Series representations of the Volterra function and the Fransén-Robinson constant

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Abstract

The Volterra function $\mu(t,\beta,\alpha)$ was introduced by Vito Volterra in 1916 as the solution to certain integral equations with a logarithmic kernel. Despite the large number of applications of the Volterra function, the only known analytic representations of this function are given in terms of integrals. In this paper we derive several convergent expansion of $\mu(t,\beta,\alpha)$ in terms of incomplete gamma functions. These expansions may be used to implement numerical evaluation techniques for this function. As a particular application, we derive a numerical series representation of the Fransén-Robinson constant $F:=\mu(1,1,0)=\int_0^\infty \frac{1}{\Gamma(x)}\,dx$. Some numerical examples illustrate the accuracy of the approximations.

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1 Introduction

The Volterra function is defined by means of the following definite integral [3]:

$$\mu(t,\beta,\alpha):=\frac{1}{\Gamma(\beta+1)}\int_0^\infty \frac{t^{u+\alpha}u^\beta}{\Gamma(u+\alpha+1)}\,du, \tag{1}$$

with $\Re\beta>-1$ and $t>0$. Some particular notations are usually adopted in the following special cases:

$$\nu(t):=\mu(t,0,0)=\int_0^\infty \frac{t^u}{\Gamma(u+1)}\,du, \tag{2}$$

$$\nu(t,\alpha) := \mu(t,0,\alpha) = \int_0^\infty \frac{t^{u+\alpha}}{\Gamma(u+\alpha+1)}\,du, \tag{3}$$

$$\mu(t,\beta) := \mu(t,\beta,0) = \frac{1}{\Gamma(\beta+1)}\int_0^\infty \frac{t^u\,u^\beta}{\Gamma(u+1)}du. \tag{4}$$

The four functions defined above are analytic functions of t with branch-points at $t=0$ and $t=\infty$ and no other singularity [4]. Also, $\nu(t,\alpha)$ and $\mu(t,\beta,\alpha)$ are entire functions of α and the definition of μ can be extended to the entire $\beta-$plane by repeated integration by parts:

$$\mu(t,\beta,\alpha) = \frac{(-1)^m}{\Gamma(\beta+m+1)}\int_0^\infty u^{\beta+m}\frac{d^m}{du^m}\left[\frac{t^{\alpha+u}}{\Gamma(u+\alpha+1)}\right]du, \qquad \Re\beta > -m-1. \tag{5}$$

The right hand side above is an explicit representation of the analytical continuation of $\mu(t,\beta,\alpha)$ from $\Re\beta > -1$ to the whole complex $\beta-$plane (for large enough m).

These functions were introduced by Vito Volterra in 1916 as solutions to certain integral equations with a logarithmic kernel [14]. Several other famous mathematicians have considered these functions along the twentieth century, in the study of prime numbers, the Laplace transform, several types of integral equations, and other important fields in mathematics (see for example [7], [9], [13] and references there in). At the beginning of the twenty-first century, Volterra functions were considered by Mainardi et al. [10], [11] as solutions to fractional relaxation/diffusion equations of distributed order. Other applications in fractional calculus may be found in [4, Chap. 18].

More recently, A. Apelblat has collected a comprehensive set of information about the Volterra function, providing a historical perspective, abundant bibliography, important identities and several integral transforms related to this function [1]. In particular, we have the following recurrence relation,

$$t\,\mu(t,\beta,\alpha) = (\beta+1)\,\mu(t,\beta+1,\alpha+1) + (\alpha+1)\,\mu(t,\beta,\alpha+1), \tag{6}$$

that we show here for convenience in our later analysis. Recurrence (6) may be derived from (1) and the duplication formula of the gamma function $\Gamma(u+1) = u\,\Gamma(u)$.

A specially interesting particular case of the Volterra function is the so called *Fransén-Robinson constant* F, defined in the form

$$F := \int_0^\infty \frac{1}{\Gamma(x)}\,dx. \tag{7}$$

Therefore, *the Fransén-Robinson constant* is nothing but $F = \mu(1,0,-1) = \mu(1,1,0)$, and has several important applications in Statistics [5]: for any positive constant c, the reciprocal Gamma function $1/\Gamma(x)$ decreases faster than e^{-cx}, and thus it may be useful as a one-sided density function for certain probability models. Then, the value of F is needed for the sake of normalization.

Despite its importance in mathematics, most books on Special Functions do not consider the Volterra function. Then, in particular, convergent and asymptotic expansions of this functions have not been fully investigated. To our knowledge, convergent expansions are not known. About its asymptotics, we can find an asymptotic study of this function in [4], and in the more recent publication [6]. More precisely, an asymptotic expansion for $t \to 0$ and $t \to \infty$ may be found in [6, eq.(4.1)] and [6, eq.(4.8)] respectively. On the one hand, we have

$$\mu(t,\beta,\alpha) \sim t^\alpha \sum_{n=0}^{\infty} (\beta+1)_n D_n^{(\alpha)} \left(\log\frac{1}{t}\right)^{-\beta-1-n}, \qquad t \to 0, \quad t \in \mathbb{C} \setminus [1,+\infty), \tag{8}$$

with

$$D_n^{(\alpha)} := \frac{(-1)^n}{n!}\mu(1,-n-1,\alpha) = \frac{1}{n!}\frac{d^n}{dx^n}\left[\frac{1}{\Gamma(\alpha+x+1)}\right]_{x=0}.$$

