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Uniform convergent expansions of integral transforms

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Abstract

Several convergent expansions are available for most of the special functions of the mathematical physics, as well as some asymptotic expansions [NIST Handbook of Mathematical Functions, 2010]. Usually, both type of expansions are given in terms of elementary functions; the convergent expansions provide a good approximation for small values of a certain variable, whereas the asymptotic expansions provide a good approximation for large values of that variable. Also, quite often, those expansions are not uniform: the convergent expansions fail for large values of the variable and the asymptotic expansions fail for small values. In recent papers [Bujanda & all, 2018-2019] we have designed new expansions of certain special functions, given in terms of elementary functions, that are uniform in certain variables, providing good approximations of those special functions in large regions of the variables, in particular for large and small values of the variables. The technique used in [Bujanda & all, 2018-2019] is based in a suitable integral representation of the special function. In this paper we face the problem of designing a general theory of uniform approximations of special functions based on their integral representations. Then, we consider the following integral transform of a function $g(t)$ with kernel $h(t,z)$, $F(z) := \int_0^1 h(t,z)g(t)dt$. We require for the function $h(t,z)$ to be uniformly bounded for $z \in \mathcal{D} \subset \mathbb{C}$ by a function $H(t)$ integrable in $t \in [0,1]$, and for the function $g(t)$ to be analytic in an open region Ω that contains the open interval $(0,1)$. Then, we derive expansions of $F(z)$ in terms of the moments of the function h, $M[h(\cdot,z),n] := \int_0^1 h(t,z)t^n dt$, that are uniformly convergent for $z \in \mathcal{D}$. The convergence of the expansion is of exponential order $\mathcal{O}(a^{-n})$, $a > 1$, when $[0,1] \in \Omega$ and of power order $\mathcal{O}(n^{-b})$, $b > 0$, when $[0,1] \notin \Omega$. Most of the special functions $F(z)$ having an integral representation may be cast in this form, possibly after an appropriate change of the integration variable. Then, special interest has the case when the moments $M[h(\cdot,z),n]$ are elementary functions of z, as the uniformly convergent expansion derived for $F(z)$ is given in terms of elementary functions. We illustrate the theory with several examples of special functions different from those considered in [Bujanda & all, 2018-2019].

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1 Introduction

Most of the special functions of the mathematical physics have an integral representation that may be written in the form of an integral transform of a function $g(t)$ with kernel $h(t,z)$ of the form [17],

$$F(z)=\int_a^b h(t,z)g(t)dt.$$

In this formula, (a,b) is a bounded or unbounded interval, $h(\cdot,z)g(\cdot)$ is integrable on (a,b) and $g(t)$ is analytic in a region $\Omega\subset\mathbb{C}$ that includes the open set $(a,b)\subset\Omega$. The function $F(z)$ could also be a function of other *extra* variables that we assume that are included in h and/or g. We omit further reference to other possible *extra* variables, as we are interested in the function F as function of a single (selected) variable z.

After an affine change of variable and/or splitting the integration interval if necessary we may assume, without loss of generality, that $[a,b]=[0,1]$ when $[a,b]$ is bounded, or $[a,b)=[0,\infty)$ when (a,b) is unbounded. Then, without loss of generality,

$$F(z)=\int_0^1 h(t,z)g(t)dt, \qquad \text{or} \qquad F(z)=\int_0^\infty h(u,z)g(u)du, \tag{1}$$

with $(0,1)\subset\Omega$ in the first case and $(0,\infty)\subset\Omega$ in the second case. But moreover, after the change of variable $u=-\log t$, the second integral in (1) may be written in the form of the first one with $h(t,z)$ and $g(t)$ identified with $t^{-1}h(-\log t,z)$ and $g(-\log t)$ respectively. Therefore, without loss of generality, we consider integral transforms of a function $g(t)$ with kernel $h(t,z)$ of the form

$$F(z)=\int_0^1 h(t,z)g(t)dt, \qquad\qquad z\in\mathcal{D}, \tag{2}$$

where $\mathcal{D}$ is a certain region (bounded or unbounded) of the complex plane. It is clear that, when the integration interval of the original integral representation of the special function $F(z)$ is unbounded (second formula in (1)) then, in general, the transformation $u=-\log t$ makes the point $t=0$ in (2) a singular point of $g(t)$.

In previous papers [5, 6, 15], we have derived analytic expansions of several examples of special functions $F(z)$ having the form of the first integral in (1). That is, we have consider there several particular examples of functions $h(t,z)$ and $g(t)$; and with the following assumptions for the functions h and g:

- (i) $|h(t,z)|\le H(t)$ for $z\in\mathcal{D}$ with H integrable on $[0,1]$,

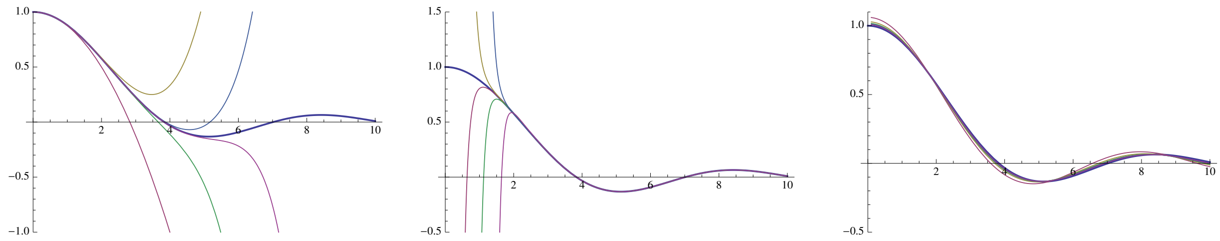


Figure 1: Approximations of $\frac{2}{x}J_1(x)$ (thicker graphics) given by the Taylor expansion [18, Sec. 2, eq. (10.2.2)] (left), the asymptotic expansion [18, Sec. 17, eq. (10.17.3)] (middle) and the uniform expansion [15, Theorem 1] (right) for $x \in [0, 10]$ and five degrees of approximation $n = 1, 2, 3, 4, 5$ (thinner graphics). The approximations are similar for complex x and other values of ν.

- (ii) $g(t)$ is analytic in a region $\Omega \subset \mathbb{C}$ that contains the open set $(0, 1) \subset \Omega$,
- (iii) the moments of h, $M[h(\cdot, z); k] := \int_0^1 h(t, z)t^k dt$ are elementary functions of z.

Then expansions derived in [5, 6, 15] have the following three properties:

- (a) The expansion is uniform for z in an unbounded subset $\mathcal{D} \subset \mathbb{C}$ that contains the point $z = 0$,
- (b) The expansion is convergent,
- (c) The terms of the expansion are elementary functions of z.

We have named those expansions *uniform expansions* because of the first property above. As an illustration of this type of approximations, we mention the following formula derived in [15] and valid for $x > 0$:

$$\frac{15\pi}{2x^3}J_3(x) = \left[\frac{3x^4 - 140x^2 + 360}{8x^6} + \theta_1(x)\right] x \sin x + \left[\frac{5(x^2 - 18)}{2x^4} + \theta_2(x)\right] \cos x, \qquad (3)$$

with $|\theta_1(x)| < 0.0062$ and $|\theta_2(x)| < 0.051$. This approximation is the particular case $n = \nu = 3$ of the general $n-$order uniform approximation of $J_\nu(x)$ given in [15, Theorem 1]. Figure 1 compares the $n-$th order approximation of $J_1(x)$ given in [15, Theorem 1] with the well-known Taylor and asymptotic approximations of $J_1(x)$.

Roughly speaking, the convergent (Taylor) expansions and the asymptotic expansions of $F(z)$ are obtained by replacing $h(t, z)$ in the integrand above by its Taylor expansion, as a function of z, at $z = 0$ (convergent) or at $z = \infty$ (asymptotic), and interchanging sum and integral. In general, the Taylor remainder in the Taylor expansion of $h(t, z)$ is not uniformly bounded in z and then, the resulting convergent or asymptotic expansions of $F(z)$ are not uniform in z. We have derived the uniform expansions given in [5, 6, 15] proceeding in the complementary way: by replacing $g(t)$ in the integrand above by its Taylor expansion at an appropriate point t_0, $g(t) = \sum_{k=0}^{n-1} a_k(t-t_0)^k + g_n(t)$, and interchanging sum and integral. Then, we have proved, in those particular examples, that the remainder term $\int_0^1 h(t,z)g_n(t)dt$ is uniformly bounded in z and vanishes in the limit $n \to \infty$.

The purpose of this paper is to generalize the above idea: the formulation of a general theory of analytic expansions of integral transforms (2) (that include the unbounded case given in the second formula in (1) after the logarithmic change of variable) that satisfy the three properties (a), (b), (c) listed above. We will see below that, when $0 \in \mathcal{D}$, the requirement (i) mentioned above assures property (a). The requirement (ii) assures property (b), and the requirement (iii) assures property (c). Although the meaning of "elementary function" in (c) must be clarified. If we want to be very general, we may just consider that "elementary" means that the moments $M[h(\cdot,z);k]$ are functions of fewer variables than $F(z)$ (this means that at least one of the "extra" variables of $F(z)$ is in $g(t)$). On the other hand, if we want to compare the uniform property of the expansion that we are going to derive in this paper with the standard Taylor or asymptotic expansions of $F(z)$ given, quite often, in terms of powers of z, we must be more demanding and establish that "elementary" means that the moments $M[h(\cdot,z);k]$ must be some of the classical elementary functions of z [20, Chap. 4].

