14) instead of $G_{\ell,n}$, thus the following scheme is obtained:

$$\begin{aligned}
K_{n,J}^{0,[r]} &= (e\otimes I_J)U_{n,J}^{0,[r]}+\tau(c\otimes I_J)V_{n,J}^{0,[r]}\\
&\quad+\tau^2\sum_{\ell=1}^m(\mathcal{A}_\ell\otimes I_J)((\mathcal{I}\otimes A_{\ell,J}^0)K_{n,J}^{0,[r]}+(\mathcal{I}\otimes S_{\ell,J})\,G_{\ell,n}^{[r]}+(\mathcal{I}\otimes P_J)\,f_{\ell,n}), &(21)\\
V_{n+1,J}^{0,[r]} &= V_{n,J}^{0,[r]}+\tau\sum_{\ell=1}^m(b_\ell^T\otimes I_J)((\mathcal{I}\otimes A_{\ell,J}^0)K_{n,J}^{0,[r]}+(\mathcal{I}\otimes S_{\ell,J})\,G_{\ell,n}^{[r]}+(\mathcal{I}\otimes P_J)\,f_{\ell,n}), &(22)\\
U_{n+1,J}^{0,[r]} &= U_{n,J}^{0,[r]}+\tau V_{n,J}^{0,[r]}\\
&\quad+\tau^2\sum_{\ell=1}^m(\beta_\ell^T\otimes I_J)((\mathcal{I}\otimes A_{\ell,J}^0)K_{n,J}^{0,[r]}+(\mathcal{I}\otimes S_{\ell,J})\,G_{\ell,n}^{[r]}+(\mathcal{I}\otimes P_J)\,f_{\ell,n}). &(23)
\end{aligned}$$

[4] $I_J:\mathcal{H}_J^0\to\mathcal{H}_J^0$ is the identity operator.

In this way, we define the global errors associated to these new boundary conditions as

$$\tilde{e}^{[r]}_{n+1,J} = P_J u'(t_{n+1}) - V^{0,[r]}_{n+1,J} \qquad \text{and} \qquad e^{[r]}_{n+1,J} = P_J u(t_{n+1}) - U^{0,[r]}_{n+1,J} \tag{24}$$

where we assume that $e^{[r]}_{0,J} = \tilde{e}^{[r]}_{0,J} = 0$, integer $r \geq 0$.

Associated to the global error, there will appear a matrix whose powers are important to bound to obtain stability in the discrete energy norm (see [2]). In the rest of paper, in order to simplify the expressions we denote $\{A^0_{i,J}\}^m_{i=1} \equiv A^0_{1,J}, \dots, A^0_{m,J}$. Thus, this stability matrix $R(\tau, B^0_J, \{A^0_{i,J}\}^m_{i=1})$ is the one given by

$$R(\tau, B^0_J, \{A^0_{i,J}\}^m_{i=1}) = \begin{bmatrix} r_{11}(\tau, B^0_J, \{A^0_{i,J}\}^m_{i=1}) & r_{12}(\tau, B^0_J, \{A^0_{i,J}\}^m_{i=1}) \\ r_{21}(\tau, B^0_J, \{A^0_{i,J}\}^m_{i=1}) & r_{22}(\tau, B^0_J, \{A^0_{i,J}\}^m_{i=1}) \end{bmatrix},$$

where

$$\begin{aligned}
r_{11}(\tau, B^0_J, \{A^0_{i,J}\}^m_{i=1}) &= (\tau B^0_J)\left(I_J + \sum_{\ell=1}^m (\beta^T_\ell \otimes \tau^2 A^0_{\ell,J})(\mathcal{I} \otimes I_J - \tau^2 \sum_{k=1}^m \mathcal{A}_k \otimes A^0_{k,J})^{-1}(e \otimes I_J)\right)(\tau B^0_J)^{-1}, \\
r_{12}(\tau, B^0_J, \{A^0_{i,J}\}^m_{i=1}) &= (\tau B^0_J)\left(I_J + \sum_{\ell=1}^m (\beta^T_\ell \otimes \tau^2 A^0_{\ell,J})(\mathcal{I} \otimes I_J - \tau^2 \sum_{k=1}^m \mathcal{A}_k \otimes A^0_{k,J})^{-1}(c \otimes I_J)\right), \\
r_{21}(\tau, B^0_J, \{A^0_{i,J}\}^m_{i=1}) &= \left(\sum_{\ell=1}^m (b^T_\ell \otimes \tau^2 A^0_{\ell,J})(\mathcal{I} \otimes I_J - \tau^2 \sum_{k=1}^m \mathcal{A}_k \otimes A^0_{k,J})^{-1}(e \otimes I_J)\right)(\tau B^0_J)^{-1}, \\
r_{22}(\tau, B^0_J, \{A^0_{i,J}\}^m_{i=1}) &= I_J + \sum_{\ell=1}^m (b^T_\ell \otimes \tau^2 A^0_{\ell,J})(\mathcal{I} \otimes I_J - \tau^2 \sum_{k=1}^m \mathcal{A}_k \otimes A^0_{k,J})^{-1}(c \otimes I_J).
\end{aligned} \tag{25}$$

Related to these functions, in order to bound the solution and the derivative, we define functions $\tilde{r}_{ij}(\tau, B^0_J, \{A^0_{i,J}\}^m_{i=1})$, $1 \leq i, j \leq 2$ given by

$$\begin{aligned}
\tilde{r}_{11}(\tau, B^0_J, \{A^0_{i,J}\}^m_{i=1}) &= (\tau B^0_J)^{-1} r_{11}(\tau, B^0_J, \{A^0_{i,J}\}^m_{i=1})(\tau B^0_J), \\
\tilde{r}_{12}(\tau, B^0_J, \{A^0_{i,J}\}^m_{i=1}) &= (\tau B^0_J)^{-1} r_{12}(\tau, B^0_J, \{A^0_{i,J}\}^m_{i=1}), \\
\tilde{r}_{21}(\tau, B^0_J, \{A^0_{i,J}\}^m_{i=1}) &= r_{21}(\tau, B^0_J, \{A^0_{i,J}\}^m_{i=1})(\tau B^0_J), \\
\tilde{r}_{22}(\tau, B^0_J, \{A^0_{i,J}\}^m_{i=1}) &= r_{22}(\tau, B^0_J, \{A^0_{i,J}\}^m_{i=1}).
\end{aligned}$$

We define by $\tilde{R}(\tau, B^0_J, \{A^0_{i,J}\}^m_{i=1})$ the matrix whose elements are functions $\tilde{r}_{ij}$, $1 \leq i, j \leq 2$. The relation between matrices $R(\tau, B^0_J, \{A^0_{i,J}\}^m_{i=1})$ and $\tilde{R}(\tau, B^0_J, \{A^0_{i,J}\}^m_{i=1})$ can be expressed as

$$\tilde{R}(\tau, B^0_J, \{A^0_{i,J}\}^m_{i=1}) = \begin{bmatrix} (\tau B^0_J)^{-1} & 0 \\ 0 & I_J \end{bmatrix} R(\tau, B^0_J, \{A^0_{i,J}\}^m_{i=1}) \begin{bmatrix} \tau B^0_J & 0 \\ 0 & I_J \end{bmatrix}. \tag{26}$$

In what follows, we will assume, for integer $k \geq 0$, that

$$\|R^k(\tau, \{A^0_{i,J}\}_{i=1}^m)\|_J \leq C. \tag{27}$$

We remind that the discrete energy norm is given by

$$\|(U,V)^T\|^2_{B^0_J} = \|B^0_J U\|^2_J + \|V\|^2_J$$

Then, the following theorems can be stated

Theorem 3.1. *Under hypotheses (H1-H4), bound (3), by assuming stability (27),* $u^{(k)}(t) \in H^\alpha(\Omega)$, $f^{(k)}_\ell(t) \in H^\alpha(\Omega)$, $k = 0, 1, \cdots, p{+}2$, $\ell = 1, \cdots, m$, *with* $\|u^{(k)}(t)\|_{H^\alpha(\Omega)}$, $\|f^{(k)}_\ell(t)\|_{H^\alpha(\Omega)}$, *uniformly bounded for* $0 \leq t \leq T$ *and with* α *such that* $\alpha - d(r+2) > \tilde{\alpha}$, *the bound for the global error in the energy norm is*

$$\left\| \begin{matrix} e^{[r]}_{n,J} \\ \tilde{e}^{[r]}_{n,J} \end{matrix} \right\|_{B^0_J} = \mathcal{O}\left(\tau^{\min\{\tilde{q}+2r,p\}} J^{\tilde{d}} + \tau^{\min\{\tilde{q}+2r-1,p\}} + \tau J^{\tilde{d}+\tilde{h}(\alpha-d(r+2))} + J^{\tilde{h}(\alpha-d(r+2))} + J^{\tilde{d}+\tilde{h}(\alpha)}\right).$$

Theorem 3.2. *Under hypotheses (H1-H4), bound (3), by assuming stability (27),* $u^{(k)}(t) \in H^\alpha(\Omega)$, $f^{(k)}_\ell(t) \in H^\alpha(\Omega)$, $k = 0, 1, \cdots, p{+}2$, $\ell = 1, \cdots, m$, *with* $\|u^{(k)}(t)\|_{H^\alpha(\Omega)}$, $\|f^{(k)}_\ell(t)\|_{H^\alpha(\Omega)}$, *uniformly bounded for* $0 \leq t \leq T$ *and with* α *such that* $\alpha - d(r+2) > \tilde{\alpha}$, *the bounds for the global error in the solution and in the derivative are*

$$\begin{aligned} \|e^{[r]}_{n,J}\|_J &= \mathcal{O}\left(\tau^{\min\{\tilde{q}+2r,p\}} + \tau J^{\tilde{h}(\alpha-d(r+2))} + J^{\tilde{h}(\alpha)}\right), \\ \|\tilde{e}^{[r]}_{n,J}\|_J &= \mathcal{O}\left(\tau^{\min\{\tilde{q}+2r-1,p-1\}} + J^{\tilde{h}(\alpha-d(r+2))} + J^{\tilde{d}+\tilde{h}(\alpha)}\right). \end{aligned}$$

From these results, assuming the spatial discretization to be good enough, we can observe that the global errors are referred to the stage order, as well as it happens with the local ones.

Remark 3.3. *Sometimes, the order which is observed in the global error is one unit greater than the one expected because of the theory. This is due to the summation-by-parts procedure, which has been deeply studied for RKN methods (see [13, 14]).*

4. Numerical experiments

To show the behavior of FSRKN methods when solving a problem like (1), we will solve equation

$$\begin{cases} u_{tt}(x,y,t) &= -u_{xxxx}(x,y,t) - u_{yyyy}(x,y,t) + f(x,y,t), & (x,y,t) \in \Omega \times [0,T], \\ u(x,y,t) &= e^{-t+x^2+2y}, & (x,y,t) \in \Gamma \times [0,T] = \partial\Omega \times [0,T], \\ u_x(x,y,t) &= 2xe^{-t+x^2+2y}, & (x,y,t) \in \Gamma_1 \times [0,T], \\ u_y(x,y,t) &= 2e^{-t+x^2+2y}, & (x,y,t) \in \Gamma_2 \times [0,T], \\ u(x,y,0) &= e^{x^2+2y} & (x,y) \in \Omega, \\ u_t(x,y,0) &= -e^{x^2+2y}, & (x,y) \in \Omega, \end{cases}$$

where $\Omega \times [0,T] = (-1,1) \times (-1,1) \times [0,1]$, with $\Gamma_1 = \{-1,1\} \times [-1,1]$ and $\Gamma_2 = [-1,1] \times \{-1,1\}$. For this problem, we split the elliptic operator as $A = A_1 + A_2$, with $A_1\, u = -u_{xxxx}$ and $A_2\, u = -u_{yyyy}$. Besides, we decompose the source term as $f(x,y,t) = f_1(x,y,t) + f_2(x,y,t)$ taking $f_1(x,y,t) = u_{xxxx} + \frac{1}{2}u_{tt}$ and $f_2(x,y,t) = u_{yyyy} + \frac{1}{2}u_{tt}$, in order to obtain $u(x,y,t) = e^{-t+x^2+2y}$ as the exact solution.

