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STRUCTURE OF THE LIPSCHITZ FREE p-SPACES $\mathcal{F}_p(\mathbb{Z}^d)$ AND $\mathcal{F}_p(\mathbb{R}^d)$ FOR $0 < p \leq 1$

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Abstract. Our aim in this article is to contribute to the theory of Lipschitz free p-spaces for $0 < p \leq 1$ over the Euclidean spaces $\mathbb{R}^d$ and $\mathbb{Z}^d$. To that end, on one hand we show that $\mathcal{F}_p(\mathbb{R}^d)$ admits a Schauder basis for every $p \in (0, 1]$, thus generalizing the corresponding result for the case $p = 1$ by Hájek and Pernecká [20, Theorem 3.1] and answering in the positive a question that was raised in [3]. Explicit formulas for the bases of both $\mathcal{F}_p(\mathbb{R}^d)$ and its isomorphic space $\mathcal{F}_p([0, 1]^d)$ are given. On the other hand we show that the well-known fact that $\mathcal{F}(\mathbb{Z})$ is isomorphic to ℓ_1 does not extend to the case when $p < 1$, that is, $\mathcal{F}_p(\mathbb{Z})$ is not isomorphic to ℓ_p when $0 < p < 1$.

1. Introduction and background

Suppose $0 < p \leq 1$. Given a pointed p-metric space $\mathcal{M}$ it is possible to construct a unique p-Banach space $\mathcal{F}_p(\mathcal{M})$ in such a way that $\mathcal{M}$ embeds isometrically in $\mathcal{F}_p(\mathcal{M})$ via a canonical map denoted $\delta_{\mathcal{M}}$, and for every p-Banach space X and every Lipschitz map $f\colon \mathcal{M} \to X$ with Lipchitz constant $\operatorname{Lip}(f)$ that maps the base point 0 in $\mathcal{M}$ to $0 \in X$ extends to a unique linear bounded map $T_f\colon \mathcal{F}_p(\mathcal{M}) \to X$ with $\|T_f\| = \operatorname{Lip}(f)$. The space $\mathcal{F}_p(\mathcal{M})$ is known as the *Lipschitz free p-space* over $\mathcal{M}$. This class of p-Banach spaces provides a canonical linearization process of Lipschitz maps between p-metric spaces: any Lipschitz map f from a p-metric space $\mathcal{M}_1$ to a p-metric space $\mathcal{M}_2$

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which maps the base point in $\mathcal{M}_1$ to the base point in $\mathcal{M}_2$ extends to a continuous linear map $L_f\colon \mathcal{F}_p(\mathcal{M}_1) \to \mathcal{F}_p(\mathcal{M}_2)$ with $\|L_f\| \le \operatorname{Lip}(f)$.

Lipschitz free p-spaces were introduced in [6], where they were used to provide for every for $0 < p < 1$ a couple of *separable* p-Banach spaces which are Lipschitz-isomorphic without being linearly isomorphic. These spaces constitute a new family of p-Banach spaces which are easy to define but whose geometry is difficult to grasp. This task was undertaken by the authors in [4] and continued in the articles [2,3].

Within this topic, it is specially interesting and challenging to understand the structure of Lipschitz free p-spaces over subsets of the metric space $\mathbb{R}$ (or more generally over $\mathbb{R}^d$ for $d \in \mathbb{N}$) endowed with the Euclidean distance (see [4, Comments at the end of section §5]). Although the papers [2, 3] do not focus on this kind of Lipschitz free p-spaces, they contain results that apply in particular to them. Let us next gather the most significant contributions to the geometry of Lipschitz free p-spaces over Euclidean spaces from those two papers. Some of them are explicitly stated either in [2] or [3] (in which case we provide the reference), while others are straightforward consequences of more general results. In the list below we assume $0 < p \le 1$.

- **(A.1)** For every $d \in \mathbb{N}$ and every net $\mathcal{N}$ in $\mathbb{R}^d$, $\mathcal{F}_p(\mathcal{N}) \simeq \mathcal{F}_p(\mathbb{Z}^d)$ ([3, Proposition 3.6]).
- **(A.2)** The space $\mathcal{F}_p([0,1])$ has a Schauder basis ([3, Theorem 5.7]).
- **(A.3)** For every $d \in \mathbb{N}$, the space $\mathcal{F}_p(\mathbb{Z}^d)$ has a Schauder basis ([3, Theorem 5.3]).
- **(A.4)** $\mathcal{F}_p(\mathbb{R}^d) \simeq \mathcal{F}_p([0,1])^d \simeq \mathcal{F}_p(\mathbb{R}^d_+) \simeq \mathcal{F}_p(S^d) \simeq \ell_p(\mathcal{F}_p(\mathbb{R}^d))$ for every $d \in \mathbb{N}$ ([2, Theorem 4.15, Corollary 4.17 and Theorem 4.21]).
- **(A.5)** $\mathcal{F}_p(\mathbb{Z}^d) \simeq \mathcal{F}_p(\mathbb{N}^d) \simeq \ell_p(\mathcal{F}_p(\mathbb{Z}^d))$ for every $d \in \mathbb{N}$ ([2, Theorems 5.8 and 5.12]).
- **(A.6)** The spaces $\mathcal{F}_p(\mathbb{Z}^d)$ and $\mathcal{F}_p(\mathbb{R}^d)$, despite being non-isomorphic, have the same local structure ([2, Corollary 5.14]).
- **(A.7)** For every $d \in \mathbb{N}$ there is a constant $C = C(p,d)$ such that for every $\mathcal{M} \subset \mathcal{N} \subset \mathbb{R}^d$, the space $\mathcal{F}_p(\mathcal{M})$ is C-complemented in $\mathcal{F}_p(\mathcal{N})$ ([2, Corollary 5.3]).
- **(A.8)** For every $0 < p \le 1$, every $d \in \mathbb{N}$, and every $\mathcal{M} \subset \mathbb{R}^d$, $\mathcal{F}_p(\mathcal{M})$ has the π-property ([2, Corollary 5.3]).
- **(A.9)** For every $\mathcal{M} \subset \mathbb{R}^d$ infinite, there is $\mathcal{N} \subset \mathcal{M}$ such that $\mathcal{F}_p(\mathcal{N}) \simeq \ell_p$ ([2, Theorem 3.2]).

Combining **(A.7)** with **(A.4)**, and using Pełczyński's decomposition method (see, e.g., [7, Theorem 2.2.3]), yields that $\mathcal{F}_p(\mathcal{N}) \simeq \mathcal{F}_p(\mathbb{R}^d)$

whenever the subset $\mathcal{N}$ of $\mathbb{R}^d$ has non-empty interior. Thus, the research on Lipschitz free p-spaces over subsets of Euclidean spaces reduces to the following two main topics (with non-empty overlapping).

(Q.a) The geometry of the spaces $\mathcal{F}_p(\mathbb{R}^d)$.
(Q.b) The contrast between the spaces $\mathcal{F}_p(\mathcal{N})$ for different subsets $\mathcal{N} \subset \mathbb{R}^d$ with empty interior.

As far as topic **(Q.a)** is concerned, the main question suggested by the preceding work on the subject is whether **(A.2)** extends to multidimensional Euclidean spaces. In the category of approximation properties (see [9]), the existence of a Schauder basis is the most demanding one. Thus, this question also connects with the result **(A.8)**, which states in particular that $\mathcal{F}_p([0,1]^d)$ has the π-property.

The study of approximation properties of Lipschitz free spaces (i.e., Lipschitz free p-spaces for $p=1$) has attracted a lot of attention since the explosion of interest in the subject in 2003. We refer the reader to [8, 11, 12, 14–18, 23, 26] for a non-exhaustive listing of papers containing contributions to this topic. In contrast, determining whether a given Lipschitz free space has a Schauder basis has shown to be a more elusive task. To the best of our knowledge, the papers that contain positive results in this direction reduces to [10, 13, 19, 20].

In the context of Lipschitz-free spaces over Euclidean spaces it is known that $\mathcal{F}(\mathbb{R}^d)$ has a Schauder basis (see [20, Theorem 3.1]). With an eye to obtaining analogous statements for the more general case of Lipschitz-free p-spaces for $p \in (0,1]$, here we extend this result by proving that $\mathcal{F}_p(\mathbb{R}^d)$ admits a Schauder basis for every $p \in (0,1]$, thus answering in the positive [3, Question 6.5]. Moreover, explcit formulas for the basis of both $\mathcal{F}_p(\mathbb{R}^d)$ and its isomorphic space $\mathcal{F}_p([0,1]^d)$ are provided (see Theorem 3.8 and Theorem 3.9).

In relation to **(Q.b)**, it is natural to initiate its study with uniformly separated subsets. Note that if $\mathcal{M} \subset \mathbb{R}^d$ is uniformly separated, then it is contained in a net $\mathcal{N}$. Therefore, by **(A.1)**, **(A.9)**, and **(A.7)**, ℓ_p is complemented in $\mathcal{F}_p(\mathcal{M})$ and $\mathcal{F}_p(\mathcal{M})$ is complemented in $\mathcal{F}_p(\mathbb{Z}^d)$. Consequently, if $\mathcal{F}_p(\mathbb{Z}^d)$ were isomorphic to ℓ_p, applying Pełczyński's decomposition method would give $\ell_p \simeq \mathcal{F}_p(\mathcal{M})$. So, our first task should be to determine whether $\mathcal{F}_p(\mathbb{Z}^d)$ is isomorphic to ℓ_p or not. It is known ([25]) that, for $d \geq 2$, $\mathcal{F}(\mathbb{Z}^d)$ is not isomorphic to ℓ_1. Taking envelopes yields that $\mathcal{F}_p(\mathbb{Z}^d)$ is not isomorphic to ℓ_p for any $0 < p \leq 1$ and any $d \geq 2$ (see [4, Corollary 4.2]). On the other hand, it is known and easy to prove, that $\mathcal{F}(\mathbb{Z}) \simeq \ell_1$. More generally, $\mathcal{F}(\mathcal{M}) \simeq \ell_1$ whenever $\mathcal{M}$ is the closure of a zero-measure subset of $\mathbb{R}$ (see [10]). This result was extended to $0 < p < 1$ by the authors replacing the Euclidean distance $|\cdot|$ on $\mathbb{R}$ with its anti-snowflaking $|\cdot|^{1/p}$. That is, we have $\mathcal{F}_p(\mathbb{Z},|\cdot|^{1/p})\simeq \ell_p$ for every $0<p\le 1$. A question that implicitly arose from [4] is whether the same holds for the Euclidean distance, i.e., whether $\mathcal{F}_p(\mathbb{Z})$ is isomorphic to ℓ_p or not for $0<p<1$. Section 2 is devoted to providing a negative answer to this problem. Note that $\mathcal{F}_p(\mathbb{Z},|\cdot|^{\alpha})\simeq \ell_p$ for $0<\alpha<1$ and $0<p\le 1$ (see [2, comments preceding Question 8]). So, this new result exhibits a surprising discontinuity in the pattern of the Lipschitz free p-spaces over the family of p-metric spaces $(\mathbb{Z},|\cdot|^{\alpha})$ for $0<\alpha\le 1/p$.

