This is a post-peer-review, pre-copyedit version of an article published in Constructive Approximation. The final authenticated version is available online at: https://doi.org/10.1007/s00365-021-09537-2.

New analytic representations of the hypergeometric functions ${}_{p+1}F_p$

José L. López[1], Pedro J. Pagola[2] and Pablo Palacios[2]

[1] *Departamento de Estadística, Informática y Matemáticas and INAMAT, Universidad Pública de Navarra, 31006-Pamplona, Spain. e-mail: jl.lopez@unavarra.es.*

[2] *Departamento de Estadística, Informática y Matemáticas, Universidad Pública de Navarra, 31006-Pamplona, Spain. e-mail: pedro.pagola@unavarra.es, palacios.129752@e.unavarra.es*

ABSTRACT. The power series expansions of the hypergeometric functions ${}_{p+1}F_p(a, b_1, ..., b_p; c_1, ..., c_p; z)$ converge either, inside the unit disk $|z| < 1$, or outside this disk $|z| > 1$. Nørlund's expansion in powers of $z/(z-1)$ converges in the half plane $\Re(z) < 1/2$. For arbitrary $z_0 \in \mathbb{C}$, Bühring's expansion in inverse powers of $z - z_0$ converges outside the disk $|z - z_0| = \max\{|z_0|, |z_0 - 1|\}$. None of them converge on the whole indented closed unit disk $|z| \le 1$, $z \neq 1$. In this paper we derive new expansions in terms of rational functions of z that converge in different regions, bounded or unbounded, of the complex plane that contain the indented closed unit disk. We give either, explicit formulas for the coefficients of the expansions or recurrence relations. The key point of the analysis is the use of multi-point Taylor expansions in appropriate integral representations of ${}_{p+1}F_p(a, b_1, ..., b_p; c_1, ..., c_p; z)$. We show the accuracy of the approximations by means of several numerical experiments.

2000 AMS Mathematics Subject Classification: 33C05; 41A58; 41A20, 65D20

Keywords & Phrases: Generalized hypergeometric functions. Approximation by rational functions. Two and three-point Taylor expansions.

1. Introduction

For $p, n \in \mathbb{N}$ and $a_k \in \mathbb{C}$, $k = 1, 2, ..., p$, we use the notation $\vec{a} := (a_1, a_2, ..., a_p)$ and $(\vec{a})_n := (a_1)_n (a_2)_n \cdots (a_p)_n$. The hypergeometric function ${}_{p+1}F_p(a, \vec{b}; \vec{c}; z)$ is defined by means of the power series expansion

$$ {}_{p+1}F_p(a, \vec{b}; \vec{c}; z) \equiv {}_{p+1}F_p\left(\begin{array}{c} a, \vec{b} \\ \vec{c} \end{array} \middle| z \right) := \sum_{n=0}^{\infty} \frac{(a)_n (\vec{b})_n}{(\vec{c})_n n!} z^n. \tag{1} $$

This series converges absolute and uniformly inside the unit disk $|z| < 1$ [[1], Chap. 16, Sec. 10, eq. 16.10.2] for any complex value of the parameters a, b_k and c_k; $k = 1, 2, 3, ..., p$, with $1 - c_k \notin \mathbb{N}$. For later convenience, we define the region $\Lambda := \{(a, \vec{b}, \vec{c}) \in \mathbb{C}^{2p+1}; 1 - c_k \notin \mathbb{N}, k = 1, 2, ..., p\}$. We define also $\gamma := (c_1 + c_2 + \cdots + c_p) - (a + b_1 + b_2 + \cdots + b_p)$. Then, on the unit circle $|z| = 1$, the series is convergent if $\Re\gamma > 0$, convergent except at the point $z = 1$ when $-1 < \Re\gamma \le 0$, and diverges when $\Re\gamma \le -1$. Outside this disk, ${}_{p+1}F_p(a, \vec{b}; \vec{c}; z)$ is defined by means of analytic continuation. The branch obtained by introducing a cut from 1 to ∞ on the real axis is the principal branch of the function. In the remaining of the paper we assume, without any further mention, that $|\arg(1-z)|<\pi$. For numerical computations we can use the right hand side of (1) to compute ${}_{p+1}F_p(a,\vec{b};\vec{c};z)$ only in the disk $|z|\le\rho<1$, with ρ depending on numerical requirements, such as precision and efficiency.

On the other hand, consider the connection formula [[1], Chap. 16, Sec. 8, eqs. 16.8.7 and 16.8.8],

$$
\begin{aligned}
{}_{p+1}F_p\left(\begin{array}{c} a_1,a_2,...,a_{p+1} \\ b_1,b_2,...,b_p \end{array}\middle| z\right) =& \sum_{k=1}^{p+1}\frac{\prod_{s=1,s\neq k}^{p+1}\Gamma(a_s-a_k)/\Gamma(a_s)}{\prod_{s=1}^{p}\Gamma(b_s-a_k)/\Gamma(b_s)}\frac{1}{(-z)^{a_k}}\times \\
& {}_{p+1}F_p\left(\begin{array}{c} a_k,b_1+1-a_k,b_2+1-a_k,...,b_p+1-a_k, \\ a_k+1-a_1,a_k+1-a_2,...,*,...,a_k+1-a_{p+1} \end{array}\middle| \frac{1}{z}\right),
\end{aligned}
\tag{2}
$$

where the symbol $*$ indicates that the entry a_k+1-a_k is omitted. From this formula and (1) we see that ${}_{p+1}F_p(a,\vec{b};\vec{c};z)$ may also be written as a series expansion in terms of inverse powers of z, convergent outside the unit disk $|z|>1$ (it also converges at the unit circle $|z|=1$ for certain values of the parameters). Then, for numerical computations we can use (2) and the right hand side of (1) to compute ${}_{p+1}F_p(a,\vec{b};\vec{c};z)$ in the exterior of the disk $|z|\ge\rho>1$, and again, the value of ρ depends on numerical requirements.

A different expansion in powers of $z/(z-1)$ was obtained by Nørlund [[10], eq. (1.21)], [[1], Chap. 16, Sec. 10, eq. 16.10.2],

$$
{}_{p+1}F_p\left(\begin{array}{c} a,\vec{b} \\ \vec{c} \end{array}\middle| z\zeta\right) = (1-z)^{-a}\sum_{k=0}^{\infty}\frac{(a)_k}{k!}\left(\frac{z}{z-1}\right)^k {}_{p+1}F_p\left(\begin{array}{c} -k,\vec{b} \\ \vec{c} \end{array}\middle| \zeta\right), \qquad \zeta\in\mathbb{C}. \tag{3}
$$

When $|\zeta-1|<1$, the series on the right-hand side converges in the half-plane $\Re z<1/2$. Also, for arbitrary complex z_0, an expansion in inverse powers of $z-z_0$, valid in the region $|z-z_0|>\max\{|z_0|,|z_0-1|\}$ and $|\arg(z_0-z)|<\pi$ (it excludes the origin $z=0$), was derived by Bühring [3], [[1], Chap. 16, Sec. 8, eq. 16.8.9],

$$
\begin{aligned}
{}_{p+1}F_p\left(\begin{array}{c} a_1,...,a_{p+1} \\ b_1,...,b_p \end{array}\middle| z\right) =& \sum_{j=1}^{p+1}\frac{(z_0-z)^{-a_j}}{\Gamma(a_j)}\sum_{n=0}^{\infty}\frac{\Gamma(a_j+n)}{n!}\prod_{k=1,k\neq j}^{p+1}\frac{\Gamma(a_k-a_j-n)}{\Gamma(a_k)}\prod_{k=1}^{p}\frac{\Gamma(b_k)}{\Gamma(b_k-a_j-n)} \\
& \times {}_{p+1}F_p\left(\begin{array}{c} a_1-a_j-n,...,a_{p+1}-a_j-n \\ b_1-a_j-n,...,b_p-a_j-n \end{array}\middle| z_0\right)\frac{1}{(z-z_0)^n}.
\end{aligned}
\tag{4}
$$

For the case that a_k-a_j is an integer $(k\neq j)$, the coefficients of expansion (4) become indeterminate, a limiting process is needed (see [3] for further details) and the convergence of the expansion becomes slower.

Other expansions in powers of $1-z$, convergent in the unit disk centered at $z=1$, can be found in [3]. The coefficients of these expansions are given in terms of an infinite series whose terms must be computed by means of intrincated formulas, see [3] form more details. Other more or less complicated expansions in terms of generalized hypergeometric functions may be found in [[1], Chap. 16, Sec. 10, eq. 16.10.1].

The Tayor expansion at $z=0$ (1), the Taylor expansion at $z=\infty$ (1)-(2), Nørlund's expansion (3) and Bühring's expansion (4) are given in terms of elementary functions of z. On the other hand, for general values of the parameters $(a,\vec{b},\vec{c})$, none of them is convergent in unbounded regions of $\mathbb{C}$ containing the indented closed unit disk $D^*:=\{z\in\mathbb{C}\,;\,|z|\le 1,\,z\neq 1\}$. Moreover, in general, the convergence of the these expansions become slower when z approaches the indented unit circle $C^*:=\{z\in\mathbb{C}\,;\,|z|=1,\,z\neq 1\}$.

In this paper we investigate new convergent expansions of ${}_{p+1}F_p(a,\vec{b};\vec{c};z)$ that, as well as the above mentioned expansions, are given in terms of elementary functions of z. On the other hand, we are seeking for expansions that are convergent in larger regions of the complex $z-$plane; in particular, unbounded regions containing in their interior the indented closed unit disk D^*, and then, expecting a faster convergence in the vicinity of the indented unit circle C^*. The starting point is the following integral representation of ${}_{p+1}F_p(a,\vec{b};\vec{c};z)$:

$$
{}_{p+1}F_p\left(\begin{array}{c} a,\vec{b} \\ \vec{c} \end{array}\bigg| z\right) = \prod_{s=1}^{p}\left(A(b_s,c_s)\int_L t_s^{b_s-1}(1-t_s)^{c_s-b_s-1}dt_s\right)(1-Tz)^{-a}, \qquad T:=\prod_{s=1}^{p} t_s, \tag{5}
$$

where, every t_s-integration path L; $s=1,2,3,...,p$, is identical, and $(1-zT)^{-a}=1$ when $z=0$. The hypersurface $\prod_{s=1}^{p} t_s = 1/z$ of $\mathbb{C}^p$ must contain in its interior the integration region $\{(t_1,t_2,...,t_p)\in\mathbb{C}^p;\ t_k\in L\}$. We consider four different possibilities for the path L and the constant $A(b,c)$ that we detail in Appendix 2. The expansions derived below do not depend on which one of these four possibilities we consider. But we need to consider the four of them in order to have at our disposal at least one integral representation for every $(a,\vec{b},\vec{c})\in\Lambda$; as each one is valid in a different region of the parameters $\vec{b}$ and $\vec{c}$ and none of them is valid in the whole region $(a,\vec{b},\vec{c})\in\Lambda$.

For reasons that will be clear in a moment, we assume that, in any of the four cases, the contour L involved in (5) is squeezed (analytically deformed) as much as possible around the real interval $[0,1]$; in such a way that we may consider that T, the product of the t_s-variables, is as close to the real interval $[0,1]$ as we wish. That is, $T\in X:=\{z\in\mathbb{C};\ |z-x|<\epsilon$ for some $x\in[0,1]$ and small enough $\epsilon>0\}$. In the remaining of the paper the parameter ϵ represents a small enough positive number.

The work presented in this paper is based on the ideas introduced in [7] and [8] for the hypergeometric functions ${}_2F_1(a,b;c;z)$ and ${}_3F_2(a,b_1,b_2;c_1,c_2;z)$, where the starting point is the Euler form of the integral (5) with $p=1$ and $p=2$ respectively, which is only valid for $\Re c_k>\Re b_k>0$ (see Appendix 2). Then, apart from the generalization from 3 or 5 parameters to $2p+1$ parameters, in this paper we derive new type of expansions that are valid in larger domains of the complex $z-$plane than the ones derived in [7] and [8]. They are also valid in a larger region Λ for the parameters a, b_k and c_k.

