Universidad Pública de Navarra

Departamento de Estadística, Informática y Matemáticas

A framework for general fusion processes under uncertainty modeling control, with an application in interval-valued fuzzy rule-based classification systems

Tiago da Cruz Asmus

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HACEN CONSTAR que el presente trabajo titulado "*A framework for general fusion processes under uncertainty modeling control, with an application in interval-valued fuzzy rule-based classification systems*" ha sido realizado bajo su dirección por D. Tiago da Cruz Asmus.

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Abstract

Information fusion is the process of combining several numerical values into a single representative one. In problems with some sort of fuzzy modeling, this process is usually computed by means of fusion functions or, their most important subclass, aggregation functions. They have been widely applied in several techniques to deal with classification problems, particularly, in Fuzzy Rule-Based Classification Systems (FRBCSs). In those types of classifiers, overlap functions (which are bivariate aggregation functions with desirable properties) and their n-dimensional generalizations have been successfully applied. When there is uncertainty regarding the modeling of membership functions in FRBCSs, usually associated with linguistic terms, one can apply interval-valued fuzzy sets. The modeling of linguistic labels via interval-valued fuzzy sets in FRBCSs gave birth to Interval-Valued Rule-Based Classification Systems (IV-FRBCSs). In those systems, the fusion processes are computed by means of aggregation functions defined in the interval context, while the widths of the assigned interval membership degrees are intrinsically related to the uncertainty with respect to the values they are approximating and, then, with the quality of the information they are carrying. However, there is not a guideline in the literature showing how to define and construct interval-valued fusion functions that takes the information quality control into consideration.

Thus, in this thesis, we develop a constructive framework to define generalized n-dimensional interval-valued fusion functions considering admissible orders and the information quality control. We apply the developed concepts in a state-of-the-art IV-FRBCS (namely, IVTURS), developing our own version of it based on overlap operators with information quality control, showing that the classification accuracy is improved by our approach. Finally, we develop a constructive framework to define n-dimensional fusion functions acting on an arbitrary closed real interval as counterparts of known classes of fusion functions acting on the unit interval, to expand the applicability of fusion functions with desirable properties to problems that do not involve fuzzy modeling.

Resumen

La fusión de información es el proceso de combinar varios valores numéricos en uno solo que los represente. En problemas con algún tipo de modelado difuso, este proceso generalmente se realiza mediante funciones de fusión o, su subclase más importante, las funciones de agregación. Estas funciones se han aplicado ampliamente en varias técnicas para resolver problemas de clasificación, en particular, en los Sistemas de Clasificación Basados en Reglas Difusas (SCBRDs). En este tipo de clasificador, se han aplicado de forma exitosa las funciones de solapamiento (que son funciones de agregación bivariadas con propiedades deseables) y sus generalizaciones n-dimensionales. Cuando hay incertidumbre con respecto al modelado de las funciones de pertenencia en los SCBRDs, generalmente asociados con términos lingüísticos, se pueden aplicar conjuntos difusos intervalo-valorados. El modelado de etiquetas lingüísticas a través de conjuntos difusos intervalo-valorados en los SCBRDs originó a los Sistemas de Clasificación Basados en Reglas Difusas Intervalo-valorados (IV-SCBRDs). En estos sistemas, los procesos de fusión se calculan mediante funciones de agregación definidas en el contexto intervalar, mientras que las amplitudes de los intervalos de pertenencia asignados están intrínsecamente relacionadas con la incertidumbre con respecto a los valores que están aproximando y, luego, con la calidad de la información que representan. Sin embargo, no existe una guía en la literatura que muestre cómo definir y construir funciones de fusión con valores intervalares que tomen en consideración el control de la calidad de la información.

Por todo ello, en esta tesis, desarrollamos un marco para definir funciones de fusión intervalo-valoradas n-dimensionales generalizadas considerando los órdenes admisibles y el control de la calidad de la información. Aplicamos los conceptos desarrollados en un IV-SCBRD considerado como estado del arte (es decir, IVTURS), desarrollando nuestra propia versión basada en operadores de solapamiento con control de la calidad de la información, demostrando que nuestro enfoque mejora el rendimiento del clasificador. Finalmente, desarrollamos un marco para definir funciones de fusión n-dimensionales que actúan en un intervalo real cerrado arbitrario como homólogas de clases conocidas de funciones

de fusión que actúan sobre el intervalo unitario, para expandir la aplicabilidad de las funciones de fusión con propiedades deseables a problemas que no involucren un modelado difuso.

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Chapter 1

INTRODUCTION

1.1 Motivation

Information fusion is the process of combining several numerical values into a single representative one, which can be modeled through fusion functions (MKB$^+$16) or, their most important subclass, aggregation functions (GMMP09). Among the many features of aggregation functions, in both theoretical and applied fields (GMMP09; BBC16; BPC07), one can highlight their capacity to model fuzzy logic operations and, in this case, appropriately fuse values from the unit interval $[0, 1]$ according to some criteria. For example, t-norms and t-conorms (KM00) are associative bivariate aggregation functions that are suitable to model, respectively, fuzzy conjunction and fuzzy disjunction operations, while uninorms (YR96) generalize both concepts. Possibly non-associative aggregation functions may be used as alternatives to t-norms, t-conorms and uninorms, such as t-seminorms or semi-copulas (Spi10), weak t-norms (Fod91), pseudo-t-norms (WY02), semi-uninorms (ZWL18), MICA operators (Yag94), and micanorms (Yan15), and, in particular, overlap functions (BFM$^+$10).

Overlap functions have captured the attention of many researchers due to their desirable properties besides not necessarily being associative. They were originally defined to measure the overlapping degree between classes in image processing problems (BPM$^+$12; JBP$^+$13; BMD$^+$21) and, since then, they have been extensively studied in the literature (BDBB13; DB15; QH18b; QH17; QH19; DBB$^+$16; WH21; CHQ18; ZWY19; hZQ20). Among their good properties, we can point out that the class of overlap functions is closed with respect to the convex sum and the aggregation by internal generalized composition. One can find clear discussions on the advantages that overlap functions have over the popular t-norms in the works of Dimuro et al. (DB15; DBF$^+$19). Furthermore, overlap functions showed good results when applied in problems where the associativity of the employed aggregation operator is not required, as in image processing (JBP$^+$13; MDLLM$^+$21), decision making (DBB$^+$17a; EGS$^+$18; GBH$^+$15), wavelet-fuzzy power quality diagnosis system (NCP$^+$19), and, of our particular interest, classification problems (EGSB16; EGS$^+$15; LSD$^+$17; LDM$^+$15; LSD$^+$20; LSD$^+$18; LDF$^+$19).

Another concept related to overlap functions is that of overlap indices (DOP00; PP99; YB94), which is a generalization of the Zadeh's consistency index (Zad78), defined to measure the degree of overlapping between two fuzzy sets in image processing (BFM$^+$09). Overlap indices can be built by means of overlap functions (GBH$^+$15; BFM$^+$09).

However, overlap functions, as originally defined, were restricted to be applied in problems where only two classes are taken into account. Since those functions may not be associative, this becomes a drawback when one needs to deal with n-dimensional problems. To address this limitation, the concept of n-dimensional overlap functions was introduced by Gómez et al. (GRM$^+$16), and a more flexible definition, that of general overlap functions, was presented by De Miguel et al. (DGR$^+$19), by relaxing their boundary conditions.

A classification problem (which is usually of the n-dimensional nature) consists in predicting the unknown class of an object, based on the values of the input attributes characterizing such object. From the many known techniques to deal with such task, like Support Vector Networks (SVN) (CV95), decisions trees (Qui93) and neural networks (Gra12), one stands out for its high interpretability, while also obtaining accurate results: that of Fuzzy Rule-Based Classification Systems (FRBCSs) (INN04; GAH11).

FRBCSs are considered interpretable classifiers because the knowledge that is learned from the data is reflected in the form of classification rules, which are composed by an antecedent, containing an intersection of linguistic variables modeled by fuzzy sets, and a consequent that specifies the class and the weight to the rule. The intersection of linguistic variables on the antecedents of the fuzzy rules have been successfully modeled by t-norms (AFAH11), n-dimensional overlap functions (GRM$^+$16) and general overlap functions (DGR$^+$19). In (EGS$^+$18), the computation of the rule weight, which is a metric of the quality of the rule, was done using fuzzy confidence values defined by overlap indices.

An important aspect of FRBCSs is the appropriate definition of the membership functions (CHV00). This may be a complex problem whenever there is uncertainty related to the modeling of such membership functions, usually associated with linguistic terms (Men07; NKZ97). One way to deal with this problem is through Interval-Valued Fuzzy Sets (IVFSs) (Zad75; Sam75; GG76; Jah75; BBP+16), which have proven to be an appropriate tool to model not only vagueness (lack of sharp class boundaries), but also uncertainty (lack of information about the membership function), as in (SFBH11; SFBH10; BB00; SFBH12; DBSR11; BBP+17). For that reason, IVFSs have been successfully applied in various problems with imperfect information (Zad08; Zad05), such as image processing (GFBB11), game theory (ADB17), multicriteria decision making (KK19), pest control (RDFF03), irrigation systems (HTZW20) and collaborative clustering (NDP18).

The modeling of linguistic labels via IVFSs in FRBCSs gave birth to Interval-Valued Rule-Based Classification Systems (IV-FRBCSs) (SFBH11; SFBH12; SFBH13; SBH+15; SFBH10; SGJ+14). In those systems, intervals are used in the whole inference process, where one has to predict the class of a new example accordingly to the learned interval-valued (iv) fuzzy rules. Thus, in the literature, intersection of linguistic variables (now represented by IVFS) is modeled by means of an iv-t-norm (BT05; BT06; DBRS08), which is a t-norm defined in the interval context.

Qiao and Hu (QH17) and Bedregal et al. (BBP+17) introduced, independently, the concept of iv-overlap functions, based on the original definition of overlap functions, which are bivariate functions. However, unlike the iv-t-norms, iv-overlap functions, as defined until this moment, cannot be applied in the n-dimensional context of IV-FRBCSs, since they are not associative. For this reason, when starting the development of this thesis, we looked for the definitions in the interval context of n-dimensional overlap functions, general overlap functions and overlap indices, since those functions have been providing good results when applied in FRBCSs. The lack of such definitions in the interval-valued context motivated the first research question faced in this thesis:

(RQ1) Is it feasible to define generalized n-dimensional overlap functions and overlap indices in the interval context so that they are suitable to be applied in n-dimensional problems, such as the ones tackled by IV-FRBCSs?

Another characteristic of iv-overlap functions is that they are increasing with respect to the usual partial order for intervals (product order (MKC09)). However, in IV-FRBCSs, the final step of the inference process consists of ranking intervals and, in this case, a partial order could lead to an undesirable stalemate (SFBH13). One possible solution is the adoption of admissible orders (BFKM13) for comparing intervals, since they are total orders that refine the usual product order. Since their introduction, several works taking into account admissible orders have appeared in the literature (ZBM+17; BBJ+15; TFF+19; BMDF+20; TUG+21; MMB+21). This leads to the second research question approached by this thesis:

(RQ2) Considering IV-FRBCSs, do admissible orders - to be used for ranking possible classification outcomes - and iv-overlap operators (which may or may not be increasing for such orders) - to be applied in the inference process of the classifier - have an impact on the whole classification process?

Now, observe that a key aspect of engineering/data science problems involving interval-valued fuzzy systems is that the widths of the assigned interval membership degrees are intrinsically related to the uncertainty/ignorance with respect to the values they are approximating (BDSR10; DBSR11) and also with the quality of the information they are carrying (DCC00; AJ94). Most iv-aggregation functions (KM11), such as the aforementioned iv-overlap functions, have no mechanism to control the information quality of the outputs, accordingly to the widths of the inputs. However, the information quality in systems with uncertainty modeling is a strong requirement claimed by scientists and engineers interested in interval-based tools (MKC09).

A first attempt in this direction was presented by Bustince et al. (BMDF$^+$20), with the concept of width-preserving interval-valued functions, which are functions whose interval outputs' width coincide with the widths of all the aggregated interval inputs, when those widths are all equal. The main drawback of this definition is that it is based on the very restrictive instance where all the interval inputs have the same width, in order to preserve the information quality.

Based on the discussion above, we arrive at our third research question:

(RQ3) Since the widths of the intervals are intrinsically related to both the uncertainty towards the value they represent and the quality of the information that they are expressing, how can one define interval-valued overlap operations in which the width of the output is controlled accordingly to a desirable threshold that depends on the widths of the inputs?

It is noteworthy that the control of the information quality in IV-FRBCSs is yet to be considered in the literature. Moreover, there is an absence in the literature of both a general approach for the control of the information quality in n-dimensional interval valued fusion processes and a theoretical framework to define classes of width controlled iv-fusion functions, based on known classes of fusion functions. From these assessments, we pose the fourth research question:

(RQ4) Is it possible to develop a general framework to define classes of width controlled n-dimensional interval fusion functions, as counterparts of known classes of fusion functions so that they can improve the accuracy of classification systems, in particular IV-FRBCSs, by the influence of the information quality control?

Finally, envisaging the enlargement of the scope of this work towards beyond the fuzzy context, it is necessary to analyze the linkage between the actual context of the developments on fusion functions and the world of current practical information fusion processes. One may observe that most of the theory on aggregation functions was developed in the context of the unit interval and applied in problems that involve some sort of fuzzy modeling. Nevertheless, there are several practical problems where the data to be aggregated are neither modeling membership degrees, nor truth values, nor some extension of them considering uncertainty modeling. Those problems could benefit from the application of fusion functions currently defined only to operate in the fuzzy context, as overlap functions and their generalizations. That is the case, for example, of the pooling process in convolutional neural networks (LBH15), which are widely applied in image processing (DBB+18; PRMFI+21), and recurrent neural networks (Gra12), such as Long Short-Term Memory (HS97), which are used in several machine learning problems with sequential information (GSK+17).

Aggregation functions, in fact, can be defined on any closed real interval, such as the ordered weighted averaging (OWA) (Yag88) operator and the Choquet integral (Cho54; DFB+20). Also, many classes of aggregation functions defined on arbitrary lattices have been studied in the literature, such as t-norms and t-conorms (EKM15; Sam06; SL21), uninorms (DHQ19; KM15), overlap and grouping functions (PSBP21; Qia21; WH21). Although some of those defined functions could operate with inputs that are not from the unit interval, there is not a guideline in the literature showing how to define and construct fusion functions beyond the unit interval, in a manner that their fundamental constitutive properties are preserved in the new context. For that reason, we raise the fifth, and last, research question to be addressed by this thesis:

(RQ5) Is it possible to develop a general framework to define classes of fusion functions acting on an arbitrary closed real interval as counterparts of known classes of fusion functions acting on the unit interval, without sacrificing their fundamental properties, so that they can be constructed and applied in practical problems that are not fuzzy in nature?

1.2 Objectives

With the aim to provide answers to each of the proposed research questions, this thesis has a two-fold general objective:

1) To develop a constructive framework to define generalized n-dimensional interval-valued fusion functions considering admissible orders and the information quality control, in order to enhance the accuracy of IV-FRBCS;

2) To develop a constructive framework to define n-dimensional fusion functions acting on an arbitrary closed real interval as counterparts of known classes of fusion functions acting on the unit interval.

From this general objective, we can discriminate the following specific objectives:

1) To define, study and introduce construction methods for n-dimensional iv-overlap functions, general iv-overlap functions and iv-overlap indices;

2) To develop a new classifier, which we call IVTURS-OV, by applying the developed theoretical concepts on crucial steps of the IVTURS algorithm (SFBH13), a state-of-the art IV-FRBCS;

3) To define, study and introduce construction methods for n-dimensional admissibly ordered iv-overlap functions, which are n-dimensional iv-overlap functions that are increasing with respect to an admissible order;

4) To analyze the effect of admissible orders and n-dimensional admissibly ordered iv-overlap functions in the classification accuracy of IV-FRBCSs, through experimentation with IVTURS-OV;

5) To introduce the concepts of width-limited interval-valued functions and width-limiting functions, which are theoretical tools to study the relation between the widths of the inputs with the width of the output of interval-valued functions, necessary for the construction of interval-valued functions with controlled information quality;

6) To define, study and introduce construction methods for width-limited interval-valued overlap functions, taking into account a width-limiting function and a pair of partial orders, which allows the definition of interval-valued overlap operations that provide output intervals that do not exceed a desirable width threshold;

7) To introduce a general framework for defining classes of iv-fusion functions with controlled information quality on the interval outputs, which we call w-iv-fusion functions, based on the extension of a set of properties from a core fusion function, presenting examples and construction methods of functions defined through the framework;

8) To apply n-dimensional w-iv-overlap functions (defined via the introduced general framework to control the information quality) in the IVTURS-OV classifier, in order to analyze if, and to which extent, such functions improve its classification accuracy;

9) To introduce a general framework for defining classes of fusion functions acting on an arbitrary closed real interval $[a, b]$ as counterparts of known classes of fusion functions acting on the unit interval $[0, 1]$, based on the proper transposition (from $[0, 1]$ to $[a, b]$) of the set of properties from such fusion functions, presenting examples and construction methods of functions defined through the framework.

1.3 Methodology

To achieve our objectives, we address each of the five proposed research questions following an incremental methodology, in which each development stage of the thesis is dedicated to a full paper (published, accepted or submitted to different prestigious journals with high impact factor) related to each one of those questions. Each stage, and its respective publication, provides a number of contributions towards each specific objective, both in theoretical and applied aspects. Although our general objective is posed in a more general framework of fusion processes, we had to chose appropriate specific fusion functions, namely, overlap functions and their generalizations/extensions, to be applied in the experiments on IV-FRBCSs.

The incremental development of the thesis follows the subsequent stages:

Stage 1: Publication to address question **(RQ1)** and achieve the specific objectives 1 and 2;

Stage 2: Publication to address question **(RQ2)** and achieve the specific objectives 3 and 4;

Stage 3: Publication to address question **(RQ3)** and achieve the specific objectives 5 and 6;

Stage 4: Publication to address question **(RQ4)** and achieve the specific objectives 7 and 8;

Stage 5: Publication to address question **(RQ5)** and achieve the specific objective 9.

We point out that specific objectives 2, 4 and 8 are all part of the process of designing the IVTURS-OV classifier, a new IV-FRBCS based on a state-of-the-art classifier (IVTURS), where we check the practical application of the introduced theoretical concepts in classification problems, and the effect of such concepts in the classification accuracy.

1.4 Structure

This thesis is organized as follows:

Chapter 2: BACKGROUND - In this chapter, we provide a background divided in two parts: i) a theoretical one, focused on aggregation functions (mainly overlap functions and their generalizations), interval mathematics, admissible orders and interval-valued aggregation functions, and ii) a practical one, dedicated to classification problems, with special attention to IV-FRBCSs.

Chapter 3: DEVELOPMENT OF THE THESIS - In this chapter, we present the main points of the development of the thesis. In Section 3.1 we discuss each of the five publications related to the research questions, highlighting their main contributions. Following that, in Section 3.2, we present a discussion around four complementary works that were derived from the development of the thesis. In Section 3.3, we analyze how each of the five main publications are connected, presenting an overview of both the theoretical and applied advancements achieved by our work, followed the representation of the relation between each complementary work with the stages of development of the thesis.

Chapter 4: CONCLUSION- In this chapter, we state our concluding remarks. First, in Section 4.1, we review the main contributions of the thesis. Finally, in Section 4.2, we discuss some possibilities for future works.

Chapter 5: PUBLICATIONS - This final chapter constitutes the collection of all the papers developed as part of this thesis, discussed in Chapter 3. Those papers are divided in two categories: i) main publications (associated with the five research questions and constituting the main body of work of this thesis) and ii) complementary contributions. In Section 5.1, we present the full manuscript of the five main publications, detailing, for each one, the journal where it was published or submitted, the impact factor of the journal and the current status of the publication. The five main publications are:

1) General interval-valued overlap functions and interval-valued overlap indices;

2) N-dimensional admissibly ordered interval-valued overlap functions and its influence in interval-valued fuzzy rule-based classification systems;

3) Towards interval uncertainty propagation control in bivariate aggregation processes and the introduction of width-limited interval-valued overlap functions;

4) A methodology for controlling the information quality in interval-valued fusion processes: theory and application;

5) A constructive framework to define fusion functions with floating domains in arbitrary closed real intervals.

In Section 5.2, we present the full manuscript of the four complementary contributions, also informing their respective publication details. The four complementary contributions are:

1) General grouping functions;

2) General interval-valued grouping functions;

3) General admissibly ordered interval-valued overlap functions;

4) Enhancing the efficiency of the interval-valued fuzzy rule-based classifier with tuning and rule selection.

Chapter 2

BACKGROUND

In this chapter, we present some preliminary concepts necessary to the development of the thesis. From the theoretical standpoint, we focus on aggregation functions, mainly overlap functions, interval mathematics and how they are combined in the field of iv-aggregation functions. On the application side, we present the basic structure of fuzzy rule-based classification systems along with their interval-valued counterpart, in which the developed theoretical concepts were applied to produce a new type of IV-FRBCS.

2.1 Fuzzy sets and aggregation functions

Fuzzy set theory was introduced by Zadeh (Zad65; Zad75) as a tool to model imprecision and represent linguistic variables. The core concept of this theory is that of membership function, which is an extension of the characteristic function from the classical set theory. We recall that, given an universe U, the characteristic function $\chi_{F_C} : U \to \{0, 1\}$ of a subset $F_C \subseteq U$, is given by

$$\chi_{F_C}(x) = \begin{cases} 1, & \text{if } x \in F \\ 0, & \text{if } x \notin F, \end{cases} \tag{2.1}$$

meaning that $x \in F_C$ when $\chi_{F_C}(x) = 1$ and $x \notin F_C$ when $\chi_{F_C}(x) = 0$.

If one admits that an element $z \in U$ may belong gradually to a set $F \subseteq U$, then F is said to be a fuzzy set. The function $\mu_F : U \to [0, 1]$ that measures the degree in which each element $z \in U$ belongs to F is called the membership function of F, where $\mu_F(z) = 0$ signifies that the element z is completely disassociated from F, $\mu_F(z) = 1$ means that x is completely belongs to F, and other values of μ_F indicate a partial belonging of z to F.

A fuzzy set $F \subseteq U$ may be represented by a set of ordered pairs $(z, \mu_F(z))$, with $z \in U$, where we assign a membership degree with respect to F to every element z of the universe U:

$$F = \{(z, \mu_F(z)) \mid z \in U\}.$$

We denote by $FS(U)$ the space of all fuzzy sets defined over U. A fuzzy set $F \in FS(U)$ is called normal if there exists $z \in U$ such that $F(z) = 1$.

Example 1. i) *Given $a, b, c \in U$, such that $a < b < c$, a triangular shaped fuzzy set F_T has its membership function $\mu_{F_T} : U \to [0, 1]$ defined, for all $z \in U$, as follows:*

$$\mu_{F_T}(z) = \begin{cases} 0, & \text{if } z \leq a \\ \frac{z-a}{b-a}, & \text{if } a \leq z \leq b \\ \frac{c-z}{c-b}, & \text{if } b \leq z \leq c \\ 0, & \text{if } z \geq c. \end{cases}$$

A graphical representation of μ_{F_T} is shown in Figure 2.1.

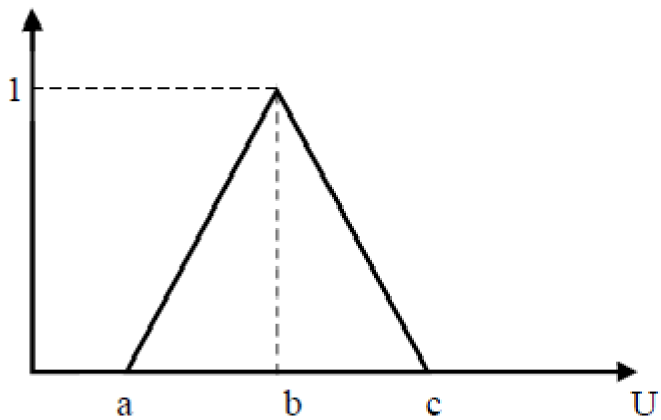

Figure 2.1: Membership function of a triangular shaped fuzzy set.

ii) *A classical (or crisp) set F_C is a particular case of a fuzzy set, when its membership function μ_{F_C} coincides with the characteristic function χ_{F_C}, given by Equation (2.1).*

In the following, we recall some important definitions of operations that are applied in fuzzy modeling, usually having membership degrees as their inputs.

Let us denote $\vec{x} = (x_1, \ldots, x_n) \in [0, 1]^n$, where $n > 1$.

Definition 1. *(KMP00) A function $N : [0,1] \to [0,1]$ is a fuzzy negation if it respects the following conditions:*

(N1) *$N(0) = 1$ and $N(1) = 0$;*

(N2) *If $x \leq y$ then $N(y) \leq N(x)$, for all $x, y \in [0,1]$.*

If the involutive property, given by

(N3) *$N(N(x)) = x$, for all $x \in [0,1]$,*

is also satisfied, then N is said to be a strong fuzzy negation.

Example 2. *The Zadeh negation given, for all $x \in [0,1]$, by*

$$N_Z(x) = 1 - x, \tag{2.2}$$

is a strong fuzzy negation.

A function $F : [0,1]^n \to [0,1]$ that merges n values from the unit interval into one value in the same interval is a *fusion function* (MKB$^+$16).

Definition 2. *(KMP00) Given a strong fuzzy negation $N : [0,1] \to [0,1]$ and a fusion function $F : [0,1]^n \to [0,1]$, then the fusion function $F^N : [0,1]^n \to [0,1]$ defined, for all $\vec{x} \in [0,1]^n$, by*

$$F^N(\vec{x}) = N(F(N(x_1), \ldots, N(x_n))), \tag{2.3}$$

is the N-dual of F.

When it is clear by the context, the N_Z-dual function (dual with respect to the Zadeh negation) of F will be just called dual of F, and will be denoted by F^d. Observe that $(F^N)^N = F$, since N is a strong negation.

A particularly important class of fusion function is that of *aggregation functions* (BBC16), defined as follows.

Definition 3. *(BBC16) An aggregation function is any fusion function $A : [0,1]^n \to [0,1]$ that respects the following conditions:*

(A1) *A is increasing;*

(A2) $A(0,\ldots,0) = 0$ *and* $A(1,\ldots,1) = 1$.

An extension of one-dimensional convexity was presented in (KMM11), in the context of aggregation functions:

Definition 4. *(KMM11) An aggregation function* $A : [0,1]^n \to [0,1]$ *is called ultramodular if, for all* $\vec{x}, \vec{y}, \vec{\epsilon} \in [0,1]^n$*, such that* $\vec{y} + \vec{\epsilon} \in [0,1]^n$ *and* $\vec{x} \leq \vec{y}$*, it holds that:*

$$A(\vec{x} + \vec{\epsilon}) - A(\vec{x}) \leq A(\vec{y} + \vec{\epsilon}) - A(\vec{y}). \tag{2.4}$$

Example 3. *The function* $AM : [0,1]^n \to [0,1]$ *(arithmetic mean), given, for all* $\vec{x} \in [0,1]^n$*, by*

$$AM(\vec{x}) = \frac{\sum_{i=1}^{n} x_i}{n}, \tag{2.5}$$

is an aggregation function that is ultramodular.

There are many classes of aggregation functions defined in the literature. Here we highlight some of them that are going to be of importance on this work.

Definition 5. *(KMP00) A t-norm is any bivariate fusion function* $T : [0,1]^2 \to [0,1]$*, that satisfies the following conditions, for all* $x, y \in [0,1]$*:*

(T1) T *is symmetric;*

(T2) T *is associative;*

(T3) T *has* 1 *as its neutral element;*

(T4) T *is increasing.*

Definition 6. *(KMP00) A t-conorm is any bivariate fusion function* $S : [0,1]^2 \to [0,1]$*, that satisfies the following conditions, for all* $x, y \in [0,1]$*:*

(S1) S *is symmetric;*

(S2) S *is associative;*

(S3) S *has* 0 *as its neutral element;*

(S4) S *is increasing.*

T-norms and t-conorms are the usually applied to model, respectively, fuzzy conjunction (generalized logic "AND") and fuzzy disjunction (generalized logic "OR") operations (KMP00). Observe that neither t-norms nor t-conorms are self-closed to the generalized composition. By duality, one can obtain t-conorms from t-norms, and vice-versa. The properties of t-norms and t-conorms have been extensively studied in the literature, such as idempotency, migrativity and homogeneity (KMP00; FR07; SBD$^+$21).

Uninorms were introduced by Yager and Rybalov (YR96) as a generalization of t-norms and t-conorms, defined as follows:

Definition 7. *(YR96) An uninorm is any bivariate fusion function $U : [0,1]^2 \to [0,1]$, that satisfies the following conditions, for all $x, y \in [0,1]$:*

(U1) *U is symmetric;*

(U2) *U is associative;*

(U3) *U has a neutral element;*

(U4) *U is increasing.*

Example 4. **i)** *The function $T_P : [0,1]^2 \to [0,1]$, given, for all $x, y \in [0,1]$, by*

$$T_P(x, y) = x \cdot y,$$

is a t-norm (product t-norm). Its dual function $S_P : [0,1]^2 \to [0,1]$, given, for all $x, y \in [0,1]$, by

$$S_P(x, y) = x + y - x \cdot y,$$

is a t-conorm (probabilistic sum);

ii) *The function $T_M : [0,1]^2 \to [0,1]$, given, for all $x, y \in [0,1]$, by*

$$T_M(x, y) = \min\{z, y\},$$

is a t-norm (minimum t-norm). Its dual function $S_M : [0,1]^2 \to [0,1]$, given, for all $x, y \in [0,1]$, by

$$S_M(x, y) = \max\{z, y\},$$

is a t-conorm (maximum t-conorm);

iii) *Consider $e \in [0,1]$. Then, the function $U_C : [0,1]^2 \to [0,1]$, given, for all $x, y \in [0,1]$, by*

$$U_C(x,y) = \begin{cases} \max\{x,y\} & \text{if } (x,y) \in [e,1]^2, \\ \min\{x,y\} & \text{otherwise}, \end{cases} \tag{2.6}$$

is an uninorm with e as its neutral element.

The definition of *n-dimensional overlap functions* is a key concept in this work:

Definition 8. *(GRM+16; EGS+15) A fusion function $On : [0,1]^n \to [0,1]$ is an n-dimensional overlap function if, for all $\vec{x} \in [0,1]^n$, the following conditions hold:*

(On1) *On is symmetric;*

(On2) $On(\vec{x}) = 0 \Leftrightarrow \prod_{i=1}^{n} x_i = 0$*;*

(On3) $On(\vec{x}) = 1 \Leftrightarrow \prod_{i=1}^{n} x_i = 1$*;*

(On4) *On is increasing;*

(On5) *On is continuous.*

A 2-dimensional overlap function is just called overlap function (BFM+10; BDBB13).

Example 5. i) *The product and minimum t-norms are also overlap functions;*

ii) *The function $GM : [0,1]^n \to [0,1]$ (geometric mean), given, for all $\vec{x} \in [0,1]^n$, by*

$$GM(\vec{x}) = \sqrt[n]{\prod_{i=1}^{n} x_i}, \tag{2.7}$$

is an n-dimensional overlap function, but not a t-norm.

Theorem 1. *(GRM+16) Consider a continuous aggregation function $A : [0,1]^m \to [0,1]$, such that*

(PA) *$A(\vec{x}) = 0$ if and only if $x_i = 0$, for some $i \in \{1, \ldots, m\}$;*

(PB) *$A(\vec{x}) = 1$ if and only if $x_i = 1$, for all $i \in \{1, \ldots, m\}$;*

and a tuple of n-dimensional overlap functions $\overrightarrow{O}n = (On_1, \ldots, On_m)$. Then, the mapping $A_{\overrightarrow{On}} : [0,1]^n \to [0,1]$, defined, for all $\vec{x} \in [0,1]^n$, by

$$A_{\overrightarrow{On}}(\vec{x}) = A(On_1(\vec{x}), \ldots, On_m(\vec{x})), \tag{2.8}$$

is an n-dimensional overlap function.