1.1. **Terminology.** Throughout this article we use standard facts and notation from quasi-Banach spaces and Lipschitz free p-spaces over quasimetric spaces as can be found in [4]. Nonetheless, we will record the notation that is most heavily used. A *quasi-norm* on a vector space X over the real field $\mathbb{R}$ is a map $\|\cdot\|\colon X\to[0,\infty)$ satisfying $\|x\|>0$ unless $x=0$, $\|t\,x\|=|t|\,\|x\|$ for all $t\in\mathbb{R}$ and all $x\in X$, and

$$\|x+y\|\le\kappa(\|x\|+\|y\|),\quad x,\,y\in X. \tag{1.1}$$

for some constant $\kappa\ge 1$. The smallest constant κ such that (1.1) holds will be called the *modulus of concavity* of X,

Let $0<p\le 1$. If $\|\cdot\|$ fulfils the condition

$$\|x+y\|^p\le\|x\|^p+\|y\|^p,\quad x,\,y\in X,$$

then it is said that $\|\cdot\|$ is a *p-norm.* Any p-norm is a quasi-norm with modulus of concavity at most $2^{1/p-1}$. A quasi-norm induces a Hausdorff vector topology on X. If X is a complete topological vector space, we say that $(X,\|\cdot\|)$ is a *quasi-Banach space.* A *p-Banach space* will be a quasi-Banach space equipped with a p-norm.

A quasi-Banach space X is said to have the *bounded approximation property* (BAP for short) if there exists a net of finite-rank linear operators $(T_\alpha)_{\alpha\in\mathcal{A}}$ with

$$\sup_{\alpha\in\mathcal{A}}\|T_\alpha\|<\infty$$

such that $(T_\alpha)_{\alpha\in\mathcal{A}}$ converges to Id_X uniformly on compact sets. If, moreover, each operator T_α is a projection we say that X has the *π-property.*

A *Schauder basis* of a quasi-Banach space X is a sequence $(x_n)_{n=1}^\infty$ in X such that for every $x\in X$ there is a unique sequence of scalars $(a_n)_{n=1}^\infty$ with $x=\sum_{n=1}^\infty a_n\,x_n$. Associated to the Schauder basis $(x_n)_{n=1}^\infty$ the partial-sum projections $P_m\colon X\to X$, $m\in\mathbb{N}$, given by

$$x=\sum_{n=1}^\infty a_n\,x_n\mapsto P_m(x)=\sum_{n=1}^m a_n\,x_n,$$

are uniformly bounded. Therefore, if a quasi-Banach space X has a Schauder basis, then it has both the BAP and the π-property. Conversely, given a sequence $(P_m)_{m=1}^\infty$ of linear maps from X into X such that $\sup_m \|P_m\| < \infty$, $\cup_{m=1}^\infty P_m(X)$ is dense in X, $\dim(P_m(X)) = m$, and $P_m \circ P_n = P_{\min\{n,m\}}$ for all n, $m \in \mathbb{N}$, then there is a Schauder basis whose associated projections are $(P_m)_{m=1}^\infty$. Namely, if for each $n \in \mathbb{N}$ we pick an arbitrary non-zero vector x_n in the one-dimensional space $P_n(X) \cap \operatorname{Ker}(P_{n-1})$, then $(x_n)_{n=1}^\infty$ is such a Schauder basis.

We say that a quasi-Banach space X is K-*complemented* in Y if there are bounded linear maps $S\colon X \to Y$ and $P\colon Y \to X$ with $P \circ S = \mathrm{Id}_X$ and $\|S\| \|T\| \le K$. If, moreover, $S \circ P = \mathrm{Id}_Y$, the spaces X and Y are said to be K-*isomorphic.* In the case when S is the inclusion map, that is, P is a projection onto X, we say that X is a K-complemented subspace of Y. If the constant K is irrelevant, we simply drop it from the notation. We can define similar notions replacing the quasi-Banach spaces X and Y with metric, or quasimetric, spaces $\mathcal{M}$ and $\mathcal{N}$, and replacing bounded linear maps with Lipschitz maps. In the "metric case", we will say that $\mathcal{M}$ is a Lipschitz retract of $\mathcal{N}$ with constant K, or that $\mathcal{M}$ and $\mathcal{N}$ are Lipschitz isomorphic with distortion at most K, respectively.

A subset $\mathcal{N}$ of a metric space $(\mathcal{M}, d)$ is said to be *uniformly separated* if

$$\inf\{d(x,y)\colon x, y \in \mathcal{N}, x \neq y\} > 0.$$

A *net* is a uniformly separated set $\mathcal{N}$ with $\sup_{x\in\mathcal{M}} d(x, \mathcal{N}) < \infty$.

We use the symbol $\mathbb{N}_*$ to denote the set of all non-negative integers, i.e., $\mathbb{N}_* = \mathbb{N} \cup \{0\}$. Given $n \in \mathbb{N}$ we will put $\mathbb{N}_n = \mathbb{Z} \cap [0, n]$.

2. The p-Banach space $\mathcal{F}_p(\mathbb{Z}^d)$ is not isomorphic to ℓ_p

Once we conjecture that two quasi-Banach spaces are not isomorphic, the best strategy for substantiating our guess is to come up with a feature that tells them apart. We find this wished-for property within the theory of locally complemented subspaces and $\mathscr{L}_p$-spaces developed by Kalton in [21]. A subspace X of a quasi-Banach space Y is said to be *locally* K-*complemented* in Y for some $K > 0$, if for every finite-dimensional space $V \subset Y$ and every $\varepsilon > 0$ there is $P\colon V \to X$ with

$$\|P\| \le K \text{ and } \|P(x) - x\| \le \varepsilon \|x\|, \quad x \in V \cap X.$$

If the constant K is irrelevant, we say that X is locally complemented in Y.

A quasi-Banach space X is a $\mathscr{L}_p$-*space,* $0 < p < \infty$, if it is isomorphic to a locally complemented subspace of $L_p(\mu)$ for some measure μ. This notion was introduced in [21] with the aim to provide a consistent (yet attractive) definition of a $\mathscr{L}_p$-space for $0 < p < 1$, since it was unclear whether L_p for $0 < p < 1$ would satisfy the analogue of the classical Lindenstrass-Pełczyński definition. For $p = 1$, the above definition is equivalent to the classical one, while for $1 < p < \infty$ the only exception to the equivalence is that, with the definition used in this paper, Hilbert spaces become $\mathscr{L}_p$-spaces.

Lemma 2.1. *Let $0 < p < \infty$, let X be a quasi-Banach space and let $(V_\alpha)_{\alpha\in A}$ be an increasing net consisting of finite-dimensional subspaces of X with $\overline{\cup_{\alpha\in A} V_\alpha} = X$. Suppose that there is $K \in (0,\infty)$ such that for every $\alpha \in A$ and every $\varepsilon > 0$ there is $S\colon V_\alpha \to \ell_p$ and $T\colon \ell_p \to X$ with*

$$\|T\|\,\|S\| \le K \text{ and } \|T(S(x)) - x\| \le \varepsilon\|x\|, \quad x \in V_\alpha.$$

Then X is an $\mathscr{L}_p$-space.

Proof. Let V be a finite-dimensional subspace of X and let $\varepsilon > 0$. Set

$$\varepsilon_0 := \min\left\{1, \frac{\varepsilon}{\kappa + 2\kappa^2}\right\}.$$

A standard argument yields $\alpha \in A$ and $J\colon V \to V_\alpha$ with $\|J(x) - x\| \le \varepsilon_0\|x\|$ for all $x \in V$. By hypothesis, there are $S\colon V_\alpha \to \ell_p$ and $T\colon \ell_p \to X$ such that $\|T\|\,\|S\| \le K$ and $\|T(S(x)) - x\| \le \varepsilon_0\|x\|$ for all $x \in V_\alpha$. Then

$$\|J\| = \|J - \mathrm{Id}_V + \mathrm{Id}_V\| \le \kappa(\varepsilon_0 + 1) \le 2\kappa,$$

so that

$$\|T\|\,\|S \circ J\| \le \|T\|\,\|S\|\,\|J\| \le 2\kappa K.$$

If $x \in V$,

$$\begin{aligned}\|T(S(J(x))) - x\| &\le \kappa(\|T(S(J(x))) - J(x)\| + \|J(x) - x\|)\\ &\le \kappa(\varepsilon_0\|J(x)\| + \varepsilon_0\|x\|)\\ &\le \kappa\varepsilon_0(2\kappa\|x\| + \|x\|) = \varepsilon\|x\|.\end{aligned}$$

Appealing to [21, Theorem 6.1] finishes the proof. □

It is known that complemented subspaces inherit the property of being $\mathscr{L}_p$-spaces (see [21, Proposition 3.3]). We state and prove the quantitative version of this result, since it is the one we will need below.

Lemma 2.2. *Let X be an $\mathscr{L}_p$-space and let $\lambda \in [1,\infty)$. There is a constant $K = K(X,\lambda)$ such that for every quasi-Banach space Y which is λ-complemented in X, every finite-dimensional subspace V of Y, and every $\varepsilon > 0$ there are bounded linear maps $S\colon V \to \ell_p$ and $T\colon \ell_p \to Y$ with $\|S\|\,\|T\| \le K$ and $\|T \circ S - \mathrm{Id}_V\| \le \varepsilon$.*

Proof. Let $J\colon Y \to X$ and $P\colon X \to Y$ be such that $\|J\|\,\|P\| \le \lambda$ and $P \circ J = \mathrm{Id}_Y$. By [21, Theorem 6.1], there are $S\colon J(V) \to \ell_p$ and $T\colon \ell_p \to X$ with $\|S\|\,\|T\| \le K_0$ and $\|T \circ S - \mathrm{Id}_{J(V)}\| \le \varepsilon/\lambda$, where $K_0 \in [1,\infty)$ depends only on X. We have

$$\|S \circ J|_V\|\,\|P \circ T\| \le \|S\|\,\|J\|\,\|P\|\,\|T\| \le \lambda K_0$$

and

$$\|P \circ T \circ S \circ J|_V - \mathrm{Id}_V\| = \|P \circ (T \circ S - \mathrm{Id}_{J(V)}) \circ J\| \le \varepsilon. \qquad \square$$

Given a quasi-Banach space X and $0 < q \le 1$, the *q-Banach envelope* of X is a pair $(X_{\mathsf{c},q}, E_{X,q})$, where $X_{\mathsf{c},q}$ is a q-Banach space and $E_{X,q}\colon X \to X_{\mathsf{c},q}$ is a linear contraction, determined by the following universal property: for every bounded linear map $T\colon X \to Y$, where Y is a q-Banach space, there is a unique linear map $T'\colon X_{\mathsf{c},q} \to Y$ with $\|T'\| \le \|T\|$ and $T = T' \circ E_{X,q}$. If X and Y are quasi-Banach spaces and $T\colon X \to Y$ is linear and bounded, there is a unique bounded linear map $T_{\mathsf{c},q}\colon X_{\mathsf{c},q} \to Y_{\mathsf{c},q}$ such that $T_{\mathsf{c},q} \circ E_{X,q} = E_{Y,q} \circ T$. Moreover, $\|T_{\mathsf{c},q}\| \le \|T\|$. For background on envelopes, see e.g. [1, §9].

