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# Allocating the costs of cleaning a river: expected responsibility versus median responsibility*

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**Abstract**

We consider the problem of cleaning a transboundary river, proposed by Ni and Wang (2007). A river is modeled as a segment divided into subsegments, each occupied by one region, from upstream to downstream. The waste is transferred from one region to the next at some rate. Since this transfer rate may be unknown, the social planner could have uncertainty over each region's responsibility. Two natural candidates to distribute the costs in this setting would be the method that assigns to each region its expected responsibility and the one that assigns to each region its median responsibility. We show that the latter is equivalent to the Upstream Responsibility method (Alcalde-Unzu et al., 2015) and the former is a new method that we call Expected Responsibility. We compare both solutions and analyze them in terms of a new property of monotonicity.

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# 1 Introduction

## Motivation and overview of results

The cleaning process of the waste present in river channels is a task that has to be undertaken frequently. Given that most rivers flow through different regions or even countries, the costs of the cleaning activities have to be divided between all governments involved, probably depending on the responsibilities that each of the regions has for the waste discharged.[1] In many cases, however, for instance when there is physical uncertainty or monitoring difficulties, it is tough to determine what region is responsible for a particular amount of pollution. For example, as stated in Segerson (1988), when there is nonpoint source pollution, it is impossible to identify with certainty the source of the waste present in each part of the river or to calculate the amount discharged by each region.

The literature has considered several solutions that allocate the costs for these problems. On the one hand, Ni and Wang (2007) proposed the Local Responsibility Sharing (LRS) and the Upstream Equal Sharing (UES) solutions. The first allocates the cost of cleaning each part of the river totally to the region located in that segment. The second considers that regions located upstream of a segment have also some responsibility over the waste present in that segment and it allocates the cost of cleaning each segment between all these responsible regions equally.[2] However, these solutions are not taking into account that the waste is transferred with the water, from upstream to downstream at a particular rate. Alcalde-Unzu et al. (2015) explicitly introduced this fact into the model and showed that if the social planner knows this transfer rate $t$, she can calculate with certainty the total amount of waste discharged by each region. If, however, there is uncertainty over the value of $t$, they showed that the social planner can infer some information on that transfer rate by analyzing the cleaning costs of all segments. Then, they proposed a new solution, the Upstream Responsibility (UR), which assigns to each region the waste of which it would be responsible if the transfer rate was equal to its expected value.

In those cases where there is uncertainty over the value of $t$, there is also uncertainty over each region's responsibility. Given that the expected value and the median are the two most important centrality measures of random variables, if the social planner wanted to distribute the costs among the regions according to their responsibilities, two natural candidates would be the method that assigns to each region its expected responsibility, and the one that assigns to each region its median responsibility. Assuming that the uncertainty on the tranfer rate takes the form of a uniform variable, we show that this second method is equivalent to the UR method, meanwhile the consideration of expected responsibilities

[1] The literature has also focused on how to distribute *fairly* the benefits of the river among the different regions. See for example Ambec and Ehlers (2008), Ambec and Sprumont (2002), Ansink et al. (2017), Ansink and Weikard (2012, 2015), van den Brink et al. (2012), van den Brink et al. (2007, 2014), Khmelnitskaya (2010), Özturk (2020), and Wang (2011).

[2] Dong et al. (2012) proposed the Downstream Equal Sharing solution (DES), which is in the same vein as UES, but considering that the regions that should pay for cleaning a segment are the ones situated downstream instead of the upstream ones.

defines a new method that we call Expected Responsibility (ER) method. We compare both methods and we show that the most upstream (respectively, downstream) region pays less (respectively, more) with the UR method than with the ER one; and an intermediate region pays more (respectively, less) with the UR method than with the ER one if the cost of cleaning the preceding region is higher (respectively, lower) than the cost of cleaning its own segment.

We introduce in this paper a new property of monotonicity. This property requires that if a region discharges more waste, the method used should guarantee that it will not pay less. We show that the UR method satisfies this basic property, but this is not the case of the ER method. The reason is that, when a region discharges more waste, two effects appear: it increases the costs of cleaning the segments, which tends to increase the costs allocated to the regions, but, at the same time, it changes the information that the social planner can deduce on the transfer rate from the cleaning cost vector, which affects how the solution distributes all costs among the regions. If the second effect dominates the first, which can occur with the ER solution but not with the UR one, a region discharging more waste could pay less. Moreover, we show that it is not only that the UR method satisfies this basic incentive compatibility property, but also that, within a general family of solutions, there is no other solution satisfying this property that assigns costs closer to regions' expected responsibilities.

**Related literature**

This paper is related to the literature that studies the allocation of the costs of transboundary rivers using game theoretical and/or axiomatic models, which has two main approaches. On the one hand, some papers assumed that the cost of cleaning each part of the river is exogenously given. In this setting, some papers study a single river and model it as a line: besides the contributions of Ni and Wang (2007) and Alcalde-Unzu et al. (2015) described above, van den Brink and van der Laan (2008) showed additional characterizations of the LRS and UES solutions, and Sun et al. (2019) characterized combinations of the LRS and UES solutions. Other papers study a river with tributaries and/or forks and model it as a network: Dong et al. (2012) proposed extensions of the LRS and UES solutions to this case and also proposed the DES solution as a dual of the UES one; and, more recently, van den Brink et al. (2018) provided different characterizations of these solutions and studied a different extension of the UES solution to this case of a river network. On the other hand, other papers such as Gengenbach et al. (2010) and van der Laan and Moes (2012) took a substantially different approach by assuming that the cost allocation method adopted may affect the decision of each region about how much waste to discharge.

**Remainder**

The paper is organized as follows. Section 2 describes the model and reviews some previous results of the literature. Section 3 analyzes two natural methods for allocating costs according to responsibilities: the expected responsibility and the median responsibility.

Section 4 is devoted to the study of the monotonicity property. Section 5 concludes with some remarks.

# 2 Notation and definitions

We study the same basic model as in Alcalde-Unzu et al. (2015). Consider a river modeled as a line divided into $n$ segments of the same size from upstream to downstream, with each region located in one of the segments. Formally, let $N = \{1, ..., n\} \subset \mathbb{N}$ be the set of regions such that $i$ is situated upstream of $i+1$. The regions discharged waste that flowed through the river. The cost of cleaning each segment $i \in N$ is $c_i \geq 0$, being $C = (c_1, ..., c_n) \in \mathbb{R}^n_+$ the cleaning cost vector. We assume that $\sum_{i=1}^n c_i > 0$ because otherwise the problem has no interest. We also assume that each of these costs depends linearly on the amount of waste present in the segment in question. The river has a transfer rate $t$ that measures the proportion of waste that has been transferred from one segment of the river to the next. The social planner knows that $t$ is situated within an interval $[\underline{t}, \overline{t}]$, where $\underline{t} \in [0, 1)$ and $\overline{t} \in (0, 1]$. It is assumed that this uncertainty on $t$ takes the form of an uniform variable.[3] Then, a *cost allocation problem* is a tuple $(C, \underline{t}, \overline{t})$.[4]

There are two solution concepts for allocating the costs among the regions. A *cost allocation rule*, proposed by Alcalde-Unzu et al. (2015), is a mapping $x$ that assigns to each problem $(C, \underline{t}, \overline{t})$ a matrix of size $n \times n$, $(x_i^j(C, \underline{t}, \overline{t}))_{i,j \in N}$, such that all its components are non–negative and $\sum\limits_{i \in N} x_i^j(C, \underline{t}, \overline{t}) = c_j$ for all $j \in N$. Then, $x_i^j(C, \underline{t}, \overline{t})$ represents the part of the cost of cleaning segment $j$ that region $i$ pays. On the other hand, a *cost allocation method*, proposed by Ni and Wang (2007), is a function $x$ that assigns to each cost allocation problem $(C, \underline{t}, \overline{t})$ a vector of size $n$, $(x_i(C, \underline{t}, \overline{t}))_{i \in N} \in \mathbb{R}^n_+$, such that $\sum\limits_{i \in N} x_i(C, \underline{t}, \overline{t}) = \sum\limits_{i \in N} c_i$. Then, $x_i(C, \underline{t}, \overline{t})$ represents the total cost that is allocated to region $i$. It is easy to see that, for any cost allocation rule, there is only one cost allocation method associated, while there could be multiple cost allocation rules associated with each cost allocation method. In this paper, we mainly focus on cost allocation methods.

The first solutions proposed in the literature for cost allocation problems were the LRS and UES methods introduced by Ni and Wang (2007) and also studied by van den Brink and van der Laan (2008).

**Definition 1** *The* Local Responsibility Sharing *(LRS) method, $\alpha$, is such that for all problems $(C, \underline{t}, \overline{t})$ and all $i \in N$, $\alpha_i(C, \underline{t}, \overline{t}) = c_i$.*

[3] In Section 5 we discuss how the results obtained along the paper depend on this assumption about the distribution of this random variable that describes the uncertainty on $t$.

[4] Several extensions of this basic model (e.g., segments of different sizes, river with tributaries and/or forks and different transfer rates across segments) can be developed as suggested in Alcalde-Unzu et al. (2015). Additionally, Ni and Wang (2007) and Alcalde-Unzu et al. (2015) included $N$ in the description of a cost allocation problem, but we omit it here because the information it contains is already included in $C$.

**Definition 2** *The* Upstream Equal Sharing *(UES) method, $\beta$, is such that for all problems $(C, \underline{t}, \overline{t})$ and all $i \in N$, $\beta_i(C, \underline{t}, \overline{t}) = \sum_{j=i}^{n} \frac{c_j}{j}$.*

Neither of these two solutions take into account any information about the transfer rate to allocate the costs and, therefore, they do not try to infer how much responsibility each region has in the waste discharged in the river. In order to understand the relevance of this information, consider first the case in which the transfer rate is known by the social planner ($\underline{t} = \overline{t} = t$). Then, she could use this information to calculate the total amount of waste that each region has discharged. This amount, which we denote by $V_i(t, C)$, is given by:[5]

$$V_i(t, C) = \begin{cases} \frac{c_i}{1-t} & \text{if } i = 1 \\ \frac{c_i}{1-t} - \frac{c_{i-1}}{1-t}t & \text{if } i \in \{2, \ldots, n-1\} \\ c_i - \frac{c_{i-1}}{1-t}t & \text{if } i = n. \end{cases} \tag{1}$$

We call $V_i(t, C)$ the *responsibility function* of region $i$. We would like to highlight some technical characteristics of this function that will be useful to prove several results along the paper: $V_i(t, C)$ is continuous in $t$ and either strictly increasing and convex in $t$ or strictly decreasing and concave in $t$. Exceptionally, $V_i(t, C)$ is constant in $t$ in the following cases: (*i*) $V_1(t, C)$ when $c_1 = 0$; (*ii*) $V_i(t, C)$, with $i \in \{2, \ldots, n-1\}$, when $c_i = c_{i-1}$; and (*iii*) $V_n(t, C)$ when $c_{n-1} = 0$.[6]

On the one hand, it would be desirable to have a method that, in these problems in which the transfer rate is known, assigns to each region $i$ a total cost equal to the value of the responsibility function. On the other hand, in the cases in which the transfer rate is unknown ($\underline{t} < \overline{t}$), a method should assign costs to each region $i$ in the interval of the limits of its possible responsibility that are, due to the strict monotonicity of the responsibility function $V_i$ in $t$, given by $V_i(\underline{t}, C)$ and $V_i(\overline{t}, C)$.[7] It is noteworthy to mention here that the information about the possible values of the transfer rate could be improved analyzing the cost vector $C$, as the following example shows.

**Example 1**: Suppose a problem in which $N = \{1, 2, 3, 4\}$, the cost vector is $C = \{100, 200, 300, 400\}$ and the social planner has the following information about the transfer rate: $\underline{t} = \frac{1}{5}$ and $\overline{t} = \frac{3}{5}$. Then, although the information a priori about the actual transfer rate allows

[5]This value is calculated in the proof of Proposition 3 in Alcalde-Unzu et al. (2015).

[6]All these characteristics apply only for the interval $t \in [0, 1)$. Observe that, when $t = 1$, $V_i$ is not well defined. This is because a transfer rate of $t = 1$ implies that all the waste is transferred to region $n$ and, then, $c_i = 0$ for all $i < n$, which provokes that $V_i$ equals an indeterminate form for all $i \in N$. In this case, the social planner could not infer any information about the waste discharged by each region and, thus, $V_i(1, C) \in [0, c_n]$ for all $i \in N$.

[7]Observe that, whenever $V_i(t, C)$ is strictly increasing (respectively, decreasing), the lower (respectively, higher) limit of the interval of possible responsibilities of region $i$ is given by $V_i(\underline{t}, C)$ and the higher (respectively, lower) limit by $V_i(\overline{t}, C)$.

any value in the interval $[\frac{1}{5}, \frac{3}{5}]$, some of these values are not compatible with $C$. If, for instante, the actual transfer rate of the river had the value $\frac{25}{42}$, we would have that $V_4(\frac{25}{42}, C)$ is negative, which is not possible. $\square$

This issue was analyzed by Alcalde-Unzu et al. (2015), who showed in its Proposition 3 a maximum limit of the actual transfer rate for each problem $(C, \underline{t}, \overline{t})$. This limit, denoted by $\overline{t}^*(\overline{t}, C)$, can be obtained with the following formula:

$$\overline{t}^*(\overline{t}, C) = \min \left\{ \left\{ \frac{c_i}{c_{i-1}} \right\}_{i \in \{2,\ldots,n-1\}}, \frac{c_n}{c_{n-1} + c_n}, \overline{t} \right\}.^{8} \tag{2}$$

Notice that $\overline{t}^*(\overline{t}, C) \leq \overline{t}$ and, therefore, this new upper limit can also reduce the uncertainty over the responsibilities of each region in the discharge of the waste present in the river.[9] Specifically, the social planner can truncate at $V_i(\overline{t}^*(\overline{t}, C), C)$ the random variable of the responsibility of each region $i$ that was defined initially over the interval limited by $V_i(\underline{t}, C)$ and $V_i(\overline{t}, C)$. We could have, for example, an extreme case in which, although $\underline{t} < \overline{t}$, $\underline{t} = \overline{t}^*(\overline{t}, C)$, allowing the social planner to apply directly the expression (1) to assign a total cost to each region. For the remaining cases in which $\underline{t} < \overline{t}^*(\overline{t}, C)$, there is still uncertainty over the responsibility of each region $i$ because it could be the case that $V_i(\underline{t}, C) \neq V_i(\overline{t}^*(\overline{t}, C), C)$.

To finish this section, consider again the LRS and UES methods defined previously. As we have mentioned before, they do not consider any information about the transfer rate, so they do not satisfy the natural requirement of assigning costs in the interval limited by $V_i(\underline{t}, C)$ and $V_i(\overline{t}^*(\overline{t}, C), C)$, as can be illustrated with the previous example:

**Example 1 (continuation)** Remember that the cost allocation problem of Example 1 is $(\{100, 200, 300, 400\}, \frac{1}{5}, \frac{3}{5})$. According to expression (2), the new upper limit for the transfer rate is $\overline{t}^*(\overline{t}, C) = \frac{4}{7}$. Observe that since the social planner knows that $t \in [\underline{t}, \overline{t}^*(\overline{t}, C)] = [\frac{1}{5}, \frac{4}{7}]$, the responsibility of region 1, $V_1(t, C)$, is in the interval $[V_1(\underline{t}, C), V_1(\overline{t}^*(\overline{t}, C), C)] = [125, 233.33]$. The LRS and the UES methods assign the solutions $\alpha(C, \underline{t}, \overline{t}) = (100, 200, 300, 400)$ and $\beta(C, \underline{t}, \overline{t}) = (400, 300, 200, 100)$, respectively. Thus, we can observe that none of them assigns to region 1 a value inside the interval of feasible responsibilities. $\square$

# 3 Alternative methods for estimating responsibilities

We have a responsibility function $V_i(t, C)$ that, knowing the transfer rate $t$, computes the actual responsibility of each region $i$ for the waste discharged in the river in any

[8]The possible quotients with the indeterminate form $\frac{0}{0}$ have to be excluded in the determination of $\overline{t}^*(\overline{t}, C)$. Obviously, $\overline{t}^*(\overline{t}, C)$ has to be not smaller than $\underline{t}$ because in other case the problem $(C, \underline{t}, \overline{t})$ would not be well-defined.

