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Multidistances and inequality measures on abstract sets: An axiomatic approach

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Abstract

Starting from the notion of a multidistance, we formalize, through a suitable system of axioms, the concept of an inequality measure defined on a nonempty set with no additional structure implemented a priori. Among inequality measures, apart from multidistances we pay special attention to dispersions, and study their main features. Classical concepts will be generalized to this abstract setting. Multidistances are then revisited, and some new methods to generate them are implemented. A wide spectrum of interdisciplinary applications is outlined in the final section.

Keywords: Multidistances; Inequality measures on general sets; Axiomatic definitions of means and dispersions

1. Introduction

Many efforts have been focused on achieving axiomatic definitions for general concepts, avoiding a priori assumptions or restrictions. In [6] it is provided an axiomatic definition of the concept of a general mean valid for any nonempty set, with no further assumptions or restrictions imposed a priori. The wide range of applications that come from the corresponding axioms for a general mean, jointly with new applications coming from a geometrical approach ([7]) suggested us to go further, now analyzing in the present manuscript some axiomatics to deal with inequality measures and related items.

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[1] Co-author Javier Martín recently passed away, while this work was in process. The manuscript is dedicated to his memory.

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In the same line, there are some axiomatics for a particular but essential case of inequality measures, namely multidistances. In fact, the abstract concept of a multidistance was introduced in [17], having in mind the idea of generalizing the usual triangle inequality in metric spaces to a higher-dimensional setting. In several earlier works (see [15,22]) terms like n-distances and multimetrics were also introduced in certain contexts, but with a different meaning.

In a way, metric spaces are generalized so that instead of pairwise comparing elements of a nonempty set X through a distance d, the notion of a multidistance works, globally, with n-tuples $(x_1, \dots, x_n)$ of elements that belong to set X, measuring how different or separated are the coordinates of that n-tuple. This is a typical situation in data analysis ([4,5,8]).

Moving to a more general framework, it is also interesting to define on an abstract nonempty set X a suitable concept of inequality measure. Thus, given an n-tuple $(x_1, \dots, x_n) \in X^n$ we wonder how its elements are scattered or concentrated, and how to measure it. It is an important task due to the nature of the different measurements, that are often disparate. Therefore, it is also crucial to distinguish and classify in a systematic way the different kinds of inequality measures (e.g., dispersions, multidistances) that can be considered (see e.g. [14,18,20]). Their properties should also be defined with rigor through a satisfactory axiomatics. Summarizing, the concept of an inequality measure should be as general as possible, to distinguish different features by adding suitable new axioms.

From the concept of an inequality measure it is possible to reach an axiomatics for dispersion measures by just adding a new axiom. Indeed, some axiomatics have already been introduced in the literature in particular contexts, but they use additional structures and are very restrictive (see e.g. [20]).

Thus, with the aim of providing axiomatics on an abstract nonempty set X for inequality measures, the present manuscript is organized accordingly, as follows: In Section 2 we give motivations about the constructions of suitable sets of axioms to deal with means and inequality measures on abstract sets. We introduce the corresponding axioms for general means and, analyzing the ones already given for multidistances, we also give axiomatics for inequality measures and dispersions. In Section 3 we furnish some historical background on inequality measures on the real line, paying attention to some axiomatics encountered in the literature. Then we analyze abstract inequality measures, and their main properties. In Section 4 we focus again on multidistances, introducing new procedures to generate them on metric spaces. Then we conclude with further comments and suggestions for future research.

2. Basic definitions

In [6] it is introduced an axiomatic definition of the concept of a general mean, valid for any nonempty set, with no further assumptions or restrictions imposed a priori. This definition is the following:

Definition 1. Let X be a nonempty set. A sequence of X-valued maps $(M_n)_{n=1}^{\infty}$, where each M_n is defined on the Cartesian product X^n, is said to be a *general mean* if it satisfies the following axioms:

- GM1 (Anonymity-neutrality) $M_n(x_1, \dots, x_n) = M_n(x_{\sigma(1)}, \dots, x_{\sigma(n)})$ for $n \in \mathbb{N}$, $(x_1, \dots, x_n) \in X^n$ and any permutation σ of the set $\{1, \dots, n\}$.
- GM2 (Unanimity) $M_n(x, \dots (n \text{ times}) \dots, x) = x$ for any $n \in \mathbb{N}$ and $x \in X$.
- GM3 (Compatibility) For any $m, n \in \mathbb{N}$ and $(x_1, \dots, x_{m+n}) \in X^{m+n}$, it holds that

$$M_{m+n}(x_1, \dots, x_{m+n}) = M_{m+n}(\bar{x}, \dots (m \text{ times}) \dots, \bar{x}, x_{m+1}, \dots, x_{m+n}),$$

where $\bar{x} = M_m(x_1, \dots, x_m)$.

If $(M_n)_{n=1}^{\infty}$ is a general mean of X, given k elements $\{x_1, \dots, x_k\}$, the element $\bar{x} = M_k(x_1, \dots, x_k) \in X$ is usualy called the *mean* of the elements $x_1, \dots, x_k$.

Once the concept of a general mean has been formalized, the next step to complete the panorama would be to extend the ideas initiated in [6], introducing a new system of axioms to deal with inequality measures, instead of means. In the same line, a set of axioms for multidistances has been introduced in [17] as follows:

Definition 2. Let X be a nonempty set. A sequence of functions $(I_n)_{n=1}^{\infty}$, with $I_n : X^n \to [0, +\infty)$ is said to be a *multidistance* if it satisfies the following axioms:

- MD1 For any $n \in \mathbb{N}$ and $(x_1 \dots, x_n) \in X^n$,

$$I_n(x_1, \dots, x_n) = 0 \Leftrightarrow x_1 = \dots = x_n.$$

  In particular, $I_1(x) = 0$ holds for every $x \in X$.
- MD2 For all $n \in \mathbb{N}$, $(x_1, \dots, x_n) \in X^n$ and a permutation σ of $\{1, \dots, n\}$ it holds that

$$I_n(x_1, \dots, x_n) = I_n(x_{\sigma(1)}, \dots, x_{\sigma(n)}).$$

- MD3 For any $n \in \mathbb{N}$ with $n \geq 2$, and $(x_1, \dots, x_n, y) \in X^{n+1}$ it holds that

$$I_n(x_1, \dots, x_n) \leq \sum_{i=1}^{n} I_2(x_i, y).$$

Moreover, the sequence $(I_n)_{n=1}^{\infty}$ is said to be a *pseudo multidistance* [17] if it satisfies the axioms MD2 and MD3 above and a *quasi multidistance* if it satisfies MD1 and MD3.

3. Inequality measures

In this section different approaches to the concept and general properties of an inequality measure in different contexts are presented. In addition, inequality measures based on means and distances are introduced.

In an abstract level, we look for some idea of inequality or unalikeability ([13]) among the coordinates of n-tuples coming from an abstract set. Thus, an inequality measure is introduced in Definition 3. Notice that only the quite natural and unavoidable conditions are required.