The BAP transfers to Banach envelopes. Let us record this fact for further reference.

Lemma 2.3. *Let X be a quasi-Banach space with the BAP. Then $X_{\mathsf{c},q}$ has the BAP for $0 < q \le 1$.*

Given $0 < p \le 1$, a subset $\mathcal{C}$ of a vector space V is said to be *absolutely p-convex* if for any x and $y \in \mathcal{C}$ and any scalars λ and μ with $|\lambda|^p + |\mu|^p \le 1$ we have $\lambda\, x + \mu\, y \in \mathcal{C}$. We will denote by $\mathrm{co}_p(Z)$ the *p-convex hull* of $Z \subset V$, i.e., the smallest absolutely p-convex set containing Z.

Lemma 2.4. *Let Z be a q-Banach space and $K \subset Z$ be relatively compact. Then the absolutely q-convex hull $\mathrm{co}_q(K)$ of K is relatively compact.*

Proof. Since the map $(t,x) \mapsto tx$ is continuous and the unit sphere of the scalar field is compact, we can suppose that $tx \in K$ whenever $x \in K$ and $|t| = 1$. If suffices to prove that $\mathrm{co}_q(K)$ possesses a finite ε-net for every $\varepsilon > 0$. Let $\mathcal{N}_0$ be a finite $(2^{-1/q}\varepsilon)$-net for K. Then

$$\mathcal{N}_1 = \left\{ \sum_{k=1}^m a_k x_k \colon x_k \in \mathcal{N}_0,\ a_k \ge 0,\ \sum_{k=1}^m a_k^q \le 1 \right\}$$

is a $(2^{-1/q}\varepsilon)$-net for $\mathrm{co}_q(K)$.

Enumerate $\mathcal{N}_0 = \{y_j \colon 1 \le j \le n\}$, and let $s \in \mathbb{N}$ be such that $2n \sup_j \|y_j\|^q \le \varepsilon^q 2^{sq}$. We will conclude the proof by showing that

$$\left\{ 2^{-s} \sum_{j=1}^n b_j y_j \colon b_j \in \mathbb{N}_*, \sum_{j=1}^n b_j^q \le 2^{sq} \right\}$$

is a (finite) $(2^{-1/q}\varepsilon)$-net for $\mathcal{N}_1$. Let $x \in \mathcal{N}_1$. There is $(a_j)_{j=1}^n$ in $[0,\infty)$ such that $x = \sum_{j=1}^n a_j y_j$ and $\sum_{j=1}^n a_j^q \le 1$. For each $j = 1, \dots, n$, let $b_j \in \mathbb{N}_*$ be such that $b_j \le 2^s a_j < b_j + 1$. Then,

$$\left\| x - 2^{-s} \sum_{j=1}^n b_j y_j \right\|^q \le n 2^{-sq} \sup_j \|y_j\|^q \le \frac{\varepsilon^q}{2}. \qquad \square$$

Theorem 2.5. *Let $T \colon X \to Y$ be a compact linear operator between quasi-Banach spaces X and Y. Then the operator $T_{\mathsf{c},q}$ is compact for any $0 < q \le 1$.*

Proof. Since the space of compact operators forms an ideal, we can assume that Y is q-Banach, so that $T_{\mathsf{c},q}$ is a map from $X_{\mathsf{c},q}$ into Y with $T_{\mathsf{c},q} \circ E_{q,X} = T$. Then, by construction,

$$\begin{aligned} \overline{T_{\mathsf{c},q}(B_{X_{\mathsf{c},q}})} &= \overline{T_{\mathsf{c},q}(\operatorname{co}_q(E_{q,X}(B_X)))} \\ &= \overline{\operatorname{co}_q(T_{\mathsf{c},q}(E_{q,X}(B_X)))} \\ &= \overline{\operatorname{co}_q(T(B_X))}, \end{aligned}$$

which is compact by Lemma 2.4. $\square$

Let X be a subspace of a quasi-Banach space Y. We say that X has the *compact extension property* (CEP for short) in Y if every compact operator $T \colon X \to Z$, where Z is a quasi-Banach space, extends to a compact operator $\widetilde{T} \colon Y \to Z$. Let $0 < q \le 1$. We say that X has the *compact extension property for q-Banach spaces* (q-CEP for short) in Y if the compact extension property holds when Z is a q-Banach space. If, moreover, we can ensure that $\|\widetilde{T}\| \le K\|T\|$ for some constant $K \in [1,\infty)$ (depending on X, Y and q), we say that X has the q-CEP in Y with constant K. Notice that, by the Aoki-Rolewicz theorem, X has the CEP in Y if and only if X has the q-CEP in Y for every $0 < q \le 1$. We have the following results in this respect.

Theorem 2.6. *Let X be a subspace of a quasi-Banach space Y. Suppose that X has the q-CEP in Y for some $0 < q \le 1$. Then there is a constant $K \in [1,\infty)$ for which X has the q-CEP in Y with constant K.*

Proof. Although the proof of [24, Theorem 2.2] was done for Banach spaces, the arguments therein apply to our case without any important modifications. □

Theorem 2.7. *Let $0 < q \le 1$ and Y be a q-Banach space with the BAP. Let X be a closed subspace of Y. The following are equivalent.*

(i) X is locally complemented in Y.
(ii) X has the BAP and the CEP in Y.
(iii) X has the BAP and the q-CEP in Y.

Proof. Although [21, Theorem 5.1] only establishes the equivalence between (i) and (ii), the very same proof gives that (iii) implies (i). So, taking into account Theorem 2.6, (iii) is equivalent to (i) and (ii). □

Theorem 2.8. *Let X be a subspace of a quasi-Banach space Y. Denote by J the inclusion of X into Y. Suppose that X has the BAP and that X has the q-CEP in Y for some $0 < q \le 1$. Then $J_{\mathsf{c},q}$ is an isomorphic embedding and $J_{\mathsf{c},q}(X_{\mathsf{c},q})$ has the q-CEP in $Y_{\mathsf{c},q}$.*

Proof. Use Theorem 2.6 to pick K such that X has the q-CEP in Y with constant K. Let Z be a q-Banach space and let $T\colon X_{\mathsf{c},q} \to Z$ be a compact operator. Since $T \circ E_{X,q}$ is also compact, the compact extension property and Theorem 2.5 yield a compact operator $S\colon Y_{\mathsf{c},q} \to Z$ with $\|S\| \le K\|T\|$ and such that the diagram

$$\begin{array}{ccccc} Y & \xrightarrow{E_{Y,q}} & Y_{\mathsf{c},q} & & \\ {\scriptstyle J}\uparrow & & & \searrow{\scriptstyle S} & \\ X & \xrightarrow[E_{X,q}]{} & X_{\mathsf{c},q} & \xrightarrow[T]{} & Z \end{array}$$

commutes. Since $E_{X,q}(X)$ is dense in $X_{\mathsf{c},q}$, merging this commutative diagram with

$$\begin{array}{ccc} Y & \xrightarrow{E_{Y,q}} & Y_{\mathsf{c},q} \\ {\scriptstyle J}\uparrow & & \uparrow{\scriptstyle J_{\mathsf{c},q}} \\ X & \xrightarrow[E_{X,q}]{} & X_{\mathsf{c},q} \end{array}$$

yields that the diagram

$$\begin{array}{ccc} Y_{\mathsf{c},q} & & \\ {\scriptstyle J_{\mathsf{c},q}}\uparrow & \searrow{\scriptstyle S} & \\ X_{\mathsf{c},q} & \xrightarrow[T]{} & Z \end{array}$$

commutes. It remains to show that $J_{\mathsf{c},q}$ is an isomorphic embedding. To that end, use Lemma 2.3 to pick C such that $X_{\mathsf{c},q}$ has that BAP with constant C. Let $x \in X_{\mathsf{c},q}$ and $\varepsilon > 0$. There is a linear operator $T\colon X_{\mathsf{c},q} \to X_{\mathsf{c},q}$ with finite-dimensional range such that $\|x - T(x)\| \le \varepsilon$ and $\|T\| \le C$. Since T is compact, there is $S\colon Y_{\mathsf{c},q} \to X_{\mathsf{c},q}$ such $S \circ J_{\mathsf{c},q} = T$ and $\|S\| \le CK$. We have

$$\|x\|^q \le \|x - T(x)\|^q + \|S(J_{\mathsf{c},q}(x))\|^q \le \varepsilon^q + C^q K^q \|J_{\mathsf{c},q}(x)\|^q.$$

Letting ε tend to zero we obtain $\|x\| \le CK\|J_{\mathsf{c},q}(x)\|$. □

Theorem 2.9. *Let $0 < p \le q \le 1$ and X be a separable $\mathscr{L}_p$-space with the BAP. Then $X_{\mathsf{c},q}$ is isomorphic to a locally complemented subspace of ℓ_q and has a Schauder basis.*

Proof. By [21, Theorem 6.4], X is isomorphic to a locally complemented subspace Y of ℓ_p, and Y has a Schauder basis. By Theorem 2.7, Y has CEP in ℓ_p. Hence, by Theorem 2.8, $X_{\mathsf{c},q}$ is isomorphic to a subspace Z of ℓ_q and has the q-CEP in ℓ_q. As it is clear that $Y_{\mathsf{c},q}$ (and so $X_{\mathsf{c},q}$) has a Schauder basis, applying once again Theorem 2.7 completes the proof. □

The following straightforward consequence of Theorem 2.9 partially solves [5, Question 4.18].

Corollary 2.10. *Let $0 < p \le q \le 1$ and X be a separable $\mathscr{L}_p$-space with the BAP. Then $X_{\mathsf{c},q}$ is a $\mathscr{L}_q$-space.*

We are now in a position to prove the main results of this section.