[9]The cost vector $C$ does not allow to infer a lower limit for the transfer rate because of the unidirectional nature of the problem. Note that we cannot even discard a value of 0 for the transfer rate in any cost vector $C$ because this cost vector could come from a situation in which each region $i$ has discharged the amount of waste present in this region, $c_i$, and no waste is transferred to next regions.

cost allocation problem $(C, \underline{t}, \overline{t})$. However, since the social planner only knows that $t$ is a random variable defined over the interval $[\underline{t}, \overline{t}^*(\overline{t}, C)]$, the responsibility function inherits this uncertainty and hence the social planner only knows that $V_i(t, C)$ is a random variable defined over the interval limited by $V_i(\underline{t}, C)$ and $V_i(\overline{t}^*(\overline{t}, C), C)$. Then, since the expected value and the median are the two most important centrality measures of random variables, if the social planner wants to distribute the costs among the regions according to their responsibilities, two natural candidates are the cost allocation method that assigns to each region its expected responsibility, $E_{(C,\underline{t},\overline{t})}(V_i(t, C))$, and the cost allocation method that assigns to each region its median responsibility, $med_{(C,\underline{t},\overline{t})}(V_i(t, C))$. As it is well-known from statistics, the latter implies a minimization of the expected sum of absolute deviations between the actual value of the responsibility of each region and its assigned cost, meanwhile the former implies a minimization of the expected sum of squared deviations between the actual value of the responsibility of each region and its assigned cost.

**Remark 1** *For all $(C, \underline{t}, \overline{t})$ and all $i \in N$:*

$$E_{(C,\underline{t},\overline{t})}(V_i(t, C)) = \arg\min_x \int_{\underline{t}}^{\overline{t}^*(\overline{t},C)} (V_i(t, C) - x)^2 dt.$$

$$med_{(C,\underline{t},\overline{t})}(V_i(t, C)) = \arg\min_x \int_{\underline{t}}^{\overline{t}^*(\overline{t},C)} |V_i(t, C) - x| dt.$$

We first analyze the cost allocation method that assigns to each region $i$, in any problem $(C, \underline{t}, \overline{t})$, the expected value of $V_i(t, C)$. The following proposition states that this cost allocation method coincides with assigning to each region $i$ the value of its responsibility function $V_i$ for a particular transfer rate, which we denote by $u(C, \underline{t}, \overline{t})$.

**Proposition 1** *For all $(C, \underline{t}, \overline{t})$ and all $i \in N$, $E_{(C,\underline{t},\overline{t})}(V_i(t, C)) = V_i(u(C, \underline{t}, \overline{t}), C)$, where*

$$u(C, \underline{t}, \overline{t}) = \begin{cases} \underline{t} & \text{if } \underline{t} = \overline{t}^*(\overline{t}, C) \text{ or } \overline{t}^*(\overline{t}, C) = 1 \\ \\ 1 - \frac{\overline{t}^*(\overline{t},C) - \underline{t}}{\ln \frac{1-\underline{t}}{1-\overline{t}^*(\overline{t},C)}} & \text{if } \underline{t} \neq \overline{t}^*(\overline{t}, C) \neq 1. \end{cases}$$

*This transfer rate $u(C, \underline{t}, \overline{t})$ is in $[\underline{t}, \overline{t}^*(\overline{t}, C)]$ for all $(C, \underline{t}, \overline{t})$.*

Proof: If $\underline{t} = \overline{t}^*(\overline{t}, C)$, it is straightforward that $E_{(C,\underline{t},\overline{t})}(V_i(t, C)) = V_i(\underline{t}, C)$ and, since $u(C, \underline{t}, \overline{t}) = \underline{t}$ for this case, it is also immediate that $u(C, \underline{t}, \overline{t}) \in [\underline{t}, \overline{t}^*(\overline{t}, C)]$.

Suppose now that $\underline{t} \neq \overline{t}^*(\overline{t}, C) = 1$. We can deduce from expression (2) that $c_i = 0$ for all $i < n$. Then, by expression (1), $V_n(t, C) = c_n$ for all $t \in [\underline{t}, \overline{t}^*(\overline{t}, C))$ and $V_n(t, C) \in [0, c_n]$ if $t = \overline{t}^*(\overline{t}, C)$. Similarly, for any $i < n$, $V_i(t, C) = 0$ for all $t \in [\underline{t}, \overline{t}^*(\overline{t}, C))$ and $V_i(t, C) \in [0, c_n]$ if $t = \overline{t}^*(\overline{t}, C)$. Therefore, given that the density of the interval $[\underline{t}, \overline{t}^*(\overline{t}, C))$ is infinitely greater than the density of the unique point $\overline{t}^*(\overline{t}, C)$, we have that $E_{(C,\underline{t},\overline{t})}(V_n(t, C)) = c_n$ and $E_{(C,\underline{t},\overline{t})}(V_i(t, C)) = 0$ for all $i < n$. Thus, $E_{(C,\underline{t},\overline{t})}(V_i(t, C)) = V_i(\underline{t}, C)$ for all $i \in N$. In this case, since we have as before that $u(C, \underline{t}, \overline{t}) = \underline{t}$, it is also immediate that $u(C, \underline{t}, \overline{t}) \in [\underline{t}, \overline{t}^*(\overline{t}, C)]$.

Suppose from now on that $\underline{t} \neq \overline{t}^*(\overline{t}, C) \neq 1$. Consider first the most upstream region; that is, $i = 1$. By expression (1), we have that $V_i(t, C) = \frac{c_i}{1-t}$. Given that the uncertainty over $t$ takes the form of an uniform variable, we have that

$$E_{(C,\underline{t},\overline{t})}(V_i(t, C)) = \frac{\int_{\underline{t}}^{\overline{t}^*(\overline{t},C)} \frac{c_i}{1-t}\,\mathrm{d}t}{\overline{t}^*(\overline{t}, C) - \underline{t}} = \frac{c_i(-\ln(1-t))\big|_{\underline{t}}^{\overline{t}^*(\overline{t},C)}}{\overline{t}^*(\overline{t}, C) - \underline{t}} = c_i \cdot \frac{\ln \frac{1-\underline{t}}{1-\overline{t}^*(\overline{t},C)}}{\overline{t}^*(\overline{t}, C) - \underline{t}} = \frac{c_i}{1 - u(C, \underline{t}, \overline{t})}. \tag{3}$$

Then, we have deduced that $E_{(C,\underline{t},\overline{t})}(V_1(t, C)) = V_1(u(C, \underline{t}, \overline{t}), C)$. Consider now any region $i \in \{2, \ldots, n-1\}$. Then, we have that

$$E_{(C,\underline{t},\overline{t})}(V_i(t, C)) = \frac{\int_{\underline{t}}^{\overline{t}^*(\overline{t},C)} \left(\frac{c_i}{1-t} - \frac{c_{i-1}\cdot t}{1-t}\right)\mathrm{d}t}{\overline{t}^*(\overline{t}, C) - \underline{t}} = \frac{\int_{\underline{t}}^{\overline{t}^*(\overline{t},C)} \frac{c_i}{1-t}\,\mathrm{d}t}{\overline{t}^*(\overline{t}, C) - \underline{t}} - \frac{c_{i-1} \cdot \int_{\underline{t}}^{\overline{t}^*(\overline{t},C)} \frac{t}{1-t}\,\mathrm{d}t}{\overline{t}^*(\overline{t}, C) - \underline{t}}. \tag{4}$$

Observe that the minuend coincides with expression (3) and, then, we concentrate on the subtrahend:

$$\frac{c_{i-1} \cdot \int_{\underline{t}}^{\overline{t}^*(\overline{t},C)} \frac{t}{1-t}\,\mathrm{d}t}{\overline{t}^*(\overline{t}, C) - \underline{t}} = \frac{c_{i-1}(-t - \ln(1-t))\big|_{\underline{t}}^{\overline{t}^*(\overline{t},C)}}{\overline{t}^*(\overline{t}, C) - \underline{t}} = c_{i-1} \cdot \frac{\ln \frac{1-\underline{t}}{1-\overline{t}^*(\overline{t},C)} + \underline{t} - \overline{t}^*(\overline{t}, C)}{\overline{t}^*(\overline{t}, C) - \underline{t}} =$$

$$c_{i-1} \cdot \frac{u(C, \underline{t}, \overline{t})}{1 - u(C, \underline{t}, \overline{t})}.$$

Thus, we obtain that $E_{(C,\underline{t},\overline{t})}(V_i(t, C)) = V_i(u(C, \underline{t}, \overline{t}), C)$ for all $i \in \{2, \ldots, n-1\}$.

Consider finally region $i = n$. Observe that $E_{(C,\underline{t},\overline{t})}(V_n(t, C))$ is equal to $c_n$ minus the subtrahend of expression (4). Then, using the same calculus we can deduce that $E_{(C,\underline{t},\overline{t})}(V_n(t, C)) = V_n(u(C, \underline{t}, \overline{t}), C)$.

To finish the proof it remains to be showed that the value $u(C, \underline{t}, \overline{t})$ is in the interval $[\underline{t}, \overline{t}^*(\overline{t}, C)]$ for these cases in which $\underline{t} \neq \overline{t}^*(\overline{t}, C) \neq 1$. First, suppose that there is at least one region $i$ for which the responsibility function $V_i$ is not constant in the entire range of $t$. In this case, observe that:

($i$) The expected value of any random variable is always between its minimum and its maximum and, applying this fact to the random variable of the responsibility function of region $i$, we get that $E_{(C,\underline{t},\overline{t})}(V_i(t, C)) \in [\min_{t \in [\underline{t}, \overline{t}^*(\overline{t},C)]} V_i(t, C), \max_{t \in [\underline{t}, \overline{t}^*(\overline{t},C)]} V_i(t, C)]$.

(*ii*) The responsibility function $V_i$ is either strictly increasing in the entire range of $t$ or strictly decreasing in the entire range of $t$ and, then, the value of the responsibility function for any value of $t$ outside the interval $[\underline{t}, \overline{t}^*(\overline{t}, C)]$ is smaller than $\min_{t \in [\underline{t}, \overline{t}^*(\overline{t}, C)]} V_i(t, C)$ or higher than $\max_{t \in [\underline{t}, \overline{t}^*(\overline{t}, C)]} V_i(t, C)$.

Therefore, we can conclude from (*i*) and (*ii*) that $V_i(\hat{t}, C)$ cannot equal the expected responsibility of region $i$ for any $\hat{t} \notin [\underline{t}, \overline{t}^*(\overline{t}, C)]$. Since we also have that the responsibility function $V_i$ is continuous in $t$, any value between $\min_{t \in [\underline{t}, \overline{t}^*(\overline{t}, C)]} V_i(t, C)$ and $\max_{t \in [\underline{t}, \overline{t}^*(\overline{t}, C)]} V_i(t, C)$ is obtained for some value of $t \in [\underline{t}, \overline{t}^*(\overline{t}, C)]$. Thus, $u(C, \underline{t}, \overline{t}) \in [\underline{t}, \overline{t}^*(\overline{t}, C)]$.[10]

Second, suppose that $V_i$ is constant in the entire range of $t$ for all $i \in N$. This case refers to those problems in which $c_1 = \ldots = c_{n-1} = 0$. Consider another problem $(C', \underline{t}, \overline{t})$ such that $c'_1 = \ldots = c'_{n-2} = 0$, $c'_{n-1} = \epsilon$, with $\epsilon$ arbitrarily small, and $c'_n = c_n$. Observe that $V_n(t, C')$ is not constant in the entire range of $t$. Then, applying the reasoning of the previous paragraph, we have that $u(C', \underline{t}, \overline{t}) \in [\underline{t}, \overline{t}^*(\overline{t}, C')]$. Given the structure of $C$ and $C'$, we also have that $\overline{t}^*(\overline{t}, C) = \overline{t}^*(\overline{t}, C') = \overline{t}$ (see expression (2)). Therefore, $u(C, \underline{t}, \overline{t}) = u(C', \underline{t}, \overline{t})$ and, thus, $u(C, \underline{t}, \overline{t}) \in [\underline{t}, \overline{t}^*(\overline{t}, C)]$. $\square$

Given Proposition 1, we can define this proposal of calculating the expected responsibility of each region as the following cost allocation method.

**Definition 3** *The Expected Responsibility method (ER), $\delta$, is such that, for all problems $(C, \underline{t}, \overline{t})$ and all $i \in N$, $\delta_i(C, \underline{t}, \overline{t}) = E_{(C,\underline{t},\overline{t})}(V_i(t, C)) = V_i(u(C, \underline{t}, \overline{t}), C)$.*

We now analyze the cost allocation method that assigns to each region $i$, in any problem $(C, \underline{t}, \overline{t})$, the median of the random variable $V_i(t, C)$. The following proposition shows that this cost allocation method coincides with assigning to each region $i$ the value of its responsibility function for the expected value of the random variable of the transfer rate $t$ defined over $[\underline{t}, \overline{t}^*(\overline{t}, C)]$. We denote this expected value by $E_{(C,\underline{t},\overline{t})}(t)$.

**Proposition 2** *For all $(C, \underline{t}, \overline{t})$ and all $i \in N$, $med_{(C,\underline{t},\overline{t})}(V_i(t, C)) = V_i(E_{(C,\underline{t},\overline{t})}(t), C)$.*

Proof: The proof comes from the combination of the following three facts:

(*i*) For any region $i \in N$, we have that $med_{(C,\underline{t},\overline{t})}(V_i(t, C)) = V_i(med_{(C,\underline{t},\overline{t})}(t), C)$, where $med_{(C,\underline{t},\overline{t})}(t)$ is the median of the random variable $t$.

[10] We have proven that the value $u(C, \underline{t}, \overline{t})$ belongs to $[\underline{t}, \overline{t}^*(\overline{t}, C)]$ doing the analysis for a region $i$ for which the responsibility function $V_i$ is not constant in the entire range of $t$. However, this analysis can be extended for the remaining regions: if $V_j(t, C)$ is constant in the entire range of $t$ for some region $j \in N$, we have that $E_{(C,\underline{t},\overline{t})}(V_j(t, C)) = V_j(\hat{t}, C)$ for all $\hat{t} \in [0, 1)$ and, therefore, we also have that $E_{(C,\underline{t},\overline{t})}(V_j(t, C)) = V_j(u(C, \underline{t}, \overline{t}), C)$ for the value $u(C, \underline{t}, \overline{t}) \in [\underline{t}, \overline{t}^*(\overline{t}, C)]$ calculated for region $i$.

This is immediate for the case in which $\overline{t}^*(\overline{t}, C) \neq 1$, where $V_i(t, C)$ is either constant in the entire range of $t$, or strictly increasing in the entire range of $t$, or strictly decreasing in the entire range of $t$.

Consider now the case in which $\overline{t}^*(\overline{t}, C) = 1$. Remember from the proof of Proposition 1 that this implies that $c_i = 0$ for all $i < n$. Then, $V_n(t, C) = c_n$ for all $t \in [\underline{t}, \overline{t}^*(\overline{t}, C))$ and $V_n(t, C) \in [0, c_n]$ if $t = \overline{t}^*(\overline{t}, C)$. Similarly, for any $i < n$, $V_i(t, C) = 0$ for all $t \in [\underline{t}, \overline{t}^*(\overline{t}, C))$ and $V_i(t, C) \in [0, c_n]$ if $t = \overline{t}^*(\overline{t}, C)$. Given that the density of the interval $[\underline{t}, \overline{t}^*(\overline{t}, C))$ is infinitely greater than the density of the unique point $\overline{t}^*(\overline{t}, C)$, we have that $med_{(C,\underline{t},\overline{t})}(V_n(t, C)) = c_n$ and $med_{(C,\underline{t},\overline{t})}(V_i(t, C)) = 0$ for all $i < n$. Thus, $med_{(C,\underline{t},\overline{t})}(V_i(t, C)) = V_i(med_{(C,\underline{t},\overline{t})}(t), C)$ for all $i \in N$.