Firstly, we have discretized in time by using the R-stable FSRKN method presented in [2] with stage order 1 and classical order 3. Thus, for each time step, we must solve four boundary value problems, one per stage $K_{n,i}$, $i = 1, \cdots, 4$. Every one of these problems is essentially one-dimensional in space, as in the odd stages only u_{xxxx} acts implicitly and in the even stages the term that acts implicitly is u_{yyyy}. We have integrated the boundary value problem that appear by imposing the boundary values given by $G^{[r]}_{\ell,n}$, $\ell = 1, \cdots, m$ for $r = 0$ and $r = 1$, i.e, by taking the classical boundary conditions for $r = 0$ and the new boundary conditions for $r = 1$.

On the other hand, after doing this, we have discretized in space by using the spectral method described in [16, 17] (and deeply studied in [14]). For our discretization we have taken 40 nodes in the interval $(-1,1)$, so we have obtained 40 decoupled systems of size 40×40 to be solved at each stage. This can be compared with the system we would have obtained when solving the same problem with an implicit RKN method and an adequate spatial discretization for solving problems like $u_{xxxx} + u_{yyyy} = F(u)$ in the square $(-1,1) \times (-1,1)$; in this case we would have obtained a $(40 \times 40) \times (40 \times 40)$ system to be solved.

In the tables, the local and global errors are given in the discrete norm associated to the spatial discretization that we are using. The global error has been calculated as the difference between the exact solution at $T = 1$ and the numerical one obtained with our method. In the figures, the error has been plotted as a function of τ, the time step-size, in double logarithmic scale, so in this way the slope of the lines corresponds to the numerical order observed.

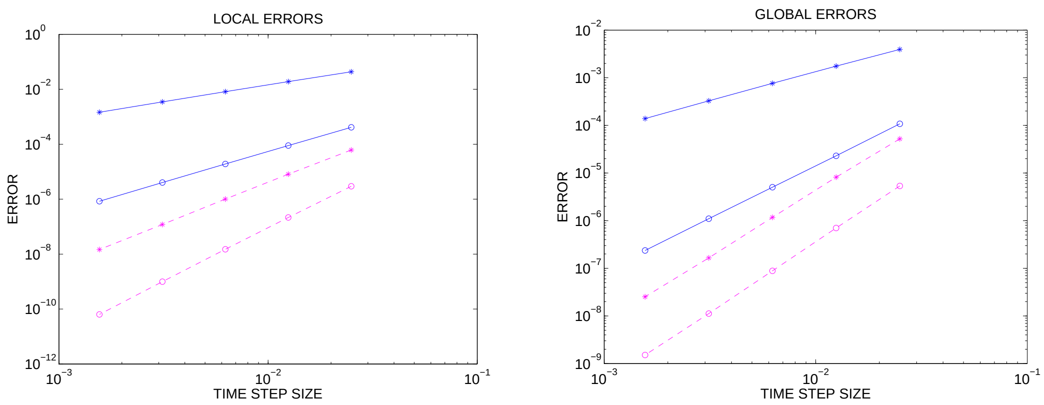

Figure 1: Graph for the local (left) and global (right) errors with order reduction (blue, continuous line) and avoiding it with one iteration (pink dashed line) for the solution (o) and the derivative (*).

	Order reduction		Avoiding order reduction	
τ	$u(t)$	$u'(t)$	$u(t)$	$u'(t)$
1/40 - 1/80	2.20728	1.19357	3.76489	2.92037
1/80 - 1/160	2.22778	1.21851	3.85445	3.01228
1/160 - 1/320	2.24447	1.23712	3.91191	3.07119
1/320 - 1/640	2.26014	1.25327	3.95459	3.02705

Table 1: Local orders

5. Proof of the main results

In order to prove the theorems, the following lemma is needed, which has been proven in [2].

Lemma 5.1. *Let us consider an FSRKN method satisfying that* $a_{\ell_i,ii} > 0$, $i = 1, \cdots, s$, $\ell_i \in \{1, \cdots, m\}$ *and let* $\{A_\ell^0\}_{\ell=1}^m$ *be a system of self-adjoint and negative definite spatial operators in* $\mathcal{H}^0$. *Then*

(i) The operator $(\mathcal{I} \otimes I - \tau^2 \sum_{\ell=1}^m \mathcal{A}_\ell \otimes A_\ell^0)$ *is invertible, and its inverse is bounded independently of* $\tau \in (0, \tau_0]$.

	Order reduction		Avoiding order reduction	
τ	$u(t)$	$u'(t)$	$u(t)$	$u'(t)$
1/40 - 1/80	2.23835	1.16461	2.93637	2.67649
1/80 - 1/160	2.18800	1.20047	2.98490	2.79997
1/160 - 1/320	2.19312	1.22326	2.98089	2.82830
1/320 - 1/640	2.21931	1.23887	2.89093	2.71291

Table 2: Global orders until T=1

(ii) The operators $\tau^2(b_\ell^T \otimes A_\ell^0)(\mathcal{I}\otimes I - \tau^2 \sum_{\ell=1}^m \mathcal{A}_\ell \otimes A_\ell^0)^{-1}$ *and* $\tau^2(\beta_\ell^T \otimes A_\ell^0)(\mathcal{I}\otimes I - \tau^2 \sum_{\ell=1}^m \mathcal{A}_\ell \otimes A_\ell^0)^{-1}$ *are bounded independently of* $\tau \in (0,\tau_0]$, $\ell = 1,\cdots,m$.

5.1. Proof of Theorem 2.5

The local errors defined by (18) can be written as

$$\xi_{n+1}^{[r]} = (u'(t_{n+1}) - V_{n+1}^{[r]}) + (V_{n+1}^{[r]} - \bar{V}_{n+1}^{[r]}), \tag{28}$$

$$\rho_{n+1}^{[r]} = (u(t_{n+1}) - U_{n+1}^{[r]}) + (U_{n+1}^{[r]} - \bar{U}_{n+1}^{[r]}), \tag{29}$$

with

$$V_{n+1}^{[r]} = u'(t_n) + \tau \sum_{\ell=1}^m (b_\ell^T \otimes I)((\mathcal{I}\otimes A_\ell)K_n^{[r]} + f_{\ell,n}), \tag{30}$$

$$U_{n+1}^{[r]} = u(t_n) + \tau\, u'(t_n) + \tau^2 \sum_{\ell=1}^m (\beta_\ell^T \otimes I)((\mathcal{I}\otimes A_\ell)K_n^{[r]} + f_{\ell,n}). \tag{31}$$

where $K_n^{[r]}$ is given by (13) for $r=0$ and (14) for integers $r \geq 1$.

Bound for $V_{n+1}^{[r]} - \bar{V}_{n+1}^{[r]}$ and $U_{n+1}^{[r]} - \bar{U}_{n+1}^{[r]}$

Let us firstly define $\delta_n^{[r]} = [\delta_{n,1}^{[r]}, \cdots, \delta_{n,s}^{[r]}]^T$, integer $r \geq 0$, as the vector that contains the errors that are committed in the quadrature formula for the stages, in the way

$$K_n^{[r]} = (e\otimes I)u(t_n) + \tau(c\otimes I)u'(t_n) + \tau^2 \sum_{\ell=1}^m (\mathcal{A}_\ell \otimes I)((\mathcal{I}\otimes A_\ell)K_n^{[r]} + f_{\ell,n}) + \delta_n^{[r]}. \tag{32}$$

By doing (32) minus (14),

$$\delta_n^{[r]} = \tau^2 \sum_{\ell=1}^m (\mathcal{A}_\ell \otimes A_\ell)(K_n^{[r-1]} - K_n^{[r]}).$$

On the other hand, from (14) it can be proven that

$$K_n^{[r-1]} - K_n^{[r]} = \tau^2 \sum_{\ell=1}^m (\mathcal{A}_\ell \otimes A_\ell)(K_n^{[r-2]} - K_n^{[r-1]}) = \ldots = \Big(\tau^2 \sum_{\ell=1}^m \mathcal{A}_\ell \otimes A_\ell\Big)^{r-1} (K_n^{[0]} - K_n^{[1]}).$$

Finally, from (32) for $r = 0$ and expression (14) for $r = 1$, we take

$$\delta_n^{[r]} = \Big(\tau^2 \sum_{\ell=1}^m \mathcal{A}_\ell \otimes A_\ell\Big)^r (K_n^{[0]} - K_n^{[1]}) = \tau^{2r}\Big(\sum_{\ell=1}^m \mathcal{A}_\ell \otimes A_\ell\Big)^r \delta_n^{[0]}. \tag{33}$$

Now, we develop $\delta_n^{[0]}$ component by component, by using Taylor developments with integral rest. In order to simplify, for functions $h(t)$ smooth enough, we introduce the following notation:

$$\mathcal{R}_{p,n,i}(h) = \int_{t_n}^{t_{n,i}} (t_{n,i} - z)^p \; h^{(p+1)}(z)dz \qquad \text{and} \qquad \mathcal{R}_{p,n}(h) = [\mathcal{R}_{p,n,1}(h), \cdots, \mathcal{R}_{p,n,s}(h)]^T.$$

Therefore,

$$\begin{aligned}
\delta_{n,i}^{[0]} &= \sum_{k=2}^{p+1} \frac{\tau^k c_i^k}{k!} \sum_{\ell=1}^m \Big(A_\ell u^{(k-2)}(t_n) + f_\ell^{(k-2)}(t_n)\Big) + \frac{1}{(p+1)!}\mathcal{R}_{p+1,n,i}(u) \\
&- \sum_{k=2}^{p+1} \frac{\tau^k}{(k-2)!} \sum_{\ell=1}^m \sum_{j=1}^i a_{\ell,ij} c_j^{k-2} \Big(A_\ell u^{(k-2)}(t_n) + f_\ell^{(k-2)}(t_n)\Big) \\
&- \frac{\tau^2}{(p-1)!} \sum_{\ell=1}^m \sum_{j=1}^i a_{\ell,ij} \Big(A_\ell \mathcal{R}_{p-1,n,j}(u) + \mathcal{R}_{p-1,n,j}(f_\ell)\Big) \\
&= \sum_{k=\tilde{q}+1}^{p+1} \frac{\tau^k}{k!} \sum_{\ell=1}^m (c_i^k - k(k-1)\sum_{j=1}^i a_{\ell,ij} c_j^{k-2}) \Big(A_\ell u^{(k-2)}(t_n) + f_\ell^{(k-2)}(t_n)\Big) \\
&+ \frac{1}{(p+1)!}\mathcal{R}_{p+1,n,i}(u) - \frac{\tau^2}{(p-1)!} \sum_{\ell=1}^m \sum_{j=1}^i a_{\ell,ij} \Big(A_\ell \mathcal{R}_{p-1,n,j}(u) + \mathcal{R}_{p-1,n,j}(f_\ell)\Big),
\end{aligned}$$

where we have used the definition of stage order given in (12) together with

$$u^{(k)}(t) = \sum_{\ell=1}^m (A_\ell u^{(k-2)}(t) + f_\ell^{(k-2)}(t)). \tag{34}$$

Thus, this expression can be written in tensorial form as

$$\begin{aligned}
\delta_n^{[0]} &= \sum_{k=\tilde{q}+1}^{p+1} \frac{\tau^k}{k!} \sum_{\ell=1}^m (c^k - k(k-1)\mathcal{A}_\ell c^{k-2}) \otimes \Big(A_\ell u^{(k-2)}(t_n) + f_\ell^{(k-2)}(t_n)\Big) + \frac{1}{(p+1)!}\mathcal{R}_{p+1,n}(u) \\
&- \frac{\tau^2}{(p-1)!} \sum_{\ell=1}^m (\mathcal{A}_\ell \otimes I)\Big((\mathcal{I} \otimes A_\ell)\mathcal{R}_{p-1,n}(u) + \mathcal{R}_{p-1,n}(f_\ell)\Big).
\end{aligned}$$