Definition 3. Let X be a nonempty set. A sequence of functions $(I_n)_{n=1}^{\infty}$, with $I_n : X^n \to [0, +\infty)$ is said to be an *inequality measure* if it satisfies:

- IN1 (= MD1) (Identity of indiscernibles) For any $n \in \mathbb{N}$ and $(x_1 \dots, x_n) \in X^n$,

$$I_n(x_1, \dots, x_n) = 0 \Leftrightarrow x_1 = \dots = x_n.$$

- IN2 (= MD2) (Anonymity-neutrality) For every $n \in \mathbb{N}$, $(x_1, \dots, x_n) \in X^n$ and any permutation σ of the set $\{1, \dots, n\}$ it holds that

$$I_n(x_1, \dots, x_n) = I_n(x_{\sigma(1)}, \dots, x_{\sigma(n)}).$$

Obviously, a multidistance is a particular case of an inequality measure. Note that classical concepts such as range, variance and percentiles satisfy the axioms IN1 and IN2 although they also satisfy other desirable properties. Thus, it makes sense to add a new axiom to formalize in an abstract setting the notion of a dispersion measure. Indeed, some axiomatics have already been introduced in the literature in particular contexts (see e.g. [20]), but using an additional structure.

On an abstract nonempty set X, with no structure given a priori, we will consider only three axioms to define a dispersion measure.

Definition 4. Let X be a nonempty set. A sequence of functions $(I_n)_{n=1}^{\infty}$, with $I_n : X^n \to [0, +\infty)$ is said to be a *dispersion measure* if it satisfies:

- DIS1 (= IN1 = MD1) (Identity of indiscernibles).
- DIS2 (= IN2 = MD2) (Anonymity-neutrality).

- DIS3 (Replication invariance) For every $n, m \in \mathbb{N}$ and $(x_1 \dots, x_n) \in X^n$, it holds that

$$I_n(x_1, \dots, x_n) =$$
$$I_{mn}(x_1, \dots (m \text{ times}) \dots, x_1, \dots, x_n, \dots (m \text{ times}) \dots, x_n).$$

Note that it is easy to check that the axioms IN1, IN2, DIS3 and MD3 are independent and consistent.

3.1. Additional axioms to cope with special features of inequality measures

Apart from the four key axioms IN1, IN2, MD3 and DIS3, we recall a few others now that have deserved attention in the specialized literature, in particular when considering multidistances ([16,17,20,21]).

Definition 5. An inequality measure $(I_n)_{n=1}^{\infty}$ on X is said to satisfy:

- FAI (Fairness) if for every $n \in \mathbb{N}$ and every $x, y \in X$ such that $x \neq y$, it holds that

$$I_{n+2}(x, \dots (n+1 \text{ times}) \dots, x, y) < I_{n+1}(x, \dots (n \text{ times}) \dots, x, y).$$

- ACC (Accumulativeness) if for every $n \in \mathbb{N}$ and every $x, y \in X$ such that $x \neq y$, it holds that

$$I_{n+2}(x, \dots (n+1 \text{ times}) \dots, x, y) > I_{n+1}(x, \dots (n \text{ times}) \dots, x, y).$$

- STR (Strength) if for every $n \in \mathbb{N}, n \geq 2$, any $i_0 = 1 < i_1 < \dots < i_k < i_{k+1} = n$ and $(x_1, \dots, x_n) \in X^n$ it holds that

$$I_n(x_1, \dots, x_n) \leq \sum_{j=1}^{k} I_{(i_j - i_{j-1})+(i_{k+1}-i_k)}(x_{i_{j-1}+1}, \dots, x_{i_j}, x_{i_k+1}, \dots, x_{i_{k+1}}),$$

- REG (Regularity) if for every $n \in \mathbb{N}, n \geq 2$, and $(x_1, \dots, x_n, y) \in X^{n+1}$ it holds that

$$I_n(x_1, \dots, x_n) \leq I_{n+1}(x_1, \dots, x_n, y).$$

Remark 1. Fairness does not imply, in general, that for any $n, m \in \mathbb{N}$ and $(x, y_1, \dots, y_m) \neq (x, \dots (m+1 \text{ times}) \dots, x) \in X^{m+1}$ it holds that

$$I_{n+m+1}(x, \dots (n+1 \text{ times}) \dots, x, y_1, \dots, y_m) <$$
$$I_{n+m}(x, \dots (n \text{ times}) \dots, x, y_1, \dots, y_m).$$

As an example, consider $(\sigma_n)_{n=1}^{\infty}$, where $\sigma_n(x_1, \dots, x_n)$ is the sample standard deviation of $\{x_1, \dots, x_n\}$. It is straightforward to prove that it constitutes a fair inequality measure on $\mathbb{R}$. Here we may notice that

$$\sigma_2(0, 100) = 70.71; \sigma_3(0, 100, 100) = 57.73; \sigma_4(0, 100, 100, 100) = 50.$$

But $\sigma_5(0, 0, 100, 100, 100) = 54.77$.

Let us see now some examples concerning inequality measures and the properties they fulfill.

Example 1.

(i) It can be straightforwardly proved that the sequence $(I_n)_{n=1}^{\infty}$, where, for any $n \in \mathbb{N}$ and $(x_1, \dots, x_n) \in \mathbb{R}^n$, the number $I_n(x_1, \dots, x_n)$ is the sample variance (respectively: the sample standard deviation, the mean absolute difference) of the set $\{x_1, \dots, x_n\}$, is indeed a dispersion measure on $\mathbb{R}$, in the sense of Definition 4.

(ii) Assume that X has at least three different elements, say a, b, c. Given $(x_1, \ldots, x_n) \in X^n$ let

$$I_n(x_1, \ldots, x_n) = \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} \delta(x_i, x_j),$$

where $\delta(x, y) = 1$ for every $x \neq y \in X$, and $\delta(z, z) = 0 \;\; (z \in X)$. The sequence $(I_n)_{n=1}^{\infty}$ is an inequality measure on X. But it is not a multidistance since

$$I_3(a, b, c) = \delta(a, b) + \delta(a, c) + \delta(b, c) = 3,$$

but

$$I_2(a, a) + I_2(b, a) + I_2(c, a) = 0 + 1 + 1 = 2.$$

(Similar examples appear in [17,21]).

(iii) Let $(\lambda_n)_{n=1}^{\infty}$ denote a strictly decreasing sequence of positive real numbers. Define $(I_n)_{n=1}^{\infty}$ by declaring that

$$I_n(x, \ldots (n \text{ times}) \ldots, x) = 0$$

for every $x \in X, \;\; n \in \mathbb{N}$ and

$$I_n(x_1, \ldots, x_n) = \lambda_n$$

if $(x_1, \ldots, x_n) \neq (x_1, \ldots (n \text{ times}) \ldots, x_1)$.
A direct checking shows that $(I_n)_{n=1}^{\infty}$ is a strong and fair multidistance that is not regular. Notice also that it fails to accomplish the replication invariance axiom DIS3. Thus, it is not a dispersion measure.

(iv) Consider $I_n(x, \ldots (n \text{ times}) \ldots, x) = 0$ for any $x \in X, \; n \in \mathbb{N}$,
and

$$I_n(x_1, \ldots, x_n) = 1$$

if $(x_1, \ldots, x_n) \neq (x_1, \ldots (n \text{ times}) \ldots, x_1)$.
In this case the sequence $(I_n)_{n=1}^{\infty}$ satisfies axioms IN1 to MD3 as well as STR and REG. However it does not satisfy FAI, nor ACC. (See also [17], p. 94).