On the other hand, we have

$$\mu(t,\beta,\alpha) \sim E(t,\beta,\alpha) + H(t,\beta,\alpha), \qquad |t| \to \infty \qquad |\arg(t)| < \pi, \tag{9}$$

where $H(t,\beta,\alpha)$ is the expansion in the right hand side of (8), and

$$E(t,\beta,\alpha) := e^t \sum_{n=0}^{\infty} \frac{E_n^{(\alpha,\beta)}}{\Gamma(\beta+1-n)} t^{\beta-n},$$

where the coefficients $E_n^{(\alpha,\beta)}$ are given by

$$E_n^{(\alpha,\beta)} := \frac{(-1)^n}{n!}\frac{\mathrm{d}^n}{\mathrm{d}x^n}\left[\frac{(1-x)^{-\alpha-1}(-x)^{\beta+1}}{\log^{\beta+1}(1-x)}\right]_{x=0}.$$

Observe that the coefficients $D_n^{(\alpha,\beta)}$ and $E_n^{(\alpha,\beta)}$ of these expansions are not given explicitly, but must be computed by evaluating derivatives of Gamma and other elementary functions.

Then, asymptotic expansions of the Volterra function for large and small $|t|$ are known. But as far as we know, convergent expansions of the Volterra function are not available. Convergent expansions could be used for its analytic approximation and, eventually, could be implemented in algorithms that would let the evaluation of this function. In this paper we provide a family of convergent series representations of the Volterra function in terms of incomplete gamma functions. As a particular case, we derive a numerical series representation of the Fransén-Robinson constant.

The paper is organized as follows: in the following section we introduce some preliminary results needed for our later analysis. In particular, we derive an integral representation of the Volterra function different from the original definition (1) that is more appropiate for our analysis. Then, we consider a convenient expansion of the incomplete gamma function and we define a family of definite integrals related to the incomplete gamma function. In Section 3 we derive the main result of the paper, a family of convergent series representations of the Volterra function. A numerical series representation of the Fransén-Robinson constant and some numerical experiments that show the accuracy of our expansion are postponed to Section 4.

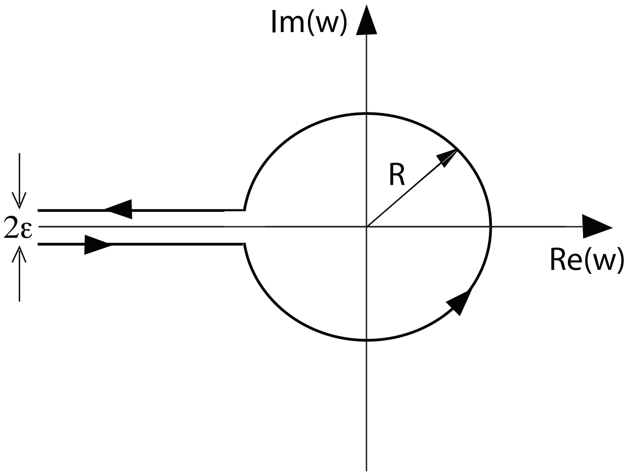

Figure 1: A possible path C is obtained joining a circle of radius $R > |t|$ with the contour of the strip $\Im w = \pm\epsilon$, $0 < \epsilon < R$; traversed in the counterclockwise direction. Eventually, we may take $\epsilon \to 0$.

2 Preliminaries

2.1 A convenient integral representation of the Volterra function

Consider the integral representation of the reciprocal gamma function [2, eq.5.9.2]:

$$\frac{1}{\Gamma(z)} = \frac{1}{2\pi i}\int_C e^w w^{-z} dw, \qquad z \in \mathbb{C},$$

where the integration contour C is the closed loop $(-\infty, 0+)$ that starts at $w = -\infty$ with $\arg(w) = -\pi$, surrounds the point $w = 0$ counterclockwise and comes back to $w = -\infty$ with $\arg(w) = \pi$ (see Figure 1). Replacing this expression into the integral (1) and interchanging the order of integration we find:

$$\mu(t,\beta,\alpha) = \frac{t^\alpha}{\Gamma(\beta+1)\, 2\pi i}\int_C e^w w^{-\alpha-1} dw \int_0^\infty \left(\frac{t}{w}\right)^u u^\beta du, \qquad \left|\frac{t}{w}\right| < 1.$$

The restriction $|w| > |t|$ is necessary for the convergence of the inner integral. This restriction imposes that the distance from the contour C to the origin $w = 0$ must be larger than $|t|$. Then, evaluating the inner integral we find

$$\mu(t,\beta,\alpha) = \frac{t^\alpha}{2\pi i}\int_C \frac{e^w w^{-\alpha-1}}{\log^{\beta+1}(w/t)} dw. \tag{10}$$