In the particular examples of special functions considered in [5, 6, 15], we have only considered standard Taylor expansions of $g(t)$ at an appropriate point $t_0 \in [0,1]$. This was enough as long as, in those examples, the disk $D_r(t_0)$ of convergence of the Taylor series is contained in Ω. But in other situations the function $g(t)$ may possess singularities located near the integration interval $(0,1)$ such that $D_r(t_0) \not\subset \Omega$ for any $t_0 \in \Omega$. It has been argued in [14] that, in this case, a multi-point Taylor expansion at conveniently chosen base points is much more appropriate, because the lemniscate of convergence of the multi-point Taylor expansion avoids the singularities of $g(t)$ more efficiently than the disk of convergence of the standard Taylor expansion. Then, in order to make the analysis more general, we do not only consider the standard Taylor expansions of $g(t)$, but multi-point Taylor expansions of $g(t)$ [11, 12].

The paper is organized as follows. In sections 2, 3 and 4 we establish the theoretical framework necessary for the derivation of the main result of the paper in Section 5. In Section 2 we briefly review the theory of multi-point Taylor expansions introduced in [11, 12]. The hypotheses required on the functions g and h in the integral (2) are established in Section 3. In Section 4 we study the speed of convergence of the multi-point Taylor expansion of $g(t)$, specially when t approaches the boundary of the convergence region of the expansion. The end points $0, 1$ of the integration interval in (2) may be regular or singular points of the function $g(t)$, and this fact is an essential aspect in the analysis. Then, in the remaining of the paper, we consider four different situations concerning the position of the end points $t = 0, 1$ of the integration interval in (2) with respect to Ω:

- Case (i) $[0,1] \subset \Omega$.
- Case (ii) $(0,1] \subset \Omega$, $[0,1] \not\subset \Omega$.
- Case (iii) $[0,1) \subset \Omega$, $[0,1] \not\subset \Omega$.

- Case (iv) $(0,1) \subset \Omega$, $[0,1] \not\subset \Omega$.

The main result of the paper is contained in Section 5, where we derive a uniformly convergent expansion of $F(z)$, with different bounds for the remainder for every one of the above four cases. Section 6 contains some examples of special functions that illustrate the theory. Finally, in appendices 1 and 2 we give some details on how to compute the coefficients of the expansion when the integration interval is unbounded and the change of variable $u = -\log t$ is required (second formula in (1)).

2 Multi-point Taylor expansions

In this section we summarize the main results about multi-point Taylor expansions of analytic functions given in [11, 12], adapted to the applications that we need in this paper. We assume that $g(w)$ is an analytic function of w in a region Ω that contains the open interval $(0,1)$. Take m arbitrary real points $t_1 < t_2 < t_3 < ... < t_m$ and define the open lemniscate

$$D_r := \{w \in \Omega,\ \ |(w-t_1)(w-t_2)\cdots(w-t_m)| < r\}, \qquad\qquad r \le \rho, \tag{4}$$

with

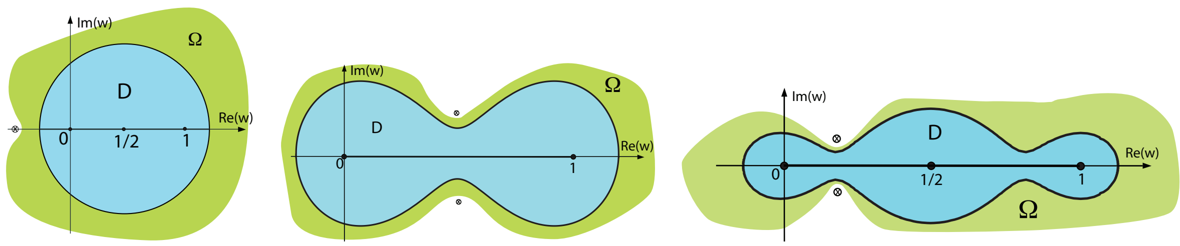
$$\rho := \mathrm{Inf}_{w\in\mathbb{C}\setminus\Omega}\left\{|(w-t_1)(w-t_2)\cdots(w-t_m)|\right\}. \tag{5}$$

The requirement $r \le \rho$ assures that the lemniscate $D_r \subset \Omega$, as D_ρ is the largest possible lemniscate with base points t_k, $k = 1, 2, ..., m$, that may be included in Ω. We assume that the points $t_1 < t_2 < t_3 < ... < t_m$ and the "radius" r of the lemniscate are chosen in such a way that $(0,1) \subset D_r$, that is, that $r \ge r_0$, with

$$r_0 := \sup_{t\in(0,1)}\left\{|(t-t_1)(t-t_2)\cdots(t-t_m)|\right\}. \tag{6}$$

Then, D_{r_0} is the smallest possible lemniscate with base points t_k, $k = 1, 2, ..., m$, that contains the interval $(0,1)$. It is explained in [14] that, the more singularities of $g(w)$ are located near the interval $(0,1)$ (the closer the border of Ω is to the interval $(0,1)$), the more base points t_1, t_2, ..., t_m located along the interval $(0,1)$ and its vicinity must be taken in order to assure that $\rho \ge r_0$. Therefore, in practice, we can always find an appropriate selection of the base points $t_1 < t_2 < t_3 < ... < t_m$ that assures $r_0 \le \rho$ and we can define a lemniscate D_r with $r_0 \le r \le \rho$, that is, $(0,1) \subset D_r \subset \Omega$.

For example, for the function $g(t) = (2t+1)^{-1}$, we may take $m = 1$, $t_1 = 1/2$ and $1/2 \le r < 1$; in this case the lemniscate D_r is nothing but a disk of radius $1/2 \le r < 1$ with center at $1/2$. For the function $g(t) = (5 - 16t + 16t^2)^{-1}$, we may take $m = 2$, $t_1 = 0$, $t_2 = 1$ and $1/4 \le r < 5/16$; in this case the lemniscate D_r is a Cassini oval of "radius" $1/4 \le r < 5/16$ with foci at the points $0, 1$. For the function $g(t) = (20t^2 - 8t + 1)^{-1}$, we may take $m = 3$, $t_1 = 0$, $t_2 = 1/2$, $t_3 = 1$ and $1/(12\sqrt{3}) \le r < \sqrt{13}/(20\sqrt{10})$; in this case D_r is a lemniscate of "radius" $1/(12\sqrt{3}) \le r < \sqrt{13}/(20\sqrt{10})$ with foci at the points $0, 1/2, 1$. See Figure 2.



(a) (b) (c)

Figure 2: In the tree pictures, the crossed points represent the singularities of the given functions $g(t)$ and define the value of ρ. (a) $g(t) = (2t+1)^{-1}$. Disk of radius $r_0 = 1/2 < r < \rho = 1$ centered at $1/2$. (b) $g(t) = (5 - 16t + 16t^2)^{-1}$. Cassini oval of radius $r_0 = 1/4 < r < \rho = 5/16$ and foci at $0, 1$. (c) $g(t) = (20t^2 - 8t + 1)^{-1}$ Lemniscate of radius $r_0 = 1/(12\sqrt{3}) < r < \rho = \sqrt{13}/(20\sqrt{10})$ and foci at $0, 1/2, 1$. In every example, the lemniscate D_r contains the interval $(0,1)$ but avoids the singularities of $g(w)$. The more base points t_1, t_2,..., t_m the lemniscate D_r has, the better it avoids the singularities of $g(w)$, as D_r becomes more and more thiner (always containing the interval $(0,1)$) as we can see in the sequence of examples (a)-(b)-(c).

The function $g(w)$ has the following multi-point Taylor expansion at the m base points t_1, t_2, ..., t_m, that converges uniformly and absolutely in D_r [11, 12]:

$$g(w) = \sum_{k=0}^{n-1} p_k(w) \left[\prod_{s=1}^{m}(w - t_s)\right]^k + g_n(w), \tag{7}$$

where $p_k(w)$ are polynomials of degree $m-1$. In [11, 12] we can find the following Lagrange representation of the polynomials $p_k(w)$:

$$p_k(w) := \sum_{j=1}^{m} a_{k,j} \frac{\prod_{s=1, s\neq j}^{m}(w - t_s)}{\prod_{s=1, s\neq j}^{m}(t_j - t_s)}, \tag{8}$$

with

$$a_{k,j} := \frac{1}{k!}\frac{d^k}{dw^k}\left[\frac{g(w)}{\prod_{s=1,s\neq j}^{m}(w-t_s)^k}\right]_{w=t_j} + \sum_{l=1, l\neq j}^{m} \frac{1}{(k-1)!}\frac{d^{k-1}}{dw^{k-1}}\left[\frac{g(w)/(w-t_j)}{\prod_{s=1,s\neq l}^{m}(w-t_s)^k}\right]_{w=t_l}.$$

On the other hand, for the computational purposes of this paper, it is more convenient to use the standard representation

$$p_k(w) := \sum_{j=0}^{m-1} A_{k,j} w^j. \tag{9}$$

The coefficients $A_{k,j}$ may be computed directly from (8) in terms of the coefficients $a_{k,j}$ just collecting equal powers of w. Alternatively, we may compute the coefficients $A_{k,j}$ using the following recurrent algorithm. For $k = 1, 2, 3, ...,$ we define the sequence of functions

$$\phi_k(w) := \frac{\phi_{k-1}(w) - p_{k-1}(w)}{U(w)}, \qquad \phi_0(w) := g(w), \qquad U(w) := \prod_{s=1}^{m}(w - t_s).$$