(v) Let (X, d) be a metric space (i.e.: X is a nonempty set endowed with a distance d). Define a sequence $(I_n)_{n=1}^{\infty}$ of functions $I_n : X^n \to [0, +\infty)$, as follows:
First we declare that $I_1(x) = 0$ for all $x \in X$;
$I_2(x_1, x_2) = d(x_1, x_2)$ for every $x_1, x_2 \in X$.
Given $(x_1, x_2, x_3) \in X^3$, we define

$$I_3(x_1, x_2, x_3) = \frac{1}{2} \cdot [d(x_2, x_3) + d(x_1, x_3) + d(x_1, x_2)].$$

Given now $(x_1, x_2, x_3, x_4) \in X^4$, define

$$I_4(x_1, x_2, x_3, x_4) = \frac{1}{3} \cdot [I_3(x_2, x_3, x_4) + I_3(x_1, x_3, x_4) + I_3(x_1, x_2, x_4) + I_3(x_1, x_2, x_3)].$$

Recurrently, given $(x_1, x_2, \ldots, x_n) \in X^n$, define

$$I_n(x_1, x_2, \ldots, x_n) = \frac{1}{n-1} \cdot \sum_{i=1}^{n} I_{n-1}(x_1, \ldots, x_{i,-1}, x_{i+1}, \ldots, x_n).$$

By definition, and taking into account that d is a distance, it is straightforward to see that $(I_n)_{n=1}^{\infty}$ satisfies IN1 and IN2.

Now observe that, by induction, given $n \geq 3 \in \mathbb{N}$ we get that, for any $x \in X$, it holds

$$I_n(x_1, x_2, \dots, x_n) = \frac{1}{n-1} \cdot \sum_{i=1}^{n} I_{n-1}(x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_n) \leq$$

$$\frac{1}{n-1} \cdot \sum_{i=1}^{n} [\sum_{j \neq i,\ j=1}^{n} d(x_j, x)] = \frac{1}{n-1} \cdot \sum_{i=1}^{n} (n-1) d(x_i, x) = \sum_{i=1}^{n} d(x_i, x).$$

Therefore $(I_n)_{n=1}^{\infty}$ also satisfies MD3, and consequently it is a multidistance on X.

3.2. Inequality measures based on means and distances

A glance at the classical examples of inequality measures on the real line shows that most of them need some suitable mean defined a priori. Moreover, it is also usual to have at hand some distance (e.g. the Euclidean one on $\mathbb{R}$).

Thus, a distance defined a priori on a nonempty given set X could help our intuition when dealing with inequality measures. In fact, from a distance we can easily define an inequality measure, as the following Proposition 1 states.

Proposition 1. *Let (X, d) denote a metric space. Given $(x_1, \dots, x_n) \in X^n$, define*

$$I_n(x_1, \dots, x_n) = \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} d(x_i, x_j).$$

The sequence $(I_n)_{n=1}^{\infty}$ is an inequality measure on X. In general, neither it is a dispersion measure nor a multidistance.

Proof. It is clear that $(I_n)_{n=1}^{\infty}$ satisfies the axioms IN1 and IN2. Given $a \neq b \in X$, notice that

$$I_4(a, a, b, b) = 4 \cdot d(a, b) > d(a, b) = I_2(a, b).$$

So $(I_n)_{n=1}^{\infty}$ does not satisfy DIS3.

Moreover, if X has three different elements a, b, c, we get

$$I_3(a, b, c) = d(a, b) + d(a, c) + d(b, c) > d(a, b) + d(a, c) =$$
$$0 + d(b, a) + d(c, a) = d(a, a) + d(b, a) + d(c, a).$$

Hence $(I_n)_{n=1}^{\infty}$ does not satisfy MD3, either. □

Remark 2. Given an abstract nonempty set X with no structure defined a priori, we may endow X, by default, with the trivial metric δ defined by $\delta(x, y) = 1 \Leftrightarrow x \neq y$, and $\delta(z, z) = 0$, for all $x, y, z \in X$.

In addition, we could also discuss about the necessity of having some kind of a mean, previously defined on a nonempty given set X. To this extent, we shall consider on X general means, in the sense of Definition 1.

Proposition 2. *Let (X, d) be a metric space that is endowed with a general mean $(M_n)_{n=1}^{\infty}$. Given $(x_1, \dots, x_n) \in X^n$, define*

$$I_n(x_1, \dots, x_n) = \sum_{i=1}^{n} d(x_i, M_n(x_1, \dots, x_n)).$$

The sequence $(I_n)_{n=1}^{\infty}$ is an inequality measure on X. In general, it fails to be a dispersion measure or a multidistance.

Proof. It is clear that $(I_n)_{n=1}^{\infty}$ satisfies IN2.

Moreover $I_n(x_1, \dots, x_n) = 0 \Leftrightarrow d(x_i, M_n(x_1, \dots, x_n)) = 0$ for all $1 \leq i \leq n$. Since d is a distance, this is equivalent to say that $x_i = M_n(x_1, \dots, x_n)$ holds for every $1 \leq i \leq n$. Hence, $(I_n)_{n=1}^{\infty}$ satisfies IN1.

By the properties of a general mean (see Theorem 3.3 in [6]), for every $n, m \in \mathbb{N}$ and $(x_1 \ldots, x_n) \in X^n$, it holds that

$$M_n(x_1, \ldots, x_n) = M_{mn}(x_1, \ldots (m \text{ times}) \ldots, x_1, \ldots, x_n, \ldots (m \text{ times}) \ldots, x_n).$$

Thus, we have that

$$I_{mn}(x_1, \ldots (m \text{ times} \ldots, x_1, \ldots, x_n, \ldots (m \text{ times}) \ldots, x_n) =$$
$$m \cdot \sum_{i=1}^{n} d(x_i, M_n(x_1, \ldots, x_n)) = m \cdot I_n(x_1, \ldots, x_n).$$

Thus, if $m, n > 1$ and $(x_1, \ldots, x_n) \neq (x_1, \ldots (n \text{ times}) \ldots, x_1)$, we see that

$$I_{mn}(x_1, \ldots (m \text{ times}) \ldots, x_1, \ldots, x_n, \ldots (m \text{ times}) \ldots, x_n) \neq I_n(x_1, \ldots, x_n).$$

Therefore $(I_n)_{n=1}^{\infty}$ does not accomplish the axiom DIS3.

To see that, in general, $(I_n)_{n=1}^{\infty}$ may fail to be a multidistance, consider the following example: Let $X = \mathbb{R}$ and $d(a, b) = |a - b|$ $(a, b \in \mathbb{R})$. Endow $\mathbb{R}$ with the mean given by $M_n(x_1, \ldots, x_n) = \max\{x_1, \ldots, x_n\}$ for any $n \in \mathbb{N}$ and $(x_1, \ldots, x_n) \in X^n$. Now observe that

$$I_3(0, 1, 4) = |0 - 4| + |1 - 4| + |4 - 4| = 7,$$

whereas

$$I_2(0, 0) + I_2(1, 0) + I_2(4, 0) = 0 + |1 - 1| + |0 - 1| + |4 - 4| + |0 - 4| = 5.$$

Therefore $(I_n)_{n=1}^{\infty}$ does not satisfy MD3. □

Paying attention to some of the classical measures, we introduce now some definitions to generalize those classical concepts to an abstract setting in which we deal with a metric space X endowed with a general mean.