Theorem 2.11. *Let $\mathcal{M}$ be a p-metric space, $0 < p < 1$. Suppose that there is a constant C such that $\mathbb{N}_n$ is a Lipschitz retract of $\mathcal{M}$ with constant C for all $n \in \mathbb{N}$. Then $\mathcal{F}_p(\mathcal{M})$ is not an $\mathscr{L}_p$-space. In particular, $\mathcal{F}_p(\mathcal{M}) \not\simeq \ell_p$.*

Theorem 2.12. *Let $\mathcal{M}$ be a metric space. Suppose that there is a constant K such that, for all $n \in \mathbb{N}$, $\mathbb{N}_n$ Lipschitz-isomorphically embeds in $\mathcal{M}$ with distortion at most K. Then $\mathcal{F}_p(\mathcal{M})$ is not an $\mathscr{L}_p$-space. In particular, $\mathcal{F}_p(\mathcal{M}) \not\simeq \ell_p$.*

Proof of Theorems 2.11 and 2.12. Since $\mathbb{R}$ is a doubling metric space, there is $D \ge 1$ such that every subset of $\mathbb{R}$, in particular $\mathbb{N}_n$ for all $n \in \mathbb{N}$, is a doubling metric space with doubling constant D. Thus, under the assumptions in Theorem 2.12, applying [2, Theorem 5.1] yields a constant K such that $\mathcal{F}_p(\mathbb{N}_n)$ is K-complemented in $\mathcal{F}_p(\mathcal{M})$ for all $n \in \mathbb{N}$. By [4, Lemma 4.19], this holds under the assumptions in Theorem 2.11 as well. Since

$$\mathcal{D}_k = \{x \in [-k, k] \colon 2^k x \in \mathbb{Z}\}$$

is a doubling metric space with constant D for all $k \in \mathbb{N}$, applying again [2, Theorem 5.1] yields a constant C such that the linearization of the inclusion of $\mathcal{D}_k$ into $\mathbb{R}$ is a C-isomorphic embedding for all $k \in \mathbb{N}$. Taking into account that $\mathcal{D}_k$ is Lipschitz isomorphic to $\mathbb{N}_{k2^{k+1}}$ with distortion constant 1 we deduce the existence of a constant K_1 such that the finite-dimensional subspace of $\mathcal{F}_p(\mathbb{R})$ given by

$$V_k := \operatorname{span}(\delta_{\mathbb{R}}(x) \colon x \in \mathcal{D}_k)$$

is K_1-complemented in $\mathcal{F}_p(\mathcal{M})$ for all $k \in \mathbb{N}$.

Suppose by contradiction that $\mathcal{F}_p(\mathcal{M})$ is an $\mathscr{L}_p$-space. Then, by Lemma 2.2, there is a constant K_2 such that for all $k \in \mathbb{N}$ and $\varepsilon > 0$ there are bounded linear maps $S \colon V_k \to \ell_p$ and $T \colon \ell_p \to V_k \subset \mathcal{F}_p(\mathbb{R})$ with $\|S\| \, \|T\| \le K_2$ and $\|T \circ S - \mathrm{Id}_{V_k}\| \le \varepsilon$. Since the set $\mathcal{D}$ consisting of all dyadic rationals is dense in $\mathbb{R}$, the subspace

$$\cup_{k=1}^{\infty} V_k = \operatorname{span}(\delta_{\mathbb{R}}(x) \colon x \in \mathcal{D})$$

is dense in $\mathcal{F}_p(\mathbb{R})$. Applying Lemma 2.1 yields that $\mathcal{F}_p(\mathbb{R})$ is an $\mathscr{L}_p$-space. Since $\mathcal{F}_p(\mathbb{R})$ has the BAP (see **(A.8)**) combining Theorem 2.9 with [4, Proposition 4.20] yields that $\mathcal{F}(\mathbb{R})$ is isomorphic to a subspace of ℓ_1. Using that $\mathcal{F}(\mathbb{R})$ is isometric to $L_1(\mathbb{R})$ and that ℓ_2 is a subspace of $L_1(\mathbb{R})$, we obtain that ℓ_2 is isomorphic to a subspace of ℓ_1, an absurdity.

For the last part of the statements, we just need to note that ℓ_p is an $\mathscr{L}_p$-space. □

Corollary 2.13. *Let $\mathcal{M}$ be a metric space containing a subset that is Lipchitz isomorphic either to $[0,1]$ or to $\mathbb{N}$. Then $\mathcal{F}_p(\mathcal{M})$ is not a $\mathscr{L}_p$-space for any $0 < p < 1$.*

Proof. Just notice that $\{k/n \colon k \in \mathbb{N}_n\} \subset [0,1]$ is Lipschitz isomorphic to $\mathbb{N}_n$ with distortion constant 1 and apply Theorem 2.12. □

Corollary 2.14. *Let X be a p-Banach space ($0 < p < 1$) with non-trivial dual. Then $\mathcal{F}_p(X)$ is not a $\mathscr{L}_p$-space.*

Proof. By Corollary 2.13, $\mathcal{F}_p(\mathbb{R})$ is not a $\mathscr{L}_p$-space. Since, by assumption, $\mathbb{R}$ is a complemented subspace of X, by [4, Lemma 4.19] it follows that $\mathcal{F}_p(\mathbb{R})$ is a complemented subspace of $\mathcal{F}_p(X)$. Consequently, $\mathcal{F}_p(X)$ is a not an $\mathscr{L}_p$-space either. □

3. Schauder bases in $\mathcal{F}_p([0,1]^d)$ and $\mathcal{F}_p(\mathbb{R}^d)$

The basic idea for building Schauder bases for $\mathcal{F}_p([0,1]^d)$ and $\mathcal{F}_p(\mathbb{R}^d)$ comes, on one hand, from [23], where the authors present a canonical way of extending linearly Lipschitz functions on d-dimensional hypercubes, and on the other hand from [3], where a method for building linear projections on $\mathcal{F}_p([0,1])$ is given.

Fix $d \in \mathbb{N}$. Given $R > 0$ and $w \in \mathbb{Z}^d$, we denote by $Q^d_{w,R}$ the hypercube of edge-length R and with vertices in the points

$$V^d_{w,R} = \{Rw + R\varepsilon \colon \varepsilon \in \{0,1\}^d\},$$

That is, if $w = (w_i)_{i=1}^d$,

$$Q^d_{w,R} = \operatorname{co}[V^d_{w,R}] = \prod_{i=1}^d [Rw_i, Rw_i + R].$$

For $R > 0$ fixed, the set of hypercubes

$$\mathcal{Q}^d_R = \{Q^d_{w,R} \colon w \in \mathbb{Z}^d\}$$

tessellates the space $\mathbb{R}^d$. Let $\mathcal{V}(Q)$ be set of vertices of $Q \in \mathcal{Q}^d_R$, i.e., $\mathcal{V}(Q^d_{w,R}) = V^d_{w,R}$ for every $w \in \mathbb{Z}^d$. We have

$$\bigcup_{Q \in \mathcal{Q}^d_R} \mathcal{V}(Q) = \mathcal{V}^d_R := \{Rw \colon w \in \mathbb{Z}^d\}.$$

We shall define a fuzzy pull back of $\mathbb{R}^d$ into the set of vertices $\mathcal{V}^d_R$. Given $x \in [0,1]$ and $w \in \mathbb{Z}$ we set

$$x^{(w)} = \begin{cases} x & \text{if } w = 1, \\ 1 - x & \text{if } w = 0, \\ 0 & \text{if } w \in \mathbb{Z} \setminus \{0,1\}. \end{cases}$$

Given $x = (x_i)_{i=1}^d \in [0,1]^d$ and $w = (w_i)_{i=1}^d \in \mathbb{Z}^d$ we put

$$x^{(w)} = \prod_{i=1}^d x_i^{(w_i)}.$$

Lemma 3.1. *Let $d \in \mathbb{N}$ and $R > 0$. There is a map*

$$\Lambda = (\Lambda(v, \cdot))_{v \in \mathcal{V}^d_R} \colon \mathbb{R}^d \to [0,1]^{\mathcal{V}^d_R}$$

such that $\Lambda(Ru, Rw + Rx) = x^{(u-w)}$ for all $x \in [0,1]^d$ and all $u, w \in \mathbb{Z}^d$.

Proof. Since the function $x \mapsto Rw + Rx$ maps $[0,1]^d$ onto $Q^d_{w,R}$, if such a function Λ exists, it is unique. By dilation, it suffices to consider the case $R = 1$. If Λ is as desired in the one-dimensional case, then

$$\Lambda(u, \cdots) = \Lambda(u_1, \cdot) \otimes \cdots \otimes \Lambda(u_i, \cdot) \otimes \cdots \otimes \Lambda(u_d, \cdot), \quad u = (u_i)_{i=1}^d \in \mathbb{Z}^d,$$

is as desired in the d-dimensional case. Hence, we can also assume that $d = 1$. To prove the result in this particular case we must check that given $w \in \mathbb{Z}$ the function given for $x \in \mathbb{R}$ by

$$x \mapsto (x-w)^{(u-w)} \text{ if } u \in \mathbb{Z} \text{ and } w \le x \le w+1$$

is well-defined. Suppose that $w \le x \le w+1$ and $v \le x \le v+1$ with v, $w \in \mathbb{Z}$. Assume without lost of generality that $v < w$. Then $x = w = v+1$. Since $x - w = 0$, we have $(x-w)^{(u-w)} = 0$ unless $u - w = 0$, in which case $(x-w)^{(u-w)} = 1$. Since $x - v = 1$, we have $(x-v)^{(u-v)} = 0$ unless $u - v = 1$, in which case $(x-v)^{(u-v)} = 1$. Since $u - w = u - v - 1$ for every $u \in \mathbb{Z}$, we are done. □

Definition 3.2. Given $R > 0$ and $d \in \mathbb{N}$, we define

$$\Lambda_R^d = (\Lambda_R^d(v,\cdot))_{v \in \mathcal{V}_R^d}$$

as the function provided by Lemma 3.1.

If $d = 1$ we simply put $\Lambda_R = \Lambda_R^1$ and $\mathcal{V}_R = \mathcal{V}_R^1$. Given a finite set $A \subset \mathbb{N}$ we can carry out the above construction replacing the set $\{1, \dots, d\}$ with the set A. We will denote by $\mathcal{V}_R^A$ the corresponding set of vertices and by $\mathcal{V}_R^A$ the corresponding function defined as in Definition 3.2.

Let us give an auxiliary lemma followed by some properties of the function Λ_R^d.

Lemma 3.3. *Let $x = (x_i)_{i=1}^d$, $y = (y_i)_{i=1}^d \in [0,1]^d$. Then*

$$\left| \prod_{i=1}^d x_i - \prod_{i=1}^d y_i \right| \le |x-y|_1.$$

Proof. We proceed by induction on d. For $d = 1$ the result is obvious. Assume that $d \in \mathbb{N}$ and that the result holds for $d-1$. Then

$$\begin{aligned} \left| \prod_{i=1}^d x_i - \prod_{i=1}^d y_i \right| &\le \left| \prod_{i=1}^d x_i - y_d \prod_{i=1}^{d-1} x_i \right| + \left| y_d \prod_{i=1}^{d-1} x_i - \prod_{i=1}^d y_i \right| \\ &= |x_d - y_d| \prod_{i=1}^{d-1} x_i + y_d \left| \prod_{i=1}^{d-1} x_i - \prod_{i=1}^{d-1} y_i \right| \\ &\le |x_d - y_d| + \sum_{i=1}^{d-1} |x_i - y_i| = |x-y|_1. \end{aligned}$$

□

Lemma 3.4. *Let $d \in \mathbb{N}$ and $S \ge R > 0$ with $S/R \in \mathbb{Z}$. We have:*

(i) $\Lambda_R^d(v,x) = 0$ if $x \in Q \in \mathcal{Q}_R^d$ and $v \notin \mathcal{V}(Q)$.