Then, for every $k = 0, 1, 2, ...,$ the coefficients $A_{k,j}$, $j = 0, 1, 2, ..., m-1$, are the solution of the following Van-Dermonde system

$$\begin{pmatrix} 1 & t_1 & t_1^2 & ... & t_1^{m-1} \\ 1 & t_2 & t_2^2 & ... & t_2^{m-1} \\ . & . & . & ... & . \\ . & . & . & ... & . \\ . & . & . & ... & . \\ 1 & t_m & t_m^2 & ... & t_m^{m-1} \end{pmatrix} \begin{pmatrix} A_{k,0} \\ A_{k,1} \\ . \\ . \\ . \\ A_{k,m-1} \end{pmatrix} = \begin{pmatrix} \phi_k(t_1) \\ \phi_k(t_2) \\ . \\ . \\ . \\ \phi_k(t_m) \end{pmatrix},$$

where, for $k = 0, 1, 2, ...,$ and $j = 1, 2, 3, ..., m$, the numbers $\phi_k(t_j)$ are computed in the form

$$\phi_k(t_j) = \lim_{w \to t_j} \frac{\phi_{k-1}(w) - p_{k-1}(w)}{U(w)} = \frac{\phi'_{k-1}(t_j) - p'_{k-1}(t_j)}{\prod_{s=1, s\neq j}^{m}(t_j - t_s)}.$$

In formula (7), $g_n(w)$ is the multi-point Taylor remainder that may be represented by means of the Cauchy integral

$$g_n(w) := \frac{\prod_{k=1}^{m}(w - t_k)^n}{2\pi i} \oint_{C_r} \frac{g(s)ds}{(s - w)\prod_{k=1}^{m}(s - t_k)^n}, \qquad w \in D_r. \tag{10}$$

In this formula, the integration contour $C_r \subset \Omega$ is the boundary of a lemniscate $D_{r-\epsilon}$, with $r - \epsilon > 0$ and small enough ϵ such that $C_r \subset \Omega$,

$$C_r := \{w \in \Omega, \ \ |(w - t_1)(w - t_2) \cdots (w - t_m)| = r - \epsilon\} \subset \Omega.$$

When $w = t \in (0, 1)$ and $[0, 1] \subset \Omega$ (case (i)) we may chose the base points t_k, $k = 1, 2, ..., m$ such that $C_r \subset \Omega$ with $\epsilon = 0$. When $[0, 1] \not\subset \Omega$ (cases (ii), (iii), (iv)) it is not possible to take $\epsilon = 0$, as the contour C_r would contain the points $s = 0$ and/or $s = 1$. In any case, for any $t \in (0, 1)$ and small enough ϵ, the contour C_r encircles, not only the points t_1, t_2,..., t_m, but also the point t.

In Section 5 we derive a convergent and uniform expansion of $F(z)$ by replacing $g(t)$ in the integrand in (2) by its multi-point Taylor expansion (7) and interchanging sum and integral. We show the convergence of the resulting expansion for $F(z)$ when the interval $(0, 1)$ is contained in the lemniscate D_r of uniform convergence of (7), that is, when $\rho \geq r_0$. In order to assure that $\rho \geq r_0$, an appropriate election of the number and location of the base points t_1,...t_m is essential, specially in the more delicate case $(0, 1) \subset \Omega$ but $[0, 1] \not\subset \Omega$. In any case, whenever $(0, 1) \subset \Omega$, it is always possible to choose appropriate points $t_1 < ... < t_m$ such that $\rho \geq r_0$.

3 Hypotheses

In this section we clearly set the hypotheses required for the two factors $h(t, z)$ and $g(t)$ in the integrand of (2), according to the four different cases (i)-(iv) mentioned in the introduction, and the hypothesis required for the base points t_1, t_2, ..., t_m.

As we have already mentioned in the introduction, we assume that $g(w)$ is analytic in an open region Ω that contains the interval $[0, 1]$ except, possibly, for an integrable singularity at $w = 0$ and/or at $w = 1$. More precisely:

Hypothesis 1. *We assume that $g(w)$ is analytic in an open region Ω that contains the interval $(0, 1)$ and the function $f(w) := w^{1-\sigma}(1-w)^{1-\gamma}g(w)$, with $0 < \sigma, \gamma \leq 1$, is bounded in Ω. More precisely:*

$$\begin{cases} \sigma = \gamma = 1 & \text{in case (i)}, \\ \sigma < 1, \gamma = 1 & \text{in case (ii)}, \\ \sigma = 1, \gamma < 1 & \text{in case (iii)}, \\ \sigma, \gamma < 1 & \text{in case (iv)}. \end{cases}$$

We have also mentioned in the introduction that $h(t, z)$ is uniformly bounded when $z \in \mathcal{D}$ by a function of t integrable on $[0, 1]$. More precisely:

Hypothesis 2. *We assume that $|h(t, z)| \leq Ht^{\alpha}(1-t)^{\beta}$ for $(t, z) \in [0, 1] \times \mathcal{D}$, with $H > 0$ independent of z and t and $\alpha + \sigma > 0$, $\beta + \gamma > 0$.*

Observe that it is natural to assume this form for the bound of the function $h(t, z)$, as the function $h(\cdot, z)g(\cdot)$ must be integrable in $[0, 1]$.

Finally, as we have mentioned in the previous section, the m base points $t_1 < t_2 < ... < t_m$ can always be chosen appropriately:

Hypothesis 3. *We choose m base points $t_1 < t_2 < ... < t_m$ such that the lemniscate D_r defined in (4) satisfies $(0, 1) \subset D_r \subset \Omega$, that is, $r_0 \leq r \leq \rho$ (see equations (5) and (6)).*

4 Analysis of the remainder $g_n(t)$

In this section we derive a bound for the remainder $g_n(t)$, $t \in (0, 1)$, of the multi-point Taylor expansion of $g(t)$ in the lemniscate D_r (see (7)) appropriate for our purposes. The analysis is more involved when one or both end points of the integration interval, $w = 0$ or $w = 1$, are singular points of $g(w)$. The analysis in this section resembles the analysis used in [8, 9] to determine the asymptotic behavior of the standard Taylor coefficients of functions $g(w)$ with integrable singularities. Here we analyze the asymptotic behavior of the remainder $g_n(t)$, and not only for the standard Taylor expansion, but for a general multi-point Taylor expansion.

For every one of the four cases (i)-(iv) mentioned in the introduction, we consider a different lemniscate D_{r_j}, $j = 1, 2, 3, 4$, with $r_j \leq \rho$ (recall the definition (5) of the maximum "radius" ρ). According to Hypotheses 1 and 3, the base points t_k, $k = 1, 2, ..., m$, and the "radius" $r \geq r_0$ of any lemniscate $D_r \subset \Omega$ with base points t_k (recall the definition (6) of the "minimal radius" r_0) must satisfy the following properties:

- Case (i) The "radius" r_1 of the lemniscate D_{r_1} must satisfy the inequality $r_1 \geq r_0 \geq \max\{\prod_{k=1}^m |t_k|, \prod_{k=1}^m |1-t_k|\}$.
- Case (ii) We have $t_1 > 0$. The "radius" r_2 of the lemniscate D_{r_2} must satisfy $r_2 := \prod_{k=1}^m |t_k| \geq r_0 > \prod_{k=1}^m |1-t_k|$. This condition assures that the interval $(0,1] \subset D_{r_2}$; the point $w = 0 \in \bar{D}_{r_2}$ but $w = 0 \notin D_{r_2}$.
- Case (iii) We have $t_m < 1$. The "radius" r_3 of the lemniscate D_{r_3} must satisfy $r_3 := \prod_{k=1}^m |1-t_k| \geq r_0 > \prod_{k=1}^m |t_k|$. This condition assures that the interval $[0,1) \subset D_{r_3}$; the point $w = 1 \in \bar{D}_{r_3}$ but $w = 1 \notin D_{r_3}$.
- Case (iv) We have $0 < t_1 < t_m < 1$. The "radius" r_4 of the lemniscate D_{r_4} must satisfy $r_4 := \prod_{k=1}^m |t_k| = \prod_{k=1}^m |1-t_k|$. This condition assures that the interval $(0,1) \subset D_{r_4}$; the points $w = 0, 1 \in \bar{D}_{r_4}$ but $w = 0, 1 \notin D_{r_4}$.

Recall the Cauchy integral representation of the remainder $g_n(w)$ given in (10), now restricted to $w = t \in (0,1)$,

$$g_n(t) := \frac{\prod_{s=1}^m (t-t_s)^n}{2\pi i} \oint_{C_r} \frac{g(w)dw}{(w-t)\prod_{s=1}^m (w-t_s)^n}, \qquad t \in (0,1). \tag{11}$$

where the $w-$integration contour C_r is the boundary of the lemniscate $D_{r-\epsilon}$ with a "radius" $r-\epsilon$, $\epsilon > 0$, such that $C_r \subset \Omega$. Then, in principle, only in case (i) we can take $\epsilon = 0$ and a "radius" $r = r_1$ such that $[0,1] \subset C_r \subset \Omega$. In the other three cases, in principle, we must take $\epsilon > 0$ and a "radius" $r = r_k - \epsilon < r_k$, $k = 2, 3, 4$, such that $(0,1) \subset D_r \subset \Omega$ but $[0,1] \not\subset D_r \subset \Omega$. In any of the four cases, in particular in the three cases (ii), (iii), (iv), the above integral is a (constant) function of ϵ that is defined for $\epsilon = 0$ ($r = r_k$) and is continuous as a function of ϵ as it is the integral of an integrable function. Therefore, in any of the cases (ii), (iii), (iv), we can take the limit $\epsilon \to 0$ ($r \to r_k$) and consider that the "radius" r of the lemniscate used in the above integral is the radius $r = r_k$, $k = 1, 2, 3, 4$, considered above. Then, in cases (ii), (iii), (iv), this limit lemniscate $D_r \not\subset \Omega$, although $D_r \setminus \{0,1\} \subset \Omega$. In any case, we may consider that the "radius" r of the lemniscate boundary C_r that defines the integration contour in (11) is such that the interval $[0,1] \subset D_r$. Figure 3 illustrates this discussion with $m = 3$ and a certain admissible selection of base points t_1, t_2 and t_3.

In the remaining of this section we derive an appropriate bound for the remainder $g_n(t)$. The analysis strongly depends on the case (i)-(iv) under consideration.