*(ii)* The uncertainty over $t$ takes the form of a uniform variable in the interval $[\underline{t}, \overline{t}^*(\overline{t}, C)]$.

This is because we assumed that, before analyzing the information on the cost vector, the uncertainty over $t$ took the form of a uniform variable in the interval $[\underline{t}, \overline{t}]$. Then, the updating of the information about $t$ from the cost vector (see expression (2)) implies that this initial random variable should be truncated at $\overline{t}^*(\overline{t}, C)$. The truncation does not change the fact that the random variable follows a uniform distribution, but only the interval of possible values from $[\underline{t}, \overline{t}]$ to $[\underline{t}, \overline{t}^*(\overline{t}, C)]$.

*(iii)* Since a uniform distribution is symmetric about the mean, the median and the expected value of the random variable that describes the uncertainty on $t$ coincide: $med_{(C,\underline{t},\overline{t})}(t) = E_{(C,\underline{t},\overline{t})}(t)$.

Then, we can conclude that $med_{(C,\underline{t},\overline{t})}(V_i(t, C)) = V_i(E_{(C,\underline{t},\overline{t})}(t), C)$. □

According to Proposition 2, the median responsibility coincides with the cost allocation method that uses the expected value of $t$ as the transfer rate in the responsibility function. Notice that therefore the median responsibility is equivalent to the Upstream Responsibility (UR) method proposed by Alcalde-Unzu et al. (2015).

**Definition 4** *The* Upstream Responsibility *(UR) method, $\gamma$, is such that for all problems $(C, \underline{t}, \overline{t})$ and all $i \in N$, $\gamma_i(C, \underline{t}, \overline{t}) = med_{(C,\underline{t},\overline{t})}(V_i(t, C)) = V_i(E_{(C,\underline{t},\overline{t})}(t), C)$.*

It can be seen with Propositions 1 and 2 that the expected responsibility of a region (*i.e.*, the cost allocated to such a region by the ER method) differs from the responsibility that this region would have if it were known that the transfer rate equals its expected value (*i.e.*, the median responsibility or, what is the same, the cost allocated to this region by the UR method). The reason of this difference is that the responsibility function $V_i$ is not linear in $t$ and, then, the expected responsibility should be calculated with a value of the transfer rate, $u(C, \underline{t}, \overline{t})$, different from its expected value, $E_{(C,\underline{t},\overline{t})}(t)$, to correct this non-linearity. The following proposition compares these two values, $u(C, \underline{t}, \overline{t})$ and $E_{(C,\underline{t},\overline{t})}(t)$.

**Proposition 3** *For all $(C, \underline{t}, \overline{t})$ with $\underline{t} \neq \overline{t}^*(\overline{t}, C) \neq 1$, $u(C, \underline{t}, \overline{t}) > E_{(C,\underline{t},\overline{t})}(t)$.*

Proof: First, we do the proof in the case in which there is at least one region $i$ for which the responsibility function $V_i$ is not constant in the entire range of $t$. In this case, we know that either $V_i$ is strictly increasing and convex in $t$ or strictly decreasing and concave in $t$. On the one hand, if $V_i$ is strictly increasing and convex in $t$, then $E_{(C,\underline{t},\overline{t})}(V_i(t,C)) > med_{(C,\underline{t},\overline{t})}(V_i(t,C))$ and, thus, $u(C,\underline{t},\overline{t}) > E_{(C,\underline{t},\overline{t})}(t)$. If, on the other hand, $V_i$ is strictly decreasing and concave in $t$, then $E_{(C,\underline{t},\overline{t})}(V_i(t,C)) < med_{(C,\underline{t},\overline{t})}(V_i(t,C))$ and, thus, we also have that $u(C,\underline{t},\overline{t}) > E_{(C,\underline{t},\overline{t})}(t)$.

Second, we do the proof in the case in which $V_i$ is constant in the entire range of $t$ for all $i \in N$. This case implies that $c_1 = \ldots = c_{n-1} = 0$. Consider another problem $(C',\underline{t},\overline{t})$ such that $c'_1 = \ldots = c'_{n-2} = 0$, $c'_{n-1} = \epsilon$, with $\epsilon$ arbitrarily small, and $c'_n = c_n$. Observe that $V_n(t,C')$ is not constant in the entire range of $t$. Then, applying the reasoning of the previous paragraph, we have that $u(C',\underline{t},\overline{t}) > E_{(C',\underline{t},\overline{t})}(t)$. Given the structure of $C$ and $C'$, we also have that $\overline{t}^*(\overline{t},C) = \overline{t}^*(\overline{t},C') = \overline{t}$ (see expression (2)). Therefore, $u(C,\underline{t},\overline{t}) = u(C',\underline{t},\overline{t})$ and $E_{(C,\underline{t},\overline{t})}(t) = E_{(C',\underline{t},\overline{t})}(t)$. Therefore, $u(C,\underline{t},\overline{t}) > E_{(C,\underline{t},\overline{t})}(t)$. $\square$

Proposition 3 shows that to compute the expected value of $V_i(t,C)$ it is necessary to take a value of the transfer rate higher than its expected value. Given that $V_1(t,C)$ is strictly increasing in $t$ (except when $c_1 = 0$ in which it is constant), $V_n(t,C)$ is strictly decreasing in $t$ (except when $c_{n-1} = 0$ in which it is constant) and, for all $i \in \{2,\ldots,n-1\}$, $V_i(t,C)$ is increasing (respectively, decreasing) in $t$ whenever $c_i \geq c_{i-1}$ (respectively, $c_i \leq c_{i-1}$), we obtain the following corollary:

**Corollary 1** *For all $(C,\underline{t},\overline{t})$ with $\underline{t} \neq \overline{t}^*(\overline{t},C) \neq 1$,*

- $V_1(E_{(C,\underline{t},\overline{t})}(t),C) < E_{(C,\underline{t},\overline{t})}(V_1(t,C))$*, except when $c_1 = 0$ in which they are equal.*
- $V_n(E_{(C,\underline{t},\overline{t})}(t),C) > E_{(C,\underline{t},\overline{t})}(V_n(t,C))$*, except when $c_{n-1} = 0$ in which they are equal.*
- *For all $i \in \{2,\ldots,n-1\}$, $V_i(E_{(C,\underline{t},\overline{t})}(t),C) \geq E_{(C,\underline{t},\overline{t})}(V_i(t,C))$ if and only if $c_i \leq c_{i-1}$.*

Corollary 1 states that, on the one hand, the most upstream (resp. downstream) region pays less (resp. more) with the UR method than with the ER method. On the other hand, an intermediate region pays more (resp. less) with the UR method than with the ER method if the cost of cleaning the preceding region is higher (resp. lower) than the cost of cleaning its own segment.

We now illustrate how the ER and UR methods work with the cost allocation problem introduced in Example 1.

**Example 1 (continuation)** Remember that the cost allocation problem of Example 1 is $(\{100,200,300,400\},\frac{1}{5},\frac{3}{5})$ and that, according to expression (2), $\overline{t}^*(\overline{t},C) = \frac{4}{7}$. On the one hand, the UR method computes the expected value of the transfer rate $E_{(C,\underline{t},\overline{t})}(t) = \frac{27}{70}$, and introduces this value as the transfer rate in expression (1) to obtain the allocation $\gamma(C,\underline{t},\overline{t}) = (V_1(\frac{27}{70},C),\ldots,V_4(\frac{27}{70},C)) = (162.8, 262.8, 362.8, 211.6)$. On the other hand, the

ER method calculates, using Proposition 1, $u(C,\underline{t},\overline{t}) = 0.405$, and it introduces this value as the transfer rate in expression (1) to obtain the allocation $\delta(C,\underline{t},\overline{t}) = (V_1(0.405, C), \ldots, V_4(0.405, C)) = (168.07, 268.07, 368.07, 195.79)$. We can now compare the solutions provided by both methods. First, as Proposition 3 states, the value $u(C,\underline{t},\overline{t}) = 0.4$ is higher than $E_{(C,\underline{t},\overline{t})}(t) = \frac{27}{70}$. As a consequence, and as Corollary 1 states, we have that the most upstream region pays more with the ER method than with the UR one ($\delta_1(C,\underline{t},\overline{t}) = 168.07 > 162.8 = \gamma_1(C,\underline{t},\overline{t})$), the most downstream region pays less with the ER method than with the UR one ($\delta_4(C,\underline{t},\overline{t}) = 195.79 < 211.6 = \gamma_4(C,\underline{t},\overline{t})$), and the intermediate regions pay in this case more with the ER method because the cost sequence is strictly increasing ($\delta_2(C,\underline{t},\overline{t}) = 268.07 > 262.8 = \gamma_2(C,\underline{t},\overline{t})$ and $\delta_3(C,\underline{t},\overline{t}) = 368.07 > 362.8 = \gamma_3(C,\underline{t},\overline{t})$). Finally, as stated by Remark 1, we can see that the UR method has a lower expected sum of absolute deviations than the ER one, and that the opposite occurs with the expected sum of squared deviations. As an example of that fact, we compute these values for region 1:

$$\int_{\underline{t}}^{\overline{t}^*(\overline{t},C)} |V_1(t,C) - \gamma_1(C,\underline{t},\overline{t})|\, dt = \int_{\frac{1}{5}}^{\frac{4}{7}} |V_1(t,C) - 162.8|\, dt = 9.59$$

$$\int_{\underline{t}}^{\overline{t}^*(\overline{t},C)} |V_1(t,C) - \delta_1(C,\underline{t},\overline{t})|\, dt = \int_{\frac{1}{5}}^{\frac{4}{7}} |V_1(t,C) - 168.07|\, dt = 9.69$$

$$\int_{\underline{t}}^{\overline{t}^*(\overline{t},C)} (V_1(t,C) - \gamma_1(C,\underline{t},\overline{t}))^2\, dt = \int_{\frac{1}{5}}^{\frac{4}{7}} (V_1(t,C) - 162.8)^2\, dt = 355.15$$

$$\int_{\underline{t}}^{\overline{t}^*(\overline{t},C)} (V_1(t,C) - \delta_1(C,\underline{t},\overline{t}))^2 dt = \int_{\frac{1}{5}}^{\frac{4}{7}} (V_1(t,C) - 168.07)^2\, dt = 344.95$$

□

# 4 A property of monotonicity

Both the UR and the ER methods are *a priori* reasonable candidates for allocating costs according to the responsibility that each region has in the discharge of the waste. We now study if they satisfy a property of monotonicity. This property reflects the natural idea that if a region discharges more waste, ceteris paribus, it should not pay less in the allocation of the costs. If a solution does not satisfy this property, regions could have incentives to discharge more waste.

In order to understand the property and its formal definition that we will state later, consider a river with an actual transfer rate $\hat{t}$, but about which the social planner only knows that $\hat{t} \in [\underline{t}, \overline{t}]$. The property that we want to study compares the performance of a method in two related problems of this river: one in which the regions have discharged some amounts of waste such that the resulting cost vector is $C$, and other in which all regions have discharged the same amounts of waste that in $C$ except for one particular region $i$ that has discharged $y > 0$ units of waste more than in the previous situation. We denote $C'$ the cost vector of the second problem. To understand the relation between $C$ and $C'$, observe first that the waste present in each region situated upstream of $i$ is the same in both problems because the additional waste discharged by $i$ in the second problem is finally located in region $i$ and its downstream regions. Thus, $c'_j = c_j$ for all $j < i$. Moreover, since the actual transfer rate between two adjacent regions of this river is $\hat{t}$, we have that $(1 - \hat{t}) \cdot y$ units of this additional discharge of region $i$ remain in region $i$ and, thus, $c'_i = c_i + y \cdot (1 - \hat{t})$. The other $\hat{t} \cdot y$ units of the additional waste discharged by region $i$ have passed to region $i + 1$, from which a proportion $\hat{t}$ has also passed to the next regions and a proportion $(1 - \hat{t})$ has remained in region $i + 1$. Then, $c'_{i+1} = c_{i+1} + y \cdot \hat{t} \cdot (1 - \hat{t})$. Applying the same reasoning with all intermediate regions situated downstream of $i$, it is obtained that $c'_j = c_j + y \cdot \hat{t}^{j-i} \cdot (1 - \hat{t})$ for all $j \in \{i + 1, \ldots, n - 1\}$. The last region in the river is different because all the waste that arrives to that region remains there and, therefore, $c'_n = c_n + y \cdot \hat{t}^{n-i}$. Then, we introduce the notation $C \to_i^{\hat{t}} C'$ whenever $C$ and $C'$ come from a river with the same actual transfer rate $\hat{t}$ and each region discharging the same amount of waste in $C$ and $C'$, except region $i$ that has discharged some additional amount $y > 0$ in $C'$ than in $C$: *i.e.*, if $c'_j = c_j$ for all $j < i$, $c'_j = c_j + y \cdot \hat{t}^{j-i} \cdot (1 - \hat{t})$ for all $j \in \{i, \ldots, n - 1\}$, and $c'_n = c_n + y \cdot \hat{t}^{n-i}$. The formal definition of the property uses this notation.

**Monotonicity** (MON): For all problems $(C, \underline{t}, \overline{t}), (C', \underline{t}, \overline{t})$ such that $C \to_i^{\hat{t}} C'$ for some $i \in N$ and some $\hat{t} \in [\underline{t}, \overline{t}^*(\overline{t}, C)]$, $x_i(C', \underline{t}, \overline{t}) \geq x_i(C, \underline{t}, \overline{t})$.

Observe that if we have $C$ and $C'$ such that $C \to_i^{\hat{t}} C'$ and the social planner knew that the actual value of the transfer rate is $\hat{t}$ (*i.e.*, $\underline{t} = \hat{t} = \overline{t}$), then she would allocate the same cost to all regions in $(C, \underline{t}, \overline{t})$ and in $(C', \underline{t}, \overline{t})$, except to region $i$, to which she would allocate the additional costs of $C'$ with respect to $C$. However, when the social planner has uncertainty over the actual value of the transfer rate, this reasoning cannot be used, because even though $C \to_i^{\hat{t}} C'$, the social planner does not know that $\hat{t}$ is the actual transfer rate. In that case, the social planner could consider that it is also possible that the additional waste in $C'$ with respect to $C$ is not only due to region $i$, but also to other regions. For instance, consider the extreme case in which $\hat{t} > 0$, but 0 is also in $[\underline{t}, \overline{t}]$. Then, the social planner cannot distinguish if $\hat{t}$ or 0 is the actual transfer rate and, therefore, she does not know if all the additional waste in $C'$ than in $C$ is responsibility of $i$ or if each region $j \in \{i, i + 1, \ldots, n\}$ is responsible for $c'_j - c_j$ additional units of waste. Thus, in the case of uncertainty over the value of the transfer rate, it is not reasonable to require to a method that all the additional waste is allocated to $i$. Then, MON requires at least that region $i$ should not pay less because the contrary would give perverse incentives to region $i$ to discharge more waste just to reduce its assigned cost in those cases in which the actual transfer rate coincides with $\hat{t}$.

The MON property is weaker than other monotonicity axioms in the literature of cost allocation. In such axioms, it is usually imposed that any agent cannot pay less if the costs increase in any way. It can be easily checked that this stronger property is violated by any method that allocates costs in terms of the responsibility function and that incorporates in its calculus the information on the transfer rate embedded in the cost vector. To see why, consider two problems $(C, \underline{t}, \overline{t}), (C', \underline{t}, \overline{t})$ such that $c'_1 = c_1$, $c'_i > c_i$ for all $i \in \{2, \dots, n\}$, and $\overline{t}^*(\overline{t}, C') < \overline{t}^*(\overline{t}, C)$. Then, since the responsibility function of region 1 equals the quotient between the cost of cleaning the first segment and 1 minus a value for the transfer rate, we would obtain a lower cost allocated to region 1 in problem $(C', \underline{t}, \overline{t})$ than in $(C, \underline{t}, \overline{t})$. Instead of requiring that all regions should not pay less in any increase of the cost vector, MON only imposes that a particular region should not pay less for some particular increases of the cost vector: those that can be the result of a unilateral discharge of waste by this region.