Corollary 1. *(GRM+16) Consider an m-dimensional overlap function $OnC : [0,1]^m \to [0,1]$ and the tuple $\overrightarrow{On} = (On_1, \ldots, On_m)$ of n-dimensional overlap functions. Then, the mapping $OnC_{\overrightarrow{On}} : [0,1]^n \to [0,1]$, defined for all $\vec{x} \in [0,1]^n$, by*

$$OnC_{\overrightarrow{On}}(\vec{x}) = OnC(On_1(\vec{x}), \ldots, On_m(\vec{x})), \tag{2.9}$$

is an n-dimensional overlap function.

By Corollary 1, one can observe that the class of overlap functions is self closed with respect to the generalized composition.

Example 6. *The function $OnB : [0,1]^n \to [0,1]$, given, for all $\vec{x} \in [0,1]^n$, by*

$$OnB(\vec{x}) = \sqrt{(\prod_{i=1}^{n} x_i) \cdot (\min\{x_1, \ldots, x_n\})} \tag{2.10}$$

is an n-dimensional overlap function, constructed through the composition of the minimum and the product by the geometric mean.

Definition 9. *(GRM+16) A fusion function $Gn : [0,1]^n \to [0,1]$ is said to be an n-dimensional grouping function if, for all $\vec{x} \in [0,1]^n$, the following conditions hold:*

(Gn1) *Gn is symmetric;*

(Gn2) *$Gn(\vec{x}) = 0 \Leftrightarrow x_i = 0$ for all $i \in \{1, \ldots, n\}$;*

(Gn3) *$Gn(\vec{x}) = 1 \Leftrightarrow$ there exists $i \in \{1, \ldots, n\}$ such that $x_i = 1$;*

(Gn4) *Gn is increasing;*

(Gn5) *Gn is continuous.*

A 2-dimensional grouping function is just called grouping function (BFM+10; BDBB13). Analogous to the relation between t-norms and t-conorms, by duality one can obtain n-dimensional grouping functions from n-dimensional overlap functions, and vice-versa.

Example 7. **i)** *The probabilistic sum and maximum t-conorms are also grouping functions;*

ii) *The function $GM^d : [0,1]^n \to [0,1]$ (dual of the geometric mean), given, for all $\vec{x} \in [0,1]^n$, by*

$$GM^d(\vec{x}) = 1 - \sqrt[n]{\prod_{i=1}^{n} 1 - x_i}, \tag{2.11}$$

is an n-dimensional grouping function, but not a t-conorm.

Overlap and grouping functions are particular classes of aggregation functions that do not need to be associative and can also be applied to model fuzzy connectives (BDBB13). They have some clear advantages over t-norms and t-conorms, since they are closed to the convex sum and the aggregation by internal generalized composition (see Theorem 1), whereas t-norms and t-conorms are not.

The theoretical development of both overlap and grouping functions was summarized by Bustince et al. in (BMD+21). Additionally, some examples of studies on overlap and grouping functions are described in the sequence. The basic properties of overlap functions and grouping functions, like homogeneity, migrativity and idempotency, were studied by Bedregal et al. in (BDBB13). Archimedian overlap functions were introduced by Dimuro et al. in (DB14). Additive generators of overlap functions and grouping functions were introduced by Dimuro et al. in (DBB+16; DBB+14), and their multiplicative generators by Qiao et al. in (QH18b). Further studies on the migrativity property of overlap functions were presented in (QH18c; QH19). Dimuro et al. developed the concept of fuzzy implication functions derived overlap and grouping functions in (DB15; DBB+17b; DBS14). The properties of such fuzzy implications were studied in (DBF+19; QH18a).

The good properties of overlap and grouping functions allowed these functions to be applied in several practical problems, in particular when associativity of the employed aggregation operator is not required, as in fuzzy rule-based classification (DFB+20; DLB+20; LSD+18), decision making (EGS+18), wavelet-fuzzy power quality diagnosis system (NCP+19) or forest fire detection (GJJP+17), among others (QH18a; Qia19; QH18c; ZY19; ZQL21; QH19; QH18b; DGR+19; DBB+16; DB14; DBB+17b).

In this thesis, we focus our attention on overlap functions and some of their generalizations, which we recall in the following.

By altering the boundary conditions of overlap functions for less restrictive ones, broader classes of aggregation functions can be defined. As introduced in (QH17) for $n = 2$, a function $O : [0,1]^n \to [0,1]$ is said to be an 0-overlap function if we loose the condition **(O2)** in Definition 8 to

$$\textbf{(On2')}\ \prod_{i=1}^{n} x_i = 0 \Rightarrow O(\vec{x}) = 0$$

without modifying any other condition.

In the same manner, a function $O : [0,1]^n \to [0,1]$ is said to be an 1-overlap function if we downgrade the condition **(O3)** in Definition 8 to

$$\textbf{(On3')}\ \prod_{i=1}^{n} x_i = 1 \Rightarrow O(\vec{x}) = 1$$

without changing the remaining conditions.

By considering both conditions **(On2')** and **(On3')** when defining a new kind of n-dimensional overlap function, De Miguel et al. (DGR+19) introduced the concept of *general overlap functions*, as follows:

Definition 10. *(DGR+19) A general overlap function is any mapping $GO : [0,1]^n \to [0,1]$ that satisfies the following conditions, for all $x_1, \ldots, x_n \in [0,1]$:*

(GO1) *GO is symmetric;*

(GO2) *If $\prod_{i=1}^{n} x_i = 0$ then $GO(x_1, \ldots, x_n) = 0$;*

(GO3) *If $\prod_{i=1}^{n} x_i = 1$ then $GO(x_1, \ldots, x_n) = 1$;*

(GO4) *GO is increasing;*

(GO5) *GO is continuous.*

Proposition 1. *(DGR+19) If $O : L([0,1])^n \to [0,1]$ is an n-dimensional overlap function, 0-overlap or 1-overlap function, then O is also a general overlap function, but the converse may not hold.*

In Table 2.1, we show some examples of general overlap functions. For more properties of general overlap functions, see (DGR+19).

Another topic related to overlap functions is that of overlap indices, which are used to measure the overlapping degree between fuzzy sets, defined as follows:

Definition 11. *(GBH+15) A mapping $\mathcal{O} : FS(U) \times FS(U) \to [0,1]$ is said to be an overlap index if it satisfies the following conditions, for all $A, B, C \in FS(U)$:*

$(\mathcal{O}1)$ *$\mathcal{O}(A,B) = 0$ if and only if, for all $z \in U, A(z) \cdot B(z) = 0$;*

$(\mathcal{O}2)$ *$\mathcal{O}(A,B) = \mathcal{O}(B,A)$;*

$(\mathcal{O}3)$ *If $B \leq C$, meaning that $B(z) \leq C(z)$ for all $z \in U$, then $\mathcal{O}(A,B) \leq \mathcal{O}(A,C)$.*

For an overlap index to be called normal, it also has to satisfy the following condition:

$(\mathcal{O}4)$ *If there exists $z \in U$ such that $A(z) \cdot B(z) = 1$, then $\mathcal{O}(A,B) = 1$.*

Table 2.1: Examples of General Overlap Functions

Definition	Type
$On_{MIN}(x_1,\ldots,x_n)=\min(x_1,\ldots,x_n)$	overlap
$On_P(x_1,\ldots,x_n)=\prod_{i=1}^n x_i$	overlap
$On_{mM}(x_1,\ldots,x_n)=\min(x_1,\ldots,x_n)\cdot\max(x_1^2,\ldots,x_n^2)$	overlap
$On_{GM}(x_1,\ldots,x_n)=\sqrt[n]{\prod_{i=1}^n x_i}$	overlap
$On_{HM}(x_1,\ldots,x_n)=\begin{cases}\frac{n}{\frac{1}{x_1}+\ldots+\frac{1}{x_n}} & \text{if } x_i>0, \text{for all } i\in\{1,\ldots,n\}\\ 0 & \text{otherwise.}\end{cases}$	overlap
$GO_L(x_1,\ldots,x_n)=\max\left(\left(\sum_{i=1}^n x_i\right)-(n-1),0\right)$	0-overlap
$GO_U(x_1,\ldots,x_n)=\begin{cases}n\cdot\prod_{i=1}^n x_i & \text{if } \prod_{i=1}^n x_i\leq 1/n,\\ 1 & \text{otherwise.}\end{cases}$	1-overlap
$GO_G(x_1,\ldots,x_n)=\begin{cases}n\cdot GO_L(x_1,\ldots,x_n) & \text{if } GO_L(x_1,\ldots,x_n)\leq 1/n,\\ 1, & \text{otherwise.}\end{cases}$	general overlap

In the following, we present some examples of overlap indices.

Example 8. *Considering the Definition 11, it holds that:*

(1) *The function $\mathcal{O}_Z : FS(U)\times FS(U)\to[0,1]$ defined, for all $A,B\in FS(U)$, by:*

$$\mathcal{O}_Z(A,B)=\max_{z\in U}\min(A(z),B(z)),$$

is a normal overlap index, which is known as the Zadeh's consistency index (Zad78).

(2) *The function $\mathcal{O}_\pi : FS(U)\times FS(U)\to[0,1]$ given, for all $A,B\in FS(U)$, by:*

$$\mathcal{O}_\pi(A,B)=\frac{1}{n}\sum_{i=1}^n A(z_i)\cdot B(z_i), \tag{2.12}$$

for $U=\{z_1,\ldots,z_n\}$, is an overlap index.

(3) *The function $\mathcal{O}_x : FS(U)\times FS(U)\to[0,1]$ defined, for all $A,B\in FS(U)$, by:*

$$\mathcal{O}_x(A,B)=\begin{cases}0 & \textit{if } \forall z\in U: A(z)\cdot B(z)=0,\\ x & \textit{otherwise},\end{cases}$$

for a given $z\in(0,1]$, is an overlap index.

The following theorem shows a construction method for overlap indices:

Theorem 2. *(GBH+15) Consider an aggregation function $M : [0,1]^n \to [0,1]$ such that*

$$M(x_1, \ldots, x_n) = 0 \Leftrightarrow x_1 = \ldots = x_n = 0,$$

and an overlap function $O : [0,1]^2 \to [0,1]$. Then, the function $\mathcal{O}_M^O : FS(U) \times FS(U) \to [0,1]$, given, for all $A, B \in FS(U)$, $U = \{z_1, \ldots, z_n\}$, by

$$\mathcal{O}_M^O(A, B) = M(O(A(z_1), B(z_1)), \ldots, O(A(z_n), B(z_n))),$$

is an overlap index.

2.2 Interval-valued fuzzy sets and admissible orders

When facing problems with imperfect information (Zad08; Zad05), there may be uncertainty regarding the values of the membership degrees or even in the definition of the membership functions to be used in a fuzzy modeling (Men07; Lod04; NKZ97). A viable and popular solution is the adoption of Interval-Valued Fuzzy Sets (IVFSs) (BBP+16; GG76; Zad75; DP05), where the membership degrees are represented by intervals. In the following, we recall their definition and related concepts.

Denote by $L([0,1])$ the set of all closed subintervals of the unit interval $[0,1]$ and $\vec{X} = (X_1, \ldots, X_n) \in L([0,1])^n$. For any $X = [x_1, x_2] \in L([0,1])$, the left and right projections of X are denoted, respectively, by $\underline{X} = x_1$ and $\overline{X} = x_2$. The width of X is denoted $w(X)$, which is given by $w(X) = \overline{X} - \underline{X}$.

Definition 12. *(Zad75) Given an universe U, an interval-valued fuzzy set on U is a function $IF : U \to L([0,1])$ such that $IF(z) = [F_l(z), F_u(z)]$, for all $z \in U \neq \emptyset$, where $F_l(z) = \underline{IF(z)}$, $F_u(z) = \overline{IF(z)}$, $F_l \leq F_u$ and $F_l, F_u \in FS(U)$.*

We denote by $IFS(U)$ the space of all interval-valued fuzzy sets defined over U.

Clearly, $IF(z) \in L([0,1])$, and it is the interval membership degree of an element $z \in U$. One key aspect of IVFS is informed by the width of the assigned intervals, given by $w(IF(z)) = F_u(z) - F_l(z)$, since they represent the uncertainty/ignorance in the modeling of fuzzy sets (BDSR10; DBSR11; SFBH11). The width of an interval can also be a measure of the quality of information (DCC00; AJ94) carried by it. In a general sense, the larger the width of an interval, the higher the difficulty in estimating the exact value it is approximating and, thus, the lesser the information quality.

By Definition 12, one can observe that an IVFS $\mathcal{F}$ can be represented by a pair of fuzzy sets: the lower fuzzy set F_l and the upper fuzzy set F_u. If $F_l(z) = F_u(z)$, for every $z \in U$, then $\mathcal{F}$ is a fuzzy set, which means that fuzzy sets are particular cases of interval-valued fuzzy sets.

An graphical representation of an IVFS can be seen in Figure 2.2.

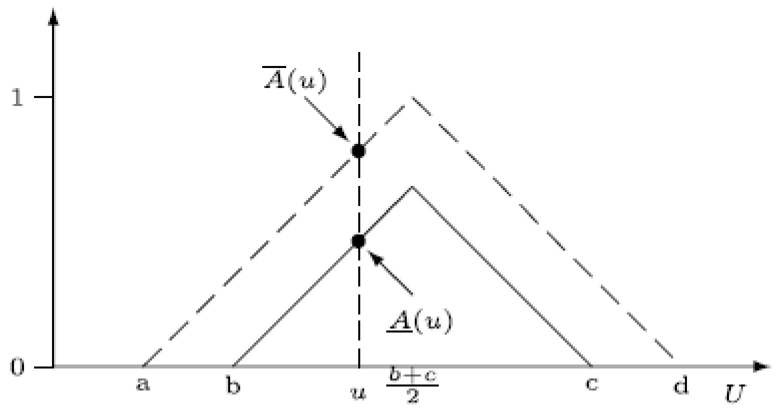

Figure 2.2: Membership function of an IVFS.

In practical problems that involve interval-valued fuzzy modeling, it is expected that the calculations have intervals as inputs and/or outputs. So, here, we recall some basic concepts on interval mathematics.

We call by *iv-fusion function* any interval-valued function $IF : L([0,1])^n \to L([0,1])$ that merges n intervals from $L([0,1])$ into a single interval in $L([0,1])$.

Definition 13. *(BMDF$^+$20) An iv-fusion function $IF : L([0,1])^n \to L([0,1])$ is called width-preserving if, for any $\vec{X} \in L([0,1])^n$ such that $w(X_1) = \ldots = w(X_n)$, it holds that $w(IF(\vec{X})) = w(X_1)$.*

An iv-fusion function $IF : L([0,1])^n \to L([0,1])$ is said to be increasing with respect to a partial order $\leq$ on $L([0,1])$ (or, simply, $\leq$-increasing) if, for all $\vec{X}, \vec{Y} \in L([0,1])^n$, the following condition holds:

$$X_i \leq Y_i \text{ for all } i \in \{1, \ldots, n\} \Rightarrow IF(\vec{X}) \leq IF(\vec{Y}).$$

The *product order* (MKC09), denoted by $\leq_{Pr}$, is a partial order relation, defined, for all $X, Y \in L([0,1])$, by:

$$X \leq_{Pr} Y \quad \Leftrightarrow \quad \underline{X} \leq \underline{Y} \wedge \overline{X} \leq \overline{Y}.$$

The projections $F^-, F^+ : [0,1]^n \to [0,1]$ of $F : L([0,1])^n \to L([0,1])$ are defined, respectively, by:

$$\begin{aligned} F^-(x_1, \ldots, x_n) &= \underline{F([x_1, x_1], \ldots, [x_n, x_n])}; \\ F^+(x_1, \ldots, x_n) &= \overline{F([x_1, x_1], \ldots, [x_n, x_n])}. \end{aligned}$$

Let $f, g : [0,1]^n \to [0,1]$ be two fusion functions such that $f \le g$. Then, the iv-fusion function $\widehat{f,g} : L([0,1])^n \to L([0,1])$ is given by:

$$\widehat{f,g}(\vec{X}) = [f(\underline{X_1}, \ldots, \underline{X_n}), g(\overline{X_1}, \ldots, \overline{X_n})]. \tag{2.13}$$

Definition 14. *(DBSR11) Let $IF : L([0,1])^n \to L([0,1])$ be a $\le_{Pr}$-increasing iv-fusion function. Then, IF is said to be representable if there exist increasing fusion functions $f, g : [0,1]^n \to [0,1]$ such that $f \le g$ and $IF = \widehat{f,g}$.*

The fusion functions f and g are called the *representatives* of IF. When $IF = \widehat{f,f}$, we denote simply as $\widehat{f}$, and, in this case, IF is said to be *the best interval representation of* f (DBSR11).

An iv-fusion function IF is said to be Moore-continuous if it is continuous with respect to the Moore metric (MKC09) $d_M : L([0,1])^2 \to \mathbb{R}$, defined, for all $X, Y \in L([0,1])$, by:

$$d_M(X,Y) = \max(|\underline{X} - \underline{Y}|, |\overline{X} - \overline{Y}|).$$

The Moore-metric can be extended to $L([0,1])^n$ as follows:

$$d_M^n(\vec{X}, \vec{Y}) = \sqrt{d_M(X_1, Y_1)^2 + \ldots + d_M(X_n, Y_n)^2}.$$

Here, we show some interval operations that are used throughout our work, considering $X, Y \in L([0,1])$: (MKC09; Ste10)

Sum: $X + Y = [\underline{X} + \underline{Y}, \overline{X} + \overline{Y}]$, with $\overline{X} + \overline{Y} \le 1$;

Limited Sum $X \dot{+} Y = [\min(1, \underline{X} + \underline{Y}), \min(1, \overline{X} + \overline{Y})]$;

Product: $X \cdot Y = [\underline{X} \cdot \underline{Y}, \overline{X} \cdot \overline{Y}]$;

Exponential: $X^p = [\underline{X}^p, \overline{X}^p]$, for any $p \in \mathbb{R}$;

Division: $X/Y = [\underline{X}/\overline{Y}, \overline{X}/\underline{Y}]$ with $\underline{Y} \neq 0$.

Generalized Hukuhara Division:

$$X \div_H Y = [\min\{\underline{X}/\underline{Y}, \overline{X}/\overline{Y}\}, \max\{\underline{X}/\underline{Y}, \overline{X}/\overline{Y}\}]$$

with $\underline{Y} \neq 0$ and $X \le_{Pr} Y$.

In the following, we recall an important concept for the development of the thesis, that of *admissible orders*:

Definition 15. *(BFKM13) Let $(L([0,1]), \leq_{AD})$ be a partially ordered set. The order $\leq_{AD}$ is an admissible order if*

(i) $\leq_{AD}$ *is a total order on* $(L([0,1]), \leq_{AD})$*;*

(ii) $X \leq_{Pr} Y \Rightarrow X \leq_{AD} Y$*, for all* $X, Y \in L([0,1])$.

Thus, an order $\leq_{AD}$ on $L([0,1])$ is said to be admissible if it is a total order that refines the product order $\leq_{Pr}$ (BFKM13).

Remark 1. *Since every admissible order $\leq_{AD}$ refines $\leq_{Pr}$, it is immediate that every $\leq_{AD}$-increasing function is also $\leq_{Pr}$-increasing.*

Example 9. *The following relations on $L([0,1])$ are examples of admissible orders:*

(i) The lexicographical orders with respect to the first and second coordinate, defined, respectively, by:

$$X \leq_{Lex1} Y \Leftrightarrow \underline{X} < \underline{Y} \vee (\underline{X} = \underline{Y} \wedge \overline{X} \leq \overline{Y});$$

$$X \leq_{Lex2} Y \Leftrightarrow \overline{X} < \overline{Y} \vee (\overline{X} = \overline{Y} \wedge \underline{X} \leq \underline{Y}).$$

(ii) The order $\leq_{XY}$ introduced by Xu and Yager in (XY06), defined by:

$$X \leq_{XY} Y \Leftrightarrow \underline{X} + \overline{X} < \underline{Y} + \overline{Y} \text{ or } (\underline{X} + \overline{X} = \underline{Y} + \overline{Y} \text{ and } \overline{X} - \underline{X} \leq \overline{Y} - \underline{Y}).$$

(iii) Whenever one considers the comparison of the information quality provided by the intervals X and Y in the order of Xu and Yager, it is possible to define it as in (SFBH13):

$$X \leq_{IQ} Y \Leftrightarrow \underline{X} + \overline{X} < \underline{Y} + \overline{Y} \text{ or } (\underline{X} + \overline{X} = \underline{Y} + \overline{Y} \text{ and } \overline{Y} - \underline{Y} \leq \overline{X} - \underline{X}). \tag{2.14}$$

Next, we recall the definition of the admissible order $\leq_{\alpha,\beta}$:

Definition 16. *(BFKM13) For $\alpha, \beta \in [0,1]$ such that $\alpha \neq \beta$, the relation $\leq_{\alpha,\beta}$ is defined, for all $X, Y \in L([0,1])$, by*

$$X \leq_{\alpha,\beta} Y \Leftrightarrow K_\alpha(\underline{X}, \overline{X}) < K_\alpha(\underline{Y}, \overline{Y}) \text{ or}$$
$$(K_\alpha(\underline{X}, \overline{X}) = K_\alpha(\underline{Y}, \overline{Y}) \text{ and } K_\beta(\underline{X}, \overline{X}) \leq K_\beta(\underline{Y}, \overline{Y})),$$

where $K_\alpha, K_\beta : [0,1]^2 \to [0,1]$ *are aggregation functions defined, for all* $x, y \in [0,1]$*, respectively, by*

$$K_\alpha(x,y) = x + \alpha \cdot (y - x), \tag{2.15}$$
$$K_\beta(x,y) = x + \beta \cdot (y - x).$$

Observe that the operator K_α corresponds to Hurwicz's criterion (Hur51) for adjusting pessimism and optimism under uncertainty, when working in contexts of imperfect information.

Remark 2. *By varying the values of* α *and* β *one can recover some of the defined admissible orders, e.g., the lexicographical orders* $\leq_{Lex1}$ *and* $\leq_{Lex2}$*, and the orders* $\leq_{XY}$ *and* $\leq_{IQ}$ *are recovered, respectively, by* $\leq_{0,1}$, $\leq_{1,0}$, $\leq_{0.5,1}$ *and* $\leq_{0.5,0}$.

Whenever we apply the mapping K_α on the endpoints of an interval $X \in [0,1]$, we denote $K_\alpha(\underline{X}, \overline{X})$ simply as $K_\alpha(X)$.

Proposition 2. *(BFKM13) For any* $\alpha, \beta \in [0,1]$, $\alpha \neq \beta$*, it holds that:*

i) $\beta > \alpha \Rightarrow \leq_{\alpha,\beta} = \leq_{\alpha,1}$;

ii) $\beta < \alpha \Rightarrow \leq_{\alpha,\beta} = \leq_{\alpha,0}$.

2.3 Interval-valued aggregation functions

Here, we recall some important interval-valued operations that are used throughout the development of the thesis.

Definition 17. *(Bed10) A function* $IN : L([0,1]) \to L([0,1])$ *is called an iv-fuzzy negation if it is* $\leq_{Pr}$*-decreasing and respects the following conditions:*

(IN1) $IN([1,1]) = [0,0]$;

(IN2) $IN([0,0]) = [1,1]$.

If $IN(IN(X)) = X$*, for all* $X \in L([0,1])$*, then* N *is said to be a strong iv-fuzzy negation.*

Definition 18. *(JPP*$^+$*09) A bivariate iv-fusion function* $IR^{IN} : L([0,1])^2 \to L([0,1])$ *is called an Interval-Valued Restricted Equivalence Function (IV-REF) associated with a strong iv-fuzzy negation* $IN : L([0,1]) \to L([0,1])$ *is it satisfies the following conditions, for all* $X, Y, Z \in L([0,1])$*:*

(IR1) IR^{IN} *is symmetric;*

(IR2) $IR^{IN}(X,Y) = [1,1] \Leftrightarrow X = Y$;

(IR3) $IR^{IN} = [0,0] \Leftrightarrow X = [0,0]$ *and* $Y = [1,1]$, *or* $X = [1,1]$ *and* $Y = [0,0]$;

(IR4) $IR^{IN}(X,Y) = IR^{IN}(IN(X), IN(Y))$;

(IR5) $\forall X, Y, Z \in L([0,1]), X \leq_{Pr} Y \leq_{Pr} Z \Rightarrow IR^{IN}(X,Y) \geq_{Pr} IR^{IN}(X,Z)$ *and* $IR^{IN}(Y,Z) \geq_{Pr} IR^{IN}(X,Z)$.

A construction method for IV-REFs (Proposition 3) based on automorphisms (see Definition 19) was introduced in (SFBH13), as follows:

Definition 19. *An automorphism of the unit interval is any continuous and strictly increasing function* $\varphi : [0,1] \to [0,1]$ *so that* $\varphi(0) = 0$ *and* $\varphi(1) = 1$.

Proposition 3. *Let* φ_1 *and* φ_2 *be two automorphisms of the unit interval,* $T : [0,1]^2 \to [0,1]$ *be a t-norm and* $S : [0,1]^2 \to [0,1]$ *be a t-conorm. Then, the function* $IR : L([0,1])^2 \to L([0,1])$*, given, for all* $X, Y \in L([0,1])$*, by*

$$\begin{aligned} IR(X,Y) &= [T(\varphi_1^{-1}(1 - |\varphi_2(\underline{X}) - \varphi_2(\underline{Y})|), \varphi_1^{-1}(1 - |\varphi_2(\overline{X}) - \varphi_2(\overline{Y})|)), \\ & \quad S(\varphi_1^{-1}(1 - |\varphi_2(\underline{X}) - \varphi_2(\underline{Y})|), \varphi_1^{-1}(1 - |\varphi_2(\overline{X}) - \varphi_2(\overline{Y})|))]. \end{aligned} \tag{2.16}$$

is an IV-REF.

Example 10. *Consider the* $a, b \in (0, +\infty)$ *and a fuzzy negation* $IN^b : L([0,1]) \to L([0,1])$*, defined, for all* $X \in L([0,1])$*, by*

$$IN^b(X) = \left[(1 - \overline{X}^b)^{\frac{1}{b}}, (1 - \underline{X}^b)^{\frac{1}{b}}\right]. \tag{2.17}$$

Then, the iv-fusion function $IR^{IN^b} : L([0,1])^2 \to L([0,1])$*, given, for all* $X, Y \in L([0,1])$*, by*

$$IR^{IN^b}(X,Y) = \left[\min\{(1 - |\underline{X}^b - \underline{Y}^b|)^{\frac{1}{a}}, (1 - |\overline{X}^b - \overline{Y}^b|)^{\frac{1}{a}}\}, \max\{(1 - |\underline{X}^b - \underline{Y}^b|)^{\frac{1}{a}}, (1 - |\overline{X}^b - \overline{Y}^b|)^{\frac{1}{a}}\}\right] \tag{2.18}$$

is an IV-REF associated with IN^b*. One may observe that Equation (2.18) was obtained by applying the construction method in Proposition 3, by taking the automorphisms* $\varphi_1, \varphi_2 : [0,1] \to [0,1]$*, defined, for all* $x \in [0,1]$*, respectively, by* $\varphi_1(x) = x^a$ *and* $\varphi_2(x) = x^b$*, and considering the minimum t-norm and the maximum t-conorm.*

IV-REFs are interval operators that are suitable to model the similarity between intervals (JPP+09). In particular, the IV-REF IR^{IN^b}, defined in Example 10, was applied in IV-FRBCSs (SFBH13).

Definition 20. *(KM11) An iv-fusion function $IA : L([0,1])^n \to L([0,1])$ is called an iv-aggregation function if the following conditions are satisfied:*

(IA1) *IA is $\leq_{Pr}$-increasing;*

(IA2) *$IA([0,0],\dots,[0,0]) = [0,0]$ and $IA([1,1],\dots,[1,1]) = [1,1]$.*

Besides the usual way of representing an interval X by its endpoints, $X = [\underline{X}, \overline{X}]$, in (BMDF$^+$20), Bustince et al. introduced a new form of representation for intervals based on K_α points (see Equation (2.16)) and the concept of maximal possible width, which we recall in the following:

Definition 21. *(BMDF$^+$20) Consider $c \in [0,1]$ and $\alpha \in [0,1]$. Then, the maximal possible width of an interval $Z \in L([0,1])$ is denoted by $d_\alpha(c)$, such that $K_\alpha(Z) = c$. Also, define, for any $X \in L([0,1])$,*

$$\lambda_\alpha(X) = \frac{w(X)}{d_\alpha(K_\alpha(X))}, \tag{2.19}$$

where we set $\frac{0}{0} = 1$.

Proposition 4. *(BMDF$^+$20) For all $\alpha \in [0,1]$ and $X \in L([0,1])$, it one has that*

$$d_\alpha(K_\alpha(X)) = \min\left\{\frac{K_\alpha(X)}{\alpha}, \frac{1-K_\alpha(X)}{1-\alpha}\right\}, \tag{2.20}$$

where we set $\frac{r}{0} = 1$, for all $r \in [0,1]$.

Thus, an interval X can be represented by $X = (K_\alpha(X), \lambda_\alpha(X))$. A construction method for iv-aggregation functions, based on this representation, was also introduced:

Theorem 3. *(BMDF$^+$20) Let $\alpha, \beta \in [0,1]$ be such that $\alpha \neq \beta$. Let $A_1, A_2 : [0,1]^n \to [0,1]$ be two aggregation functions where A_1 is strictly increasing. Then $IF^\alpha : L([0,1])^n \to L([0,1])$ defined by:*

$$IF^\alpha_{A1,A2}(\vec{X}) = R, \ where, \begin{cases} K_\alpha(R) = A_1(K_\alpha(X_1), \dots, K_\alpha(X_n)), \\ \lambda_\alpha(R) = A_2(\lambda_\alpha(X_1), \dots, \lambda_\alpha(X_n)), \end{cases}$$

for all $\vec{X} \in L([0,1])^n$, is an $\leq_{\alpha,\beta}$-increasing iv-aggregation function.

Since one of the main interests in our work is to apply interval-valued overlap operations in practical problems, here we recall the concept of *iv-overlap functions*, which was defined, independently, by Qiao and Hu (QH17) and Bedregal et al. (BBP$^+$17), as an extension of overlap functions to the interval-valued context:

Definition 22. *(BBP+17; QH17) An iv-overlap function is a mapping* $IO : L([0,1])^2 \to L([0,1])$ *that respects the following conditions:*

(IO1) IO *is symmetric;*

(IO2) $IO(X,Y) = [0,0]$ *if and only if* $X \cdot Y = [0,0]$*;*

(IO3) $IO(X,Y) = [1,1]$ *if and only if* $X \cdot Y = [1,1]$*;*

(IO4) IO *is* $\leq_{Pr}$*-increasing;*

(IO5) IO *is Moore continuous.*

In the following, we recall the definition of iv-t-norms:

Definition 23. *(DBRS08) An interval-valued (iv) t-norm is a mapping* $IT : L([0,1])^2 \to L([0,1])$ *that respects the following conditions:*

(IT1) IT *is symmetric;*

(IT2) IT *is associative;*

(IT3) IT *has* $[1,1]$ *as its neutral element;*

(IT4) IT *is increasing.*

2.4 Classification Problems

Until this point, we presented some preliminary concepts from which our theoretical contributions in the thesis were based on or inspired by. Now, we review the main characteristics of classification problems and some specific methods for tackling these types of problems, since one of our objectives is to improve the classification accuracy of those methods by applying our developed concepts.

A classification problem, under the supervised point of view, is characterized by the task of predicting the unknown class of some object, which we call *example*, by applying a model learned using other examples whose classes are previously known. This set of correctly classified examples is called *training set*, denoted by E. In this set, each example $e \in E$ is described by the values of N features (also called variables, characteristics or attributes) $X(e) = (e_1, \ldots, e_N)$, and belongs to one of M classes in $C=\{C_1, \ldots, C_M\}$.

Thus, the objective is to build a model $D : X(e) \rightarrow C$ that can predict the class of new examples with the minimal cost possible. The performance of the model can be measured based on a ratio of correctly classified examples from a testing set that is not part of the learning process of the system. So, the prediction of the test data is performed by the learned classifier, which can be also used to classify the training data to verify if the model has a good generalization capability. The steps of a supervised classification problem can be seen in Figure 2.3.