Definition 6. Let (X, d) be a metric space endowed with a general mean $(M_n)_{n=1}^{\infty}$. Consider a sequence $(I_n)_{n=1}^{\infty}$, where I_n maps X^n into $[0, +\infty)$.

(i) $(I_n)_{n=1}^{\infty}$ is said to be the *variance* of $(x_1, \ldots, x_n) \in X^n$ if

$$I_n(x_1, \ldots, x_n) = \frac{1}{n} \cdot \sum_{i=1}^{n} (d(x_i, M_n(x_1, \ldots, x_n)))^2$$

holds for every $(x_1, \ldots, x_n) \in X^n$,

(ii) $(I_n)_{n=1}^{\infty}$ is called the *average distance* of $(x_1, \ldots, x_n) \in X^n$ provided that

$$I_n(x_1, \ldots, x_n) = \frac{1}{n^2} \cdot \sum_{i=1}^{n} \sum_{j=1}^{n} d(x_i, x_j)$$

holds for any $(x_1, \ldots, x_n) \in X^n$.

Proposition 3. *Let (X, d) be a metric space, $(M_n)_{n=1}^{\infty}$ a general mean defined on X and $(I_n)_{n=1}^{\infty}$ its associated variance. Then $(I_n)_{n=1}^{\infty}$ is a dispersion measure on X. In general, it is not a multidistance.*

Proof. By its own definition it is plain that $(I_n)_{n=1}^{\infty}$ accomplishes IN2.

Furthermore, if $I_n(x_1, \ldots, x_n) = 0$ it follows that $d(x_i, M_n(x_1, \ldots, x_n)) = 0$ holds for every $i = 1, \ldots, n$. Therefore $x_1 = \ldots = x_n = M_n(x_1, \ldots, x_n)$. Conversely, if $x_1 = \ldots = x_n$ holds, then $M_n(x_1, \ldots, x_n) = x_i$ $(i = 1, \ldots, n)$ by GM2. Hence $d(x_i, M_n(x_1, \ldots, x_n)) = 0$ holds for any $i = 1, \ldots, n$. Therefore $I_n(x_1, \ldots, x_n) = 0$ and $(I_n)_{n=1}^{\infty}$ also satisfies IN1.

Finally, using the properties GM1 to GM3 of the general mean we obtain that

$$M_{mn}(x_1, \ldots (m \text{ times}) \ldots, x_1, \ldots, x_n, \ldots (m \text{ times}) \ldots, x_n) =$$
$$M_{mn}(x_1, \ldots, x_n, \ldots (m \text{ times}) \ldots, x_1, \ldots, x_n) =$$
$$M_{mn}(\bar{x}, \ldots (mn \text{ times}) \ldots, \bar{x}) = \bar{x},$$

with $\bar{x} = M_n(x_1, \ldots, x_n)$. Hence, it follows that

$$I_{mn}(x_1, \ldots (m \text{ times}) \ldots, x_1, \ldots, x_n, \ldots (m \text{ times}) \ldots, x_n) =$$
$$\frac{1}{mn}\sum_{i=1}^{n}[m \cdot d(x_i, \bar{x})] = \frac{1}{n}\sum_{i=1}^{n} d(x_i, \bar{x}) = I_n(x_1, \ldots, x_n).$$

So, $(I_n)_{n=1}^{\infty}$ also satisfies the replication invariance axiom DIS3.

Consequently, it constitutes a dispersion measure on X.

To see that $(I_n)_{n=1}^{\infty}$ may fail to be a multidistance, let $X = \mathbb{R}$ and $d(a, b) = |a - b|$ $(a, b \in \mathbb{R})$. As in Proposition 2, we consider the general mean

$$M_n(x_1, \ldots, x_n) = \max\{x_1, \ldots, x_n\}$$

for any $n \in \mathbb{N}$ and $(x_1, \ldots, x_n) \in X^n$. Now notice that

$$I_3(0, 1, 5) = \frac{1}{3} \cdot (|0 - 5|^2 + |1 - 5|^2 + |5 - 5|^2) = \frac{41}{3},$$

whereas

$$I_2(0, 0) + I_2(1, 0) + I_2(5, 0) = \frac{1}{2} \cdot (0 + |1 - 1|^2 + |0 - 1|^2 + |5 - 5|^2 + |0 - 5|^2) = 13.$$

But $\frac{41}{3} > 13$. Therefore $(I_n)_{n=1}^{\infty}$ does not satisfy MD3. □

Remark 3. The dispersion measure considered in Proposition 3 may fail to be fair. Consider the following example: Let $X = \mathbb{R}$ and let δ be the trivial distance. Endow $\mathbb{R}$ with M_n. Notice that $I_2(0, 1) = 1 = I_3(0, 0, 1) = I_4(0, 0, 0, 1)$, and so on. Hence $(I_n)_{n=1}^{\infty}$ does not satisfy the condition of fairness.

Proposition 4. *Let (X, d) be a metric space. Define*

$$I_n(x_1, \ldots, x_n) = \frac{1}{n^2} \cdot \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} [d(x_i, x_j)]^2$$

for any $(x_1, \ldots, x_n) \in X^n$. Then $(I_n)_{n=1}^{\infty}$ is a fair dispersion measure. However, it may fail to be a multidistance.

Proof. By definition, $(I_n)_{n=1}^{\infty}$ satisfies IN2.

The fact $I_n(x_1, \ldots, x_n) = 0$ implies that $d(x_i, x_j) = 0$ holds for every $i < j$ with $i, j \in \{1, \ldots, n\}$. Hence $x_1 = \ldots = x_n$. Conversely, if $x_1 = \ldots = x_n$, we have that $d(x_i, x_j) = 0$ holds for all $i, j \in \{1, \ldots, n\}$ and $I_n(x_1, \ldots, x_n) = 0$. Thus $(I_n)_{n=1}^{\infty}$ also accomplishes IN1.

Moreover, given $m, n \in \mathbb{N}$ it is easy to see that

$$I_{mn}(x_1, \ldots (m \text{ times}) \ldots, x_1, \ldots, x_n, \ldots (m \text{ times}) \ldots, x_n) =$$
$$\frac{1}{m^2n^2}\sum_{i=1}^{n-1}\sum_{j=i+1}^{n} m^2 \cdot (d(x_i, x_j))^2 = \frac{1}{n^2}\sum_{i=1}^{n-1}\sum_{j=i+1}^{n} (d(x_i, x_j))^2 = I_n(x_1, \ldots, x_n).$$

Thus, $(I_n)_{n=1}^{\infty}$ satisfies DIS3 and is a dispersion measure.

Finally, given $n \in \mathbb{N}$ and $x \neq y \in X$, it follows that

$$I_{n+2}(x, \ldots (n + 1 \text{ times}) \ldots, x, y) = \frac{n + 1}{(n + 2)^2} \cdot (d(x, y))^2.$$

But $\frac{n+1}{(n+2)^2} < \frac{n}{(n+1)^2}$. Thus

$$\frac{n+1}{(n+2)^2} \cdot (d(x,y))^2 < \frac{n}{(n+1)^2} \cdot (d(x,y))^2 = I_{n+1}(x, \ldots (n \text{ times}) \ldots, x, y).$$

So $(I_n)_{n=1}^{\infty}$ is fair.