We now analyze if the methods proposed in the previous section satisfy the MON property. Unfortunately, the following result shows that the ER solution violates it.

**Theorem 1** *The ER method $\delta$ does not satisfy MON.*

Proof: Consider the problem $(C, \underline{t}, \overline{t})$ such that $C = (11, 10, 20, 500)$, $\underline{t} = 0$ and $\overline{t} = 1$. Assume also that the actual transfer rate is $t = \frac{1}{2}$. The upper limit of the transfer rate for the social planner is $\overline{t}^*(\overline{t}, C) = \frac{10}{11}$ and, then, $u(C, \underline{t}, \overline{t}) = 0.621$. Therefore, region 1 should pay 29.024 monetary units with the ER method. If, however, region 1 had decided to discharge one additional unit of waste, we would have had the cost cleaning vector $C' = (11.5, 10.25, 20.125, 500.125)$, in which the upper limit of the transfer rate would have been $\overline{t}^*(\overline{t}, C') = 0.891$ and, then, $u(C, \underline{t}, \overline{t}) = 0.598$. Thus, region 1 would have paid 28.606 monetary units, less than what it would have paid if it had not discharged this additional waste. □

The reason underlying this result is that if a region $i$ discharges more waste, it affects $\delta_i$ in two ways. On the one hand, this discharge weakly increases $c_i$, which tends to increase $\delta_i$. On the other hand, this discharge may modify the upper limit of the transfer rate from $\overline{t}^*(\overline{t}, C)$ to $\overline{t}^*(\overline{t}, C')$, transforming also the value $u(C, \underline{t}, \overline{t})$ into $u(C', \underline{t}, \overline{t})$, which also affects $\delta_i$. There are cases in which the second effect tends to decrease $\delta_i$ and in some of them this effect is greater than the first effect, leading to an overall decrease in $\delta_i$ when $i$ discharges more waste.

We now show that the UR method does satisfy MON. Observe for example the problem defined in the proof of Theorem 1. The original problem was $(C, \underline{t}, \overline{t})$ such that $C = (11, 10, 20, 500)$, $\underline{t} = 0$ and $\overline{t} = 1$, and the actual transfer rate was $t = \frac{1}{2}$. Then, $\overline{t}^*(\overline{t}, C) = \frac{10}{11}$ and, thus, $E_{(C,\underline{t},\overline{t})}(t) = \frac{5}{11}$. Therefore, region 1 should pay $V_1(\frac{5}{11}, C) =$ 20.166 monetary units with the UR method. If, however, region 1 had decided to discharge one additional unit of waste, we would have had the cost cleaning vector $C' = (11.5, 10.25, 20.125, 500.125)$, in which the upper limit of the transfer rate would have been $\overline{t}^*(\overline{t}, C') = 0.891$ and then, $E_{(C',\underline{t},\overline{t})}(t) = 0.445$. Therefore, the UR method would allocate to region 1 $V_1(0.445, C') = 20.72$ monetary units, more than what it would have paid if it had not discharged this additional waste. The following theorem, whose long proof is shown in Appendix A, states that the fulfilling of MON by the UR method is not limited to this case.

**Theorem 2** *The UR method $\gamma$ satisfies MON.*

To sum up, we have two main issues that differentiate both methods. On the one hand, fixing any cost vector, the UR method minimizes the expected sum of absolute deviations between the actual value of the responsibility of each region and the assigned cost, and the ER one minimizes the expected sum of squared deviations between the actual value of the responsibility of each region and the assigned cost. On the other hand, the ER method does not satisfy MON, which implies that regions could have incentives to discharge more waste and increase the values of the cost vector, meanwhile this problem is not present with the UR method.

A social planner who ideally prefers to minimize the expected sum of squared deviations between the actual value of each region's responsibility and the assigned cost has a trade-off: The method that produces her preferred result in each cost vector, the ER method, tends to increase the discharged waste by the regions and, then, the values of the cost vector. Then, a natural question for this social planner is whether there exists a method closer than the UR method to the ER one that satisfies MON. The following result shows that this is not the case in the family of solutions that assign to each region a cost using a weighted combination of $\underline{t}$ and $\overline{t}^*(\overline{t}, C)$ as the transfer rate in the responsibility function.

**Theorem 3** *Consider the family of functions $f_\alpha(\underline{t}, \overline{t}^*(\overline{t}, C)) = (1-\alpha)\cdot \underline{t} + \alpha \cdot \overline{t}^*(\overline{t}, C)$, with $\alpha \in [0,1]$. Then, the method that allocates to each region $V_i(f_\alpha(\underline{t}, \overline{t}^*(\overline{t}, C)), C)$ satisfies MON if and only if $\alpha \le \frac{1}{2}$.*

Proof: The proof is done by steps.

*Step 1: We show that the method that allocates to each region $V_i(f_\alpha(\underline{t}, \overline{t}^*(\overline{t}, C)), C)$ does not satisfy MON if $\alpha > \frac{1}{2}$.*

We know that $V_1(f_\alpha(\underline{t}, \overline{t}^*(\overline{t}, C)), C) = \frac{c_1}{1-\alpha\cdot\overline{t}^*(\overline{t},C)-(1-\alpha)\cdot\underline{t}}$. Consider the class of problems $(C, \underline{t}, \overline{t})$ such that there are more than two regions, $\underline{t} = 0$, $\overline{t} = 1$, $\overline{t}^*(\overline{t}, C) = \frac{c_2}{c_1} \to 1$ and where the actual transfer rate is $t \to 0$. We have that $c_1 = V_1(t, C)\cdot(1-t)$ and $c_2 = V_1(t, C)\cdot t\cdot(1-t) + V_2(t, C)\cdot(1-t)$. To simplify notation, denote $V_1(t, C)$ by $x$ and $V_2(t, C)$ by $y$. Then, $\overline{t}^*(\overline{t}, C) = \frac{xt+y}{x}$. Therefore, $V_1(f_\alpha(0, \overline{t}^*(\overline{t}, C)), C) = \frac{x(1-t)}{1-\alpha\cdot\frac{xt+y}{x}}$. If the method satisfies MON, the derivative of $V_1(f_\alpha(0, \overline{t}^*(\overline{t}, C)), C)$ with respect to $x$ should be non-negative. This derivative is:

$$\frac{(1-t)(1-\alpha\frac{xt+y}{x}) - (1-t)\alpha\frac{y}{x}}{(1-\alpha\frac{xt+y}{x})^2}.$$

The denominator of the derivative is always positive and, then, we have to guarantee that the numerator is non-negative. After doing some calculus, we obtain that this fact is equivalent to

$$\alpha \leq \frac{x}{xt + 2y}.$$

Given that $\overline{t}^*(\overline{t}, C) \to 1$ and $t \to 0$, we obtain that $y \to x(1-t)$. Then, in the limit

$$\alpha \leq \frac{1}{2-t}.$$

Since $t \to 0$, we obtain that $\alpha \leq \frac{1}{2}$. Then, we have that $\alpha \leq \frac{1}{2}$ is a necessary condition to satisfy MON. Or, equivalently, that any method $V_i(f_\alpha(\underline{t}, \overline{t}^*(\overline{t}, C)), C)$, with $\alpha > \frac{1}{2}$, does not satisfy MON.

*Step 2: We show that if the method that allocates to each region $V_i(f_\alpha(\underline{t}, \overline{t}^*(\overline{t}, C)), C)$ satisfies MON, then any method that allocates to each region $V_i(f_{\alpha'}(\underline{t}, \overline{t}^*(\overline{t}, C)), C)$, with $\alpha^{'} < \alpha$, also satisfies MON.*

Suppose by contradiction that the method that allocates to each region $V_i(f_\alpha(\underline{t}, \overline{t}^*(\overline{t}, C)), C)$ satisfies MON for some $\alpha \in (0, 1]$, but $V_i(f_{\alpha'}(\underline{t}, \overline{t}^*(\overline{t}, C)), C)$, with $\alpha^{'} < \alpha$, does not satisfy MON. Then, there is a region $i \in N$ and two problems $(C, \underline{t}, \overline{t}), (C', \underline{t}, \overline{t})$ such that $C \to_i^{\hat{t}} C'$ for some $\hat{t} \in [\underline{t}, \overline{t}^*(\overline{t}, C)]$ and $V_i(f_{\alpha'}(\underline{t}, \overline{t}^*(\overline{t}, C')), C') < V_i(f_{\alpha'}(\underline{t}, \overline{t}^*(\overline{t}, C)), C)$.

By construction, we have that $f_\alpha(\underline{t}, \overline{t}^*(\overline{t}, C)) > f_{\alpha'}(\underline{t}, \overline{t}^*(\overline{t}, C))$ and $|f_{\alpha'}(\underline{t}, \overline{t}^*(\overline{t}, C')) - f_{\alpha'}(\underline{t}, \overline{t}^*(\overline{t}, C))| < |f_\alpha(\underline{t}, \overline{t}^*(\overline{t}, C')) - f_\alpha(\underline{t}, \overline{t}^*(\overline{t}, C))|$. It is also easy to see that $\frac{\partial^2 V_i(t,C)}{\partial t \, \partial c_i} \geq 0$ for any $i \in N$. Then, by the union of these three facts, it can be deduced that $V_i(f_\alpha(\underline{t}, \overline{t}^*(\overline{t}, C')), C') < V_i(f_\alpha(\underline{t}, \overline{t}^*(\overline{t}, C)), C)$. Then, the method that allocates to each region $V_i(f_\alpha(\underline{t}, \overline{t}^*(\overline{t}, C))$ does not satisfy MON and this is a contradiction.

*Step 3: We complete the proof.*

Since $E_{(C,\underline{t},\overline{t})}(t) = \frac{1}{2} \cdot \overline{t}^*(\overline{t}, C) + \frac{1}{2} \cdot \underline{t}$, the UR method coincides with the method that allocates to each region $V_i(f_{\frac{1}{2}}(\underline{t}, \overline{t}^*(\overline{t}, C)), C)$. Then, Theorem 2 states that $V_i(f_{\frac{1}{2}}(\underline{t}, \overline{t}^*(\overline{t}, C)), C)$ satisfies MON. By Step 2, we can then deduce that $V_i(f_\alpha(\underline{t}, \overline{t}^*(\overline{t}, C)), C)$, with $\alpha < \frac{1}{2}$ also satisfies MON. Finally, Step 1 shows that $V_i(f_\alpha(\underline{t}, \overline{t}^*(\overline{t}, C)), C)$, with $\alpha > \frac{1}{2}$, does not satisfy MON. □

We can deduce from Theorem 3, jointly with Proposition 3 and the fact that $E_{(C,\underline{t},\overline{t})}(t) = f_{\frac{1}{2}}(\underline{t}, \overline{t}^*(\overline{t}, C)), C)$, the following corollary.

**Corollary 2** *The UR method is the one closest to the ER method that satisfies MON in the family of methods that allocates to each region $V_i(f_\alpha(\underline{t}, \overline{t}^*(\overline{t}, C)), C)$ for any $\alpha \in [0, 1]$.*

# 5 Concluding remarks

There are several legislations that regulate the problem of river pollution. For instance, the European Union Framework Directive (Directive 2000/60/EC) establishes that "The European Parliament and the Council shall adopt specific measures against pollution of water by individual pollutants or groups of pollutants presenting a significant risk to or via the aquatic environment" and, more specifically, it states that "for those bodies of groundwater which cross the boundary between two or more Member States [...] the following information shall, where relevant be collected [...] ($e$) the rates of discharge at such points". In order to compute the rates of discharge by each region, this regulation states that monitoring points along the river should be selected such that each of them is "required to estimate the pollutant load which is transferred across member State boundaries, and which is transferred into the marine environment".

We have modeled the problem of allocating the costs of cleaning a transboundary river using this approach: if the transfer rate between any two neighboring regions was known, each region's responsibility for the waste present in the river could be directly determined by using the responsibility function. However, it could be the case that the estimation of the transfer rate in the monitoring points were not totally precise (*i.e.*, it could vary from one day to another depending on factors such as the volume of flow) and, thus, there would be inaccuracy about the exact responsibility of each region for the waste discharged. We have represented this uncertainty with a random variable over the possibles values of the transfer rate and, thus, the responsibility of each region is also a random variable. Given that the expected value and the median are the most important centrality measures of random variables, we have analyzed two cost allocation methods that assign to each region its expected responsibility and its median responsibility, respectively. Meanwhile the first of these proposals defines a new method in the literature (that we have called the ER method), we have shown that the second one coincides with the UR method analyzed in Alcalde-Unzu et al. (2015). We have found that the allocations proposed by these two methods present systematic differences. For example, we have proved that the most upstream region pays always more with the ER method than with the UR one, while the opposite occurs with the most downstream region.

It is well-known that the expected value and the median, and hence the ER and UR methods, try to minimize the errors in the assignment of regions' responsibilities with respect to their actual values. In particular, fixing any cost vector, the UR method minimizes the expected sum of absolute deviations between the actual value of each region's responsibility and the assigned cost, meanwhile the ER method minimizes the expected sum of squared deviations between the actual value of each region's responsibility and the assigned cost. At this point, if the social planner only knew these differences between these two methods, the choice would depend on the objective function that she wanted to minimize with respect to the errors the method would make assigning responsibilities: the expected sum of absolute deviations or the expected sum of squared deviations. However, we have found another interesting difference between these two methods in terms of a property of monotonicity. The UR method satisfies the property of MON, which requires that if a region discharges more waste, it should not pay less. The ER method does not satisfy it. Then, if the social planner favors the minimization of the expected sum of squared deviations between the actual value of each region's responsibility and the assigned cost, she confronts a trade-off: The method that optimizes her objective function could occasionally give incentives to regions to discharge more waste in order to reduce their allocated costs. In the search for other methods satisfying MON that assign costs closer to the ER method to solve this trade-off, we have found that the closest method in a general family to the ER method that satisfies this property is precisely the UR method.

It is interesting to remark two issues about our results. First, all our analysis has been performed focusing on cost allocation methods, and not on cost allocation rules. Since there are many cost allocation rules associated to each of these cost allocation methods, it is interesting to define which one is the appropriate for each method. Alcalde-Unzu et al. (2015) provided an axiomatic characterization of a cost allocation rule consistent with the UR method. It is possible to adapt this characterization modifying only one axiom to characterize a rule consistent with the ER method. We perform this analysis in Appendix B.