4.1 Case (i)

From the definition of D_{r_1} we have that $\prod_{k=1}^m |t-t_k| < \prod_{k=1}^m |w-t_k| = r_1$ for any $t \in [0,1]$ and $w \in C_{r_1}$. Therefore,

$$|g_n(t)| \leq \frac{1}{2\pi a^n} \oint_{C_{r_1}} \frac{|g(w)dw|}{|w-t|} = \frac{M}{a^n}, \qquad t \in [0,1],$$

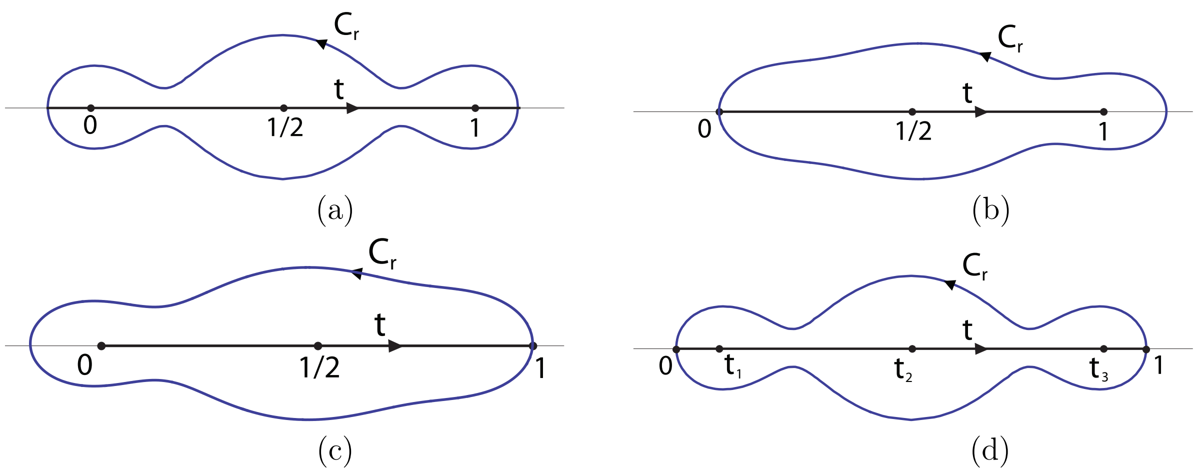


Figure 3: Consider three base points: $m = 3$. (a) Case (i) A possible selection of appropriate base points is $t_1 = 0$, $t_2 = 1/2$, $t_3 = 1$ and $r_1 = 1/20 > r_0 = 1/(12\sqrt{3})$. (b) Case (ii) A possible selection of appropriate base points is $t_1 = 1/10$, $t_2 = 1/2$, $t_3 = 1$ and $r_2 = t_1t_2t_3 = 1/20 = r_0$. (c) Case (iii) A possible selection of appropriate base points is $t_1 = 0$, $t_2 = 1/2$, $t_3 = 9/10$ and $r_3 = (1-t_1)(1-t_2)(1-t_3) = 1/20 = r_0$. (d) Case (iv) A possible selection of appropriate base points is $t_1 = 1/14$, $t_2 = 1/2$, $t_3 = 1 - t_1$ and $r_4 = t_1t_2t_3 = 13/392 = r_0$.

with the obvious definition of $M > 0$ and $a := r_1 / \sup_{t\in(0,1)} \prod_{k=1}^m |t - t_k| > 1$. Both constants a and M are independent of t and n.

4.2 Case (ii)

From the definition of D_{r_2} we have that $\prod_{k=1}^m |t-t_k| < \prod_{k=1}^m |w-t_k| = r_2$ for any $t \in [t_1, 1]$ and $w \in C_{r_2}$, but $\prod_{k=1}^m |t-t_k| \le \prod_{k=1}^m |w-t_k| = r_2 := \prod_{k=1}^m |t_k|$ for any $t \in [0, t_1]$ and $w \in C_{r_2}$. Then, for any $t \in [t_1, 1]$ we may derive a similar bound to the one derived in case (i):

$$|g_n(t)| \le \frac{1}{2\pi a^n} \oint_{C_{r_2}} \frac{|g(w)dw|}{|w-t|} = \frac{M}{a^n}, \qquad t \in [t_1, 1],$$

with $M > 0$ and $a > 1$ independent of t and n.

But for $t \in [0, t_1]$ we must be more careful. From Hypothesis 1 we have that $g(w) = w^{\sigma-1} f(w)$, with $f(w)$ bounded in D_{r_2}, and $\prod_{k=1}^m |w-t_k| = \prod_{k=1}^m |t_k|$ and $|t-t_k| \le t_k$ for $k = 2, 3, 4, ..., m$:

$$|g_n(t)| \le \frac{M(t_1-t)^n}{2\pi t_1^n} \oint_{C_{r_2}} \frac{|w^{\sigma-1} f(w)dw|}{|w-t|}, \qquad t \in [0, t_1].$$

After the change of variable $w \to tw$ we find

$$|g_n(t)| \le \frac{M(t_1-t)^n t^{\sigma-1}}{2\pi t_1^n} \oint_{C_{r_2}/t} \frac{|w^{\sigma-1} f(tw)dw|}{|w-1|}, \qquad t \in [0, t_1],$$

where now, the integration contour is the scaled lemniscate boundary C_{r_2}/t. For any $t > 0$, the most left point of this scaled lemniscate is the point $w = 0$ and the most right point is the point $w = t_0/t$, where t_0 is the most right point of the lemniscate boundary C_{r_2}. In the limit $t \to 0$ the scaled lemniscate boundary C_{r_2}/t becomes the imaginary axis traversed downwards. In this new path we have that $|f(wt)| \le M_0$, with $M_0 > 0$ independent of $w \in C_{r_2}/t$ and $t > 0$, and the integral

$$\oint_{C_{r_2}/t} \frac{|w^{\sigma-1}dw|}{|w-1|}, \qquad 0 < \sigma < 1,$$

is finite and independent of t. Therefore,

$$|g_n(t)| \le \frac{M(t_1-t)^n t^{\sigma-1}}{t_1^n}, \qquad t \in [0, t_1],$$

with the obvious definition of $M > 0$ independent of t and n.

4.3 Case (iii)

It is similar to the case (ii), but interchanging the roles of the points $w = 0$ and $w = 1$ and the lemniscate D_{r_2} by D_{r_3}. In other words, case (iii) becomes case (ii) after the change of variable $w \to 1 - w$, considering the factorization $g(w) = (1-w)^{\gamma-1} f(w)$ instead of the factorization $g(w) = w^{\sigma-1} f(w)$, r_3 instead of r_2 and reversing the order of the points t_1, t_2,..., t_m. Then, similarly we obtain that, for any $t \in [0, t_m]$ we may derive a similar bound to that one derived in case (i):

$$|g_n(t)| \le \frac{M}{a^n}, \qquad t \in [0, t_m],$$

with $M > 0$ and $a > 1$ independent of t and n. On the other hand,

$$|g_n(t)| \le \frac{M(t-t_m)^n (1-t)^{\gamma-1}}{(1-t_m)^n}, \qquad t \in [t_m, 1),$$

with $0 < \gamma < 1$, and a certain $M > 0$ independent of t and n.

4.4 Case (iv)

From the definition of D_{r_4} we have that $\prod_{k=1}^m |t - t_k| < \prod_{k=1}^m |w - t_k| = r_4$ for any $t \in [t_1, t_m]$ and $w \in C_{r_4}$, but $\prod_{k=1}^m |t - t_k| \le \prod_{k=1}^m |w - t_k| = r_4 := \prod_{k=1}^m |t_k| = \prod_{k=1}^m |1 - t_k|$ for any $w \in C_{r_4}$ and $t \in [0, t_1]$ or $t \in [t_m, 1]$. Then, for any $t \in [t_1, t_m]$ we may derive a similar bound to that one derived in case (i):

$$|g_n(t)| \le \frac{M}{a^n}, \qquad t \in [t_1, t_m],$$

with $M > 0$ and $a > 1$ independent of t and n.

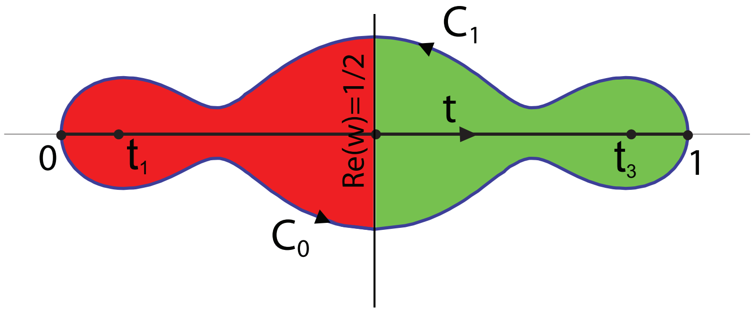


Figure 4: Case (iv). Half-lemniscates C_0 and C_1 for the example $m = 3$, certain $t_1 > 0$, $t_2 = 1/2$ and $t_3 = 1 - t_1 < 1$.

But for $t \in [0, t_1]$ and $t \in [t_m, 1]$ we must be more careful. We consider only the case $t \in [0, t_1]$, as the case $t \in [t_m, 1]$ is similar (the symmetry between cases (ii) and (iii) applies here as well). We assume that the base points t_k are symmetrically distributed with respect to the middle point of the integration interval $t = 1/2$. This condition is superfluous and could be eliminated, but then the analysis would be more cumbersome without providing more generality. Then, for the seek of simplicity in the analysis, and without loss of generality, we assume this symmetric distribution of the base points t_k. We divide the lemniscate C_{r_4} into two mirror half-lemniscates C_0 and C_1, obtained after cutting C_{r_4} with the vertical line $\Re w = 1/2$; $C_{r_4} = C_0 \cup C_1$, (see Figure 4).