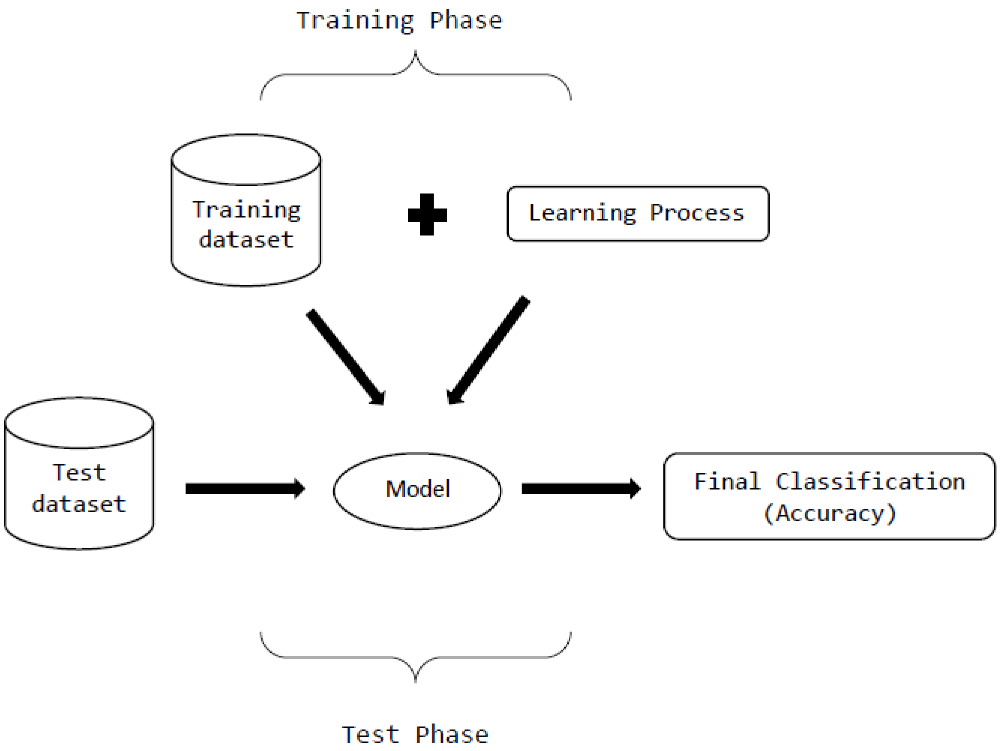

Figure 2.3: Representation of the supervised learned method for classification.

There are several methods to deal with classification problems, such as Support Vector Networks (SVN) (CV95), decisions trees (Qui93) and neural networks (Gra12). In particular, when the objective is to design a robust classifier that is both accurate and interpretable, a viable alternative is the use of Fuzzy Rule-Based Classification Systems (FRBCSs) (INN04).

2.5 Fuzzy rule-based classification systems

In classical rule-based classification systems (Tun09), the rules that compose the system are of the following type:

IF condition THEN decision.

The IF part is known as the antecedent of the rule, where one or more features (attributes) are linked by logical connectives, such as conjunctions (AND) or disjunctions (OR). The THEN part is known as the consequent of the rule and it shows one of the classes from C.

The antecedents of the rules in a rule-based system are usually categorical or numerical, and have only TRUE of FALSE as a result, since they are based on boolean logic. That is, the rules only take into account two possible scenarios: either the example completely has a given feature or such feature is completely absent from the example. This rigid approach does not consider cases in which a example may have a degree of a given feature, like when using linguistic labels to represent the states of a variable (e.g., HEIGHT is TALL or AGE is YOUNG).

Thus, FRBCSs were developed as an extension of the rule-based classification systems, with the main difference between them being that the FRBCSs allows for the modeling of linguistic variables through fuzzy sets, while its rules are composed by propositions based on fuzzy logic. For that reason, some of the advantages of FRBCSs are their flexibility and high level of interpretability (SBH$^+$15).

The most known FRBCSs are the ones defined by Takagi-Sugeno-Kang (TSK) (TS85) and Mamdani (Mam74), which is the one used in this thesis and whose generic structure can be seen in Figure 2.4.

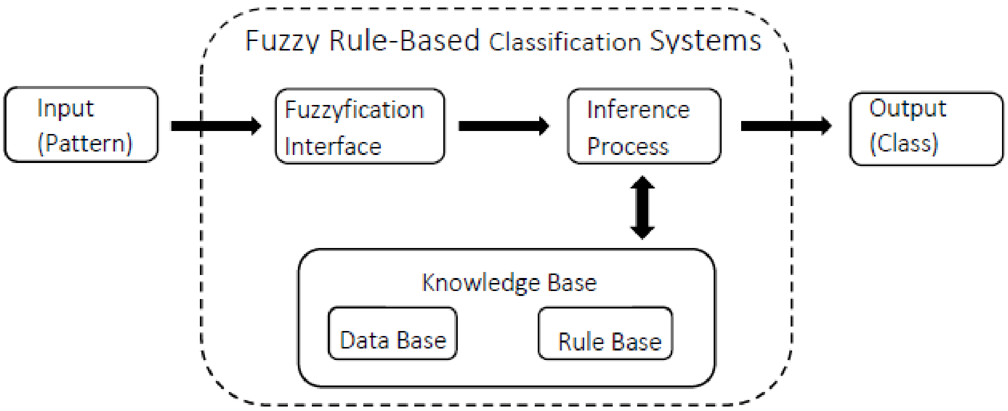

Figure 2.4: A structure of FRBCS of the Mamdani type.

The Knowledge Base (KB) stores the available information about the problem at hand. There are two levels of information stored in the KB: the semantics of the fuzzy rules (based on fuzzy sets) and the linguistic rules that represent the expert knowledge. This conceptual distinction is made clear by the two components that constitutes the KB:

i) The Data Base (DB) - It stores the definition of the membership functions associated with the linguistic labels whose features are considered in the fuzzy rules;

ii) The Rule Base (RB) - It is composed by a collection of linguistic fuzzy rules that are linked by a connective (operator "OR"). This configuration enables that multiple rules may be activated by the same input. In this thesis, we consider that the fuzzy rules have the following structure:

$$\text{Rule } R_j : \text{If } x_1 \text{ is } A_{j1} \text{ and } \ldots \text{ and } x_n \text{ is } A_{jn} \text{ then } Class = C_j' \text{ with } RW_j, \tag{2.21}$$

where $j \in \{1, \ldots, L\}$, L is the number of rules in the RB, R_j is the label of the j-th rule, $x = (x_1, \ldots, x_n)$ is a feature vector (input variables), A_{ji} is the fuzzy set representing the linguistic term of the j-th rule in the i-th antecedent, $C_j' \in C$ is a class label, and $RW_j \in [0, 1]$ is the rule weight, which measures the importance/relevance of the rule for the classification task.

The fuzzyfication interface transforms the numerical input data into fuzzy values. Once such conversion is complete, the inference system parse the stored information in the KB to predict the class of any data pattern that is admitted by the system, in a process known as Fuzzy Reasoning Method (FRM).

Here, we discuss each step of the FRM, considering P training examples $\vec{x_p} = (x_{p1}, \ldots, x_{pn}), p \in \{1, \ldots, P\}$ where x_{pi} is the value of the i-th variable of the p-th example:

1) Matching degree ($A_j(x_p)$) - It measures the strength of the IF-part of a rule R_j for the example x_p to be classified, calculated through a fuzzy conjunction operator $\mathfrak{c}$ (generalized AND), as follows:

$$A_j(x_p) = \mathfrak{c}(A_{j1}(x_{p1}), \cdots, A_{jn}(x_{pn})). \tag{2.22}$$

Usually, $\mathfrak{c}$ is represented by either an extended t-norm or an n-dimensional overlap function.

2) Association degree (b_j^k) - It weights the matching degree $A_j(x_p)$ by the rule weight RW_j, through a product operation:

$$b_j^k = A_j(x_p) \cdot RW_j, \tag{2.23}$$

where $k \in \{1, \ldots, M\}$ corresponds to the class C_k in the consequent of the Rule R_j. Specifically, we consider the computation of the rule weight using the fuzzy confidence value or certainty factor (CdJH99), given by:

$$RW_j = \frac{\sum_{x_p \in C_j'} A_j(x_p)}{\sum_{p=1}^{P} A_j(x_p)}. \tag{2.24}$$

3) Example classification soundness degree for all classes (Y_k) - For each class $C_k \in C$, we aggregate all the positive association degrees b_j^k that were obtained in the previous step with respect to C_k, through an aggregation function A:

$$Y_k = A(b_j^k, j = 1, \ldots, L, \text{ and } b_j^k > 0). \tag{2.25}$$

Some of the functions that are usually considered in this step are the maximum and the normalized sum, which define, respectively, the wining rule and the additive combination FRMs (CdJH99). The Choquet integral and its generalizations has also been successfully applied as the aggregation operator in this stage (LDF+19).

4) Classification - The final decision is made in this step. For that, a function $F : [0, 1]^M \to C$ is applied over all example classification soundness degrees calculated in the previous step:

$$F(Y_1, \ldots, Y_M) = \arg \max_{k=1,\ldots,M} (Y_k). \tag{2.26}$$

2.6 Interval-valued fuzzy rule-based classification systems

An Interval-Valued Fuzzy Rule-Based Classification System (IV-FRBCS) is an extension of a FRBCS when some of the linguistic labels (or all of them) are modelled using IVFSs. This means that the fuzzy reasoning method must work with intervals instead of numbers, being called as Interval-Valued Fuzzy Reasoning Method (IV-FRM), to take into account the interval widths (uncertainty) throughout the whole inference process (SFBH12; SBH+15).

The IV-FRM with Tuning and Rule Selection (IVTURS) (SFBH13) is a state-of-the-art IV-FRBCS based on the concept of minimum distance between the interval matching degrees and the ideal interval $[1, 1]$ (which symbolizes a "perfect match" between a pattern and the antecedent of a rule). The motivation behind this approach is to strengthen the relevance of the rules with a higher equivalence degree with respect to the new pattern to be classified. Those equivalence degrees are measured through IV-REFs (Definition 18) that are constructed in a parameterized manner, meaning that a genetic tuning of such parameters may be employed to find the appropriate IV-REFs that better suit a given problem.

A depiction of the IVTURS method can be seen in Figure 2.5.

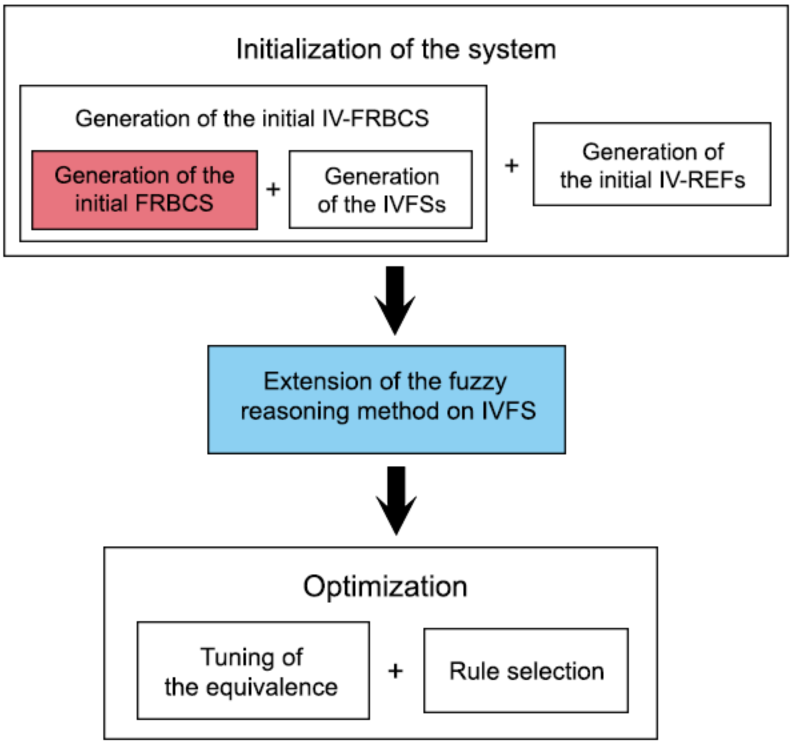

Figure 2.5: A flowchart of the IVTURS method.

Here, we present an overview of each of its steps:

1) Initialization of the IV-FRBCS, involving three tasks:

- **i)** Generation of an initial FRBCS by means of the FARCHD method (AFAH11);
- **ii)** Modeling of the linguistic labels by means of IVFSs;
- **iii)** Construction of the initial IV-REF for each variable of the problem.

2) Definition of the IV-FRM according to the DB composed by IVFSs.

3) Application of an optimization approach with two tasks:

i) Genetic tuning to find best values for the IV-REFs' parameters;

ii) Rule selection process in order to decrease the system's complexity.

In Figure 2.5, we highlight in color the generation of the initial FRBCS (in red) and the extension of the fuzzy reasoning method on IVFS (in blue), since those are the ones where most modifications took place when developing our own IV-FRBCS. In the following, we review each step from the IVTURS algorithm, since it served as a basis for the IV-FRBCS where most of the experimentations took place in the development of the thesis.

2.6.1 Initialization of the interval-valued fuzzy rule-based classification system

There are two distinct approaches when designing an interval fuzzy model: one is a partial dependent one, where an initial fuzzy model is learned and then used as a smart initialization of the parameters of the interval fuzzy model (JFW$^+$09) and the other is a totally independent method, where the interval fuzzy model is learned without the help of a base fuzzy model (WW06). The IVTURS method is based on the first approach, generating its base FRBCS by means of the FARC-HD algorithm (AFAH11) (Fuzzy Association Rule-based Classification model for High Dimensional problems), which is based on three steps:

1) Fuzzy association rule extraction for classification - An initial RB is generated, through the construction a search tree (AS94) for each class, with its maximal depth being limited by a parameter. For each variable, the system considers five linguistic labels modeled by triangular fuzzy sets, by performing a linear partitioning of the input domain, as presented on Figure 2.6.

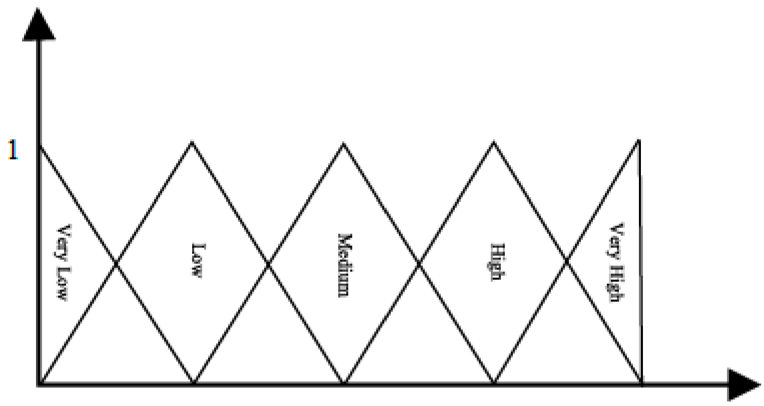

Figure 2.6: Modeling of the linguistic labels in FARCHD.

2) Candidate rule prescreening - Since the number of rules in the initial RB is usually too large, the algorithm preselects the most promising rules through subgroup discovery to reduce the computational cost of the system (BLB+03). This selection is done through a pattern weighting scheme, an iterative process in which, initially, each example is assigned the same the weight, and then, examples that have been covered by one or more rules decrease their weights in each iteration. Then, each rule is evaluated based on the values of the weights of all examples covered by it.

3) Genetic rule selection and lateral tuning - The KB is optimized by means of a combination between the tuning of the lateral position of the membership functions and a rule selection process (AAFH07). This stage is done differently in the IVTURS method, and it is not part of the generation of the initial FRBCS.

Only the first two steps of the FARCHD algortithm are performed in order to generate the base FRBCS. From that, the linguistic labels are modeled by means of IVFSs. This process is done by following two steps:

1) The lower bound of each IVFS (lower fuzzy set) is defined based on the initial membership functions that were generated by the FARCHD algorithm;

2) The upper bound of each IVFS (upper fuzzy set) is defined so that its support has a 50% larger width than its lower bound counterpart;

An example of these constructed IVFS can be seen in Figure 2.7.

The expression of the initial IV-REF IR for each variable of the problem is given on Equation (2.18), from Example 10. In this initial stage, we set $a = b = 1$, so Equation (2.18) can be rewritten as:

$$IR(X, Y) = [\min\{1 - |\underline{X} - \underline{Y}|, 1 - |\overline{X} - \overline{Y}|\}, \max\{1 - |\underline{X} - \underline{Y}|, 1 - |\overline{X} - \overline{Y}|\}]. \tag{2.27}$$

2.6.2 Definition of the IV-FRM

This stage is quite similar to the FRM described previously, but now its four steps are designed to operate with the interval membership degrees obtained accordingly to the constructed IVFS. Here we review each of those steps:

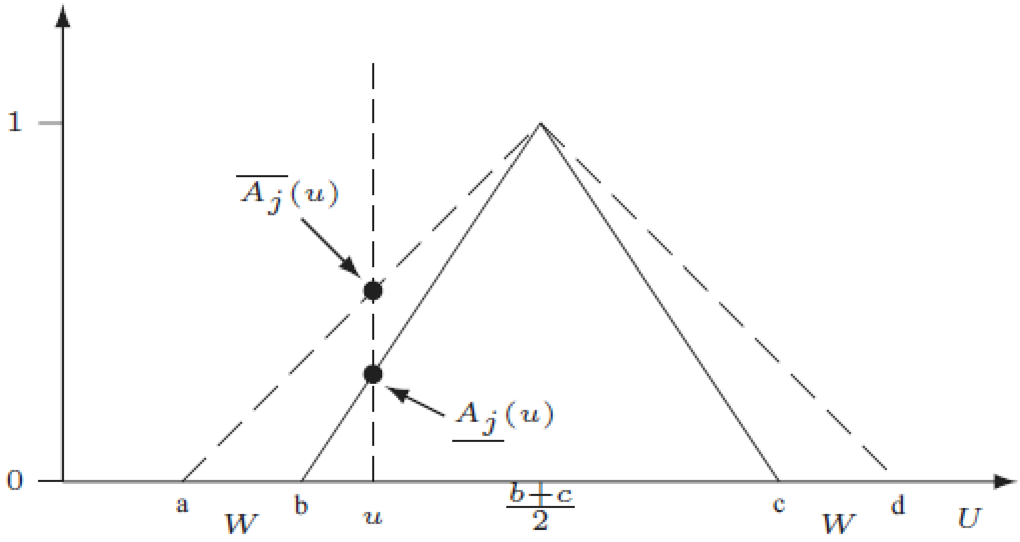

Figure 2.7: Example of a constructed IVFS in the IVTURS method.

(1) Interval matching degree: The similarity between the interval membership degrees (of each variable to the corresponding IVFS) and the ideal membership degree $[1, 1]$ is measured, for $j \in \{1, \ldots, L\}$, and, then, those similarities are aggregated through a representable iv-t-norm $IT : L([0, 1])^n \to L([0, 1])$ (see Definitions 14 and 23), as follows:

$$\left[\underline{\mathcal{A}_j(x_p)}, \overline{\mathcal{A}_j(x_p)}\right] = IT\left(IR\left(\left[\underline{\mathcal{A}_{j1}(x_{p1})}, \overline{\mathcal{A}_{j1}(x_{p1})}\right], [1,1]\right), \ldots, IR\left(\left[\underline{\mathcal{A}_{jn}(x_{pn})}, \overline{\mathcal{A}_{jn}(x_{pn})}\right], [1,1]\right)\right).$$

The interval matching degree represents the strength of the activation of the if-part of the rules for each x_p.

(2) Interval association degree: For the class of each rule, the interval matching degree is weighted with the corresponding interval-valued (iv) rule weight $IRW_j^k \in L([0, 1])$, resulting in the following expression:

$$\left[\underline{b_j^k}, \overline{b_j^k}\right] = \left[\underline{\mathcal{A}_j(x_p)}, \overline{\mathcal{A}_j(x_p)}\right] \cdot \left[\underline{IRW_j^k}, \overline{IRW_j^k}\right] \text{ with } k = 1, \ldots, M \text{ and } j = 1, \ldots, L,$$

The iv-rule weight is defined by the iv-confidence value, which is the interval extension of the confidence value introduced in (IY05), given by:

$$IRW_j = \sum_{x_p \in C_j'} \mathcal{A}_j(x_p) \div_H \sum_{p=1}^{P} \mathcal{A}_j(x_p),$$

with $\div_H$ being the generalized Hukuhara division, defined in Equation (2.14).

(3) Interval pattern classification soundness degree for all classes: The interval association degrees of each class (obtained in the last step) with the upper bounds that are greater than 0 are aggregated, by applying an iv-aggregation function IA, as follows:

$$\left[\underline{Y_k}, \overline{Y_k}\right] = IA\left(\left[\underline{b_j^k}, \overline{b_j^k}\right], j = 1, \ldots, L \text{ and } \overline{b_j^k} > 0\right) \text{ with } k = 1, \ldots, M.$$

(4) Classification: A decision function F is applied over the interval soundness degree of the system for the pattern classification for all classes, given by:

$$F\left(\left[\underline{Y_1},\overline{Y_1}\right],\ldots,\left[\underline{Y_M},\overline{Y_M}\right]\right)=\arg\max_{k=1,\ldots,M}\left(\left[\underline{Y_k},\overline{Y_k}\right]\right).$$

The last step of the IV-FRM consists of selecting the maximum interval soundness degree. To achieve that, the system needs to compare the intervals with a total order relation (e.g. the order of Xu and Yager (XY06)).

2.6.3 Tuning of the equivalence and rule selection

At this point, an evolutionary process is applied to: (i) tune the values of the parameters a and b used in the construction of the IV-REFs (See Example 10) and (ii) perform a rule selection process with the goal of removing redundant or contradictory rules from the RB.

Regarding the tuning of the parameters a and b, in order to cover as much search space as possible in the optimization problem, Sanz et al. (SFBH13) suggest varying these values in the interval $[0.01, 100]$. In Figure 2.8, the shadow surface represents the space that is covered when using the proposed variation interval, which is almost the entirety of the search space. This selection of suitable IV-REFs to measure the equivalence degrees in each variable has the capability of improving the system reasoning accuracy.

From the many known strategies that deal with the rule reduction process (GAH11), the IVTURS method applies a rule selection approach developed through a simple binary codification that expresses whether or not the fuzzy rules should belong to the RB.

To accomplish both tasks, a CHC evolutionary model (Esh91) is applied, since it provides good results in classification systems (AAFH07). In the particular case of the IVTURS method, the evolutionary model has the following characteristics:

1) Coding scheme - Each chromosome is composed by two distinct parts, which implies a double codification scheme: real codification for the tuning of the parameters a and b of the IV-REFs (CE) and binary coding for the rule selection process (CR).

a) Tuning of the equivalence: Considering n as the number of features, the part of the chromosome to carry out the tuning of the IV-REFs is a vector CE with dimension $2\times n$, given by $CE=\{a_1,b_1,\ldots,a_n,b_n\}$, where $a_i,b_i\in[0.01,1.99]$ for all $i\in\{1,\ldots,n\}$. Each pair (a_i,b_i) represents the values of the parameters a and b that are used in the construction of the IV-REF associated with the i-th feature.

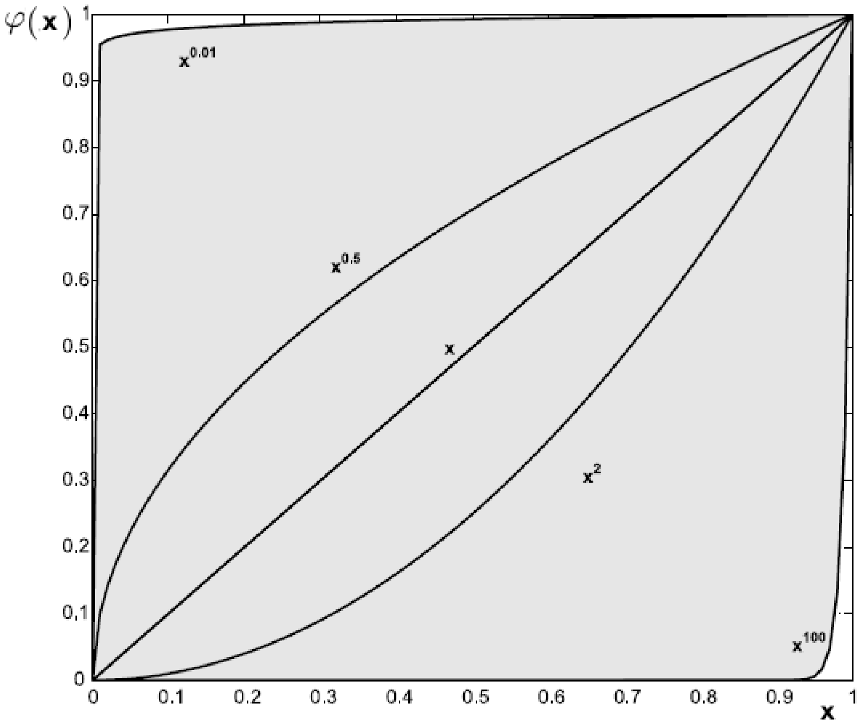

Figure 2.8: Representation of the search space by varying the values of the parameter a.

To construct the IV-REF, the system has to adapt the gene values to the interval in which the automorphisms φ_1 and φ_2 can vary ([0.01, 100]). This is done by adapting the value of each parameter a according to the following expression:

$$a = \begin{cases} a & \text{if } 0 < a \leq 1 \\ \frac{1}{2-a} & \text{otherwise.} \end{cases} \tag{2.28}$$

The same adaptation is considered for the parameter b.

b) Rule selection: The part of the chromosome that performs the rule selection is a vector CR of dimension L (number of rules), given by $CR = \{r_1, \ldots, r_L\}$, where $r_i \in \{0, 1\}$ and $i \in \{1, \ldots, L\}$. This vector determines the subset of fuzzy rules that compose the final RB as follows: if $r_i = 1$ then $R_i \in RB$, otherwise $R_i \notin RB$.

Thus, the chromosome scheme is given by $C_{E+R} = \{CE, CR\}$;

2) Initial gene pool - An individual with all genes with value 1 is initialized, in order to include all the fuzzy rules from the initial RB and set all the IV-REFs to the one expressed by Equation (2.27);

3) Chromosome evaluation - The quality of the chromosome is evaluated through the accuracy rate, that is, the ratio between the number of correctly classified examples by the total number of examples;

4) Crossover operator - It depends on the considered chromosome part (CE or CR), due to the double coding scheme. For CE, the Parent Centrix BLX operator (HLS03) is applied, which is based on the concept of neighborhood, in the sense that it allows the offspring genes to be around the genes of one parent or around a zone determined by both parents. For CR, the half uniform crossover scheme (ES93) is considered, where the genes to be crossed are randomly selected among those that are dissimilar in the parents, to further differentiate the offspring from the parents;

5) Restart approach - To get away from local optima, a restarting approach is considered when no offsprings are incorporated to the current population. To restart this population, the current best global solution found is included the following population, with the remaining individuals being generated randomly, which indicates an elitist scheme.

2.7 Aggregation in non-fuzzy classification systems

Although most of our application developments are focused on IV-FRBCSs, there are other classification techniques with aggregation processes that could benefit from known (interval-valued) fusion functions, but the aggregated values do not come from the unit interval. We highlight two of those techniques, both from the field of Deep Learning (LBH15).

2.7.1 Convolutional neural networks

Convolutional Neural Networks (CNN) (LBH15) are a type of neural network designed for processing data in which the local/neighboring information is relevant, such as in image classification (DBB^{+}18; PRMFI^{+}21). The main stages that characterizes this type of system are:

1) Convolution: the local features of a given image are extracted;

2) Pooling: the extracted features are downsampled, sequentially, by aggregating the local data.

In most CNNs, the information fusion process in the pooling stage is carried out by the maximum or the arithmetic mean. However, good classification results have been achieved by applying a convex combination of those aggregation functions in the pooling process (LGT18).

2.7.2 Long Short-term memory recurrent neural networks

Long Short-term Memory (LSTM) are a type of recurrent neural networks (Gra12) applied in several classification problems with sequential information (GSK^{+}17), such as speak recognition and machine translation. Its prominent feature is the capacity of storing knownledge/information to be used in latter stages of its sequential architecture. This stored knowledge, constituted by a short-term memory and a long-term memory, influences the output of a time-step of the system, when aggregating the short-term information of a previous step with the current one. This fusion process is usually carried out by a summation, but, recently, the Choquet integral has been applied as the aggregation operator, since it adequately captures the possible coalitions present in the data (FJTH^{+}21).

Chapter 3

DEVELOPMENT OF THE THESIS

In this chapter, we present a discussion on how we dealt with the five research questions proposed in the Introduction, each one associated with a corresponding publication. Following that, we present some the complementary contributions that derived from the development of the thesis. Finally, we present a summary of all stages of the development of the thesis, both from the theoretical and practical point of view.

3.1 Discussion

3.1.1 (RQ1) Is it feasible to define generalized overlap functions and overlap indices in the interval context so that they are suitable to be applied in n-dimensional problems, such as the ones tackled by IV-FRBCSs?

This question was addressed by the following paper (available in Chapter 5, Section 5.1.1):

- T. Asmus, G. Dimuro, B. Bedregal, J. Sanz, S. Pereira Jr, and H. Bustince, "General interval-valued overlap functions and interval-valued overlap indices", Information Sciences 527 (2020) 27–50.

Our initial interest was to explore if n-dimensional overlap functions, once properly defined in the interval context, could benefit the performance of IV-FRBCSs, since this was the case when applying n-dimensional and general overlap functions in FRBCSs. Although iv-overlap functions had already been introduced, independently, by Qiao and Hu (QH17) and Bedregal et al. (BBP$^+$17), they could only be applied in problems with two classes, which is a severe limitation if one intend to use them in

IV-FRBCSs. To overcome this drawback, in this paper, we introduced the definition of n-dimensional iv-overlap functions:

Definition 24. *A function $IOn : L([0,1])^n \to L([0,1])$ is called an n-dimensional interval-valued (iv) overlap function if the following conditions are satisfied, for all $\vec{X} \in L([0,1])^n$:*

(IOn1) *IOn is symmetric;*

(IOn2) *$IOn(\vec{X}) = [0,0] \Leftrightarrow \prod_{i=1}^{n} X_i = [0,0]$;*

(IOn3) *$IOn(\vec{X}) = [1,1] \Leftrightarrow \prod_{i=1}^{n} X_i = [1,1]$;*

(IOn4) *IOn is $\leq_{Pr}$-increasing;*

(IOn5) *IOn is Moore continuous.*

One interesting studied aspect was the representation of n-dimensional iv-overlap functions. Here, we highlight some contributions on this topic:

Theorem 4. *Let $On_1, On_2 : [0,1]^n \to [0,1]$ be n-dimensional overlap functions such that $On_1 \leq On_2$. Then, the function $\widehat{On_1, On_2}$ is an n-dimensional iv-overlap function.*

We discussed that there may be representable n-dimensional iv-overlap functions in which some (or neither) of their representatives are n-dimensional overlap functions. For that reason, we introduced the following definition to denote representable n-dimensional iv-overlap functions which both representatives are n-dimensional overlap functions:

Definition 25. *An n-dimensional iv-overlap function $IOn : L([0,1])^n \to L([0,1])$ is said to be o-representable if there exist n-dimensional overlap functions $On_1, On_2 : [0,1]^n \to [0,1]$, $On_1 \leq On_2$, such that $IOn = \widehat{On_1, On_2}$.*

In particular, we showed the conditions in which n-dimensional iv-overlap functions have to satisfy in order for them to be *o-representable*, that is, when they are representable functions with n-dimensional overlap functions as their representatives, while also being inclusion monotonic:

Theorem 5. *Let $IOn : L([0,1])^n \to L([0,1])$ be an n-dimensional iv-overlap function. Then, IOn is o-representable if and only if IOn is inclusion monotonic and the following conditions are satisfied, for all $X_1, \ldots, X_n \in L([0,1])$:*

(i) $\underline{IOn(X_1, \ldots, X_n)} = 0 \Leftrightarrow \prod_{i=1}^{n} \underline{X_i} = 0$;

(ii) $\overline{IOn(X_1, \ldots, X_n)} = 1 \Leftrightarrow \prod_{i=1}^{n} \overline{X_i} = 1$.