Second, we have assumed along the paper that the uncertainty on the transfer rate takes the form of a uniform distribution. To finish the paper, we discuss how the results depend on this assumption. Our first results, Propositions 1 and 2, establish that the Expected Responsibility and the Median Responsibility methods coincide with assigning to each region $i$ the value of its responsibility function $V_i$ for a particular transfer rate, $u(C, \underline{t}, \overline{t})$ and $E_{(C,\underline{t},\overline{t})}(t), C)$, respectively. If other distributions different from the uniform one were adopted, the existence of such transfer rates would be guaranteed, but the exact value would obviously depend on the distribution. To see why, observe first that all the reasoning in the proof of Proposition 1 to show that the value $u(C, \underline{t}, \overline{t})$ is unique and in the interval $[\underline{t}, \overline{t}^*(\overline{t}, C)]$ does not rely neither on the exact value taken by $u(C, \underline{t}, \overline{t})$ under the assumption of the uniform distribution, nor on the own distribution itself. With respect to Proposition 2, observe that one of the arguments of its proof is that, independently of the distribution function assumed on the transfer rate, we always have that the median responsibility coincides with the responsibility for the median transfer rate and, therefore, it is always possible to compute the median responsibility of each region $i$ by calculating $V_i(med_{(C,\underline{t},\overline{t})}(t), C)$.[11] Proposition 3 shows, under the assumption of a uniform distribution, that the value of the transfer rate needed to implement the Expected Responsibility is higher than the one needed to implement the Median Responsibility. Since the responsibility functions are increasing and convex or decreasing and concave, one should analyze

[11] In the case of the uniform distribution, we also have that $med_{(C,\underline{t},\overline{t})}(t) = E_{(C,\underline{t},\overline{t})}(t)$ and, therefore, the median responsibility can be calculated with $V_i(E_{(C,\underline{t},\overline{t})}(t), C)$. However, this equality may not occur with other distributions because the other arguments in the proof of Proposition 2 do not hold for all distributions. This could happen either because the distribution before updating the information from the cost vector is not symmetric about the mean, or because after updating this information and truncating the random variable at $\overline{t}^*(\overline{t}, C)$ its distribution changes to one not symmetric about the mean.

the skewness of the distribution function to determine this relation in the case of other distributions: if it is positively skewed, we would have the same inequality as with the uniform distribution, and the inequality only reverts if the distribution function is sufficiently negatively skewed. Finally, the analysis of the fulfilment of the MON property in the two methods done in Theorems 1 and 2 does depend on the assumption of a uniform distribution and any other distribution would need an independent analysis. We consider this as an interesting topic for further research.

# Appendix A: Proof of Theorem 2

We start introducing new notation: $C \rightarrow^{\hat{t}}_{i,y} C'$ if $C \rightarrow^{\hat{t}}_{i} C'$ and $\sum_{i\in N} c'_i = \sum_{i\in N} c_i + y$. That is, $C \rightarrow^{\hat{t}}_{i,y} C'$ when $C'$ has been constructed from $C$ by an additional discharge of $y$ units of waste by region $i$. We also define for each $i \in \{2,\ldots,n\}$ a mapping $q_i : \mathbb{R}^n_+ \rightarrow (\mathbb{R}_+ \cup \{\frac{0}{0}\})$ such that $q_i(C) = \frac{c_i}{c_{i-1}}$ if $i < n$ and $q_i(C) = \frac{c_i}{c_i + c_{i-1}}$ if $i = n$. Then, $\overline{t}^*(\overline{t}, C) = \min\{\{q_i(C)\}_{i\in\{2,\ldots,n\}}, \overline{t}\}$. Finally, we define a mapping $d : (0,1] \times \mathbb{R}^n_+ \rightarrow 2^{N\setminus\{1\}}$ such that for all $\overline{t} \in (0,1]$ and all $\bar{C} \in \mathbb{R}^n_+$, $d(\overline{t}, \bar{C}) = \{i \in (N \setminus \{1\}) \,|\, q_i(N) = \overline{t}^*(\overline{t}, \bar{C})\}$. Observe that whenever $d(\overline{t}, \bar{C}) = \emptyset$, this means that $\overline{t}^*(\overline{t}, \bar{C}) = \overline{t}$.

Then, consider a problem $(C, \underline{t}, \overline{t})$, an agent $i \in N$, a transfer rate $\hat{t} \in [\underline{t}, \overline{t}^*(\overline{t}, C)]$, and a vector $C' \in \mathbb{R}^n_+$ such that $C \rightarrow^{\hat{t}}_{i} C'$. Suppose by contradiction that $\gamma_i(C', \underline{t}, \overline{t}) < \gamma_i(C, \underline{t}, \overline{t})$. Let $y > 0$ be the value such that $C \rightarrow^{\hat{t}}_{i,y} C'$.

If $\hat{t} = 1$, then $c_j = c'_j = 0$ for all $j < n$ and $c'_n = c_n + y$. Therefore, $\gamma_j(C', \underline{t}, \overline{t}) = \gamma_j(C, \underline{t}, \overline{t}) = 0$ for all $j < n$ and $\gamma_n(C', \underline{t}, \overline{t}) = c_n + y > c_n = \gamma_n(C, \underline{t}, \overline{t})$. Thus, the previous contradiction cannot occur and we can assume from now on that $\hat{t} < 1$.

If $\overline{t}^*(\overline{t}, C') = \overline{t}^*(\overline{t}, C)$, then $E_{(C', \underline{t}, \overline{t})}(t) = E_{(C, \underline{t}, \overline{t})}(t)$. Since $c'_i > c_i$ and $c'_{i-1} = c_{i-1}$, we obtain that $\gamma_i(C', \underline{t}, \overline{t}) > \gamma_i(C, \underline{t}, \overline{t})$. Then, the previous contradiction cannot occur and, therefore, assume from now on that $\overline{t}^*(\overline{t}, C') \neq \overline{t}^*(\overline{t}, C)$. We proceed now with a set of lemmas. The objective of these lemmas is to show that there exist $\hat{C}, \hat{C}' \in \mathbb{R}^n_+$ such that $\hat{C} \rightarrow^{\hat{t}}_{i} \hat{C}'$, $\gamma_i(\hat{C}', \underline{t}, \overline{t}) < \gamma_i(\hat{C}, \underline{t}, \overline{t})$ and $d(\overline{t}, \hat{C}) \cap d(\overline{t}, \hat{C}') \in \{\{i\}, \{i+1\}\}$. The first lemma is a simple algebra result, whose proof is straightforward and we omit it.

**Lemma 1** *Let $a, b, c, d, e, f \in \mathbb{R}_{++}$ such that $\frac{a}{b} \leq \frac{c}{d}$. Then,*

- $\frac{a}{b} \leq \frac{a+c}{b+d} \leq \frac{c}{d}$
- $e \geq f \Leftrightarrow \frac{ea+fc}{eb+fd} \leq \frac{a+c}{b+d}$.

The second lemma describes the direction of the change of the functions $q_j$, with $j \in \{2,\ldots,n\}$, from $C$ to $C'$.

**Lemma 2** *$q_i(C') \geq q_i(C)$, $q_j(C') = q_j(C)$ for all $j < i$ and $q_j(C') \leq q_j(C)$ for all $j > i$.*[12]

[12]This statement only applies if the function $q$ does not equal the indeterminate form $\frac{0}{0}$.

Proof: By construction, $c'_j = c_j$ for all $j < i$ and, then, $q_j(C') = q_j(C)$ for all $j < i$. If $\hat{t} = 0$, then $c'_j = c_j$ for all $j \neq i$ and $c'_i = c_i + y > c_i$. Then, $q_j(C') = q_j(C)$ for all $j \in \{i+2, \ldots, n\}$, $q_i(C') \geq q_i(C)$ and $q_{i+1}(C') \leq q_{i+1}(C)$. Then, assume from now on that $\hat{t} \in (0, 1)$. We divide the rest of the proof into two cases.

*Case 1: $i = n$*

We have that $c'_n = c_n + y > c_n$. Observe then that $q_n(C) = \frac{c_n}{c_n + c_{n-1}}$ and $q_n(C') = \frac{c_n + y}{c_n + c_{n-1} + y}$. If we denote $c_n$ by $a$, $(c_n + c_{n-1})$ by $b$, $y$ by $c = d$, $q_n(C) = \frac{a}{b}$ and $q_n(C') = \frac{a+c}{b+d}$. Since $\hat{t} > 0$, then $c_n > 0$ and, thus, $a, b, c, d \in \mathbb{R}_{++}$. Since $q_n(C) \in (0, 1]$, then $\frac{a}{b} \leq 1$ and, thus, $\frac{a}{b} \leq \frac{c}{d}$. Then, by Lemma 1, $q_n(C') \geq q_n(C)$.

*Case 2: $i < n$.*

By construction, $c'_i = c_i + y \cdot (1 - \hat{t}) > c_i$, $c'_j = c_j + y \cdot \hat{t}^{j-i} \cdot (1 - \hat{t}) > c_j$ for all $j \in \{i+1, \ldots, n-1\}$ and $c'_n = c_n + y \cdot \hat{t}^{n-i} > c_n$. If $i > 1$, we obtain that $q_i(C') \geq q_i(C)$ since $c'_i > c_i$ and $c'_{i-1} = c_{i-1}$.

Consider now any $j \in \{i+1, \ldots, n-1\}$. We have that $q_j(C') = \frac{c_j + y \cdot \hat{t}^{j-i} \cdot (1-\hat{t})}{c_{j-1} + y \cdot \hat{t}^{j-i-1} \cdot (1-\hat{t})}$. Note that if $c_j = c_{j-1} = 0$, $q_j(C) = \frac{0}{0}$ and the lemma does not apply. Similarly, $c_j = 0$ and $c_{j-1} > 0$ is not possible since $\hat{t} > 0$. If, however, $c_j > 0$ and $c_{j-1} = 0$, $q_j(C) = \infty$ and $q_j(C') < \infty$ leading to $q_j(C') \leq q_j(C)$. Finally, if $c_j > 0$ and $c_{j-1} > 0$, denote $[y \cdot \hat{t}^{j-i} \cdot (1 - \hat{t})]$ by $a$, $[y \cdot \hat{t}^{j-i-1} \cdot (1 - \hat{t})]$ by $b$, $c_j$ by $c$ and $c_{j-1}$ by $d$. Note that $a, b, c, d \in \mathbb{R}_{++}$, $\hat{t} = \frac{a}{b}$, $q_j(C) = \frac{c}{d}$ and $q_j(C') = \frac{a+c}{b+d}$. Given that $q_j(C) \geq \overline{t}^*(\overline{t}, C)$ by definition, and that $\overline{t}^*(\overline{t}, C) \geq \hat{t}$, $q_j(C) \geq \hat{t}$. Then, $\frac{a}{b} \leq \frac{c}{d}$ and we can apply Lemma 1 to obtain that $q_j(C') \leq q_j(C)$.

We also have that $q_n(C') = \frac{c_n + y \cdot \hat{t}^{n-i}}{c_n + c_{n-1} + y \cdot \hat{t}^{n-i-1}}$. If we denote $[y \cdot \hat{t}^{n-i}]$ by $a$, $[y \cdot \hat{t}^{n-i-1}]$ by $b$, $c_n$ by $c$ and $(c_n + c_{n-1})$ by $d$, $\hat{t} = \frac{a}{b}$, $q_n(C) = \frac{c}{d}$ and $q_n(C') = \frac{a+c}{b+d}$. Since $\hat{t} > 0$, then $c_n > 0$ and, thus, $a, b, c, d \in \mathbb{R}_{++}$. As before, $q_n(C) \geq \overline{t}^*(\overline{t}, C)$ and, then, $q_n(C) \geq \hat{t}$. Then, $\frac{a}{b} \leq \frac{c}{d}$ and we can apply Lemma 1 to obtain that $q_n(C') \leq q_n(C)$. □

The next two lemmas study the case in which $\overline{t}^*(\overline{t}, C') < \overline{t}^*(\overline{t}, C)$. The last one, which needs the first one as an auxiliary result, shows that if $\overline{t}^*(\overline{t}, C') < \overline{t}^*(\overline{t}, C)$, then there exist $\hat{C}, \hat{C}' \in \mathbb{R}^n_+$ such that $\hat{C} \rightarrow^{\hat{t}}_i \hat{C}'$, $\gamma_i(\hat{C}', \underline{t}, \overline{t}) < \gamma_i(\hat{C}, \underline{t}, \overline{t})$ and $d(\overline{t}, \hat{C}) \cap d(\overline{t}, \hat{C}') = \{i + 1\}$.

**Lemma 3** *Suppose that $\overline{t}^*(\overline{t}, C') < \overline{t}^*(\overline{t}, C)$. Then, there exist $\hat{C}, \hat{C}' \in \mathbb{R}^n_+$ such that $\hat{c}_i = c_i$, $\hat{c}_{i-1} = c_{i-1}$, $\hat{C} \rightarrow^{\hat{t}}_{i,y} \hat{C}'$, $\overline{t}^*(\overline{t}, \hat{C}) = \overline{t}^*(\overline{t}, C)$, $\overline{t}^*(\overline{t}, \hat{C}') \leq \overline{t}^*(\overline{t}, C')$, and $d(\overline{t}, \hat{C}) \cap d(\overline{t}, \hat{C}') = \{i + 1\}$.*

Proof: Since $\overline{t}^*(\overline{t}, C') < \overline{t}^*(\overline{t}, C)$, then $d(\overline{t}, C') \neq \emptyset$ and, given Lemma 2, $d(\overline{t}, C') \subseteq \{i + 1, \ldots, n\}$. Consider $k = \min d(\overline{t}, C')$, that is, $k$ is the closest region to $i$ of $d(\overline{t}, C')$. Then, construct $\bar{C} \in \mathbb{R}^n_+$ such that $\bar{c}_j = c_j$ for all $j < k$, $\bar{c}_j = M \cdot \bar{c}_{j-1}$ for all $j > k$, with $M$ sufficiently big, $\bar{c}_k = \overline{t}^*(\overline{t}, C) \cdot \bar{c}_{k-1}$ if $k < n$ and $\bar{c}_k = \frac{\overline{t}^*(\overline{t}, C)}{1 - \overline{t}^*(\overline{t}, C)} \cdot \bar{c}_{k-1}$ if $k = n$. Observe that $q_k(\bar{C}) = \overline{t}^*(\overline{t}, C)$ and that $q_j(\bar{C}) \geq \overline{t}^*(\overline{t}, C)$ for all $j \neq k$. Then, $k \in d(\overline{t}, \bar{C})$ and $\overline{t}^*(\overline{t}, \bar{C}) = \overline{t}^*(\overline{t}, C)$.

Consider now $\bar{C}' \in \mathbb{R}^n_+$ such that $\bar{C} \rightarrow^{\hat{t}}_{i,y} \bar{C}'$. We know that $\bar{c}_j = c_j$ for all $j < k$ and, then, $\bar{c}'_j = c'_j$ for all $j < k$. Since $q_k(\bar{C}) = \bar{t}^*(\bar{t}, C)$, $q_k(\bar{C}) \leq q_k(C)$. Then, since $\bar{c}_{k-1} = c_{k-1}$, $\bar{c}_k \leq c_k$. Thus, $\bar{c}'_k \leq c'_k$. Then, since $\bar{c}'_{k-1} = c'_{k-1}$, $q_k(\bar{C}') \leq q_k(C')$. Since $k \in d(\bar{t}, C')$, we obtain that $\bar{t}^*(\bar{t}, \bar{C}') \leq \bar{t}^*(\bar{t}, C')$.

The selection of $M$ sufficiently big guarantees that $q_k(\bar{C}') < q_j(\bar{C}')$ for all $j > k$. Moreover, given that $q_k(\bar{C}') \leq q_k(C')$, that $q_k(C') < q_j(C')$ for all $j < k$ (because $k = \min d(\bar{t}, C')$) and that $q_j(C') = q_j(\bar{C}')$ for all $j < k$, $q_k(\bar{C}') < q_j(\bar{C}')$ for all $j < k$. Therefore, $d(\bar{t}, \bar{C}') = \{k\}$. If $k = (i+1)$, then denote $\bar{C}$ and $\bar{C}'$ by $\hat{C}$ and $\hat{C}'$, respectively, and the proof is done.

Suppose then that $k > (i+1)$ and observe that this is only possible if $\hat{t} > 0$ (otherwise $\bar{c}'_k = \bar{c}_k$ and $\bar{c}'_{k-1} = \bar{c}_{k-1}$, contradicting that $q_k(\bar{C}') < q_k(\bar{C})$). Consider $\hat{C} \in \mathbb{R}^n_+$ such that $\hat{c}_j = \bar{c}_j$ for all $j \leq i$, $\hat{c}_j = \bar{t}^*(\bar{t}, \bar{C}) \cdot \hat{c}_{j-1}$ for all $j \in \{i+1, \ldots, n-1\}$ and $\hat{c}_n = \frac{\bar{t}^*(\bar{t}, \bar{C})}{1-\bar{t}^*(\bar{t}, \bar{C})} \cdot \hat{c}_{n-1}$. Observe that $q_j(\hat{C}) = \bar{t}^*(\bar{t}, \bar{C})$ for all $j > i$ by construction. We also know that $q_j(\hat{C}) = q_j(\bar{C}) \geq \bar{t}^*(\bar{t}, \bar{C})$ for all $j \leq i$. Thus, $\bar{t}^*(\bar{t}, \hat{C}) = \bar{t}^*(\bar{t}, \bar{C}) = \bar{t}^*(\bar{t}, C)$ and $(i+1) \in d(\bar{t}, \hat{C})$.