Therefore, by substituting in (33)

$$\begin{aligned}
\delta_n^{[r]} &= \sum_{k=\tilde{q}+1}^{p+1} \frac{\tau^{k+2r}}{k!} \Big(\sum_{\ell_1=1}^m \mathcal{A}_{\ell_1} \otimes A_{\ell_1}\Big)^r \sum_{\ell_2=1}^m (c^k - k(k-1)\mathcal{A}_{\ell_2} c^{k-2}) \otimes (A_{\ell_2} u^{(k-2)}(t_n) + f_{\ell_2}^{(k-2)}(t_n)) \\
&+ \frac{\tau^{2r}}{(p+1)!} \Big(\sum_{\ell=1}^m \mathcal{A}_\ell \otimes A_\ell\Big)^r \mathcal{R}_{p+1,n}(u) \\
&- \frac{\tau^{2r+2}}{(p-1)!} \Big(\sum_{\ell_1=1}^m \mathcal{A}_{\ell_1} \otimes A_{\ell_1}\Big)^r \sum_{\ell_2=1}^m (\mathcal{A}_{\ell_2} \otimes I)\Big((\mathcal{I} \otimes A_{\ell_2})\mathcal{R}_{p-1,n}(u) + \mathcal{R}_{p-1,n}(f_{\ell_2})\Big).
\end{aligned} \tag{35}$$

Apart from this, we define $\Delta_n^{[r]}$ as the difference between $K_n^{[r]}$ and $\bar{K}_n^{[r]}$. Then, when subtracting (15) to (32), taking into account that $(\partial_1, \cdots, \partial_m)\bar{K}_n^{[r]} = (\partial_1, \cdots, \partial_m)K_n^{[r]}$ we obtain

$$\begin{aligned} \Delta_n^{[r]} &= \tau^2 \sum_{\ell=1}^{m} (\mathcal{A}_\ell \otimes A_\ell)\Delta_n^{[r]} + \delta_n^{[r]}, \\ (\partial_1, \cdots, \partial_m)\Delta_n^{[r]} &= (0, \cdots, 0). \end{aligned}$$

By using Lemma 5.1 we can solve for $\Delta_n^{[r]}$, (notice that, now, $A_\ell \equiv A_\ell^0$)

$$\Delta_n^{[r]} = (\mathcal{I} \otimes I - \tau^2 \sum_{\ell=1}^{m} \mathcal{A}_\ell \otimes A_\ell^0)^{-1} \delta_n^{[r]}. \tag{36}$$

Now, we subtract (30) to (16), by using that now $\tau(b_k^T \otimes A_k)\Delta_n^{[r]} \equiv \tau(b_k^T \otimes A_k^0)\Delta_n^{[r]}$ together with (35) and (36). Then, we obtain

$$\begin{aligned} V_{n+1}^{[r]} &- \bar{V}_{n+1}^{[r]} = \tau \sum_{\ell=1}^{m} (b_\ell^T \otimes A_\ell^0)\Delta_n^{[r]} = \tau \sum_{\ell=1}^{m} (b_\ell^T \otimes A_\ell^0)(\mathcal{I} \otimes I - \tau^2 \sum_{j=1}^{m} \mathcal{A}_j \otimes A_j^0)^{-1} \delta_n^{[r]} \\ &= \sum_{k=\tilde{q}+1}^{p+1} \frac{\tau^{k+2r-1}}{k!} \sum_{\ell_1,\cdots,\ell_{r+2}=1}^{m} \tilde{R}_{k,\ell_1,\cdots,\ell_{r+2}}(A_1^0, \cdots, A_m^0) A_{\ell_2} \cdots A_{\ell_{r+1}} (A_{\ell_{r+2}} u^{(k-2)}(t_n) + f_{\ell_{r+2}}^{(k-2)}(t_n)) \\ &+ \frac{\tau^{2r-1}}{(p+1)!} \sum_{\ell_1,\cdots,\ell_{r+1}=1}^{m} b_{\ell_1}(A_1^0, \cdots, A_m^0)(\mathcal{A}_{\ell_2} \cdots \mathcal{A}_{\ell_{r+1}} \otimes A_{\ell_2} \cdots A_{\ell_{r+1}})\mathcal{R}_{p+1,n}(u) \\ &- \frac{\tau^{2r+1}}{(p-1)!} \sum_{\ell_1,\cdots,\ell_{r+2}=1}^{m} b_{\ell_1}(A_1^0, \cdots, A_m^0)(\mathcal{A}_{\ell_2} \cdots \mathcal{A}_{\ell_{r+2}} \otimes A_{\ell_2} \cdots A_{\ell_{r+1}})(\mathcal{I} \otimes A_{\ell_{r+2}})\mathcal{R}_{p-1,n}(u) \\ &- \frac{\tau^{2r+1}}{(p-1)!} \sum_{\ell_1,\cdots,\ell_{r+2}=1}^{m} b_{\ell_1}(A_1^0, \cdots, A_m^0)(\mathcal{A}_{\ell_2} \cdots \mathcal{A}_{\ell_{r+2}} \otimes A_{\ell_2} \cdots A_{\ell_{r+1}})\mathcal{R}_{p-1,n}(f_{\ell_{r+2}}), \end{aligned} \tag{37}$$

where we have used notation

$$\begin{aligned} b_\ell(A_1^0, \cdots, A_m^0) &= (b_\ell^T \otimes \tau^2 A_\ell^0)(\mathcal{I} \otimes I - \tau^2 \sum_{k=1}^{m} \mathcal{A}_k \otimes A_k^0)^{-1}, \\ \tilde{R}_{k,\ell_1,\cdots,\ell_d}(A_1^0, \cdots, A_m^0) &= b_{\ell_1}(A_1^0, \cdots, A_m^0)(\mathcal{A}_{\ell_2} \cdots \mathcal{A}_{\ell_{d-1}}(c^k - k(k-1)\mathcal{A}_{\ell_d} c^{k-2}) \otimes I), \end{aligned}$$

Similarly, from (17) and (31), by using again (35) and (36),

$$\begin{aligned} U_{n+1}^{[r]} &- \bar{U}_{n+1}^{[r]} = \tau^2 \sum_{\ell=1}^{m} (\beta_\ell^T \otimes A_\ell^0)\Delta_n^{[r]} = \tau^2 \sum_{\ell=1}^{m} (\beta_\ell^T \otimes A_\ell^0)(\mathcal{I} \otimes I - \tau^2 \sum_{j=1}^{m} \mathcal{A}_j \otimes A_j^0)^{-1} \delta_n^{[r]} \\ &= \sum_{k=\tilde{q}+1}^{p+1} \frac{\tau^{k+2r}}{k!} \sum_{\ell_1,\cdots,\ell_{r+2}=1}^{m} R_{k,\ell_1,\cdots,\ell_{r+2}}(A_1^0, \cdots, A_m^0) A_{\ell_2} \cdots A_{\ell_{r+1}} (A_{\ell_{r+2}} u^{(k-2)}(t_n) + f_{\ell_{r+2}}^{(k-2)}(t_n)) \\ &+ \frac{\tau^{2r}}{(p+1)!} \sum_{\ell_1,\cdots,\ell_{r+1}=1}^{m} \beta_{\ell_1}(A_1^0, \cdots, A_m^0)(\mathcal{A}_{\ell_2} \cdots \mathcal{A}_{\ell_{r+1}} \otimes A_{\ell_2} \cdots A_{\ell_{r+1}})\mathcal{R}_{p+1,n}(u) \end{aligned} \tag{38}$$

$$
\begin{aligned}
&- \frac{\tau^{2r+2}}{(p-1)!} \sum_{\ell_1,\cdots,\ell_{r+2}=1}^{m} \beta_{\ell_1}(A_1^0,\cdots,A_m^0)(\mathcal{A}_{\ell_2}\cdots\mathcal{A}_{\ell_{r+2}} \otimes A_{\ell_2}\cdots A_{\ell_{r+1}})(\mathcal{I}\otimes A_{\ell_{r+2}})\,\mathcal{R}_{p-1,n}(u)\\
&- \frac{\tau^{2r+2}}{(p-1)!} \sum_{\ell_1,\cdots,\ell_{r+2}=1}^{m} \beta_{\ell_1}(A_1^0,\cdots,A_m^0)(\mathcal{A}_{\ell_2}\cdots\mathcal{A}_{\ell_{r+2}} \otimes A_{\ell_2}\cdots A_{\ell_{r+1}})\mathcal{R}_{p-1,n}(f_{\ell_{r+2}}),
\end{aligned}
$$

where we have used now notation

$$
\begin{aligned}
\beta_\ell(A_1^0,\cdots,A_m^0) &= (\beta_\ell^T \otimes \tau^2 A_\ell^0)(\mathcal{I}\otimes I - \tau^2 \sum_{k=1}^{m} \mathcal{A}_k \otimes A_k^0)^{-1},\\
R_{k,\ell_1,\cdots,\ell_d}(A_1^0,\cdots,A_m^0) &= \beta_{\ell_1}(A_1^0,\cdots,A_m^0)(\mathcal{A}_{\ell_2}\cdots\mathcal{A}_{\ell_{d-1}}(c^k - k(k-1)\mathcal{A}_{\ell_d}c^{k-2})\otimes I),
\end{aligned}
$$

From Lemma 5.1 we have that $\tilde{R}_{k,\ell_1,\cdots,\ell_{r+2}}(A_1^0,\cdots,A_m^0)$, $R_{k,\ell_1,\cdots,\ell_{r+2}}(A_1^0,\cdots,A_m^0)$, $b_{\ell_1}(A_1^0,\cdots,A_m^0)$ and $\beta_{\ell_1}(A_1^0,\cdots,A_m^0)$ are well defined and bounded for integers $r \geq 0$, $k \geq \tilde{q}$, $\ell_1,\cdots,\ell_{r+2} \in \{1,\cdots,m\}$. Furthermore, when bounding the integral terms, if $h^{(p+2)}(t)$ are bounded independently of τ for $t \in [t_n, t_{n,i}]$, we have

$$
\|\mathcal{R}_{p+1,n,i}(h)\| \;\leq\; |c_i|\tau \max_{t_n \leq t \leq t_{n,i}} |t_{n,i}-t|^{p+1}\|h^{(p+2)}(t)\| = \mathcal{O}(\tau^{p+2}). \tag{39}
$$

Moreover, for any (B,∂), closed with $B\,h^{(p)}(t)$ bounded independently of τ for $t \in [t_n, t_{n,i}]$ (see [18]), we also have

$$
\|B\,\mathcal{R}_{p-1,n,i}(h)\| = \left\|\int_{t_n}^{t_{n,i}} (t_{n,i}-z)^{p-1} B\,h^{(p)}(z)dz\right\| \leq |c_i|\tau \max_{t_n \leq t \leq t_{n,i}} |t_{n,i}-t|^{p-1}\|B\,h^{(p)}(t)\| = \mathcal{O}(\tau^p).
$$

Thus, from hypothesis (B1), as (A_ℓ, ∂_ℓ) is closed for $\ell = 1,\cdots,m,$, if $A_1 \ldots A_k\,h^{(p)}(t)$ are bounded independently of τ for $t \in [t_n, t_{n,i}]$, we deduce that

$$
\begin{aligned}
\|A_1\ldots A_k\mathcal{R}_{p-1,n,i}(h)\| &= \left\|\int_{t_n}^{t_{n,i}} (t_{n,i}-z)^{p-1} A_1\ldots A_k h^{(p)}(z)dz\right\|\\
&\leq |c_i|\tau \max_{t_n \leq t \leq t_{n,i}} |t_{n,i}-t|^{p-1}\|A_1\ldots A_k h^{(p)}(t)\| = \mathcal{O}(\tau^p).
\end{aligned} \tag{40}
$$