Then, inspired by the work of De Miguel et al. (DGR+19), where general overlap functions were defined with promising results in FRBCSs, we developed the concepts of n-dimensioanl iv-0-overlap function, n-dimensioanl iv-1-overlap function and, by combining these two definitions, we introduced general iv-overlap functions, as follows:

Definition 26. *A general iv-overlap function is any mapping $IGO : L([0,1])^n \rightarrow L([0,1])$ that satisfies the following conditions, for all $\vec{X} \in L([0,1])^n$:*

(IGO1) *IGO is symmetric;*

(IGO2) *If $\prod_{i=1}^{n} X_i = [0,0]$ then $IGO(\vec{X}) = [0,0]$;*

(IGO3) *If $\prod_{i=1}^{n} X_i = [1,1]$ then $IGO(\vec{X}) = [1,1]$;*

(IGO4) *IGO is $\leq_{Pr}$-increasing;*

(IGO5) *IGO is Moore continuous.*

We studied their characterization and presented three construction methods. One constitutes the characterization of any general overlap function IGO, based on two iv-fusion functions F and G, with some conditions:

Theorem 6. *The mapping $IGO : L([0,1])^n \rightarrow L([0,1])$ is an general iv-overlap function if and only if, for all $X_1, \ldots, X_n \in L([0,1])$, it holds that:*

$$IGO(X_1, \ldots, X_n) = F(X_1, \ldots, X_n) \div_H (F(X_1, \ldots, X_n) \dot{+} G(X_1, \ldots, X_n)),$$

for some $F, G : L([0,1])^n \rightarrow L([0,1])$ such that

(i) *F and G are symmetric;*

(ii) *If $\prod_{i=1}^{n} X_i = [0,0]$ then $F(X_1, \ldots, X_n) = [0,0]$;*

(iii) *If $\prod_{i=1}^{n} X_i = [1,1]$ then $G(X_1, \ldots, X_n) = [0,0]$;*

(iv) *F is $\leq_{Pr}$-increasing and G is $\leq_{Pr}$-decreasing;*

(v) *F and G are Moore continuous;*

(vi) $\underline{F(X_1, \ldots, X_n) \dot{+} G(X_1, \ldots, X_n)} \neq 0$,

for any $X_1, \ldots, X_n \in L([0,1])$.

The other two are based on composition of aggregation functions. Here, we highlight the construction method, of the second type, that was applied in our illustrative example in IV-FRBCSs:

Proposition 5. *Given a general iv-overlap function $IGO : L([0,1])^n \to L([0,1])$ and a symmetric, Moore continuous n-dimensional interval-valued aggregation function $IA : L([0,1])^n \to L([0,1])$, then the function $IGO_{IA} : L([0,1])^n \to L([0,1])$, defined, for all $\vec{X} \in L([0,1])^n$, by, $IGO_{IA}(\vec{X}) = IGO(\vec{X}) \cdot IA(\vec{X})$ is a general iv-overlap function.*

Proposition 5 indicates that general iv-overlap functions can be obtained by the composition of other general iv-overlap functions by the product operation, which shows that general iv-overlap functions are very flexible and adaptable to be applied in practical problems.

In the work of Elkano et al. (EGS+18), the rule weight, which measures the acuity of the classification system rules, was defined by means of overlap indices. Inspired by this approach, we introduced the concept of iv-overlap indices and presented some construction methods for them. Here, we show their definition:

Definition 27. *A mapping $\mathcal{IO} : IFS(U) \times IFS(U) \to L([0,1])$ is said to be an iv-overlap index if it respects the following conditions, for all $\mathcal{A}, \mathcal{B}, \mathcal{C} \in IFS(U)$:*

$(\mathcal{IO}1)$ *$\mathcal{IO}(\mathcal{A}, \mathcal{B}) = [0,0]$ if and only if for all $z \in U, \mathcal{A}(z) \cdot \mathcal{B}(z) = [0,0]$;*

$(\mathcal{IO}2)$ *$\mathcal{IO}(\mathcal{A}, \mathcal{B}) = \mathcal{IO}(\mathcal{B}, \mathcal{A})$;*

$(\mathcal{IO}3)$ *If $\mathcal{B} \leq \mathcal{C}$, meaning that $\mathcal{B}(z) \leq_{Pr} \mathcal{C}(z)$ for every $z \in U$, then $\mathcal{IO}(\mathcal{A}, \mathcal{B}) \leq_{Pr} \mathcal{IO}(\mathcal{A}, \mathcal{C})$,*

An iv-overlap index is said to be normal, whenever it also satisfies the following condition:

$(\mathcal{IO}4)$ *If there exists $z \in U$ such that $\mathcal{A}(z) \cdot \mathcal{B}(z) = [1,1]$, then $\mathcal{IO}(\mathcal{A}, \mathcal{B}) = [1,1]$.*

Finally, we presented an illustrative example in IV-FRBCS, in which n-dimensional and general iv-overlap functions were applied to model the conjunction operator on the first stage of the IV-FRM of the IVTURS algorithm (interval matching degree), while the iv-rule weight was defined based on iv-overlap indices. The experiment showed that the newly defined IV-FRM produced competitive results, statistically equivalent to the ones from the original IVTURS algorithm. This was the first step in developing our own version of the IVTURS algorithm, which we call here as the IVTURS-OV, since it is based on overlap operators.

3.1.2 (RQ2) Considering IV-FRBCSs, do admissible orders - to be used for ranking possible classification outcomes - and iv-overlap operators (which may or may not be increasing for such orders) - to be applied in the inference process of the classifier - have an impact on the whole classification process?

This question was addressed by the following paper (available in Chapter 5, Section 5.1.2):

- T. Asmus, J. Sanz, G. Dimuro, B. Bedregal, J. Fernandez, and H. Bustince, "N-dimensional admissibly ordered interval-valued overlap functions and its influence in interval-valued fuzzy rule-based classification systems.", IEEE Transactions on Fuzzy Systems (In press, early access).

When further researching IV-FRBCSs, we observed that both the iv-aggregation function used to compute the interval matching degree and the adopted total order in the final classification step play a key role in the behaviour of the system. Moreover, there were neither previous studies concerning the relation between those aspects in IV-FRBCSs, nor it was considered cases in which the chosen iv-aggregation function was increasing with respect to the total order, and how this could impact the whole classification process.

For that reason, in this work, we decided to explore this gap in the literature, by defining n-dimensional iv-overlap functions that are increasing with respect to an admissible order (admissibly ordered) and apply this new concept in IV-FRBCSs. Before delving into the classification problem, first, we presented some new results on admissible orders, in particular, $\leq_{\alpha,\beta}$ orders (Definition 16), to aid the study of n-dimensional admissibly ordered iv-overlap function. We summarize these results in the following remark:

Remark 3. *For all $\alpha_1, \alpha_2, \beta_1, \beta_2 \in [0, 1]$ such that $\alpha_1 \neq \beta_1$ and $\alpha_2 \neq \beta_2$, one has that $\leq_{\alpha_1,\beta_1} \neq \leq_{\alpha_2,\beta_2}$, except when $\alpha_1 = \alpha_2 = \alpha$, $\beta_1 < \alpha$ and $\beta_2 < \alpha$ or when $\alpha_1 = \alpha_2 = \alpha$, $\alpha < \beta_1$ and $\alpha < \beta_2$.*

Then, we presented the main concept of this work:

Definition 28. *A function $AOn : L([0, 1])^n \rightarrow L([0, 1])$ is an n-dimensional admissibly ordered* *iv-overlap function for an admissible order $\leq_{AD}$ (n-dimensional $\leq_{AD}$-overlap function) if it satisfies the conditions* **(IOn1)**, **(IOn2)** *and* **(IOn3)** *of Def. 24 and:*

(AOn4) AOn $\leq_{AD}$*-increasing.*

Condition **(IOn5)** from Def. 24 is not needed, since the continuity was only a requirement in the original definition of overlap functions in order to enable them to be applied in image processing (BFM$^+$10), which is not the focus of our work. Henceforward, in the following developments of the thesis, we disregard the continuity when introducing new definitions of interval-valued overlap functions.

The most common way of defining interval-valued aggregation operations (such as iv-overlap functions) is through representable functions (Definition 14), usually with both representatives being the same aggregation function. For example, one can define an interval-valued geometric mean $IO_{GM} : L([0,1])^n \to L([0,1])$, given, for all $\vec{X} \in L([0,1])^n$, by

$$IO_{GM}(\vec{X}) = [GM(\underline{X_1}, \ldots, \underline{X_n}), GM(\overline{X_1}, \ldots, \overline{X_n})],$$

with GM being the geometric mean, given by Equation (2.7). In this case, IO_{GM} is an o-representable n-dimensional iv-overlap function, and, thus, it is $\leq_{Pr}$ increasing. However, it is not necessarily increasing with respect to an admissible order $\leq_{\alpha,\beta}$. In the following result, we showed when an o-representable n-dimensional iv-overlap function is $\leq_{\alpha,\beta}$-increasing:

Theorem 7. *Let $IOn : L([0,1])^n \to L([0,1])$ be an o-representable n-dimensional iv-overlap function and $\alpha, \beta \in [0,1]$, $\alpha \neq \beta$. Then, IOn is $\leq_{\alpha,\beta}$-increasing if and only if $\alpha = 1$ and IOn^+ (upper projection) is a strict n-dimensional overlap function.*

So, it is not trivial to obtain expressions for n-dimensional $\leq_{\alpha,\beta}$-overlap functions when $\alpha \neq 1$. To address that, we presented a construction method for n-dimensional $\leq_{\alpha,\beta}$-overlap functions, as follows:

Theorem 8. *Let On be a strict n-dimensional overlap function, $\alpha \in (0,1)$ and $\beta \in [0,1]$ such that $\alpha \neq \beta$. Then $AOn^\alpha : L([0,1])^n \to L([0,1])$ defined, for all $\vec{X} \in L([0,1])^n$, by*

$$AOn^\alpha(\vec{X}) = [On(K_\alpha(X_1), \ldots, K_\alpha(X_n)) - \alpha m, On(K_\alpha(X_1), \ldots, K_\alpha(X_n)) + (1-\alpha)m],$$

where

$$m = \min\{\overline{X_1} - \underline{X_1}, \ldots, \overline{X_n} - \underline{X_n}, On(K_\alpha(X_1), \ldots, K_\alpha(X_n)), 1 - On(K_\alpha(X_1), \ldots, K_\alpha(X_n))\},$$

is an n-dimensional $\leq_{\alpha,\beta}$-overlap function.

The construction method introduced in Theorem 8 allows us to obtain different n-dimensional $\leq_{\alpha,\beta}$-overlap functions with respect to any $\leq_{\alpha,\beta}$ order with $\alpha \in (0,1)$, $\beta \in [0,1]$ and $\alpha \neq \beta$. Thus, its adaptability allows for it to be employed in various applications with different approaches to the ranking of intervals, determined by the choice of different α and β.

For the experiment in IV-FRBCSs, we analyzed the behaviour of different admissible orders $\leq_{\alpha,\beta}$ and n-dimensional (admissibly ordered) iv-overlap functions applied on a newly developed IV-FRM based on the IVTURS algorithm, further developing our IVTURS-OV method, with the following key modifications:

- When generating the initial FRBCS through the FARCHD method, we replace the product, which is applied as the conjunction operator to calculate the matching degree, by an n-dimensional overlap function;
- An n-dimensional (admissibly ordered) iv-overlap function is applied as the conjunction operator when calculating the interval matching degree (first stage of the IV-FRM), based on the same n-dimensional overlap function applied on the generation of the initial FRBCS;
- In the second step of the IV-FRM (interval association degree), the interval matching degree is weighted by the corresponding iv-rule weight through either a representable interval product or an admissibly ordered interval product obtained by the construction method in Theorem 8, depending on the chosen conjunction operation applied in the previous step;
- The final classification task is carried accordingly to a chosen $\leq_{\alpha,\beta}$ order.

Then, we proceeded to test different configurations of this new IV-FRBCS, experimenting with twelve different n-dimensional (admissibly ordered) iv-overlap functions and three admissible orders - $\leq_{Lex1}$, $\leq_{Lex2}$ and $\leq_{IQ}$ (see Section 2.2) - comparing their classification accuracy when applied to 31 real world data-sets from the KEEL repository (AFSG$^+$09).

In first place, we studied if there were differences in the accuracy for a given method (associated with an interval overlap operator) when we varied the chosen admissible order. In summary, we can concluded that $\leq_{Lex1}$ is not a suitable choice and $\leq_{IQ}$ provides a robust behaviour regardless of the configuration. For these reasons, we decided to investigate the behaviour of our classifiers by varying the n-dimensional (admissibly ordered) iv-overlap functions used in the IV-FRM, taking in consideration the admissible order $\leq_{IQ}$.

We observed that the interval-valued conjunction operators based non-associative overlap functions (such as the geometric mean) produced the best results, which allowed us to recommend them, along with the order $\leq_{IQ}$, to be applied in IV-FRBCSs, more so when they are n-dimensional admissibly ordered iv-overlap functions constructed via Theorem 8. One particular property of this construction method is that it produces functions that are width-preserving (Definition 13), a concept that we study and expand in our following work.

3.1.3 (RQ3) Since the widths of the intervals are intrinsically related to both the uncertainty towards the value they represent and the quality of the information that they are expressing, how can one define interval-valued overlap operations in which the widths of the outputs are controlled accordingly to a desirable threshold that depends on the widths of the inputs?

This question was addressed by the following paper (available in Chapter 5, Section 5.1.3):

- T. Asmus, G. Dimuro, B. Bedregal, J. Sanz, R. Mesiar and H. Bustince, “Towards interval uncertainty propagation control in bivariate aggregation processes and the introduction of width-limited interval-valued overlap functions.”, Fuzzy Sets and Systems (In press, Corrected Proof).

In our works discussed in Sections 3.1.1 and 3.1.2, we studied different definitions of interval-valued overlap functions, their representability and relation with different interval orders, analyzing the effect of them in IV-FRBCSs. However, there is one aspect that is intrinsically related to the modeling of uncertainty when working with interval fuzzy systems, that being the widths of operated intervals (BDSR10; DBSR11; SFBH11). We observed that most interval-valued aggregation operations were defined and studied without taking into account the relation between the widths of the operated intervals and the width of the output interval. Additionally, in many interval-valued processes, the output intervals’ widths becomes larger than a desirable threshold, which may be imposed by applications constraints concerning the quality of the information required for the interval results (DCC00). In those cases, the interval outputs may carry no meaningful information about the value they are approximating, and we point out that the information quality of interval-valued results is a strong requirement claimed by scientists and engineers interested in interval-based tools (MKC09).

Thus, the study of the relation between the width of the inputs and the output of interval-valued fuzzy operations coupled with adaptable tools to conserve the information quality in the output of such operations was still a challenge to overcome in the literature, especially regarding interval-valued aggregation and interval-valued overlap functions. So, as an initial study in this direction, in this paper we developed a theoretical approach to aid the analysis of bivariate interval-valued operations with respect to the width of the operated intervals in order to control the information quality deterioration, with special attention to interval-valued overlap functions, admissibly ordered or not.

The main concepts introduced in this work are that of width-limited and width-limiting functions:

Definition 29. *Consider an interval-valued function $IF : L([0,1])^2 \to L([0,1])$ and a mapping $B : [0,1]^2 \to [0,1]$. Then, IF is said to be width-limited by B if $w(IF(X,Y)) \leq B(w(X), w(Y))$, for all $X, Y \in L([0,1])$. B is called a width-limiting function of IF.*

Based on that, we proceeded to analyze how to obtain the least width-limiting function for a given iv-fusion function. First, let us clarify the notation:

$$\mathcal{IF} = \{IF : L([0,1])^2 \to L([0,1]) \mid IF \text{ is a binary interval-valued function}\}$$

and

$$\mathcal{F} = \{F : [0,1]^2 \to [0,1] \mid F \text{ is binary function}\}.$$

Then, our first contribution in this analysis came from the following result:

Theorem 9. *The mapping $\mathfrak{L} : \mathcal{IF} \to \mathcal{F}$ defined for all $IF \in \mathcal{IF}$ and $\epsilon, \delta \in [0,1]$, by*

$$\mathfrak{L}(IF)(\epsilon, \delta) = \sup_{\substack{u \in [0, 1-\epsilon] \\ v \in [0, 1-\delta]}} \{w(IF([u, u+\epsilon], [v, v+\delta]))\}$$

provides the least width-limiting function $\mathfrak{L}(IF) : [0,1]^2 \to [0,1]$ for IF.

In the case of the best interval representation of aggregation functions, we showed that the least width-limiting function is also an aggregation function.

To further analyze the behaviour of representable iv-aggregation functions, we introduced a less restrictive extension of one-dimension convexity for bivariate aggregation functions:

Definition 30. *Consider $a, b \in [0,1]$. An aggregation function $A : [0,1]^2 \to [0,1]$ is called (a,b)-ultramodular if, for all $x, y, \epsilon, \delta \in [0,1]$ and $x+\epsilon, y+\delta, a-\epsilon, b-\delta \in [0,1]$, it holds that:*

$$A(x+\epsilon, y+\delta) - A(x,y) \leq A(a,b) - A(a-\epsilon, b-\delta). \tag{3.1}$$

In particular, the best interval representation of an $(1,1)$-ultramodular aggregation function ($a = b = 1$) have a predictable relation its least width-limiting function, as follows:

Theorem 10. *Let $A : [0,1]^2 \to [0,1]$ be an aggregation function, $\mathfrak{L}(\widehat{A}), \mathfrak{L}(\widehat{A^d}) : [0,1]^2 \to [0,1]$ be the least width-limiting functions for $\widehat{A}$ and $\widehat{A^d}$, respectively. Then, $\mathfrak{L}(\widehat{A}) = \mathfrak{L}(\widehat{A^d}) = A^d$ if and only if A is an $(1,1)$-ultramodular aggregation function.*

For example, the product overlap $O_P : [0,1]^2 \to [0,1]$, given, for all $x, y \in [0,1]$, by $O_P(x,y) = x{\cdot}y$, is an $(1,1)$-ultramodular aggregation function, so, its best interval representation $\widehat{O_P}$ has the dual of the product O_P^d (the probabilistic sum) as its least width-limiting function. The same hold for the best interval representation of O_P^d.

To enable more flexible definitions of interval-valued functions, we introduced the concept of increasingness with respect to a pair of partial orders:

Definition 31. *Let $IF : L([0,1])^2 \to L([0,1])$ be an interval-valued function and $\leq_1$, $\leq_2$ be two partial order relations on $L([0,1])$. Then, IF is said to be $(\leq_1, \leq_2)$-increasing if the following condition holds, for all $X_1, X_2, Y_1, Y_2 \in L([0,1])$:*

$$X_1 \leq_1 X_2 \;\wedge\; Y_1 \leq_1 Y_2 \Rightarrow IF(X_1, Y_1) \leq_2 IF(X_2, Y_2).$$

Then, we presented the main definition of this work:

Definition 32. *Let $B : [0,1]^2 \to [0,1]$ be a symmetric and increasing function and $\leq_1$, $\leq_2$ be two partial order relations on $L([0,1])$. Then, the mapping $IOw : L([0,1])^2 \to L([0,1])$ is said to be a width-limited interval-valued overlap function (w-iv-overlap function) with respect to the tuple ($\leq_1$, $\leq_2$, B), if the following conditions hold for all $X, Y \in L([0,1])$:*

(IOw1) *IOw is symmetric;*

(IOw2) *$IOw(X,Y) = [0,0] \Leftrightarrow X \cdot Y = [0,0]$;*

(IOw3) *$IOw(X,Y) = [1,1] \Leftrightarrow X \cdot Y = [1,1]$;*

(IOw4) *IOw is $(\leq_1, \leq_2)$-increasing;*

(IOw5) *IOw is width-limited by B.*

As an example, the best interval representation of the product overlap, $\widehat{O_P}$, is a w-iv-overlap function with respect to the tuple ($\leq_{Pr}$, $\leq_{Pr}$, O_P^d), since O_P is $(1,1)$-ultramodular and its best interval representation is $\leq_{Pr}$-increasing.

Even in our strictly theoretical papers, which is the case of this one, we always have an interest in presenting construction methods for the newly defined functions to enable them to be easily applied in practical applications. Our aim, here, was to construct interval-valued overlap functions in which the width of the interval output does not surpass a desirable threshold, according to the width-limiting function applied to the widths of the interval inputs. This desirable maximal threshold may be determined by the application requirement, concerning the extent of the necessity to conserve the information quality of the results, with respect to the information quality of the inputs.

The following definition introduced a key concept to be applied in two of the construction methods presented in this paper:

Definition 33. *Consider a function $B : [0,1]^2 \to [0,1]$ and let $IF : L([0,1])^2 \to L([0,1])$ be an interval-valued function. Then, the function $m_{IF,B} : L([0,1])^2 \to [0,1]$, defined for all $X, Y \in L([0,1])$ by:*

$$m_{IF,B}(X,Y) = \min\{w(IF(X,Y)), w(IF(Y,X)), B(w(X), w(Y)), B(w(Y), w(X))\}, \quad (3.2)$$

is called the maximal width threshold for the pair (IF, B). Whenever B and IF are both symmetric, then Equation (3.2) can be reduced to:

$$m_{IF,B}(X,Y) = \min\{w(IF(X,Y)), B(w(X), w(Y))\}.$$

Then, we proceeded to introduce, study and compare three construction methods for w-iv-overlap functions. The first one was based on constructing the best interval representation of an overlap function and, then, “narrow” the output if it surpasses a given maximal threshold. This narrowing occurs in the direction of a K_α point, determined by the same α that is chosen for the order $\leq_{\alpha,\beta}$ that is part of the construction method:

Theorem 11. *Consider a symmetric and increasing function $B : [0,1]^2 \to [0,1]$, a strict overlap function $O : [0,1]^2 \to [0,1]$ and take $\alpha \in (0,1]$ and $\beta \in [0,\alpha)$. Then, the interval-valued function $IOw_B^\alpha : L([0,1])^2 \to L([0,1])$ defined, for all $X, Y \in L([0,1])$, by*

$$IOw_B^\alpha(X,Y) = [K_\alpha(\widehat{O}(X,Y)) - \alpha \cdot m_{\widehat{O},B}(X,Y), K_\alpha(\widehat{O}(X,Y)) + (1-\alpha) \cdot m_{\widehat{O},B}(X,Y)], \quad (3.3)$$

is a w-iv-overlap function for the tuple $(\leq_{Pr}, \leq_{\alpha,\beta}, B)$.

The second method was based on applying an overlap function on the K_α points of the interval inputs to generate the K_α point of the interval output, and then, apply a conjunctive operator between a series of parameters (among them are the widths of the interval inputs) to generate the width of the output around the previously obtained K_α point:

Theorem 12. *Let* $O : [0,1]^2 \to [0,1]$ *be a strict overlap function,* $B : [0,1]^2 \to [0,1]$ *be a symmetric, increasing and conjunctive function and* $\alpha \in (0,1)$, $\beta \in [0,1]$ *such that* $\alpha \neq \beta$. *Then* $IOw_B^\alpha : L([0,1])^2 \to L([0,1])$ *defined, for all* $X, Y \in L([0,1])$, *by*

$$IOw_B^\alpha(X,Y) = [O(K_\alpha(X), K_\alpha(Y)) - \alpha\theta, O(K_\alpha(X), K_\alpha(Y)) + (1-\alpha)\theta],$$

where

$$\theta = B(B(w(X), w(Y)), B(O(K_\alpha(X), K_\alpha(Y)), 1 - O(K_\alpha(X), K_\alpha(Y))))$$

is a w-iv-overlap function for the tuple $(\leq_{\alpha,\beta}, \leq_{\alpha,\beta}, B)$.

The third method was a combination of the two previous approaches, with its own particularities. The K_α point of the interval output is calculated similarly as in the second method, but the width is generated by means of the "maximal possible width of an interval" d_α (Definition 4). A similar "narrowing" of the output as the one in the first method occurs, if the maximal width threshold is surpassed:

Theorem 13. *Consider a strict overlap function* $O : [0,1]^2 \to [0,1]$, *a symmetric aggregation function* $B : [0,1]^2 \to [0,1]$, *an iv-aggregation function* $IF_{O,B}^\alpha : L([0,1])^2 \to L([0,1])$ *defined as in Theorem 3, the maximal width threshold* $m_{IF_{O,B}^\alpha, B} : L([0,1])^2 \to L([0,1])$ *for the pair* $(IF_{O,B}^\alpha, B)$, $\alpha \in (0,1)$ *and* $\beta \in [0,1]$ *with* $\alpha \neq \beta$. *Then, the interval-valued function* $IOw_B^\alpha : L([0,1])^2 \to L([0,1])$ *defined by*

$$IOw_B^\alpha(X,Y) = R,$$

where:

(i) $K_\alpha(R) = O(K_\alpha(X), K_\alpha(Y))$;

(ii) $w(R) = m_{IF_{O,B}^\alpha, B}(X,Y)$.

is a w-iv-overlap function for the tuple $(\leq_{\alpha,\beta}, \leq_{\alpha,\beta}, B)$.

By analyzing advantages and drawbacks from each method, we observed that the second method presented itself as the most restrictive one, since the width-limiting function must be conjunctive, meaning that one degenerate interval input ($w(X) = 0$) is enough to cause the interval output to also be degenerate, as it was the case with the construction method (see Theorem 8) that we introduced in the work discussed in Section 3.1.2 .

As these construction methods are all based on choices of overlap functions, width-limiting functions and admissible orders, it was made clear the adaptability of the developed concepts, as one can obtain an interval-valued overlap operations that best satisfy the restrictions of the context regarding the acceptable amount of width propagation and/or the ordering of intervals to be applied. Our next intent was to generalize adequately the presented theoretical approach to encompass other classes of fusion functions and apply the new concepts in IV-FRBCSs, which we reserved for our following paper.

3.1.4 (RQ4) Is it possible to develop a general framework to define classes of width controlled n-dimensional interval fusion functions, as counterparts of known classes of fusion functions so that they can improve the accuracy of classification systems, in particular IV-FRBCSs, by the influence of the information quality control?

This question was addressed by the following paper (available in Chapter 5, Section 5.1.4):

- T. Asmus, G. Dimuro, B. Bedregal, J. Sanz, R. Mesiar and H. Bustince, "A methodology for controlling the information quality in interval-valued fusion processes: theory and application.", Knowledge Based Systems (submitted).

Motivated by the initial developments on w-iv-overlap functions, presented in our work discussed in Section 3.1.3, we observed that (i) other fusion functions, including n-dimensional ones, could benefit from a similar approach regarding the information quality control when defined in the interval-context, and that (ii) IV-FRBCSs accuracy could improve from the application of adaptable width-limited interval functions, since the best performing configurations of our IVTURS-OV method had functions whose widths were limited by the minimum of the inputs' widths (see Theorem 8, in Section 3.1.2), which is a very strict width limitation.

Thus, in this paper, we extended the notion of width-limitation (and related concepts) to the n-dimensional context, introduced a methodology for defining classes of n-dimensional width-limited iv-fusion functions as counterparts of known classes of n-dimensional fusion functions, based on the representation of classes of fusion functions through their set of constitutive properties. Also, we presented some very flexible construction methods for functions from those classes (with choices of width limiting functions and pair of admissible orders). Finally, we applied n-dimensional w-iv-overlap functions (defined by the new methodology) in IV-FRBCSs, showing that the classification accuracy was improved by our approach.

Concepts such as width-limiting and width-limited functions (Definition 29) and maximal width-threshold (Definition 33) were naturally extended to the n-dimensional context, so let us focus on the new developed methodology for defining classes of width-limited iv-fusion functions (w-iv-fusion functions). First, inspired by the approach of directional increasing fusion functions developed by Bustince et al. (BMK+20), we presented a characterization of any subclass $\mathcal{F}$ of increasing n-dimensional fusion functions through a set of properties $P_{\mathcal{F}}$ such that: (i) includes boundary conditions for any $F \in \mathcal{F}$ and (ii) possibly includes some other constraints not related to the monotonicity. Such subclass of fusion functions is given by:

$$\mathcal{F} = \{F : [0,1]^n \to [0,1] |\, F \text{ is increasing and satisfies all the properties in } P_{\mathcal{F}}\}. \tag{3.4}$$

Given the set $P_{\mathcal{F}}$ of properties of a fusion function $F \in \mathcal{F}$, denote by $IP_{\mathcal{F}}$ the set of interval extensions of the properties in $P_{\mathcal{F}}$. Usually, there are more than one way to extend a given property of a function to the interval context, so $IP_{\mathcal{F}}$ varies accordingly to how one extends such properties. This means that from a set of properties $P_{\mathcal{F}}$ one can obtain several sets of interval-context properties $IP_{\mathcal{F}}$. If we define classes of functions based on their set of properties, as expressed by Equaqion (3.4), we can see that from one class of function $\mathcal{F}$ (with its respective set $P_{\mathcal{F}}$), we can define its interval counterpart $\mathcal{IF}$ by associating an appropriate $IP_{\mathcal{F}}$ to it.

Thus, the main idea of our methodology when defining a class $\mathcal{IF}$ of iv-fusion function as interval counterpart of a class $\mathcal{F}$ of fusion functions that have controlled widths was to establish that, besides being associated with a set of extended properties $IP_{\mathcal{F}}$, the functions from the defined class were width-limited by an increasing fusion function (Definition 29) and increasing with respect to a pair of partial orders (Definition 31).

Formally, consider the function $B \in \mathcal{B}$, where a $\mathcal{B}$ is a subclass of increasing fusion functions (with its corresponding set of properties $P_{\mathcal{B}}$) and let $\leq_1, \leq_2$ be partial orders on $L([0,1])$. Then, denote the class of w-iv-fusion functions for the tuple $(\leq_1, \leq_2, B)$ by $\mathcal{IFW}^B_{\leq_1,\leq_2}$, which is given by:

$$\begin{aligned} \mathcal{IF}w^B_{\leq_1,\leq_2} = \{IF : L([0,1])^n \to L([0,1]) |\ IF \text{ is } (\leq_1, \leq_2)\text{-increasing,} \\ \text{width-limited by } B \text{ and satisfies all the properties in } IP_{\mathcal{F}}\} \end{aligned} \tag{3.5}$$

Our framework for defining w-iv-fusion functions by Eq. (3.5) is general enough so that different iv-aggregation functions defined in the literature may be retrieved, such as iv-t-norms (DBSR11) and iv-overlap functions (QH17; BBP+17), by restricting to the case where $\leq_1 = \leq_2 = \leq_{Pr}$ and $B(\vec{x}) = 1$, for all $\vec{x} \in [0,1]^n$. However, those functions clearly have no limitation regarding their output widths and may not be applicable in problems where admissible orders must be considered.

Concerning the development of the thesis, our previously defined n-dimensional iv-overlap functions (Definition 24, in Section 3.1.1), general iv-overlap functions (Definition 26, in Section 3.1.1), n-dimensional admissibly ordered iv-overlap functions (Definition 28, in Section 3.1.2) and w-iv-overlap functions (Definition 32, in Section 3.1.3) could all be retrieved by our general framework. We also applied this methodology to define new w-iv-fusion functions, but we highlight here the one that was prominently featured in the experimentation part of this work, that of n-dimensional w-iv-overlap functions:

Definition 34. *Consider a function $B \in \mathcal{B}$, where $\mathcal{B}$ is the subclass of increasing fusion functions, such that $P_{\mathcal{B}} = \{simmetry\}$, and two partial orders $\leq_1, \leq_2$ on $L([0,1])$. Then, $\mathcal{IO}nw^{B}_{\leq_1,\leq_2}$ is the class of width-limited n-dimensional interval-valued overlap functions (w-iv-overlap functions) for the tuple $(\leq_1, \leq_2, B)$, given by:*

$$\mathcal{IO}nw^{B}_{\leq_1,\leq_2} = \{IOnw : L([0,1])^n \rightarrow L([0,1]) \mid IOnw \text{ is width-limited by } B, \quad (\leq_1, \leq_2)\text{-increasing and satisfies all the properties in } IP_{\mathcal{O}n'}\} \tag{3.6}$$

*where $IP_{\mathcal{O}n'} = \{$**(IOn1)**, **(IOn2)**, **(IOn3)**$\}$. (see Definition 24, in Section 3.1.1).*

One thing is to define a class of interval functions with interesting properties, another one is to provide examples of functions that respects those properties and are suitable to be applied in practical problems. For that reason, inspired on the best construction methods for w-iv-overlap functions (Theorems 11 and 13), presented in our work discussed in Section 3.1.3, we introduced new construction methods for w-iv-fusion functions defined through our methodology. They were organized into two main groups: construction methods based on representable functions (CMR) and construction methods based on admissibly ordered functions (CMA). We showed examples of constructed w-iv-fusion functions, based on different known aggregation functions (e.g., n-dimensional overlap and grouping functions), for both CMR and CMA, and presented numerical examples to make clear the effect of the width limitation on the outputs of the constructed functions.