(ii) $\Lambda_R^d(v,u) = \delta_{u,v}$ for every u, $v \in \mathcal{V}_R^d$.

(iii) $\sum_{v\in\mathcal{V}_R}\Lambda^d_R(v,x)=1$ *for every* $x\in\mathbb{R}^d$*.*

(iv) If there is $Q\in\mathcal{Q}^d_R$ *such that* x*,* $y\in Q$*, then*

$$|\Lambda^d_R(v,x)-\Lambda^d_R(v,y)|\le R^{-1}|x-y|_1$$

for every $v\in\mathcal{V}^d_R$*.*

(v) Let (A,B) *be a partition of* $\{1,\dots,d\}$*. Then*

$$\Lambda^d_R((u,v),(x,y))=\Lambda^A_R(u,x)\Lambda^B_R(v,y)$$

for every $u\in\mathcal{V}^A_R$*,* $v\in\mathcal{V}^B_R$*,* $x\in\mathbb{R}^A$ *and* $y\in\mathbb{R}^B$*.*

(vi) $\Lambda^d_S(v,x)=\sum_{u\in\mathcal{V}^d_R}\Lambda^d_S(v,u)\Lambda^d_R(u,x)$ *for every* $x\in\mathbb{R}^d$ *and* $v\in\mathcal{V}^d_S$*.*

Proof. (i) is clear from the definition. (ii) follows from the equality $0^{(0)}=1$. A straightforward induction on d yields

$$\sum_{w\in\mathbb{Z}^d}x^{(w)}=1,\quad x\in[0,1]^d,$$

and (iii) is clear from this identity. (iv) is a consequence of Lemma 3.3. (v) is clear from the definition. In light of (v), in order to prove (vi) it suffices to consider the case $d=1$. Given $x\in\mathbb{R}$ there are u_0, $u_1\in\mathcal{V}_S$ and v_0, $v_1\in\mathcal{V}_R$ with $u_0\le v_0\le x<v_1\le u_1$, and we have $v_1=v_0+R$ and $u_1=u_0+S$. Suppose that $u\in\mathcal{V}_S\setminus\{u_0,u_1\}$. Then $\Lambda_S(u,v)=0$ for $v\in\{v_0,v_1\}$. Since $\Lambda_R(v,x)=0$ for $v\in\mathcal{V}_R\setminus\{v_0,v_1\}$ we have

$$\Gamma(u,x):=\sum_{v\in\mathcal{V}_R}\Lambda_S(u,v)\Lambda_R(v,x)=0=\Lambda_S(u,x).$$

Hence, considering also the symmetry $x\mapsto -x$, if suffices to prove the result in the case when $u=u_1$. We have

$$\begin{aligned}\Gamma(u_1,x)&=\Lambda_S(u_1,v_0)\Lambda_R(v_0,x)+\Lambda_S(u_1,v_1)\Lambda_R(v_1,x)\\&=\eta(x):=\frac{v_0-u_0}{S}\frac{v_1-x}{R}+\frac{v_1-u_0}{S}\frac{x-v_0}{R}.\end{aligned}$$

Since $\eta(u_0)=0$ we have $\eta(y)=\gamma(y-u_0)$ for all $y\in\mathbb{R}$, where

$$\gamma=\left(-\frac{v_0-u_0}{SR}+\frac{v_1-u_0}{SR}\right)=\frac{1}{S}.$$

Since $\Lambda_S(u_1,x)=(x-u_0)/S$ we are done. □

Although the previous auxiliary results are stated in terms of the ℓ_1-norm, in this section we will consider $\mathbb{R}^d$ and its subsets equipped with the supremum norm $\|\cdot\|_\infty$.

Theorem 3.5. *Let* $d\in\mathbb{N}$ *and* $0<p\le 1$*. There is a constant* $C=C(p,d)$ *such that for every* $R>0$ *and every* $\mathcal{R}\subset\mathcal{Q}^d_R$*, if we set* $K=\cup_{Q\in\mathcal{R}}Q$ *and* $V=\cup_{Q\in\mathcal{R}}\mathcal{V}(Q)$*, and we choose an arbitrary point of* V

as base point of both metric spaces, there is a C-Lipschitz map $r = r_{K,V}\colon K \to \mathcal{F}_p(V)$ such that

$$r(x) = \sum_{v\in V} \Lambda_R^d(v,x)\,\delta_V(v), \quad x \in K.$$

Proof. Let $Q \in \mathcal{R}$ and x, $y \in Q$. Pick $u \in \mathcal{V}(Q)$. By Lemma 3.4,

$$\begin{aligned}\|r(x)-r(y)\|^p &= R\left\|\sum_{v\in\mathcal{V}(Q)\setminus\{u\}} (\Lambda_R^d(v,x)-\Lambda_R^d(v,y))\frac{\delta_V(v)-\delta_V(u)}{|v-u|_\infty}\right\|^p\\ &\le R\sum_{v\in\mathcal{V}(Q)\setminus\{u\}} |\Lambda_R^d(v,x)-\Lambda_R^d(v,y)|^p\\ &\le (2^d-1)|x-y|_1^p.\end{aligned}$$

Let $x = (x_i)_{i=1}^d \in K$ and $y = (y_i)_{i=1}^d \in K$. Pick $u = (u_i)_{i=1}^d$ and $w = (w_i)_{i=1}^d \in \mathbb{Z}^d$ such that $x \in Q_{R,u}^d$ and $y \in Q_{R,w}^d$. Define

$$F = \{i \in \{1,\dots,d\}\colon u_i = w_i\}.$$

For each $i \in G = \{1,\dots,d\} \setminus F$ there is $m_i \in \{u_i, u_i+1\}$ and $n_i \in \{w_i, w_i+1\}$ such that

$$|y_i - x_i| = |y_i - Rn_i| + |Rn_i - Rm_i| + |Rm_i - x_i|.$$

Suppose that $n_i = m_i$ for every $i \in G$. Define $z = (z_i)_{i=1}^d$ by

$$z_i = \begin{cases} x_i & \text{if } i \in F,\\ Rn_i = Rm_i & \text{if } i \in G.\end{cases}$$

We have $z \in Q_{R,u}^d \cap Q_{R,w}^d$. Consequently,

$$\begin{aligned}\|r(x)-r(y)\|^p &\le \|r(x)-r(z)\|^p + \|r(z)-r(y)\|^p\\ &\le (2^d-1)(|x-z|_1^p + |z-y|_1^p)\\ &\le 2^{1-p}(2^d-1)(|x-z|_1 + |z-y|_1)^p\\ &= 2^{1-p}(2^d-1)|x-y|_1^p.\end{aligned}$$

Suppose that $m_i \ne n_i$ for some i. Define $x' = (x'_i)_{i=1}^d$ and $y' = (y'_i)_{i=1}^d$ by

$$x'_i = \begin{cases} Ru_i = Rw_i & \text{if } i \in F,\\ Rm_i & \text{if } i \in G,\end{cases} \qquad y'_i = \begin{cases} Ru_i = Rw_i & \text{if } i \in F,\\ Rn_i & \text{if } i \in G.\end{cases}$$

We have $x' \in V_{R,u}^d$, $y' \in V_{R,w}^d$, and $1 \le \mu := \max_{i\in G}|m_i - n_i|$. Hence

$$\begin{aligned}&\|r(x)-r(y)\|^p\\ &\quad\le \|r(x)-r(x')\|^p + \|r(x')-r(y')\|^p + \|r(y')-r(y)\|^p\end{aligned}$$

$$\begin{aligned}
&\leq (2^d-1)|x-x'|_1^p + |x'-y'|_\infty^p + (2^d-1)|y-y'|_1^p \\
&\leq 2(2^d-1)dR^p + \mu^p R^p \\
&\leq (1+2d(2^d-1))\mu^p R^p \\
&\leq (1+2d(2^d-1))|x-y|_\infty.
\end{aligned}$$

This way the result holds with $C(p,d) = (1+2d(2^d-1))^{1/p}$. □

We are almost ready to prove that $\mathcal{F}_p(\mathbb{R}^d)$ has a Schauder basis. Let us record first a couple of auxiliary lemmas.

Lemma 3.6. *Let $\mathcal{M}$ and $\mathcal{N}$ be quasimetric spaces. For $i = 1, 2$, let $f_i \colon \mathcal{M}_i \subset \mathcal{M} \to \mathcal{N}$ be a Lipschitz function. Assume that*

(i) $\mathcal{M} = \mathcal{M}_1 \cup \mathcal{M}_2$,
(ii) $f_1|_{\mathcal{M}_1\cap\mathcal{M}_2} = f_2|_{\mathcal{M}_1\cap\mathcal{M}_2}$, *and*
(iii) There is a constant C such that $d(x_1, \mathcal{M}_1 \cap \mathcal{M}_2) \leq C d(x_1, x_2)$ for all $(x_1, x_2) \in \mathcal{M}_1 \times \mathcal{M}_2$.

Then the map $f \colon \mathcal{M} \to \mathcal{N}$ defined by $f|_{\mathcal{M}_i} = f_i$ for $i = 1, 2$ is Lipschitz. Moreover, if $k_\mathcal{N}$ and $k_\mathcal{M}$ are the quasimetric constants of $\mathcal{N}$ and $\mathcal{M}$, respectively, we have

$$\operatorname{Lip}(f) \leq k_\mathcal{N}(C + k_\mathcal{M} + Ck_\mathcal{M}) \max_{i=1,2} \operatorname{Lip}(f_i).$$

Proof. Put $L = \max_{i=1,2} \operatorname{Lip}(f_i)$. Let $(x_1, x_2) \in \mathcal{M}_1 \times \mathcal{M}_2$ and pick $x \in \mathcal{M}_1 \cap \mathcal{M}_2$. Since $f_1(x) = f_2(x)$ we have

$$\begin{aligned}
d_\mathcal{N}(f_1(x_1), f_2(x_2)) &\leq k_\mathcal{N}(d_\mathcal{N}(f_1(x_1), f_1(x)) + d_\mathcal{N}(f_2(x), f_2(x_2))) \\
&\leq k_\mathcal{N} L (d_\mathcal{M}(x_1, x) + d_\mathcal{M}(x, x_2)) \\
&\leq k_\mathcal{N} L ((1+k_\mathcal{M}) d_\mathcal{M}(x_1, x) + k_\mathcal{M} d_\mathcal{M}(x_1, x_2)).
\end{aligned}$$

Since the element x can be chosen so that $d_\mathcal{M}(x_1, x)$ is arbitrarily close to $d_\mathcal{M}(x_1, \mathcal{M}_1 \cap \mathcal{M}_2)$, using the assumption (iii) yields the desired result. □

The following lemma exhibits a situation that will occur several times throughout this section in which Lemma 3.6 is useful.