To prove that $(I_n)_{n=1}^{\infty}$ may fail to be a multidistance, consider on $\mathbb{R}$ the usual metric. Notice then that $I_2(0,10) = \frac{100}{4} = 25$, whilst

$$I_2(0,5) + I_2(10,5) = \frac{25}{4} + \frac{25}{4} = \frac{25}{2}.$$

So $(I_n)_{n=1}^{\infty}$ does not satisfy MD3. □

4. Multidistances revisited

In this section we go further on the study of some general results related to multidistances.

4.1. Further results on multidistances

Proposition 5. *Let (X,d) be a metric space and $(I_n)_{n=1}^{\infty}$ denote its average distance. Then $(I_n)_{n=1}^{\infty}$ is a fair dispersion measure and a multidistance.*

Proof. To prove that $(I_n)_{n=1}^{\infty}$ is a fair dispersion measure, first observe that

$$\frac{1}{n^2} \cdot \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} d(x_i, x_j) = \frac{1}{2} \cdot \frac{1}{n^2} \cdot \sum_{i=1}^{n} \sum_{j=1}^{n} d(x_i, x_j) = \frac{1}{2} I_n(x_1, \ldots, x_n).$$

Then repeat the arguments in the proof of Proposition 4, but using always $d(x_i, x_j)$ instead of $(d(x_i, x_j))^2$. Given $(x_1, \ldots, x_n, y) \in X^{n+1}$, we have that

$$I_n(x_1, \ldots, x_n) \leq \frac{2}{n^2} \cdot \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} [d(x_i, y) + d(x_j, y)] = \frac{2n-2}{n^2} \cdot \sum_{i=1}^{n} d(x_i, y) = \frac{4n-4}{n^2} \cdot \sum_{i=1}^{n} I_2(x_i, y).$$

If $n > 2$ we have that $4n - 4 < n^2$, so that

$$I_n(x_1, \ldots, x_n) \leq \sum_{i=1}^{n} I_2(x_i, y).$$

If $n = 2$, by definition of I_2 it is clear that $I_2(x_1, x_2) \leq I_2(x_1, y) + I_2(x_2, y)$ because d is a distance. Finally, if $n = 1$, $I_1(x_1) = 0 \leq I_2(x_1, y) = \frac{d(x_1,y)}{2}$. Thus we conclude that $(I_n)_{n=1}^{\infty}$ satisfies the generalized triangle inequality MD3, too. So it is a multidistance on X. □

Definition 7. Let (X,d) denote a metric space. Let $\alpha = (\alpha_n)_1^{\infty}$ stand for a sequence of positive real numbers. Consider the sequence of functions $(I_n)_{n=1}^{\infty}$, with I_n defined on X^n and taking values on $[0, +\infty)$, given by:

(i) $I_1(x) = 0$ for any $x \in X$,
(ii) $I_n(x_1, \ldots, x_n) = \alpha_n \cdot \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} d(x_i, x_j)$, for all $(x_1, \ldots, x_n) \in X^n$.

This sequence $(I_n)_{n=1}^{\infty}$ is said to be the *basic sequence* associated to α and α is said to be the *generator* of this basis sequence.

We will see at the next proposition that, under some conditions on the generator, we can obtain a multidistance from any distance d such that it coincides with d for the case $n = 2$ just considering the basic sequence associated to the generator and the distance. Notice that d will not be any given distance fixed a priori, but, instead, the property holds true for every distance d (all at the same time, with the same construction of the corresponding basic sequence of functions). This nuance is crucial.

Proposition 6. *Let X be a set with at least two different points. Then, for every distance d defined on X, the basic sequence of functions $(I_n)_{n=1}^{\infty}$ associated to a generator α and the distance d is a multidistance such that I_2 agrees with d if, and only if, $\alpha_2 = 1$ and $0 < \alpha_n \le \frac{1}{n-1}$ $(n \ge 2)$ hold.*

Proof. Suppose that $(I_n)_{n=1}^{\infty}$ is a multidistance for any metric d on X, and $I_2(x_1, x_2) = d(x_1, x_2)$ for all $(x_1, x_2) \in X^2$. In particular, for any given $x_1 \neq x_2 \in X$ we have that $0 < d(x_1, x_2) = I_2(x_1, x_2) = \alpha_2 d(x_1, x_2)$. So $\alpha_2 = 1$.

Let us suppose that $\alpha_k > \frac{1}{k-1}$ holds for some $k \ge 2$. From the statement of this result, we have that the property on these constructions must hold for every distance d, all at the same time, so it must hold also for the trivial distance δ.

Given $x_1 \neq x_2 \in X$ we get

$$I_k(x_1, \ldots (k-1 \text{ times}) \ldots x_1, x_2) = \alpha_k \cdot [(k-1)\delta(x_1, x_2)] = (k-1)\alpha_k > 1.$$

However, since $(I_n)_{n=1}^{\infty}$ is a multidistance,

$$I_k(x_1, \ldots (k-1 \text{ times}) \ldots x_1, x_2) \le$$
$$\delta(x_1, x_1) + \ldots (k-1 \text{ times}) \ldots \delta(x_1, x_1) + \delta(x_1, x_2) = 1.$$

This leads to a contradiction, so $\alpha_n \le \frac{1}{n-1}$ holds for every $n > 2$.

Moreover, if $\alpha_k \le 0$, given $x_1 \neq x_2 \in X$ we get that

$$I_k(x_1, \ldots (k-1 \text{ times}) \ldots x_1, x_2) = \alpha_k \cdot [(k-1)d(x_1, x_2)].$$

Since $d(x_1, x_2) > 0$ because d is a distance, it follows that

$$I_k(x_1, \ldots (k-1 \text{ times}) \ldots x_1, x_2) \le 0.$$

Actually, since I_k takes values in $[0, +\infty)$ we arrive at

$$I_k(x_1, \ldots (k-1 \text{ times}) \ldots x_1, x_2) = 0.$$

This contradicts the fact of $(I_n)_{n=1}^{\infty}$ satisfying IN1.

For the converse observe that, given $n \in \mathbb{N}, n \ge 2$, it holds that

$$\sum_{i=1}^{n-1} \sum_{j=i+1}^{n} d(x_i, x_j) = \frac{1}{2} \cdot \sum_{i,j=1}^{n} d(x_i, x_j),$$

and this expression does not depend on the order in which the coordinates x_i $(i = 1, \ldots, n)$ appear in the n-tuple $(x_1, \ldots, x_n)$. Thus the basic sequence of functions satisfies the unanimity-anonymity axiom IN2.

Given $x_1, x_2 \in X$ we observe now that

$$I_2(x_1, x_2) = \alpha_2 \cdot d(x_1, x_2) = d(x_1, x_2),$$

because $\alpha_2 = 1$. Notice also that, by definition,

$$I_n(x_1, \ldots, x_n) = 0 \Leftrightarrow x_1 = \ldots = x_n,$$

since d is a distance and $\alpha_n > 0$ for every $n \ge 2$. Thus the basic sequence accomplishes the identity of indiscernibles IN1.