Therefore, by using in expressions (37) and (38) these results together with (3), we take

$$
\|V_{n+1}^{[r]} - \bar{V}_{n+1}^{[r]}\| = \mathcal{O}(\tau^{\min\{\tilde{q}+2r,p+2r+1\}}) \qquad \text{and} \qquad \|U_{n+1}^{[r]} - \bar{U}_{n+1}^{[r]}\| = \mathcal{O}(\tau^{\min\{\tilde{q}+2r+1,p+2r+2\}}). \tag{41}
$$

Bound for $u'(t_{n+1}) - V_{n+1}^{[r]}$ and $u(t_{n+1}) - U_{n+1}^{[r]}$

Differences $u'(t_{n+1}) - V_{n+1}^{[r]}$ and $u(t_{n+1}) - U_{n+1}^{[r]}$ can be written as

$$
u'(t_{n+1}) - V_{n+1}^{[r]} = u'(t_{n+1}) - V_{n+1}^{[0]} + \sum_{i=0}^{r-1}(V_{n+1}^{[i]} - V_{n+1}^{[i+1]}), \tag{42}
$$

$$
u(t_{n+1}) - U_{n+1}^{[r]} = u(t_{n+1}) - U_{n+1}^{[0]} + \sum_{i=0}^{r-1}(U_{n+1}^{[i]} - U_{n+1}^{[i+1]}). \tag{43}
$$

From (30-31) together with (14-15) and (32) we obtain

$$
\begin{aligned}
V_{n+1}^{[i]} - V_{n+1}^{[i+1]} &= \tau \sum_{\ell=1}^{m} (b_\ell^T \otimes A_\ell)(K_n^{[i]} - K_n^{[i+1]}) = \tau \sum_{\ell=1}^{m} (b_\ell^T \otimes A_\ell)\delta_n^{[i]}, \\
U_{n+1}^{[i]} - U_{n+1}^{[i+1]} &= \tau^2 \sum_{\ell=1}^{m} (\beta_\ell^T \otimes A_\ell)(K_n^{[i]} - K_n^{[i+1]}) = \tau^2 \sum_{\ell=1}^{m} (\beta_\ell^T \otimes A_\ell)\delta_n^{[i]}.
\end{aligned}
$$

Then, because of (35) and by reorganizing the operators

$$
\begin{aligned}
V_{n+1}^{[i]} &- V_{n+1}^{[i+1]} \\
&= \sum_{\substack{k=\tilde{q}+1,\\ k+2i\geq p}}^{p+1} \frac{\tau^{k+2i+1}}{k!} \sum_{\ell_1,\cdots,\ell_{i+2}=1}^{m} b_{\ell_1}^T \mathcal{A}_{\ell_2}\cdots\mathcal{A}_{\ell_{i+1}}(c^k - k(k-1)\mathcal{A}_{\ell_{i+2}}c^{k-2})A_{\ell_1}\cdots A_{\ell_{i+2}}u^{(k-2)}(t_n) \\
&+ \sum_{\substack{k=\tilde{q}+1,\\ k+2i\geq p}}^{p+1} \frac{\tau^{k+2i+1}}{k!} \sum_{\ell_1,\cdots,\ell_{i+2}=1}^{m} b_{\ell_1}^T \mathcal{A}_{\ell_2}\cdots\mathcal{A}_{\ell_{i+1}}(c^k - k(k-1)\mathcal{A}_{\ell_{i+2}}c^{k-2})A_{\ell_1}\cdots A_{\ell_{i+1}}f_{\ell_{i+2}}^{(k-2)}(t_n) \\
&+ \frac{\tau^{2i+1}}{(p+1)!} \sum_{\ell_1,\cdots,\ell_{i+1}=1}^{m} (b_{\ell_1}^T \mathcal{A}_{\ell_2}\cdots\mathcal{A}_{\ell_{i+1}} \otimes A_{\ell_1}\cdots A_{\ell_{i+1}})\mathcal{R}_{p+1,n}(u) \\
&- \frac{\tau^{2i+3}}{(p-1)!} \sum_{\ell_1,\cdots,\ell_{i+2}=1}^{m} (b_{\ell_1}^T \mathcal{A}_{\ell_2}\cdots\mathcal{A}_{\ell_{i+2}} \otimes A_{\ell_1}\cdots A_{\ell_{i+1}})((\mathcal{I}\otimes A_{\ell_{i+2}})\,\mathcal{R}_{p-1,n}(u) + \mathcal{R}_{p-1,n}(f_{\ell_{i+2}})),
\end{aligned}
$$

where we have used that some terms vanish because of the order p conditions $\forall\, i = 0,\ldots,r-1$

$$
b_{\ell_0}^T \mathcal{A}_{\ell_1}\cdots\mathcal{A}_{\ell_i}c^k = \frac{1}{(k+2i+1)(k+2i)\cdots(k+1)}, \tag{44}
$$

for $0 \leq k+2i \leq p-1$ and $\ell_0, \ell_1, \ldots, \ell_i \in \{1,\ldots,m\}$.

Similarly, we consider now the order p conditions $\forall\, i = 0,\ldots,r-1$

$$
\beta_{\ell_0}^T \mathcal{A}_{\ell_1}\cdots\mathcal{A}_{\ell_i}c^k = \frac{1}{(k+2i+2)(k+2i+1)\cdots(k+1)}, \tag{45}
$$

for $0 \leq k+2i \leq p-2$ and $\ell_0, \ell_1, \ldots, \ell_i \in \{1,\ldots,m\}$. Then, again, because of (35) and by reorganizing the operators

$$
\begin{aligned}
U_{n+1}^{[i]} &- U_{n+1}^{[i+1]} \\
&= \sum_{\substack{k=\tilde{q}+1,\\ k+2i\geq p-1}}^{p+1} \frac{\tau^{k+2i+2}}{k!} \sum_{\ell_1,\cdots,\ell_{i+2}=1}^{m} \beta_{\ell_1}^T \mathcal{A}_{\ell_2}\cdots\mathcal{A}_{\ell_{i+1}}(c^k - k(k-1)\mathcal{A}_{\ell_{i+2}}c^{k-2})A_{\ell_1}\cdots A_{\ell_{i+2}}u^{(k-2)}(t_n)
\end{aligned}
$$

$$+ \sum_{\substack{k=\tilde{q}+1,\\ k+2i\geq p-1}}^{p+1} \frac{\tau^{k+2i+2}}{k!} \sum_{\ell_1,\cdots,\ell_{i+2}=1}^{m} \beta_{\ell_1}^T \mathcal{A}_{\ell_2}\cdots\mathcal{A}_{\ell_{i+1}}(c^k - k(k-1)\mathcal{A}_{\ell_{i+2}}c^{k-2})A_{\ell_1}\cdots A_{\ell_{i+1}} f_{\ell_{i+2}}^{(k-2)}(t_n)$$

$$+ \frac{\tau^{2i+2}}{(p+1)!} \sum_{\ell_1,\cdots,\ell_{i+1}=1}^{m} (\beta_{\ell_1}^T \mathcal{A}_{\ell_2}\cdots\mathcal{A}_{\ell_{i+1}} \otimes A_{\ell_1}\cdots A_{\ell_{i+1}})\mathcal{R}_{p+1,n}(u)$$

$$- \frac{\tau^{2i+4}}{(p-1)!} \sum_{\ell_1,\cdots,\ell_{i+2}=1}^{m} (\beta_{\ell_1}^T \mathcal{A}_{\ell_2}\cdots\mathcal{A}_{\ell_{i+2}} \otimes A_{\ell_1}\cdots A_{\ell_{i+1}})((\mathcal{I}\otimes A_{\ell_{i+2}})\,\mathcal{R}_{p-1,n}(u) + \mathcal{R}_{p-1,n}(f_{\ell_{i+2}})).$$

Therefore, by using (3) we have

$$\|V_{n+1}^{[i]} - V_{n+1}^{[i+1]}\| = \mathcal{O}(\tau^{p+1}) \qquad \text{and} \qquad \|U_{n+1}^{[i]} - U_{n+1}^{[i+1]}\| = \mathcal{O}(\tau^{p+1}). \tag{46}$$

In order to bound $u'(t_{n+1}) - V_{n+1}^{[0]}$ we use (30) for $r = 0$ and that $K_{n,i}^{[0]} \equiv u(t_{n,i})$. By developing by Taylor,

$$\begin{aligned} u'(t_{n+1}) \;\; - \;\; & V_{n+1}^{[0]} = \sum_{k=1}^{p+1} \frac{\tau^k}{k!} u^{(k+1)}(t_n) + \frac{1}{(p+1)!}\int_{t_n}^{t_{n+1}} (t_{n+1}-z)^{p+1}\, u^{(p+3)}(z)dz \\ - \;\; & \sum_{k=0}^{p} \frac{\tau^{k+1}}{k!} \sum_{\ell=1}^{m} b_\ell^T c^k (A_\ell u^{(k)}(t_n) + f_\ell^{(k)}(t_n)) - \frac{\tau}{p!}\sum_{\ell=1}^{m}(b_\ell^T \otimes I)((\mathcal{I}\otimes A_\ell)\mathcal{R}_{p,n}(u) + \mathcal{R}_{p,n}(f_\ell)) \\ = \;\; & \frac{\tau^{p+1}}{(p+1)!}\sum_{\ell=1}^{m}(1-(p+1)b_\ell^T c^p)(A_\ell u^{(p)}(t_n) + f_\ell^{(p)}(t_n)) \\ & + \frac{1}{(p+1)!}\int_{t_n}^{t_{n+1}} (t_{n+1}-z)^{p+1}\, u^{(p+3)(z)dz} \\ - \;\; & \frac{\tau}{p!}\sum_{\ell=1}^{m}(b_\ell^T \otimes I)((\mathcal{I}\otimes A_\ell)\mathcal{R}_{p,n}(u) + \mathcal{R}_{p,n}(f_\ell)), \end{aligned} \tag{47}$$

where, in the last equality, we have used (34) and the order conditions given by (44).

With a similar argument to bound $u(t_{n+1}) - U_{n+1}^{[0]}$, we subtract (31) for $r = 0$ to $u(t_{n+1})$,

$$\begin{aligned} u(t_{n+1}) - U_{n+1}^{[0]} \;\; = \;\; & \frac{\tau^{p+1}}{(p+1)!}\sum_{\ell=1}^{m}(1-(p+1)p\beta_\ell^T c^{p-1})(A_\ell u^{(p-1)}(t_n) + f_\ell^{(p-1)}(t_n)) \\ & + \frac{1}{(p+1)!}\int_{t_n}^{t_{n+1}} (t_{n+1}-z)^{p+1}\, u^{(p+2)}(z)dz \\ & - \frac{\tau^2}{(p-1)!}\sum_{\ell=1}^{m}(\beta_\ell^T \otimes I)((\mathcal{I}\otimes A_\ell)\mathcal{R}_{p-1,n}(u) + \mathcal{R}_{p-1,n}(f_\ell)), \end{aligned} \tag{48}$$

where (34) has been used again together with (45).