Lemma 3.7. *Let $\mathcal{M}$ and $\mathcal{N}$ be quasi-metric spaces. For $i = 1, 2$, let $f_i \colon \mathcal{M}_i \subset \mathcal{M} \to \mathcal{N}$ be a Lipschitz function. Assume that $\mathcal{M} = \mathcal{M}_1 \cup \mathcal{M}_2$ and that $f_1|_{\mathcal{M}_1\cap\mathcal{M}_2} = f_2|_{\mathcal{M}_1\cap\mathcal{M}_2}$. Suppose that there are constants $\lambda > 0$ and $C \geq 1$ such that*

(i) $\mathcal{M}$ is λ-separated, and
(ii) $d(x_1, \mathcal{M}_1 \cap \mathcal{M}_2) \leq C$ for every $x_1 \in \mathcal{M}_1$.

Then the map $f\colon \mathcal{M} \to \mathcal{N}$ defined by $f|_{\mathcal{M}_i} = f_i$ for $i = 1, 2$ is Lipschitz. Moreover, if $k_{\mathcal{N}}$ and $k_{\mathcal{M}}$ are the quasimetric constants of $\mathcal{N}$ and $\mathcal{M}$, respectively,

$$\operatorname{Lip}(f) \le k_{\mathcal{N}} \left(\frac{C}{\lambda}(1 + k_{\mathcal{M}}) + k_{\mathcal{M}} \right) \max_{i=1,2} \operatorname{Lip}(f_i).$$

Proof. Let $(x_1, x_2) \in \mathcal{M}_1 \times \mathcal{M}_2$ with $x_1 \ne x_2$. We have

$$d(x_1, \mathcal{M}_1 \cap \mathcal{M}_2) \le \frac{C}{\lambda} d(x_1, x_2).$$

Hence, the result follows from Lemma 3.6. □

Theorem 3.8 (cf. [3, Theorem 5.7])**.** *Let $0 < p \le 1$ and $d \in \mathbb{N}$. Then $\mathcal{F}_p([0,1]^d)$ has a Schauder basis. In fact, if $V_n = [0,1]^d \cap 2^{-n}\mathbb{Z}^d$ for all $n \in \mathbb{N}_*$ and $V_{-1} = \{0\}$, and we put $\alpha(x) = n$ if $x \in V_n \setminus V_{n-1}$, then any arrangement $(f(x_j))_{j=1}^\infty$ of the family*

$$f(x) = \delta_{[0,1]^d}(x) - \sum_{v \in V_{n-1}} \Lambda^d_{2^{-n+1}}(v, x) \delta_{[0,1]^d}(v) \quad n \in \mathbb{N}_*,\ x \in V_n \setminus V_{n-1}$$

such that $(\alpha(x_j))_{j=1}^\infty$ is non-decreasing is a Schauder basis of $\mathcal{F}_p([0,1]^d)$.

Proof. Set $\delta = \delta_{[0,1]^d}$ and $\delta_n = \delta_{V_n}$ for $n \in \mathbb{N}_*$. By Theorem 3.5, there exist a constant C and linear maps $T_n\colon \mathcal{F}_p([0,1]^d) \to \mathcal{F}_p(V_n)$ such that $\|T_n\| \le C$ and

$$T_n(\delta(x)) = \sum_{v \in V_n} \Lambda^d_{2^{-n}}(v, x)\, \delta_n(v), \quad x \in [0,1]^d.$$

Let m, $n \in \mathbb{N}_*$ with $m \le n$. Since $V_m \subset V_n$, we can consider the canonical map $L_{m,n}\colon \mathcal{F}_p(V_m) \to \mathcal{F}_p(V_n)$ associated to the inclusion. Consider also the canonical map $L_n\colon \mathcal{F}_p(V_n) \to \mathcal{F}_p([0,1]^d)$ associated to the inclusion of V_n into $[0,1]^d$. Applying Proposition 3.4 (vi) yields

- $T_m \circ L_n \circ T_n = T_m$ and $T_n \circ L_m \circ T_m = L_{m,n} \circ T_m$.

Moreover, since $\cup_{n=0}^\infty V_n$ is dense in $[0,1]^d$,

- if $X_n = L_n(T_n(\mathcal{F}_p([0,1]^d)))$, $\cup_{n=0}^\infty X_n$ is dense is $\mathcal{F}_p([0,1]^d)$.

Hence, there is a Schauder decomposition $(Y_n)_{n=1}^\infty$ of $\mathcal{F}_p([0,1]^d)$ with associated projections $L_n \circ T_n$. Since, with the convention $L_{-1} \circ T_{-1} = 0$,

$$Y_n = \{x - L_{n-1}(T_{n-1}(x))\colon x \in X_n\},$$

and $X_n = \operatorname{span}\{\delta(x)\colon x \in V_n\}$, the family of nonzero vectors

$$\mathcal{B}_n := (f(x))_{x \in V_n \setminus V_{n-1}}$$

generates the space Y_n. We shall prove that $\mathcal{B}_n$ is an unconditional basis of Y_n with uniformly bounded unconditional basis constant. Let $F \subset V_n \setminus V_{n-1}$. We define $r_{n,F} \colon V_n \to \mathcal{F}_p([0,1]^d)$ by

$$r_{n,F}(x) = \begin{cases} L_{n-1} \circ T_{n-1}(\delta(x)) & \text{if } x \in V_n \setminus F, \\ \delta(x) & \text{if } x \in F. \end{cases}$$

If $z \in V_{n-1}$, then $L_{n-1}(T_{n-1}(\delta(z))) = L_{n-1}(\delta_{n-1}(z)) = \delta(z)$. For every $x \in V_n$ there is $z \in V_{n-1}$ such that $|x - z|_\infty = 2^{-n}$, and V_n is 2^{-n}-separated. By Lemma 3.7, $r_{n,F}$ is C_1-Lipschitz with $C_1 = (1 + 2^{1/p})2^{1/p-1}C$. We infer that there is $U_{n,F} \colon \mathcal{F}_p(V_n) \to \mathcal{F}_p([0,1]^d)$ such that $\|U_{n,F}\| \le C_1$ and $U_{n,F} \circ \delta_n = r_{n,F}$. Put $Q_{n,F} = U_{n,F} \circ T_n|_{Y_n}$. We have $\|Q_{n,F}\| \le CC_1$ and

$$Q_{n,F}(f(x)) = \begin{cases} f(x) & \text{if } x \in F, \\ 0 & \text{if } x \in (V_n \setminus V_{n-1}) \setminus F. \end{cases}$$

Thus, the mappings $(Q_{n,F})_{F \subset V_n \setminus V_{n-1}}$ are commuting projections satisfying $Q_{n,F}(Y_n) = \operatorname{span}\{f(x) \colon x \in F\}$ and $Q_{n,F}^{-1}(0) = \operatorname{span}\{f(x) \colon x \in (V_n \setminus V_{n-1}) \setminus F\}$, which implies that $\mathcal{B}_n$ is unconditional basis of Y_n with basis constant at most CC_1. □

We have explicitly constructed a Schauder basis of $\mathcal{F}_p([0,1]^d)$. Using the isomorphism $\mathcal{F}_p([0,1]^d) \simeq \mathcal{F}_p(\mathbb{R}^d)$ we infer that $\mathcal{F}_p(\mathbb{R}^d)$ has a Schauder basis, but now our proof is not constructive. So, it is worth it to point out that an argument slightly more involved than the one we have used to prove Theorem 3.8 yields an explicit Schauder basis of $\mathcal{F}_p(\mathbb{R}^d)$.

Theorem 3.9. *Let $0 < p \le 1$ and $d \in \mathbb{N}$. Given an increasing sequence of natural numbers $(k_n)_{n=-1}^\infty$, put*

$$V_n = \{x \in \mathbb{R}^d \colon |x|_\infty \le k_n\} \cap 2^{-n}\mathbb{Z}^d, \text{ and}$$
$$W_n = \{x \in \mathbb{R}^d \colon |x|_\infty \le k_{n-1}\} \cap 2^{-n}\mathbb{Z}^d$$

if $n \in \mathbb{N}_*$, and set $V_{-1} = \{0\}$. Define

$$s_n(x) = \left(\min\left\{-2^{-n} + |x|_\infty, |x_i|\right\} \operatorname{sgn}(x_i)\right)_{i=1}^d, \; x = (x_i)_{i=1}^d \in V_n \setminus W_n,$$

and put $\eta(x) = (n, |x|_\infty - k_{n-1})$ if $x \in V_n \setminus W_n$ and $\eta(x) = (n, 0)$ if $x \in W_n \setminus V_{n-1}$. Define $f(x)$ for $x \in \cup_{n=0}^\infty V_n$ by

$$f(x) = \begin{cases} \delta_{\mathbb{R}^d}(x) - \delta_{\mathbb{R}^d}(s_n(x)) & \text{if } x \in V_n \setminus W_n, \\ \delta_{\mathbb{R}^d}(x) - \sum_{v \in V_{n-1}} \Lambda^d_{2^{-n+1}}(v, x)\delta_{\mathbb{R}^d}(v) & \text{if } x \in W_n \setminus V_{n-1}. \end{cases}$$

Let $(x_j)_{j=1}^\infty$ be an arrangement of $\cup_{n=0}^\infty V_n$ such that $(\eta(x_j))_{j=1}^\infty$ is non-decreasing with respect to the lexicographical order. Then $(f(x_j))_{j=1}^\infty$ is a Schauder basis of $\mathcal{F}_p(\mathbb{R}^d)$.

Proof. Given $t \in (0,\infty)$, for $d=1$ define $r_t \colon \mathbb{R} \to \mathbb{R}$ by

$$r_t(x) = \min\left\{1, \frac{t}{|x|}\right\} x, \quad x \in \mathbb{R}.$$

In general, we define $r_t \colon \mathbb{R}^d \to \mathbb{R}^d$ by

$$r_t((x_i)_{i=1}^d) = (r_t(x_i))_{i=1}^d.$$

The map r_t is 1-Lipschitz and its range is $B_t := \{x \in \mathbb{R}^d \colon \|x\|_\infty \le t\}$. Let $S_t \colon \mathcal{F}_p(\mathbb{R}^d) \to \mathcal{F}_p(B_t)$ be obtained by linearization of r_t. Given $R>0$ such that $t/R \in \mathbb{N}$, denote $V_{t,R} = \mathcal{V}_R^d \cap B_t$, and let $T_R \colon \mathcal{F}_p(\mathbb{R}^d) \to \mathcal{F}_p(\mathcal{V}_R^d)$ and $T_{t,R} \colon \mathcal{F}_p(B_t) \to \mathcal{F}_p(V_{t,R})$ be the linear maps provided by Theorem 3.5. Notice that Theorem 3.5 yields a constant C such that $\|T_{t,R}\| \le C$ for all $t>0$ and R with $t/R \in \mathbb{N}$. Let $L_{\mathcal{M},\mathcal{N}}$ denote the linearization of the inclusion of $\mathcal{M}$ into $\mathcal{N}$, and put $L_R = L_{\mathcal{V}_R^d,\mathbb{R}^d}$, $L_{t,R} = L_{V_{t,R},B_t}$ and $L'_{t,R} = L_{V_{t,R},\mathbb{R}^d}$. We shall prove that

$$L_{t,R} \circ T_{t,R} \circ S_t = S_t \circ L_R \circ T_R, \quad \frac{t}{R} \in \mathbb{N}. \tag{3.2}$$

Set $\delta = \delta_{\mathbb{R}^d}$, $\delta_t = \delta_{B_t}$, and $\delta_{t,R} = \delta_{V_{t,R}}$.