We use, in both half-lemniscates, C_0 and C_1, that $|t - t_k| \le |t_k|$ for $k = 2, 3, 4, ..., m$ and that $|w - t_1| \ge t_1$. On the other hand, in C_0 we use the factorization $g(w) = w^{\sigma-1} f_0(w)$, with $f_0(w) := (1 - w)^{\gamma-1} f(w)$ bounded in C_0; whereas in C_1 we just use that $g(w)$ is integrable:

$$|g_n(t)| \le \frac{(t_1 - t)^n}{2\pi t_1^n} \oint_{C_0} \frac{|w^{\sigma-1} f_0(w) dw|}{|w - t|} + \frac{(t_1 - t)^n}{2\pi t_1^n} \oint_{C_1} \frac{|g(w) dw|}{|w - t|}.$$

In the second integral we have that $|w - t| \ge c > 0$ for any $w \in C_1$, with c independent of t. In the first integral we perform the change of variable $w \to tw$. We find

$$|g_n(t)| \le \frac{M_0 (t_1 - t)^n t^{\sigma-1}}{2\pi t_1^n} \oint_{C_0/t} \frac{|w^{\sigma-1} dw|}{|w - 1|} + \frac{(t_1 - t)^n}{2\pi t_1^n} \oint_{C_1} \frac{|g(w) dw|}{|w - t|}.$$

where now, the integration contour in the first integral is the scaled half-lemniscate contour C_0/t, and M_0 is a bound for $f_0(w)$ in C_0. For any $t > 0$, the most left point of this scaled half-lemniscate is the point $w = 0$ and the most right points are the two points w that satisfy $\Re w = 1/(2t)$. In the limit $t \to 0$ the scaled half-lemniscate C_0/t becomes the imaginary axis traversed downwards. Both integrals above are bounded by a constant independent of t and n and then,

$$|g_n(t)| \le \frac{M'(t_1 - t)^n}{t_1^n} [t^{\sigma-1} + 1] \le \frac{M(t_1 - t)^n t^{\sigma-1}}{t_1^n}, \qquad t \in (0, t_1],$$

with $0 < \sigma < 1$ and a certain constant $M > 0$ independent of t and n.

A similar analysis shows that, for $t \in [t_m, 1)$, we have

$$|g_n(t)| \leq \frac{M(t-t_m)^n(1-t)^{\gamma-1}}{(1-t_m)^n}, \qquad 0 < \gamma < 1$$

with $M > 0$ independent of t and n.

5 The uniform expansion of the integral $F(z)$

After all the preliminary results of the previous sections, we can formulate the main result of the paper in the following theorem.

Theorem 1. *Assume hypotheses 1-3 of Section 3 for the functions $h(t,z)$ and $g(t)$ and the base points t_1, t_2, ..., t_m. Then, for $n = 1, 2, 3, ...,$*

$$F(z) = \int_0^1 h(t,z)g(t)dt = \sum_{k=0}^{n-1}\sum_{s=0}^{m-1} A_{k,s} M[h(\cdot,z); s, k] + R_n(z), \tag{12}$$

where $A_{k,s}$ are the multi-point Taylor coefficients of the function $g(t)$ at the base points t_1, t_2, ..., t_m (see formulas (7)-(9)) that may be computed in either of the two forms described in Section 2, and $M[h(\cdot,z); s,k]$ are the multi-point moments:

$$M[h(\cdot,z); s,k] := \int_0^1 h(t,z)t^s \left[\prod_{l=1}^{m}(t-t_l)\right]^k dt. \tag{13}$$

On the other hand, the remainder $R_n(z)$ may be bounded in the form

$$|R_n(z)| \leq MH\left[\frac{1}{a^n} + A\, t_1^{\alpha+\sigma}\frac{n!\Gamma(\alpha+\sigma)}{\Gamma(n+\alpha+\sigma+1)} + B\,(1-t_m)^{\gamma+\beta}\frac{n!\Gamma(\gamma+\beta)}{\Gamma(n+\beta+\gamma+1)}\right], \tag{14}$$

with

$$(A,B) := \begin{cases} (0,0) & \textit{in case (i)}, \\ (1,0) & \textit{in case (ii)}, \\ (0,1) & \textit{in case (iii)}, \\ (1,1) & \textit{in case (iv)}, \end{cases} \tag{15}$$

where the constants $a > 1$, $H, M > 0$ independent on n and z where introduced in Hypothesis 2 and Section 4. The parameters α, β, σ and γ are defined in hypotheses 1 and 2. Therefore, expansion (12) is uniformly convergent for $z \in \mathcal{D}$ in any of the four cases (i)-(iv); the convergence is exponential in case (i) and of power type in the other three cases. More precisely, when $n \to \infty$,

$$R_n(z) = \mathcal{O}(a^{-n} + A\, n^{-\sigma-\alpha} + B\, n^{-\gamma-\beta}).$$

Proof. Consider the multi-point Taylor expansion of the function $g(t)$ at the base points $t_1 < t_2 < ... < t_m$, with the representation (9) of $p_k(t)$, that converges in the lemniscate D_r, with $(0,1) \subset D_r$ and $r = r_1$, r_2, r_3 or r_4 according to the case (i)-(iv) considered. When we replace the expansion (7) of $g(t)$ into the integral in the right hand side of (2) and interchange sum and integral we obtain (12), with

$$R_n(z) := \int_0^1 h(t,z) g_n(t) dt. \tag{16}$$

The multi-point moments (13) of the function $h(\cdot, z)$ exists because of Hypothesis 2. From here the analysis is different in the four cases (i)-(iv):

- Case (i). From Section 4.1 it is clear that the remainder $g_n(t)$ may be bounded in the form $|g_n(t)| \le M a^{-n}$, with M a positive constant independent on n and t and $a > 1$. When we introduce this bound and use Hypothesis 2 in (16) we get (14), case (i).

- Case (ii). Write

$$R_n(z) = \int_0^{t_1} h(t,z) g_n(t) dt + \int_{t_1}^1 h(t,z) g_n(t) dt.$$

  From Section 4.2 we have that $|g_n(t)| \le M a^{-n}$, with $a > 1$, in the second integral and $|g_n(t)| \le M (t_1 - t)^n t^{\sigma-1} t_1^{-n}$ in the first one. Introducing these bounds and using Hypothesis 2 in the above formula we get

$$|R_n(z)| \le \frac{MH}{t_1^n} \int_0^{t_1} t^{\alpha+\sigma-1} (t_1 - t)^n dt + \frac{MH}{a^n} = \frac{MH}{a^n} + MH t_1^{\alpha+\sigma} \int_0^1 t^{\alpha+\sigma-1} (1-t)^n dt.$$

  Formula (14) for case (ii) follows immediately.

- Case (iii). Write

$$R_n(z) = \int_0^{t_m} h(t,z) g_n(t) dt + \int_{t_m}^1 h(t,z) g_n(t) dt.$$

  From Section 4.3 we have that $|g_n(t)| \le M a^{-n}$, with $a > 1$, in the first integral and $|g_n(t)| \le M (t - t_m)^n (1-t)^{\gamma-1} (1 - t_m)^{-n}$ in the second one. Introducing these bounds and using Hypothesis 2 in the above formula we find

$$\begin{aligned}
|R_n(z)| &\le \frac{MH}{a^n} + \frac{MH}{(1-t_m)^n} \int_{t_m}^1 (1-t)^{\beta+\gamma-1} (t - t_m)^n dt \\
&= \frac{MH}{a^n} + \frac{MH}{(1-t_m)^n} \int_0^{1-t_m} t^{\gamma+\beta-1} (1 - t_m - t)^n dt \\
&= \frac{MH}{a^n} + MH (1-t_m)^{\gamma+\beta} \int_0^1 t^{\gamma+\beta-1} (1-t)^n dt.
\end{aligned}$$

  Formula (14) for case (iii) follows immediately.

- Case (iv). Write
$$R_n(z)=\int_0^{t_1}h(t,z)g_n(t)\,dt+\int_{t_1}^{t_m}h(t,z)g_n(t)\,dt\,+\int_{t_m}^1 h(t,z)g_n(t)dt.$$
From Section 4.4 we have that $|g_n(t)|\le Ma^{-n}$, with $a>1$, in the second integral; $|g_n(t)|\le M(t_1-t)^n t^{\sigma-1}t_1^{-n}$ in the first one, and $|g_n(t)|\le M(t-t_m)^n(1-t)^{\gamma-1}(1-t_m)^{-n}$ in the third one. Introducing these bounds and using Hypothesis 2 in the above formula and after similar steps to those given in cases (ii) and (iii) we find (14) for case (iv).$\odot$

Remark 1. The proof of Theorem 1 in cases (ii)-(iv) is more involved than in case (i). We could have repeated step by step the simpler proof of case (i) for the other three cases, but with $a=1$. Then that proof would not have shown the convergence of (12) in cases (ii)-(iv) as the parameter a in formula (14) would not be large enough (>1). Therefore, the more involved proof in cases (ii)-(iv) is necessary to show the convergence.

Remark 2. In the best scenario, when the end points of the integration interval $[0,1]$ are contained in the region Ω of analyticity of the function $g(w)$ (case (i)) the convergence of the expansion (12) is of exponential type. In the worse scenario that one or both end points are not contained in Ω, then expansion (12) is still convergent, although the convergence is only of power type.

Remark 3. It follows from (14) that, the larger α and β are, the faster the convergence of the expansion of $F(z)$ in cases (ii)-(iv) is. We have only considered the possibility $\alpha=\beta=0$ in case (i) because the bound $|h(t,z)|\le Ht^\alpha(1-t)^\beta$ with α and/or $\beta>0$ does not mean any improvement in the convergence speed of expansion (12): regardless α and/or β vanish or not, we would derive formula (14) for the remainder in case (i). On the other hand, as we will see in the examples below, the bound $|h(t,z)|\le Ht^\alpha(1-t)^\beta$ with α and/or $\beta>0$ is not uncommon in practice, and then it is worth it to consider the possibility α and/or $\beta\ge 0$ in any of the four cases (i)-(iv).