For $\Re\alpha \geq -1$ we can deform the contour C to the vertical contour $C' := \{w = R + iu;\ -\infty < u < \infty\}$, with $R > |t|$, and then

$$\mu(t,\beta,\alpha) = \frac{e^R t^\alpha}{2\pi}\int_{-\infty}^\infty \frac{e^{iu}(R+iu)^{-\alpha-1}}{\log^{\beta+1}[(R+iu)/t]} du. \tag{11}$$

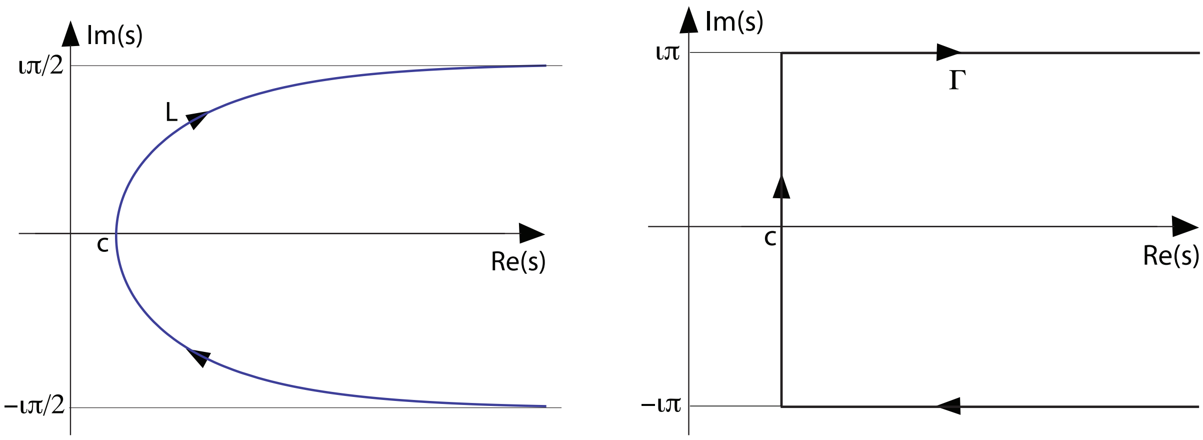

Figure 2: Integration contour L in integral (12) (left) and integration contour Γ in integral (13) (right). Both paths intersect the real axis at an arbitrary point $c > 0$.

After the change of variable $u \to s$ defined by the equality $R + iu = te^s$, or $s = \log\left(\frac{R+iu}{t}\right)$ with $|\Im s| < \pi$, the Volterra function is written in the form

$$\mu(t, \beta, \alpha) = \frac{1}{2\pi i} \int_L e^{-\alpha s} s^{-\beta-1} e^{te^s} ds, \tag{12}$$

where the contour $L := \{s = \log\left(\frac{R+iu}{t}\right); -\infty < u < \infty, |\Im s| \leq \pi/2\}$ is depicted in Figure 2 (left). The right hand side of (12) is an analytic function of β. Therefore, it is an explicit expression for the analytic continuation of $\mu(t, \beta, \alpha)$ defined in (1) from the half-plane $\Re\beta > -1$ to the entire complex $\beta-$plane. The path L in formula (12), Figure 2 (left), may be further deformed to the path Γ in Figure 2 (right): $L \to \Gamma := \{s = u - i\pi; c < u < \infty\} \cup \{s = c + iu; -\pi < u < \pi\} \cup \{s = u + i\pi; c < u < \infty\}$, where $c := \log(R/t)$ is any positive constant. Then, for any arbitrary $c > 0$, the Volterra function may be written as the sum of the integrals

$$\mu(t, \beta, \alpha) = \mu_0(t, \beta, \alpha, c) + \mu_\infty(t, \beta, \alpha, c), \tag{13}$$

with

$$\begin{aligned} \mu_\infty(t, \beta, \alpha, c) &:= \frac{e^{-i\alpha\pi}}{2\pi i} \int_c^\infty \frac{e^{-\alpha u} e^{-te^u}}{(u + i\pi)^{\beta+1}} du - \frac{e^{i\alpha\pi}}{2\pi i} \int_c^\infty \frac{e^{-\alpha u} e^{-te^u}}{(u - i\pi)^{\beta+1}} du, \\ \mu_0(t, \beta, \alpha, c) &:= \frac{e^{-\alpha c}}{2\pi} \int_{-\pi}^{\pi} \frac{e^{-i\alpha u} e^{te^c e^{iu}}}{(c + iu)^{\beta+1}} du. \end{aligned} \tag{14}$$

2.2 An expansion of the incomplete Gamma function

For later convenience, in the following proposition we derive a convergent expansion of the incomplete Gamma function in terms of Laguerre polynomials.

Proposition 1. *Let $a \in \mathbb{C}$ and $x \in \mathbb{R}$ such that $x > 0 > \Re a$. Then, for any $n \in \mathbb{N}$,*

$$\Gamma(a,x) = e^{-x}\, x^a \sum_{m=0}^{n-1} \mathcal{L}_m(x)\, \beta(1-a, m+1) + r_n(a,x), \tag{15}$$

where $\mathcal{L}_n(x)$ are the Laguerre polynomials [8], $\beta(a,b)$ is the beta function [2, Sec. 5.12] and the remainder term is bounded in the form

$$|r_n(a,x)| \le e^{-x/2}\, x^{\Re a}\, \beta(-\Re a, n+1) \le e^{-x/2}\, x^{\Re a}\, \Gamma(-\Re a)\, n^{\Re a}. \tag{16}$$

The expansion is convergent for any complex a in the semi-plane $\Re a \le \Lambda < 0$, and the convergence rate is of power order: $r_n(a,x) = \mathcal{O}(n^{\Re a})$ when $n \to \infty$.