Finally, given $n \in \mathbb{N}$ and $(x_1, \ldots, x_n, y) \in X^{n+1}$, we have that

$$I_n(x_1, \ldots, x_n) = \alpha_n \cdot \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} d(x_i, x_j).$$

Given $i, j \in \{1, \ldots, n\}$ we have that $d(x_i, x_j) \leq d(x_i, y) + d(x_j, y)$ since d satisfies the triangle inequality. Therefore, since $\alpha_n \leq \frac{1}{n-1}$, we get:

$$I_n(x_1, \ldots, x_n) \leq \alpha_n \cdot \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} [d(x_i, y) + d(x_j, y)] =$$

$$\alpha_n \cdot (n-1) \cdot \sum_{i=1}^{n} d(x_i, y) \leq \sum_{i=1}^{n} d(x_i, y).$$

Hence $(I_n)_{n=1}^{\infty}$ satisfies the generalized triangle inequality MD3, too. So it is a multidistance. □

Remark 4. Each point that belongs to

$$F(x_1, \ldots, x_n) = \{x \in X : \sum_{i=1}^{n} d(x_i, x) \leq \sum_{i=1}^{n} d(x_i, a) \text{ for every } a \in X\}$$

is said to be a *Fermat point* of the set $\{x_1, \ldots, x_n\}$ Fermat points may fail to exist: Consider on the plane $\mathbb{R}^2$ the points $x_1 = (0, 0)$, $x_2 = (1, 0)$ and $x_3 = (\frac{1}{2}, \frac{\sqrt{3}}{2})$. These points are the vertices of an equilateral triangle whose barycenter is the point $B = (\frac{1}{2}, \frac{\sqrt{3}}{6})$. It is well-known that, in the real plane $\mathbb{R}^2$ endowed with the usual Euclidean distance, B is also the Fermat point of the set $\{x_1, x_2, x_3\}$. However, if we consider $X = \mathbb{R}^2 \setminus \{B\}$, endowed with the restriction of the usual distance (i.e.: $d(a, b) = |a - b| \quad (a, b \in X)$), now the set $\{x_1, x_2, x_3\}$ has no Fermat point in X. As a consequence, the statement of Proposition 2 in [17] –whose proof is omitted in that paper– is false in general. In order for Fermat points to exist on a metric space (X, d) we need some additional topological conditions (e.g. compactness). In addition, a Fermat point, if any, is not unique in general. For instance, on $X = [0, 1] \subset \mathbb{R}$ endowed with the usual distance, any x such that $0 \leq x \leq 1$ is a Fermat point of the set $\{x_1 = 0, \ x_2 = 1\}$.

Proposition 7. *Let (X, d) be a compact metric space. Then any function $f : X \to \mathbb{R}$ that is continuous with respect to metric topology on X and the usual on $\mathbb{R}$, attains its maximum and minimum. Consequently, Fermat points exist for any $n \in \mathbb{N}$, $(x_1, \ldots, x_n) \in X^n$.*

Proof. For the existence of extrema, see e.g. [11], Theorem 2.3 on p. 227. Then notice that the map $f : X \to \mathbb{R}$ given by $f(x) = d(x_1, x) + \ldots + d(x_n, x)$ is continuous (see e.g. [11], Theorem 4.3 on p. 185). □

Proposition 8. *Let (X, d) be a compact metric space. Given $(x_1, \ldots, x_n) \in X^n$, define*

$$I_n(x_1, \ldots, x_n) = \sum_{i=1}^{n} d(x_i, F_n(x_1, \ldots, x_n)),$$

where $F_n(x_1, \ldots, x_n)$ is a Fermat point of $\{x_1, \ldots, x_n\}$. Then $(I_n)_{n=1}^{\infty}$ is a multidistance on X, and I_2 coincides with d. However, $(I_n)_{n=1}^{\infty}$ is not a dispersion measure, in general.

Proof. The fact of $(I_n)_{n=1}^{\infty}$ being an inequality measure can be proved as in Proposition 2.

Concerning MD3, for any given $(x_1, \ldots, x_n, y) \in X^{n+1}$ notice that considering the definition of a Fermat point

$$\sum_{i=1}^{n} d(x_i, F_n(x_1, \ldots, x_n)) \leq \sum_{i=1}^{n} d(x_i, y).$$

Therefore $(I_n)_{n=1}^{\infty}$ is actually a multidistance on X.

Moreover, given $x_1, x_2 \in X$ we have that $d(x_1, x_2) \leq d(x_1, z) + d(x_2, z)$ for every $z \in X$, since d is a distance. This is true, in particular, for any Fermat point of the set $\{x_1, x_2\}$. Hence $d(x_1, x_2) \leq I_2(x_1, x_2)$. Also, $I_2(x_1, x_2) = \min_{z \in X}[d(x_1, z) + d(x_2, z)] \leq d(x_1, x_2) + d(x_2, x_2) = d(x_1, x_2)$. Therefore I_2 coincides with d.

To see that, in general, $(I_n)_{n=1}^{\infty}$ is not a dispersion measure, consider the unit interval $[0, 1] \subset \mathbb{R}$ endowed with the usual distance d. Now notice that $I_2(0, 1) = 1$ but $I_4(0, 0, 1, 1) = 2$. So $(I_n)_{n=1}^{\infty}$ does not satisfy DIS3. □

Definition 8. Let (X, d) be a compact metric space. The multidistance $(I_n)_{n=1}^{\infty}$ defined in Proposition 8 is said to be the *Fermat multidistance* associated to d.

Remark 5. The concept of a Fermat multidistance was launched and analyzed in Section 3 of [17].

4.2. Generating multidistances on metric spaces

Given a multidistance $(I_n)_{n=1}^{\infty}$ on a nonempty set X we immediately notice that I_2 is a distance on X (see also [17]). Due to this fact, in this section we will start with a metric space (X, d), and directly from suitable manipulations that involve the metric d we will define multidistances $(I_n)_{n=1}^{\infty}$ on X such that, unless otherwise stated, will satisfy that I_2 agrees with d. Several studies have already been done to study how a multidistance on a metric space (X, d) can be expressed through functions that directly depend on the given distance d. (See [19] for a further account).

4.2.1. From classical inequalities to multidistances

Taking into account several classical inequalities (see Lemma 2 below) we may generate new multidistances on a metric space (X, d).

Lemma 1. *(see [17]) Define, for all $n \in \mathbb{N}$ and $(x_1, \ldots, x_n) \in X^n$,*

$$I_n(x_1, \ldots, x_n) = \max\{d(x_i, x_j) : 1 \leq i, j \leq n\}.$$

Then the sequence $(I_n)_{n=1}^{\infty}$ is a multidistance on X.

Lemma 2. *(Classical inequalities) For any real number $p \geq 1$ it holds that*

$$\max\{d(x_i, x_j) : 1 \leq i, j \leq n\} \leq [\sum_{i=1}^{n-1} \sum_{j=i+1}^{n} (d(x_i, x_j))^p]^{\frac{1}{p}}$$
$$\leq [\frac{n(n-1)}{2}]^{\frac{1}{p}} \cdot \max\{d(x_i, x_j) : 1 \leq i, j \leq n\}.$$

Proposition 9. *Define, for all $n \in \mathbb{N}$ and $(x_1, \ldots, x_n) \in X^n$,*

$$I_n(x_1, \ldots, x_n) = [\frac{2}{n(n-1)}]^{\frac{1}{p}} \cdot [\sum_{i=1}^{n-1} \sum_{j=i+1}^{n} (d(x_i, x_j))^p]^{\frac{1}{p}}.$$

Then the sequence $(I_n)_{n=1}^{\infty}$ is a multidistance on X.

Proof. It is straightforward to see that $(I_n)_{n=1}^{\infty}$ accomplishes IN1 and IN2.