Consider now $\hat{C}' \in \mathbb{R}^n_+$ such that $\hat{C} \rightarrow^{\hat{t}}_{i,y} \hat{C}'$. We now prove that $\bar{t}^*(\bar{t}, \hat{C}') \leq \bar{t}^*(\bar{t}, \bar{C}')$ by showing that $q_{i+1}(\hat{C}') \leq q_k(\bar{C}')$. If $k < n$ (if $k = n$ the proof is similar and thus omitted), $q_{i+1}(\hat{C}') = \frac{\hat{c}_{i+1} + y \cdot \hat{t} \cdot (1-\hat{t})}{\hat{c}_i + y \cdot (1-\hat{t})}$ and $q_k(\bar{C}') = \frac{\bar{c}_k + y \cdot \hat{t}^{k-i} \cdot (1-\hat{t})}{\bar{c}_{k-1} + y \cdot \hat{t}^{k-i-1} \cdot (1-\hat{t})}$. Observe that $\frac{\hat{c}_{i+1}}{\hat{c}_i} = \frac{\bar{c}_k}{\bar{c}_{k-1}} = \bar{t}^*(\bar{t}, C) \geq \hat{t} = \frac{y \cdot \hat{t} \cdot (1-\hat{t})}{y \cdot (1-\hat{t})} = \frac{y \cdot \hat{t}^{k-i} \cdot (1-\hat{t})}{y \cdot \hat{t}^{k-i-1} \cdot (1-\hat{t})}$. If we denote $[y \cdot \hat{t} \cdot (1-\hat{t})]$ by $a$, $[y \cdot (1-\hat{t})]$ by $b$, $\hat{c}_{i+1}$ by $c$ and $\hat{c}_i$ by $d$, $a, b, c, d \in \mathbb{R}_{++}$, $\frac{a}{b} \leq \frac{c}{d}$, $q_{i+1}(\hat{C}') = \frac{a+c}{b+d}$ and $q_k(\bar{C}') = \frac{ea+fc}{eb+fd}$, with $e = \hat{t}^{k-i-1}$ and $f = \frac{\bar{c}_k}{\hat{c}_{i+1}} = \frac{\bar{c}_{k-1}}{\hat{c}_i}$. Note that $\bar{c}_j \geq \bar{t}^*(\bar{t}, \bar{C}) \cdot \bar{c}_{j-1}$ for all $j \in \{i+2, \ldots, k\}$ and, then, $\bar{c}_k \geq [\bar{t}^*(\bar{t}, \bar{C})]^{k-i-1} \cdot \bar{c}_{i+1}$. Since $\bar{t}^*(\bar{t}, \bar{C}) \geq \hat{t}$, $\bar{c}_k \geq \hat{t}^{k-i-1} \cdot \bar{c}_{i+1}$. Since, by construction, $\hat{c}_{i+1} \leq \bar{c}_{i+1}$, we conclude that $\bar{c}_k \geq \hat{t}^{k-i-1} \cdot \hat{c}_{i+1}$. Therefore, $f \geq e$ and applying Lemma 1 $q_{i+1}(\hat{C}') \leq q_k(\bar{C}')$, as desired. Then, $\bar{t}^*(\bar{t}, \hat{C}') \leq \bar{t}^*(\bar{t}, \bar{C}')$ and, therefore, $\bar{t}^*(\bar{t}, \hat{C}') \leq \bar{t}^*(\bar{t}, C')$.

Finally, we are going to prove that $(i+1) = d(\bar{t}, \hat{C}')$ by showing that $q_j(\hat{C}') > q_{i+1}(\hat{C}')$ for all $j \neq (i+1)$. If $j \leq i$, by Lemma 2 $q_j(\hat{C}') \geq q_j(\hat{C})$. Then, given that $\bar{t}^*(\bar{t}, \hat{C}') < \bar{t}^*(\bar{t}, \hat{C})$, $q_j(\hat{C}') > q_{i+1}(\hat{C}')$. If $j \in \{i+2, \ldots, n-1\}$ (if $j = n$, the proof is similar and thus omitted), note that $q_j(\hat{C}') = \frac{\hat{c}_j + y \cdot \hat{t}^{j-i} \cdot (1-\hat{t})}{\hat{c}_{j-1} + y \cdot \hat{t}^{j-i-1} \cdot (1-\hat{t})} = \frac{e'a+f'c}{e'b+f'd}$, with $e' = \hat{t}^{j-i-1}$ and $f' = \frac{\hat{c}_j}{\hat{c}_{i+1}}$. By construction, $f' = [\bar{t}^*(\bar{t}, \bar{C})]^{j-i-1}$. Since $\bar{t}^*(\bar{t}, \bar{C}) > \bar{t}^*(\bar{t}, \bar{C}')$ and $\bar{t}^*(\bar{t}, \bar{C}') \geq \hat{t}$, $\bar{t}^*(\bar{t}, \bar{C}) > \hat{t}$. Then, $f' > e'$ and we can apply Lemma 1 to obtain that $q_j(\hat{C}') > q_{i+1}(\hat{C}')$ for all $j \in \{i+2, \ldots, n-1\}$. $\square$

**Lemma 4** *If $\bar{t}^*(\bar{t}, C') < \bar{t}^*(\bar{t}, C)$, then there exist $\hat{C}, \hat{C}' \in \mathbb{R}^n_+$ such that $\hat{C} \rightarrow^{\hat{t}}_{i,y} \hat{C}'$, $d(\bar{t}, \hat{C}) \cap d(\bar{t}, \hat{C}') = \{i+1\}$, and $\gamma_i(\hat{C}', \underline{t}, \bar{t}) < \gamma_i(\hat{C}, \underline{t}, \bar{t})$.*

<u>Proof</u>: Since $\bar{t}^*(\bar{t}, C') < \bar{t}^*(\bar{t}, C)$, we know by Lemma 3 that there exist $\hat{C}, \hat{C}' \in \mathbb{R}^n_+$ such that $\hat{c}_i = c_i$, $\hat{c}_{i-1} = c_{i-1}$, $\hat{C} \rightarrow^{\hat{t}}_{i,y} \hat{C}'$, $\bar{t}^*(\bar{t}, \hat{C}) = \bar{t}^*(\bar{t}, C)$, $\bar{t}^*(\bar{t}, \hat{C}') \leq \bar{t}^*(\bar{t}, C')$, and $d(\bar{t}, \hat{C}) \cap d(\bar{t}, \hat{C}') = \{i+1\}$. Given that $\bar{t}^*(\bar{t}, \hat{C}) = \bar{t}^*(\bar{t}, C)$, $E_{(C, \underline{t}, \bar{t})]}(t) = E_{(\hat{C}, \underline{t}, \bar{t})}(t)$. Then, since we also have that $\hat{c}_i = c_i$ and $\hat{c}_{i-1} = c_{i-1}$, we obtain that $\gamma_i(\hat{C}, \underline{t}, \bar{t}) = \gamma_i(C, \underline{t}, \bar{t})$. Moreover, given that $\hat{C} \rightarrow^{\hat{t}}_{i,y} \hat{C}'$ and $C \rightarrow^{\hat{t}}_{i,y} C'$, we obtain that $\hat{c}'_i = c'_i$ and $\hat{c}'_{i-1} = c'_{i-1}$.

On the other hand, since $V_i(t, C)$ is a monotone function in $t$ and $\overline{t}^*(\overline{t}, C') < \overline{t}^*(\overline{t}, C)$, we can deduce from $\gamma_i(C', \underline{t}, \overline{t}) < \gamma_i(C, \underline{t}, \overline{t})$ that $V_i(t, C)$ is strictly increasing in $t$. Then, given that $\overline{t}^*(\overline{t}, \hat{C}') \leq \overline{t}^*(\overline{t}, C')$, $\gamma_i(\hat{C}', \underline{t}, \overline{t}) \leq \gamma_i(C', \underline{t}, \overline{t})$. Therefore, $\gamma_i(\hat{C}', \underline{t}, \overline{t}) < \gamma_i(\hat{C}, \underline{t}, \overline{t})$. $\square$

The next lemma shows that if $\overline{t}^*(\overline{t}, C') > \overline{t}^*(\overline{t}, C)$, there exist $\hat{C}, \hat{C}' \in \mathbb{R}^n_+$ such that $\hat{C} \rightarrow^{\hat{t}}_i \hat{C}'$, $\gamma_i(\hat{C}', \underline{t}, \overline{t}) < \gamma_i(\hat{C}, \underline{t}, \overline{t})$ and $d(\overline{t}, \hat{C}) \cap d(\overline{t}, \hat{C}') = \{i\}$.

**Lemma 5** *If $\overline{t}^*(\overline{t}, C') > \overline{t}^*(\overline{t}, C)$, then there exist $C'' \in \mathbb{R}^n_+$ such that $C \rightarrow^{\hat{t}}_i C''$, $d(\overline{t}, C) \cap d(\overline{t}, C'') = \{i\}$, and $\gamma_i(C'', \underline{t}, \overline{t}) < \gamma_i(C, \underline{t}, \overline{t})$.*

Proof: By Lemma 2 $q_i(C') \geq q_i(C)$ and $q_j(C') \leq q_j(C)$ for all $j \neq i$. Then, since $\overline{t}^*(\overline{t}, C') > \overline{t}^*(\overline{t}, C)$, $\{i\} = d(\overline{t}, C)$. Consider now $C'' \in \mathbb{R}^n_+$ such that $C \rightarrow^{\hat{t}}_{i,z} C''$, with $z \in \mathbb{R}_{++}$ such that $q_i(C'') = \overline{t}^*(\overline{t}, C')$. We show now that $z \leq y$. To do it, observe that since $q_i(C') \geq \overline{t}^*(\overline{t}, C')$, $q_i(C'') \leq q_i(C')$. If $i < n$ (otherwise, the proof is very similar and thus omitted), $c''_{i-1} = c'_{i-1}$, $c''_i = c_i + z \cdot (1 - \hat{t})$, and $c'_i = c_i + y \cdot (1 - \hat{t})$. Therefore, $z \leq y$ and then $c''_i \leq c'_i$.

We now show that $\overline{t}^*(\overline{t}, C'') = \overline{t}^*(\overline{t}, C')$ and that, then, $i \in d(\overline{t}, C'')$. By construction, $q_i(C'') = \overline{t}^*(\overline{t}, C')$ and, then, we only need to prove that $q_j(C'') \geq q_j(C')$ for all $j \neq i$ because $q_j(C') \geq \overline{t}^*(\overline{t}, C')$ for all $j \neq i$. Observe first that $q_j(C'') = q_j(C')$ for all $j < i$ by construction. Consider now any $j > i$ (suppose that $j < n$, otherwise the proof is similar and thus omitted). If $\hat{t} = 0$, then $q_j(C'') = q_j(C')$ for all $j > i + 1$ and $q_{i+1}(C'') = \frac{c_{i+1}}{c_i+z}$ and $q_{i+1}(C') = \frac{c_{i+1}}{c_i+y}$. Since $y \geq z$, $q_{i+1}(C'') \geq q_{i+1}(C')$. If $\hat{t} \in (0, 1)$, observe first that since $\{i\} = d(\overline{t}, C)$, $c_{i-1} > 0$ and, then, $c_j > 0$ for all $j > i$. Then, $q_j(C'') = \frac{c_j + z \cdot \hat{t}^{j-i} \cdot (1-\hat{t})}{c_{j-1} + z \cdot \hat{t}^{j-i-1} \cdot (1-\hat{t})}$ and $q_j(C') = \frac{c_j + y \cdot \hat{t}^{j-i} \cdot (1-\hat{t})}{c_{j-1} + y \cdot \hat{t}^{j-i-1} \cdot (1-\hat{t})}$. If we denote $[z \cdot \hat{t}^{j-i} \cdot (1 - \hat{t})]$ by $a$, $[z \cdot \hat{t}^{j-i-1} \cdot (1 - \hat{t})]$ by $b$, $c_j$ by $c$ and $c_{j-1}$ by $d$, $a, b, c, d \in \mathbb{R}_+$, $\frac{a}{b} = \hat{t} \leq \frac{c}{d} = q_j(C)$, $q_j(C'') = \frac{a+c}{b+d}$ and $q_j(C') = \frac{ea+fc}{eb+fd}$, where $f = 1$ and $e = \frac{y}{z}$. Since $z \leq y$, $e \geq f$ and, by Lemma 1, $q_j(C'') \geq q_j(C')$ for all $j > i$.

Finally, given that $c''_i \leq c'_i$, $c''_{i-1} = c'_{i-1}$ and $\overline{t}^*(\overline{t}, C'') = \overline{t}^*(\overline{t}, C')$, $\gamma_i(C'', \underline{t}, \overline{t}) \leq \gamma_i(C', \underline{t}, \overline{t})$. Since $\gamma_i(C', \underline{t}, \overline{t}) < \gamma_i(C, \underline{t}, \overline{t})$ by assumption, we obtain that $\gamma_i(C'', \underline{t}, \overline{t}) < \gamma_i(C, \underline{t}, \overline{t})$. $\square$

Then, we have deduced from Lemmas 4 and 5 that we can assume that $d(\overline{t}, C) \cap d(\overline{t}, C') \in \{\{i\}, \{i+1\}\}$. To do the proof of the theorem, consider a family of vectors $\{C^{(z)}\}_{z \in \mathbb{R}_+}$ such that $C \rightarrow^{\hat{t}}_{i,z} C^{(z)}$. We are going to show that $\gamma_i(C^{(z)}, \underline{t}, \overline{t})$ is weakly increasing on $z$, which would imply that $\gamma_i(C, \underline{t}, \overline{t}) \leq \gamma_i(C', \underline{t}, \overline{t})$, a contradiction. The proof is developed in four cases.

Case 1: $i = n$.

Given that $d(\overline{t}, C^{(z)}) = \{n\}$ and, therefore, $\overline{t}^*(\overline{t}, C^{(z)}) = \frac{c^{(z)}_n}{c^{(z)}_n + c^{(z)}_{n-1}}$. Then, $E_{(C^{(z)}, \underline{t}, \overline{t})}(t) = \frac{(\underline{t}+1)(c_n+z) + \underline{t} \cdot c_{n-1}}{2 \cdot (c_n+z+c_{n-1})}$. Therefore,

$$\gamma_n(C^{(z)}, \underline{t}, \overline{t}) = V_n(E_{(C^{(z)}, \underline{t}, \overline{t})}(t), C^{(z)}) = c_n + z - c_{n-1} \cdot \frac{(\underline{t}+1)(c_n+z) + \underline{t} \cdot c_{n-1}}{(1-\underline{t})(c_n+z) + (2-\underline{t})c_{n-1}}.$$

By doing some basic calculus,

$$\frac{\partial V_n(E_{(C^{(z)},\underline{t},\overline{t})}(t),C^{(z)})}{\partial z} = 1 - \frac{2c_{n-1}^2}{((1-\underline{t})(c_n+z)+(2-\underline{t})c_{n-1})^2}.$$

Given that $\overline{t}^*(\overline{t},C) = \frac{c_n}{c_n+c_{n-1}}$, $\underline{t} \leq \frac{c_n}{c_n+c_{n-1}}$ and, therefore, $c_n \geq \frac{\underline{t}}{1-\underline{t}} \cdot c_{n-1}$. Thus,

$$\frac{\partial V_n(E_{(C^{(z)},\underline{t},\overline{t})}(t),C^{(z)})}{\partial z} \geq 1 - \frac{2c_{n-1}^2}{(\underline{t}\cdot c_{n-1}+(1-\underline{t})\cdot z+(2-\underline{t})c_{n-1})^2} = 1 - \frac{2c_{n-1}^2}{((1-\underline{t})\cdot z+2c_{n-1})^2}.$$

It is easy to see that the numerator of the last fraction is strictly lower than the denominator and, then, the derivative is strictly positive. Then, $\gamma_n(C^{(z)},\underline{t},\overline{t})$ is strictly increasing on $z$.