Proof. From the integral representation of the incomplete Gamma function [12, Eq. 8.2.2], and after a straightforward change of variable, we obtain

$$\Gamma(a,x) = x^a \int_0^\infty e^{a\,u} e^{-x\,e^u}\, du = x^a \int_0^1 t^{-a-1} e^{-x/t}\, dt. \tag{17}$$

On the other hand, the Laguerre polynomials $\mathcal{L}_n(x)$, have the following generating function [8, eq.18.12.13],

$$\frac{e^{x\frac{w}{w-1}}}{1-w} = \sum_{n=0}^\infty \mathcal{L}_n(x)\, w^n, \qquad |w| < 1.$$

When we set $w = 1-t$ we obtain

$$\frac{1}{t}\, e^{-\frac{x}{t}} = e^{-x} \sum_{n=0}^\infty \mathcal{L}_n(x)\,(1-t)^n, \qquad |1-t| < 1. \tag{18}$$

Replacing the exponential factor in the last integral in (17) by the right hand side of (18) and interchanging summation and integration we obtain (15) with

$$r_n(a,x) := e^{-x}\, x^a \int_0^1 t^{-a} \sum_{m=n}^\infty \mathcal{L}_m(x)\,(1-t)^m\, dt. \tag{19}$$

Taking into account the bound $e^{-x/2}\,|\mathcal{L}_n(x)| \le 1$ [8, eq. 18.14.8], the integral representation of the beta function, the bound $|\Gamma(z+a)/\Gamma(z+b)| \le |z|^{a-b}$ valid for $\Re z > 0$ and $0 \le a \le b-1$ [2, Eq. 5.6.8], and formula $\sum_{m=n}^\infty (1-x)^m = \frac{(1-x)^n}{x}$, valid for $|1-x| < 1$, we derive the two bounds in (16). $\square$

2.3 A family of definite integrals

For later convenience we define the following family of definite integrals:

$$\phi(a,b,c) := \int_{-\pi}^{\pi} \frac{e^{-a\,(c+iu)}}{(c+iu)^{b+1}} du; \qquad a,b,c \in \mathbb{C}, \tag{20}$$

with the restriction $\Re b < 0$ if $\Re c = 0$ and $\Im c \in [-\pi, \pi]$.

They admit the following integral representation if $\Re c > 0$ and $\Re b > -1$:

$$\phi(a,b,c) = \frac{2e^{-c\,a}}{\Gamma(b+1)} \int_{0}^{\infty} \frac{x^b e^{-c\,x} \sin(\pi(a+x))}{a+x} dx, \tag{21}$$

which follows from the well known integral representation of the Gamma function [2, Eq. 5.9.1], also valid for $\Re c > 0$ and $\Re b > -1$,

$$\frac{\Gamma(b+1)}{z^{b+1}} = \int_{0}^{\infty} w^b e^{-z\,w}\, dw. \tag{22}$$

In the following proposition we compute the integrals $\phi(a,b,c)$ in terms of incomplete gamma functions.

Proposition 2. *Let $a,b,c \in \mathbb{C}$ with the restrictions $\Re b < 0$ if $\Re c = 0$ and $\Im c \in [-\pi, \pi]$. Then, the following formulas hold true,*

1. *If $a \neq 0$,*

   (1.1) and $b \notin \mathbb{Z}$,

   $$\phi(a,b,c) = \frac{i\,\pi}{\Gamma(b+1)\,\sin(\pi b)} \left[\frac{\gamma^*(-b, a\,(c+i\,\pi))}{(c+i\,\pi)^b} - \frac{\gamma^*(-b, a\,(c-i\,\pi))}{(c-i\,\pi)^b} \right],$$

   where $\gamma^*(\alpha, z)$ is the regularized incomplete gamma function $\gamma^*(\alpha, z) := \frac{z^{-\alpha}}{\Gamma(\alpha)}\gamma(\alpha, z)$.

   (1.2) When $b \in \mathbb{Z}$,

   $$\begin{aligned}\phi(a,b,c) =& i\,a^b \left[\Gamma(-b, a\,(c+i\pi)) + \frac{(-1)^b(\log(a\,(c+i\pi)) - \log(c+i\pi))}{\Gamma(b+1)} - \right. \\ & \left. \Gamma(-b, a\,(c-i\pi)) - \frac{(-1)^b(\log(a\,(c-i\pi)) - \log(c-i\pi))}{\Gamma(b+1)} \right].\end{aligned}$$

2. *If $a = 0$,*

   (2.1) and $b \neq 0$,

   $$\phi(0,b,c) = \frac{2}{b} \frac{\sin[\,b\,\arctan(\pi/c)\,]}{\left(\sqrt{c^2+\pi^2}\right)^b}.$$