In addition, I_2 coincides with d, and $I_1(x) = 0$ holds for every $x \in X$.

To see that $(I_n)_{n=1}^{\infty}$ also satisfies MD3, notice that for any $x \in X$ and $n \in \mathbb{N}, n \geq 2$ we have, because d is a distance, that

$$I_n(x_1, \ldots, x_n) \leq \max\{d(x_i, x_j) : 1 \leq i, j \leq n\} \leq$$
$$\max\{d(x_i, x) + d(x_j, x) : 1 \leq i, j \leq n\} \leq \sum_{i=1}^{n} d(x_i, x). \quad \square$$

4.2.2. From product distances to multidistances

A metric d defined on a set X generates new distances on the Cartesian products $X^n \quad (n \in \mathbb{N})$. They are called *product distances*. (See e.g. [10], or [11], pp. 189-191). To put just an example, for every $p \geq 1$, given $n \in \mathbb{N}$ the map $d_{n,p} : X^n \times X^n \to [0, +\infty)$ given by

$$d_{n,p}[(x_1, \ldots, x_n), (y_1, \ldots, y_n)] = [\sum_{i=1}^{n} (d(x_i, y_i))^p]^{\frac{1}{p}},$$

for every $(x_1,\dots,x_n),\ (y_1,\dots,y_n)\in X^n$ is indeed a product distance on X^n.

Definition 9. Given a metric space (X,d), a sequence $(d_n)_{n=1}^{\infty}$ of product distances, such that each d_n is a distance on X^n $(n\in\mathbb{N})$, and $d_1=d$, is said to be *neat* if for every $n\in\mathbb{N}$, $x\in X$ and $(x_1,\dots,x_n)\in X^n$ it holds that

$$d_n[(x_1,\dots,x_n),(x,\dots(n \text{ times})\dots,x)]\le\sum_{i=1}^{n}d(x_i,x).$$

Example 2. Let (X,d) be a metric space.

Given $n\in\mathbb{N}$ and $(x_1,\dots,x_n),(y_1,\dots,y_n)\in X^n$, define

$$d_n[(x_1,\dots,x_n),(y_1,\dots,y_n)]=\sum_{i=1}^{n}\frac{d(x_i,y_i)}{2^{i-1}}.$$

This sequence of product distances $(d_n)_{n=1}^{\infty}$ is obviously neat.

Definition 10. Given a metric space (X,d), a sequence $(d_n)_{n=1}^{\infty}$ of product distances is said to be *symmetric* if for every $n\in\mathbb{N}$, $x\in X$ and $(x_1,\dots,x_n),(y_1,\dots,y_n)\in X^n$ it holds that, for any permutation σ of the set $\{1,\dots,n\}$,

$$d_n[(x_1,\dots,x_n),(y_1,\dots,y_n)]=d_n[(x_{\sigma(1)},\dots,x_{\sigma(n)}),(y_{\sigma(1)},\dots,y_{\sigma(n)})].$$

Example 3. In general, a sequence $(d_n)_{n=1}^{\infty}$ of product distances may fail to be symmetric. For instance, given $X=\mathbb{R}$, $n\in\mathbb{N}$ and $(x_1,\dots,x_n),(y_1,\dots,y_n)\in\mathbb{R}^n$, define

$$\begin{aligned}d_n[(x_1,\dots,x_n),&(y_1,\dots,y_n)]=\sum_{\{1\le i\le n,\ i\ is\ odd\}}d(x_i,y_i)+\\&[\sum_{\{1\le j\le n,\ j\ is\ even\}}[d(x_j,y_j)]^2]^{\frac{1}{2}}.\end{aligned}$$

Notice that $d_4[(1,0,1,0),(1,1,1,1)]=\sqrt{2}$, but $d_4[(0,1,0,1),(1,1,1,1)]=2$.

Remark 6. Given a sequence $(d_n)_{n=1}^{\infty}$ of product distances on a metric space (X,d), define now for any $n\in\mathbb{N}$ and $(x_1,\dots,x_n),(y_1,\dots,y_n)\in X^n$,

$$\begin{aligned}&d_n^*[(x_1,\dots,x_n),(y_1,\dots,y_n)]=\\&\frac{1}{n!}\cdot\sum_{\sigma\in S(n)}d_n[(x_{\sigma(1)},\dots,x_{\sigma(n)}),(y_{\sigma(1)},\dots,y_{\sigma(n)})],\end{aligned}$$

where $S(n)$ stands for the set of permutations of $\{1,\dots,n\}$. Observe now that $(d_n^*)_{n=1}^{\infty}$ is a symmetric sequence of product distances on X.

Proposition 10. *Let $(d_n)_{n=1}^{\infty}$ be a neat and symmetric sequence of product distances on a metric space (X,d). Let the sequence of functions $(I_n)_{n=1}^{\infty}$, where $I_n:X^n\to[0,+\infty)$, be defined as $I_1(x_1)=0,\ I_2(x_1,x_2)=d(x_1,x_2)$ and*

$$I_n(x_1,\dots,x_n)=\inf\{d_n[(x_1,\dots,x_n),(x,\dots(n\ times)\dots,x)]:x\in X\}$$

for any $n\ge 3\in\mathbb{N}$ and $(x_1,\dots,x_n,x)\in X^{n+1}$. Then the sequence $(I_n)_{n=1}^{\infty}$ is a multidistance on X.

Proof. Since $(d_n)_{n=1}^{\infty}$ is symmetric, it is clear that $(I_n)_{n=1}^{\infty}$ satisfies IN2.

Moreover, given $(a,b)\in X^2$, we have that $d(a,b)=0\Leftrightarrow a=b$ because d is a distance on X. If $n\ge 3$ we have that $I_n(x_1,\dots,x_n)=0$ implies that $(x_1,\dots,x_n)$ belongs to the closure of the diagonal $\Delta^n=\{(x,\dots(n \text{ times})\dots,x):x\in X\}\subset X^n$. But the diagonal Δ^n in a Cartesian product X^n of n copies of a given metric space X is a closed set as regards the product distance. (See e.g. [11], p. 138 or [12], p. 65). So $I_n(x_1,\dots,x_n)=0\Leftrightarrow(x_1,\dots,x_n)\in\Delta^n\Leftrightarrow x_1=\ldots=x_n$ and IN1 holds.