When we replace $f(T):=(1-zT)^{-a}$ in integral (5) by the standard Taylor series expansion of $f(T)$ at $T=0$, interchange summation and integration, and use the integral representations of the beta function given in [[2], Chap. 5, Sec. 12], we obtain the power series expansion (1) (see Appendix 2). The Taylor series expansion of $f(T)$ at $T=0$ converges uniformly in z for any $T\in X$ (for any $\mathbf{t}:=(t_1,t_2,...,t_p)$ in any of the four multiple integration domains considered in (5)) if $|z|<1$. Then, expansion (1) is convergent in the unit disk $|z|<1$.

For purposes that will become clear later, it is more convenient to consider the above argument about the region of convergence of the right hand side of (1) from a different point of view, which is the following. The domain of convergence of (1) (the disk $|z|<1$) is determined by the two following requirements: (i) For any $\mathbf{t}$ in the domain of integration in (5), the domain X for the variable T, must be contained in the domain of convergence of the series expansion of $f(T)$: a disk D of center $T=0$ and radius $r>1$, $D=\{T\in\mathbb{C},\ |T|<r\}$. (ii) The branch point $T=1/z$ of $f(T)$ must be located outside that domain D, which means that z must be located in a region $S=$ *the inverse to the exterior of* D: $S=(D^{\rm EXT})^{-1}=\{z\in\mathbb{C},\ |z|<r^{-1}\}$. Therefore, the smaller D is (the smaller r), the bigger the domain S of validity of (1) is. But D must satisfy (i) and then the smallest possible r is $r=1+\epsilon$ and $S=\{z\in\mathbb{C},\ |z|<1\}$ (see Fig. 1).

In this paper we explore the following idea. Instead of the Taylor series expansion of $f(T)$ at $T=0$, we consider the Taylor series expansion of $f(T)$ at a different base point $T=w$; moreover, we consider two and three-point Taylor expansions of $f(T)$ at conveniently chosen base points [4], [5]. The mentioned base points must be chosen in such a way that the domain D of convergence of the expansions satisfy the two above mentioned requirements:

(i) $X\subset D$ (the domain of the variable T must be contained in D);

(ii) $z \in S := (D^{\rm EXT})^{-1}$ (z must be located in a region $S =$ *the inverse to the exterior of* D).

Then, replacing $f(T)$ in (5) by these new expansions and interchanging summation and integration, we will obtain new expansions for ${}_{p+1}F_p(a,\vec{b};\vec{c};z)$ convergent for $z \in S$. The larger S is, the better, and one expects that, the smaller D is (containing X), the bigger S will be. The first possibility that we explore in Section 2 is an expansion of $f(T)$ at a generic point $T = w$. We re-derive in this way Nørlund's expansion, with recurrence relations for the coefficients, an error bound and new domains of convergence. In Section 3 we explore a two-point Taylor expansion of $f(T)$; the simplest expansion is obtained when we choose $T = 0$ and $T = 1$ as base points. On the other hand, in order to enlarge the domain of convergence S as much as possible, it is more convenient to use a two-point Taylor expansion of $f(T)$ at two conveniently chosen points of the interval $[0,1]$. We obtain in this way unbounded regions of convergence that contain the indented closed unit disk D^* in their interior. Section 4 is analog to section 3, but considering a three-point Taylor expansion instead of a two-point expansion. For simplicity in the exposition, we only consider in Section 4 two particularly interesting selections of base points for the three-point Taylor expansion, relegating the general case to Appendix 1. We obtain in this way convergence regions larger than those obtained in Section 3. Some final remarks and comments, as well as some numerical experiments are given in Section 5. Finally, in Appendix 2 we derive the four integral representations of the form (5) mentioned above.

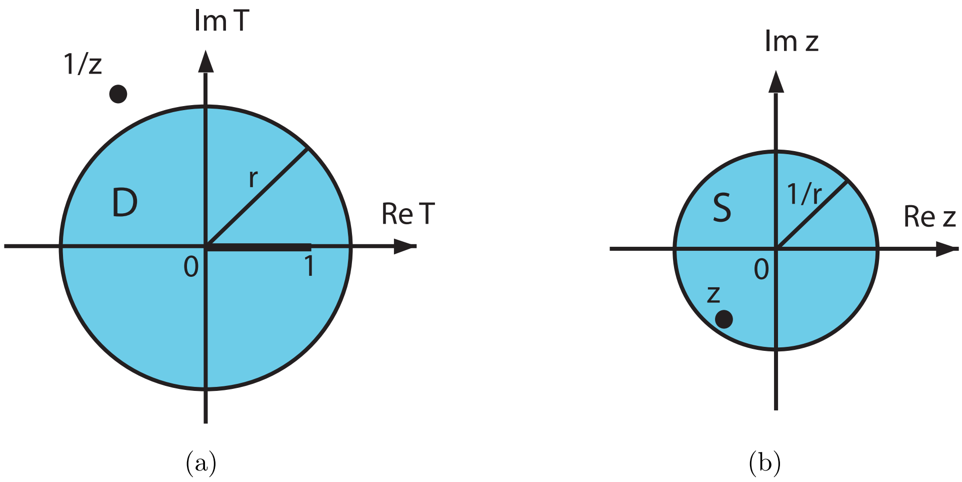


Figure 1. *The disk of convergence D of the Taylor expansion of $f(T)$ at $T = 0$ is shown in figure (a) for a certain r (> 1), and the region S, inverse of the exterior of D, is shown in figure (b). The smaller D is, the larger S is. The smallest possible value of r for which $X \subset D$ is any $r = 1 + \epsilon$.*

2. An expansion for $2\Re(wz) < 1$ with arbitrary $w \in \mathcal{C}$

In this section we re-derive expansion (3), analyzing new convergence regions and obtaining a recurrence relation for the coefficients of the expansion and an error bound. The results are summarized in the following theorem.

Theorem 1. *For arbitrary $w \in \mathcal{C}$, define $W := \max\{|w|, |1-w|\}$ and the region $S := \{z \in \mathcal{C},\ |1-wz| > |z|W\}$ (see Figure 2 (b) and (d)). Then, for any $z \in S$ and $(a,\vec{b},\vec{c}) \in \Lambda$,*

$$ {}_{p+1}F_p\left(\begin{array}{c} a,\vec{b} \\ \vec{c} \end{array}\middle|\, z\right) = (1-wz)^{-a}\sum_{k=0}^{n-1}\frac{(a)_k}{k!}\left(\frac{wz}{wz-1}\right)^k {}_{p+1}F_p\left(\begin{array}{c} -k,\vec{b} \\ \vec{c} \end{array}\middle|\, \frac{1}{w}\right) + R_n\left(\begin{array}{c} a,\vec{b} \\ \vec{c} \end{array}\middle|\, z\right), \qquad (6)$$

where the remainder term is uniformly bounded in compacts of S in the form

$$\left| R_n\left(\begin{array}{c} a,\vec{b} \\ \vec{c} \end{array}\middle| z\right)\right| \leq M(a,\vec{b},\vec{c},z)\left|\frac{zW}{1-wz}\right|^n, \tag{7}$$

with $M(a,\vec{b},\vec{c},z)>0$ independent on n. Moreover, for $\Re c_k > \Re b_k > 0$; $k=1,2,3,...,p$,

$$\left| R_n\left(\begin{array}{c} a,\vec{b} \\ \vec{c} \end{array}\middle| z\right)\right| \leq \frac{e^{\pi|\Im a|}\Gamma(n+\Re a)}{|\Gamma(a)|n!}\frac{|zW|^n}{(|1-wz|-|zW|)^{n+\Re a}} \sim n^{\Re a-1}\left(\frac{|zW|}{|wz-1|-|zW|}\right)^n, \qquad n\to\infty. \tag{8}$$

The terms of the expansion may be computed by means of the recurrence relation

$${}_{p+1}F_p\left(\begin{array}{c} -k,\vec{b} \\ \vec{c} \end{array}\middle| \frac{1}{w}\right) = {}_{p+1}F_p\left(\begin{array}{c} 1-k,\vec{b} \\ \vec{c} \end{array}\middle| \frac{1}{w}\right) - \frac{1}{w}\left(\prod_{s=1}^p \frac{b_s}{c_s}\right){}_{p+1}F_p\left(\begin{array}{c} 1-k,\overrightarrow{b+1} \\ \overrightarrow{c+1} \end{array}\middle| \frac{1}{w}\right), \qquad k=1,2,3,..., \tag{9}$$

with

$${}_{p+1}F_p\left(\begin{array}{c} 0,\vec{b} \\ \vec{c} \end{array}\middle| \frac{1}{w}\right) = 1,$$

where we have defined $\overrightarrow{a+1} := (a_1+1, a_2+1, ..., a_p+1)$.

Proof. Consider the standard Taylor expansion of the function $f(T)=(1-zT)^{-a}$ at a generic point $w=u+vi\in\mathbb{C}$, $u,v\in\mathbb{R}$:

$$f(T)=\sum_{k=0}^{n-1}\frac{(a)_k\,z^k}{k!}(1-wz)^{-a-k}(T-w)^k + r_n(T), \tag{10}$$

with

$$r_n(T) := (1-wz)^{-a}\sum_{k=n}^{\infty}\frac{(a)_k}{k!}\left(\frac{z(w-T)}{wz-1}\right)^k = (1-wz)^{-a}\frac{(a)_n}{n!}\left(\frac{z(T-w)}{1-wz}\right)^n {}_2F_1\left(\begin{array}{c} a+n,1 \\ n+1 \end{array}\middle| z\frac{T-w}{1-wz}\right). \tag{11}$$

Expansion (10) satisfies condition (i) for $D=\{T\in\mathbb{C},\ |T-w|<W+\epsilon\}$. It also satisfies condition (ii), that is, $1/z\notin D$, for $z\in S$. A straightforward computation shows that, for $u=\Re w\geq 1/2$, the domain S is the half-plane $S=\{z=x+iy; x,y\in\mathbb{R},\ 2\Re(wz)=2ux-2vy<1\}$; whereas for $u<1/2$, it is the disk $S=\{z\in\mathbb{C},\ |z+(1-2u)^{-1}\bar{w}|<(1-2u)^{-1}|w-1|\}$ (see Figure 2 (b) and (d)).

Then, for $z\in S$, we can introduce the expansion (10) in (5) and interchange summation and integration to obtain (6) with

$$R_n\left(\begin{array}{c} a,\vec{b} \\ \vec{c} \end{array}\middle| z\right) := \left(\prod_{s=1}^p A(b_s,c_s)\int_L t_s^{b_s-1}(1-t_s)^{c_s-b_s-1}dt_s\right) r_n(T). \tag{12}$$

On the one hand, from the Cauchy's integral form for the remainder $r_n(T)$, we have that