(2.2) When $b = 0$,

$$\phi(0,0,c) = 2\arctan\left(\frac{\pi}{c}\right).$$

Proof. Replacing the power series expansion of the exponential function in the integral in (20) and interchanging sum and integral we obtain

$$\phi(a,b,c) = \frac{1}{i(c+i\pi)^b}\sum_{n=0}^{\infty}\frac{(-a(c+i\pi))^n}{n!\,(n-b)} - \frac{1}{i(c-i\pi)^b}\sum_{n=0}^{\infty}\frac{(-a(c-i\pi))^n}{n!\,(n-b)}. \tag{23}$$

Now, from the series representation [12, eq.8.7.1] of the regularized incomplete gamma function $\gamma^*(\alpha, z)$ we obtain *(1.1)*. Taking the limit $b \to n$, with n integer, we get *(1.2)*. Finally, formulas *(2.1)* and *(2.2)* follow after a straightforward computation of the integral (20). □

3 A family of series representations of the Volterra function

In this section we derive our main result: a convergent expansion of the Volterra function in terms of incomplete gamma functions and elementary functions. The starting point is the integral representation given in (13)-(14) with $\Re\alpha \geq -1$, $\Re\beta > -1$, $t > 0$ and arbitrary $c > 0$. In the two following subsections we analyze the respective integrals $\mu_\infty(t,\beta,\alpha,c)$ and $\mu_0(t,\beta,\alpha,c)$ and derive a series representation of each one. Because of the recurrence (6), without loss of generality, we restrict ourselves to $\Re\alpha > 0$ in the remaining of the paper.

3.1 A family of series representations of $\mu_\infty(t,\beta,\alpha,c)$

From the well known integral definition of the Gamma function (22), and using Fubini's theorem in the definition (14) of $\mu_\infty(t,\beta,\alpha,c)$, we find

$$\mu_\infty(t,\beta,\alpha,c) = \frac{1}{\Gamma(\beta+1)\,2\pi i}\times$$

$$\left[\int_c^\infty e^{-\alpha(u+i\pi)}e^{-te^u}\int_0^\infty v^\beta e^{-(u+i\pi)\,v}\,dv\,du - \int_c^\infty e^{-\alpha(u-i\pi)}e^{-te^u}\int_0^\infty v^\beta e^{-(u-i\pi)\,v}\,dv\,du\right] =$$

$$-\frac{1}{\Gamma(\beta+1)\,\pi}\int_0^\infty v^\beta\,\sin(\pi(v+\alpha))\int_c^\infty e^{-u(v+\alpha)}e^{-te^u}\,du\,dv.$$

The later integral is an incomplete Gamma function and then

$$\mu_\infty(t,\beta,\alpha,c) = -\frac{1}{\Gamma(\beta+1)\,\pi}\int_0^\infty v^\beta\,\sin(\pi(v+\alpha))\,t^{v+\alpha}\,\Gamma(-v-\alpha, te^c)\,dv. \tag{24}$$

Using formula (15) in Proposition 1 with $a = -v - \alpha$ and $x = te^c$, and (21) we obtain that, for any $n \in \mathbb{N}$,

$$\mu_\infty(t, \beta, \alpha, c) = \frac{e^{-t\,e^c}\,e^c}{2\,\pi} \sum_{m=0}^{n-1} \mathcal{L}_m(t\,e^c) \sum_{k=0}^{m} \binom{m}{k} e^{c\,k}\,\phi(\alpha + k + 1, \beta, c) + R_n^\infty(t, c, \beta, \alpha), \quad (25)$$

where the functions $\phi(a, b, c)$ have been defined in Section 2.3. On the other hand,

$$R_n^\infty(t, c, \beta, \alpha) := -\frac{1}{\Gamma(\beta+1)\,\pi} \int_0^\infty v^\beta \, \sin(\pi(v + \alpha))\, t^{v+\alpha}\, r_n(-v - \alpha, te^c)\, dv, \qquad (26)$$

with $r_n(a, z)$ defined in (19). Using the first inequality in (16) we find

$$\left| t^{v+\alpha}\, r_n(-v - \alpha, te^c) \right| \le e^{-te^c/2}\, e^{-c(v+\Re\alpha)} \beta(v + \Re\alpha, n+1).$$

Due to the fact that the above beta function is a decreasing function of v and inequality $|\Gamma(z+a)/\Gamma(z+b)| \le |z|^{a-b}$, valid for $\Re z > 0$ and $b \ge a + 1 \ge 1$ [2, Eq. 5.6.8], we find

$$\left| t^{v+\alpha}\, r_n(-v - \alpha, te^c) \right| \le e^{-te^c/2}\, e^{-c(v+\Re\alpha)} \frac{\Gamma(\Re\alpha)}{n^{\Re\alpha}}.$$