Remark 4. We could make the theory a little bit more general by relaxing Hypothesis 2: replacing "H independent on z" by "H an integrable function of the variable t". On the one hand, the price to pay would be a more involved derivation of the uniform bounds of $R_n(z)$. On the other hand, the theses of Theorem 1 would be essentially the same ones, with a slight modification of the form of (14). But moreover, as we can see in the next section, the requirement "H constant" is usually enough in practice. Therefore, we do not consider here that possible generalization of Theorem 1.

6 Examples

In this section we illustrate the applications of Theorem 1 in the derivation of expansions of special functions $F(z)$ in terms of elementary functions of z that are uniformly convergent in large domains $\mathcal{D}$ of the variable z. Some of them are already known, other ones are new.

Example 1. Define $\mathcal{D} = \{z \in \mathbb{C};\ \Re z \geq \delta > 0\}$. Then, for $z_1, z_2 \in \mathcal{D}$ and $u \in \mathbb{C}$, consider the confluent hypergeometric function

$$M(z_1, z_1 + z_2; u) := \frac{\Gamma(z_1 + z_2)}{\Gamma(z_1)\Gamma(z_2)} \int_0^1 t^{z_1-1}(1-t)^{z_2-1} e^{ut} dt.$$

It has the form considered in the above theorem with $g(t) = e^{ut}$, $h(t, z_1, z_2) = t^{z_1-1}(1-t)^{z_2-1}$ (we consider two uniform variables z_1 and z_2 instead of only one). We may take $\alpha = \beta = \delta - 1$ and $\sigma = \gamma = 1$ (case (i)).

From (12)-(13) we derive the standard Taylor expansion of the the confluent M function in powers of u [16, Sec. 13.2, eq. 13.2.3], with a remainder of the order $\mathcal{O}(a^{-n})$, for a certain $a > 1$ uniformly in $z_1, z_2 \in \mathcal{D}$.

Example 2. Define $\mathcal{D} = \{z \in \mathbb{C};\ \Re z \geq \delta > 0\}$. Then, for $z_1, z_2 \in \mathcal{D}$, $d \in \mathbb{C}$ and $u \in \mathbb{C} \setminus [1, \infty)$, consider the hypergeometric function

$$\frac{\Gamma(z_1)\Gamma(z_2)}{\Gamma(z_1 + z_2)} {}_2F_1(d, z_1, z_1 + z_2; u) = \int_0^1 t^{z_1-1}(1-t)^{z_2-1}(1-ut)^{-d}\, dt,$$

We can apply Theorem 1 with $g(t) = (1-ut)^{-d}$ and $h(t, z_1, z_2) = t^{z_1-1}(1-t)^{z_2-1}$, considering z_1 and z_2 as uniform variables. We may take $\alpha = \beta = \delta - 1$, $\sigma = \gamma = 1$ (case (i)). Then, we take the points $t_1 = 0$ and $t_2 = 1$ as base points ($m = 2$) in order to better avoid the singularity at $t = 1/u$. We obtain expansion [13, eq.(15)] with $R_n(z) = \mathcal{O}(a^{-n}), a > 1$, uniformly in $z_1, z_2 \in \mathcal{D}$.

Example 3. Define $\mathcal{D} = \{z \in \mathbb{C};\ \Re z \geq \delta \in \mathbb{R}\}$ and consider the parameter $c \in \mathbb{C} \setminus \mathbb{N}$, $\Re c > 0$. Consider the following integral representation of the incomplete gamma function

$$\gamma(c, z) = z^c \int_0^1 e^{-zt} t^{c-1} dt.$$

(We do not consider natural values of the parameter c, as the incomplete gamma function is an elementary function in this case.) We consider z as the uniform variable, so that we can apply Theorem 1 with $g(t) := t^{c-\lceil \Re c \rceil}$ and $h(t, z) = e^{-zt} t^{\lceil \Re c \rceil - 1}$, where $\lceil x \rceil$ denotes the least integer greater or equal to x. We may take $\sigma = \Re c - \lceil \Re c \rceil + 1$, $\gamma = 1$, $\alpha = \lceil \Re c \rceil - 1$ and $\beta = 0$ (case (ii)). We take only one base point $t_1 = 1$ ($m = 1$). We obtain a uniform expansion similar to expansion [5, eq. (8)] with $R_n(z) = \mathcal{O}(n^{-\Re c})$ uniformly in $z \in \mathcal{D}$.

Example 4. Define $\mathcal{D} = \{z \in \mathbb{C};\ \Re z \leq \delta \in \mathbb{R}\}$ and consider the parameters $(b, c) \in \mathbb{C} \times \mathbb{C} \setminus \mathbb{N} \times \mathbb{N}$, with $\Re b > \Re c > 0$. Consider the following integral representation of the confluent hypergeometric function

$$M(b, c; z) = \int_0^1 t^{c-1}(1-t)^{b-c-1} e^{zt} dt.$$

(We do not consider natural values for both parameters b and c, as the confluent hypergeometric function is an elementary function in this case.) We can apply Theorem 1 with

$g(t) = t^{c-\lceil \Re c \rceil}(1-t)^{b-c-\lceil \Re b - \Re c \rceil}$ and $h(t,z) = e^{zt} t^{\lceil \Re c \rceil - 1}(1-t)^{\lceil \Re(b-c) \rceil - 1}$, considering z as the uniform variable. We may take $\alpha = \lceil \Re c \rceil - 1$, $\beta = \lceil \Re(b-c) \rceil - 1$, $\sigma = \Re c - \lceil \Re c \rceil + 1$ and $\gamma = \Re(b-c) - \lceil \Re(b-c) \rceil + 1$. We are in case (ii) if $b-c \in \mathbb{N}$ and $c \notin \mathbb{N}$, case (iii) if $b-c \notin \mathbb{N}$ and $c \in \mathbb{N}$ or case (iv) if $c, b-c \notin \mathbb{N}$. In any case, we take only one base point $t_1 = 1/2$ ($m = 1$). We obtain a uniform expansion similar to expansion [6, eq. (21)] with the uniform bound for $R_n(z)$ given in Theorem 1.

Example 5. Define $\mathcal{D} = \{z \in \mathbb{C};\ \Re z \geq \delta > 0\}$. Consider the following integral representation of the exponential integral

$$e^z E(z) = \int_0^\infty \frac{e^{-zu}}{1+u} du = \int_0^1 \frac{t^{z-1}}{1-\log t} dt.$$

We can apply Theorem 1 with $g(t) = (1-\log t)^{-1}$ and $h(t,z) = t^{z-1}$, considering z as the uniform variable. We may take $\alpha = \delta - 1$, $\beta = 0$, $\gamma = 1$ and any $0 < \sigma < 1$ (case (ii)). We take only one base point $t_1 = 1$ ($m = 1$):

$$g(t) = \sum_{k=0}^{n-1} A_k (t-1)^k + g_n(t).$$

The coefficients A_k may be computed in the form (see Appendix B)

$$A_0 = 1, \qquad A_n = \sum_{k=0}^{n} \frac{k}{n} \widetilde{B}_{n-k}(n), \qquad n \geq 1, \tag{17}$$

where $\widetilde{B}_m(\alpha)$ are the normalized Nörlund polynomials (see Appendix A). Then, from Theorem 1,

$$e^z E(z) = \sum_{k=0}^{n-1} A_k H_k(z) + R_n(z),$$

where $H_k(z)$ are the elementary functions

$$H_k(z) := \int_0^1 t^{z-1}(t-1)^k dt = (-1)^k \frac{\Gamma(z)\Gamma(k+1)}{\Gamma(z+k+1)} = \sum_{j=0}^{k} \binom{k}{j} \frac{(-1)^{k-j}}{z+j}.$$

The remainder term behaves as $R_n(z) = \mathcal{O}(n^{1-\delta-\sigma})$, with $0 < \sigma < 1$ as close to 1 as we wish, uniformly in $z \in \mathcal{D}$.

Example 6. Define $\mathcal{D} = \{z \in \mathbb{C};\ \Re z \geq \delta > 0\}$ and consider the parameters $b, c \in \mathbb{C}$ with $\Re c > 0$. Consider the following integral representation of the confluent hypergeometric U,

$$U(c,b,z) = \frac{1}{\Gamma(c)} \int_0^\infty e^{-zu} u^{c-1}(1+u)^{b-c-1} du = \frac{1}{\Gamma(c)} \int_0^1 t^{z-1}(-\log t)^{c-1}(1-\log t)^{b-c-1} dt.$$