The final part of this work was dedicated to apply the developed w-iv-fusion functions, in particular n-dimensional w-iv-overlap functions, in our IV-FRBCS, IVTURS-OV, to check if the control of the information quality can improve the classification acuity of the system. We followed a similar approach as done in our work discussed in Section 3.1.2, where we adapted the IVTURS algorithm to take into account the chosen overlap operators from the start of the rules learning process. The key difference now is that we considered a choice of width limiting function B, in which the user can determine accordingly to how strict the width limitation must be when applying interval overlap operators on the first two stages of the newly defined IV-FRM. To analyze the effect of the width limitation on classification accuracy, we defined the function B, for all $\vec{x} \in [0,1]^n$, by means of a parameter $\rho \in [0,1]$, as follows:

$$B^{\rho}(w(x_1),\ldots,w(x_n)) = \min\{w(x_1),\ldots,w(x_n)\} + \rho \cdot (\max\{w(x_1),\ldots,w(x_n)\} - \min\{w(x_1),\ldots,w(x_n)\}). \tag{3.7}$$

Specifically, we test each configuration with five possible values for ρ: $\rho = 0$ ($B = \min$); $\rho = 0.25$; $\rho = 0.5$; $\rho = 0.75$; $\rho = 1$ ($B = \max$). In this manner, the parameter ρ indicates the amount of width control that we are imposing on the system. When $\rho = 0$, the output's width is limited by the minimum of the inputs' widths, representing the most strict width limitation. Conversely, when $\rho = 1$, the output's width is limited by the maximum of the inputs' widths, representing the less width control.

Since, in the work discussed in Section 3.1.2, we concluded that the order $\leq_{IQ}$ (Definition 2.14) was a recommended choice of total order to be applied in IV-FRBCSs, we considered it as the $\leq_{\alpha,\beta}$ order needed to construct the employed n-dimensional w-iv-overlaps and as the total order used in the last classification step of the IV-FRM. With all the discussed modifications, this is the final version of our IVTURS-OV method.

The general goal of our experiment was to analyze the classification performance of the system when applying different n-dimensional w-iv-overlap functions obtained by either the construction method based on representable fusion functions (CMR) or the construction method based on $\leq_{IQ}$-increasing fusion functions (CMA). To conduct our experiment, we have selected the same 31 real-world datasets from the KEEL repository (AFSG$^+$09) as the ones we tested on the works discussed on Sections 3.1.1 and 3.1.2. The selected n-dimensional overlap functions (On) to be used as the core of the construction methods were based on the best performing operations (as observed in our work, discussed in Section 3.1.2) for this kind of classifier, namely, GM and OnB (Eq. (2.7) and (2.10), respectively), as well as the product, since it was the operation used on the original IVTURS. To further analyze the effect of information quality preservation on the classification performace, we also considered configurations of the system based on the best interval representation of each of those n-dimensional overlap functions, which do not have any width limitation.

In total, there are 33 configurations of the system, which we organized in 3 groups:

1. Group REP, with one configuration for each best interval representation of the considered n-dimensional overlap functions (3 in total);

2. Group CONR, with one configuration for each combination of considered n-dimensional w-iv-overlap function (constructed via CMR) and ρ (15 in total);

3. Group CONA, with one configuration for each combination of considered n-dimensional w-iv-overlap function (constructed via CMA) and ρ (15 in total).

Once we had the test result for all methods, we proceeded to compare the effect of the parameter ρ in each configuration. Obviously, methods from the group REP were not considered in this stage, since they are not affected by ρ (no width limitation). By fixing each considered n-dimensional w-iv-overlap function and comparing the performance of each configuration with that function by varying the values of ρ, we could observe that for some functions, such as the ones from Group CONR based on the product and OnB, a rigid width limitation ($\rho = 0$) produced a significantly better performance then a less strict width control ($\rho = 1$). On the other hand, methods based on the geometric mean seemed to benefit from a less strict control on the information quality. In particular, methods from the group CONA based on the n-dimensional overlap function OnB (Equation (2.10)) produced excellent results for every considered ρ, with the best one being with $\rho = 0.75$.

From there, we decided to carry out 3 tests, to identify the best performing method from each group. We observed that the configuration based on the n-dimensional overlap function OnB was the best performing one in each and every group (with $\rho = 0$ in CONR and $\rho = 0.75$ in CONA). Then, we proceed to statistically compare those three winning methods. The results showed that the method based on the best interval representation of OnB with no width limitation does not achieve the same level of performance of the other two compared methods, being significantly less accurate than the control method. As those three methods are all based on the same core n-dimensional overlap function (OnB), which was used throughout all the components of those algorithms, the main difference between them lies on the construction of the interval-valued operations that take place in the IV-FRM, which may or may not control the widths of the outputs of such operations. Thus, we concluded that controlling the width of the intervals, which implies having intervals with better information quality, is beneficial for the system's performance.

Finally, to further analyze the benefits of the new proposed methods, we carried out three pairwise comparisons between the best performing method from each group with the original configuration of the IVTURS algorithm. These results showed, clearly, that the configurations of the best methods from CONR and CONA improve significantly the performance of the IVTURS algorithm, whereas the best method from REP does not improve the accuracy of IVTURS in the same manner. Therefore, we concluded that the exchange from the product to the n-dimensional overlap function OnB was not the sole reason for the better performance of those improved methods, indicating that these new configuration benefited from a certain amount of width limitation. All in all, our IVTURS-OV method with the aforementioned configurations surpassed the accuracy of the original IVTURS algorithm and could be recommended as viable choice of IV-FRBCS for dealing with classification problems.

3.1.5 (RQ5) - Is it possible to develop a general framework to define classes of fusion functions acting on an arbitrary closed real interval as counterparts of known classes of fusion functions acting on the unit interval, without sacrificing their fundamental properties, so that they can be constructed and applied in practical problems that are not fuzzy in nature?

This question was addressed by the following paper (available in Chapter 5, Section 5.1.5):

- T. Asmus, G. Dimuro, B. Bedregal, J. Sanz, J. Fernandez, R. Mesiar and H. Bustince, "A constructive framework to define fusion functions with floating domains in arbitrary closed real intervals", Information Sciences (submitted).

Since we obtained an improvement of the classification accuracy our IV-FRBCSs by applying novel definitions of interval-valued overlap operations, we estimated that other classification techniques, such as convolutional neural networks (LBH15), could benefit from the application of overlap or other fusion operators. However, the data to be aggregated in such problems are not modeling membership degrees, nor truth values, and, thus, are not from the unit interval.

So, inspired in our previously introduced methodology for defining w-iv-fusion functions based on extended set of properties from a known fusion function (see Section 3.1.4), in this work we developed a similar methodology to define fusion functions on arbitrary intervals $[a, b]$, with $a, b \in \mathbb{R}$ and $a < b$, which we called (a, b)-fusion functions. The fundamental aspect of this framework is that the necessary and sufficient properties (constitutive properties) of the core fusion function, defined in the context of the unit interval, are preserved in the context of an arbitrary interval $[a, b]$ when defining an analogous (a, b)-fusion function.

Before developing the framework, we had to introduce some concepts and to establish the terminology:

Definition 35. *An (a, b)-fusion function is an arbitrary function $F^{a,b} : [a, b]^n \to [a, b]$.*

It is clear that every fusion function is an (a, b)-fusion function for $a = 0$ and $b = 1$. Then, henceforward, every $(0, 1)$-fusion function is called here just as fusion function.

The action of *shifting* a property **(P1)** of a function $F_1 : [a_1, b_1]^n \to [a_1, b_1]$ from $[a_1, b_1]$ to $[a_2, b_2]$ is to "rewrite" **(P1)** so that it conveys the same concept in the context of $[a_2, b_2]$, resulting in a property **(P2)** of a function $F_2 : [a_2, b_2]^n \to [a_2, b_2]$. In other words, **(P2)** is the counterpart in $[a_2, b_2]$ for the property **(P1)**. Some properties can be shifted without any rewriting (e.g., monotonicity, continuity, associativity and idempotency). However, boundary conditions, in general, have to be rewritten when shifted.

Definition 36. *Let $\mathcal{F}$ be the subclass of fusion functions $F : [0, 1]^n \to [0, 1]$ determined by the set of constitutive properties $P_{\mathcal{F}}$. Then, a set of constitutive properties $P_{\mathcal{F}^{a,b}}$ of a class of (a, b)-fusion functions $\mathcal{F}^{a,b}$ is said to be $\mathcal{F}$-shiftable if $P_{\mathcal{F}}$ coincides with the set composed of all the properties obtained by shifting each property of $P_{\mathcal{F}^{a,b}}$ from $[a, b]$ to $[0, 1]$.*

Definition 37. *Let $P_{\mathcal{F}}$ be the set of constitutive properties of a class of fusion functions $\mathcal{F}$. Then, $\mathcal{F}^{a,b}$, given by*

$$\mathcal{F}^{a,b} = \{F^{a,b} : [a, b]^n \to [a, b] |\ F^{a,b} \text{ satisfies all the properties in } P_{\mathcal{F}}^{a,b}\}, \tag{3.8}$$

is said to be $\mathcal{F}$-shifted if $P_{\mathcal{F}}^{a,b}$ is $\mathcal{F}$-shiftable.

Thus, a $\mathcal{F}$-shifted class of (a, b)-fusion functions $\mathcal{F}^{a,b}$ is a counterpart (in $[a, b]$) of a class of fusion function $\mathcal{F}$ (in $[0, 1]$).

In (GMMP09), aggregation functions were already defined in the context of a domain $[a, b]^n$. But here, to avoid confusion, we call them aggregation functions only when $a = 0$ and $b = 1$ (Definition 3). Otherwise, we call them (a, b)-aggregation functions, just to standardize the notation.

In the following, we showed how to define the class of n-dimensional (a, b)-overlap functions $\mathcal{O}^{a,b}$, based on the class $\mathcal{O}$ of n-dimensional overlap functions, through the framework. In other words, we intended to define an $\mathcal{O}$-shifted subclass $\mathcal{O}^{a,b}$ of (a, b)-aggregation functions as the counterpart in $[a, b]$ for the class of n-dimensional overlap functions $\mathcal{O}$ (Definition 8). For that, we had to define the set of constitutive properties $P_{\mathcal{O}^{a,b}}$ in a way for it to be $\mathcal{O}$-shiftable, that is, so that $P_{\mathcal{O}^{a,b}} = P_{\mathcal{O}}$ when shifting the properties of $P_{\mathcal{O}^{a,b}}$ from $[a, b]$ to $[0, 1]$.

From Definition 8, we see that the set $P_{\mathcal{O}}$ has three properties that can be shifted without rewriting them: **(On1)**, **(On4)** and **(On5)**. So, these three properties can be part of the set $P_{\mathcal{O}^{a,b}}$. However, properties **(On2)** and **(On3)** are the lower and upper boundary conditions, respectively, and, thus, they depend on the values of such boundaries (0 and 1). Also, they are defined by means of the product operation which, in the context of the interval $[0, 1]$, has the lower boundary as its annihilator element and the upper boundary as its neutral element. This characteristic is not carried when defining such boundary conditions on a different interval $[a, b]$.

So, it was clear that we could not simply exchange 0 for the left endpoint (a) on condition **(On2)** and 1 for right endpoint (b) on condition **(On3)** to obtain the analogous boundary conditions for $P_{\mathcal{O}^{a,b}}$. There are more than one way to define such boundary conditions so that they are equivalent to **(On2)** and **(On3)** when $a = 0$ and $b = 1$. Here we presented a viable alternative. Considering an (a, b)-fusion function $On^{a,b} : [a, b]^n \to [a, b]$, the following properties complete the set $P_{\mathcal{O}^{a,b}}$:

(OAB1) $On^{a,b}$ is symmetric;

(OAB2) $On^{a,b}(x_1, \ldots, x_n) = a$ if and only if $\prod_{i=1}^{n}(x_i - a) = 0$;

(OAB3) $On^{a,b}(x_1, \ldots, x_n) = b$ if and only if $\prod_{i=1}^{n}(\frac{x_i - a}{b - a}) = 0$;

(OAB4) $On^{a,b}$ is increasing;

(OAB5) $On^{a,b}$ is continuous.

One can observe that **(OAB2)** and **(OAB3)** are equivalent to **(On2)** and **(On3)**, respectively, when $a = 0$ and $b = 1$, since the relevant properties of the product operation are respected in $[0, 1]$. The other three properties were just relabelled to not mix the notation. Thus, the set of properties $P_{\mathcal{O}^{a,b}} = \{$**(OAB1)**, **(OAB2)**, **(OAB3)**, **(OAB4)**, **(OAB5)**$\}$ is $\mathcal{O}$-shiftable.

Based on the set of properties $P_{\mathcal{O}^{a,b}}$, we defined the class of n-dimensional (a, b)-overlap functions:

Definition 38. *The class $\mathcal{O}^{a,b}$ of n-dimensional (a, b)-overlap functions $On^{a,b}$ is given by:*

$$\mathcal{O}^{a,b} = \{On^{a,b} : [a, b]^n \to [a, b] |\ On^{a,b} \textit{ satisfies all the properties in } P_{\mathcal{O}^{a,b}}\} \tag{3.9}$$

*where $P_{\mathcal{O}^{a,b}} = \{$**(OAB1), (OAB2),(OAB3), (OAB4), (OAB5)**$\}$.*

Similar definitions were presented for (a, b)-t-norms and (a, b)-unimorms, to showcase how other (a, b)-aggregation functions can also be defined through our general framework, by shifting the constitutive properties of the core aggregation functions. Since the motivation came from an application standpoint, we presented some construction methods for these newly defined (a, b)-aggregation functions, guaranteeing that the constructed function behaves in $[a, b]$ in a similar manner as the core function does in $[0, 1]$.

Consider a fusion function $F : [0, 1]^n \to [0, 1]$ and an increasing and bijective function $\phi : [a, b] \to [0, 1]$ and the (a, b)-fusion function $F_\phi^{a,b} : [a, b]^n \to [a, b]$ given, for all $x_1, \dots, x_n \in [a, b]$, by

$$F_\phi^{a,b}(x_1, \dots, x_n) = \phi^{-1} \left(F \left(\phi(x_1), \dots, \phi(x_n) \right) \right). \tag{3.10}$$

Then, F is said to be the core function of $F_\phi^{a,b}$. In the remainder of the paper, we denote $F_\phi^{a,b}$ simply by $F^{a,b}$. Equation (3.10) place an important role in the following construction methods.

Theorem 14. *Consider a fusion function $A : [0, 1]^n \to [0, 1]$, an increasing and bijective function $\phi : [a, b] \to [0, 1]$ and an (a, b)-fusion function $A^{a,b} : [a, b]^n \to [a, b]$ given, for all $x_1, \dots, x_n \in [a, b]$, by*

$$A^{a,b}(x_1, \dots, x_n) = \phi^{-1} \left(A \left(\phi(x_1), \dots, \phi(x_n) \right) \right). \tag{3.11}$$

Then, $A^{a,b}$ is an (a, b)-aggregation function if and only if A is an aggregation function.

Theorem 15. *Consider a fusion function $On : [0, 1]^n \to [0, 1]$, an increasing and bijective function $\phi : [a, b] \to [0, 1]$ and an (a, b)-fusion function $On^{a,b} : [a, b]^n \to [a, b]$ given, for all $x_1, \dots, x_n \in [a, b]$, by*

$$On^{a,b}(x_1, \dots, x_n) = \phi^{-1} \left(On \left(\phi(x_1), \dots, \phi(x_n) \right) \right), \tag{3.12}$$

Then, $On^{a,b}$ is an n-dimensional (a, b)-overlap function if and only if O is an n-dimensional overlap function.

One may observe that the geometric mean GM, given by Equation (2.7), is only an n-dimensional (a, b)-overlap function when $a = 0$ and $b > 0$. In the following, we applied the construction method from Theorem 15 to obtain an n-dimensional (a, b)-overlap function $GM^{a,b}$ based on the geometric mean GM, for any arbitrary $a, b \in \mathbb{R}$, such that $a < b$.

Example 11. *Consider increasing bijection $\phi_A : [b, a] \to [0, 1]$, for all $x \in [a, b]$, by*

$$\phi_A(x) = \left(\frac{x - a}{b - a} \right). \tag{3.13}$$

Then, let $GM : [0, 1]^n \to [0, 1]$ be the geometric mean, given by Equation (2.7). Thus, the (a, b)-fusion function $GM^{a,b} : [a, b]^n \to [a, b]$, given, for all $x_1, \dots, x_n \in [a, b]$, by

$$GM^{a,b}(x_1, \dots, x_n) = {\phi_A}^{-1} \left(GM \left(\phi_A(x_1), \dots, \phi_A(x_n) \right) \right), \tag{3.14}$$

is an n-dimensional (a, b)-overlap function. We can rewrite Equation (3.14) as follows:

$$GM^{a,b}(x_1, \ldots, x_n) = GM\left(\frac{x_1 - a}{b - a}, \ldots, \frac{x_n - a}{b - a}\right) \cdot (b - a) + a.$$

Similar construction methods were presented for (a, b)-t-norms and (a, b)-uninorms. The presented construction methods are based on the choice of core aggregation function and an increasing bijective function, both able to be defined with parameters that can be manipulated/adapted/learned, accordingly to the application at hand, without sacrificing the main properties of the desired constructed function.

Then, we proceed to study some interesting properties of aggregation functions, namely, idempotency, a kind of generalized migrativity (introduced in this work) and abstract homogeneity (SBD$^+$21), and showed that those properties are preserved when our construction methods for (a, b)-aggregation functions are applied.

Finally, we presented the main concepts to develop a similar framework to define fusion functions whose inputs come from an interval $[a, b]$ but the output is mapped on a possibly different interval $[c, d]$. This type of function was introduced as follows:

Definition 39. *An (a, b, c, d)-fusion function is an arbitrary function $F^{c,d}_{a,b} : [a, b]^n \to [c, d]$.*

It is immediate that every fusion function is an (a, b, c, d)-fusion function for $a = c = 0$ and $b = d = 1$. Also, every (a, b)-fusion function is an (a, b, c, d)-fusion function when $a = c$ and $b = d$. So, every $(0, 1, 0, 1)$-fusion function is called just as fusion function and every (a, b, a, b)-fusion function is called just as (a, b)-fusion function.

Properties from either fusion functions or (a, b)-fusion functions can be shifted to the context of (a, b, c, d)-fusion functions, by taking into consideration the domain $[a, b]^n$ and codomain $[c, d]$. Thus, one can define subclasses of (a, b, c, d)-fusion functions, by the same methodology presented for (a, b)-fusion functions, that is, by appropriately shifting the constitutive properties of the core class of functions.

Then, based on this framework, subclasses of (a, b, c, d)-fusion functions were defined and construction methods for them were presented. We showed that, under some constraints, when a constructed (a, b, c, d)-aggregation function is based on an (a, b)-aggregation function, which, in turn, is based on a core aggregation function defined on $[0, 1]^n$, it is equivalent to the (a, b, c, d)-aggregation function obtained directly from the same core aggregation function defined in $[0, 1]$.

In conclusion, the theoretical contributions of this paper can benefit applications with aggregation processes where the data are not restricted to the unit interval, in particular with the assurance that the advantageous properties of known aggregation functions can be preserved (shifted) when applying the newly developed functions, even on problems that do not necessarily involve fuzzy modeling.

3.2 Complementary contributions

In this section, we present four works that, although they do not conform the main discussion around our research questions, were directly related to the development of this thesis.

3.2.1 General grouping functions

The associated publication (available in Chapter 5, Section 5.2.1) related to this contribution is the following:

- H. Santos, G. Dimuro, T. Asmus, G. Lucca, E. Bueno, B. Bedregal, J. Sanz and H. Bustince, "General grouping functions", Information Process- ing and Management of Uncertainty in Knowledge-Based Systems 1238 (2020) 481–495.

Inspired by the work of De Miguel et al. (DGR+19) on general overlap functions, this work introduced the concept of general grouping functions:

Definition 40. *A general grouping function is any mapping $GG : [0,1]^n \to [0,1]$ that satisfies the following conditions, for all $x_1, \ldots, x_n \in [0,1]$:*

(GG1) *GG is symmetric;*

(GG2) *If $\sum_{i=1}^{n} x_i = 0$ then $GG(x_1, \ldots, x_n) = 0$;*

(GG3) *If there exist $i \in \{1, \ldots, n\}$ such that $x_i = 1$ then $GG(x_1, \ldots, x_n) = 1$;*

(GG4) *GG is increasing;*

(GG5) *GG is continuous.*

General grouping functions generalize n-dimensional grouping functions (Definition 9), by having less restrictive boundary conditions. This flexibility allowed us to present several construction methods for this kind of function, which we summarize here (the numeration of each result corresponds with the ones from the associated publication):

Theorem 2 - Characterization of general grouping functions by two fusion functions with some conditions;

Theorem 3 - Construction of a general grouping function as a truncated version of an n-dimensional grouping functions;

Proposition 6 - Construction of a general grouping function through the product of an n-dimensional grouping function by a symmetric and continuous aggregation function that respects **(GG3)**;

Theorem 5 - Construction of a general grouping function by generalized composition of general grouping functions by a continuous aggregation function;

Theorem 6 - Construction of a general grouping function by generalized composition of continuous, symmetric and disjunctive aggregation functions by a general grouping function;

Proposition 7 - Construction of a general grouping function as the N-dual of a general overlap functions, considering a fuzzy negation N.

The presented construction methods show the adaptability and applicability of the developed concepts. Also, many of the developed theory for general overlap functions can be adapted for general grouping functions, since the duality between them is verified.

3.2.2 General interval-valued grouping functions

The associated publication (available in Chapter 5, Section 5.2.2) related to this contribution is the following:

- T. Asmus, G. Dimuro, H. Bustince, B. Bedregal, H. Santos and J. A. Sanz, "General interval-valued grouping functions", IEEE International Conference on Fuzzy Systems (FUZZ-IEEE) (2020) 1–8.

Our paper on general iv-overlap functions (Section 3.1.1) and the aforementioned related work on general grouping functions (Section 3.2.1) lead to the development of the concepts of n-dimensional iv-grouping functions and their generalization, namely, general iv-grouping functions:

Definition 41. *A function $IGn : L([0,1])^n \to L([0,1])$ is called an n-dimensional interval-valued (iv) grouping function if the following conditions are satisfied, for all $\vec{X} \in L([0,1])^n$:*

(IGn1) *IGn is symmetric;*

(IGn2) $IGn(\vec{X}) = [0,0] \Leftrightarrow \sum_{i=1}^{n} X_i = [0,0]$;

(IGn3) $IGn(\vec{X}) = [1,1] \Leftrightarrow$ *there exist* $i \in \{1,\ldots,n\}$ *such that* $X_i = [1,1]$;

(IGn4) IGn *is* $\leq_{Pr}$*-increasing;*

(IGn5) IGn *is Moore continuous.*

We showed that an n-dimensional grouping function can be obtained, by duality, from a n-dimensional iv-overlap function (and vice versa):

Proposition 6. *Let* $IOn : L([0,1])^n \to L([0,1])$ *be an* n*-dimensional iv-overlap function. Then, the mapping* $IGn_{IOn} : L([0,1])^n \to L([0,1])$ *defined by*

$$IGn_{IOn}(\vec{X}) = [1,1] - IOn([1,1] - X_1, \ldots, [1,1] - X_n)$$

is an n-dimensional iv-grouping function.

The concept of g-representable functions was also introduced, that is, representable interval functions that have n-dimensional grouping functions as both their representatives, and the conditions for a n-dimensional iv-grouping function to be g-representable were presented:

Theorem 16. *Let* $IGn : L([0,1])^n \to L([0,1])$ *be an* n*-dimensional iv-grouping function. Then,* IGn *is* g*-representable if and only if* IGn *is inclusion monotonic and the following conditions are satisfied: (i)* $\underline{IGn(\vec{X})} = 0 \Leftrightarrow \underline{X_1} = \ldots = \underline{X_n} = 0$*; (ii)* $\overline{IGn(\vec{X})} = 1 \Leftrightarrow \max(\overline{\vec{X}}) = 1$.

By replacing conditions **(IGn2)** and **(IGn3)** on Definition 41 for less restrictive ones, we obtained the following definition:

Definition 42. *A general iv-grouping function is any mapping* $IGG : L([0,1])^n \to L([0,1])$ *that satisfies the following conditions, for all* $\vec{X} \in L([0,1])^n$*:*

(IGG1) IGG *is symmetric;*

(IGG2) *If* $\sum_{i=1}^{n} X_i = [0,0]$ *then* $IGG(\vec{X}) = [0,0]$;

(IGG3) *If there exist* $i \in \{1,\ldots,n\}$ *such that* $X_i = [1,1]$ *then* $IGG(\vec{X}) = [1,1]$;

(IGG4) IGG *is* $\leq_{Pr}$*-increasing;*

(IGG5) IGG *is Moore continuous.*

It is immediate that general iv-grouping functions generalize n-dimensional iv-grouping functions. Then, construction methods for general iv-grouping functions were presented (the numeration of each result corresponds with the ones from the associated publication):

Theorem 4.2 - Characterization of general iv-grouping functions by two iv-fusion functions with some conditions;

Proposition 4.3 - Construction of a general iv-grouping function through the supremum of a general iv-grouping function and a Moore continuous iv-aggregation function;

Theorem 4.3 - Construction of a general iv-grouping function by generalized composition of general iv-grouping functions by a Moore continuous aggregation function.

The theoretical contributions presented in this paper allow for a more flexible approach when dealing with decision making problems with multiple alternatives and interval-valued data, which we suggested as a potential application for the introduced concepts.

3.2.3 General admissibly ordered interval-valued overlap functions

The associated publication (available in Chapter 5, Section 5.2.3) related to this contribution is the following:

- T. Asmus, G. Dimuro, J. A. Sanz, J. Wieczynski, G. Lucca and H. Bustince, "General admissibly ordered interval-valued overlap functions", The 13th International Workshop on Fuzzy Logic and Applications (WILF) (2021) (accepted).

In our work on n-dimensional admissibly ordered iv-overlap functions (Section 3.1.2), we presented a construction method (Theorem 8) for this type of function considering $\leq_{\alpha,\beta}$ orders, with $\alpha \in (0, 1)$. This means that neither of the lexicographical orders can be considered when applying this method. Although this is not a serious problem, with the initial motivation to overcome this drawback, in this complementary work we combined the concepts of general iv-overlap functions (Definition 26, in Section 3.1.1) and n-dimensional admissibly ordered iv-overlap functions (Definition 28, in Section 3.1.2) to introduce general admissibly ordered iv-overlap functions, as follows:

Definition 43. *A function $AGO : L([0,1])^n \to L([0,1])$ is a general admissibly ordered iv-overlap function for an admissible order $\leq_{AD}$ (general $\leq_{AD}$-overlap function) if it satisfies the conditions **(IGO1)**, **(IGO2)** and **(IGO3)** of Definition 26 (see Section 3.1.1), and the following condition holds:*

(AGO4) *AGO is $\leq_{AD}$-increasing.*

The following result is immediate:

Proposition 7. *If $AOn : L([0,1])^n \to L([0,1])$ is an n-dimensional $\leq_{AD}$-overlap function, then it is also a general $\leq_{AD}$-overlap function, but the converse may not hold.*

In the sequence, we presented three construction methods for general $\leq_{AD}$-overlap function (the numeration of each result corresponds with the ones from the associated publication):

Theorem 7 - An adaptation of Theorem 8 (in Section 3.1.2), by considering $\alpha \in [0,1]$, which produces a general $\leq_{\alpha,\beta}$-overlap function;

Theorem 9 - An adaptation of Theorem 3 (in Section 2.3) to obtain a general $\leq_{\alpha,\beta}$-overlap functions, with $\alpha, \beta \in [0,1]$, by applying a strict n-dimensional overlap function to aggregate the K_α points of the inputs to generate the K_α point of the output of the constructed function;

Theorem 4.3 - Construction of general $\leq_{AD}$-overlap functions by generalized composition of general $\leq_{AD}$-overlap functions by an $\leq_{AD}$-increasing iv-aggregation function.

Thus, the introduced concept of general admissibly ordered iv-overlap functions proved to very flexible and adaptable, allowing for the development of different construction methods, and even the composition of functions constructed through those methods, which makes them suitable to be applied in practical problems. In particular, we highlight that even the lexicographical orders are contemplated by the presented construction methods.

3.2.4 Enhancing the efficiency of the interval-valued fuzzy rule-based classifier with tuning and rule selection

The associated publication (available in Chapter 5, Section 5.2.4) related to this contribution is the following:

- H. Santos, G. Dimuro, T. Asmus, G. Lucca, E. Bueno, B. Bedregal, J. Sanz and H. Bustince, "Enhancing the efficiency of the interval-valued fuzzy rule-based classifier with tuning and rule selection", Information Processing and Management of Uncertainty in Knowledge-Based Systems 1238 (2020) 463–478.

The first three complementary works, discussed in Sections 3.2.1, 3.2.2 and 3.2.3, were focused on the theoretical developments of functions inspired by the works on generalized/extended (interval-valued) overlap operations. Nevertheless, the following work was motivated entirely from an application standpoint, in an attempt to enhance the efficiency of the IVTURS algorithm (the state-of-the-art IV-FRBCS used as a basis throughout the development of this thesis) for it to be less time consuming and maintaining the performance results.

The first strategy involved a simple mathematical simplification on the expressions of the IV-REFs applied in step 1 of the IV-FRM. Since the IV-REF, given by Equation (2.18), is used to compute the similarity between the interval membership degree $X \in L([0,1])$ with the perfect membership expressed by the interval $[1,1]$, and $a, b \in (0, +\infty)$, one has that:

$$IR(X,[1,1]) = \left[(1-|\underline{X}^b - \underline{1}^b|)^{\frac{1}{a}}, (1-|\overline{X}^b - \overline{1}^b|)^{\frac{1}{a}}\right] = \left[\underline{X}^{\frac{b}{a}}, \overline{X}^{\frac{b}{a}}\right] \tag{3.15}$$

This simplification results in an economy of computational cost, since the power operations, which are very taxing on the system, are cut in half.

The second optimization was done when computing the interval matching degree (see Section 2.6.2) and how the system manages "do not care" labels, that is, when some variables (attributes) have to be disregarded on the rule at hand. On IVTURS's original code, since the conjunction operation in this step is calculated through an interval product, do not care labels are assigned the value $[1,1]$ (neutral element of the interval product), so that they do not interfere in the final output of the matching degree. Even so, a lot of calculations are carried out, even if the final result remains unaltered.

To avoid such trivial but costly operations with do not care labels, we proposed the addition of an initial iteration where the system check whether the example is compatible with the rule or not. Then, the interval matching degree is only computed when they are compatible, meaning that do not care labels are disregarded in this process. This extra initial iteration may appear to incur in an extra charge for the run time of the algorithm, but the fact is that it provided a great reduction on the run time, by avoiding many calculations in the IV-FRM.

The experimentation that was executed in this work showed that these two modifications provides a system that is about eight times faster than the original IVTURS algorithm. Thus, this modified version of IVTURS can be applied in a wider range of classification problems, since its efficiency has been notably enhanced.

Another aspect that was analyzed was the evolutionary process, since the parameters a and b applied in Equation (3.15) can be exchanged by a single parameter $c = \frac{b}{a}$, reducing the search space. This modification could affect the accuracy of the system, so the authors carried out an experimentation, comparing the classification acuity of the system with other evolutionary approaches. The results showed that the configuration of the original IVTURS in this regard provided competitive results, so the evolutionary process does not need to be modified.