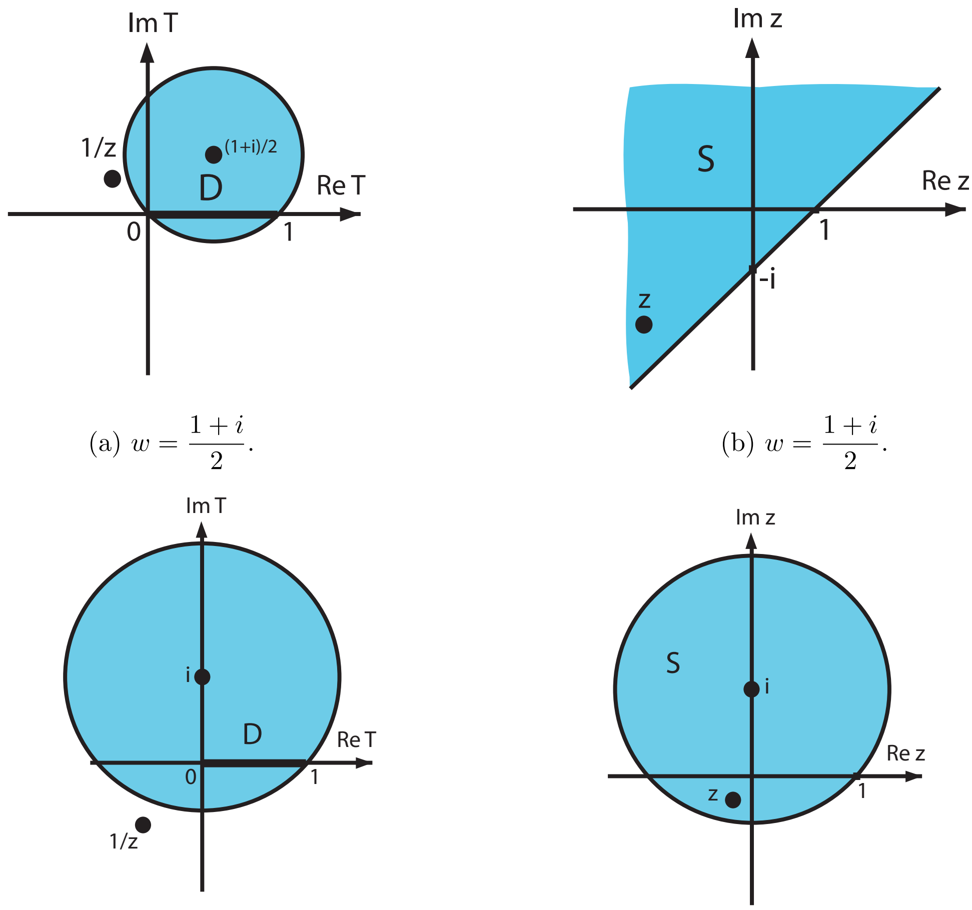
$$|r_n(T)|\leq C(a,z)\left|\frac{zW}{1-wz}\right|^n, \tag{13}$$

with $C(a,z)>0$ independent on n. On the other hand, from the Euler integral representation of the Gauss hypergeometric function [[11], Chap. 15, Sec. 6, eq. 15.6.1] we find that

$$|r_n(T)|\leq \left|(1-wz)^{-a}\right|\frac{|(a)_n|}{n!}\left|\frac{zW}{1-wz}\right|^n {}_2F_1\left(\begin{array}{c} \Re a+n,1 \\ n+1 \end{array}\middle| \left|\frac{zW}{1-wz}\right|\right). \tag{14}$$

Introducing (13) into (12) we obtain (7). Introducing (14) into the Euler form of L in (12) (see Appendix 2, case (a)), and after straightforward computations, we obtain (8).

The three-terms recurrence relation (9) follows straightforwardly from the integral representation (5) with $a = -k$, k integer. □



(a) $w = \dfrac{1+i}{2}$.

(b) $w = \dfrac{1+i}{2}$.

(c) $w = i$.

(d) $w = i$.

Figure 2. *The minimal domain of convergence D of the standard Taylor expansion of $f(T)$ at $T = w$ containing X is a disk of center at $T = w$ and radius $r = max\{|w|, |1-w|\} + \epsilon$ (figures (a) and (c)). The region S, inverse of the exterior of D is: the half-plane $S = \{z = x + iy; x, y \in \mathbb{R},\ 1 - 2\Re(wz) > 0\}$ if $\Re w \geq 1/2$ (figure (b)) or the disk of center $\bar{w}/(2\Re w - 1)$ and radius $|w-1|/(1-2\Re w)$ if $\Re w < 1/2$ (figure (d)).*

3. An expansion for $|(1-qz)(1+qz-z)| > |(1/2-q)z|^2$ with $0 \leq q < (2-\sqrt{2})/4$

As it has been pointed out in [6] (in a different context), the use of a two-point Taylor expansion [4], [5] with base points in the interval $[0, 1]$ is preferable to the use of a standard Taylor expansion. With a two-point Taylor expansion we can avoid the singularity $T = 1/z$ of $f(T)$ in its domain of convergence and, at the same time, include X in its interior, in a more efficient way (see Figure 3(a)).

Therefore, in this section, we consider the two-point Taylor expansion of the function $f(T) = (1 - zT)^{-a}$ at two base points $T = q$ and $T = 1 - q$, for a certain positive number $q \in [0, 1]$. The results are summarized in the following theorem.

Theorem 2. *For arbitrary $q \in [0, q_0]$, with $q_0 := (2-\sqrt{2})/4$, define the region (see Figure 3(b))*

$$S_q := \{z \in \mathbb{C}; (1-qz)(1+qz-z)| > (1/2-q)^2|z|^2\}. \tag{15}$$

Then, for any $z \in S_q$ and $(a, \vec{b}, \vec{c}) \in \Lambda$,

$$\begin{aligned} {}_{p+1}F_p\left(\begin{matrix} a, \vec{b} \\ \vec{c} \end{matrix} \middle| z\right) = \sum_{n=0}^{N-1}\sum_{k=0}^{n}\binom{n}{k}(q(1-q))^{n-k}(-1)^k\prod_{s=1}^{p}\frac{(b_s)_k}{(c_s)_k}\left[A_n(a,z)\,{}_{p+1}F_p\left(\begin{matrix} -k, \overrightarrow{b+k} \\ \overrightarrow{c+k} \end{matrix} \middle| 1\right)+\right. \\ \left.\prod_{s=1}^{p}\frac{b_s+k}{c_s+k}B_n(a,z)\,{}_{p+1}F_p\left(\begin{matrix} -k, \overrightarrow{b+k+1} \\ \overrightarrow{c+k+1} \end{matrix} \middle| 1\right)\right] + R_N\left(\begin{matrix} a, \vec{b} \\ \vec{c} \end{matrix} \middle| z\right), \end{aligned} \tag{16}$$

where, for $n = 0, 1, 2, \ldots$, the coefficients $A_n(a, z)$ and $B_n(a, z)$ are obtained from the following recurrence relation:

$$\begin{aligned} A_{n+1}(a,z) =& \frac{M_{11}(z,n,a,q)A_n(a,z) + M_{12}(z,n,a,q)B_n(a,z)}{(n+1)(4q^2-4q+1)(z-1-q(1-q)z^2)}, \\ B_{n+1}(a,z) =& \frac{M_{21}(z,n,a,q)A_n(a,z) + M_{22}(z,n,a,q)B_n(a,z)}{(n+1)(4q^2-4q+1)(z-1-q(1-q)z^2)}, \end{aligned} \tag{17}$$

$$\begin{aligned} M_{11}(z,n,a,q) :=& z(a+2n)[1-(1+2q^2-2q)z], \\ M_{12}(z,n,a,q) :=& z^2(-n+q(q-1)(a+1)) + z(2aq(1-q)+1+3n) - 2n - 1, \\ M_{21}(z,n,a,q) :=& -z(2-z)(a+2n), \\ M_{22}(z,n,a,q) :=& z^2(n(1+4q-4q^2)+2(1+a)q(1-q)) - (a+2+6n)z + 4n + 2, \end{aligned} \tag{18}$$

and

$$A_0(a,z) := \frac{(1-q)(1-zq)^{-a} - q(1-z(1-q))^{-a}}{1-2q}, \qquad B_0(a,z) := \frac{(1-z(1-q))^{-a} - (1-zq)^{-a}}{1-2q}. \tag{19}$$

The coefficients ${}_{p+1}F_p(-k, \vec{b}; \vec{c}; 1)$ may be recursively computed by using (9). The rate of convergence of (16) is of power type:

$$\left|R_N\left(\begin{matrix} a, \vec{b} \\ \vec{c} \end{matrix} \middle| z\right)\right| \leq M(a, \vec{b}, \vec{c}, z)\left|\frac{(1-2q)^2z^2}{4(1-qz)(1+(q-1)z)}\right|^N, \tag{20}$$

with $M(a, \vec{b}, \vec{c}, z) > 0$ independent on N. When $\Re c_k > \Re b_k > 0$; $k = 1, 2, \ldots, p$; $M(a, \vec{b}, \vec{c}, z)$ is also independent on $\vec{b}$ and $\vec{c}$.

Proof. From Theorem 1 of [4] we have that:

$$f(T) = \sum_{n=0}^{N-1}[A_n(a,z) + B_n(a,z)T][(T-q)(T+q-1)]^n + r_N(T), \qquad T \in D_q, \tag{21}$$

where D_q is a Cassini's oval with foci at $T = q$ and $T = 1-q$ and radius $r > 0$ that we determine below: $D_q = \{T \in \mathbb{C}, |(T-q)(T+q-1)| < r\}$, and $r_N(T)$ is the two-point Taylor remainder [[4], Theorem 1].

An explicit formula for the coefficients $A_n(a, z)$ and $B_n(a, z)$ may be derived from [4], but we omit it here for simplicity. Instead, we derive the recurrence relation (17) from the differential equation satisfied by $f(T)$: $(1-zT)f' = azf$. To this end we introduce expansion (21) and

$$\begin{aligned} f'(T) =& \sum_{n=0}^{\infty}\{[(2n+1)B_n(a,z) - (n+1)(A_{n+1}(a,z) + 2q(1-q)B_{n+1}(a,z))] + \\ & (n+1)[2A_{n+1}(a,z) + B_{n+1}(a,z)]T\}(T-q)^n(T+q-1)^n \end{aligned}$$

into the differential equation $(1-zT)f' = azf$. Equating coefficients of $(T-q)^n(T+q-1)^n$ and $T(T-q)^n(T+q-1)^n$ we obtain (17)-(19).

Now we determine the radius r. The interval $[0,1]$ is contained in the Cassini oval D_q when its radius $r \geq \text{Sup}_{T\in[0,1]}\{|(T-q)(T-1+q)|\} = \text{Max}\{q(1-q),(1/2-q)^2\}$. This happens for $r \geq (1/2-q)^2$ when $0 \leq q \leq q_0 := (2-\sqrt{2})/4$, where q_0 is the solution of the equation $q(1-q) = (1/2-q)^2$. Then, expansion (21) satisfies condition (i) for $r > (1/2-q)^2$ and $q \in [0,q_0]$. On the other hand, it satisfies condition (ii) if $1/z \notin D_q$ [4], that is, for any

$$r < \left|\left(\frac{1}{z}-q\right)\left(\frac{1}{z}+q-1\right)\right|.$$

The smallest r that we can take is $r = (1/2-q)^2+\epsilon$ and then, the largest S_q we can choose is (15). Then, for $z \in S_q$, we can introduce expansion (21) in (5) and interchange summation and integration to obtain (16), with

$$R_N\left(\begin{array}{c} a,\vec{b} \\ \vec{c} \end{array}\middle| z\right) := \prod_{s=1}^{p}\left(A(b_s,c_s)\int_L t_s^{b_s-1}(1-t_s)^{c_s-b_s-1}dt_s\right) r_N(T). \tag{22}$$

From Theorem 1 of [4] we have that the remainder in the expansion (21) may be bounded in the form $|r_N(T)| \leq C(a,z)|(T-q)(T+q-1)/r|^N$, $T \in D_q$, where $C(a,z) > 0$ is independent of T and N. When $T \in X$, we have that $|r_N(T)| \leq C(a,z)[(1/2-q)^2/r]^N$. Introducing this bound in (22) and after straightforward computations we obtain (20). When $\Re c_k > \Re b_k > 0$; $k = 1,2,...,p$, we may use the Euler form of (5) (see Appendix 2, case (a)) to see that $M(a,\vec{b},\vec{c},z)$ is independent on $\vec{b}$ and $\vec{c}$. □

Two particularly interesting corollaries of Theorem 2 are obtained for $q=0$ and for $q=q_0$. In the first case, the analytic form of (16) is the simplest possible one; in the second case, the convergence region S_q is the largest possible one. They are analyzed in the following two subsections.