Then, by bounding (47) and (48), by using hypothesis (3), and bounds (39-40),

$$\|u'(t_{n+1}) - V_{n+1}^{[0]}\| = \mathcal{O}(\tau^{p+1}) \qquad \text{and} \qquad \|u(t_{n+1}) - U_{n+1}^{[0]}\| = \mathcal{O}(\tau^{p+1}), \tag{49}$$

where we can conclude, by using bounds (46) and (49) in (42) and (43) that

$$\|u'(t_{n+1}) - V_{n+1}^{[r]}\| \leq \|v(t_{n+1}) - V_{n+1}^{[0]}\| + \sum_{i=0}^{r-1} \|V_{n+1}^{[i]} - V_{n+1}^{[i+1]}\| = \mathcal{O}(\tau^{p+1}), \quad (50)$$

$$\|u(t_{n+1}) - U_{n+1}^{[r]}\| \leq \|u(t_{n+1}) - U_{n+1}^{[0]}\| + \sum_{i=0}^{r-1} \|U_{n+1}^{[i]} - U_{n+1}^{[i+1]}\| = \mathcal{O}(\tau^{p+1}). \quad (51)$$

Finally, we bound (28) and (29) by using (41), (50) and (51), so

$$\begin{aligned}
\|\xi_{n+1}^{[r]}\| &\leq \|u'(t_{n+1}) - V_{n+1}^{[r]}\| + \|V_{n+1}^{[r]} - \bar{V}_{n+1}^{[r]}\| \\
&= \mathcal{O}(\tau^{\min\{\tilde{q}+2r, p+2r+1\}}) + \mathcal{O}(\tau^{p+1}) = \mathcal{O}(\tau^{\min\{\tilde{q}+2r, p+1\}}), \\
\|\rho_{n+1}^{[r]}\| &\leq \|u(t_{n+1}) - U_{n+1}^{[r]}\| + \|U_{n+1}^{[r]} - \bar{U}_{n+1}^{[r]}\| \\
&= \mathcal{O}(\tau^{\min\{\tilde{q}+2r+1, p+2r+2\}}) + \mathcal{O}(\tau^{p+1}) = \mathcal{O}(\tau^{\min\{\tilde{q}+2r+1, p+1\}}).
\end{aligned}$$

5.2. Proof of Theorem 3.1

In order to study these global errors, let us define $\bar{K}_{n,J}^{0,[r]}$, $\bar{U}_{n+1,J}^{0,[r]}$ and $\bar{V}_{n+1,J}^{0,[r]}$ as the vectors that satisfy

$$\begin{aligned}
\bar{K}_{n,J}^{0,[r]} &= (e \otimes R_J)u(t_n) + \tau(c \otimes R_J)u'(t_n) \\
&\quad + \tau^2 \sum_{\ell=1}^{m} (\mathcal{A}_\ell \otimes I_J)((\mathcal{I} \otimes A_{\ell,J}^0)\bar{K}_{n,J}^{0,[r]} + (\mathcal{I} \otimes S_{\ell,J})\, G_{\ell,n}^{[r]} + (\mathcal{I} \otimes R_J)\, f_{\ell,n}), \quad (52) \\
\bar{V}_{n+1,J}^{0,[r]} &= R_J u'(t_n) + \tau \sum_{\ell=1}^{m} (b_\ell^T \otimes I_J)((\mathcal{I} \otimes A_{\ell,J}^0)\bar{K}_{n,J}^{0,[r]} + (\mathcal{I} \otimes S_{\ell,J})\, G_{\ell,n}^{[r]} + (\mathcal{I} \otimes R_J)\, f_{\ell,n}), \quad (53) \\
\bar{U}_{n+1,J}^{0,[r]} &= R_J u(t_n) + \tau R_J u'(t_n) \\
&\quad + \tau^2 \sum_{\ell=1}^{m} (\beta_\ell^T \otimes I_J)((\mathcal{I} \otimes A_{\ell,J}^0)\bar{K}_{n,J}^{0,[r]} + (\mathcal{I} \otimes S_{\ell,J})\, G_{\ell,n}^{[r]} + (\mathcal{I} \otimes R_J)\, f_{\ell,n}). \quad (54)
\end{aligned}$$

Taking into account the expressions for $\bar{V}_{n+1}^{[r]}$, $\bar{U}_{n+1}^{[r]}$, $\xi_{n+1}^{[r]}$ and $\rho_{n+1}^{[r]}$ given by (16-18), the global errors can be decomposed as

$$\begin{aligned}
\tilde{e}_{n+1,J}^{[r]} &= (P_J - R_J)u'(t_{n+1}) + R_J \xi_{n+1}^{[r]} + (R_J \bar{V}_{n+1}^{[r]} - \bar{V}_{n+1,J}^{0,[r]}) + (\bar{V}_{n+1,J}^{0,[r]} - V_{n+1,J}^{0,[r]}), \quad (55) \\
e_{n+1,J}^{[r]} &= (P_J - R_J)u(t_{n+1}) + R_J \rho_{n+1}^{[r]} + (R_J \bar{U}_{n+1}^{[r]} - \bar{U}_{n+1,J}^{0,[r]}) + (\bar{U}_{n+1,J}^{0,[r]} - U_{n+1,J}^{0,[r]}). \quad (56)
\end{aligned}$$

If we apply operator R_J to the expressions given by (15-17), for integer $r \geq 0$, we take

$$\begin{aligned}
(\mathcal{I} \otimes R_J)\bar{K}_n^{[r]} &= (e \otimes R_J)u(t_n) + \tau(c \otimes R_J)u'(t_n) \\
&\quad + \tau^2 \sum_{\ell=1}^{m} (\mathcal{A}_\ell \otimes R_J)((\mathcal{I} \otimes A_\ell)\bar{K}_n^{[r]} + f_{\ell,n}), \quad (57) \\
(\partial_1, \cdots, \partial_m)\bar{K}_n^{[r]} &= (G_{1,n}^{[r]}, \cdots, G_{m,n}^{[r]}),
\end{aligned}$$

$$R_J\bar{V}_{n+1}^{[r]} = R_J u'(t_n) + \tau \sum_{\ell=1}^{m} (b_\ell^T \otimes R_J)((\mathcal{I} \otimes A_\ell)\bar{K}_n^{[r]} + f_{\ell,n}), \quad (58)$$

$$R_J\bar{U}_{n+1}^{[r]} = R_J u(t_n) + \tau R_J u'(t_n) + \tau^2 \sum_{\ell=1}^{m} (\beta_\ell^T \otimes R_J)((\mathcal{I} \otimes A_\ell)\bar{K}_n^{[r]} + f_{\ell,n}). \quad (59)$$

Doing now (57) minus (52), and by using (19) we obtain

$$\begin{aligned}(\mathcal{I} \otimes R_J)\bar{K}_n^{[r]} &- (\mathcal{I} \otimes I_J)\bar{K}_{n,J}^{0,[r]} \\ &= \tau^2 \sum_{\ell=1}^{m} (\mathcal{A}_\ell \otimes I_J)((\mathcal{I} \otimes R_J\, A_\ell)\bar{K}_n^{[r]} - (\mathcal{I} \otimes A_{\ell,J}^0)\bar{K}_{n,J}^{0,[r]} - (\mathcal{I} \otimes S_{\ell,J})G_{\ell,n}^{[r]}) \\ &= \tau^2 \sum_{\ell=1}^{m} (\mathcal{A}_\ell \otimes A_{\ell,J}^0)((\mathcal{I} \otimes R_J)\, \bar{K}_n^{[r]} - (\mathcal{I} \otimes I_J)\bar{K}_{n,J}^{0,[r]})) \\ &\quad + \tau^2 \sum_{\ell=1}^{m} (\mathcal{A}_\ell \otimes (R_J - P_J)A_\ell)\bar{K}_n^{[r]}.\end{aligned}$$

Therefore, by applying hypothesis (H3) we obtain

$$(\mathcal{I} \otimes R_J)\bar{K}_n^{[r]} - (\mathcal{I} \otimes I_J)\bar{K}_{n,J}^{0,[r]} = (\mathcal{I} \otimes I_J - \tau^2 \sum_{\ell=1}^{m} (\mathcal{A}_\ell \otimes A_{\ell,J}^0))^{-1} \tau^2 \sum_{j=1}^{m} (\mathcal{A}_j \otimes (R_J - P_J)A_j)\bar{K}_n^{[r]}. \quad (60)$$

On the other hand, subtracting (54) to (59), and by using again (19),

$$\begin{aligned}R_J\bar{U}_{n+1}^{[r]} - \bar{U}_{n+1,J}^{0,[r]} &= \tau^2 \sum_{\ell=1}^{m} (\beta_\ell^T \otimes (R_J - P_J)\, A_\ell)\bar{K}_n^{[r]} \\ &\quad + \tau^2 \sum_{\ell=1}^{m} (\beta_\ell^T \otimes A_{\ell,J}^0)((\mathcal{I} \otimes R_J)\, \bar{K}_n^{[r]} - (\mathcal{I} \otimes I_J)\bar{K}_{n,J}^{0,[r]}).\end{aligned}$$

Now, by substituting in this expression (60) and by using the following notation

$$T\,(\tau, v_{\ell_1}, \mathcal{A}_{\ell_2}, \{A_{i,J}^0\}_{i=1}^m) = \tau^2 (v_{\ell_1}^T \otimes A_{\ell_1,J}^0)(\mathcal{I} \otimes I_J - \tau^2 \sum_{k=1}^{m} \mathcal{A}_k \otimes A_{k,J}^0)^{-1} (\mathcal{A}_{\ell_2} \otimes (R_J - P_J)), \quad (61)$$

we take

$$R_J\bar{U}_{n+1}^{[r]} - \bar{U}_{n+1,J}^{0,[r]} = \tau^2 \sum_{\ell=1}^{m} (\beta_\ell^T \otimes (R_J - P_J)A_\ell)\bar{K}_n^{[r]} + \tau^2 \sum_{\ell_1,\ell_2=1}^{m} T\,(\tau, \beta_{\ell_1}, \mathcal{A}_{\ell_2}, \{A_{i,J}^0\}_{i=1}^m)(\mathcal{I} \otimes A_{\ell_2})\bar{K}_n^{[r]}. \quad (62)$$

In an analogous way, doing (58) minus (53), by using (19) and (60) together with (61) we obtain

$$R_J\bar{V}_{n+1}^{[r]} - \bar{V}_{n+1,J}^{0,[r]} = \tau \sum_{\ell=1}^{m} (b_\ell^T \otimes (R_J - P_J)A_\ell)\bar{K}_n^{[r]} + \tau \sum_{\ell_1,\ell_2=1}^{m} T\,(\tau, b_{\ell_1}, \mathcal{A}_{\ell_2}, \{A_{i,J}^0\}_{i=1}^m)(\mathcal{I} \otimes A_{\ell_2})\bar{K}_n^{[r]}. \quad (63)$$

To bound $\bar{U}^{0,[r]}_{n+1,J} - U^{0,[r]}_{n+1,J}$ (as well as $\bar{V}^{0,[r]}_{n+1,J} - V^{0,[r]}_{n+1,J}$) we must subtract (21) to (52), and by applying notation (24) together with Lemma 5.1 we take

$$\begin{aligned}\bar{K}^{0,[r]}_{n,J} \quad - \quad K^{0,[r]}_{n,J} &= (\mathcal{I}\otimes I_J - \tau^2\sum_{\ell=1}^{m}\mathcal{A}_\ell\otimes A^0_{\ell,J})^{-1}\Big((e\otimes I_J)e^{[r]}_{n,J} + \tau(c\otimes I_J)\tilde{e}^{[r]}_{n,J} \\ &+\tau^2\sum_{\ell=1}^{m}(\mathcal{A}_\ell\otimes(R_J-P_J))f_{\ell,n} + (e\otimes(R_J-P_J))u(t_n) + \tau(c\otimes(R_J-P_J))u'(t_n)\Big).\end{aligned} \tag{64}$$