3.3 Summary

Here, we present an overview of the development of the thesis, both from a theoretical and practical point of view. Figure 3.1 shows a flowchart of such developments.

Each stage of development is associated with one of the main papers from the collection, as described in the Methodology. In each of those stages, we have a row dedicated to the main theoretical concepts (left nodes) and another row to the development of the new IV-FRBCS based on IVTURS, named IVTURS-OV (right nodes).

Notice that each node is connected to another from a following stage and has a $+$ symbol, indicating that the concepts and features from past nodes are not disregarded in latter stages of development, which characterizes the incremental methodology. For instance, the structure of the general framework for w-iv-fusion functions introduced in stage 4 was heavily influential in the introduced framework for (a, b)-fusion functions in stage 5.

The reason for some of the right nodes to have different colors is to match the color of the particular step of the IVTURS algorithm (see Figure 2.5, in Section 2.6) in which the modification was made. For example, in stage 2, both the generation of the initial FRBCS (red) and the IV-FRM (blue) were affected by the introduced theoretical developments from this stage. In stage 4, the new concepts were only applied in the IV-FRM, but we kept the red color on the top part of this node to reinforce that the previous modifications on the generation of the initial FRBCS are still being considered.

In Figure 3.2, we present the relation between the complementary contributions (discussed in Section 3.2) with the stages of development of the thesis. The square nodes signify each stage (with its respective publication, discussed in Section 3.1) and the circle nodes represent each complementary work, as follows:

- CW1 - General grouping functions (Section 3.2.1);
- CW2 - General interval-valued grouping functions (Section 3.2.2);
- CW3 - General admissibly ordered interval-valued overlap functions (Section 3.2.3);
- CW4 - Enhancing the efficiency of the interval-valued fuzzy rule-based classifier with tuning and rule selection (Section 3.2.4).

So, complementary works CW1 and CW2 are related to the first stage of development of the thesis

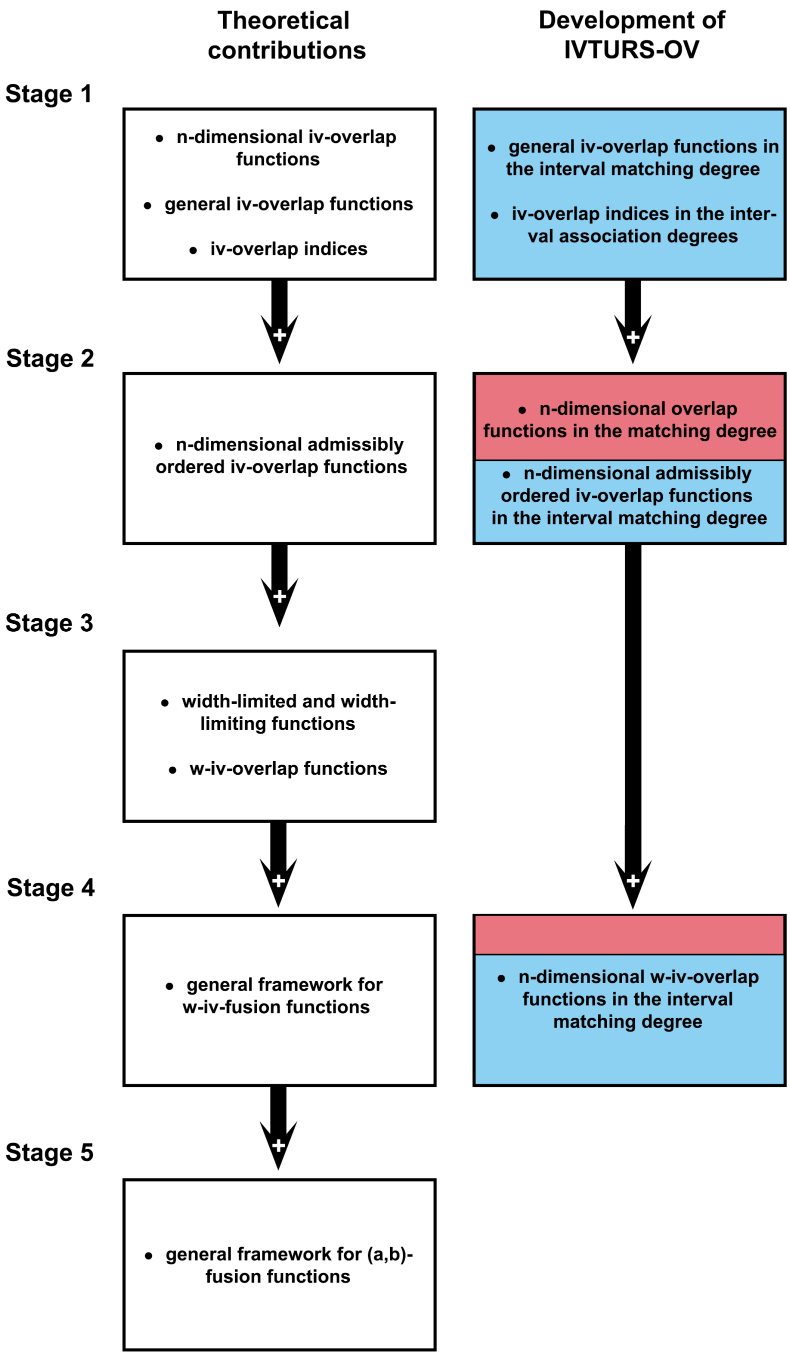

Figure 3.1: A flowchart of development of the thesis.

(Section 3.1.1), while CW3 and CW4 derived from the first two stages (Sections 3.1.1 and 3.1.2). At this point, we have no further published works related to the remaining stages of development, which is why we omitted the last three stages on Figure 3.2.

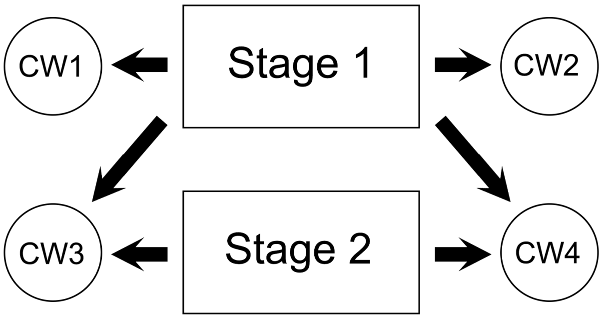

Figure 3.2: A flowchart for the complementary works.

Chapter 4

CONCLUSION

4.1 Final remarks and contributions

Overlap functions and their generalizations have been successfully applied as fusion operators in problems where associativity is not a requirement, such as FRBCSs. Despite their desirable properties, the application of overlap operators in either IV-FRBCSs or other non-fuzzy classifiers could not be found in the literature previous to this thesis. Most of this absence derives from how overlap functions were originally defined (as aggregation functions acting on the unit interval), which limit their application to fuzzy contexts, and how they were defined in the context of intervals (as bivariate functions that are only increasing with respect to a partial order).

Generalizations on the concept of overlap functions have been successfully applied in FRBCSs, namely, n-dimensional overlap functions and general overlap functions. Motivated by this fact, we could introduce similarly defined functions in the interval context, in order to be applied in IV-FRBCSs. To avoid stalemate situations in the classification task carried out by IV-FRBCSs, admissible orders could be considered, since they refine the usual product order for intervals.

When operating with intervals in fusion processes, their widths are associated with the uncertainty of the value they are approximating and the quality of the information they are carrying. So, by controlling the widths of the interval outputs of such operations accordingly to the widths of the interval inputs, one could prevent the deterioration of such information quality. However, this type of width control has only been studied, in the literature previous to this thesis, based on a very restrictive instance were all the interval inputs have the same width. Also, there was no framework to define and construct width controlled iv-fusion functions, based on known fusion functions with desirable properties, like n-dimensional overlap functions. Furthermore, the effect any kind of information quality control in IV-FRBCSs has also not being studied in the literature.

In the case of non-fuzzy classifiers, fusion processes are performed with data that are not limited to the unit interval. Still, in the literature previous to this thesis, one cannot find a framework to properly define and construct fusion functions acting on a closed real interval, based on known fusion functions acting on the unit interval, such as n-dimensional overlap functions, in a manner that the desirable properties of such functions are adequately transposed in this new arbitrary domain.

All the considerations above inspired the proposal of the five research questions posed in the Introduction. Following the incremental methodology, we dedicated a full article to address each one of those questions, introducing new, general theoretical concepts, always with the intent that those concepts could be applied in practical problems. That could be observed by our ongoing development of the IVTURS-OV classifier throughout the thesis, in which we applied most of the developed concepts, achieving fulfilling results.

Here, we review the main contributions of this thesis:

- The definition and introduction of construction methods of n-dimensional iv-overlap functions, general interval valued overlap functions and iv-overlap indices, allowing for the construction of generalized interval overlap operators that can be applied in n-dimensional problems, such as the ones tackled by IV-FRBCSs;
- The definition and introduction of construction methods of n-dimensional admissibly ordered iv-overlap functions, allowing for the construction of interval overlap operators that are increasing with respect to an admissible order;
- The analysis of the effect of admissible orders and n-dimensional admissibly ordered iv-overlap functions in the classification accuracy of IV-FRBCSs, showing that non-associative interval overlap operators and the admissible order which favors the information quality are recommended for such classification systems;
- The introduction of the concepts of width-limited interval-valued functions and width-limiting functions, which, besides being broader than the concept of width-preservation, are the core of our developments on the control of the information quality in interval fusion processes;

- The definition and introduction of construction methods of w-iv-overlap functions, taking into account a width-limiting function and a pair of partial orders, which allows the construction of interval overlap operators with controlled information quality;
- The introduction of a general framework for defining classes of w-iv-fusion functions, based on the extension of a set of properties from a core fusion function, along with several construction methods for w-iv-fusion functions defined through the framework, which allows the construction of several fusion operators with controlled information quality;
- The analysis of the application of interval overlap operators with information quality control in IV-FRBCSs, showing that their application can improve the classification accuracy of the system;
- The development of a new IV-FRBCS based on interval overlap operators that consider both the admissible order applied in the system and has controlled information quality, named IVTURS-OV, which surpasses the classification accuracy of the state-of-the-art classifier IVTURS;
- The introduction of a general framework for defining classes of (a, b)-fusion function as counterparts of known classes of fusion functions, based on the shifting of the constitutive properties of such fusion functions. We also provide several construction methods for (a, b)-fusion functions, which allows the construction of several operators that may fuse data that is not necessarily from the unit interval and preserve the desirable properties of a given fusion function.

4.2 Future lines of work

In this section, we present some possible futures lines of work that could be motivated by our contributions. One could observe that the last two stages of development of the thesis, dedicated to address the research questions **(RQ4)** and **(RQ5)**, were a culmination of all the previous development stages. For that reason, we first discuss some works that could be motivated by our introduced general framework for w-iv-fusion functions and their application in IV-FRBCSs. Following that, we point out potential works that could benefit from the introduced general framework for (a, b)-fusion functions that preserve the desirable properties of a given fusion function. Finally, we present a possible future work derived from the combined contributions from these stages of development of the thesis.

4.2.1 Motivated by our general framework for w-iv-fusion functions and their application in IV-FRBCSs

In the works of Lucca et al. (LSD+17; LSD+18; LDF+19), several generalizations of the discrete Choquet integral were introduced and successfully applied in FRBCSs to aggregate the information given by the fired fuzzy rules. Such functions could be redefined in the interval context, by means of our general framework with information quality control and admissible orders, and be similarly applied on newly developed IV-FRBCSs based on IVTURS-OV.

Recently, other generalizations of the discrete Choquet integral based on dissimilarities were introduced. First, Bustince et al. (BMF+21) presented the notion d-Choquet integral, by replacing the difference operation in the definition of the Choquet integral by a restricted dissimilarity function (BBP08). After that, Takáč et al. (TUG+21) further generalized this concept, introducing the d_G-Choquet integral. An analogous definition for interval-valued d_G-Choquet integrals that are increasing with respect to an admissible order was also presented, which allowed them to be applied to combine the predictions of an ensemble of IVTURS classifiers. So, one possible development of such concept would be to consider the construction of width-limited interval-valued d_G-Choquet integrals, through our general framework, and carry out similar experiments with an ensemble of IVTURS-OV classifiers, possibly considering different levels of information quality control.

One could also consider the development of a general framework for w-iv-fusion functions that are not monotonic. For example, interval-valued implications (ZBM+17) could be defined taking into account the concept of width-limitation. More generally, R-implications-like functions (Ouy12) and material implications (PBBD16), which can be defined by means of aggregation functions, could also be defined in the interval context through this framework, that is, with control of the interval outputs' widths.

Other interesting possibility is to develop a new IV-FRBCS where the width limitation is considered even in the generation of the fuzzy rules and the modeling of the IVFS, accordingly to the application at hand.

4.2.2 Motivated by our general framework for (a, b)-fusion functions and their application in non fuzzy classification systems

Our general framework for (a, b)-fusion functions could provide the definitions of several (n-dimensional) (a, b)-fusion functions (e.g., (a, b)-t-norms, (a, b)-t-conorms, (a, b)-uninorms, n-dimensional (a, b)-

overlap functions and n-dimensional (a, b)-grouping functions) to be applied as the fusion operator in non fuzzy systems, e.g., the aggregation of forces in image processing problems based on a gravitational approach (LMBF$^+$10), the pooling stage of convolutional neural networks (RLS$^+$) and the fusion processes on LSTM neural networks (MFJB).

In the theoretical space, several concepts could be developed based our general framework and the notion of property shifting. For instance, counterparts for the notions of fuzzy negations and restricted dissimilarity functions could be introduced in the form of (a, b)-negations and restricted (a, b)-dissimilarity functions, which would be a first step in the development of several other concepts to be applied beyond the unit interval.

4.2.3 General framework for width-limited interval-valued (a, b)-fusion functions and their application in non-fuzzy classification systems with uncertainty modeling

An immediate possible theoretical development could arise from the amalgamation of the concepts introduced by this thesis: a general constructive framework for defining w-iv-fusion functions on a set L of closed real intervals $[a, b]$, by appropriately shifting and extending the constitutive properties of a core fusion function to the context of L. This would allow for the application of iv-fusion functions with information quality control in problems with imperfect information and uncertainty modeling, but that are not fuzzy in nature, such as in classification via deep learning.

4.3 Conclusión (versión en español)

Las funciones de solapamiento y sus generalizaciones se han aplicado con éxito como operadores de fusión en problemas donde la asociatividad no es un requisito, como los SCBRDs. A pesar de sus buenas propiedades, la aplicación de operadores de solapamiento en IV-SCBRDs u otros clasificadores no difusos no se pudo encontrar en la literatura previa a esta tesis. Las grades razones de esta ausencia son la definición original de las funciones de solapamiento (como funciones de agregación que actúan sobre el intervalo unitario), que limitan su aplicación a contextos difusos, y su definición en el contexto de intervalos (como funciones bivariadas que solo son crecientes con respecto a un órden parcial).

Las generalizaciones sobre el concepto de funciones de solapamiento se han aplicado con éxito en los SCBRDs, a saber, funciones de solapamiento n-dimensionales y funciones de solapamiento generales. Motivados por este hecho, podríamos introducir funciones definidas de manera similar en el contexto de intervalos, para poder aplicarlas en IV-SCBRDs. Para evitar situaciones en las que no se pueda tomar una decisión sobre la clase a asignar a un ejemplo a la hora de realizar la clasificación realizada por IV-SCBRDs, se podrían considerar los órdenes admisibles, ya que refinan el órden habitual del producto para intervalos.

Cuando se opera con intervalos en procesos de fusión, sus amplitudes están asociadas con la incertidumbre del valor que están aproximando y la calidad de la información que representan. El control de las amplitudes de los intervalos de salida de tales operaciones de acuerdo con las amplitudes de los intervalos de entrada podría evitar el deterioro de la dicha calidad de la información. Sin embargo, este tipo de control de la amplitud solo ha sido estudiado, en la literatura previa a esta tesis, a partir de una situación muy restrictiva donde todos los intervalos de entrada tienen la misma amplitud. Además, no había un marco para definir y construir funciones de fusión intervalo-valoradas con amplitudes controladas, basadas en funciones de fusión conocidas con propiedades deseables, como funciones de solapamiento n-dimensionales. Además, el efecto de cualquier tipo de control de la calidad de la información en IV-SCBRDs tampoco se había estudiado en la literatura.

En el caso de clasificadores no difusos, los procesos de fusión se realizan con datos que no están limitados al intervalo unitario. Sin embargo, en la literatura previa a esta tesis, no se podía encontrar un marco para definir y construir correctamente funciones de fusión que actúen en un intervalo real cerrado, basándose en funciones de fusión conocidas que actúen sobre el intervalo unitario, como las funciones de solapamiento n-dimensionales, de manera que las propiedades deseables de tales funciones se traspongan adecuadamente al nuevo dominio arbitrario.

Todas las consideraciones anteriores inspiraron la propuesta de las cinco preguntas de investigación planteadas en la Introducción. Siguiendo una metodología incremental, dedicamos un artículo completo a abordar cada una de esas preguntas, introduciendo nuevos conceptos teóricos generales y siempre con la intención de que pudieran aplicarse en problemas prácticos. Este hecho se puede observar en el continuo desarrollo del clasificador IVTURS-OV a lo largo de la tesis, en el que aplicamos la mayoría de los conceptos desarrollados, logrando resultados satisfactorios.

A continuación, repasamos las principales aportaciones de esta tesis:

- La definición e introducción de métodos de construcción de funciones de solapamiento intervalo-valoradas n-dimensionales, funciones de solapamiento intervalo-valoradas generales e índices de solapamiento intervalo-valorados, permitiendo la construcción de operadores generalizados de solapamiento de intervalos que se pueden aplicar en problemas n -dimensionales, como los que abordan los IV-SCBRDs;
- La definición e introducción de métodos de construcción de funciones de solapamiento intervalo-valoradas n-dimensionales ordenadas admisiblemente, lo que permite la construcción de operadores de solapamiento de intervalos que son crecientes con respecto a un orden admisible;
- El análisis del efecto de los órdenes admisibles y las funciones de solapamiento intervalo-valoradas n-dimensionales ordenadas admisiblemente en el rendimiento de clasificación de los IV-SCBRDs, mostrando que los operadores de solapamiento de intervalos no asociativos y el orden admisible que favorece la calidad de la información son recomendables para tales sistemas de clasificación;
- La introducción de los conceptos de funciones de intervalo con amplitudes limitadas y funciones de limitación de la amplitud, que además de ser más amplios que el concepto de preservación de la amplitud, son el núcleo de nuestros desarrollos sobre el control de la calidad de la información en los procesos de fusión de intervalos;
- La definición e introducción de métodos de construcción de funciones de solapamiento intervalo-valoradas con amplitudes limitadas, teniendo en cuenta una de limitación de la amplitud y un par de órdenes parciales, lo que permite la construcción de operadores de solapamiento de intervalos con calidad de información controlada;
- La introducción de un marco general para definir clases de funciones de fusión intervalo-valoradas con amplitudes limitadas, basado en la extensión de un conjunto de propiedades de una función

de fusión base, junto con varios métodos de construcción para funciones de fusión intervalo-valoradas con amplitudes limitadas definidas a través de dicho marco, que permite la construcción de varios operadores de fusión con calidad de información controlada;

- El análisis de la aplicación de operadores de solapamiento de intervalos con control de calidad de la información en IV-SCBRDs, mostrando que su aplicación puede mejorar el rendimiento de clasificación del sistema;

- El desarrollo de un nuevo IV-SCBRD basado en operadores de solapamiento de intervalos que consideran tanto el orden admisible aplicado en el sistema como la calidad de la información controlada, denominado IVTURS-OV, que supera el rendimiento de clasificación del clasificador IVTURS, que es considerado como estado del arte;

- La introducción de un marco general para definir clases de funciones de (a, b)-fusión como homólogas de clases conocidas de funciones de fusión, basado en el cambio de las propiedades constitutivas de tales funciones de fusión. También proporcionamos varios métodos de construcción de funciones de (a, b)-fusión, lo que permite la construcción de varios operadores que pueden fusionar datos que no son necesariamente del intervalo unitario y la preservación de las propiedades deseables de una función de fusión dada.

Chapter 5

PUBLICATIONS

This final chapter constitutes the collection of all the papers developed as part of this thesis, discussed in Chapter 3. Those papers are divided in two categories: i) main publications (associated with the five research questions and constituting the main body of work of this thesis) and ii) complementary contributions.

5.1 Main publications

Here, we present the five papers discussed in Section 3.1. For each one, we inform the journal where it was published or submitted, the impact factor of the journal and the current status of the publication.

5.1.1 General interval-valued overlap functions and interval-valued overlap indices

Related publication:

- T. Asmus, G. Dimuro, B. Bedregal, J. Sanz, S. Pereira Jr, and H. Bustince, “General interval-valued overlap functions and interval-valued overlap indices”, Information Sciences 527 (2020) 27–50.
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General interval-valued overlap functions and interval-valued overlap indices

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A B S T R A C T

Overlap functions are aggregation functions that express the overlapping degree between two values. They have been used both as a conjunction in several practical problems (e.g., image processing and decision making), and to generate overlap indices between two fuzzy sets, which can be used to construct fuzzy confidence values to be applied in fuzzy rule based classification systems. Some generalizations of overlap functions were recently proposed, such as n-dimensional and general overlap functions, which allowed their application in n-dimensional problems. More recently, the concept of interval-valued overlap functions was presented, mainly to deal with uncertainty in providing membership functions. In this paper, we introduce: (i) the concept of n-dimensional interval-valued overlap functions, studying their representability, (ii) the definition of general interval-valued overlap functions, providing their characterization and some construction methods. Moreover, we also define the concept of interval-valued overlap index, as well as some constructing methods. In addition, we show an illustrative example where those new concepts are applied.

1. Introduction

The concept of overlap function, as a special type of aggregation function [9] that is not required to be associative, was introduced by Bustince et al. in [12] in order to measure the degree of overlapping between two different classes or objects. In the literature, one can find several works on overlap functions and related concepts, as, e.g., [6,18,19,42,43]. Overlap functions have been commonly used in different applications where the associative property is not required during

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the information aggregation process, such as image processing [32], decision making [20,24,27], wavelet-fuzzy power quality diagnosis system [41], and classification [22,23,34–37].

Notice that overlap functions are defined as bivariate functions, and, as such, can only be used in problems that consider just two classes or objects. This is a drawback when dealing with n-dimensional problems, as overlap functions are not required to be associative. To address this limitation, the concept of n-dimensional overlap functions was introduced by Gómez et al. [28] and, more recently, De Miguel et al. [15] defined general overlap functions by relaxing the boundary conditions of n-dimensional overlap functions.

On the other hand, Fuzzy Rule Based Classification Systems (FRBCSs) have been largely used in classification problems, mainly because they are consisted of a linguistic model that may be easily interpretable by end users. In fact, FRBCSs are defined based on a set of rules composed by linguistic variables, which are qualified by linguistic terms modelled as fuzzy sets [25]. Weights are associated to those rules, according to their effectiveness in the classification task.

In [24], the computation of rule weights of FRBCSs was done using fuzzy confidence values or certainty factors defined by overlap indices, which are a generalization of the Zadeh's consistency index [49]. The concept of overlap index was initially introduced to measure the overlapping degree between two fuzzy sets in image processing [11]. In [11,27], overlap indices were built by means of overlap functions.

Now, observe that one important aspect in the modelling of any kind of fuzzy system is the appropriate definition of the membership functions [14]. This is a complex problem due to the uncertainty related to the modeling of such membership functions, usually associated with linguistic terms [39]. One way to deal with such problem is through interval-valued fuzzy sets (IVFSs) [50], which have proven to be an adequate tool to model both vagueness (soft class boundaries) and uncertainty (with respect to the membership function), as discussed in [4,21,45]. For that reason, IVFSs have been successfully applied in different kinds of problems such as game theory [2], decision making [3], pest control [44] and classification [45]. For example, by modeling the membership functions via IVFSs in a FRBCS, one can obtain an interval-valued FRBCSs (IV-FRBCSs), where the ignorance represented by the IVFSs is taken into account throughout the reasoning process, as proposed by Sanz et al. [46,47].

Interval-valued overlap functions (iv-overlap functions, for short) were introduced independently by Qiao and Hu [42] and Bedregal et al. [4]. However, both definitions can only be applied in problems with two classes. For several applications dealing with n-dimensional problems (e.g., IV-FRBCSs), this is a limitation. To overcome this drawback, in this paper we have the following objectives:

1. To define n-dimensional iv-overlap functions, and study some properties and their representation;
2. To define general iv-overlap functions, study their characterization and introduce some construction methods;
3. To define interval-valued (iv) overlap indices and introduce some construction methods;
4. To show an illustrative example in IV-FRBCS, where we apply the new theoretical concepts.

The paper is organized as follows. Section 2 presents some preliminary concepts necessary for the development of the paper. In Section 3, we define n-dimensional iv-overlap functions, and study their representability. The introduction of general iv-overlap functions as well as their characterization, representation and construction methods are presented in Section 4. In Section 5, we introduce the definition of iv-overlap indices and provide some construction methods. An illustrative example of an application of general iv-overlap functions and iv-overlap indices is shown in Section 6. In Section 7 we draw the main conclusions.

2. Preliminary concepts

In this section, we recall some concepts on interval mathematics [40], overlap [6,12], n-dimensional [28] and general [15] overlap functions, overlap indices [27] and interval-valued overlap functions [4,42].

2.1. Interval mathematics, interval-valued fuzzy sets and related concepts

In this subsection, firstly we briefly present some important concepts about interval mathematics and related concepts. For that, let us denote as $L([0,1])$ the set of all closed subintervals of the unit interval [0,1], that is:

$$L([0,1]) = \{[x_1, x_2] | 0 \leq x_1 \leq x_2 \leq 1\}.$$

For any $X = [x_1, x_2]$, the left and right endpoints of X are denoted, respectively, by $\underline{X}$ and $\overline{X}$. Thus, one has that $\underline{X} = x_1$ and $\overline{X} = x_2$.

In the literature, there are many different definitions of partial orders in $L([0,1])$. In this paper, we are going to use the product $\leq_{Pr}$ and the inclusion $\subseteq$ orders, defined for all $X, Y \in L([0,1])$, respectively, by:

$$X \leq_{Pr} Y \Leftrightarrow \underline{X} \leq \underline{Y} \wedge \overline{X} \leq \overline{Y};$$

$$X \subseteq Y \Leftrightarrow \underline{X} \geq \underline{Y} \wedge \overline{X} \leq \overline{Y}.$$

We call as $\leq_{Pr}$-increasing ($\leq_{Pr}$-decreasing) a function that is increasing (decreasing) with respect to the product order $\leq_{Pr}$.

Given an interval-valued function F: $L([0, 1])^n \to L([0, 1])$, we define the projections $F^-, F^+ : [0, 1]^n \to [0, 1]$ of F, for all $x_1, \ldots, x_n \in [0, 1]$, respectively, by:

$$F^-(x_1, \ldots, x_n) = \underline{F([x_1, x_1], \ldots, [x_n, x_n])};$$

$$F^+(x_1, \ldots, x_n) = \overline{F([x_1, x_1], \ldots, [x_n, x_n])}.$$

Given two functions f, g: $[0, 1]^n \to [0, 1]$ such that $f \leq g$, we define the function $\widehat{f,g} : L([0, 1])^n \to L([0, 1])$, for all $X_1, \ldots, X_n \in L([0, 1])$, as

$$\widehat{f,g}(X_1, \ldots, X_n) = [f(\underline{X_1}, \ldots, \underline{X_n}), g(\overline{X_1}, \ldots, \overline{X_n})].$$

The concept of representability of interval-valued functions is important in the context of this work:

Definition 2.1 [21]**.** An $\leq_{Pr}$-increasing interval-valued function F: $L([0, 1])^n \to L([0, 1])$ is said to be representable if there exist increasing functions f, g: $[0, 1]^n \to [0, 1]$ such that $f \leq g$ and $F = \widehat{f,g}$.

In the context of Definition 2.1, f and g are the *representatives* of the interval-valued function F. When $F = \widehat{f,f}$, we denote simply as $\widehat{f}$.

Proposition 2.1 [42]**.** *Let F: $L([0, 1])^n \to L([0, 1])$ be an $\leq_{Pr}$-increasing interval-valued function. Then, F is inclusion monotonic if and only if $F = \widehat{F^-, F^+}$.*

Definition 2.2 [16]**.** A mapping N: $L([0, 1]) \to L([0, 1])$ is said to be an interval-valued fuzzy negation if the following conditions hold:

(N1) N is $\leq_{Pr}$-decreasing;
(N2) N satisfies the boundaries conditions: $N([1, 1]) = [0, 0]$ and $N([0, 0]) = [1, 1]$.

N is said to be involutive if $N(N(X)) = X$, for all $X \in L([0, 1])$.

Definition 2.3 [33]**.** A mapping IR: $L([0, 1])^2 \to L([0, 1])$ is an interval-valued restricted equivalence function (IV-REF) associated with an interval-valued fuzzy negation N: $L([0, 1]) \to L([0, 1])$ if it satisfies the following conditions, for all X, Y, $Z \in L([0, 1])$:

(IR1) $IR(X, Y)$ is commutative;
(IR2) $IR(X, Y) = [1, 1]$ if and only if $X = Y$;
(IR3) $IR(X, Y) = [0, 0]$ if and only if $X = [0, 0]$ and $Y = [1, 1]$ or $X = [1, 1]$ and $Y = [0, 0]$;
(IR4) $IR(X, Y) = IR(N(X), N(Y))$ with N being an involutive interval-valued fuzzy negation;
(IR5) If $X \leq_{Pr} Y \leq_{Pr} Z$, then $IR(X, Y) \geq_{Pr} IR(X, Z)$ and $IR(Y, Z) \geq_{Pr} IR(X, Z)$.

Some interval operations used in this paper are defined, for all X, $Y \in L([0, 1])$ as:

Infimum: $\inf(X, Y) = [\min(\underline{X}, \underline{Y}), \min(\overline{X}, \overline{Y})]$;
Supremum: $\sup(X, Y) = [\max(\underline{X}, \underline{Y}), \max(\overline{X}, \overline{Y})]$;
Sum: $X + Y = [\underline{X} + \underline{Y}, \overline{X} + \overline{Y}]$;
Limited Sum: $X \dot{+} Y = [\min(1, \underline{X} + \underline{Y}), \min(1, \overline{X} + \overline{Y})]$;
Product: $X \cdot Y = [\underline{X} \cdot \underline{Y}, \overline{X} \cdot \overline{Y}]$;
Exponential: $X^p = [\underline{X}^p, \overline{X}^p]$, for any $p \in \mathbb{R}$;
Division: $X/Y = [\underline{X}/\overline{Y}, \overline{X}/\underline{Y}]$ with $\underline{Y} \neq 0$;
Generalized Hukuhara Division: $X \div_H Y = [\min(\underline{X}/\underline{Y}, \overline{X}/\overline{Y}), \max(\underline{X}/\underline{Y}, \overline{X}/\overline{Y})]$ with $\underline{Y} \neq 0$.

Remark 2.1. For any X, $Y \in L([0, 1])$ such that $X \leq_{Pr} Y$ one has that $X \div_H Y \in L([0, 1])$.

For more details on these interval operations on a more general framework, see [7,40,48].

Now, we recall some concepts on continuity and metric spaces that are important for some developments in this paper. Observe that the notion of continuity of a function in mathematical analysis intends to capture, in a rigorous way, the common sense of a function that varies without jumps or abrupt breaks. Continuity of functions, in real analysis, can be defined in term of either limits, topology or metrics. Here we adopt the latter approach.

A function $d : A \times A \to \mathbb{R}$ is said to be a metric if, for each a, b, $c \in A$, we have that:

(i) $d(a, a) = 0$;
(ii) If $a \neq b$, then $d(a, b) > 0$;
(iii) $d(a, b) = d(b, a)$;

(iv) $d(a, c) \leq d(a, b) + d(b, c)$.