3.1. Case $q=0$: an expansion for $|z|^2 < 4|1-z|$

From Theorem 2 we see that the simplest form of the analytic expansion (16) is obtained for $q=0$, that is, when we consider the two point Taylor expansion of the function $f(T)$ at the end points of the $T-$ domain: $T=0$ and $T=1$. In this case, a simple explicit formula for the coefficients $A_n(a,z)$ and $B_n(a,z)$ is given in [4]. We formulate the result in the form of a corollary:

Corollary 1. *Define the region (see Figure 3(b))*

$$S_0 := \{z \in \mathbb{C}; |z|^2 < 4|1-z|\} = \{x+iy; x,y \in \mathbb{R},\ \ y^4 + (2x^2-16)y^2 + (x^4-16x^2+32x-16) < 0\}.$$

Then, for any $z \in S_0$ and $(a,\vec{b},\vec{c}) \in \Lambda$,

$$\begin{aligned} {}_{p+1}F_p\left(\begin{array}{c} a,\vec{b} \\ \vec{c} \end{array}\middle| z\right) = \sum_{n=0}^{N-1}(-1)^n \prod_{s=1}^{p}\frac{(b_s)_n}{(c_s)_n}\Bigg[A_n(a,z)\,{}_{p+1}F_p\left(\begin{array}{c} -n,\overrightarrow{b+n} \\ \overrightarrow{c+n} \end{array}\middle| 1\right) + \\ \prod_{s=1}^{p}\frac{b_s+n}{c_s+n}B_n(a,z)\,{}_{p+1}F_p\left(\begin{array}{c} -n,\overrightarrow{b+n+1} \\ \overrightarrow{c+n+1} \end{array}\middle| 1\right)\Bigg] + R_N\left(\begin{array}{c} a,\vec{b} \\ \vec{c} \end{array}\middle| z\right), \end{aligned} \tag{23}$$

where, for $n = 0,1,2,...$, the coefficients $A_n(a,z)$ and $B_n(a,z)$ are given explicitly by

$$\begin{aligned} A_n(a,z) &:= \frac{1}{n!}\sum_{k=0}^{n}\frac{(n+k-1)!}{k!(n-k)!}[(-1)^n n - (-1)^k k(1-z)^{k-a-n}](a)_{n-k}z^{n-k}, \\ B_n(a,z) &:= \frac{1}{n!}\sum_{k=0}^{n}\frac{(n+k)!}{k!(n-k)!}[(-1)^k(1-z)^{k-a-n} + (-1)^{n+1}](a)_{n-k}z^{n-k}, \end{aligned} \tag{24}$$

and may also be obtained from the following recurrence relation:

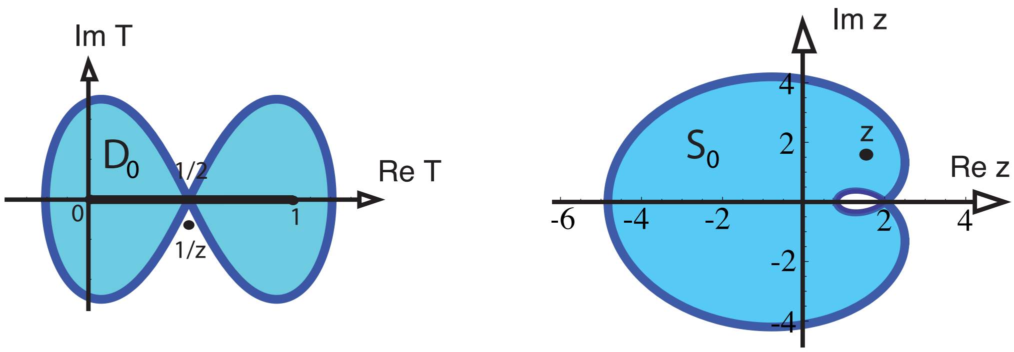
$$\begin{aligned} A_{n+1}(a,z) &= \frac{-z(a+2n)A_n(a,z)+[1+n(2-z)]B_n(a,z)}{n+1}, \\ B_{n+1}(a,z) &= \frac{z(2-z)(a+2n)A_n(a,z)+[z(a+2)+n(6z-z^2-4)-2]B_n(a,z)}{(n+1)(1-z)}, \end{aligned} \tag{25}$$

with $A_0(a,z)=1$ *and* $B_0(a,z)=(1-z)^{-a}-1$.

The coefficients ${}_{p+1}F_p(-n,\vec{b};\vec{c};1)$ *may be recursively computed using (9). The rate of convergence of (23) is of power type:*

$$\left| R_N\left(\begin{array}{c} a,\vec{b} \\ \vec{c} \end{array} \middle| z \right) \right| \le M(a,\vec{b},\vec{c},z) \left| \frac{z^2}{4(1-z)} \right|^N, \tag{26}$$

with $M(a,\vec{b},\vec{c},z)>0$ *independent on* N. *When* $\Re c_k > \Re b_k > 0$; $k=1,2,...,p$; $M(a,\vec{b},\vec{c},z)$ *is also independent on* $\vec{b}$ *and* $\vec{c}$.



(a) (b)

Figure 3. *The minimal domain of convergence* $D_0=\{T\in\mathbb{C},\ |T(T-1)|<r\}$ *of the two-point Taylor expansion (21) containing* X *is a Cassini oval of radius* $r=1/4+\epsilon$ *and foci* $T=0$ *and* $T=1$ *(figure (a)). The region* S_0, *inverse of the exterior of* D_0, *is the region shown in figure (b):* $S_0=\{z\in\mathbb{C};\ \ |z|^2<4|1-z|\}$.

Proof. Formula (24) is derived in [4]. Formulas (23), (25) and (26) follow from (16), (17) and (20) setting $q=0$. □

3.2. Case $q=(2-\sqrt{2})/4$: an expansion for $|z^2-8z+8|>|z|^2$

When we want to maximize the size of the convergence region S_q, the optimal election of the base points q and $1-q$ is that one that minimizes the region D_q. In the minimal region D_q, the end points of the interval $[0,1]$, as well as the middle point $T=1/2$ are on the boundary of the Cassini's oval D_q (see Fig. 4(a)). As we have seen above, this happens for $q=q_0:=(2-\sqrt{2})/4$. With this choice, the Cassini's oval D_{q_0} of convergence of the two-point Taylor expansion is the smallest possible two-point Cassini's oval that contains X. We formulate the result in the form of a corollary:

Corollary 2. *Define the region (see Figure 4(b))*

$$S_{q_0}=\{z\in\mathbb{C};|z^2-8z+8|>|z|^2\}=\{x+iy;x,y\in\mathbb{R},\ \ 4+5x^2-x^3+3y^2-x(8+y^2)>0\}.$$

Then, for any $z \in S_{q_0}$ and $(a,\vec b,\vec c) \in \Lambda$,

$$
{}_{p+1}F_p\left(\begin{matrix} a,\vec b \\ \vec c\end{matrix}\middle| z\right) = \sum_{n=0}^{N-1}\sum_{k=0}^{n}\binom{n}{k} 8^{k-n}(-1)^k \prod_{s=1}^{p}\frac{(b_s)_k}{(c_s)_k}\left[A_n(a,z)\,{}_{p+1}F_p\left(\begin{matrix} -k,\overrightarrow{b+k} \\ \overrightarrow{c+k}\end{matrix}\middle| 1\right) + \prod_{s=1}^{p}\frac{b_s+k}{c_s+k}B_n(a,z)\,{}_{p+1}F_p\left(\begin{matrix} -k,\overrightarrow{b+k+1} \\ \overrightarrow{c+k+1}\end{matrix}\middle| 1\right)\right] + R_N\left(\begin{matrix} a,\vec b \\ \vec c\end{matrix}\middle| z\right), \tag{27}
$$

where, for $n = 0, 1, 2, \ldots$, the coefficients $A_n(a,z)$ and $B_n(a,z)$ are obtained from the recurrence relation

$$
\begin{aligned}
A_{n+1}(a,z) =& \frac{M_{11}(z,n,a,q_0)A_n(a,z) + M_{12}(z,n,a,q_0)B_n(a,z)}{(n+1)(8z-8-z^2)},\\
B_{n+1}(a,z) =& \frac{M_{21}(z,n,a,q_0)A_n(a,z) + M_{22}(z,n,a,q_0)B_n(a,z)}{(n+1)(8z-8-z^2)},
\end{aligned}\tag{28}
$$

with

$$
\begin{aligned}
M_{11}(z,n,a,q_0) :=& 4z(a+2n)[4-3z],\\
M_{12}(z,n,a,q_0) :=& -z^2(16n+2(a+1)) + 4z(a+4+12n) - 16(2n+1),\\
M_{21}(z,n,a,q_0) :=& 16z(z-2)(a+2n),\\
M_{22}(z,n,a,q_0) :=& 4z^2(6n+1+a) - 16z(a+2+6n) + 32(2n+1),
\end{aligned}
$$

and

$$
A_0(a,z) = \sqrt{2}\,[(1-q_0)(1-z\,q_0)^{-a} - q_0(1-z(1-q_0))^{-a}], \qquad B_0(a,z) = \sqrt{2}\,[(1-z(1-q_0))^{-a} - (1-z\,q_0)^{-a}].
$$

The functions ${}_{p+1}F_p(-k,\vec b;\vec c;1)$ may be recursively computed using (9). The rate of convergence of (27) is of power type:

$$
\left| R_N\left(\begin{matrix} a,\vec b \\ \vec c\end{matrix}\middle| z\right)\right| \le M(a,\vec b,\vec c,z)\left|\frac{z^2}{z^2-8z+8}\right|^N,
$$

with $M(a,\vec b,\vec c,z) > 0$ independent on N. When $\Re c_k > \Re b_k > 0$; $k = 1, 2, \ldots, p$; $M(a,\vec b,\vec c,z)$ is also independent on $\vec b$ and $\vec c$.

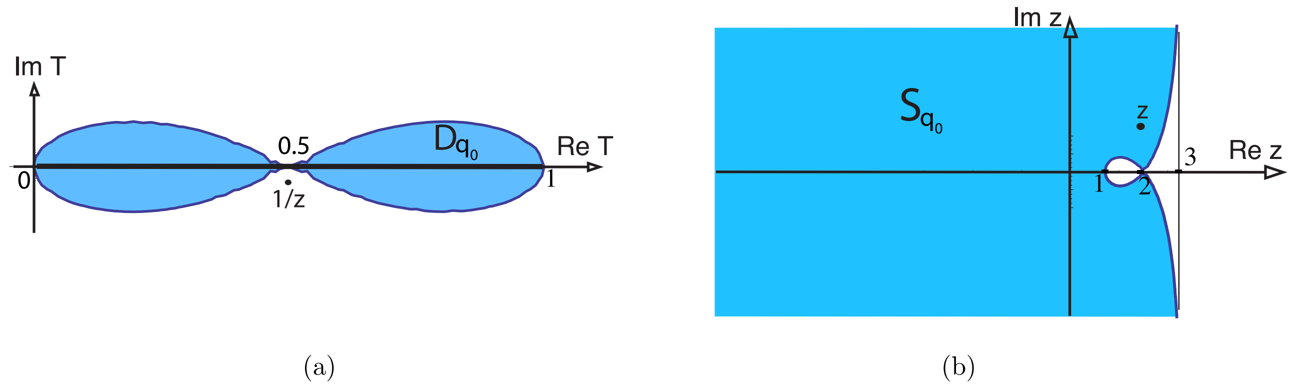


Figure 4. *The minimal domain of convergence D_{q_0} of the two-point Taylor expansion of $f(T)$ at $T = q_0 = (2-\sqrt{2})/4$ and $T = 1-q_0 = (2+\sqrt{2})/4$ containing X is a Cassini's oval of radius $r = 1/8+\epsilon$ and foci $T = q_0$ and $T = 1-q_0$ (figure (a)). The region S_{q_0}, inverse of the exterior of D_{q_0}, is the region shown in figure (b): $S_{q_0} = \{z \in \mathbb{C};\ \ |z^2-8z+8| > |z|^2\}$.*

Proof. Just set $q = q_0$ in the proof of Theorem 2. $\square$

4. Expansions in larger domains

As it has been pointed out in [6] (in a different context), the use of a three-point Taylor expansion [5] with base points in the interval $[0,1]$ is preferable to the use of a two-point Taylor expansion in order to better avoid the singularity $T = 1/z$ of $f(t)$ in its domain of convergence, and, at the same time, to include X in its interior (see Figure 5(a)). Therefore, in this section we consider the three-point Taylor expansion of the function $f(T) = (1 - zT)^{-a}$. For simplicity in the exposition, we consider here the three base points $T = 0$, $T = 1/2$ and $T = 1$ (subsection 4.1) and the three base points $T = q_0 := (2 - \sqrt{3})/4$, $T = 1/2$ and $T = 1 - q_0$ (subsection 4.2). Because of the long expressions involved in the expansion, we relegate the general case $T = q$, $T = 1/2$ and $T = 1 - q$, $q \in [0, 1/2)$, to Appendix 1.