Now, on the one hand we do (54) minus (23) and on the other (53) minus (22) in order to calculate $\bar{U}^{0,[r]}_{n+1,J} - U^{0,[r]}_{n+1,J}$ and $\bar{V}^{0,[r]}_{n+1,J} - V^{0,[r]}_{n+1,J}$, respectively. From here, by using (64), we obtain

$$\begin{aligned}\bar{U}^{0,[r]}_{n+1,J} - U^{0,[r]}_{n+1,J} &= e^{[r]}_{n,J} + (R_J-P_J)u(t_n) + \tau\tilde{e}^{[r]}_{n,J} + \tau(R_J-P_J)u'(t_n) \\ &+\tau^2\sum_{\ell=1}^{m}(\beta_\ell^T\otimes A^0_{\ell,J})(\bar{K}^{0,[r]}_{n,J} - K^{0,[r]}_{n,J}) + \tau^2\sum_{\ell=1}^{m}(\beta_\ell^T\otimes(R_J-P_J))\,f_{\ell,n}\end{aligned}$$

expression that by using (25) together with (61) and (63) we reorganize as

$$\begin{aligned}\bar{U}^{0,[r]}_{n+1,J} - U^{0,[r]}_{n+1,J} &= (B^0_J)^{-1}r_{11}(\tau, B^0_J, \{A^0_{i,J}\}_{i=1}^m)B^0_J(e^{[r]}_{n,J} + (R_J-P_J)u(t_n)) \\ &+(B^0_J)^{-1}r_{12}(\tau, B^0_J, \{A^0_{i,J}\}_{i=1}^m)(\tilde{e}^{[r]}_{n,J} + (R_J-P_J)u'(t_n)) \\ &+\tau^2\sum_{\ell_1,\ell_2=1}^{m}T\,(\tau,\beta_{\ell_1},\mathcal{A}_{\ell_2},\{A^0_{i,J}\}_{i=1}^m)f_{\ell_2,n} + \tau^2\sum_{\ell=1}^{m}(\beta_\ell^T\otimes(R_J-P_J))f_{\ell,n}\end{aligned} \tag{65}$$

and similarly

$$\begin{aligned}\bar{V}^{0,[r]}_{n+1,J} - V^{0,[r]}_{n+1,J} &= r_{21}(\tau, B^0_J, \{A^0_{i,J}\}_{i=1}^m)B^0_J(e^{[r]}_{n,J} + (R_J-P_J)u(t_n)) \\ &+r_{22}(\tau, B^0_J, \{A^0_{i,J}\}_{i=1}^m)(\tilde{e}^{[r]}_{n,J} + (R_J-P_J)u'(t_n)) \\ &+\tau\sum_{\ell_1,\ell_2=1}^{m}T\,(\tau,b_{\ell_1},\mathcal{A}_{\ell_2},\{A^0_{i,J}\}_{i=1}^m)f_{\ell_2,n} + \tau\sum_{\ell=1}^{m}(b_\ell^T\otimes(R_J-P_J))f_{\ell,n}.\end{aligned} \tag{66}$$

From expressions (62) and (65) in (56) and the ones given by (63) and (66) in (55), we have

$$\begin{aligned}e^{[r]}_{n+1,J} + (R_J-P_J)u(t_{n+1}) &= R_J\rho^{[r]}_{n+1} + \tau^2\sum_{\ell=1}^{m}(\beta_\ell^T\otimes(R_J-P_J))\,((\mathcal{I}\otimes A_\ell)\bar{K}^{[r]}_n + f_{\ell,n}) \\ &+\tau^2\sum_{\ell_1,\ell_2=1}^{m}T\,(\tau,\beta_{\ell_1},\mathcal{A}_{\ell_2},\{A^0_{i,J}\}_{i=1}^m)((\mathcal{I}\otimes A_{\ell_2})\bar{K}^{[r]}_n + f_{\ell_2,n}) \\ &+(B^0_J)^{-1}r_{11}(\tau, B^0_J, \{A^0_{i,J}\}_{i=1}^m)B^0_J(e^{[r]}_{n,J} + (R_J-P_J)u(t_n)) \\ &+(B^0_J)^{-1}r_{12}(\tau, B^0_J, \{A^0_{i,J}\}_{i=1}^m)(\tilde{e}^{[r]}_{n,J} + (R_J-P_J)u'(t_n))\end{aligned}$$

and

$$
\begin{aligned}
\tilde{e}_{n+1,J}^{[r]}+(R_J-P_J)u'(t_{n+1}) &= R_J\xi_{n+1}^{[r]}+\tau\sum_{\ell=1}^{m}(b_\ell^T\otimes(R_J-P_J))\left((\mathcal{I}\otimes A_\ell)\bar{K}_n^{[r]}+f_{\ell,n}\right)\\
&+\tau\sum_{\ell_1,\ell_2=1}^{m}T\left(\tau,b_{\ell_1},\mathcal{A}_{\ell_2},\{A_{i,J}^0\}_{i=1}^m\right)((\mathcal{I}\otimes A_{\ell_2})\bar{K}_n^{[r]}+f_{\ell_2,n})\\
&+r_{21}(\tau,B_J^0,\{A_{i,J}^0\}_{i=1}^m)B_J^0(e_{n,J}^{[r]}+(R_J-P_J)u(t_n))\\
&+r_{22}(\tau,B_J^0,\{A_{i,J}^0\}_{i=1}^m)(\tilde{e}_{n,J}^{[r]}+(R_J-P_J)u'(t_n))
\end{aligned}
$$

If we define matrix $M(\tau,B_J^0,\{A_{i,J}^0\}_{i=1}^m)$ as

$$
M(\tau,B_J^0,\{A_{i,J}^0\}_{i=1}^m) = \begin{bmatrix} (B_J^0)^{-1}r_{11}(\tau,B_J^0,\{A_{i,J}^0\}_{i=1}^m)B_J^0 & (B_J^0)^{-1}r_{12}(\tau,B_J^0,\{A_{i,J}^0\}_{i=1}^m) \\ r_{21}(\tau,B_J^0,\{A_{i,J}^0\}_{i=1}^m)B_J^0 & r_{22}(\tau,B_J^0,\{A_{i,J}^0\}_{i=1}^m) \end{bmatrix},
$$

then the global errors can be written in matrix form as follows

$$
\begin{aligned}
\begin{bmatrix} e_{n+1,J}^{[r]}+(R_J-P_J)u(t_{n+1}) \\ \tilde{e}_{n+1,J}^{[r]}+(R_J-P_J)u'(t_{n+1}) \end{bmatrix} &= M(\tau,B_J^0,\{A_{i,J}^0\}_{i=1}^m)\begin{bmatrix} e_{n,J}^{[r]}+(R_J-P_J)u(t_n) \\ \tilde{e}_{n,J}^{[r]}+(R_J-P_J)u'(t_n) \end{bmatrix}\\
&+\begin{bmatrix} \tau^2\sum_{\ell_1,\ell_2=1}^m T\left(\tau,\beta_{\ell_1},\mathcal{A}_{\ell_2},\{A_{i,J}^0\}_{i=1}^m\right)((\mathcal{I}\otimes A_{\ell_2})\bar{K}_n^{[r]}+f_{\ell_2,n}) \\ \tau\sum_{\ell_1,\ell_2=1}^m T\left(\tau,b_{\ell_1},\mathcal{A}_{\ell_2},\{A_{i,J}^0\}_{i=1}^m\right)((\mathcal{I}\otimes A_{\ell_2})\bar{K}_n^{[r]}+f_{\ell_2,n}) \end{bmatrix}\\
&+\begin{bmatrix} \tau^2\sum_{\ell=1}^m(\beta_\ell^T\otimes(R_J-P_J))((\mathcal{I}\otimes A_\ell)\bar{K}_n^{[r]}+f_{\ell,n}) \\ \tau\sum_{\ell=1}^m(b_\ell^T\otimes(R_J-P_J))((\mathcal{I}\otimes A_\ell)\bar{K}_n^{[r]}+f_{\ell,n}) \end{bmatrix}+\begin{bmatrix} R_J\rho_{n+1}^{[r]} \\ R_J\xi_{n+1}^{[r]} \end{bmatrix}
\end{aligned}
$$

and, in a recursive way, we get that

$$
\begin{aligned}
\begin{bmatrix} e_{n,J}^{[r]}+(R_J-P_J)u(t_n) \\ \tilde{e}_{n,J}^{[r]}+(R_J-P_J)u'(t_n) \end{bmatrix} &= M^n(\tau,B_J^0,\{A_{i,J}^0\}_{i=1}^m)\begin{bmatrix} (R_J-P_J)u(0) \\ (R_J-P_J)u'(0) \end{bmatrix} \qquad (67)\\
+\sum_{k=1}^{m}M^{n-k}(\tau,B_J^0,\{A_{i,J}^0\}_{i=1}^m)&\begin{bmatrix} \tau^2\sum_{\ell_1,\ell_2=1}^m T\left(\tau,\beta_{\ell_1},\mathcal{A}_{\ell_2},\{A_{i,J}^0\}_{i=1}^m\right)((\mathcal{I}\otimes A_{\ell_2})\bar{K}_{k-1}^{[r]}+f_{\ell_2,k-1}) \\ \tau\sum_{\ell_1,\ell_2=1}^m T\left(\tau,b_{\ell_1},\mathcal{A}_{\ell_2},\{A_{i,J}^0\}_{i=1}^m\right)((\mathcal{I}\otimes A_{\ell_2})\bar{K}_{k-1}^{[r]}+f_{\ell_2,k-1}) \end{bmatrix}\\
+\sum_{k=1}^{m}M^{n-k}(\tau,B_J^0,\{A_{i,J}^0\}_{i=1}^m)&\left(\begin{bmatrix} \tau^2\sum_{\ell=1}^m(\beta_\ell^T\otimes(R_J-P_J))((\mathcal{I}\otimes A_\ell)\bar{K}_{k-1}^{[r]}+f_{\ell,k-1}) \\ \tau\sum_{\ell=1}^m(b_\ell^T\otimes(R_J-P_J))((\mathcal{I}\otimes A_\ell)\bar{K}_{k-1}^{[r]}+f_{\ell,k-1}) \end{bmatrix}+\begin{bmatrix} R_J\rho_k^{[r]} \\ R_J\xi_k^{[r]} \end{bmatrix}\right)
\end{aligned}
$$

Apart from this, notice that

$$
M(\tau,B_J^0,\{A_{i,J}^0\}_{i=1}^m) = \begin{bmatrix} (B_J^0)^{-1} & 0 \\ 0 & I_J \end{bmatrix} R(\tau,B_J^0,\{A_{i,J}^0\}_{i=1}^m)\begin{bmatrix} B_J^0 & 0 \\ 0 & I_J \end{bmatrix} \qquad (68)
$$

and therefore,

$$
M^k(\tau,B_J^0,\{A_{i,J}^0\}_{i=1}^m) = \begin{bmatrix} (B_J^0)^{-1} & 0 \\ 0 & I_J \end{bmatrix} R^k(\tau,B_J^0,\{A_{i,J}^0\}_{i=1}^m)\begin{bmatrix} B_J^0 & 0 \\ 0 & I_J \end{bmatrix}.
$$

Then, when we bound in the energy norm, by using bound (27), we get that

$$
\begin{aligned}
\|M^k(\tau, B_J^0, \{A_{i,J}^0\}_{i=1}^m)\|_{B_J^0} &= \left\| \begin{bmatrix} B_J^0 & 0 \\ 0 & I_J \end{bmatrix} M^k(\tau, B_J^0, \{A_{i,J}^0\}_{i=1}^m) \begin{bmatrix} (B_J^0)^{-1} & 0 \\ 0 & I_J \end{bmatrix} \right\|_J \\
&= \left\| R^k(\tau, B_J^0, \{A_{i,J}^0\}_{i=1}^m) \right\|_J \leq C
\end{aligned}
$$