Now, let d_A and d_B be metrics on the sets A and B, respectively. A function f: $A \to B$ is said to be (d_A, d_B)-continuous if, for each $x \in A$ and $\epsilon > 0$, there exists $\delta > 0$ such that for, each $y \in A$ with $d_A(x, y) \leq \delta$, we have that $d_B(f(x), f(y)) \leq \epsilon$. In particular, when $A, B \subseteq \mathbb{R}$, $d_A(x, y) = |x - y|$ and $d_B(u, v) = |u - v|$, f is a continuous function.

Example 2.1. Consider $a \in (0, 1]$ and $A = [a, 1]$. The function Div_a: $A \to A$ defined by $Div_a(x) = \frac{a}{x}$ is well defined and continuous. In fact, given $x \in A$ and $\epsilon > 0$, take $\delta = x\epsilon$. Thus, if $y \in A$ is such that $|x - y| \leq \delta$, then it holds that $\frac{a}{xy}|x - y| \leq \frac{a}{y}\epsilon$. But, since $\frac{a}{y} \in [a, 1]$, one has that $|Div_a(x) - Div_a(y)| \leq \epsilon$, because $\frac{a}{y}\epsilon \leq \epsilon$ and $|Div_a(x) - Div_a(y)| = \frac{a}{xy}|x - y|$.

Taking $A \subseteq L([0, 1])$ and $d_M : A \times A \to \mathbb{R}$ defined, for all $[x_1, y_1], [x_2, y_2] \in A$, by:

$$d_M([x_1, y_1], [x_2, y_2]) = \max(|x_1 - x_2|, |y_1 - y_2|),$$

it holds that (A, d_M) is a metric space, where d_M is a Moore metric (in fact, it is a restriction to A of Moore metrics [40]). The (d_M, d_M)-continuous functions in this context are called Moore-continuous.

The Moore-metric can be extended to A^n as follows:

$$d_M^n((X_1, \ldots, X_n), (Y_1, \ldots, Y_n)) = \sqrt{d_M(X_1, Y_1)^2 + \ldots + d_M(X_n, Y_n)^2}.$$

In what follows, we recall the concept of interval-valued fuzzy sets. For that, for a given universe U, we denote by $FS(U)$ the space of all fuzzy sets defined over U. A fuzzy set $F \in FS(U)$ is called normal if there exists $z \in U$ such that $F(z) = 1$.

Definition 2.4 [50]**.** Given an universe U, an interval-valued fuzzy set (IVFS) on U is a function $\mathcal{F} : U \to L([0, 1])$ such that $\mathcal{F}(z) = [F_l(z), F_u(z)]$, for all $z \in U \neq \emptyset$, where $F_l(z) = \underline{\mathcal{F}(z)}, F_u(z) = \overline{\mathcal{F}(z)}, F_l \leq F_u$ and F_l, $F_u \in FS(U)$.

By Definition 2.4, one can observe that an IVFS $\mathcal{F}$ can be represented by a pair of fuzzy sets: the lower fuzzy set F_l and the upper fuzzy set F_u. If $F_l(z) = F_u(z)$, for every $z \in U$, then $\mathcal{F}$ is a fuzzy set, which means that fuzzy sets are particular cases of interval-valued fuzzy sets.

We denote by $IFS(U)$ the space of all interval-valued fuzzy sets defined over U.

2.2. Overlap, general overlap and interval-valued overlap functions, and related concepts

This subsection brings the main concepts related to overlap, general overlap and interval-valued overlap functions, starting from the definition of aggregation function:

Definition 2.5 [9]**.** An aggregation function is any function A: $[0, 1]^n \to [0, 1]$ that is increasing in each argument and satisfies $A(0, \ldots, 0) = 0$ and $A(1, \ldots, 1) = 1$.

An important type of aggregation function are the overlap functions:

Definition 2.6 [6,12]**.** An overlap function is any bivariate function O: $[0, 1]^2 \to [0, 1]$ that satisfies the following conditions, for all x, $y \in [0, 1]$:

(O1) O is commutative;
(O2) $O(x, y) = 0$ if and only if $xy = 0$;
(O3) $O(x, y) = 1$ if and only if $xy = 1$;
(O4) O is increasing;
(O5) O is continuous.

As introduced in [42], a function O: $[0, 1]^2 \to [0, 1]$ is said to be an 0-overlap function if we loose the condition **(O2)** in Definition 2.6 to

$$(\mathbf{O2'})\ xy = 0 \Rightarrow O(x, y) = 0$$

without modifying any other condition.

In the same manner, a function O: $[0, 1]^2 \to [0, 1]$ is said to be an 1-overlap function if we downgrade the condition **(O3)** in Definition 2.6 to

$$(\mathbf{O3'})\ xy = 1 \Rightarrow O(x, y) = 1$$

without changing the remaining conditions.

Example 2.2. The following are some examples of overlap, 0-overlap and 1-overlap functions:

(1) For any $p > 0$, the function O_p: $[0, 1]^2 \to [0, 1]$ given, for any x, $y \in [0, 1]$, by

$$O_p(x, y) = x^p y^p$$

is an overlap function;

(2) For any $t>0$, the function O_t: $[0,1]^2 \to [0,1]$ defined, $x, y \in [0,1]$, by

$$O_t(x,y) = x^t y^t \max(x^t + y^t - 1, 0)$$

is a 0-overlap function;

(3) The function O_U: $[0,1]^2 \to [0,1]$ given, $x, y \in [0,1]$, by

$$O_U(x,y) = \begin{cases} 2xy & \text{if } xy \leq 0.5, \\ 1 & \text{otherwise.} \end{cases}$$

is a 1-overlap function.

For properties and related concepts on overlap, 0-overlap and 1-overlap functions, see also: [17–19,42,43].

Definition 2.7 [28]**.** An n-dimensional overlap function is any mapping On: $[0,1]^n \to [0,1]$ that satisfies the following conditions, for all $x_1, \ldots, x_n \in [0,1]$:

(On1) On is commutative;
(On2) $On(x_1, \ldots, x_n) = 0$ if and only if $\prod_{i=1}^n x_i = 0$;
(On3) $On(x_1, \ldots, x_n) = 1$ if and only if $\prod_{i=1}^n x_i = 1$;
(On4) On is increasing;
(On5) On is continuous.

Here, we show that the concept of 0-overlap and 1-overlap functions can be easily extended to n-dimensional interval-valued functions. Considering an n-dimensional overlap function On: $[0,1]^n \to [0,1]$, one can loose conditions **(On2)** to

$$(\mathbf{On2'})\ \prod_{i=1}^n x_i = 0 \Rightarrow On(x_1, \ldots, x_n) = 0$$

without changing any other condition, obtaining an n-dimensional 0-overlap function.

Analogously, a function On: $[0,1]^n \to [0,1]$ is considered an n-dimensional 1-overlap function if the condition **(On3)** is loosened to

$$(\mathbf{On3'})\ \prod_{i=1}^n x_i = 1 \Rightarrow On(x_1, \ldots, x_n) = 1$$

while the other conditions remain unchanged.

By broadening and combining the concepts of n-dimensional 0-overlap and 1-overlap functions, the concept of general overlap functions is defined as follows:

Definition 2.8 [15]**.** A general overlap function is any mapping GO: $[0,1]^n \to [0,1]$ that satisfies the following conditions, for all $x_1, \ldots, x_n \in [0,1]$:

(GO1) GO is commutative;
(GO2) If $\prod_{i=1}^n x_i = 0$ then $GO(x_1, \ldots, x_n) = 0$;
(GO3) If $\prod_{i=1}^n x_i = 1$ then $GO(x_1, \ldots, x_n) = 1$;
(GO4) GO is increasing;
(GO5) GO is continuous.

The following proposition is adapted from Proposition 1 in [15], to include n-dimensional 0-overlap and 1-overlap functions:

Proposition 2.2. *If On: $L([0,1])^n \to [0,1]$ is an n-dimensional overlap, 0-overlap or 1-overlap function, then On is also a general overlap function.*

Theorem 2.1 [15]**.** *The n-ary function GO: $[0,1]^n \to [0,1]$ is a general overlap function if and only if, for all $x_1, \ldots, x_n \in [0,1]$, it holds that:*

$$GO(x_1, \ldots, x_n) = \frac{f(x_1, \ldots, x_n)}{f(x_1, \ldots, x_n) + g(x_1, \ldots, x_n)},$$

for some f, g: $[0,1]^n \to [0,1]$ such that

(i) *f and g are commutative;*
(ii) *If $\prod_{i=1}^n x_i = 0$ then $f(x_1, \ldots, x_n) = 0$;*
(iii) *If $\prod_{i=1}^n x_i = 1$ then $g(x_1, \ldots, x_n) = 0$;*
(vi) *f is increasing and g is decreasing;*
(v) *f and g are continuous functions;*
(vi) *$f(x_1, \ldots, x_n) + g(x_1, \ldots, x_n) \neq 0$.*

Table 1
Examples of General Overlap Functions.

Definition	Type
$GO_{MIN}(x_1,\ldots,x_n)=\min(x_1,\ldots,x_n)$	overlap
$GO_P(x_1,\ldots,x_n)=\prod_{i=1}^n x_i$	overlap
$GO_{mM}(x_1,\ldots,x_n)=\min(x_1,\ldots,x_n)\max(x_1^2,\ldots,x_n^2)$	overlap
$GO_{GM}(x_1,\ldots,x_n)=\sqrt[n]{\prod_{i=1}^n x_i}$	overlap
$GO_{HM}(x_1,\ldots,x_n)=\begin{cases}\frac{n}{\frac{1}{x_1}+\ldots+\frac{1}{x_n}} & \text{if } x_i>0, \text{ for all } i\in\{1,\ldots,n\}\\ 0 & \text{otherwise.}\end{cases}$	overlap
$GO_L(x_1,\ldots,x_n)=\max\left(\left(\sum_{i=1}^n x_i\right)-(n-1),0\right)$	0-overlap
$GO_U(x_1,\ldots,x_n)=\begin{cases} n\prod_{i=1}^n x_i & \text{if } \prod_{i=1}^n x_i\le 1/n,\\ 1 & \text{otherwise.}\end{cases}$	1-overlap
$GO_G(x_1,\ldots,x_n)=\begin{cases} nGO_L(x_1,\ldots,x_n) & \text{if } GO_L(x_1,\ldots,x_n)\le 1/n,\\ 1, & \text{otherwise.}\end{cases}$	general overlap

In Table 1, we show some examples of general overlap functions. For more properties of general overlap functions, see [15].

Finally, we present the concept of interval-valued overlap functions, starting from the generalization of the concept of aggregation function to the interval context:

Definition 2.9 [38]**.** An interval-valued function IA: $L([0,1])^n\to L([0,1])$ is said to be an interval-valued aggregation function if it is an $\le_{Pr}$-increasing function satisfying $IA([0,0],\ldots,[0,0])=[0,0]$ and $IA([1,1],\ldots,[1,1])=[1,1]$.

Definition 2.10 [4,42]**.** An interval-valued (iv) overlap function is a mapping IO: $L([0,1])^2\to L([0,1])$ that respects the following conditions:

(IO1) IO is commutative;
(IO2) $IO(X,Y)=[0,0]$ if and only if $X\cdot Y=[0,0]$;
(IO3) $IO(X,Y)=[1,1]$ if and only if $X\cdot Y=[1,1]$;
(IO4) IO is $\le_{Pr}$-increasing in the first component: $IO(Y,X)\le_{Pr}IO(Z,X)$ when $Y\le_{Pr}Z$.
(IO5) IO is Moore continuous.

Note that, by **(IO1)** and **(IO4)**, iv-overlap functions are also monotonic in the second component.

Theorem 2.2 [42]**.** *Let* IO: $L([0,1])^2\to L([0,1])$ *be an inclusion monotonic interval-valued function. Then,* IO *is an iv-overlap function if and only if there exist a 0-overlap function* O_1 *and an 1-overlap function* O_2 *such that* $O_1\le O_2$ *and* $IO=\widehat{O_1,O_2}$*. Also, it holds that* $O_1=IO^-$ *and* $O_2=IO^+$*.*

Remark 2.2. Observe that, when considering two overlap functions O_1 and O_2 such that $O_1\le O_2$, the function $\widehat{O_1,O_2}$ is a representable iv-overlap function [4]. However, from Theorem 2.2, it is clear that not every representable iv-overlap function has overlap functions as its representatives. In Section 3, we present some new results and definitions regarding this characteristic of iv-overlap functions.

2.3. Overlap index

An important concept in this work is the one of overlap index, whose related concepts we recall below.

Definition 2.11 [27]**.** A mapping $\mathcal{O}:FS(U)\times FS(U)\to[0,1]$ is said to be an overlap index if it satisfies the following conditions, for all $A, B, C\in FS(U)$:

($\mathcal{O}$1) $\mathcal{O}(A,B)=0$ if and only if, for all $z\in U, A(z)\cdot B(z)=0$;
($\mathcal{O}$2) $\mathcal{O}(A,B)=\mathcal{O}(B,A)$;
($\mathcal{O}$3) If $B\le C$, meaning that $B(z)\le C(z)$ for all $z\in U$ [10], then $\mathcal{O}(A,B)\le\mathcal{O}(A,C)$.

For an overlap index to be called normal, it also has to satisfy the following condition:

($\mathcal{O}$4) If there exists $z\in U$ such that $A(z)\cdot B(z)=1$, then $\mathcal{O}(A,B)=1$.

In the following, we present some examples of overlap indices.

Example 2.3. Considering the Definition 2.11, it holds that:

(1) The function $\mathcal{O}_Z:FS(U)\times FS(U)\to[0,1]$ defined, for all $A, B\in FS(U)$, by:

$$\mathcal{O}_Z(A,B)=\max_{z\in U}\min(A(z),B(z)),$$

is a normal overlap index, which is known as the Zadeh's consistency index [49].

(2) The function $\mathcal{O}_\pi : FS(U) \times FS(U) \to [0, 1]$ given, for all $A, B \in FS(U)$, by:

$$\mathcal{O}_\pi(A, B) = \frac{1}{n}\sum_{i=1}^{n} A(z_i) \cdot B(z_i), \tag{1}$$

for $U = \{z_1, \dots, z_n\}$, is an overlap index.

(3) The function $\mathcal{O}_x : FS(U) \times FS(U) \to [0, 1]$ defined, for all $A, B \in FS(U)$, by:

$$\mathcal{O}_x(A, B) = \begin{cases} 0 & \text{if } \forall z \in U : A(z) \cdot B(z) = 0, \\ x & \text{otherwise}, \end{cases}$$

for a given $z \in (0, 1]$, is an overlap index.

Theorem 2.3 [27]. *Consider an aggregation function M: $[0, 1]^n \to [0, 1]$ such that*

$$M(x_1, \dots, x_n) = 0 \Leftrightarrow x_1 = \dots = x_n = 0,$$

and an overlap function O: $[0, 1]^2 \to [0, 1]$. Then, the function $\mathcal{O}_M^O : FS(U) \times FS(U) \to [0, 1]$, given, for all $A, B \in FS(U)$, $U = \{z_1, \dots, z_n\}$, by

$$\mathcal{O}_M^O(A, B) = M(O(A(z_1), B(z_1)), \dots, O(A(z_n), B(z_n))),$$

is said to be an overlap index.

3. N-dimensional interval-valued overlap functions

As shown previously in Section 2.2, the theoretical developments on interval-valued overlap functions have been done considering bivariate functions. However, many problems require the use of n-dimensional interval-valued functions to aggregate the information, such as obtaining the matching degree for each rule in a IV-FRBCS. To address this kind of situation, which is not trivial since interval-valued overlap functions are not required to be associative, in this section we introduce the concept of n-dimensional interval-valued overlap functions and study some properties of such functions.

The following proposition follows directly from Proposition 2.1:

Proposition 3.1. *An $\leq_{Pr}$-increasing interval-valued function F: $L([0, 1])^n \to L([0, 1])$ is representable if and only if F is inclusion monotonic.*

Now, the following proposition is derived from Proposition 7 in [21]:

Proposition 3.2. *If an $\leq_{Pr}$-increasing interval-valued function F: $L([0, 1])^n \to L([0, 1])$ is inclusion monotonic, then it holds that*

$$\underline{F(X_1, \dots, X_n)} = F^-(\underline{X_1}, \dots, \underline{X_n})$$
$$\overline{F(X_1, \dots, X_n)} = F^+(\overline{X_1}, \dots, \overline{X_n}),$$

for all $X_1, \dots, X_n \in L([0, 1])$.

Proof. Since F is $\leq_{Pr}$-increasing and inclusion monotonic and, for each $i \in \{1, 2, \dots, n\}$, one has that $[\underline{X_i}, \underline{X_i}] \leq_{Pr} X_i$ and $[\underline{X_i}, \underline{X_i}] \subseteq X_i$, for $X_1, \dots, X_n \in L([0, 1])$, then it holds that

$$F([\underline{X_1}, \underline{X_1}], \dots, [\underline{X_n}, \underline{X_n}]) \leq_{Pr} F(X_1, \dots, X_n)$$
$$F([\underline{X_1}, \underline{X_1}], \dots, [\underline{X_n}, \underline{X_n}]) \subseteq F(X_1, \dots, X_n),$$

respectively. So, we have that

$$F^-(\underline{X_1}, \dots, \underline{X_n}) \leq_{Pr} \underline{F(X_1, \dots, X_n)}$$
$$\underline{F(X_1, \dots, X_n)} \leq_{Pr} F^-(\underline{X_1}, \dots, \underline{X_n}),$$

and, thus,

$$\underline{F(X_1, \dots, X_n)} = F^-(\underline{X_1}, \dots, \underline{X_n}).$$

The proof for $\overline{F(X_1, \dots, X_n)} = F^+(\overline{X_1}, \dots, \overline{X_n})$ can be obtained analogously. □ □

Then we define n-dimensional interval-valued overlap functions:

Definition 3.1. An n-dimensional interval-valued (iv) overlap function is a mapping IOn: $L([0, 1])^n \to L([0, 1])$ that satisfies the following conditions, for all $X_1, \dots, X_n \in L([0, 1])$):

(IOn1) IOn is commutative;
(IOn2) $IOn(X_1, \dots, X_n) = [0, 0]$ if and only if $\prod_{i=1}^{n} X_i = [0, 0]$;
(IOn3) $IOn(X_1, \dots, X_n) = [1, 1]$ if and only if $\prod_{i=1}^{n} X_i = [1, 1]$;

(IOn4) IOn is increasing in the first component: $IOn(X_1,\ldots,X_n) \leq_{Pr} IOn(Y, X_2,\ldots,X_n)$ when $X_1 \leq_{Pr} Y$;
(IOn5) IOn is Moore continuous.

Example 3.1. Some examples of n-dimensional iv-overlap functions are the following, where $X_1,\ldots,X_n \in L([0,1])$:

1. $IOn_M(X_1,\ldots,X_n) = \inf(X_1,\ldots,X_n)$;
2. $IOn_p(X_1,\ldots,X_n) = \prod_{i=1}^n X_i^p$, for $p > 0$;
3. $IOn_{mM}(X_1,\ldots,X_n) = [\min(\underline{X_1},\ldots,\underline{X_n})\cdot\max(\underline{X_1^2},\ldots,\underline{X_n^2}), \min(\overline{X_1},\ldots,\overline{X_n})\cdot\max(\overline{X_1^2},\ldots,\overline{X_n^2})]$.

The following result is the adaptation of Theorem 2.2 to n-dimensional iv-overlap functions.

Theorem 3.1. *Let IOn: $L([0,1])^n \to L([0,1])$ be an inclusion monotonic interval-valued function. Then, IOn is an n-dimensional iv-overlap function if and only if there exist an n-dimensional 0-overlap function On_1: $[0,1]^n \to [0,1]$ and an n-dimensional 1-overlap function On_2: $[0,1]^n \to [0,1]$ such that $On_1 \leq On_2$ and $IOn = \widehat{On_1, On_2}$. Also, it holds that $On_1 = IOn^-$ and $On_2 = IOn^+$.*

Proof. One has that: ($\Rightarrow$) Suppose that IOn: $L([0,1])^n \to L([0,1])$ is an inclusion monotonic n-dimensional iv-overlap function. Let On_1, On_2: $[0,1]^n \to [0,1]$ be such that

$$\begin{aligned} On_1(x_1,\ldots,x_n) &= \underline{IOn([x_1,x_1],\ldots,[x_n,x_n])},\\ On_2(x_1,\ldots,x_n) &= \overline{IOn([x_1,x_1],\ldots,[x_n,x_n])},\end{aligned}$$

for all $x_1,\ldots,x_n \in [0,1]$. Thus, it is immediate that On_1 and On_2 are well defined. Now, let us show that On_1 is an n-dimensional 0-overlap function, according to Definition 3.1:

(On1) It is immediate, since IOn is commutative.
(On2) Considering $\prod_{i=1}^n x_i = 0$, one can assume, without loss of generality, that $x_1 = 0$. Thus, for all $x_2,\ldots,x_n \in [0,1]$, one has that:

$$On_1(0,\ldots,x_n) = \underline{IOn([0,0],[x_2,x_2],\ldots,[x_n,x_n])} = \underline{[0,0]} = 0.$$

(On3) Suppose $\prod_{i=1}^n x_i = 1$. Then, $x_1 = x_2 = \ldots = x_n = 1$ and

$$On_1(1,\ldots,1) = \underline{IOn([1,1],\ldots,[1,1])} = \underline{[1,1]} = 1.$$

Now, consider $On_1(x_1,\ldots,x_n) = 1$, for some $(x_1,\ldots,x_n) \in [0,1]^n$. Thus, one has that:

$$\begin{aligned}\underline{IOn([x_1,x_1],\ldots,[x_n,x_n])} = 1 &\Rightarrow IOn([x_1,x_1],\ldots,[x_n,x_n]) = [1,1]\\ &\Rightarrow [x_1,x_1]\cdot[x_2,x_2]\cdot\ldots\cdot[x_n,x_n] = [1,1]\\ &\Rightarrow \prod_{i=1}^n x_i = 1.\end{aligned}$$

(On4) As IOn is $\leq_{Pr}$-increasing in the first component, one has that

$$IOn([x_1,x_1],\ldots,[x_n,x_n]) \leq_{Pr} IOn([y,y],[x_2,x_2],\ldots,[x_n,x_n])$$

for any $y, x_1,\ldots,x_n \in [0,1]$ with $x_1 \leq y$. Then, one has that

$$On_1(x_1,\ldots,x_n) = \underline{IOn([x_1,x_1],\ldots,[x_n,x_n])} \leq \underline{IOn([y,y],[x_2,x_2],\ldots,[x_n,x_n])} = On_1(y,x_2,\ldots,x_n).$$

(On5) Since IOn is increasing and inclusion monotonic, by Proposition 2.1 in [42], it holds that $IOn = \widehat{IOn^-, IOn^+}$. Furthermore, as IOn is Moore-continuous and $On_1 = IOn^-$, then, by Theorem 11 in [21], one has that On_1 is continuous.

The proof that On_2 is an n-dimensional 1-overlap function can be obtained analogously. Also, it is immediate that $On_1 \leq On_2$, $On_1 = IOn^-$, $On_2 = IOn^+$, and that $IOn = \widehat{On_1, On_2}$.

($\Leftarrow$) Let On_1 be an n-dimensional 0-overlap function and On_2 an n-dimensional 1-overlap function such that $On_1 \leq On_2$ and $IOn = \widehat{On_1, On_2}$. Then, let us verify if IOn is an n-dimensional iv-overlap function:

(IOn1) It is immediate as both On_1 and On_2 are commutative.
(IOn2) It holds that:

$$IOn(X_1,\ldots,X_n) = [0,0] \Leftrightarrow \widehat{On_1, On_2} = [0,0].$$

Thus, it follows that:

$$[On_1(\underline{X_1},\ldots,\underline{X_n}), On_2(\overline{X_1},\ldots,\overline{X_n})] = [0,0] \Leftrightarrow On_1(\underline{X_1},\ldots,\underline{X_n}) = 0 \wedge On_2(\overline{X_1},\ldots,\overline{X_n}) = 0.$$

Therefore, one has that:

$$\underline{X_1}\cdot\underline{X_2}\cdot\ldots\cdot\underline{X_n} = 0 \wedge \overline{X_1}\cdot\overline{X_2}\cdot\ldots\cdot\overline{X_n} = 0 \Leftrightarrow X_1\cdot X_2\cdot\ldots\cdot X_n = [0,0].$$

(IOn3) One has that:

$$IOn(X_1,\ldots,X_n)=[1,1]\Leftrightarrow \widehat{On_1,On_2}=[1,1].$$

Then, it holds that:

$$[On_1(\underline{X_1},\ldots,\underline{X_n}),On_2(\overline{X_1},\ldots,\overline{X_n})]=[1,1]\Leftrightarrow On_1(\underline{X_1},\ldots,\underline{X_n})=1\wedge On_2(\overline{X_1},\ldots,\overline{X_n})=1.$$

It follows that:

$$\underline{X_1}\cdot\underline{X_2}\cdot\ldots\cdot\underline{X_n}=1\wedge\overline{X_1}\cdot\overline{X_2}\cdot\ldots\cdot\overline{X_n}=1\Leftrightarrow X_1\cdot X_2\cdot\ldots\cdot X_n=[1,1].$$

(IOn4) For any $Y,X_1,X_2,\ldots,X_n\in L([0,1])$ such that $X_1\leq_{Pr}Y$, as On_1 and On_2 are increasing, one has that

$$\begin{aligned}IOn(X_1,\ldots,X_n) &= [On_1(\underline{X_1},\ldots,\underline{X_n}),On_2(\overline{X_1},\ldots,\overline{X_n})]\\ &\leq_{Pr}[On_1(\underline{Y},\ldots,\underline{X_n}),On_2(\overline{X_1},\ldots,\overline{X_n})]=IOn(Y,\ldots,X_n),\end{aligned}$$

meaning that IOn is $\leq_{Pr}$-increasing in the first component.

(IOn5) As On_1 and On_2 are both increasing and continuous, then, by Theorem 11 in [21], one has that IOn is Moore-continuous. □

Since general overlap functions are a generalization of n-dimensional 0-overlap functions and n-dimensional 1-overlap functions, the following result is immediate:

Corollary 3.1. *An inclusion monotonic interval-valued function $IOn\colon L([0,1])^n\to L([0,1])$ is an n-dimensional iv-overlap function if and only if there exist general overlap functions $GO_1\colon[0,1]^n\to[0,1]$ and $GO_2\colon[0,1]^n\to[0,1]$ such that $GO_1\leq GO_2$ and $IOn=\widehat{GO_1,GO_2}$. In particular, one has that $GO_1=IOn^-$ and $GO_2=IOn^+$.*

Observe that the result presented in Corollary 3.1 also applies to bivariate iv-overlap functions, complementing Theorem 2.2 as well. Furthermore, Corollary 3.1 reinforces the idea that not every representable n-dimensional iv-overlap function has n-dimensional overlap functions as its representatives. In order to make a clear differentiation, in what follows we present some definitions and results regarding the representation of n-dimensional iv-overlap functions.

Theorem 3.2. *Let $On_1,On_2\colon[0,1]^n\to[0,1]$ be n-dimensional overlap functions such that $On_1\leq On_2$. Then, the function $\widehat{On_1,On_2}$ is an n-dimensional iv-overlap function.*

Proof. Conditions **(IOn1)-(IOn4)** are trivially obtained. Condition **(IOn5)** is verified by Theorem 4.2 in [5] and the fact that its extension to n-dimensional functions is trivial, as stated in [8]. □

Definition 3.2. An n-dimensional iv-overlap function $IOn\colon L([0,1])^n\to L([0,1])$ is said to be o-representable if there exist n-dimensional overlap functions $On_1,On_2\colon[0,1]^n\to[0,1]$, $On_1\leq On_2$, such that $IOn=\widehat{On_1,On_2}$.

Clearly, by considering Definition 3.2 for bi-variate functions ($n=2$), we have the same concept of o-representability for iv-overlap functions.

It is noteworthy that Theorem 3.1 and Corollary 3.1 result from the fact that there are some n-dimensional iv-overlap functions that are inclusion monotonic but are not o-representable. To address this situation, we added conditions in which inclusion monotonic n-dimensional iv-overlap functions must satisfy in order to also be o-representable, as stated in the following theorem:

Theorem 3.3. *Let $IOn\colon L([0,1])^n\to L([0,1])$ be an n-dimensional iv-overlap function. Then, IOn is o-representable if and only if IOn is inclusion monotonic and the following conditions are satisfied, for all $X_1,\ldots,X_n\in L([0,1])$:*

(i) $\underline{IOn(X_1,\ldots,X_n)}=0\Leftrightarrow\prod_{i=1}^n\underline{X_i}=0$;
(ii) $\overline{IOn(X_1,\ldots,X_n)}=1\Leftrightarrow\prod_{i=1}^n\overline{X_i}=1$.

Proof. $(\Rightarrow)$ If IOn is o-representable, then by Theorem 3.1, IOn is inclusion monotonic. Also, by Proposition 3.2, for all $X_1,\ldots,X_n\in L([0,1])$, one has that

$$\begin{aligned}IOn^-(\underline{X_1},\ldots,\underline{X_n}) &= \underline{IOn(X_1,\ldots,X_n)},\\ IOn^+(\overline{X_1},\ldots,\overline{X_n}) &= \overline{IOn(X_1,\ldots,X_n)}.\end{aligned}$$

Furthermore, by Proposition 2.1, $IOn=\widehat{IOn^-,IOn^+}$, which means that IOn^- and IOn^+ are both n-dimensional overlap functions. Thus, for all $X_1,\ldots,X_n\in L([0,1])$, it holds that :

$$\underline{IOn(X_1,\ldots,X_n)}=0\Leftrightarrow IOn^-(\underline{X_1},\ldots,\underline{X_n})=0\Leftrightarrow\prod_{i=1}^n\underline{X_i}=0$$

and

$$\overline{IOn(X_1,\ldots,X_n)}=1\Leftrightarrow IOn^+(\overline{X_1},\ldots,\overline{X_n})=1\Leftrightarrow\prod_{i=1}^n\overline{X_i}=1.$$

(⇐) If IOn is an n-dimensional iv-overlap function that is inclusion monotonic and satisfies conditions (i) and (ii), then, by Proposition 3.1, there exist f, g: $[0, 1]^n \to [0, 1]$ (both increasing) such that, for all $X_1, \dots, X_n \in L([0, 1])$, it holds that:

$$IOn(X_1, \dots, X_n) = [f(\underline{X_1}, \dots, \underline{X_n}), g(\overline{X_1}, \dots, \overline{X_n})].$$

By Proposition 3.2, one has that

$$IOn^-(\underline{X_1}, \dots, \underline{X_n}) = \underline{IOn(X_1, \dots, X_n)} = f(\underline{X_1}, \dots, \underline{X_n})$$

and

$$IOn^+(\overline{X_1}, \dots, \overline{X_n}) = \overline{IOn(X_1, \dots, X_n)} = g(\overline{X_1}, \dots, \overline{X_n}),$$

for all $X_1, \dots, X_n \in L([0, 1])$. Thus, it follows that $f = IOn^-$ and $g = IOn^+$.

Now, let us verify that IOn^- and IOn^+ satisfy the conditions of Definition 2.7:

(On1) It is trivial as IOn is commutative.

(On2) From condition (i), for all $x_1, \dots, x_n \in [0, 1]$, we have that:

$$IOn^-(x_1, \dots, x_n) = 0 \Leftrightarrow \underline{IOn([x_1, x_1], \dots, [x_n, x_n])} = 0 \Leftrightarrow \prod_{i=1}^{n} \underline{[x_i, x_i]} = 0 \Leftrightarrow \prod_{i=1}^{n} x_i = 0.$$

(On3) From condition (ii), for all $x_1, \dots, x_n \in [0, 1]$, it holds that:

$$IOn^+(x_1, \dots, x_n) = 1 \Leftrightarrow \overline{IOn([x_1, x_1], \dots, [x_n, x_n])} = 1 \Leftrightarrow \prod_{i=1}^{n} \overline{[x_i, x_i]} = 1 \Leftrightarrow \prod_{i=1}^{n} x_i = 1.$$

(On4) From Proposition 3.1 and the conclusion in this context that $f = IOn^-$ and $g = IOn^+$, then both IOn^- and IOn^+ are increasing.