4.1. An expansion for $|z|^3 < 6\sqrt{3}|(1-z)(2-z)|$

The simplest form of the general analytic expansion derived in Appendix 1 is obtained for $q = 0$:

Corollary 3. *Define the region (see Figure 5(b))*

$$S_0 := \{z \in \mathbb{C}; |z|^3 < 6\sqrt{3}|(1-z)(2-z)|\} = \{x + iy; x, y \in \mathbb{R},\ \ 108[(1-x)^2 + y^2][(2-x)^2 + y^2] > (x^2 + y^2)^3\}.$$

Then, for any $z \in S_0$ and $(a, \vec{b}, \vec{c}) \in \Lambda$,

$$\begin{aligned}
{}_{p+1}F_p\left(\begin{matrix} a, \vec{b} \\ \vec{c} \end{matrix}\middle|\, z\right) = \sum_{n=0}^{N-1} (-1)^n \sum_{k=0}^{n} \binom{n}{k} (-2)^{k-n} \prod_{s=1}^{p} \left(\frac{(b_s)_k}{(c_s)_k}\frac{(b_s+k)_n}{(c_s+k)_n}\right) \Bigg[A_n(a,z)\, {}_{p+1}F_p\left(\begin{matrix} -n, \overrightarrow{b+n+k} \\ \overrightarrow{c+n+k} \end{matrix}\middle|\, 1\right) + \\
\prod_{s=1}^{p} \frac{b_s+n+k}{c_s+n+k} B_n(a,z)\, {}_{p+1}F_p\left(\begin{matrix} -n, \overrightarrow{b+n+k+1} \\ \overrightarrow{c+n+k+1} \end{matrix}\middle|\, 1\right) + \\
\prod_{s=1}^{p} \frac{(b_s+n+k)(b_s+n+k+1)}{(c_s+n+k)(c_s+n+k+1)} C_n(a,z)\, {}_{p+1}F_p\left(\begin{matrix} -n, \overrightarrow{b+n+k+2} \\ \overrightarrow{c+n+k+2} \end{matrix}\middle|\, 1\right)\Bigg] + R_N\left(\begin{matrix} a, \vec{b} \\ \vec{c} \end{matrix}\middle|\, z\right),
\end{aligned} \tag{29}$$

where, for $n = 0, 1, 2, \ldots$, the coefficients $A_n(a,z)$, $B_n(a,z)$ and $C_n(a,z)$ are obtained from the recurrence relation

$$\begin{aligned}
A_{n+1}(a,z) =& \frac{1}{2(n+1)}\{2[3n(z-2) - 2]B_n(a,z) + 4z(3n+a)A_n(a,z) + n(5z-6)C_n(a,z)\}, \\
B_{n+1}(a,z) =& \frac{1}{2(n+1)(z^2-3z+2)}\{4z(3n+a)(26z - 3z^2 - 24)A_n(a,z) + 2[48 - 4z(18+5a) + 6z^2(4+3a) + \\
& 3n(48 - 96z + 50z^2 - 3z^3)]B_n(a,z) + [4(20 - 6z(5+a) + 5z^2(2+a)) + \\
& n(264 - 516z + 262z^2 - 15z^3)]C_n(a,z)\}, \\
C_{n+1}(a,z) =& \frac{1}{(n+1)(z^2-3z+2)}\{4z(3n+a)(12 - 12z + z^2)A_n(a,z) + 2[2(6(3+a)z - (6+5a)z^2 - 12) + \\
& 3n(z^3 - 24z^2 + 48z - 24)]B_n(a,z) + [4(2z(9+2a) - 3z^2(2+a) - 12) + \\
& n(5z^3 - 132z^2 + 276z - 144)]C_n(a,z)\},
\end{aligned} \tag{30}$$

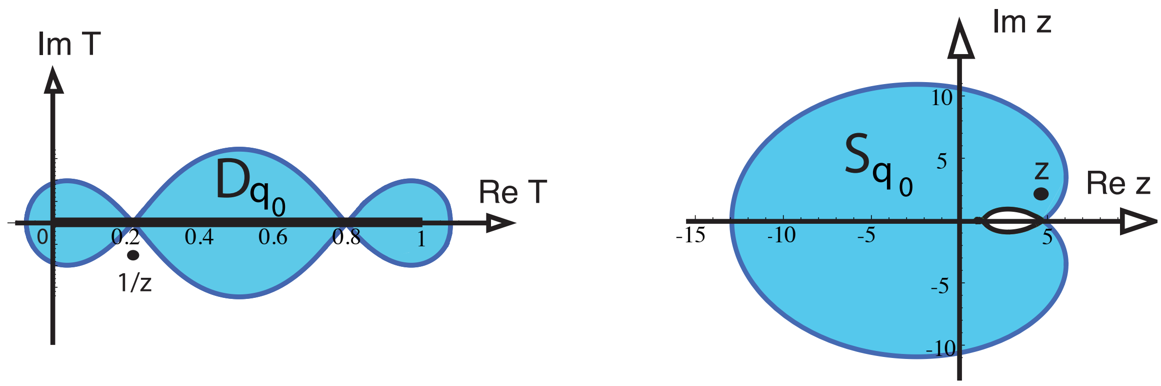
with $A_0(a,z) = 1$, $B_0(a,z) = 4(1-z/2)^{-a} - (1-z)^{-a} - 3$ and $C_0(a,z) = 2 + 2(1-z)^{-a} - 4(1-z/2)^{-a}$.

The functions ${}_{p+1}F_p(-n, \vec{b}; \vec{c}; 1)$ may be recursively computed using (9). The rate of convergence of (29) is of power type:

$$\left| R_N\left(\begin{matrix} a, \vec{b} \\ \vec{c} \end{matrix}\middle|\, z\right)\right| \le M(a, \vec{b}, \vec{c}, z)\left|\frac{z^3}{6\sqrt{3}(1-z)(2-z)}\right|^N,$$

with $M(a,\vec{b},\vec{c},z)>0$ independent on N. When $\Re c_k > \Re b_k > 0$; $k=1,2,...,p$; $M(a,\vec{b},\vec{c},z)$ is also independent on $\vec{b}$ and $\vec{c}$.

Proof. Set $q=0$ in the proof of Theorem 3 in Appendix 1. □



(a) (b)

Figure 5. *The minimal domain of convergence D_0 of the three-point Taylor expansion of $f(T)$ at $T=0$, $T=1/2$ and $T=1$ containing X is a Cassini oval of radius $r=(12\sqrt{3})^{-1}+\epsilon$ and foci at $T=0$, $T=1/2$ and $T=1$ (figure (a)). The region S_0, inverse of the exterior of D_0, is the region shown in figure (b): $S_0=\{z\in\mathbb{C};\ \ |z|^3<6\sqrt{3}|(1-z)(2-z)|\}$.*

4.2. An expansion for $|z|^3<|(2-z)(z^2-16z+16)|$

When we want to maximize the size of the convergence region S_q for the variable z, the optimal election of the base points q and $1-q$ is that one that minimizes the region D_q (see Fig. 6(a)). As we derive in Appendix 1, this happens for $q=q_0:=(2-\sqrt{3})/4$. With this choice, the Cassini's oval D_{q_0} is the smallest possible three-point Cassini's oval containing X. We formulate the result in the form of a corollary.

Corollary 4. *Define the region (see Figure 6(b))*

$$\begin{aligned}S_{q_0}:=&\{z\in\mathbb{C};|z|^3<|(2-z)|\,|z^2-16z+16|\,\}=\{x+iy;x,y\in\mathbb{R},\\ &256-x(768-864x+448x^2-105x^3+9x^4)+6y^2(48-64x+27x^2-3x^3)+3y^4(19-3x)>0\}.\end{aligned}$$

Then, for any $z\in S_{q_0}$ and $(a,\vec{b},\vec{c})\in\Lambda$,

$$\begin{aligned}{}_{p+1}F_p\left(\begin{matrix}a,\vec{b}\\ \vec{c}\end{matrix}\middle|\,z\right)=&\sum_{n=0}^{N-1}\sum_{k=0}^{n}\binom{n}{k}(-2)^{k-n}\prod_{s=1}^{p}\frac{(b_s)_k}{(c_s)_k}\left[A_n(a,z)\,H_n^q\left(\begin{matrix}\overrightarrow{b+k}\\ \overrightarrow{c+k}\end{matrix}\right)+\right.\\ &\prod_{s=1}^{p}\frac{b_s+k}{c_s+k}B_n(a,z)\,H_n^q\left(\begin{matrix}\overrightarrow{b+k+1}\\ \overrightarrow{c+k+1}\end{matrix}\right)+\left.\prod_{s=1}^{p}\frac{(b_s+k)(b_s+k+1)}{(c_s+k)(c_s+k+1)}C_n(a,z)\,H_n^q\left(\begin{matrix}\overrightarrow{b+k+2}\\ \overrightarrow{c+k+2}\end{matrix}\right)\right]+R_N\left(\begin{matrix}a,\vec{b}\\ \vec{c}\end{matrix}\middle|\,z\right),\end{aligned}\tag{31}$$

with

$$H_n^q\left(\begin{matrix}\vec{b}\\ \vec{c}\end{matrix}\right):=\sum_{m=0}^{n}\binom{n}{m}16^{n-m}(-1)^m\prod_{s=1}^{p}\frac{(b_s)_m}{(c_s)_m}\,{}_{p+1}F_p\left(\begin{matrix}-m,\overrightarrow{b+m}\\ \overrightarrow{c+m}\end{matrix}\middle|\,1\right).\tag{32}$$

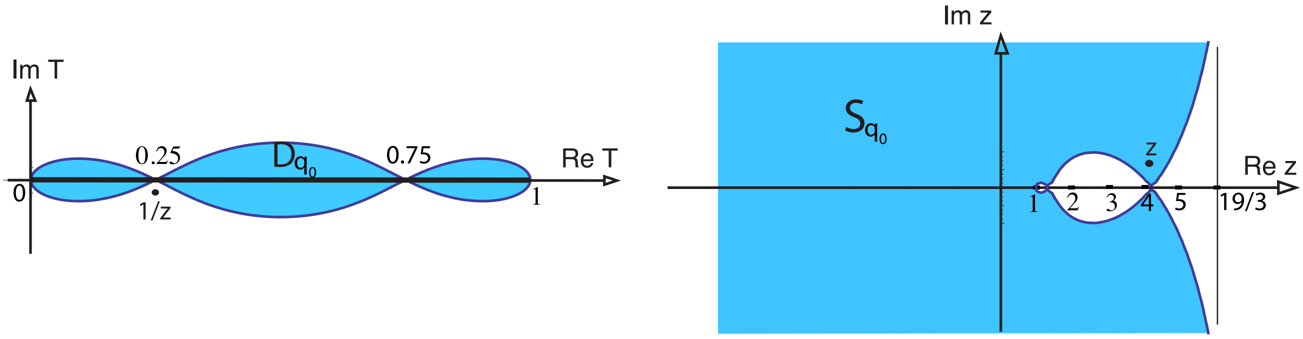
For $n=0,1,2,...,$ the coefficients $A_n(a,z)$, $B_n(a,z)$ and $C_n(a,z)$ are obtained from the recurrence (36) with $q=(2-\sqrt{3})/4$.

The functions ${}_{p+1}F_p(-n,\vec{b};\vec{c};1)$ may be recursively computed using (9). The rate of convergence of (31) is of power type:

$$\left| R_N\left(\begin{array}{c} a,\vec{b} \\ \vec{c} \end{array}\middle| z\right)\right| \le M(a,\vec{b},\vec{c},z)\left|\frac{z^3}{(z^2-16z+16)(2-z)}\right|^N,$$

with $M(a,\vec{b},\vec{c},z)>0$ independent on N. When $\Re c_k > \Re b_k > 0$; $k=1,2,...,p$; $M(a,\vec{b},\vec{c},z)$ is also independent on $\vec{b}$ and $\vec{c}$.