Using this bound in (26) we obtain

$$|R_n^\infty(t, c, \beta, \alpha)| \le \frac{e^{-te^c/2 - c\,\Re\alpha + \pi|\Im\alpha|}\,\Gamma(\Re\beta + 1)\,\Gamma(\Re\alpha)}{\pi\,|\Gamma(\beta+1)|\,c^{\Re\beta+1}} \frac{1}{n^{\Re\alpha}}, \qquad (27)$$

that proves the convergence of expansion (25). Moreover, taking into account the bound $\Gamma(\Re\beta + 1)/|\Gamma(\beta+1)| \le \sqrt{\cosh(\pi\Im\beta)}$, valid for $\Re\beta \ge -1/2$ [2, Eq. 5.6.7], we find

$$|R_n^\infty(t, c, \beta, \alpha)| \le \frac{e^{-te^c/2 - c\,\Re\alpha + \pi|\Im\alpha|}\,\Gamma(\Re\alpha)}{\pi\,\sqrt{\operatorname{sech}(\pi\Im\beta)}\,c^{\Re\beta+1}} \frac{1}{n^{\Re\alpha}}, \qquad \Re\beta \ge -\frac{1}{2}. \qquad (28)$$

This last bound shows that expansion (25) is uniformly convergent in $\Re\beta$ in the semi-plane $\Re\beta \ge -1/2$.

Then, in principle, for a fixed $c > 0$, the speed of convergence of expansion (25) is only of power order. However, a convenient election of the arbitrary constant c let us improve the convergence speed: if we let c depend on n in the form $te^c = \lambda n$ for any fixed positive parameter λ, that is, $c = \log(\lambda n/t)$, then (28) becomes

$$|R_n^\infty(t, \log(\lambda n/t), \beta, \alpha)| \le \frac{e^{\pi|\Im\alpha| - \lambda n/2}\, t^{\Re\alpha}}{\lambda^{\Re\alpha}\pi \left[\log\left(\frac{\lambda n}{t}\right)\right]^{\Re\beta+1} \sqrt{\operatorname{sech}(\pi\Im\beta)}} \frac{\Gamma(\Re\alpha)}{n^{2\Re\alpha}}, \qquad (29)$$

which is valid for $\Re\beta \ge -\frac{1}{2}$, whenever $n > t/\lambda$, for any fixed $\lambda > 0$. This formula shows that

$$R_n^\infty(t, \log(\lambda n/t), \beta, \alpha) = \mathcal{O}\left(\frac{e^{-\lambda n/2}}{(\log n)^{\Re\beta+1} n^{2\Re\alpha}}\right), \quad \text{when } n \to \infty.$$

3.2 A family of series representations of $\mu_0(t,\beta,\alpha,c)$

Consider the well-known series expansion representation of the exponential function

$$e^{te^{c+iu}} = \sum_{k=0}^{n-1} \frac{t^k}{k!} e^{k(c+iu)} + r_n(t,c,u), \tag{30}$$

valid for any $n \in \mathbb{N}$, with

$$|r_n(t,c,u)| \le \sum_{k=n}^{\infty} \frac{(te^c)^k}{k!} = e^{te^c} \frac{\gamma(n,te^c)}{\Gamma(n)}. \tag{31}$$

Replacing the exponential function in the integral definition of $\mu_0(t,\beta,\alpha,c)$ in (14) by the right hand side of (30) we obtain that, for any $n \in \mathbb{N}$,

$$\mu_0(t,\beta,\alpha,c) = \sum_{k=0}^{n-1} \frac{t^k}{2\pi\, k!} \phi(\alpha - k, \beta, c) + R_n^0(t,c,\beta,\alpha), \tag{32}$$

where the functions $\phi(a,b,c)$ were defined in Section 2.3 and

$$R_n^0(t,c,\beta,\alpha) := \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{e^{-\alpha(c+iu)}}{(c+iu)^{\beta+1}} r_n(t,c,u) du.$$

Using the bound (31), we find that the remainder $R_n^0(t,c,\beta,\alpha)$ may be bounded in the form

$$|R_n^0(t,c,\beta,\alpha)| \le \frac{e^{-c\Re\alpha + \pi|\Im\alpha| + te^c}}{2\pi} \frac{\gamma(n,te^c)}{\Gamma(n)} \int_{-\pi}^{\pi} \frac{du}{|(c+iu)^{\beta+1}|} \le \frac{e^{-c\Re\alpha + \pi(|\Im\alpha| + |\Im\beta|/2) + te^c}}{c^{\Re\beta+1}} \frac{\gamma(n,te^c)}{\Gamma(n)}. \tag{33}$$

From the integral definition of the incomplete gamma function [12, Eq. 8.2.1], it is clear that $\gamma(n,te^c) \le (te^c)^n/n$ and then,

$$|R_n^0(t,c,\beta,\alpha)| \le \frac{e^{-c\Re\alpha + \pi(|\Im\alpha| + |\Im\beta|/2) + te^c}}{c^{\Re\beta+1}} \frac{(te^c)^n}{n!}. \tag{34}$$

From Stirling formula for the factorial $n!$ we find $|R_n^0(t,c,\beta,\alpha)| = \mathcal{O}\left((te^{c+1})^n n^{-1/2-n}\right)$, as $n \to \infty$, that proves the convergence of expansion (32) for any $c > 0$.