We can apply Theorem 1 with $g(t) = (-\log t)^{c-1}(1-\log t)^{b-c-1}$ and $h(t,z) = t^{z-1}$, considering z as the uniform variable. We may take $\alpha = \delta - 1$, $\beta = 0$, any $0 < \sigma < 1$ and $\gamma = 1$ if $\Re c \geq 1$ or $\gamma = \Re c$ if $0 < \Re c < 1$ (in order to ensure that $f(w) := w^{1-\sigma}(1-w)^{1-\gamma}g(w)$ is bounded in C_r). We are in case (ii) if $c \in \mathbb{N}$ or in case (iv) otherwise. In the first case we could consider the Taylor expansion of the function $g(t)$ at the base point $t_1 = 1$ and apply the results of Appendix B. Instead, we give only the expansion in the more general case $c \in \mathbb{C}$ and consider $t_1 = 1/2$ as the only base point:

$$g(t) = \sum_{k=0}^{n-1} A_k(c,b)\left(t - \frac{1}{2}\right)^k + g_n(t).$$

The coefficients $A_k(c,b)$ are elementary functions of c and b:

$$A_0(c,b) = (\log 2)^{c-1}(1+\log 2)^{b-c-1},$$

$$A_n(c,b) = \frac{A_0(c,b)}{n!}\sum_{k=1}^{n}\frac{(-1)^k b(n,k)}{(1+\log 2)^k}(b-c-k)_k\, {}_2F_1\left(1-c,-k;-c+b-k;1+\frac{1}{\log 2}\right), n \geq 1 \tag{18}$$

where $b(n,k)$ are partial ordinary Bell polynomials [19] that, for $n = 1,2,3,...$, can be computed recursively in the form

$$b(0,0) = 1, \quad b(n,0) = 0, \quad b(0,k) = 0,$$

$$b(n,k) = \sum_{j=1}^{n-k+1}\frac{(n-1)!}{(n-j)!}(-1)^{j+1}2^j b(n-j,k-1).$$

Therefore, from Theorem 1 we find

$$U(c,b,z) = \frac{1}{\Gamma(c)}\left[\sum_{k=0}^{n-1} A_k(c,b)G_k(z) + R_n(z)\right], \tag{19}$$

with $A_k(c,b)$ defined in (18) and $G_k(z)$ given by

$$G_k(z) := \int_0^1 t^{z-1}\left(t-\frac{1}{2}\right)^k dt = \left(\frac{1}{2}\right)^k \frac{1}{z}\, {}_2F_1(-k,1;z+1;2) = \sum_{j=0}^{k}\binom{k}{j}\left(\frac{-1}{2}\right)^{k-j}\frac{1}{z+j}.$$

Moreover, $R_n(z) = \mathcal{O}\left(n^{1-\delta-\sigma} + Bn^{-\gamma}\right)$, with σ as close to 1 as we wish and $B = 0$ if $c \in \mathbb{N}$, uniformly in $z \in \mathcal{D}$.

As an illustration, we derive the following formula from (19) with $n = 3$, valid for $x > 0$:

$$U\!\left(2,\frac{3}{2},x\right) \simeq \frac{8\left(x^2+3x+2\right)\log^3 2 + 2\left(9x^2+27x+10\right)\log^2 2 + \left(9x^2+15x+34\right)\log 2 - 16x^2}{8x(x+1)(x+2)\left(1+\log 2\right)^{\frac{7}{2}}}.$$

Figure 5 compares the well-known Taylor and asymptotic approximations of the confluent $U(2,3/2,x)$ function with the approximation given by Theorem 1.

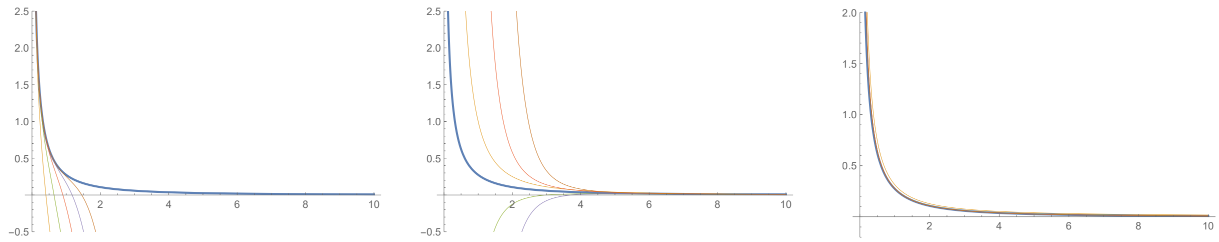


Figure 5: Approximations of $U(2, \frac{3}{2}, x)$ (thicker graphics) given by the Taylor expansion [16, Sec. 13, eq. (13.2.2) and (13.2.42)] (left), the asymptotic expansion [16, Sec. 13, eq. (13.7.3)] (middle) and the uniform expansion (19) (right) for $x \in [0, 10]$ and five degrees of approximation $n = 1, 2, 3, 4, 5$ (thinner graphics). The approximations are similar for complex x and other values of c, b.

7 Appendix A. Computation of the Nörlund polynomials

Consider the unbounded case given in the second formula in (1); see examples 5 and 6. In this case, the above theory requires, as a previous step, the change of variable $u \to t$ given by $u = -\log t$ to convert the second integral in (1) into the integral (2). Many times in applications, the function $g(u)$ in the integrand in (1) is analytic at $u = 0$, that is, $g(-\log t)$ is analytic at $t = 1$ and then it is convenient to consider the standard Taylor expansion of $g(-\log t)$ at $t_1 = 1$ Then, the theory given above requires the computation of the Taylor coefficients of the composite function $g(-\log t)$ at $t = 1$. But in practice, only the Taylor coefficients of $g(u)$ at $u = 0$ are available. Then, it is worth to consider the computation of the Taylor coefficients A_k of the composite function $g(-\log t)$ at $t = 1$,

$$g(-\log t) = \sum_{k=0}^{\infty} A_k (t-1)^k, \tag{20}$$

in terms of the coefficients a_k of the Taylor expansion of $g(u)$ at $u = 0$: $g(u) = \sum_{k=0}^{\infty} a_k u^k$. Two possible algorithms are given in propositions 1 and 2 in Appendix B. The formulation of Proposition 1 requires the use of the *normalized* Nörlund polynomials $\widetilde{B}_m(\alpha)$ that we study in this Appendix.

Consider the normalized Nörlund polynomials $\widetilde{B}_m(\alpha) := B_m(\alpha)/m!$, where $B_m(\alpha)$ are the standard Nörlund polynomials [7, Sec. 24.16, eq. 24.16.9]. The polynomials $\widetilde{B}_m(\alpha)$ are generated by the function

$$F_\alpha(t) := \left(\frac{t}{e^t - 1}\right)^\alpha = \sum_{m=0}^{\infty} \widetilde{B}_m(\alpha) t^m, \qquad |t| < 2\pi, \quad \alpha \in \mathbb{C}. \tag{21}$$

Lemma 1. *The normalized Nörlund polynomials $\widetilde{B}_m(\alpha)$ are polynomials in α of degree m, $\widetilde{B}_0(\alpha) = 1$ and, for $m = 1, 2, 3, \ldots$, they may be recursively computed in either of the following* *three forms:*

$$(i) \qquad \widetilde{B}_m(\alpha) = \frac{1}{m}\sum_{k=0}^{m-1}\frac{\alpha(k-m)-k}{(m+1-k)!}\widetilde{B}_k(\alpha), \tag{22}$$

$$(ii) \qquad \widetilde{B}_m(\alpha+1) = \frac{\alpha-m}{m}\sum_{k=1}^{m}\frac{\widetilde{B}_{m-k}(\alpha+1)}{(k+1)!} - \frac{\alpha}{m}\widetilde{B}_{m-1}(\alpha), \tag{23}$$

or

$$(iii) \qquad \widetilde{B}_m(\alpha) = \sum_{k=1}^{m}\sigma_k^m\alpha^k, \tag{24}$$

with

$$\sigma_0^0 := 1; \qquad \sigma_0^m := 0, \quad m = 1,2,3,...; \qquad \sigma_m^m := \frac{(-1)^m}{2^m m!}$$

and, for k=m-1,m-2,m-3,...,1; m=2,3,4,...,

$$\sigma_k^m = -\frac{1}{m+k}\left[\sigma_{k-1}^{m-1} + \sum_{n=k+1}^{m}\binom{n}{k-1}\sigma_n^m\right]. \tag{25}$$

Proof. It is clear that $\widetilde{B}_0(\alpha) = 1$. The recurrence relation (22) is proved in [4, Theorem 2] introducing the expansion (21) into the differential equation $t(e^t-1)F'_\alpha(t) = \alpha[(1-t)e^t - 1]F_\alpha(t)$ and equating the coefficients of equal powers of t.

In order to prove (23), consider the Cauchy integral representation that follows from (21):

$$\widetilde{B}_m(\alpha) = \frac{1}{2\pi i}\oint\left(\frac{t}{e^t-1}\right)^\alpha\frac{dt}{t^{m+1}}, \tag{26}$$

where the integration contour is a closed loop around the point $t=0$, contained inside the disk $D_0(2\pi)$ and traversed once counterclockwise. Then, on the one hand, a simple algebra shows that

$$\begin{aligned}\widetilde{B}_m(\alpha) =& \frac{1}{2\pi i}\oint\left(\frac{t}{e^t-1}\right)^{\alpha+1}(e^t-1)\frac{dt}{t^{m+2}} = \\ &\sum_{k=1}^{m+1}\frac{1}{k!}\frac{1}{2\pi i}\oint\left(\frac{t}{e^t-1}\right)^{\alpha+1}\frac{dt}{t^{m+2-k}} = \sum_{k=0}^{m}\frac{\widetilde{B}_{m-k}(\alpha+1)}{(k+1)!}.\end{aligned} \tag{27}$$

On the other hand, integrating by parts in (26) we find that, for $m = 1,2,3,...$,

$$m\widetilde{B}_m(\alpha) = \alpha\left[\widetilde{B}_m(\alpha) - \widetilde{B}_{m-1}(\alpha) - \widetilde{B}_m(\alpha+1)\right]. \tag{28}$$

Solving the two equations (27) and (28) for $\{\widetilde{B}_m(\alpha), \widetilde{B}_m(\alpha+1)\}$ we find (23).

In order to prove (24)-(25), replace the right hand side of (24) into formula (28), writing

$$\widetilde{B}_m(\alpha+1) = \sum_{k=1}^{m} \sigma_k^m (\alpha+1)^k = \sum_{k=0}^{m} \sigma_k^m \sum_{n=0}^{k} \binom{k}{n} \alpha^n = \sum_{k=0}^{m} \alpha^k \sum_{n=k}^{m} \binom{n}{k} \sigma_n^m .$$

We obtain, for $m = 0, 1, 2, ...,$

$$-\sigma_0^{m-1}\,\alpha + \sum_{k=1}^{m} \left[m\,\sigma_k^m + \sigma_{k-1}^{m-1} \right] \alpha^k = -\sigma_0^m\,\alpha \sum_{k=1}^{m+1} \left[\sigma_{k-1}^m - \sum_{n=k-1}^{m} \binom{n}{k-1} \sigma_n^m \right] \alpha^k .$$

When we identify the coefficients of every power α^k, $k = 0, 1, 2, ..., m+1$, we obtain formulas (24)-(25).