Proof. Set $q=q_0$ in the proof of Theorem 3. □



(a) (b)

Figure 6. *The minimal domain of convergence D_{q_0} of the three-point Taylor expansion of $f(T)$ at $T=q_0=(2-\sqrt{3})/4$, $T=1/2$ and $T=1-q_0=(2+\sqrt{3})/4$ containing X is a Cassini oval of radius $r=1/32+\epsilon$ and foci $T=q_0$, $T=1/2$ and $T=1-q_0$ (figure (a)). The region S_{q_0}, inverse of the exterior of D_{q_0}, is the region shown in figure (b): $S_{q_0}=\{z\in \mathbb{C};\ \ |z|^3<|(2-z)(z^2-16z+16)|\,\}$.*

5. Concluding remarks and numerical experiments

In Section 2 we have used a standard one-point Taylor expansion of $f(T)=(1-zT)^{-a}$ with the smallest possible convergence region D containing X (the domain of T): an expansion at the point $t=w$ and convergence radius r =max$\{|w|,|w-1|\}+\epsilon$. Then, the inverse of the complement of this disk D is the largest possible region S that we can obtain with a one-point Taylor expansion of $f(T)$: $\Re z<1$ or, in general, the semi-plane $2\Re(zw)<1$ (when $\Re w\ge 1/2$).

In Section 3 we have improved this idea using a two-point Taylor expansion of $f(T)$ with two base points located in the interval $[0,1]$ and, in Section 4 and Appendix 1, a three-point Taylor expansion. In this way, we have obtained expansions with larger domains of convergence S, some of them unbounded, and containing the indented closed unit disk $|z|\le 1$, $z\ne 1$. In order to enlarge the convergence region S, we may consider the possibility of expanding $f(T)$ at four or more points located in the interval $[0,1]$. In fact, one gets new approximations valid in regions S larger than the ones shown in Figs. 3(b), 4(b), 5(b) or 6(b); but the integrals defining the coefficients of the expansion, as well as the recurrences for the functions $A_n(a,z)$, $B_n(a,z)$,..., become more and more complicated.

In this paper we have generalized our previous work [7] and [8] for the ${}_2F_1(a,b;c;z)$ and ${}_3F_2(a,b_1,b_2;c_1,c_2;z)$ functions to arbitrary ${}_{p+1}F_p(a,\vec{b};\vec{c};z)$ functions in three different directions. On the one hand, obviously, we have extended the number of parameters (3 or 5) to an arbitrary number $2p+1$. On the other hand, we have considered new base points for the two and three-point Taylor expansions of the function $f(T)$. We have optimized in this way the shape of the domain of convergence of the expansions of $f(T)$, obtaining new and larger domains for the variable z in the expansion of ${}_{p+1}F_p(a,\vec{b};\vec{c};z)$. We have also considered multiple loop integral representations (5), (cases (b), (c) and (d) detailed in Appendix 2), instead of the multiple Euler integral representation used in [7] and [8] that are particular cases of (5) (see Appendix 2, case (a)). These integral representations are necessary to extend the range of validity of the parameters a, $\vec{b}$ and $\vec{c}$ to any complex value (except $1-c_k\in\mathbb{N}$).

All the expansions are given in terms of rational functions of z and the parameters b_k and c_k, multiplied by factors of the form $(1-uz)^{-a}$ for different constants u. We have derived a precise bound for the remainder only for the expansion given in Section 2. We have not given a precise bound for the remainder in the expansions given in sections 3, 4 and Appendix 1 because: (i) we do not have at our disposal an appropriate expression for the remainder $r_N(T)$ of the two and three-point Taylor expansions of $f(T)$ that would let the derivation of an accurate constant $C(a,z)$ and (ii) the loop integrals considered in (5) (cases (b), (c) and (d) in Appendix 2) are less appropriate than the Euler integral (5) (case (a) in Appendix 2) when we want to find and accurate form of the bounding constant $M(a,\vec{b},\vec{c},z)$.

The following tables show some numerical experiments for the ${}_4F_3(a,\vec{b};\vec{c};z)$ and ${}_8F_7(a,\vec{b};\vec{c};z)$ functions. We compare the accuracy of the approximations given in sections 2, 3 and 4 with the power series definition (1), the connection formula (1)-(2) and Bühring's formula (4).

The first row in the tables represents the number n of terms used in either of the expansions. The remaining rows represent the relative error obtained with the indicated approximations. In Bühring's formula (4) and formula (6) of Theorem 1 we have taken the base point $w=1/2$ for the Taylor expansion.

These numerical experiments confirm that the approximations derived from a three-point Taylor expansion ((29) and (31)) are more accurate than the approximations derived from a two-point Taylor ((23) and (27)) or a standard Taylor expansion (6): three-point Taylor expansions provide more uniform approximations than two-point or standard Taylor series.

When we take values of z close to 0, our 5 new approximations are more competitive than Bühring's formula or the standard Taylor series definition. Also, when the difference between any couple of parameters a_j is close or equal to an integer, our 5 approximations are more competitive than Bühring's approximation. We may observe that formula (31) is the most accurate one.

Parameter values: $p=3$, $a=1,\vec{b}=\left(\frac{1}{2},\frac{4}{3},\frac{5}{6}\right),\vec{c}=\left(\frac{5}{3},\frac{7}{5},\frac{5}{7}\right),\ z=-\frac{1+i}{5}$.

Approximation	0	2	4	6
Power series definition (1)	8.971e-2	2.829e-3	1.368e-4	7.742e-6
Connection formula (1)-(2)	—	—	—	—
Bühring's formula $z_0=1/2$ (4)	6.371e-1	1.844e-1	6.933e-2	2.828e-2
Theorem 1: formula (6)	4.702e-2	5.770e-4	7.870e-6	1.127e-7
Corollary 1: formula (23)	8.932e-3	1.551e-6	3.325e-10	7.676e-14
Corollary 2: formula (27)	7.280e-4	3.829e-8	2.201e-12	2.376e-16
Corollary 3: formula (29)	1.943e-4	8.069e-11	4.189e-17	2.345e-23
Corollary 4: formula (31)	2.155e-6	4.879e-13	1.190e-19	3.003e-26

Parameter values: $p = 3,\ a = 1, \vec{b} = \left(\frac{1}{2}, \frac{4}{3}, \frac{5}{6}\right), \vec{c} = \left(\frac{5}{3}, \frac{7}{5}, \frac{5}{7}\right), z = -3 + i$.

Approximation	0	2	4	6
Power series definition (1)	—	—	—	—
Connection formula (1)-(2)	8.182e-2	2.442e-3	1.384e-4	9.510e-6
Bühring's formula $z_0 = 1/2$ (4)	1.101e-1	1.361e-3	2.099e-5	3.474e-7
Theorem 1: formula (6)	3.474e-1	1.021e-1	3.301e-2	1.115e-2
Corollary 1: formula (23)	2.705e-1	5.812e-2	1.645e-2	5.089e-3
Corollary 2: formula (27)	2.799e-2	1.134e-3	5.157e-5	2.476e-6
Corollary 3: formula (29)	4.963e-2	6.413e-4	1.053e-5	1.873e-7
Corollary 4: formula (31)	3.102e-3	1.518e-5	8.371e-8	4.880e-10

Parameter values: $p = 3,\ a = 1, \vec{b} = \left(\frac{1}{2}, \frac{4}{3}, \frac{5}{6}\right), \vec{c} = \left(\frac{5}{3}, \frac{7}{5}, \frac{5}{7}\right), z = \exp(\frac{i\pi}{4})$.

Approximation	0	2	4	6
Power series definition (1)	3.701e-1	1.598e-1	9.907e-2	7.070e-2
Connection formula (1)-(2)	4.657e-1	1.722e-1	1.029e-1	7.22e-2
Bühring's formula $z_0 = 1/2$ (4)	6.007e-1	1.757e-1	6.512e-2	2.606e-2
Theorem 1: formula (6)	2.393e-1	7.884e-2	2.952e-2	1.171e-2
Corollary 1: formula (23)	1.959e-1	1.416e-2	1.222e-3	1.125e-4
Corollary 2: formula (27)	4.168e-2	1.167e-3	3.610e-5	1.169e-6
Corollary 3: formula (29)	2.256e-2	1.085e-4	6.332e-7	3.959e-9
Corollary 4: formula (31)	4.522e-3	1.232e-5	3.713e-8	1.172e-10

Parameter values: $p = 3,\ a = 1,\ \vec{b} = (2, 3, 1),\ \vec{c} = (2, 3, 4),\ z = \frac{1}{2}\exp(\frac{7i\pi}{6})$.

Approximation	0	2	4	6
Power series definition (1)	1.179e-1	5.519e-3	4.757e-4	5.429e-5
Connection formula (1)-(2)	—	—	—	—
Bühring's formula $z_0 = 1/2$ (4)	—	—	—	—
Theorem 1: formula (6)	1.081e-1	2.746e-3	8.254e-5	2.695e-6
Corollary 1: formula (23)	2.492e-2	2.843e-5	4.064e-8	6.312e-11
Corollary 2: formula (27)	3.416e-3	1.276e-6	5.058e-10	2.032e-13
Corollary 3: formula (29)	1.823e-3	1.454e-8	1.322e-13	3.370e-15
Corollary 4: formula (31)	6.574e-4	2.401e-9	7.157e-15	3.265e-15

6. Appendix 1. An expansion for $6\sqrt{3}\,|(1-qz)(2-z)(1+qz-z)| > |(1-2q)z|^3$

We generalize in this section the expansion given in Section 4: we consider the three-point Taylor expansion of the function $f(T) = (1 - zT)^{-a}$ at three-points $T = q$, $T = 1/2$ and $T = 1 - q$, $q \in [0, 1/2)$. The results are summarized in the following theorem.

Parameter values: $p=7$, $a=1, \vec{b}=\left(\frac{1}{2},\frac{1}{3},\frac{3}{4},\frac{5}{7},\frac{3}{8},\frac{7}{6},\frac{5}{9}\right)$, $\vec{c}=\left(\frac{12}{11},\frac{2}{3},\frac{5}{4},\frac{6}{7},\frac{6}{5},\frac{9}{10},\frac{5}{12}\right)$, $z=\frac{2}{3}\exp(\frac{2i\pi}{3})$.

Approximation	0	2	4	6
Power series definition (1)	3.748e-2	2.369e-3	3.921e-4	8.987e-5
Connection formula (1)-(2)	—	—	—	—
Bühring's formula $z_0 = 1/2$ (4)	4.515e-2	2.352e-3	2.963e-4	5.377e-5
Theorem 1: formula (6)	2.464e-1	1.715e-2	1.236e-3	9.052e-5
Corollary 1: formula (23)	1.200e-2	3.011e-5	1.244e-7	5.885e-10
Corollary 2: formula (27)	2.640e-2	3.445e-5	4.698e-8	6.538e-11
Corollary 3: formula (29)	1.797e-3	6.212e-8	3.113e-12	2.047e-16
Corollary 4: formula (31)	2.923e-3	7.305e-8	1.908e-12	1.419e-16

Parameter values: $p=7$, $a=1, \vec{b}=\left(\frac{1}{2},\frac{1}{3},\frac{3}{4},\frac{5}{7},\frac{3}{8},\frac{7}{6},\frac{5}{9}\right)$, $\vec{c}=\left(\frac{12}{11},\frac{2}{3},\frac{5}{4},\frac{6}{7},\frac{6}{5},\frac{9}{10},\frac{5}{12}\right)$, $z=-2-2i$.