By Proposition 3.4 (v), a similar argument works for the general case, hence for notational ease we will deal with the case $d=1$. Let $x \in \mathbb{R}$. In the case when $|x| \le t$, we have $r_t(x) = x$ and $\Lambda_R(v,x) = 0$ unless $|v| \le t$, in which case $r_t(v) = v$. Consequently,

$$\begin{aligned} S_t(L_R(T_R(\delta(x)))) &= \sum_{v \in \mathcal{V}_R} \Lambda_R(v,x)\delta_t(r_t(v)) \\ &= \sum_{v \in \mathcal{V}_R} \Lambda_R(v,x)\delta_t(v) \\ &= L_{t,R}(T_{t,R}(\delta_t(x))) \\ &= L_{t,R}(T_{t,R}(\delta_t(r_t(x)))). \end{aligned}$$

In the case when $|x| > t$ we have $\Lambda_R(v,x) = 0$ unless $|v| \ge t$ and $\operatorname{sgn}(v) = \operatorname{sgn}(x)$, in which case $r_t(v) = r_t(x)$. Then we have

$$\begin{aligned} S_t(L_R(T_R(\delta(x)))) &= \sum_{v \in \mathcal{V}_R} \Lambda_R(v,x)\delta_t(r_t(v)) \\ &= \sum_{v \in \mathcal{V}_R} \Lambda_R(v,x)\delta_t(r_t(x)) \\ &= \delta_t(r_t(x)) \end{aligned}$$

$$= L_{t,R}(T_{t,R}(\delta_t(r_t(x)))).$$

Since $\delta_t(r_t(x)) = S_t(\delta(x))$ and $\{\delta(x)\colon x \in \mathbb{R}\}$ spans $\mathcal{F}_p(\mathbb{R})$, (3.2) holds. Note that the range of $U_{t,R} := T_{t,R} \circ S_t$ is $\operatorname{span}\{\delta_{t,R}(x)\colon x \in V_{t,R}\}$. Put $P_{t,R} := L'_{t,R} \circ T_{t,R} \circ S_t$. By (3.2), for every $x \in \mathbb{R}^d$ we have

$$P_{t,R}(\delta(x)) = \sum_{v\in\mathcal{V}_R^d} \Lambda_R^d(v,x)\delta(r_t(v)), \quad \frac{t}{R} \in \mathbb{N}. \tag{3.3}$$

Thus, for every $v \in \mathcal{V}_R^d$ we have $P_{t,R}(\delta(v)) = \delta(r_t(v))$, which in combination with (3.3) implies that $P_{t,R}$ is a projection with range equal to $\operatorname{span}\{\delta(x)\colon x \in V_{t,R}\}$. We infer that

$$P_{t,R} \circ P_{t',R'} = P_{t',R'} \circ P_{t,R} = P_{t,R}, \quad \frac{R}{R'}, \frac{t'}{R'}, \frac{t}{R} \in \mathbb{N},\ t' \geq t > 0. \tag{3.4}$$

Indeed, $P_{t',R'} \circ P_{t,R} = P_{t,R}$ follows from the fact that the range of $P_{t,R}$ is contained in the range of $P_{t',R'}$ and we have

$$\begin{aligned} P_{t,R} \circ P_{t',R'} &= L'_{t,R} \circ T_{t,R} \circ S_t \circ L_{B_{t'},\mathbb{R}^d} \circ S_{t'} \circ L_{R'} \circ T_{R'} \\ &= L'_{t,R} \circ T_{t,R} \circ S_t \circ L_{R'} \circ T_{R'} \\ &= P_{t,R}, \end{aligned}$$

where in the first equality we used (3.2), in the second we used the observation that $S_t \circ L_{B_{t'},\mathbb{R}^d} \circ S_{t'} = S_t$ and the third equality follows from Proposition 3.4 (vi).

Hence, if for $n \in \mathbb{N}_*$ we put

$$P_{2n} = P_{k_n, 2^{-n}} \text{ and } P_{2n-1} = P_{k_{n-1}, 2^{-n}},$$

we have $P_j \circ P_{j'} = P_{j'} \circ P_j = P_j$ whenever $-1 \leq j \leq j'$. Therefore, there is Schauder decomposition $(Y_j)_{j=-1}^\infty$ of $\mathcal{F}_p(\mathbb{R}^d)$ whose associated projections are $(P_j)_{j=-1}^\infty$. Moreover, for all $n \in \mathbb{N}_*$ the range of P_{2n} is $\operatorname{span}\{\delta(x)\colon x \in V_n\}$ and the range of P_{2n-1} is $\operatorname{span}\{\delta(x)\colon x \in W_n\}$.

Similar arguments as in the proof of Theorem 3.8 show that there is a constant $C = C(p,d)$ such that

$$(f(x))_{x\in W_n \setminus V_{n-1}}$$

is a C-unconditional basis of Y_{2n-1} for all $n \in \mathbb{N}_*$. Let $n \in \mathbb{N}_*$. Note that $r_t(x) = r_t(s_n(x))$ for all t with $2^n t \in \mathbb{N}$ and all $x \in V_n \setminus W_n$ with $\|x\|_\infty > t$. In particular,

$$r_{k_{n-1}}(x) = r_{k_{n-1}}(s_n(x)), \quad x \in V_n \setminus W_n.$$

Therefore, $P_{2n-1}(f(x)) = 0$ for every $x \in V_n \setminus W_n$, which in turn implies that $f(x) \in Y_{2n}$. An inductive argument yields that for every $x \in V_n \setminus W_n$,

$$g(x) := \delta(x) - \delta(r_{k_{n-1}}(x))$$

is a linear combination of the family (of nonzero vectors)

$$\mathcal{B}_n = (f(y))_{y\in V_n\setminus W_n}.$$

Since, by an argument similar to that used in the proof of Theorem 3.8, $(g(x))_{x\in V_n\setminus W_n}$ generates Y_{2n}, $\mathcal{B}_n$ also generates Y_{2n}. Let $F \subset V_n \setminus W_n$ be such that

$$V_{t-2^{-n},2^{-n}} \setminus W_n \subset F \subset V_{t,2^{-n}} \setminus W_n$$

for some $t \in (k_{n-1}, k_n] \cap 2^{-n}\mathbb{Z}$. Set $\alpha = (t, 2^{-n})$ and let $\widehat{\alpha}$ be the “predecesor” of α given by $\widehat{\alpha} = (t-2^{-n}, 2^{-n})$. Define $r_{\alpha,F} \colon V_\alpha \to \mathbb{R}^d$ by

$$r_{\alpha,F}(x) = \begin{cases} r_{t-2^{-n}}(x) & \text{if } x \in V_\alpha \setminus F, \\ x & \text{if } x \in F. \end{cases}$$

If $x \in V_{\widehat{\alpha}}$ then $r_{t-2^{-n}}(x) = x$. Given $x \in V_\alpha$ there is $z \in V_{\widehat{\alpha}}$ with $|x-z|_\infty = 2^{-n}$, and V_α is 2^{-n}-separated. Hence, by Lemma 3.7, $r_{\alpha,F}$ is C_p-Lipschitz, where $C_p = (1+2^{1/p})$. Let $U_{\alpha,F} \colon \mathcal{F}_p(V_\alpha) \to \mathcal{F}_p(\mathbb{R}^d)$ be the linear map defined by $\delta_\alpha(x) \mapsto \delta(x)$ if $x \in F$ and $\delta_\alpha(x) \mapsto \delta(r_{t-2^{-n}}(x))$ if $x \in V_\alpha \setminus F$. Set $U_\alpha = T_\alpha \circ S_t$. We infer that, if

$$Q_{\alpha,F} = U_{\alpha,F} \circ U_\alpha|_{Y_{2n}},$$

then $\|Q_{\alpha,F}\| \le CC_p$. Note that $U_\alpha(\delta(x)) = \delta_\alpha(x)$ for all $x \in V_\alpha$ and that $U_\alpha(\delta(x)) = U_\alpha(\delta(s_n(x)))$ for all $x \in V_n \setminus V_\alpha$. If $x \in V_\alpha$, then $s_n(x) \in V_{\widehat{\alpha}}$, and so $Q_{\alpha,F}(f(x)) = f(x)$ for every $f \in F$. If $x \in V_\alpha \setminus V_{\widehat{\alpha}}$, then $s_n(x) = r_{t-2^{-n}}(x)$. Consequently, $Q_{\alpha,F}(f(x)) = 0$ for all $x \in V_\alpha \setminus F$. Finally, we deduce that $Q_{\alpha,F}(f(x)) = 0$ for all $x \in V_n \setminus V_\alpha$. We infer that, if $(x_j)_{j=1}^{|V_n|-|W_n|}$ is an arrangement of $V_n \setminus W_n$ with $(|x_j|)_{j=1}^{|V_n|-|W_n|}$ non-decreasing, then $(f(x_j))_{j=1}^{|V_n|-|W_n|}$ is a Schauder basis of Y_{2n-1} with basis constant at most CC_p. □

4. Open problems

By [22, Theorem 5.2], there exists a subspace Z of ℓ_p whose Banach envelope is isomorphic to L_1, so we would like to know whether $\mathcal{F}_p(\mathbb{R})$ is different from the subspace Z, whose existence is guaranteed by a general abstract construction.

Question 4.1. Let $p \in (0,1)$. Is $\mathcal{F}_p(\mathbb{R})$ isomorphic to a subspace of ℓ_p?

Once we know that $\mathcal{F}_p(\mathbb{N})$ is not isomorphic to ℓ_p for $p < 1$, it is natural to further the topic **(Q.b)** by trying to determine how many non-isomorphic Lipchitz free p-spaces one can obtain from subsets of $\mathbb{N}$. Note that if $\mathcal{N}$ is a subset of $\mathbb{N}$ then $\mathcal{F}_p(\mathcal{N})$ is a complemented subspace of $\mathcal{F}_p(\mathbb{N})$ by **(A.7)**. So, this problem connects with the problem of characterizing the complemented subspaces of $\mathcal{F}_p(\mathbb{N})$. Let us illustrate the issue with an example. Suppose that $\mathcal{N} \subset \mathbb{N}$ contains arbitrarily long chains of consecutive integers. Then there is $(a_k)_{k=0}^{\infty}$ in $\mathbb{N}$ such that $a_k + \mathbb{N}_{2^k-1} \subset \mathcal{N}$ and $2(a_k + 2^k - 1) \le a_{k+1}$ for all $k \in \mathbb{N}_*$. Set

$$\mathcal{N}_0 = \bigcup_{k=0}^{\infty} a_k + \mathbb{N}_{2^k-1}.$$

Combining [3, Lemma 2.1] with [2, Theorem 5.8] yields $\mathcal{F}_p(\mathcal{N}_0) \simeq \mathcal{F}_p(\mathbb{N})$. Then, by **(A.7)**, $\mathcal{F}_p(\mathcal{N})$ is complemented in $\mathcal{F}_p(\mathbb{N})$ and, the other way around, $\mathcal{F}_p(\mathbb{N})$ is complemented in $\mathcal{F}_p(\mathcal{N})$. Taking into account **(A.5)**, Pełczyński's decomposition method yields $\mathcal{F}_p(\mathcal{N}) \simeq \mathcal{F}_p(\mathbb{N})$.