Therefore, when we bound in the energy norm expression (67), we get that

$$
\begin{aligned}
\left\| \begin{bmatrix} e_{n,J}^{[r]} \\ \tilde{e}_{n,J}^{[r]} \end{bmatrix} \right\|_{B_J^0} &\leq \|R^n(\tau, B_J^0, \{A_{i,J}^0\}_{i=1}^m)\|_J \left(\|B_J^0(R_J - P_J)u(0)\|_J + \|(R_J - P_J)u'(0)\|_J \right) \\
&+ \tau^2 \sum_{k=1}^m \|R^{n-k}(\tau, B_J^0, \{A_{i,J}^0\}_{i=1}^m)\|_J \| \sum_{\ell_1,\ell_2=1}^m B_J^0 T(\tau, \beta_{\ell_1}, \mathcal{A}_{\ell_2}, \{A_{i,J}^0\}_{i=1}^m)((\mathcal{I} \otimes A_{\ell_2})\bar{K}_{k-1}^{[r]} + f_{\ell_2,k-1})\|_J \\
&+ \tau \sum_{k=1}^m \|R^{n-k}(\tau, B_J^0, \{A_{i,J}^0\}_{i=1}^m)\|_J \| \sum_{\ell_1,\ell_2=1}^m T(\tau, b_{\ell_1}, \mathcal{A}_{\ell_2}, \{A_{i,J}^0\}_{i=1}^m)((\mathcal{I} \otimes A_{\ell_2})\bar{K}_{k-1}^{[r]} + f_{\ell_2,k-1})\|_J \\
&+ \sum_{k=1}^m \|R^{n-k}(\tau, B_J^0, \{A_{i,J}^0\}_{i=1}^m)\|_J \left(\tau^2 \| \sum_{\ell=1}^m B_J^0(\beta_\ell^T \otimes (R_J - P_J))((\mathcal{I} \otimes A_\ell)\bar{K}_{k-1}^{[r]} + f_{\ell,k-1})\|_J + \|B_J^0 R_J \rho_k^{[r]}\|_J \right) \\
&+ \sum_{k=1}^m \|R^{n-k}(\tau, B_J^0, \{A_{i,J}^0\}_{i=1}^m)\|_J \left(\tau \| \sum_{\ell=1}^m (b_\ell^T \otimes (R_J - P_J))((\mathcal{I} \otimes A_\ell)\bar{K}_{k-1}^{[r]} + f_{\ell,k-1})\|_J + \|R_J \xi_k^{[r]}\|_J \right) \\
&+ \|B_J^0(R_J - P_J)u(t_n)\|_J + \|(R_J - P_J)u'(t_n)\|_J
\end{aligned}
$$

From here, by taking into account the expression for functions $T(\tau, v_{\ell_1}, \mathcal{A}_{\ell_2}, \{A_{i,J}^0\}_{i=1}^m)$, $1 \leq \ell_1, \ell_2 \leq m$, the bound (27) together with hypotheses (H2) and (H4), we get that

$$
\begin{aligned}
\left\| \begin{bmatrix} e_{n,J}^{[r]} \\ \tilde{e}_{n,J}^{[r]} \end{bmatrix} \right\|_{B_J^0} &= \mathcal{O}(J^{\tilde{d}} \|(R_J - P_J)u(0)\|_J + J^{\tilde{d}} \|(R_J - P_J)u(t_n)\|_J) \\
&+ \mathcal{O}(\|(R_J - P_J)u'(t_n)\|_J + \|(R_J - P_J)u'(0)\|_J) \\
&+ \tau^2 \sum_{k=1}^m \mathcal{O}(J^{\tilde{d}} \| \sum_{\ell=1}^m (e \otimes (R_J - P_J))((\mathcal{I} \otimes A_\ell)\bar{K}_{k-1}^{[r]} + f_{\ell,k-1})\|_J) \\
&+ \tau \sum_{k=1}^m \mathcal{O}(\| \sum_{\ell=1}^m (e \otimes (R_J - P_J))((\mathcal{I} \otimes A_\ell)\bar{K}_{k-1}^{[r]} + f_{\ell,k-1})\|_J) \\
&+ \sum_{k=1}^m \mathcal{O}(J^{\tilde{d}} \tau^2 \| \sum_{\ell=1}^m (\beta_\ell^T \otimes (R_J - P_J))((\mathcal{I} \otimes A_\ell)\bar{K}_{k-1}^{[r]} + f_{\ell,k-1})\|_J + J^{\tilde{d}} \|R_J \rho_k^{[r]}\|_J) \\
&+ \sum_{k=1}^m \mathcal{O}(\tau \| \sum_{\ell=1}^m (b_\ell^T \otimes (R_J - P_J))((\mathcal{I} \otimes A_\ell)\bar{K}_{k-1}^{[r]} + f_{\ell,k-1})\|_J + \|R_J \xi_k^{[r]}\|_J)
\end{aligned}
$$

From hypothesis (H1) and the uniform boundedness of u and f_ℓ, $\ell = 1, \cdots, m$, we have[5]

$$
\begin{aligned}
\|(P_J - R_J)u(t_n)\|_J &= \mathcal{O}(J^{\tilde{h}(\alpha)}\|u(t_n)\|_{H^\alpha(\Omega)}) = \mathcal{O}(J^{\tilde{h}(\alpha)}\|u\|_{\infty,H^\alpha(\Omega)}) = \mathcal{O}(J^{\tilde{h}(\alpha)}), \quad (69)\\
\|(P_J - R_J)u'(t_n)\|_J &= \mathcal{O}(J^{\tilde{h}(\alpha)}\|u'(t_n)\|_{H^\alpha(\Omega)}) = \mathcal{O}(J^{\tilde{h}(\alpha)}\|u'\|_{\infty,H^\alpha(\Omega)}) = \mathcal{O}(J^{\tilde{h}(\alpha)}), \quad (70)\\
\|(R_J - P_J)f_\ell(t_{k,i})\|_J &= \mathcal{O}(J^{\tilde{h}(\alpha-d)}\|f_\ell(t_{k,i})\|_{H^{\alpha-d}(\Omega)})\\
&= \mathcal{O}(J^{\tilde{h}(\alpha-d)}\|f_\ell\|_{\infty,H^{\alpha-d}(\Omega)}) = \mathcal{O}(J^{\tilde{h}(\alpha-d)}). \quad (71)
\end{aligned}
$$

The bounds obtained for $\|(R_J - P_J)u(0)\|_J$ and $\|(R_J - P_J)u'(0)\|_J$ are similar to the ones given by (69) and (70). From (71) we deduce, for $v \in \mathbb{R}^s$, $\ell = 1, \cdots, m$ and $k \geq 0$ that

$$
\|(v^T \otimes (R_J - P_J))f_{\ell,k}\|_J = \mathcal{O}(J^{\tilde{h}(\alpha-d)}). \quad (72)
$$

On the other hand, we have defined $\Delta_k^{[r]} = K_k^{[r]} - \bar{K}_k^{[r]}$, so we have that $\bar{K}_k^{[r]} = K_k^{[r]} - \Delta_k^{[r]}$. From (13) and (14) it can be proven in a recursive way that

$$
\begin{aligned}
K_k^{[r]} &= \sum_{j=0}^{r-1}\tau^{2j}(\sum_{\ell=1}^m \mathcal{A}_\ell \otimes A_\ell)^j (e \otimes I)u(t_k) + \sum_{j=0}^{r-1}\tau^{2j+1}(\sum_{\ell=1}^m \mathcal{A}_\ell \otimes A_\ell)^j (c \otimes I)u'(t_k)\\
&+ \sum_{j=0}^{r-1}\tau^{2j+2}(\sum_{\ell=1}^m \mathcal{A}_\ell \otimes A_\ell)^j \sum_{i=1}^m (\mathcal{A}_i \otimes I)f_{i,k} + \sum_{j=0}^{r}\tau^{2j}(\sum_{\ell=1}^m \mathcal{A}_\ell \otimes A_\ell)^j K_k^{[0]},
\end{aligned}
$$

so $(\mathcal{I} \otimes A_\ell)K_k^{[r]} \in H^{\alpha-d(r+1)}(\Omega)$ for $\ell = 1, \ldots, m$. Apart from that, from (32) for $r = 0$, (33) and (36), we take

$$
\begin{aligned}
\Delta_k^{[r]} &= \tau^{2r}\left(\mathcal{I} \otimes I - \tau^2 \sum_{\ell=1}^m \mathcal{A}_\ell \otimes A_\ell^0\right)^{-1} (\sum_{\ell=1}^m \mathcal{A}_\ell \otimes A_\ell)^r \Big[K_k^{[0]} - \tau^2 \sum_{\ell=1}^m (\mathcal{A}_\ell \otimes I)((\mathcal{I} \otimes A_\ell)K_k^{[0]} + f_{\ell,k})\\
&\qquad - (e \otimes I)u(t_k) - \tau(c \otimes I)u'(t_k)\Big],
\end{aligned}
$$

which permits us to deduce that $\Delta_k^{[r]} \in H^{\alpha-(r+1)d}(\Omega)$. Therefore $(\mathcal{I}\otimes A_\ell)\Delta_k^{[r]} \in H^{\alpha-(r+2)d}(\Omega)$. From this, we have that $\bar{K}_k^{[r]} \in H^{\alpha-(r+2)d}(\Omega)$, so by bounding, for $v \in \mathbb{R}^s$

$$
\|(v^T \otimes (R_J - P_J)A_\ell)\bar{K}_k^{[r]}\|_J = \mathcal{O}(J^{\tilde{h}(\alpha-d(r+2))}). \quad (73)
$$

Finally,

$$
\begin{aligned}
\|R_J\rho_k^{[r]}\|_J &\leq \|P_J\rho_k^{[r]}\|_J + \|(R_J - P_J)\rho_k^{[r]}\|_J \leq \|P_J\rho_k^{[r]}\|_J + \|(R_J - P_J)(u(t_k) - U_k^{[0]})\|_J\\
&+ \sum_{i=0}^{r-1}\|(R_J - P_J)(U_k^{[i]} - U_k^{[i+1]})\|_J + \|(R_J - P_J)(U_k^{[r]} - \bar{U}_k^{[r]})\|_J,\\
\|R_J\xi_k^{[r]}\|_J &\leq \|P_J\xi_k^{[r]}\|_J + \|(R_J - P_J)(u'(t_k) - V_k^{[0]})\|_J\\
&+ \sum_{i=0}^{r-1}\|(R_J - P_J)(V_k^{[i]} - V_k^{[i+1]})\|_J + \|(R_J - P_J)(V_k^{[r]} - \bar{V}_k^{[r]})\|_J.
\end{aligned}
$$

[5]The notation $\|u\|_{\infty,H^\alpha(\Omega)} \equiv \max_{0\leq t\leq T}\|u(t)\|_{H^\alpha(\Omega)}$.