Furthermore, given $n \geq 3$, for every $t \in X$ we have that

$$I_n(x_1, \dots, x_n) = \inf\{d_n[(x_1, \dots, x_n), (x, \dots (n \text{ times}) \dots, x)] : x \in X\} \leq \sum_{i=1}^{n} d(x_i, t)$$

since $(d_n)_{n=1}^{\infty}$ is neat. Therefore, $(I_n)_{n=1}^{\infty}$ also satisfies MD3. □

4.2.3. Multidistances based on rearrangements and weights

Let (X, d) denote a metric space. Given $n \geq 3 \in \mathbb{N}$ consider a fixed set $W_n = \{w_{n,1}, w_{n,2}, \dots, w_{n,n}\}$ of real numbers, called *n-weights*, such that

$$0 \leq w_{n,i} \ (1 \leq i \leq n); \ \sum_{i=1}^{n} w_{n,i} > 0.$$

Now we define a sequence $(I_n)_{n=1}^{\infty}$ of maps $I_n : X^n \to [0, +\infty)$. To do so, we declare that $I_1(x_1) = 0$ and also $I_2(x_1, x_2) = d(x_1, x_2)$ for every $x_1, x_2 \in X$. Then, recurrently, for every $n \geq 3$, $(x_1, \dots, x_n) \in X^n$, and $1 \leq i \leq n$, let

$$t_{n,i} = I_{n-1}(x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_n),$$

where $(x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_n)$ stands for the $n-1$-tuple that remains from $(x_1, \dots, x_n)$ after suppressing the i-th coordinate x_i. Once we get the n-tuple of real numbers $(t_{n,1}, \dots, t_{n,n})$ we rearrange its coordinates in increasing order, so getting a new n-tuple $(t_{n,\sigma(1)}, \dots, t_{n,\sigma(n)})$ in which

$$t_{n,\sigma(1)} \leq t_{n,\sigma(2)} \leq \dots \leq t_{n,\sigma(n)}.$$

Now, we define

$$I_n(x_1, x_2, \dots, x_n) = \sum_{i=1}^{n} w_{n,i} \cdot t_{n,\sigma(i)} \quad .$$

Proposition 11. *With the above construction, if $w_{n,i} \leq \frac{1}{n-1}$ holds for every $n \geq 3 \in \mathbb{N}$; $1 \leq i \leq n$, the sequence $(I_n)_{n=1}^{\infty}$ is a pseudo multidistance on X. If, in addition, for any $n \geq 3$ there exist i, j with $1 \leq i \neq j \leq n$ such that $0 < w_{n,i}$ and also $0 < w_{n,j}$, then $(I_n)_{n=1}^{\infty}$ is a multidistance.*

Proof. It is straightforward to see that $(I_n)_{n=1}^{\infty}$ satisfies IN2 by its own definition.

In addition, for any $x_1, x_2, x \in X$ and $n = 1, 2$ we have that $I_1(x_1) = 0 \leq d(x_1, x)$ and also $I_2(x_1, x_2) = d(x_1, x_2) \leq d(x_1, x) + d(x_2, x)$. Assume now, by induction, that for some $k \geq 2$ it holds that $I_k(x_1, \dots, x_k) \leq \sum_{i=1}^{k} d(x_i, x)$ holds for any $(x_1, \dots, x_k, x) \in X^{k+1}$.

Let us prove that now, for any $x_{k+1} \in X$,

$$I_{k+1}(x_1, \dots, x_k, x_{k+1}) \leq \sum_{i=1}^{k+1} d(x_i, x)$$

also holds. Indeed,

$$I_{k+1}(x_1, \dots, x_k, x_{k+1}) = \sum_{i=1}^{k+1} (w_{k+1,i} \cdot t_{k+1,\sigma(i)}) \leq \sum_{i=1}^{k+1} (w_{k+1,i} \cdot [\sum_{j \neq i, j=1}^{k+1} d(x_j, x)]) \leq$$
$$\frac{1}{k} \cdot \sum_{i=1}^{k+1} [\sum_{j \neq i, j=1}^{k+1} d(x_j, x)] = \frac{1}{k} \cdot \sum_{i=1}^{k+1} [k \cdot d(x_i, x)] = \sum_{i=1}^{k+1} d(x_i, x).$$

So, $(I_n)_{n=1}^{\infty}$ accomplishes MD3, and it is a pseudo multidistance.

Finally, assume now that for any $n \geq 3$ there exist i, j with $1 \leq i \neq j \leq n$ such that $0 < w_{n,i}$ and also $0 < w_{n,j}$. If $n = 2$ and $I_2(x_1, x_2) = 0$, then $d(x_1, x_2) = 0$ so $x_1 = x_2$ because d is a metric. Let us prove, by induction, that if

$$I_k(x_1,\ldots,x_k)=0 \Leftrightarrow x_1=\ldots=x_k$$

holds for some $k\in\mathbb{N}$ then

$$I_{k+1}(y_1,\ldots,y_k,y_{k+1})=0 \Leftrightarrow y_1=\ldots=y_k=y_{k+1}$$

also holds.

To do so, assume now that $k\geq 2$ and $I_{k+1}(y_1,\ldots,y_{k+1})=0$. In that case we have that there exist $i\neq j$ such that

$$I_k(y_1,\ldots,\hat{y}_i,\ldots,x_n)=I_{n-1}(x_1,\ldots,\hat{x}_j,\ldots,x_n)=0.$$

By induction, all the coordinates in the tuples $(x_1,\ldots,x_{i-1},x_{i+1},\ldots,y_{k+1})$ and $(y_1,\ldots,y_{j-1},y_{j+1},\ldots,y_{k+1})$ coincide. Therefore $y_1=y_2=\ldots=y_{k+1}$. Conversely, if $x_1=\ldots=x_n$, then $I_n(x_1,\ldots,x_n)=0$, by definition of I_n. Hence $(I_n)_{n=1}^{\infty}$ satisfies IN1, and we may conclude that it is actually a multidistance. □

5. Final remarks

The introduction of an axiomatic approach to deal with the concept of inequality on abstract sets allows us to deal with generalizations of classical concepts. When working with abstract sets, the lack of a structure defined a priori forces us to state suitable axioms to define what is an inequality measure. Among them, dispersion measures and multidistances play important and independent roles. In the present paper we furnish a concise axiomatics to define inequality measures, dispersions and multidistances. We show that the corresponding axioms are independent one-another. Then we analyze inequality measures that come from a distance, so assuming that the given set was a metric space. In addition, we also pay a particular attention to general means defined on a nonempty abstract set. The use of distances and general means allows us to generalize in a natural way several classical dispersion measures arising in Statistics, as well as to define some suitable multidistances, as the sum-based ones or the Fermat multidistances. A revision of some previous results recently introduced has been made accordingly.

It is also important to remark that some possible applications of this setting are possible. In the context of Image Processing dispersion measures or multidistances can be used for comparison issues. For instance, when analyzing a sequence of pictures in a movie, so that they only differ a little from each one to the next frame, so giving the impression of motion when they are sequentially sent to the screen at the suitable speed for our eye and brain not to distinguish from one another and perceive the sequence as something in motion.

Multidistances constitute an important tool in real-life situations in order to build networks and systems of communication (e.g.: transmitters and receptors). For example, to locate a transmitter providing signal to different points it is natural to choose the Fermat point, because the total energy, understood as the sum of distances or routes operated by the signal is kept to a minimum (see [7]). Again, this minimum value could indeed be given by means of a suitable multidistance. The Steiner's problem and the traveling salesman are problems also related to the Fermat point [3,9]. These problems can also be analyzed in terms of a multidistance: the shortest path to be traveled by the salesman who should visit k towns, as well as the total length of the minimal system of roads joining n points can be possibly understood as a multidistance among those k towns or n points.

Economics is another field of application as concepts of poverty and social welfare are defined in terms of inequalities (see e.g. [1,2]). In general, multidistances can be applied to model or solve problems involving comparison of sets [11].

Once the abstract definition of inequality measures, dispersions and multidistances have been given it seems interesting to search for new examples that are not based on previously defined distances or general means. Indeed, in the particular case of multidistances (see e.g. [17,21]) most of the classes analyzed depend on the definition, a priori, of a distance on the given set, so assumed to be a metric space. Thus, it could be interesting to define new classes of inequality measures that, at least a priori, do not depend on any distance and are defined "per se".

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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