Question 4.2. Let $0 < p < 1$. Does there exist $\mathcal{N} \subset \mathbb{N}$ such that $\mathcal{F}_p(\mathcal{N})$ is neither isomorphic to ℓ_p nor to $\mathcal{F}_p(\mathbb{N})$?

A well-known problem in Geometric Functional Analysis is whether $\mathcal{F}(\mathbb{N}^2)$ is isomorphic to $\mathcal{F}(\mathbb{N}^3)$ or, more generally, whether $\mathcal{F}(\mathbb{N}^d)$ is isomorphic to $\mathcal{F}(\mathbb{N}^{d+1})$ for $d \ge 2$. By [4, Proposition 4.20], if $\mathcal{F}_p(\mathbb{N}^d) \simeq \mathcal{F}_p(\mathbb{N}^{d+1})$ for some $p < 1$ it would follow that $\mathcal{F}(\mathbb{N}^d) \simeq \mathcal{F}(\mathbb{N}^{d+1})$. Hence, investigating in more depth the geometry of $\mathcal{F}_p(\mathbb{N}^d)$ for $p < 1$ and $d \ge 2$ could shed some light on that important problem. Distinguishing $\mathcal{F}_p(\mathbb{R}^d)$ from $\mathcal{F}_p(\mathbb{R}^{d+1})$ for $p < 1$ could also be easier to tackle than telling apart $\mathcal{F}(\mathbb{R}^d)$ from $\mathcal{F}(\mathbb{R}^{d+1})$, and by [2, Theorem 4.21], showing that $\mathcal{F}_p(\mathbb{R}^d)$ is not isomorphic to $\mathcal{F}_p(\mathbb{R}^{d+1})$ is equivalent to proving that the Lipschitz free p-spaces over Euclidean spaces are not isomorphic to the Lipschitz free p-spaces over their spheres.

Question 4.3. Let $0 < p \le 1$ and $d \ge 2$. Is $\mathcal{F}_p(\mathbb{N}^d)$ is isomorphic to $\mathcal{F}_p(\mathbb{N}^{d+1})$?

Question 4.4. Let $0 < p \le 1$ and $d \ge 3$. Is $\mathcal{F}_p(\mathbb{R}^d)$ is isomorphic to $\mathcal{F}_p(S_{\mathbb{R}^d})$?

Corollary 2.14 leads naturally to wonder about the geometry of $\mathcal{F}_p(X)$ for $0 < p < 1$ in the case when X is a p-Banach space with trivial dual. In order to take further this line of research, the first space to look at is the Lipschitz free p-space over $X = L_p = L_p([0, 1])$. To the best of our knowledge, all that is known about $\mathcal{F}_p(L_p)$ is that its q-Banach envelope is trivial for all $p < q \le 1$ and that it contains a complemented subspace isometric to L_p. Indeed, the former assertion follows from [4, Proposition 3.7 and Proposition 4.20], and the latter can be deduced from [4, Theorem 4.13], which gives that L_p is isometric to a certain Lipschitz free p-space. Note that for every p-Banach space X there is a natural bounded linear map $\beta_X := T_{\mathrm{Id}_X} : \mathcal{F}_p(X) \to X$ given by $\delta_X(x) \mapsto x$ for all $x \in X$. In the particular case that $X = \mathcal{F}_p(\mathcal{M})$ for some p-metric space $\mathcal{M}$, the linearization $L_{\delta_\mathcal{M}} \colon X \to \mathcal{F}_p(X)$ of the canonical isometric embedding of $\mathcal{M}$ into X is a linear lifting of β_X.

Question 4.5. Let $0 < p < 1$. Is $\mathcal{F}_p(L_p)$ a $\mathscr{L}_p$-space?

Note that since $\mathcal{F}(\mathbb{R}^2)$ is not a $\mathscr{L}_1$-space, we infer that $\mathcal{F}(L_1)$ is not a $\mathscr{L}_1$-space either (see [25]). Thus, a positive answer to Question 4.5 would evince an important structural difference between the *Kalton zone* $p < 1$ and the case $p = 1$.

A detailed look at the proofs of Theorems 3.8 and 3.9 shows that the basis constant of the Schauder basis $\mathcal{B} := (f(x_j))_{j=1}^{\infty}$ of $\mathcal{F}_p(\mathbb{R}^d, \|\cdot\|_\infty)$ is bounded above by $C(p)\, C(p,d)$, where

$$C(p) = (1 + 2^{1/p})^2 2^{1/p-1}$$

and $C(p,d)$ is the constant from Theorem 3.5, which can be estimated by

$$C(p,d) \le (1 + 2d(2^d - 1))^{1/p}.$$

It is known that if we replace the ℓ_∞-norm on $\mathbb{R}^d$ with the ℓ_1-norm, Theorem 3.5 in the case when $p = 1$ holds with $C = 1$, which gives an estimate independent of the dimension for the basis constant of $\mathcal{B}$ regarded in $\mathcal{F}(\mathbb{R}^d, \|\cdot\|_1)$ (see [20, 23]). However, our arguments do not yield an estimate independent of the dimension for the basis constant of $\mathcal{B}$ in $\mathcal{F}_p(\mathbb{R}^d, \|\cdot\|_1)$. Hence, it might be an interesting problem to determine whether such an estimate exists:

Question 4.6. Let $r \colon K \to \mathcal{F}_p(V)$ be the Lipschitz map defined in Theorem 3.5. Suppose that both K and V are equipped with the ℓ_1-distance. Is there a constant C depending on p but not on d, K, or V, such that $\mathrm{Lip}(r) \le C$?

References

[1] F. Albiac, J. L. Ansorena, P. M. Berná, and P. Wojtaszczyk, *Greedy approximation for biorthogonal systems in quasi-banach spaces*, arXiv e-prints (2019), available at `1903.11651`. Accepted for Publication in Dissertationes Math.

[2] F. Albiac, J. L. Ansorena, M. Cúth, and M. Doucha, *Lipschitz free spaces isomorphic to their infinite sums and geometric applications*, arXiv e-prints (2020), available at `2005.06555`.

[3] F. Albiac, J. L. Ansorena, M. Cúth, and M. Doucha, *Embeddability of ℓ_p and bases in Lipschitz free p-spaces for* $0 < p \le 1$, J. Funct. Anal. **278** (2020), no. 4, 108354, 33.

[4] F. Albiac, J. L. Ansorena, M. Cúth, and M. Doucha, *Lipschitz free p-spaces for* $0 < p < 1$, Israel J. Math. **240** (2020), no. 1, 65–98.

[5] F. Albiac, J. L. Ansorena, and P. a. Wojtaszczyk, *On certain subspaces of ℓ_p for $0 < p \leq 1$ and their applications to conditional quasi-greedy bases in p-Banach spaces*, Math. Ann. **379** (2021), no. 1-2, 465–502.
[6] F. Albiac and N. J. Kalton, *Lipschitz structure of quasi-Banach spaces*, Israel J. Math. **170** (2009), 317–335.
[7] ———, *Topics in Banach space theory*, Second, Graduate Texts in Mathematics, vol. 233, Springer, [Cham], 2016. With a foreword by Gilles Godefroy.
[8] L. Ambrosio and D. Puglisi, *Linear extension operators between spaces of Lipschitz maps and optimal transport*, J. Reine Angew. Math. **764** (2020), 1–21.
[9] P. G. Casazza, *Approximation properties*, Handbook of the geometry of Banach spaces, Vol. I, North-Holland, Amsterdam, 2001, pp. 271–316.
[10] M. Cúth and M. Doucha, *Lipschitz-free spaces over ultrametric spaces*, Mediterr. J. Math. **13** (2016), no. 4, 1893–1906.
[11] A. Dalet, *Free spaces over countable compact metric spaces*, Proc. Amer. Math. Soc. **143** (2015), no. 8, 3537–3546.
[12] ———, *Free spaces over some proper metric spaces*, Mediterr. J. Math. **12** (2015), no. 3, 973–986.
[13] M. Doucha and P. L. Kaufmann, *Approximation properties in Lipschitz-free spaces over groups*, arXiv e-prints (2020), available at `2005.09785`.
[14] V. P. Fonf and P. Wojtaszczyk, *Properties of the Holmes space*, Topology Appl. **155** (2008), no. 14, 1627–1633.
[15] G. Godefroy and N. J. Kalton, *Lipschitz-free Banach spaces*, Studia Math. **159** (2003), no. 1, 121–141. Dedicated to Professor Aleksander Pełczyński on the occasion of his 70th birthday.
[16] G. Godefroy, *Extensions of Lipschitz functions and Grothendieck's bounded approximation property*, North-West. Eur. J. Math. **1** (2015), 1–6.
[17] ———, *A survey on Lipschitz-free Banach spaces*, Comment. Math. **55** (2015), no. 2, 89–118.
[18] G. Godefroy and N. Ozawa, *Free Banach spaces and the approximation properties*, Proc. Amer. Math. Soc. **142** (2014), no. 5, 1681–1687.
[19] P. Hájek and M. Novotný, *Some remarks on the structure of Lipschitz-free spaces*, Bull. Belg. Math. Soc. Simon Stevin **24** (2017), no. 2, 283–304.
[20] P. Hájek and E. Pernecká, *On Schauder bases in Lipschitz-free spaces*, J. Math. Anal. Appl. **416** (2014), no. 2, 629–646.
[21] N. J. Kalton, *Locally complemented subspaces and $\mathcal{L}_p$-spaces for $0 < p < 1$*, Math. Nachr. **115** (1984), 71–97.
[22] ———, *Banach envelopes of nonlocally convex spaces*, Canad. J. Math. **38** (1986), no. 1, 65–86.
[23] G. Lancien and E. Pernecká, *Approximation properties and Schauder decompositions in Lipschitz-free spaces*, J. Funct. Anal. **264** (2013), no. 10, 2323–2334.
[24] J. Lindenstrauss, *Extension of compact operators*, Mem. Amer. Math. Soc. **48** (1964), 112.
[25] A. Naor and G. Schechtman, *Planar earthmover is not in L_1*, SIAM J. Comput. **37** (2007), no. 3, 804–826.
[26] E. Pernecká and R. J. Smith, *The metric approximation property and Lipschitz-free spaces over subsets of $\mathbb{R}^N$*, J. Approx. Theory **199** (2015), 29–44.

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