However, if te^c is large, we may need a large number of terms n to obtain a result numerically satisfactory. Then, as in the previous subsection, we let c depend on n in the form $te^c = \lambda n$, for any fixed parameter $\lambda > 0$. Then, from (33), the asymptotic behaviour of the incomplete gamma function $\gamma(n,\lambda n)$ given in [12, Eq. 8.11.6], that is valid for $0 < \lambda < 1$, and the Stirling approximation of the gamma function, we find

$$|R_n^0(t,\log(\lambda n/t),\beta,\alpha)| \le \frac{t^{\Re\alpha} e^{\pi(|\Im\alpha| + |\Im\beta|/2)} \sqrt{2\pi}}{(1-\lambda)\lambda^{\Re\alpha} \left[\log\left(\frac{\lambda n}{t}\right)\right]^{\Re\beta+1}} \frac{(e\lambda)^n}{n^{\Re\alpha+1/2}}, \tag{35}$$

which shows that

$$R_n^0(t,\log(\lambda n/t),\beta,\alpha) = \mathcal{O}\left(\frac{(e\lambda)^n}{(\log n)^{\Re\beta+1} n^{\Re\alpha+1/2}}\right), \quad \text{when } n \to \infty.$$

Therefore, for any number $0 < \lambda < e^{-1}$ and $c = \log(\lambda n/t)$, the rate of convergence of expansion (32) is exponential.

3.3 A family of series representations of $\mu(t,\beta,\alpha)$

A series representation of $\mu(t,\beta,\alpha)$ follows from (13), (25) and (32), for any $c>0$. The remainder $R_n(t,c,\beta,\alpha)=R_n^{\infty}(t,c,\beta,\alpha)+R_0^{\infty}(t,c,\beta,\alpha)$ can be bounded by means of (28) and (34). Since c is a free parameter, we may take, as in the previous subsections, a varying parameter c defined in the form $te^c=\lambda n$, with $0<\lambda<e^{-1}$. Then

$$R_n(t,\log(\lambda n/t),\beta,\alpha)=\mathcal{O}\left(\frac{n^{-\Re\alpha}}{(\log n)^{\Re\beta+1}}\left[\frac{e^{\frac{-\lambda n}{2}}}{n^{\Re\alpha}}+\frac{(e\lambda)^n}{n^{1/2}}\right]\right),\quad \text{when } n\to\infty. \tag{36}$$

For any number $0<\lambda<e^{-1}$, both terms inside the brackets in (36) decay faster than a negative exponential. The optimal value of the parameter λ is the unique solution λ_0 of the trascendent equation $\log(e\lambda)+\lambda/2=0$, $\lambda=\lambda_0:=0.31436990296762807\ldots$; because in this case both terms inside the brackets in (36) are equal. This makes the expansion (25) of $\mu_\infty(t,\beta,\alpha,c)$ and the expansion (32) of $\mu_0(t,\beta,\alpha,c)$ to have a similar convergence rate. Hence, a numerically satisfactory approximation of $\mu(t,\beta,\alpha)$ requires to truncate expansions (25) and (32) at a number of terms $n>t/\lambda$, with $0<\lambda<e^{-1}$ (ideally $\lambda=\lambda_0$) and set $c=\log\left(\frac{\lambda n}{t}\right)$.

We summarize all the calculations above in the following theorem.

Theorem 1. *Let $\alpha,\beta\in\mathbb{C}$, $t\in\mathbb{R}$ with $\Re\alpha>0$, $\Re\beta>-1$ and $t>0$. Then, for any positive integer n, and any arbitrary positive number $c>0$,*

$$\begin{aligned}\mu(t,\beta,\alpha)=\frac{1}{2\pi}\sum_{m=0}^{n-1}\Bigg[&e^{-t\,e^c}\mathcal{L}_m(t\,e^c)\sum_{k=0}^{m}\binom{m}{k}e^{c\,(k+1)}\,\phi(\alpha+k+1,\beta,c)\\ &+\frac{t^m}{m!}\,\phi(\alpha-m,\beta,c)\Bigg]+R_n(t,c,\beta,\alpha),\end{aligned} \tag{37}$$

where $\mathcal{L}_m(x)$ are the Laguerre polynomials and $\phi(a,b,c)$ are the functions defined in Proposition 2 in terms of incomplete gamma functions. The remainder $R_n(t,c,\beta,\alpha)$ is bounded in the form

$$|R_n(t,c,\beta,\alpha)|\le\frac{e^{-c\Re\alpha+\pi|\Im\alpha|}}{c^{\Re\beta+1}}\left[\frac{e^{-te^c/2}M(\beta)\Gamma(\Re\alpha)}{\pi n^{\Re\alpha}}+\frac{e^{\pi|\Im\beta|/2+te^c}\gamma(n,te^c)}{\Gamma(n)}\right], \tag{38}$$

with

$$M(\beta):=\frac{\Gamma(\Re\beta+1)}{|\Gamma(\beta+1)|}.$$

For $\Re\beta\ge-1/2$, $M(\beta)$ may be replaced by $[sech(\pi\Im\beta)]^{-1/2}$ and therefore, expansion (37) is uniformly convergent in $\Re\beta$ in the semi-plane $\Re\beta\ge-1/2$. When $n\to\infty$,

$$R_n(t,c,\beta,\alpha)=\mathcal{O}\left(\frac{1}{n^{\Re\alpha}}+\frac{(te^{c+1})^n}{n^{n+1/2}}\right),$$

and then, the expansion (37) is convergent, with a convergence order of power type.