<u>Case 2: $i = 1$.</u>

We have that $d(\overline{t},C^{(z)}) = \{2\}$ and, therefore, $\overline{t}^*(\overline{t},C^{(z)}) = \frac{c_2^{(z)}}{c_1^{(z)}}$. Then, $E_{(C^{(z)},\underline{t},\overline{t})}(t) = \frac{c_2+z\cdot\hat{t}\cdot(1-\hat{t})+\underline{t}\cdot c_1+\underline{t}\cdot z\cdot(1-\hat{t})}{2c_1+2z\cdot(1-\hat{t})}$. Therefore,

$$\gamma_1(C^{(z)},\underline{t},\overline{t}) = V_1(E_{(C^{(z)},\underline{t},\overline{t})}(t),C^{(z)}) = \frac{2(c_1+(1-\hat{t})\cdot z)^2}{(2-\underline{t})\cdot c_1-c_2+(2-\underline{t}-\hat{t})\cdot(1-\hat{t})\cdot z}.$$

By doing some calculus[13]

$$\frac{\partial V_1(E_{(C^{(z)},\underline{t},\overline{t})}(t),C^{(z)})}{\partial z} = \frac{(2-2\hat{t})\cdot(c_1+z(1-\hat{t}))\cdot(c_1\cdot(\hat{t}+2-\underline{t})-2c_2+(2-\underline{t}-\hat{t})\cdot(1-\hat{t})\cdot z)}{((2-\underline{t})\cdot c_1-c_2+(2-\underline{t}-\hat{t})\cdot(1-\hat{t})\cdot z)^2}.$$

The denominator is clearly positive. With respect to the numerator, the first two terms of the product are also positive given that $\hat{t} \in [0,1)$. To determine the sign of the last term of the product, observe that $\overline{t}^*(\overline{t},C) = \frac{c_2}{c_1}$, which implies that $c_2 < c_1$. Observe additionally that $\underline{t} \leq \hat{t}$ and that both $\underline{t}$ and $\hat{t}$ are not greater than 1. Therefore, this term is also positive. Thus, we have arrived at the desired result.

<u>Case 3: $i \in \{2,\ldots,n-2\}$.</u>

We have that $d(\overline{t},C^{(z)}) \in \{\{i\},\{i+1\}\}$. Suppose first that $d(\overline{t},C^{(z)}) = \{i+1\}$ and, then, $\overline{t}^*(\overline{t},C^{(z)}) = \frac{c_{i+1}^{(z)}}{c_i^{(z)}}$. Then, $E_{(C^{(z)},\underline{t},\overline{t})}(t) = \frac{\underline{t}\cdot c_i+c_{i+1}+(\underline{t}+\hat{t})\cdot z\cdot(1-\hat{t})}{2c_i+2z\cdot(1-\hat{t})}$. Therefore, $V_i(E_{(C^{(z)},\underline{t},\overline{t})}(t),C^{(z)}) = \frac{c_i+z\cdot(1-\hat{t})}{1-E_{(C^{(z)},\underline{t},\overline{t})}(t)} - c_{i-1}\cdot\frac{E_{(C^{(z)},\underline{t},\overline{t})}(t)}{1-E_{(C^{(z)},\underline{t},\overline{t})}(t)}$. As in the previous cases, we are going to show that the derivative of $V_i(E_{(C^{(z)},\underline{t},\overline{t})}(t),C')$ with respect to $z$ is strictly positive. Observe that the new $E_{(C^{(z)},\underline{t},\overline{t})}(t)$ coincides with the one of Case 2 and that, therefore, the minuend of the expression $V_i(E_{(C^{(z)},\underline{t},\overline{t})}(t),C^{(z)})$ has the same structure as $V_1(E_{(C^{(z)},\underline{t},\overline{t})}(t),C^{(z)})$ in Case 2, whose derivative is strictly positive as we showed there. On the other hand, given that $\hat{t} \leq \frac{c_{i+1}}{c_i} < 1$, it is easy see that $\frac{\partial E_{(C^{(z)},\underline{t},\overline{t})}(t)}{\partial z} < 0$ and that the subtrahend depends positively

[13] The details of the calculus can be provided upon request.

on $E_{(C^{(z)},\underline{t},\overline{t})}(t)$. Therefore, if the minuend depends positively on $z$ and the subtrahend depends negatively on $z$, $\frac{\partial V_i(E_{(C^{(z)},\underline{t},\overline{t})}(t),C^{(z)})}{\partial z} > 0$, as we wanted to prove.

Consider now the case in which $d(\overline{t}, C^{(z)}) = \{i\}$ and, then, $\overline{t}^*(\overline{t}, C^{(z)}) = \frac{c_i^{(z)}}{c_{i-1}^{(z)}}$. Then, $E_{(C^{(z)},\underline{t},\overline{t})}(t) = \frac{\underline{t}c_{i-1}+c_i+z\cdot(1-\hat{t})}{2c_{i-1}}$ and, therefore,

$$\gamma_i(C^{(z)}, \underline{t}, \overline{t}) = V_i(E_{(C^{(z)},\underline{t},\overline{t})}(t), C^{(z)}) = \frac{c_{i-1}\cdot(c_i+z\cdot(1-\hat{t})-\underline{t}\cdot c_{i-1})}{(2-\underline{t})\cdot c_{i-1}-c_i-z\cdot(1-\hat{t})}.$$

By doing some calculus[14],

$$\frac{\partial V_i(E_{(C^{(z)},\underline{t},\overline{t})}(t),C^{(z)})}{\partial z} = \frac{(2-2\underline{t})\cdot c_{i-1}^2\cdot(1-\hat{t})}{((2-\underline{t})\cdot c_{i-1}-c_i-z\cdot(1-\hat{t}))^2}.$$

The denominator is positive and, given that $\underline{t} \leq \hat{t} < 1$, the numerator too. Thus, the derivative is strictly positive, as desired.

Case 4: $i = n-1$.

We have that $d(\overline{t}, C^{(z)}) \in \{\{n-1\}, \{n\}\}$. If $d(\overline{t}, C^{(z)}) = \{n-1\}$, the analysis is the same as the one followed at the second part of Case 3. Then, suppose from now on that $d(\overline{t}, C^{(z)}) = \{n\}$. Then, $E_{(C^{(z)},\underline{t},\overline{t})}(t) = \frac{\underline{t}\cdot c_{n-1}+(\underline{t}+1)\cdot c_n+(\underline{t}+\hat{t})\cdot z}{2c_{n-1}+2c_n+2z}$. Observe that $E_{(C^{(z)},\underline{t},\overline{t})}(t)$ is strictly decreasing with $z$. We also know that $V_{n-1}(E_{(C^{(z)},\underline{t},\overline{t})}(t), C^{(z)}) = \frac{c_{n-1}+z\cdot(1-\hat{t})}{1-E_{(C^{(z)},\underline{t},\overline{t})}(t)} - c_{n-2}\cdot\frac{E_{(C^{(z)},\underline{t},\overline{t})}(t)}{1-E_{(C^{(z)},\underline{t},\overline{t})}(t)}$. As in Case 3, the subtrahend of $V_i(E_{(C^{(z)},\underline{t},\overline{t})}(t), C^{(z)})$ depends positively on $E_{(C^{(z)},\underline{t},\overline{t})}(t)$, so if we prove that the minuend depends positively on $z$, we will have the desired result. Then, by doing some calculus, the derivative of the minuend with respect to $z$ is

$$\frac{2\cdot(c_n^2\cdot(1-\underline{t})\cdot(1-\hat{t})+2c_n\cdot(1-\underline{t})\cdot(1-\hat{t})\cdot(c_{n-1}+z)+c_{n-1}^2\cdot(\underline{t}\cdot(\hat{t}-1)-\hat{t}+2)+2c_{n-1}\cdot(2-\underline{t})\cdot(1-\hat{t})\cdot z+(1-\hat{t})\cdot z^2\cdot(2-\underline{t}-\hat{t}))}{((\underline{t}-2)\cdot c_{n-1}+(\underline{t}-1)\cdot c_n+(\underline{t}+\hat{t}-2)\cdot z)^2}.$$

It is easy to check that both the numerator and the denominator are strictly positive. Therefore, we have arrived at the desired result.

# Appendix B: A rule compatible with the ER method

The literature already includes cost allocation rules associated with the different methods (see Alcalde-Unzu et al. (2015)). We now present an axiomatic characterization of a cost allocation rule consistent with the ER method. In particular, adapting a characterization of the UR rule, we characterize a new rule compatible with the ER method by modifying one axiom. This axiom is the one that leads in the characterization of the UR rule to consider $E_{(C,\underline{t},\overline{t})}(t)$ as the value of the transfer rate in the responsibility function.

[14]The details of the calculus can be provided upon request.

The three axioms that coincide with the characterization of the UR rule are Limits of Responsibility, No Downstream Responsibility and Consistent Responsibility. We are going to present them briefly.[15] To introduce the first one, it is important to note that Proposition 4 of Alcalde-Unzu et al. (2015) establishes that the knowledge that $t$ is situated within the interval $[\underline{t}, \overline{t}^*(\overline{t}, C)]$ allows to deduce some limits of the responsibility that each region $i$ has on the waste present in its own segment. These limits, denoted by $\underline{l}_i^i$ and $\overline{l}_i^i$, are the following:

$$\underline{l}_i^i(C, \underline{t}, \overline{t}) = \begin{cases} c_i & \text{if } i = 1 \\ c_i - c_{i-1} \cdot \overline{t}^*(\overline{t}, C) & \text{if } i \in \{2, \ldots, n-1\} \\ c_i - \frac{c_{i-1} \cdot \overline{t}^*(\overline{t}, C)}{1 - \overline{t}^*(\overline{t}, C)} & \text{if } i = n \text{ and } \overline{t}^*(\overline{t}, C) < 1 \\ 0 & \text{if } i = n \text{ and } \overline{t}^*(\overline{t}, C) = 1. \end{cases}$$

$$\overline{l}_i^i(C, \underline{t}, \overline{t}) = \begin{cases} c_i & \text{if } i = 1 \\ c_i - c_{i-1} \cdot \underline{t} & \text{if } i \in \{2, \ldots, n-1\} \\ c_i - \frac{c_{i-1} \cdot \underline{t}}{1 - \underline{t}} & \text{if } i = n \end{cases}$$

Then, the first of these axioms requires that the cost paid by each region for cleaning its own segment should always be within the interval established by these limits.

**Limits of Responsibility** (LR): For all problems $(C, \underline{t}, \overline{t})$, and for all $i \in N$, $x_i^i(C, \underline{t}, \overline{t}) \in [\underline{l}_i^i(C, \underline{t}, \overline{t}), \overline{l}_i^i(C, \underline{t}, \overline{t})]$.

The second of these axioms requires that no region has responsibility for the waste present in another region situated upstream from it.

**No Downstream Responsibility** (NDR): For all problems $(C, \underline{t}, \overline{t})$ and all $i, j \in N$ such that $i < j$, $x_j^i(C, \underline{t}, \overline{t}) = 0$.

The third of these axioms imposes that the rule should establish the same degree of responsibility of a region $i$ relative to the responsibility of a region $j$ for the waste present in the most downstream region of this pair as the relative responsibilities established for these regions in the waste present in a region $k$ situated downstream from them.

**Consistent Responsibility** (CR): For all problems $(C, \underline{t}, \overline{t})$ and all $i, j, k \in N$ such that $i < j < k$,

$$x_j^j(C, \underline{t}, \overline{t}) \cdot x_i^k(C, \underline{t}, \overline{t}) = x_j^k(C, \underline{t}, \overline{t}) \cdot x_i^j(C, \underline{t}, \overline{t}).$$

Finally, the last axiom, Corrected Monotonicity with respect to Information on the Transfer rate (CMIT), adapts the MIT property that is included in the characterization of the UR rule. Both axioms study situations in which, ceteris paribus, the information improves in such a way that some previously possible values for the transfer rate can be ruled out with

[15] Extended explanations are included in Alcalde-Unzu et al. (2015).

the new information. That is, it compares the cost allocations in two problems, $(C, \underline{t}, \overline{t})$ and $(C, \underline{u}, \overline{u})$, such that $[\underline{u}, \overline{u}^*(\overline{u}, C)] \subset [\underline{t}, \overline{t}^*(\overline{t}, C)]$. To see their differences, we first present both of them formally:[16]

**Monotonicity with respect to Information on the Transfer rate** (MIT): For all problems $(C, \underline{t}, \overline{t})$ and $(C, \underline{u}, \overline{u})$ such that $[\underline{u}, \overline{u}^*(\overline{u}, C)] \subset [\underline{t}, \overline{t}^*(\overline{t}, C)]$ and for all $j \in N$,

$$\underline{u} - \underline{t} > \overline{t}^*(\overline{t}, C) - \overline{u}^*(\overline{u}, C) \Rightarrow x_j^j(C, \underline{u}, \overline{u}) \leq x_j^j(C, \underline{t}, \overline{t})$$

$$\underline{u} - \underline{t} < \overline{t}^*(\overline{t}, C) - \overline{u}^*(\overline{u}, C) \Rightarrow x_j^j(C, \underline{u}, \overline{u}) \geq x_j^j(C, \underline{t}, \overline{t}).$$

**Corrected Monotonicity with respect to Information on the Transfer rate** (CMIT): For all problems $(C, \underline{t}, \overline{t}), (C, \underline{u}, \overline{u})$ such that $[\underline{u}, \overline{u}^*(\overline{u}, C)] \subset [\underline{t}, \overline{t}^*(\overline{t}, C)]$ and for all $j \in N$,

$$\int_{\underline{t}}^{\underline{u}} [x_j^j(C, v, v) - x_j^j(C, \underline{u}, \overline{u})]\, dv > \int_{\overline{u}^*(\overline{u}, C)}^{\overline{t}^*(\overline{t}, C)} [x_j^j(C, \underline{u}, \overline{u}) - x_j^j(C, v, v)]\, dv \Rightarrow x_j^j(C, \underline{u}, \overline{u}) \leq x_j^j(C, \underline{t}, \overline{t})$$

$$\int_{\underline{t}}^{\underline{u}} [x_j^j(C, v, v) - x_j^j(C, \underline{u}, \overline{u})]\, dv < \int_{\overline{u}^*(\overline{u}, C)}^{\overline{t}^*(\overline{t}, C)} [x_j^j(C, \underline{u}, \overline{u}) - x_j^j(C, v, v)]\, dv \Rightarrow x_j^j(C, \underline{u}, \overline{u}) \geq x_j^j(C, \underline{t}, \overline{t}).$$

MIT and CMIT impose conditions on the comparison between how much each region $j$ pays for cleaning its own segment in each of the two problems (that is, on the comparison between $x_j^j(C, \underline{t}, \overline{t})$ and $x_j^j(C, \underline{u}, \overline{u})$).

On the one hand, MIT states that the improvement in information will increase or decrease $x_j^j(\cdot)$ depending on the masses of probability of the intervals that are discarded for the transfer rate. For example, if the lowest part of the interval that has been discarded [*i.e.*, $(\underline{u} - \underline{t})$] is bigger than the highest part of the interval that has been discarded [*i.e.*, $(\overline{t}^*(\overline{t}, C) - \overline{u}^*(\overline{u}, C))$], then $x_j^j(C, \underline{u}, \overline{u})$ should be not higher than $x_j^j(C, \underline{t}, \overline{t})$.