From either of the three recursive formulas (i), (ii) or (iii), it is straightforward to show that $B_m(\alpha)$ is a polynomial in α of degree m. $\odot$

An alternative recursive formula to (24)-(25) may be found in [10], although it is more involved. Nörlund polynomials are the particular case $x = 0$ of the generalized Bernoulli polynomials $B_n^{(\alpha)}(x)$. Several (more or less involved) explicit formulas, recurrences relation and properties of the Bernoulli polynomials may be found in [1, 3, 10, 21].

In the following table we indicate the first polynomials $\widetilde{B}_m(n+\alpha)$; $n = 0, 1, 2, ...$; $m = 0, 1, 2, .., n$ and how to compute them recursively using algorithms (i) or (ii).

$\widetilde{B}_m(n+\alpha-1)$	$m=0$	$m=1$	$m=2$	$m=3$	$\cdot$	$\cdot$
$n=0$	1					
$n=1$	1	$-\frac{\alpha}{2}$				
$n=2$	1	$-\frac{\alpha+1}{2}$	$\frac{(3\alpha+2)(\alpha+1)}{24}$			
$n=3$	1	$-\frac{\alpha+2}{2}$	$\frac{(3\alpha+5)(\alpha+2)}{24}$	$-\frac{(\alpha+1)(\alpha+2)^2}{48}$		
$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	
$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$

Table 1: Infinite triangle of the Nörlund polynomials $\widetilde{B}_m(n+\alpha)$; $n = 0, 1, 2, ...$; $m = 0, 1, 2, .., n$. The recurrence relation (22) let us compute every polynomial $\widetilde{B}_m(n+\alpha)$ from the previous polynomials $\widetilde{B}_k(n+\alpha)$, $k = 0, 1, 2, ..., m-1$, located in its same row. The recurrence relation (23) let us compute every polynomial $\widetilde{B}_m(n+\alpha)$ from the previous polynomials $\widetilde{B}_k(n+\alpha)$, $k = 0, 1, 2, ..., m-1$, located in its same row and the polynomial $\widetilde{B}_{m-1}(n+\alpha-1)$ located in the previous row and column.

8 Appendix B. Computation of the Taylor coefficients of the composite function $g(-\log t)$

In this appendix we compute the Taylor coefficients A_k of the composite function $g(-\log t)$ at $t = 1$,

$$g(-\log t) = \sum_{k=0}^{\infty} A_k (t-1)^k, \tag{29}$$

in terms of the coefficients a_k of the Taylor expansion of $g(u)$ at $u = 0$: $g(u) = \sum_{k=0}^{\infty} a_k u^k$. Two possible algorithms are given in the following two propositions.

Proposition 1. *We have that* $A_0 = a_0 := g(0)$ *and, for* $n = 1, 2, 3, ...$,

$$A_n = \sum_{k=0}^{n} \frac{k}{n} a_k (-1)^k \widetilde{B}_{n-k}(n), \qquad a_k := \frac{g^{(k)}(0)}{k!}. \tag{30}$$

Proof. We have that

$$A_n := \frac{f^{(n)}(1)}{n!}, \qquad f(u) := g(-\log u).$$

Therefore,

$$A_n := \frac{1}{n!} \frac{d^n}{du^n} \left[g(-\log u) \right]_{u=1} = \frac{1}{2\pi i} \oint_C \frac{f(u)}{(u-1)^{n+1}} du, \tag{31}$$

where the integration path is a circle C of radius $r < 1$ centered at $u = 1$, $|u - 1| = r$, traversed once counterclockwise, see Figure 6(a). After the change of integration variable $u \to t$ given by $u = e^{-t}$ we have that

$$A_n = \frac{-1}{2\pi i} \oint_\Gamma \frac{g(t) e^{-t}}{(e^{-t}-1)^{n+1}} dt, \tag{32}$$

where the integration contour Γ is the path $|e^{-t} - 1| = r$ depicted in Figure 6(b). Then we have that

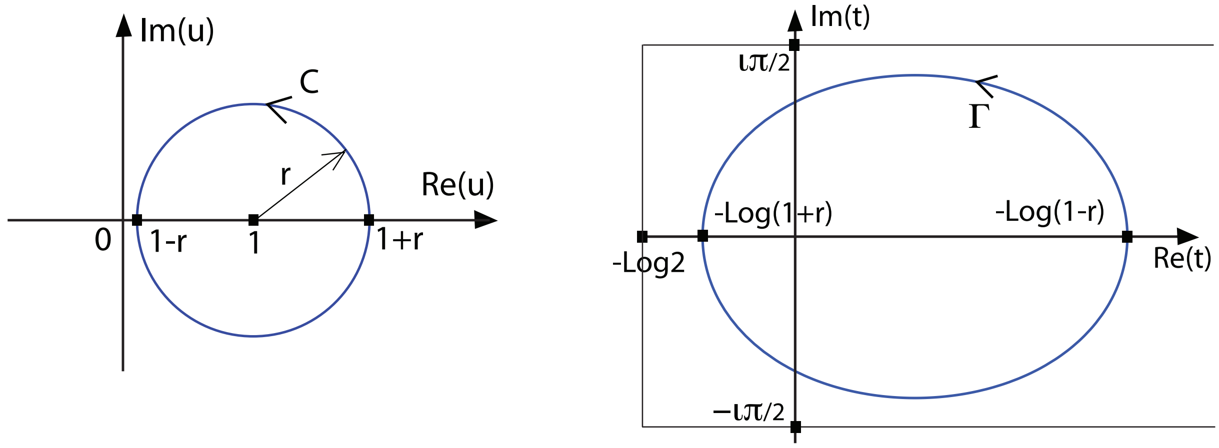
$$A_n := \frac{-1}{n!} \frac{d^n}{dt^n} \left[\left(\frac{t}{e^{-t}-1} \right)^{n+1} g(t) e^{-t} \right]_{t=0}. \tag{33}$$

We write

$$\left(\frac{t}{e^{-t}-1} \right)^{n+1} g(t) e^{-t} = \left(\frac{t}{e^{-t}-1} \right)^{n+1} g(t) + \left(\frac{t}{e^{-t}-1} \right)^{n} t g(t).$$

Then, from this formula, equation (33), the expansions $g(t) = \sum_{k=0}^{\infty} a_k t^k$ and $t g(t) = \sum_{k=1}^{\infty} a_{k-1} t^k$ and (21) we find that $A_0 = a_0$ and, for $n = 1, 2, 3, ...$,

$$A_n = \sum_{k=0}^{n} (-1)^k a_k [\widetilde{B}_{n-k-1}(n) + \widetilde{B}_{n-k}(n+1)], \tag{34}$$



(a) (b)

Figure 6: (a) The integration contour in (31) is a circle $|u-1| = r$ centered at $u = 1$ and radius $r < 1$ contained inside the analyticity region of $f(u)$. (b) The integration contour Γ in (32) is the path $|e^{-t} - 1| = r$ contained inside the analyticity region of $g(t) = f(e^{-t})$.

with $\widetilde{B}_{-1}(1) := 0$. Formula (30) follows from this one and (28). $\odot$

Formula (30) is an explicit formula for the computation of the coefficients A_n. In the following proposition we give some alternative recursive algorithms for the computation of these coefficients.

Proposition 2. *The coefficients A_n in expansion* (29) *may be computed in the following form: $A_0 = a_0$ and, for $n = 1, 2, 3, ...,$*

$$A_n = \sum_{k=0}^{n-1} b_{n,k+1} a_{k+1}, \qquad a_k := \frac{g^{(k)}(0)}{k!}. \tag{35}$$

where $b_{n,m}$ are the partial ordinary Bell numbers [19]. They may be computed recursively as follows: $b_{0,0} = 1$, $b_{n+1,0} = 0$ for $n = 0, 1, 2, ...,$ and the remaining ones are computed by means of the formula

$$b_{n,m} = \sum_{k=m-1}^{n-1} (-1)^{n-k} \frac{b_{k,m-1}}{n-k}, \qquad m = 1, 2, 3, ..., n, \qquad n = 1, 2, 3, ... \tag{36}$$

Proof. In order to derive formula (35) we invoke Faá di Bruno's formula [2] for the successive derivatives of a composite function: the coefficients A_n of the Taylor expansion of the composite function $f(t(u))$ at $u = u_0$ are given by the formula

$$f(t(u)) = \sum_{n=0}^{\infty} A_n (u-u_0)^n, \qquad A_n = \sum_{m=1}^{n} \frac{b_{n,m}}{m!} f^{(m)}(t_0), \qquad t_0 := t(u_0).$$

Formula (35) is a particular case of this formula with $t(u) = -\log u$ and $u_0 = 1$.

In order to derive formula (36) we consider the recursive algorithm given in [19] for the partial ordinary Bell polynomials $b_{n,m}$: $b_{0,0} = 1$, $b_{n+1,0} = 0$ for $n = 0, 1, 2, ...$, and the remaining ones are, for $m = 1, 2, 3, ..., n$, $n = 1, 2, 3, ...$,

$$b_{n,m} = \sum_{k=m-1}^{n-1} \frac{t^{(n-k)}(u_0)}{(n-k)!} b_{k,m-1}.$$

Formula (36) is the particular case of this one with $t(u) = -\log u$, $u_0 = 1$. $\odot$

9 Acknowledgments

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