Moreover, let $0 < \lambda < e^{-1}$ *(ideally* $\lambda = \lambda_0 := 0.31436990296762807\ldots$*, the unique solution of the trascendent equation* $\log(e\lambda) + \lambda/2 = 0$*) and take* $c = \log(\lambda n/t)$*. Then, the rate of convergence of expansion* (37) *is of exponential type:*

$$R_n(t, \log(\lambda n/t), \beta, \alpha) = \mathcal{O}\left(\frac{n^{-\Re\alpha}}{(\log n)^{\Re\beta+1}}\left[\frac{e^{\frac{-\lambda n}{2}}}{n^{\Re\alpha}} + \frac{(e\lambda)^n}{n^{1/2}}\right]\right), \quad \textit{when } n \to \infty. \tag{39}$$

4 Final comments and Numerical experiments

The *Fransén-Robinson constant* is the particular case $F = \mu(1, 0, -1) = \mu(1, 1, 0) = 2\,\mu(1, 2, 1) + \mu(1, 1, 1)$, where the last equality follows from the recurrence relation (6). Then, a family of numerical series representations of F can be obtained from Theorem 1: for any $c > 0$,

$$\begin{aligned} F = 2\mu(1,2,1) + \mu(1,1,1) = \frac{1}{2\pi}\sum_{n=0}^{\infty}\Bigg\{ & \frac{1}{n!}\left[2\,\phi\,(1-n, 2, c\,) + \phi\,(1-n, 1, c\,)\right] \\ + e^{-e^c}\,\mathcal{L}_n(e^c)\sum_{k=0}^{n}\binom{n}{k} e^{c(k+1)} & \left[2\,\phi\,(k+2, 2, c\,) + \phi\,(k+2, 1, c\,)\right]\Bigg\}. \end{aligned} \tag{40}$$

As pointed out in the previous section, for λ as close to $\lambda_0 := 0.31436990296762807\ldots$ as possible, a more appropriate numerical approximation of F may be derived by truncating expansion (40) at a given number of terms $n > 1/\lambda$, and setting $c = \log(n\lambda)$.

Figure 3 and Table 1 are some numerical experiments that show the accuracy of the expansion (37) given in Theorem 1, for certain values of the parameters α and β and different values of the variable t. We have used the command *NIntegrate* of Wolfram Mathematica 11.3 to compute the "exact" value of the Volterra function.

Moreover, in Figure 3 we compare the approximation supplied by expansion (37) with the asymptotic approximations supplied by (8) and (9). On the one hand, the number of terms used in (8) and (9) is such that these divergent expansions are numerically useful in the largest possible range of the variable t. On the other hand, expansion (37) is convergent and then, the more terms used the better (we have taken a moderate number of terms). From a numerical point of view, expansion (8) is more appropriate for small values of t, expansion (9) is more appropriate for large values of t, whereas expansion (37) is more appropriate for intermediate values of t. In any case, expansion (37) produces a globally more satisfactory approximation.

5 Acknowledgments

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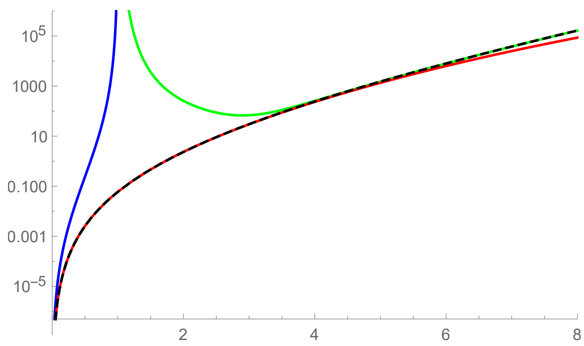

Figure 3: Plot, in logarithmic scale, of the Volterra function $\mu(t, 3.1, 2.5)$ (black, dashed), the first term of the asymptotic expansion for small t (8) (blue), the 4 first terms of the asymptotic expansion for large t (9) (green) and the 12 first terms of the expansion given by (37) with $c = 0.1$ (red).

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t	$n=4$	$n=7$	$n=10$	$n=20$
0.1	3.42493	0.0938748	0.00898929	0.00006397
0.5	6.89635	0.119798	0.0103995	0.0000612484
2	—	0.22662	0.00777211	0.0000275787
6	—	—	—	0.0000375823

t	$n=4$	$n=7$	$n=10$	$n=20$
0.1	180.876	9.63543	6.77157	2.30558
0.5	7.24844	0.0891918	0.0215027	0.00272701
2	6.13141	0.211356	0.00473969	0.00000289376
6	0.734252	1.17147	0.492865	0.0000637623

Table 1: Absolute value of the relative error provided by formula (37) with the optimum value $c = \log\left(\frac{\lambda n}{t}\right)$, $\lambda = 0.3143699$ (top) and $c = 0.3$ (bottom), for different values of the variable t, and several orders n of the approximation. Parameter values: $\alpha = 2.4 + 1.1i$, $\beta = 0.8 + 2.9i$.