On the other hand, CMIT states that the comparison between $x_j^j(C, \underline{t}, \overline{t})$ and $x_j^j(C, \underline{u}, \overline{u})$ depends on the responsibilities that each region would have in each of the cost allocation problems defined with the values of the transfer rate that have been discarded. If the responsibility function was linear in $t$, both proposals would be equal. However, given that this is not the case, the distinction is important.

We now define formally the new cost allocation rule. This rule is similar to the UR one, with the only difference that it considers the value $u(C, \underline{t}, \overline{t})$ for the transfer rate instead of $E_{(C, \underline{t}, \overline{t})}(t)$.

[16] In the original formulation of MIT included in Alcalde-Unzu et al. (2015), the condition is not imposed directly on the comparison between each $x_j^j(\cdot)$, but on the comparison between each $\sum_{i<j} x_i^j(\cdot)$. Given NDR, these sums are inversely related to each $x_j^j(\cdot)$.

**Definition 5** *The* Expected Responsibility *(ER) rule, $\delta$, is given by:*

$$\delta_i^j(C,\underline{t},\overline{t}) = \begin{cases} 0 & \text{if } i > j, \\ c_i \cdot u(C,\underline{t},\overline{t})^{j-i} - c_{i-1} \cdot u(C,\underline{t},\overline{t})^{j+1-i} & \text{if } i \leq j < n, \\ c_i - \frac{c_{i-1} \cdot u(C,\underline{t},\overline{t})}{1-u(C,\underline{t},\overline{t})} & \text{if } i = j = n, \\ \frac{c_i \cdot u(C,\underline{t},\overline{t})^{j-i} - c_{i-1} \cdot u(C,\underline{t},\overline{t})^{j+1-i}}{1-u(C,\underline{t},\overline{t})} & \text{if } i < j = n, \end{cases}$$

*where $c_0$ is set to $0$ and the indeterminate form $0^0$ is set to $1$.*

The following result states that the ER rule is characterized by the same axioms as the UR one with the unique change of CMIT by MIT.

**Theorem 4** *A rule satisfies LR, NDR, CR and CMIT if and only if it is the Expected Responsibility rule $\delta$.*

The proof technique parallels that of the characterization of the UR rule in Alcalde-Unzu et al. (2015). Although the huge similarities, we have opted for including it. However, we have opted for omitting the proof of the independence of this set of axioms. This issue can be proven also in the same way as in Alcalde-Unzu et al. (2015).

Proof: First, it is easy to see that the ER rule $\delta$ satisfies LR, NDR, CR and CMIT. To prove the other implication, consider a problem $(C,\underline{t},\overline{t})$ and its corresponding $\overline{t}^*(\overline{t},C)$ inferred from expression (2). Let $x$ be a rule satisfying LR, NDR, CR and CMIT. We are going to show that $x$ has to correspond to $\delta$.

We will calculate the assignment given by $x$ in $n$ steps. In the $j-$th step, we calculate the values of $x_i^j(\cdot)$ for all $i \in \{1,\ldots,n\}$.

- Step 1: We distribute the cost $c_1$. In this case, by NDR, $x_i^1(\cdot) = 0$ for all $i > 1$. Then, by definition of a rule, $x_1^1(\cdot) = c_1$. If $n = 1$, the proof is finished. If $n > 1$, go to step 2.
- Step $j$, with $j \in \{2,\ldots,n\}$: We distribute the cost $c_j$. By the application of NDR, $x_i^j(C,\underline{t},\overline{t}) = 0$ for all $i > j$. Consider other problem $(C, u(C,\underline{t},\overline{t}), u(C,\underline{t},\overline{t}))$, where $u(C,\underline{t},\overline{t})$ is the value determined in Proposition 1. Now, we have two cases:
  - If $j < n$, by LR $x_j^j(C, u(C,\underline{t},\overline{t}), u(C,\underline{t},\overline{t})) = c_j - c_{j-1} \cdot u(C,\underline{t},\overline{t})$. We now prove that $x_j^j(C, u(C,\underline{t},\overline{t}), u(C,\underline{t},\overline{t})) = x_j^j(C,\underline{t},\overline{t})$. The equality holds trivially when $\underline{t} = u(C,\underline{t},\overline{t}) = \overline{t}^*(\overline{t},C)$, so suppose instead that $\underline{t} \neq \overline{t}^*(\overline{t},C)$ and then, $\underline{t} <$

$u(C,\underline{t},\overline{t}) < \overline{t}^*(\overline{t},C)$. Consider all problems $(C,r,r)$ such that $r \in [\underline{t}, u(C,\underline{t},\overline{t}))$. Then, by LR $x_j^j(C,r,r) = c_j - c_{j-1} \cdot r$. Observe that

$$\int_{\underline{t}}^{r} [x_j^j(C,v,v) - x_j^j(C,r,r)]\, dv < \int_{r}^{\overline{t}^*(\overline{t},C)} [x_j^j(C,r,r) - x_j^j(C,v,v)]\, dv.$$

Then, by CMIT $x_j^j(C,r,r) \geq x_j^j(C,\underline{t},\overline{t})$. Therefore, $x_j^j(C,\underline{t},\overline{t}) \leq c_j - c_{j-1} \cdot (u(C,\underline{t},\overline{t}) - \varepsilon)$ for all $\varepsilon \geq 0$. Similarly, we can deduce that $x_j^j(C,r,r) \leq x_j^j(C,\underline{t},\overline{t})$ for all $r \in (u(C,\underline{t},\overline{t}), \overline{t}^*(\overline{t},C)]$ and, then, $x_j^j(C,\underline{t},\overline{t}) \geq c_j - c_{j-1} \cdot (u(C,\underline{t},\overline{t}) + \varepsilon)$ for all $\varepsilon \geq 0$. Then, the unique possibility is that $x_j^j(C,\underline{t},\overline{t}) = x_j^j(C, u(C,\underline{t},\overline{t}), u(C,\underline{t},\overline{t}))$. Therefore, $x_j^j(C,\underline{t},\overline{t}) = c_j - c_{j-1} \cdot u(C,\underline{t},\overline{t})$.
Observe that, by definition, $u(C,\underline{t},\overline{t}) > 0$. Let us concentrate first in the case of $j = 2$. Then, by definition $x_1^2(C,\underline{t},\overline{t}) = c_1 \cdot u(C,\underline{t},\overline{t})$ and the proof of step 2 is finished. Now, go to step 3.
If $j \geq 3$, $\sum_{i=1}^{j-1} x_i^j(C,\underline{t},\overline{t}) = c_{j-1} \cdot u(C,\underline{t},\overline{t})$. By CR $x_k^k(C,\underline{t},\overline{t}) \cdot x_i^j(C,\underline{t},\overline{t}) = x_i^k(C,\underline{t},\overline{t}) \cdot x_k^j(C,\underline{t},\overline{t})$ for all $i,k < j$. Similarly, by CR $x_k^k(C,\underline{t},\overline{t}) \cdot x_i^{j-1}(C,\underline{t},\overline{t}) = x_i^k(C,\underline{t},\overline{t}) \cdot x_k^{j-1}(C,\underline{t},\overline{t})$ for all $i,k < j-1$. Then, we can deduce that $x_i^j(C,\underline{t},\overline{t}) \cdot x_k^{j-1}(C,\underline{t},\overline{t}) = x_k^j(C,\underline{t},\overline{t}) \cdot x_i^{j-1}(C,\underline{t},\overline{t})$ for all $i,k < j-1$. Therefore, we can obtain that $x_i^j(C,\underline{t},\overline{t}) \cdot \sum_{i=1}^{j-1} x_i^{j-1}(C,\underline{t},\overline{t}) = x_i^{j-1}(C,\underline{t},\overline{t}) \cdot \sum_{i=1}^{j-1} x_i^j(C,\underline{t},\overline{t})$ for all $i < j$.

Given that $\sum_{i=1}^{j-1} x_i^{j-1}(C,\underline{t},\overline{t}) = c_{j-1}$ and that we also know from step $j-1$ that $x_i^{j-1}(C,\underline{t},\overline{t}) = c_i \cdot (u(C,\underline{t},\overline{t}))^{j-1-i} - c_{i-1} \cdot (u(C,\underline{t},\overline{t}))^{j-i}$, we have that for all $i \in \{1,\ldots,j-1\}$,

$$x_i^j(C,\underline{t},\overline{t}) = \frac{c_i \cdot (u(C,\underline{t},\overline{t}))^{j-1-i} - c_{i-1} \cdot (u(C,\underline{t},\overline{t}))^{j-i}}{c_{j-1}} \cdot c_{j-1} \cdot u(C,\underline{t},\overline{t}).$$

Therefore, for all $i \in \{1,\ldots,j-1\}$,

$$x_i^j(C,\underline{t},\overline{t}) = c_i \cdot (u(C,\underline{t},\overline{t}))^{j-i} - c_{i-1} \cdot (u(C,\underline{t},\overline{t}))^{j+1-i}.$$

Now, go to step $j+1$.

– If $j = n$, by LR $x_n^n(C, u(C,\underline{t},\overline{t}), u(C,\underline{t},\overline{t})) = c_n - \frac{c_{n-1} \cdot u(C,\underline{t},\overline{t})}{1 - u(C,\underline{t},\overline{t})}$. We now prove that $x_n^n(C, u(C,\underline{t},\overline{t}), u(C,\underline{t},\overline{t})) = x_n^n(C,\underline{t},\overline{t})$. The equality holds trivially when $\underline{t} = u(C,\underline{t},\overline{t}) = \overline{t}^*(\overline{t},C)$, so suppose instead that $\underline{t} \neq \overline{t}^*(\overline{t},C)$ and then, $\underline{t} <$

$u(C,\underline{t},\overline{t}) < \overline{t}^*(\overline{t},C)$. Consider all problems $(C,r,r)$ such that $r \in [\underline{t}, u(C,\underline{t},\overline{t}))$. Then, by LR $x_n^n(C,r,r) = c_n - \frac{c_{n-1}\cdot r}{1-r}$. Observe that

$$\int_{\underline{t}}^{r} [x_n^n(C,v,v) - x_n^n(C,r,r)]\, dv < \int_{r}^{\overline{t}^*(\overline{t},C)} [x_n^n(C,r,r) - x_n^n(C,v,v)]\, dv.$$

Then, by CMIT $x_n^n(C,r,r) \geq x_n^n(C,\underline{t},\overline{t})$. Therefore, $x_n^n(C,\underline{t},\overline{t}) \leq c_n - \frac{c_{n-1}\cdot(u(C,\underline{t},\overline{t})-\varepsilon)}{1-(u(C,\underline{t},\overline{t})-\varepsilon)}$ for all $\varepsilon \geq 0$. Similarly, we can deduce that $x_n^n(C,v,v) \leq x_n^n(C,\underline{t},\overline{t})$ for all $v \in (u(C,\underline{t},\overline{t}), \overline{t}^*(\overline{t},C)]$ and, then, $x_n^n(C,\underline{t},\overline{t}) \geq c_n - \frac{c_{n-1}\cdot(u(C,\underline{t},\overline{t})+\varepsilon)}{1-(u(C,\underline{t},\overline{t})+\varepsilon)}$ for all $\varepsilon \geq 0$. Then, the unique possibility is that $x_n^n(C,\underline{t},\overline{t}) = x_n^n(C, u(C,\underline{t},\overline{t}), u(C,\underline{t},\overline{t}))$. Therefore, $x_n^n(C,\underline{t},\overline{t}) = c_n - \frac{c_{n-1}\cdot u(C,\underline{t},\overline{t})}{1-u(C,\underline{t},\overline{t})}$ and, by definition, $\sum_{i=1}^{n-1} x_i^n(C,\underline{t},\overline{t}) = \frac{c_{n-1}\cdot u(C,\underline{t},\overline{t})}{1-u(C,\underline{t},\overline{t})}$. Observe that, by definition, $u(C,\underline{t},\overline{t}) > 0$. Consider first the case of $n = 2$. It implies that $x_1^2(C,\underline{t},\overline{t}) = \frac{c_1\cdot u(C,\underline{t},\overline{t})}{1-u(C,\underline{t},\overline{t})}$. If $n \geq 3$, by CR $x_k^k(C,\underline{t},\overline{t}) \cdot x_i^n(C,\underline{t},\overline{t}) = x_i^k(C,\underline{t},\overline{t}) \cdot x_k^n(C,\underline{t},\overline{t})$ for all $i,k < n$. Similarly, by CR $x_k^k(C,\underline{t},\overline{t}) \cdot x_i^{n-1}(C,\underline{t},\overline{t}) = x_i^k(C,\underline{t},\overline{t}) \cdot x_k^{n-1}(C,\underline{t},\overline{t})$ for all $i,k < n-1$. Then, we can deduce that $x_i^n(C,\underline{t},\overline{t}) \cdot x_k^{n-1}(C,\underline{t},\overline{t}) = x_k^n(C,\underline{t},\overline{t}) \cdot x_i^{n-1}(C,\underline{t},\overline{t})$ for all $i,k < n-1$. Therefore, we can obtain that $x_i^n(C,\underline{t},\overline{t}) \cdot \sum_{i=1}^{n-1} x_i^{n-1}(C,\underline{t},\overline{t}) = x_i^{n-1}(C,\underline{t},\overline{t}) \cdot \sum_{i=1}^{n-1} x_i^n(C,\underline{t},\overline{t})$ for all $i < n$.

Given that $\sum_{i=1}^{n-1} x_i^{n-1}(C,\underline{t},\overline{t}) = c_{n-1}$ and that we also know from step $j-1$ that $x_i^{n-1}(C,\underline{t},\overline{t}) = c_i \cdot (u(C,\underline{t},\overline{t}))^{n-1-i} - c_{i-1} \cdot (u(C,\underline{t},\overline{t}))^{n-i}$, we have that for all $i \in \{1,\ldots,n-1\}$,

$$x_i^n(C,\underline{t},\overline{t}) = \frac{c_i \cdot (u(C,\underline{t},\overline{t}))^{n-i-1} - c_{i-1} \cdot (u(C,\underline{t},\overline{t}))^{n-i}}{c_{n-1}} \cdot \frac{c_{n-1}\cdot u(C,\underline{t},\overline{t})}{1-u(C,\underline{t},\overline{t})}.$$

Therefore, for all $i \in \{1,\ldots,n-1\}$,

$$x_i^n(C,\underline{t},\overline{t}) = \frac{c_i \cdot (u(C,\underline{t},\overline{t}))^{n-i} - c_{i-1} \cdot (u(C,\underline{t},\overline{t}))^{n-i+1}}{1-u(C,\underline{t},\overline{t})}.$$

$\square$

We have shown in the main text that MON is satisfied by the UR method, but not by of the ER method. We have also defined in Alcalde-Unzu et al. (2015) and here two rules that are associated with these methods and whose characterizations only differ in one axiom related to an idea of monotonicity: MIT in the case of the UR rule and CMIT in the case of the ER rule. Then, an interesting question is if the UR rule can be characterized replacing MIT by MON. The answer is negative because for instance the following rule, which differs with the UR and ER rules only in considering the value $\underline{t}$ for the transfer rate instead of $E_{(C,\underline{t},\overline{t})}(t)$ or $u(C,\underline{t},\overline{t})$, satisfies the three common axioms (LR, NDR and CR) and also MON:

$$\theta_i^j(C,\underline{t},\overline{t}) = \begin{cases} 0 & \text{if } i > j, \\ c_i \cdot \underline{t}^{j-i} - c_{i-1} \cdot \underline{t}^{j+1-i} & \text{if } i \leq j < n, \\ c_i - \frac{c_{i-1} \cdot \underline{t}}{1-\underline{t}} & \text{if } i = j = n, \\ \frac{c_i \cdot \underline{t}^{j-i} - c_{i-1} \cdot \underline{t}^{j+1-i}}{1-\underline{t}} & \text{if } i < j = n, \end{cases}$$

where $c_0$ is set to 0 and the indeterminate form $0^0$ is set to 1. Then, an interesting question for further research is to characterize the rules that satisfy LR, NDR, CR and MON.

**Conflict of Interest**

The authors declare that they have no conflict of interest.

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