By bounding, using the results (47-49) together with (20), we have

$$\begin{aligned}\|(R_J - P_J)(u'(t_k) - V_k^{[0]})\|_J &= \mathcal{O}(\tau^{p+1} J^{\tilde{h}(\alpha-d)}),\\ \|(R_J - P_J)(u(t_k) - U_k^{[0]})\|_J &= \mathcal{O}(\tau^{p+1} J^{\tilde{h}(\alpha-d)}).\end{aligned}$$

Apart form this, from the definition of $V_k^{[i]}$ and $U_k^{[i]}$, given by (30) and (31) respectively, as $(\mathcal{I} \otimes A_\ell)K_k^{[i]} \in H^{\alpha-d(i+1)}(\Omega)$, we have that $V_k^{[i]} - V_k^{[i+1]} \in H^{\alpha-d(i+2)}(\Omega)$ and $U_k^{[i]} - U_k^{[i+1]} H^{\alpha-d(i+2)}(\Omega)$. Then, by using this together with bounds (46), we get

$$\begin{aligned}\|(R_J - P_J)(U_k^{[i]} - U_k^{[i+1]})\|_J &= \mathcal{O}(\tau^{p+1} J^{\tilde{h}(\alpha-d(i+2))}),\\ \|(R_J - P_J)(V_k^{[i]} - V_k^{[i+1]})\|_J &= \mathcal{O}(\tau^{p+1} J^{\tilde{h}(\alpha-d(i+2))}).\end{aligned}$$

Finally, as $U_k^{[r]} - \bar{U}_k^{[r]} = \tau^2 \sum_{\ell=1}^m (\beta_\ell^T \otimes A_\ell^0)\Delta_{k-1}^{[r]}$ and $V_k^{[r]} - \bar{V}_k^{[r]} = \tau \sum_{\ell=1}^m (b_\ell^T \otimes A_\ell^0)\Delta_{k-1}^{[r]}$, we deduce that $U_k^{[r]} - \bar{U}_k^{[r]} \in H^{\alpha-d(r+2)}(\Omega)$ and $V_k^{[r]} - \bar{V}_k^{[r]} \in H^{\alpha-d(r+2)}(\Omega)$; thus by using (41), we get that

$$\begin{aligned}\|(R_J - P_J)(U_k^{[r]} - \bar{U}_k^{[r]})\|_J &= \mathcal{O}(\tau^{\min\{\tilde{q}+2r+1,p+2r+2\}} J^{\tilde{h}(\alpha-d(r+2))}),\\ \|(R_J - P_J)(V_k^{[r]} - \bar{V}_k^{[r]})\|_J &= \mathcal{O}(\tau^{\min\{\tilde{q}+2r,p+2r+1\}} J^{\tilde{h}(\alpha-d(r+2))}).\end{aligned}$$

Therefore, Theorem 2.5 together with these bounds leads to

$$\begin{aligned}\|R_J \rho_k^{[r]}\|_J &= \mathcal{O}(\tau^{\min\{\tilde{q}+2r+1,p+1\}}) + \mathcal{O}(\tau^{p+1} J^{\tilde{h}(\alpha-d)})\\ &\quad + \sum_{i=0}^{r-1} \mathcal{O}(\tau^{p+1} J^{\tilde{h}(\alpha-d(i+2))}) + \mathcal{O}(\tau^{\min\{\tilde{q}+2r+1,p+2r+2\}} J^{\tilde{h}(\alpha-d(r+2))})\\ &= \mathcal{O}(\tau^{\min\{\tilde{q}+2r+1,p+1\}}), \end{aligned} \tag{74}$$

and, similarly

$$\|R_J \xi_k^{[r]}\|_J = \mathcal{O}(\tau^{\min\{\tilde{q}+2r,p+1\}}). \tag{75}$$

Then, by using bounds (69-75) we get

$$\begin{aligned}\left\| \begin{bmatrix} e_{n,J}^{[r]} \\ \tilde{e}_{n,J}^{[r]} \end{bmatrix} \right\|_{B_J^0} &= \mathcal{O}(J^{\tilde{d}+\tilde{h}(\alpha)} + J^{\tilde{h}(\alpha)}) + \tau^2 \sum_{k=1}^m \mathcal{O}(J^{\tilde{d}+\tilde{h}(\alpha-d(r+2))}) + \tau \sum_{k=1}^m \mathcal{O}(J^{\tilde{h}(\alpha-d(r+2))})\\ &\quad + \tau^2 \sum_{k=1}^m \mathcal{O}(J^{\tilde{d}+\tilde{h}(\alpha-d(r+2))}) + \tau \sum_{k=1}^m \mathcal{O}(J^{\tilde{h}(\alpha-d(r+2))})\\ &\quad + \sum_{k=1}^m \mathcal{O}(\tau^{\min\{\tilde{q}+2r+1,p+1\}} J^{\tilde{d}} + \tau^{\min\{\tilde{q}+2r,p+1\}}) + \mathcal{O}(J^{\tilde{d}+\tilde{h}(\alpha)} + J^{\tilde{h}(\alpha)})\\ &= \mathcal{O}(\tau^{\min\{\tilde{q}+2r,p\}} J^{\tilde{d}} + \tau^{\min\{\tilde{q}+2r-1,p\}} + \tau J^{\tilde{d}+\tilde{h}(\alpha-d(r+2))} + J^{\tilde{h}(\alpha-d(r+2))} + J^{\tilde{d}+\tilde{h}(\alpha)})\end{aligned}$$

5.3. Proof of Theorem 3.2

To calculate the error in the solution and in the derivative, we use that the powers of matrix $R(\tau, B_J^0, \{A_{i,J}^0\}_{i=1}^m)$ can be expressed as

$$R^k(\tau, B_J^0, \{A_{i,J}^0\}_{i=1}^m) = \begin{bmatrix} r_{11,k}(\tau, B_J^0, \{A_{i,J}^0\}_{i=1}^m) & r_{12,k}(\tau, B_J^0, \{A_{i,J}^0\}_{i=1}^m) \\ r_{21,k}(\tau, B_J^0, \{A_{i,J}^0\}_{i=1}^m) & r_{22,k}(\tau, B_J^0, \{A_{i,J}^0\}_{i=1}^m) \end{bmatrix},$$

where the elements $r_{ij,k}$, $1 \leq i, j \leq 2$, integer $k \geq 1$, are formed by products and sums of the elements r_{ij}, $1 \leq i, j \leq 2$. From (27) we take that $\|R(\tau, B_J^0, \{A_{i,J}^0\}_{i=1}^m)\|_J \leq C$, with C constant independent of the power k. Therefore, we can deduce that, (for C again constants independent of k)

$$\|r_{ij,k}(\tau, B_J^0, \{A_{i,J}^0\}_{i=1}^m)\|_J \leq C, \quad 1 \leq i, j \leq 2. \tag{76}$$

On the other hand, by using (26) together with (68), we get that

$$M^k(\tau, B_J^0, \{A_{i,J}^0\}_{i=1}^m) = \begin{bmatrix} \tilde{r}_{11,k}(\tau, B_J^0, \{A_{i,J}^0\}_{i=1}^m) & \tau\tilde{r}_{12,k}(\tau, B_J^0, \{A_{i,J}^0\}_{i=1}^m) \\ \tau^{-1}\tilde{r}_{21,k}(\tau, B_J^0, \{A_{i,J}^0\}_{i=1}^m) & \tilde{r}_{22,k}(\tau, B_J^0, \{A_{i,J}^0\}_{i=1}^m) \end{bmatrix}.$$

Therefore, by using this in (67), together with $\tau^{-1}\tilde{r}_{21,k}(\tau, \{A_{i,J}^0\}_{i=1}^m) = r_{21,k}(\tau, \{A_{i,J}^0\}_{i=1}^m)B_J^0$ and bounding adequately, we obtain

$$\begin{aligned}
\|e_{n,J}^{[r]}\|_J &\leq \|(R_J - P_J)u(t_n)\|_J + \|\tilde{r}_{11,n}(\tau, B_J^0, \{A_{i,J}^0\}_{i=1}^m)\|_J\|(R_J - P_J)u(0)\|_J \\
&+\tau\|\tilde{r}_{12,n}(\tau, B_J^0, \{A_{i,J}^0\}_{i=1}^m)\|_J\|(R_J - P_J)u'(0)\|_J \\
&+\tau^2\sum_{k=1}^m \|\tilde{r}_{11,n-k}(\tau, B_J^0, \{A_{i,J}^0\}_{i=1}^m)\|_J\| \sum_{\ell_1,\ell_2=1}^m T\left(\tau, \beta_{\ell_1}, \mathcal{A}_{\ell_2}, \{A_{i,J}^0\}_{i=1}^m\right)((\mathcal{I}\otimes A_{\ell_2})\bar{K}_{k-1}^{[r]} + f_{\ell_2,k-1})\|_J \\
&+\tau^2\sum_{k=1}^m \|\tilde{r}_{12,n-k}(\tau, B_J^0, \{A_{i,J}^0\}_{i=1}^m)\|_J\| \sum_{\ell_1,\ell_2=1}^m T\left(\tau, b_{\ell_1}, \mathcal{A}_{\ell_2}, \{A_{i,J}^0\}_{i=1}^m\right)((\mathcal{I}\otimes A_{\ell_2})\bar{K}_{k-1}^{[r]} + f_{\ell_2,k-1})\|_J \\
&+\sum_{k=1}^m \|\tilde{r}_{11,n-k}(\tau, B_J^0, \{A_{i,J}^0\}_{i=1}^m)\|_J\left(\tau^2\|\sum_{\ell=1}^m(\beta_\ell^T\otimes(R_J - P_J))((\mathcal{I}\otimes A_\ell)\bar{K}_{k-1}^{[r]} + f_{\ell,k-1})\|_J + \|R_J\rho_k^{[r]}\|_J\right) \\
&+\tau\sum_{k=1}^m \|\tilde{r}_{12,n-k}(\tau, B_J^0, \{A_{i,J}^0\}_{i=1}^m)\|_J\left(\tau\|\sum_{\ell=1}^m(b_\ell^T\otimes(R_J - P_J))((\mathcal{I}\otimes A_\ell)\bar{K}_{k-1}^{[r]} + f_{\ell,k-1})\|_J + \|R_J\xi_k^{[r]}\|_J\right)
\end{aligned}$$

and

$$\begin{aligned}
\|\tilde{e}_{n,J}^{[r]}\|_J &\leq \|(R_J - P_J)u'(t_n)\|_J + \|r_{21,n}(\tau, B_J^0, \{A_{i,J}^0\}_{i=1}^m)\|_J\|B_J^0(R_J - P_J)u(0)\|_J \\
&+\|\tilde{r}_{22,n}(\tau, B_J^0, \{A_{i,J}^0\}_{i=1}^m)\|_J\|(R_J - P_J)u'(0)\|_J \\
&+\tau\sum_{k=1}^m \|\tilde{r}_{21,n-k}(\tau, B_J^0, \{A_{i,J}^0\}_{i=1}^m)\|_J\| \sum_{\ell_1,\ell_2=1}^m T\left(\tau, \beta_{\ell_1}, \mathcal{A}_{\ell_2}, \{A_{i,J}^0\}_{i=1}^m\right)((\mathcal{I}\otimes A_{\ell_2})\bar{K}_{k-1}^{[r]} + f_{\ell_2,k-1})\|_J
\end{aligned}$$

$$+\tau^2 \sum_{k=1}^{m} \|\tilde{r}_{22,n-k}(\tau, B_J^0, \{A_{i,J}^0\}_{i=1}^m)\|_J \| \sum_{\ell_1,\ell_2=1}^{m} T(\tau, b_{\ell_1}, \mathcal{A}_{\ell_2}, \{A_{i,J}^0\}_{i=1}^m)((\mathcal{I} \otimes A_{\ell_2})\bar{K}_{k-1}^{[r]} + f_{\ell_2,k-1})\|_J$$

$$+ \sum_{k=1}^{m} \|\tau^{-1}\tilde{r}_{21,n-k}(\tau, B_J^0, \{A_{i,J}^0\}_{i=1}^m)\|_J \tau^2 \| \sum_{\ell=1}^{m} (\beta_\ell^T \otimes (R_J - P_J))((\mathcal{I} \otimes A_\ell)\bar{K}_{k-1}^{[r]} + f_{\ell,k-1})\|_J$$

$$+ \sum_{k=1}^{m} \|\tau^{-1}\tilde{r}_{21,n-k}(\tau, B_J^0, \{A_{i,J}^0\}_{i=1}^m)\|_J \|R_J \rho_k^{[r]}\|_J$$

$$+ \sum_{k=1}^{m} \|\tau^{-1}\tilde{r}_{21,n-k}(\tau, B_J^0, \{A_{i,J}^0\}_{i=1}^m)\|_J \tau^2 \| \sum_{\ell=1}^{m} (\beta_\ell^T \otimes (R_J - P_J))((\mathcal{I} \otimes A_\ell)\bar{K}_{k-1}^{[r]} + f_{\ell,k-1})\|_J$$

$$+ \sum_{k=1}^{m} \|\tilde{r}_{22,n-k}(\tau, B_J^0, \{A_{i,J}^0\}_{i=1}^m)\|_J \|R_J \xi_k^{[r]}\|_J.$$

By using the results obtained in [2], together with the stability bound (27), we obtain that

$$\|\tilde{r}_{ij,k}(\tau, B_J^0, \{A_{i,J}^0\}_{i=1}^m)\| \le C, \qquad 1 \le i, j, \le 2.$$

Then, by using this result together with bounds (76) and (72-75), we get the result.

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