Approximation	0	2	4	6
Power series definition (1)	—	—	—	—
Connection formula (1)-(2)	9.630e-3	1.115e-4	4.577e-6	2.786e-7
Bühring's formula $z_0 = 1/2$ (4)	2.122e-2	1.591e-4	2.206e-6	3.710e-8
Theorem 1: formula (6)	5.809e-1	2.132e-1	8.042e-2	3.075e-2
Corollary 1: formula (23)	6.942e-2	7.813e-3	1.608e-3	3.893e-4
Corollary 2: formula (27)	1.779e-1	8.749e-3	4.497e-4	2.359e-5
Corollary 3: formula (29)	2.684e-2	2.381e-4	3.198e-6	4.812e-8
Corollary 4: formula (31)	1.057e-1	1.809e-3	3.248e-5	5.975e-7

Theorem 3. *For arbitrary $q\in[0,q_0]$, with $q_0:=(2-\sqrt{3})/4$, define the region (see Figures 5(b) or 6(b))*

$$S_q := \{z\in\mathbb{C}; 6\sqrt{3}\,|(1-qz)(2-z)(1+qz-z)| > |(1-2q)z|^3\}. \tag{33}$$

Then, for any $z\in S_q$ and $(a,\vec{b},\vec{c})\in\Lambda$,

$$\begin{aligned}{}_{p+1}F_p\left(\begin{matrix}a,\vec{b}\\ \vec{c}\end{matrix}\middle| z\right) = \sum_{n=0}^{N-1}\sum_{k=0}^{n}\binom{n}{k}(-2)^{k-n}\prod_{s=1}^{p}\frac{(b_s)_k}{(c_s)_k}\left[A_n(a,z)\,H_n^q\left(\begin{matrix}\overrightarrow{b+k}\\ \overrightarrow{c+k}\end{matrix}\right)+\right.\\ \left.\prod_{s=1}^{p}\frac{b_s+k}{c_s+k}B_n(a,z)\,H_n^q\left(\begin{matrix}\overrightarrow{b+k+1}\\ \overrightarrow{c+k+1}\end{matrix}\right)+\prod_{s=1}^{p}\frac{(b_s+k)(b_s+k+1)}{(c_s+k)(c_s+k+1)}C_n(a,z)\,H_n^q\left(\begin{matrix}\overrightarrow{b+k+2}\\ \overrightarrow{c+k+2}\end{matrix}\right)\right]+R_N\left(\begin{matrix}a,\vec{b}\\ \vec{c}\end{matrix}\middle| z\right),\end{aligned} \tag{34}$$

with

$$H_n^q\left(\begin{matrix}\vec{b}\\ \vec{c}\end{matrix}\right) := \sum_{m=0}^{n}\binom{n}{m}(q(1-q))^{n-m}(-1)^m\prod_{s=1}^{p}\frac{(b_s)_m}{(c_s)_m}\,{}_{p+1}F_p\left(\begin{matrix}-m,\overrightarrow{b+m}\\ \overrightarrow{c+m}\end{matrix}\middle| 1\right). \tag{35}$$

For $n=0,1,2,\ldots$, the coefficients $A_n(a,z)$ and $B_n(a,z)$ are obtained from the following recurrence relation:

$$\begin{aligned}A_{n+1}(a,z) &= \frac{M_{11}(z,n,a,q)\,A_n(a,z)+M_{12}(z,n,a,q)\,B_n(a,z)+M_{13}(z,n,a,q)\,C_n(a,z)}{2(n+1)(1-2q)^4(z-2)\,((q-1)qz^2+z-1)},\\ B_{n+1}(a,z) &= \frac{M_{21}(z,n,a,q)\,A_n(a,z)+M_{22}(z,n,a,q)\,B_n(a,z)+M_{23}(z,n,a,q)\,C_n(a,z)}{2(n+1)(1-2q)^4(z-2)\,((q-1)qz^2+z-1)},\\ C_{n+1}(a,z) &= \frac{M_{31}(z,n,a,q)\,A_n(a,z)+M_{32}(z,n,a,q)\,B_n(a,z)+M_{33}(z,n,a,q)\,C_n(a,z)}{(n+1)(1-2q)^4(z-2)\,((q-1)qz^2+z-1)},\end{aligned} \tag{36}$$

with

$$
\begin{aligned}
M_{11}(z,n,a,q) :=& 4z(a+3n)\left((2(q-1)q(8(q-1)q+1)+1)z^2+3(4(q-1)q-1)z-16(q-1)q+2\right),\\
M_{12}(z,n,a,q) :=& 2\left(2(a+1)(q-1)q(8(q-1)q-1)z^3-2z^2(2(q-1)q(-6a+8(q-1)q-5)+1)\right.\\
&-24(a+2)(q-1)qz+3n(z-1)\left(\left(32(q-1)^2q^2+1\right)z^2+4(8(q-1)q-1)z-32(q-1)q+4\right)\\
&\left.+32(q-1)q+6z-4\right),\\
M_{13}(z,n,a,q) :=& 4(q-1)q\left(z^2\left(-4(a+6)q^2+4(a+6)q+5a+12\right)+6(a+2)(q-1)qz^3-6(a+6)z+24\right)\\
&+n\left((4(q-1)q(2(q-1)q(16(q-1)q+27)+3)+5)z^3-3(8(q-1)q(6(q-1)q-7)+7)z^2-\right.\\
&\left.8(q-1)q(16(q-1)q+53)z+240(q-1)q+4(7z-3)\right),\\
M_{21}(z,n,a,q) :=& 4z(a+3n)(z(4(q-1)q(3z+2)-3z+26)-24),\\
M_{22}(z,n,a,q) :=& 2(2z(a(4(q-1)q(3(z-1)z+2)+9z)+12z((q-1)q(z-2)+1))-4(5a+18)z+\\
&3n(z(z(4(q-1)q(9z-10)-3z+50)-96)+48)+48),\\
M_{23}(z,n,a,q) :=& z^3\left(4(a+2)q\left(-4q^3+8q^2+q-5\right)+3n(8(q-1)q(2(q-1)q+7)-5)\right)+2z^2\left(16q^4(a+5n+2)-\right.\\
&\left.32q^3(a+5n+2)+12q^2(a+4n-2)+4q(a+8n+14)+10a+131n+20\right)-\\
&12z(2a+n(4(q-1)q+43)-8(q-1)q+10)+8(3n(11-4(q-1)q)-8(q-1)q+10),\\
M_{31}(z,n,a,q) :=& 4z(a+3n)(z((1-8(q-1)q)z-12)+12),\\
M_{32}(z,n,a,q) :=& 2(2(z(a(z(-2(q-1)q(3z-2)-5)+6)-6z((q-1)q(z-2)+1)+18)-12)+\\
&3n(z(z(4(q-1)q(6-5z)+z-24)+48)-24)),\\
M_{33}(z,n,a,q) :=& 4z\left(a\left((q-1)q\left(4-3z^2\right)-3z+4\right)-6z((q-1)q(z-2)+1)+18\right)+\\
&n\left(z\left((5-8(q-1)q(8(q-1)q+13))z^2+48(q-1)qz+96(q-1)q-132z+276\right)-144\right)-48,\\
A_0(a,z) :=& \frac{4q^2(1-z/2)^{-a}+(1-qz)^{-a}+q(-4(1-z/2)^{-a}+(1+(-1+q)z)^{-a}-(1-qz)^{-a})}{(1-2q)^2},\\
B_0(a,z) :=& \frac{4(1-z/2)^{-a}-(1+(-1+q)z)^{-a}-2q(1+(-1+q)z)^{-a}-3(1-qz)^{-a}+2q(1-qz)^{-a}}{(1-2q)^2},\\
C_0(a,z) :=& \frac{-2^{(2+a)}(2-z)^{-a}+2(1+(-1+q)z)^{-a}+2(1-qz)^{-a}}{(1-2q)^2}.
\end{aligned}
$$

The functions ${}_{p+1}F_p(-m,\vec{b};\vec{c};1)$ *may be recursively computed using (9). The rate of convergence of (34) is of power type:*

$$
\left|R_N\left(\begin{array}{c}a,\vec{b}\\ \vec{c}\end{array}\middle|\, z\right)\right|\le M(a,\vec{b},\vec{c},z)\left|\frac{(1-2q)z^3}{6\sqrt{3}(1-qz)(1+(q-1)z)(2-z)}\right|^N, \tag{37}
$$

with $M(a,\vec{b},\vec{c},z)>0$ *independent on* N*. When* $\Re c_k>\Re b_k>0$*;* $k=1,2,...,p$*;* $M(a,\vec{b},\vec{c},z)$ *is also independent on* $\vec{b}$ *and* $\vec{c}$*.*

Proof. From [5] we have that:

$$
f(T)=\sum_{n=0}^{N-1}[A_n(a,z)+B_n(a,z)T+C_n(a,z)T^2][(T-q)(T-1/2)(T+q-1)]^n+r_N(T), \qquad T\in D_q, \tag{38}
$$

where D_q is a Cassini's oval with foci at $T=q$, $T=1/2$ and $T=1-q$ and radius $r>0$ that we determine below: $D_q=\{T\in\mathbb{C}, |(T-q)(T-1/2)(T+q-1)|<r\}$, and $r_N(T)$ is the three-point Taylor remainder [[5], Theorem 1].

An explicit formula for the coefficients $A_n(a,z)$, $B_n(a,z)$ and $C_n(a,z)$ may be derived from [5], but it is very involved and we omit them here for simplicity. Instead, we derive the recurrence relation (36) from the differential equation satisfied by $f(T)$: $(1-zT)f'=azT$. We have that

$$f'(T)=\sum_{n=0}^{\infty}[A'_n(a,z)+B'_n(a,z)T+C'_n(a,z)T^2][(T-q)(T-1/2)(T-1+q)]^n, \tag{39}$$

with

$$\begin{aligned}A'_n(a,z):=&(3n+1)B_n(a,z)+\frac{3n}{2}C_n(a,z)+\\ &(n+1)\left[\frac{A_{n+1}(a,z)}{2}+q(1-q)\left(A_{n+1}(a,z)+\frac{3}{2}B_{n+1}(a,z)+\frac{3}{4}C_{n+1}(a,z)\right)\right],\\ B'_n(a,z):=&(3n+2)C_n(a,z)-3(n+1)A_{n+1}(a,z)-(n+1)(1+2q(1-q))B_{n+1}(a,z)-\frac{3(n+1)}{4}C_{n+1}(a,z),\\ C'_n(a,z):=&3(n+1)A_{n+1}(a,z)+\frac{3(n+1)}{2}B_{n+1}(a,z)+\left(\frac{5(n+1)}{4}-2q(1-q)\right)C_{n+1}(a,z).\end{aligned}$$

Introducing (38) and (39) into the differential equation $(1-zT)f'=azf$ and equating coefficients of $[(T-q)(T-1/2)(T+q-1)]^n$, $T[(T-q)(T-1/2)(T+q-1)]^n$ and $T^2[(T-q)(T-1/2)(T+q-1)]^n$, we obtain (36).

Expansion (38) converges inside a Cassini's oval with foci at $T=q$, $T=1/2$ and $T=1-q$ and radius $r>0$ of the form $D_q=\{T\in\mathbb{C},\ |(T-q)(T-1/2)(T+q-1)|<r\}$. The interval $[0,1]$ is contained in the Cassini oval D_q when its radius $r\geq \mathrm{Sup}_{T\in[0,1]}\{|(T-q)(T-1+q)|\}=\mathrm{Max}\{q(1-q)/2,(1-2q)^3/(12\sqrt{3})\}$. This happens for $r\geq(1-2q)^3/(12\sqrt{3})$ when $0\leq q\leq q_0:=(2-\sqrt{3})/4$, where q_0 is the solution of the equation $q(1-q)/2=(1-2q)^3/(12\sqrt{3})$. Then, expansion (38) satisfies condition (i) for $r>|(1-2q)^3/(12\sqrt{3})$ with $q\in[0,q_0]$. On the other hand, it satisfies condition (ii) if $1/z\notin D_q$ [4], that is, for any

$$r<\left|\left(\frac{1}{z}-q\right)\left(\frac{1}{z}-\frac{1}{2}\right)\left(\frac{1}{z}+q-1\right)\right|.$$

The smallest r we can take is $r=(1-2q)^3/(12\sqrt{3})+\epsilon$ and then, the largest S_q we can choose is (33). Then, for $z\in S_q$, we can introduce the expansion (38) in (5) and interchange summation and integration. We obtain (34)-(35) with

$$R_N\left(\begin{array}{c}a,\vec{b}\\ \vec{c}\end{array}\middle| z\right):=\prod_{s=1}^{p}\left(A(b_s,c_s)\int_L t_s^{b_s-1}(1-t_s)^{c_s-b_s-1}dt_s\right)r_N(T). \tag{40}$$

From Theorem 1 of [5] we have that the remainder in the expansion (38) may be bounded in the form $|r_N(T)|\leq C(a,z)|(T-q)(T-1/2)(T+q-1)/r|^N$, $T\in D_q$, where $C(a,z)>0$ is independent of T and N. When $T\in X$, we have that $|r_N(T)|\leq C(a,z)[(1-2q)^3/(12\sqrt{3})]^{2N}/r^N$. Introducing this bound in (40) and after straightforward computations we obtain (37). When $\Re c_k>\Re b_k>0$; $k=1,2,...,p$, we may use the Euler form of (5) (see Appendix 2, case (a)) to see that $M(a,\vec{b},\vec{c},z)$ is independent on $\vec{b}$ and $\vec{c}$. □

7. Appendix 2

In this section we derive four versions of formula (5),

$${}_{p+1}F_p\left(\begin{array}{c}a,\vec{b}\\ \vec{c}\end{array}\middle| z\right)=\left(\prod_{s=1}^{p}A(b_s,c_s)\int_L t_s^{b_s-1}(1-t_s)^{c_s-b_s-1}dt_s\right)(1-Tz)^{-a},\qquad T:=\prod_{s=1}^{p}t_s, \tag{41}$$

according to four different possibilities for the path L and the constant $A(b,c)$:

(a) The path is $L=[0,1]$ and

$$A(b,c):=\frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)}. \tag{42}$$

In this case, formula (41) is valid for $\Re(c_k - b_k) > \Re(b_k) > 0$; $k = 1, 2, 3, ..., p$.

(b) The path L starts and terminates at $t = 0$ and encircles the point $t = 1$ once in the positive direction. At the point where L cuts the positive real axis we have that $\arg(t) = 0$ and $\arg(1 - t) = \pi$, (see Figure 7(b)). On the other hand,

$$A(b, c) := \frac{e^{i\pi(b-c)}\Gamma(1 + b - c)\Gamma(c)}{2\pi i\Gamma(b)}. \tag{43}$$

In this case, formula (41) is valid for $c_k - b_k \notin \mathbb{N}$ and $\Re(b_k) > 0$; $k = 1, 2, 3, ..., p$.

(c) The path L starts and terminates at $t = 1$ and encircles the point $t = 0$ once in the positive direction. At the point where L cuts the negative real axis we have that $\arg(t) = \pi$ and $\arg(1 - t) = 0$, (see Figure 7(c)). On the other hand,

$$A(b, c) := \frac{e^{-i\pi b}\Gamma(1 - b)\Gamma(c)}{2\pi i\Gamma(c - b)}. \tag{44}$$

In this case, formula (41) is valid for $b_k \notin \mathbb{N}$ and $\Re(c_k - b_k) > 0$; $k = 1, 2, 3, ..., p$.

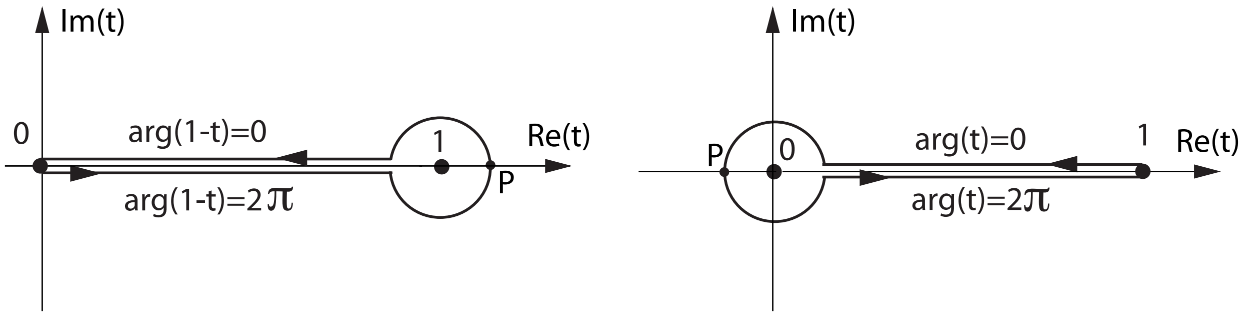
(d) The path L starts and terminates at an arbitrary point P on the real axis between $t = 0$ and $t = 1$, it encircles the points 0 and 1 once in the positive direction and then once in the negative direction. At the point P of L we have that $\arg(t)$ =$\arg(1 - t) = 0$ (see Figure 8). On the other hand,

$$A(b, c) := \frac{e^{i\pi(1-c)}\Gamma(c)}{4\Gamma(b)\Gamma(c - b)\sin(\pi b)\sin(\pi(c - b))}. \tag{45}$$

In this case, formula (41) is valid for b_k and $c_k - b_k \notin \mathbb{N}$; $k = 1, 2, 3, ..., p$.

In any case we have that $(1 - zT)^{-a} = 1$ when $z = 0$ and the $p-$dimensional (complex) integration region L^p must be bounded by the hypersurface $T = 1/z$ of real dimension $2p - 2$. In any of the four cases, the multiple integral representation (41) is a generalization of the respective integral representation of the Gauss hypergeometric function ${}_2F_1(a, b; c; z)$ given in [[11], Chap. 15, Sec. 6, eqs. 15.6.1, 15.6.2, 15.6.4 or 15.6.5]. None of them covers all the possible values of the parameters $(a, \vec{b}, \vec{c}) \in \Lambda$, but all together do.

As we have indicated in the introduction, we assume that the contours L described in cases (b), (c) and (d) are squeezed (analytically deformed) as much as possible around the real interval $[0, 1]$, as it is indicated in figures 7 and 8.



(b) (c)

Figure 7. *The p identical integration contours L in (41), cases (b) and (c), may be deformed to the respective contours (b) and (c) depicted in this figure. The horizontal segments are assumed to be sticked to the real interval* $[0, 1]$ *and the radius* ϵ *of the circles is infinitesimally small. At the point P of figure (b) we have that* $arg(t) = 0$ *and* $arg(1 - t) = \pi$. *At the point P of figure (c) we have that* $arg(t) = \pi$ *and* $arg(1 - t) = 0$.

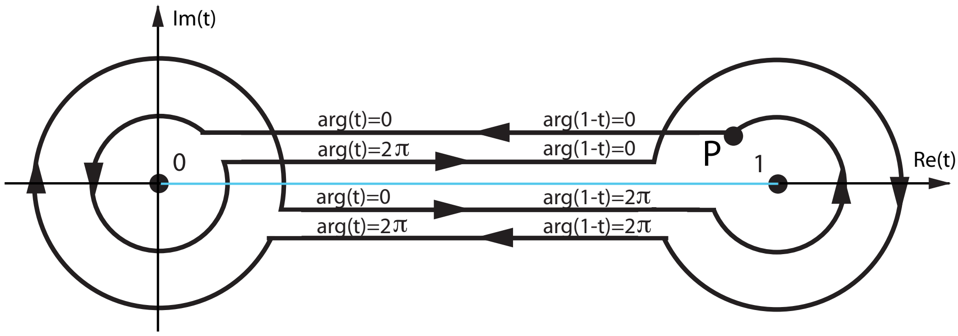


Figure 8. *The p identical integration contours L in (41), case (d), may be deformed to the contour depicted in this figure. The horizontal segments are assumed to be sticked to the real interval* $[0, 1]$ *and the radius* ϵ *of the circles is infinitesimally small. At the point P we have that arg*$(t) =$*arg*$(1 - t) = 0$.

The derivation of any of the integral representations [[11], Chap. 15, Sec. 6, eqs. 15.6.1, 15.6.2, 15.6.4 or 15.6.5] of the ${}_2F_1$ function is straightforward; for example, [[11], Chap. 15, Sec. 6, eq. 15.6.5] is derived in [[12], Chap. 4, Sec. 5], in particular in [[12], p. 153, eq. (7)] (see also [[9], p.58, eq. (8)]). But we are not aware of any reference for the multiple integral representations (5) of the ${}_{p+1}F_p$ function in any or the four cases (a)-(d). Nevertheless, the derivation of these multiple integral representations is a straightforward generalization of the derivation of the corresponding simple integral representations of the ${}_2F_1$ function. In order to derive (5) (for any version (a)-(d)) we replace the expansion

$$(1 - Tz)^{-a} = \sum_{n=0}^{\infty} \frac{(a)_n}{n!}(zT)^n, \qquad\qquad |zT| < 1,$$

into the right hand side of (41), interchange sum and integral and use the reflection formula of the gamma function. We also use different integral representations of the beta function [[2], Chap. 5, Sec. 12, eqs. 5.12.1, 5.12.10 or 5.12.12] depending on the case (a), (b)-(c) or (d) under consideration:

$$\int_L t^{a-1}(1-t)^{b-1}dt = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)} \times \begin{cases} 1, & \text{case} \quad (a), \\ 2ie^{i\pi b}\sin(\pi b), & \text{case} \quad (b), \\ 2ie^{i\pi a}\sin(\pi a), & \text{case} \quad (c), \\ -4e^{i\pi(a+b)}\sin(\pi a)\sin(\pi b), & \text{case} \quad (d). \end{cases} \tag{46}$$

We obtain the series expansion (1) of the function ${}_{p+1}F_p(a, \vec{b}; \vec{c}; z)$. Therefore, the right hand side of (41) represents the analytic continuation in the variable z of ${}_{p+1}F_p(a, \vec{b}; \vec{c}; z)$, defined by the right hand side of (1), from the disk $|z| < 1$ to the region $|\arg(1 - z)| < \pi$. □

8. Acknowledgments

Professors Nico M. Temme and Adri Olde Daalhuis are acknowledged by their useful comments and suggestions. This research was supported by the *Ministerio de Economía y Competitividad* (MTM2017-83490-P). The *Universidad Pública de Navarra* is acknowledged by its financial support.

References

[1] R. A. Askey and A. B. Olde Daalhuis, Chapter 16, Generalized Hypergeometric Functions and Meijer G-Function, NIST Handbook of Mathematical Functions, *NIST and Cambridge Univ. Press*, 2010. http://dlmf.nist.gov/16

[2] R. A. Askey and R. Roy, Chapter 5, Gamma Functions, NIST Handbook of Mathematical Functions, *NIST and Cambridge Univ. Press*, 2010. http://dlmf.nist.gov/5

[3] W. Bühring, Generalized hypergeometric functions at unit argument, *Proc. Amer. Math. Soc.*, **114** n. 1 (1992) 145-153.

[4] J. L. Lopez and N. M. Temme, Two-point Taylor expansions of analytic functions, *Stud. Appl. Math.*, **109** (2002) 297-311.

[5] J. L. Lopez and N. M. Temme, Multi-point Taylor expansions of analytic functions, *Trans. Amer. Math. Soc.*, **356** n. 11 (2004) 4323-4342.

[6] J. L. Lopez, E. Pérez and N. M. Temme, Multi-point Taylor approximations in one-dimensional linear boundary value problems, *Appl. Math. Comput.*, **207** n. 2 (2009) 519-527.

[7] J. L. Lopez and N. M. Temme, New series expansions of the Gauss hypergeometric function, *Advances in Computational Mathematics*, **39** n.2 (2013) 349-365.

[8] J. L. Lopez, P. Pagola and E. Pérez, New series expansions of the 3F2 function, *J. Math. Anal. Appl.*, **421** n.2 (2015) 982-995.

[9] Y. L. Luke, The Special Functions and their Approximations, Vol I, *Academic Press*, 1969.

[10] E. Nørlund, Hypergeometric functions, *Acta Math.*, **94** (1955) 289-349.

[11] A. B. Olde Daalhuis, Chapter 15, Hypergeometric Function, NIST Handbook of Mathematical Functions, *NIST and Cambridge Univ. Press*, 2010. http://dlmf.nist.gov/15

[12] Z. X. Wang and D. R. Guo, Special Functions, *World Scientific, New York*, 1989.