(On5) From Corollary 12 in [21], IOn^- and IOn^+ are continuous.

As it was proven that IOn^- and IOn^+ are n-dimensional overlap functions, then IOn is o-representable. □

From Theorem 3.3, when considering $n = 2$ the following result is immediate:

Corollary 3.2. *Let IO: $L([0, 1])^2 \to L([0, 1])$ be an iv-overlap function. Then, IO is o-representable if and only if IO is inclusion monotonic and the following conditions are satisfied:*

(i) $\underline{IO(X, Y)} = 0 \Leftrightarrow X \cdot Y = 0$;

(ii) $\overline{IO(X, Y)} = 1 \Leftrightarrow X \cdot Y = 1$.

4. General interval-valued overlap function

In this section, we introduce the concept of general interval-valued overlap functions, as well as some construction methods, properties and characterization. Aside from being the broader approach for n-dimensional iv-overlap functions, this new definition shows a suitable behaviour in classification problems, as we present in Section 6.

But first, by loosening certain conditions in Definition 3.1, one can obtain n-dimensional iv-0-overlap and iv-1-overlap functions in the same way as it was done with n-dimensional overlap functions.

A function IOn: $L([0, 1])^n \to L([0, 1])$ is called an n-dimensional iv-0-overlap function if the condition **(IOn2)** in Definition 3.1 is loosened to

$$(\mathbf{IOn2'}) \prod_{i=1}^{n} X_i = [0, 0] \Rightarrow IOn(X_1, \dots, X_n) = [0, 0]$$

without changing any other condition.

Example 4.1. The function defined, for all $X_1, \dots, X_n \in L([0, 1])$, as

$$IOn_L(X_1, \dots, X_n) = \widehat{GO_L}(X_1, \dots, X_n)$$

with GO_L shown in Table 1, is an n-dimensional iv-0-overlap function, which is not an n-dimensional iv-overlap function.

Analogously, a function IOn: $L([0, 1])^n \to L([0, 1])$ is considered an n-dimensional iv-1-overlap function if the condition **(IOn3)** is loosened to

$$(\mathbf{IOn3'}) \prod_{i=1}^{n} X_i = [1, 1] \Rightarrow IOn(X_1, \dots, X_n) = [1, 1]$$

while the other conditions remain unchanged.

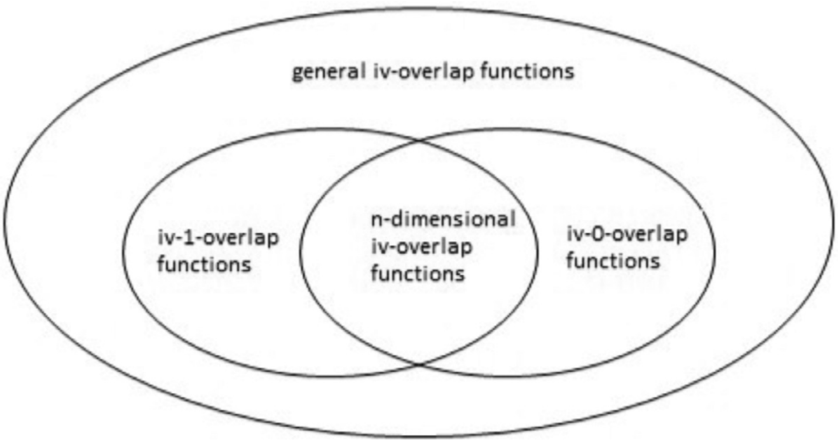

Fig. 1. Relations between iv-overlap functions, iv-0-overlap functions, iv-1-overlap functions and general iv-overlap functions.

Example 4.2. The function defined, for all $X_1, \ldots, X_n \in L([0,1])$, as

$$IOn_U(X_1,\ldots,X_n) = \begin{cases} n\prod_{i=1}^n X_i & \text{if } \prod_{i=1}^n \underline{X_i} \le 1/n \text{ and } \prod_{i=1}^n \overline{X_i} \le 1/n, \\ \left[n\prod_{i=1}^n \underline{X_i}, 1\right] & \text{if } \prod_{i=1}^n \underline{X_i} \le 1/n \text{ and } \prod_{i=1}^n \overline{X_i} > 1/n, \\ [1,1], & \text{otherwise.} \end{cases}$$

is an n-dimensional iv-1-overlap function, which is not an n-dimensional iv-overlap function.

Now, by combining the concepts of n-dimensional iv-0-overlap and iv-0-overlap functions, we present the definition of general interval-valued overlap function.

Definition 4.1. A general interval-valued (iv) overlap function is any mapping $IGO\colon L([0,\ 1])^n \to L([0,\ 1])$ that satisfies the following conditions, for all $X_1, \ldots, X_n \in L([0,1])$:

(IGO1) IGO is commutative;
(IGO2) If $\prod_{i=1}^n X_i = [0,0]$ then $IGO(X_1,\ldots,X_n) = [0,0]$;
(IGO3) If $\prod_{i=1}^n X_i = [1,1]$ then $IGO(X_1,\ldots,X_n) = [1,1]$;
(IGO4) IGO is increasing in the first component: $IGO(X_1,\ldots,X_n) \le_{Pr} IGO(Y, X_2,\ldots,X_n)$ when $X_1 \le_{Pr} Y$;
(IGO5) IGO is Moore continuous.

Example 4.3. The function defined, for all $X_1, \ldots, X_n \in L([0,1])$, as

$$IGO_1(X_1,\ldots,X_n) = \begin{cases} n\cdot IOn_L(X_1,\ldots,X_n) & \text{if } On_L(\overline{X_1},\ldots,\overline{X_n}) \le 1/n, \\ \left[n\cdot GO_L(\underline{X_1},\ldots,\underline{X_n}), 1\right] & \text{if } GO_L(\underline{X_1},\ldots,\underline{X_n}) \le 1/n \text{ and } GO_L(\overline{X_1},\ldots,\overline{X_n}) > 1/n, \\ [1,1], & \text{otherwise.} \end{cases}$$

is a general iv-overlap function, which is neither an n-dimensional iv-0-overlap function, nor an n-dimensional iv-1-overlap function. Then, it is neither an n-dimensional iv-overlap function.

The relations between iv-overlap functions, iv-0-overlap functions, iv-1-overlap functions and general iv-overlap functions are shown in Fig. 1.

It is immediate that:

Proposition 4.1. *If $F\colon L([0,\ 1])^n \to L([0,\ 1])$ is either an n-dimensional iv-overlap, iv-0-overlap or iv-1-overlap function, then F is also a general iv-overlap function.*

Theorem 4.1. *Let $GO_1, GO_2\colon [0,\ 1]^n \to [0,\ 1]$ be two general overlap functions such that $GO_1 \le GO_2$. Then, the function $\widehat{GO_1, GO_2}$ is an general iv-overlap function.*

Proof. Analogous to the proof of Theorem 3.2. □

Based on Theorem 4.1 and Proposition 4.1, one can obtain a representable general iv-overlap function by constructing it via either n-dimensional overlap, 0-overlap, 1-overlap or general overlap functions as its representatives. On the other hand, if an iv-general overlap function is representable, then its representatives must be general overlap functions.

Example 4.4. The representable general iv-overlap function IGO_M can be constructed by considering the general overlap function GO_M (Table 1) as its two representatives, given by

$$IGO_M(X_1,\ldots,X_n) = \widehat{GO_M}(X_1,\ldots,X_n).$$

Definition 4.2. Let IA: $L([0, 1])^n \to L([0, 1])$ be an n-dimensional interval-valued aggregation function. Then, IA is said to be *conjunctive* if $IA(X_1, \dots, X_n) \leq_{Pr} \inf(X_1, \dots, Xn)$, for any $X_1, \dots, X_n \in L([0, 1])$.

Definition 4.3. Let IA: $L([0, 1])^n \to L([0, 1])$ be an n-dimensional interval-valued aggregation function. Then, IA is said to be *disjunctive* if $IA(X_1, \dots, X_n) \geq_{Pr} \sup(X_1, \dots, Xn)$, for any $X_1, \dots, X_n \in L([0, 1])$.

Proposition 4.2. *For a commutative, Moore continuous n-dimensional interval-valued aggregation function* IA: $L([0, 1])^n \to L([0, 1])$, *it holds that:*

1. *If IA is conjunctive, then it is a general iv-overlap function;*
2. *If IA is disjunctive, then it is not a general iv-overlap function.*

Proof. Let IA: $L([0, 1])^n \to L([0, 1])$ be a commutative Moore continuous n-dimensional interval-valued aggregation function. It is immediate that IA satisfies conditions **(IGO1), (IGO4)** and **(IGO5)** from Definitions 4.1. Now, let us verify if it satisfies conditions **(IGO2)** and **(IGO3)** when IA is conjunctive and, in the sequence, when IA is disjunctive.

(i) Suppose that IA is conjunctive. If $X_1, \dots, X_n \in L([0,1])$ are such that $\prod_{i=1}^n X_i = [0,0]$, then there is at least one $X_i = [0,0]$, and so $IA(X_1, \dots, X_n) \leq_{Pr} \inf(X_1, \dots, X_n) = [0,0]$. Thus, IA satisfies condition **(IGO2)** as $IA(X_1, \dots, X_n) = [0,0]$. On the other hand, if $X_1, \dots, X_n \in L([0,1])$ are such that $\prod_{i=1}^n X_i = [1,1]$, then $X_i = [1,1]$ for each $i \in \{1, \dots, n\}$. So, by Definition 2.9 it holds that $IA(X_1, \dots, X_n) = [1,1]$, which means that IA also satisfies condition **(IGO3)**. Therefore, IA is a general iv-overlap function;

(ii) Suppose that IA is disjunctive. Then, one has that

$$IA([1,1],[0,0],\dots,[0,0]) \geq_{Pr} \sup([1,1],[0,0],\dots,[0,0]) = [1,1],$$

which contradicts condition **(IGO2)**. Thus, IA cannot be a general iv-overlap function. □

4.1. Characterization and construction methods of general iv-overlap functions

Inspired by the characterization of general overlap function presented in [15] and recalled previously by Theorem 2.1, here we present the development of a characterization for general iv-overlap functions, followed by some construction methods for this type of function.

First, we introduce some new related results that are necessary in the proofs concerning the characterization for general iv-overlap functions and their construction methods.

Lemma 4.1. *Consider* $A \in L([0, 1])$ *and* $L(A) = \{B \in L([0,1]) | A \leq_{Pr} B \text{ and } \underline{B} > 0\}$. *The function* Div_A: $L(A) \to L(A)$ *defined by* $Div_A(X) = A \div_H X$ *is Moore-continuous.*

Proof. Take $X \in L(A)$, $\epsilon > 0$ and consider $\delta = \underline{X}\epsilon$ and $Y \in L(A)$. If $d_{L(A)}(X, Y) \leq \delta$, that is,

$$\max(|\underline{X} - \underline{Y}|, |\overline{X} - \overline{Y}|) \leq \underline{X}\epsilon,$$

then one has that $|\underline{X} - \underline{Y}| \leq \underline{X}\epsilon$ and $|\underline{X} - \underline{Y}| \leq \underline{X}\epsilon \leq \overline{X}\epsilon$. So, by Example 2.1, one has that

$$|Div_{\underline{A}}(\underline{X}) - Div_{\underline{A}}(\underline{Y})| \leq \epsilon,$$
$$|Div_{\overline{A}}(\overline{X}) - Div_{\overline{A}}(\overline{Y})| \leq \epsilon.$$

Therefore, it holds that:

$$\begin{aligned} & d_{L([0,1])}(Div_A(X), Div_A(Y)) \\ &= \max(|\underline{Div_A(X)} - \underline{Div_A(Y)}|, |\overline{Div_A(X)} - \overline{Div_A(Y)}|) \\ &= \max(|\underline{A \div_H X} - \underline{A \div_H Y}|, |\overline{A \div_H X} - \overline{A \div_H Y}|) \\ &= \max(|\min(\underline{A}/\underline{X}, \overline{A}/\overline{X}) - \min(\underline{A}/\underline{Y}, \overline{A}/\overline{Y})|, |\max(\underline{A}/\underline{X}, \overline{A}/\overline{X}) - \max(\underline{A}/\underline{Y}, \overline{AY})|). \end{aligned}$$

Then, we have four possible cases: □

Case 1. If $\min(\underline{A}/\underline{X}, \overline{A}/\overline{X}) = \underline{A}/\underline{X}$ and $\min(\underline{A}/\underline{Y}, \overline{A}/\overline{Y}) = \underline{A}/\underline{Y}$, then it holds that

$$\begin{aligned} \max(\underline{A}/\underline{X}, \overline{A}/\overline{X}) &= \overline{A}/\overline{X}, \\ \max(\underline{A}/\underline{Y}, \overline{A}/\overline{Y}) &= \overline{A}/\overline{Y}. \end{aligned}$$

Therefore, it follows that

$$\begin{aligned} d_{L([0,1])}(Div_A(X), Div_A(Y)) &= \max(|\underline{A}/\underline{X} - \underline{A}/\underline{Y}|, |\overline{A}/\overline{X} - \overline{A}/\overline{Y}|) \\ &= \max(|Div_{\underline{A}}(\underline{X}) - Div_{\underline{A}}(\underline{Y})|, |Div_{\overline{A}}(\overline{X}) - Div_{\overline{A}}(\overline{Y})|) \\ &\leq \epsilon. \end{aligned}$$

Case 2. If $\min(\underline{A}/\underline{X}, \overline{A}/\overline{X}) = \underline{A}/\underline{X}$ and $\min(\underline{A}/\underline{Y}, \overline{A}/\overline{Y}) = \overline{A}/\overline{Y}$ then, one has that

$$\max(\underline{A}/\underline{X}, \overline{A}/\overline{X}) = \overline{A}/\overline{X},$$

$$\max(\underline{A}/\underline{Y}, \overline{A}/\overline{Y}) = \underline{A}/\underline{Y}.$$

Therefore, it holds that

$$d_{L([0,1])}(Div_A(X), Div_A(Y)) = \max(|\underline{A}/\underline{X} - \overline{A}/\overline{Y}|, |\overline{A}/\overline{X} - \underline{A}/\underline{Y}|).$$

It follows that:

(i) If $\underline{X} \leq \underline{Y} \leq \overline{X} \leq \overline{Y}$ then $\overline{A}/\overline{X} \geq \underline{A}/\underline{X} \geq \underline{A}/\underline{Y} \geq \overline{A}/\overline{Y}$. Then it holds that:

$$\begin{aligned} L([0,1])(Div_A(X), Div_A(Y)) &\leq \overline{A}/\overline{X} - \overline{A}/\overline{Y} \\ &= \max(|\underline{A}/\underline{X} - \underline{A}/\underline{Y}|, |\overline{A}/\overline{X} - \overline{A}/\overline{Y}|) \\ &= \max(|Div_{\underline{A}}(\underline{X}) - Div_{\underline{A}}(\underline{Y})|, |Div_{\overline{A}}(\overline{X}) - Div_{\overline{A}}(\overline{Y})|) \\ &\leq \epsilon. \end{aligned}$$

(ii) If $\underline{Y} \leq \underline{X} \leq \overline{Y} \leq \overline{X}$ then $\underline{A}/\underline{Y} \geq \overline{A}/\overline{Y} \geq \overline{A}/\overline{X} \geq \underline{A}/\underline{X}$. Thus, one has that:

$$\begin{aligned} d_{L([0,1])}(Div_A(X), Div_A(Y)) &\leq \underline{A}/\underline{Y} - \underline{A}/\underline{X} \\ &= \max(|\overline{A}/\overline{X} - \overline{A}/\overline{Y}|, |\underline{A}/\underline{X} - \underline{A}/\underline{Y}|) \\ &= \max(|Div_{\overline{A}}(\overline{X}) - Div_{\overline{A}}(\overline{Y})|, |Div_{\underline{A}}(\underline{X}) - Div_{\underline{A}}(\underline{Y})|) \\ &\leq \epsilon. \end{aligned}$$

(iii) If $\underline{X} \leq \underline{Y} \leq \overline{Y} \leq \overline{X}$ then $\overline{A}/\overline{Y} \geq \overline{A}/\overline{X} \geq \underline{A}/\underline{X} \geq \underline{A}/\underline{Y}$. However, since $\min(\underline{A}/\underline{Y}, \overline{A}/\overline{Y}) = \overline{A}/\overline{Y}$, then $\overline{A}/\overline{Y} = \overline{A}/\overline{X} = \underline{A}/\underline{X} = \underline{A}/\underline{Y}$. Therefore, it holds that

$$d_{L([0,1])}(Div_A(X), Div_A(Y)) = 0 < \epsilon.$$

(iv) If $\underline{Y} \leq \underline{X} \leq \overline{X} \leq \overline{Y}$ then $\overline{A}/\overline{X} \geq \overline{A}/\overline{Y}$ and $\underline{A}/\underline{Y} \geq \underline{A}/\underline{X}$. However, since $\min(\underline{A}/\underline{X}, \overline{A}/\overline{X}) = \underline{A}/\underline{X}$ and $\min(\underline{A}/\underline{Y}, \overline{A}/\overline{Y}) = \overline{A}/\overline{Y}$, we have four possibilities:

1. $\overline{A}/\overline{X} \geq \underline{A}/\underline{Y} \geq \underline{A}/\underline{X} \geq \overline{A}/\overline{Y}$;
2. $\overline{A}/\overline{X} \geq \underline{A}/\underline{Y} \geq \overline{A}/\overline{Y} \geq \underline{A}/\underline{X}$;
3. $\underline{A}/\underline{Y} \geq \overline{A}/\overline{X} \geq \underline{A}/\underline{X} \geq \overline{A}/\overline{Y}$;
4. $\underline{A}/\underline{Y} \geq \overline{A}/\overline{X} \geq \overline{A}/\overline{Y} \geq \underline{A}/\underline{X}$.

However, in all these cases, we have that

$$\max(|\underline{A}/\underline{X} - \overline{A}/\overline{Y}|, |\overline{A}/\overline{X} - \underline{A}/\underline{Y}|) \leq \max(|\overline{A}/\overline{X} - \overline{A}/\overline{Y}|, |\underline{A}/\underline{X} - \underline{A}/\underline{Y}|) \leq \epsilon.$$

Therefore, it holds that

$$d_{L([0,1])}(Div_A(X), Div_A(Y)) \leq \epsilon.$$

Case 3. If $\min(\underline{A}/\underline{X}, \overline{A}/\overline{X}) = \overline{A}/\overline{X}$ and $\min(\underline{A}/\underline{Y}, \overline{A}/\overline{Y}) = \underline{A}/\underline{Y}$, then the proof is analogous to **Case 2**.

Case 4. If $\min(\underline{A}/\underline{X}, \overline{A}/\overline{X}) = \overline{A}/\overline{X}$ and $\min(\underline{A}/\underline{Y}, \overline{A}/\overline{Y}) = \overline{A}/\overline{Y}$, then the proof is analogous to **Case 1**.

Corollary 4.1. *Let $A, B \in L([0,1])$ be such that $\underline{A \dotplus B} > 0$. Then the function $Div_H(A, B) = A \div_H (A \dotplus B)$ is Moore continuous.*

Proof. Clearly, $X \dotplus Y = \widehat{f}(X, Y)$, where f: $[0,1]^2 \to [0,1]$ is defined, for all $x, y \in [0,1]$, by $f(x, y) = \min(1, x + y)$, which is continuous. Then, by Theorem 5 in [7], $\dotplus$ is Moore continuous. Since, for each $A, B \in L([0,1])$, such that $\underline{A \dotplus B} > 0$, we have that $C = A \dotplus B \in L(A)$, then $Div_H(A, B) = Div_A(A \dotplus B)$ and, therefore, it is clear that Div_H is Moore continuous. □

Lemma 4.2. *For $a, b, c, d \in [0,1]$ such that $a \leq b$ and $c \geq d$, one has that*

$$\frac{a}{a \dotplus c} \leq \frac{b}{b \dotplus d}.$$

Proof. Suppose that $a, b, c, d \in [0,1]$ are such that $a \leq b$ and $c \geq d$. It is trivial that $ad \leq bc$, and then it holds that:

$$\begin{aligned} ad \leq bc &\Rightarrow (ad \leq bc) \vee (a \leq ab + bc) \\ &\Leftrightarrow (ab + ad \leq ab + bc) \vee (a \leq ab + bc) \\ &\Leftrightarrow \min(ab + ad, a) \leq ba + bc \\ &\Leftrightarrow \min(ab + ad, a) \leq \min(ba + bc, b) \\ &\Leftrightarrow \min(a(b + d), a) \leq \min(b(a + c), b) \\ &\Leftrightarrow a\min(b + d, 1) \leq b\min(a + c, 1) \\ &\Leftrightarrow \frac{a}{\min(a + c, 1)} \leq \frac{b}{\min(b + d, 1)} \end{aligned}$$

$$\Leftrightarrow \frac{a}{a\dotplus c} \leq \frac{b}{b\dotplus d}.$$

□

Now, we are ready to introduce the main results of this section.

Theorem 4.2. *The mapping IGO*: $L([0,1])^n \to L([0,1])$ *is an general iv-overlap function if and only if, for all* $X_1, \ldots, X_n \in L([0,1])$, *it holds that:*

$$IGO(X_1,\ldots,X_n) = F(X_1,\ldots,X_n) \div_H (F(X_1,\ldots,X_n) \dotplus G(X_1,\ldots,X_n)),$$

for some F, G: $L([0,1])^n \to L([0,1])$ *such that*

(i) *F and G are commutative;*
(ii) *If* $\prod_{i=1}^n X_i = [0,0]$ *then* $F(X_1,\ldots,X_n) = [0,0]$*;*
(iii) *If* $\prod_{i=1}^n X_i = [1,1]$ *then* $G(X_1,\ldots,X_n) = [0,0]$*;*
(iv) *F is* $\leq_{Pr}$*-increasing in the first component and G is* $\leq_{Pr}$*-decreasing in the first component;*
(v) *F and G are Moore continuous;*
(vi) $\underline{F(X_1,\ldots,X_n)\dotplus G(X_1,\ldots,X_n)} \neq 0$,

for any $X_1,\ldots,X_n \in L([0,1])$.

Proof. One has that:

($\Rightarrow$) Suppose that *IGO* is a general iv-overlap function, and consider $F(X_1,\ldots,X_n) = IGO(X_1,\ldots,X_n)$ and $G(X_1,\ldots,X_n) = [1 - \underline{IGO(X_1,\ldots,X_n)}, 1 - \underline{IGO(X_1,\ldots,X_n)}]$, for all $X_1,\ldots,X_n \in L([0,1])$. By Definition 4.1, it is immediate that conditions **(i)** - **(v)** hold. Furthermore, it holds that

$$F(X_1,\ldots,X_n)\dotplus G(X_1,\ldots,X_n) = IGO(X_1,\ldots,X_n)\dotplus[1 - \underline{IGO(X_1,\ldots,X_n)}, 1 - \underline{IGO(X_1,\ldots,X_n)}] = [1,1],$$

which means that $\underline{F(X_1,\ldots,X_n)\dotplus G(X_1,\ldots,X_n)} = 1$, respecting condition **(vi)**. Finally, it is clear that

$$IGO(X_1,\ldots,X_n) = F(X_1,\ldots,X_n) \div_H (F(X_1,\ldots,X_n)\dotplus G(X_1,\ldots,X_n)).$$

($\Leftarrow$) Consider that *F, G*: $L([0,1])^n \to L([0,1])$ satisfy the conditions **(i)-(vi)**. Let us show that

$$IGO(X_1,\ldots,X_n) = F(X_1,\ldots,X_n) \div_H (F(X_1,\ldots,X_n)\dotplus G(X_1,\ldots,X_n))$$

is a general iv-overlap function, or, in other words, that *IGO* is well defined and satisfies each condition from Definition 4.1. As $F(X_1,\ldots,X_n) \leq_{Pr} (F(X_1,\ldots,X_n)\dotplus G(X_1,\ldots,X_n))$, by Remark 2.1 it is clear that *IGO* is well defined. Now, let us verify if it satisfies each condition from Definition 4.1:

(IGO1) It is trivial as *F* and *G* are both commutative.

(IGO2) Let $X_1,\ldots,X_n \in L([0,1])$ be such that $\prod_{i=1}^n X_i = [0,0]$. From condition (ii) one has that

$$F(X_1,\ldots,X_n) = [0,0],$$

and from condition **(vi)** it holds that $\underline{F(X_1,\ldots,X_n)\dotplus G(X_1,\ldots,X_n)} \neq 0$. Thus, we have that:

$$F(X_1,\ldots,X_n) \div_H (F(X_1,\ldots,X_n)\dotplus G(X_1,\ldots,X_n)) = [0,0] \div_H G(X_1,\ldots,X_n).$$

Since $\underline{G(X_1,\ldots,X_n)} \neq 0$, then it follows that $IGO(X_1,\ldots,X_n) = [0,0]$.

(IGO3) Let $X_1,\ldots,X_n \in L([0,1])$ be such that $\prod_{i=1}^n X_i = [1,1]$. From condition **(iii)** one has that

$$G(X_1,\ldots,X_n) = [0,0].$$

Then, it holds that

$$F(X_1,\ldots,X_n) \div_H (F(X_1,\ldots,X_n)\dotplus G(X_1,\ldots,X_n)) = F(X_1,\ldots,X_n) \div_H (F(X_1,\ldots,X_n)\dotplus[0,0]).$$

As, from condition **(vi)**, one has that $\underline{F(X_1,\ldots,X_n)\dotplus G(X_1,\ldots,X_n)} \neq 0$, then it follows that

$$IGO(X_1,\ldots,X_n) = [1,1].$$

(IGO4) Let $X_1,\ldots,X_n, Y \in L([0,1])$ such that $X_1 \leq_{Pr} Y$. To simplify the notation, consider

$$F(X_1,\ldots,X_n) = A, F(Y,X_2,\ldots,X_n) = B, G(X_1,\ldots,X_n) = C \text{ and } G(Y,X_2,\ldots,X_n) = D.$$

From condition **(iv)** one has that $A \leq_{Pr} B$ and $D \leq_{Pr} C$. Then, by Lemma 4.2 it holds that

$$\frac{\underline{A}}{\underline{A}\dotplus\underline{C}} \leq \frac{\underline{B}}{\underline{B}\dotplus\underline{D}} \text{ and } \frac{\overline{A}}{\overline{A}\dotplus\overline{C}} \leq \frac{\overline{B}}{\overline{B}\dotplus\overline{D}}.$$

Thus, one has that:

$$\min\left(\frac{\underline{A}}{\underline{A}\dotplus\underline{C}}, \frac{\overline{A}}{\overline{A}\dotplus\overline{C}}\right) \leq \min\left(\frac{\underline{B}}{\underline{B}\dotplus\underline{D}}, \frac{\overline{B}}{\overline{B}\dotplus\overline{D}}\right)$$

and

$$\max\left(\frac{\underline{A}}{\underline{A}\dotplus\underline{C}}, \frac{\overline{A}}{\overline{A}\dotplus\overline{C}}\right) \le \max\left(\frac{\underline{B}}{\underline{B}\dotplus\underline{D}}, \frac{\overline{B}}{\overline{B}\dotplus\overline{D}}\right).$$

It follows that:

$$\left[\min\left(\frac{\underline{A}}{\underline{A}\dotplus\underline{C}}, \frac{\overline{A}}{\overline{A}\dotplus\overline{C}}\right), \max\left(\frac{\underline{A}}{\underline{A}\dotplus\underline{C}}, \frac{\overline{A}}{\overline{A}\dotplus\overline{C}}\right)\right] \le_{Pr} \left[\min\left(\frac{\underline{B}}{\underline{B}\dotplus\underline{D}}, \frac{\overline{B}}{\overline{B}\dotplus\overline{D}}\right), \max\left(\frac{\underline{B}}{\underline{B}\dotplus\underline{D}}, \frac{\overline{B}}{\overline{B}\dotplus\overline{D}}\right)\right].$$

However, one has that:

$$A \div_H (A\dotplus C) = \left[\min\left(\frac{\underline{A}}{\underline{A}\dotplus\underline{C}}, \frac{\overline{A}}{\overline{A}\dotplus\overline{C}}\right), \max\left(\frac{\underline{A}}{\underline{A}\dotplus\underline{C}}, \frac{\overline{A}}{\overline{A}\dotplus\overline{C}}\right)\right]$$

and

$$B \div_H (B\dotplus D) = \left[\min\left(\frac{\underline{B}}{\underline{B}\dotplus\underline{D}}, \frac{\overline{B}}{\overline{B}\dotplus\overline{D}}\right), \max\left(\frac{\underline{B}}{\underline{B}\dotplus\underline{D}}, \frac{\overline{B}}{\overline{B}\dotplus\overline{D}}\right)\right],$$

meaning that

$$A \div_H (A\dotplus C) \le_{Pr} B \div_H (B\dotplus D),$$

or in other words, $IGO(X_1, \ldots, X_n) \le_{Pr} IGO(Y, X_2, \ldots, X_n)$, proving that IGO is $\le_{Pr}$-increasing in the first component.
(IGO5) Straightforward from Corollary 4.1 and the fact that F, G are Moore continuous. □

Example 4.5. Let us apply the construction method presented in Theorem 4.2 to characterize some general iv-overlap functions through different pairs of functions F, G: $L([0,1])^n \to L([0,1])$ that satisfies conditions (i)-(vi).

1. For any F and G such that $G(X_1, \ldots, X_n) = [1 - \underline{F(X_1, \ldots, X_n)}, 1 - \underline{F(X_1, \ldots, X_n)}]$, for all $X_1, \ldots, X_n \in L([0,1])$, we have that $IGO = F$ is an general iv-overlap function.
2. Consider F and G defined, respectively, for all $X_1, \ldots, X_n \in L([0,1])$, by

$$F(X_1, \ldots, X_n) = \begin{cases} [0,0] & \text{if } \min(m(X_1), \ldots, m(X_n)) \le 0.5 \\ [2\mathrm{m}, 2\mathrm{m}] & \text{if } 0.5 \le \min(m(X_1), \ldots, m(X_n)) \le 1 \end{cases}$$

and

$$G(X_1, \ldots, X_n) = [\max(1 - m(X_1), \ldots, 1 - m(X_n)), \max(1 - m(X_1), \ldots, 1 - m(X_n))]$$

with $m(X) = 0.5(\underline{X} + \overline{X})$ and $\mathrm{m} = \min(m(X_1), \ldots, m(X_n)) - 0.5$. Then,

$$IGO(X_1, \ldots, X_n) = F(X_1, \ldots, X_n) \div_H (F(X_1, \ldots, X_n)\dotplus G(X_1, \ldots, X_n))$$

is an general iv-overlap function.

Proposition 4.3. *Given a general iv-overlap function* IGO: $L([0,1])^n \to L([0,1])$ *and a commutative, Moore continuous n-dimensional interval-valued aggregation function* IA: $L([0,1])^n \to L([0,1])$, *then the function* IGO_{IA}: $L([0,1])^n \to L([0,1])$, *defined, for all* $X_1, \ldots, X_n \in L([0,1])$, *by*

$$IGO_{IA}(X_1, \ldots, X_n) = IGO(X_1, \ldots, X_n) \cdot IA(X_1, \ldots, X_n)$$

is a general iv-overlap function.

Proof. It is immediate that IGO_{IA} is commutative **(IGO1)**, $\le_{Pr}$-increasing in the first component **(IGO4)** and Moore continuous **(IGO5)**, since IGO, IA and the interval product share those same properties. Now, let us verify if IGO_{IA} satisfies conditions **(IGO2)** and **(IGO3)**.

(IGO2) Consider $X_1, \ldots, X_n \in L([0,1])$ such that $\prod_{i=1}^n X_i = [0,0]$. Then, one has that

$$IGO(X_1, \ldots, X_n) = [0,0],$$

and, then,

$$IGO_{IA}(X_1, \ldots, X_n) = [0,0] \cdot IA(X_1, \ldots, X_n) = [0,0];$$

(IGO3) Consider $X_1, \ldots, X_n \in L([0,1])$ such that $\prod_{i=1}^n X_i = [1,1]$. Then, one has that $X_i = [1,1]$ for each $n \in \{1, \ldots, n\}$. Since IGO satisfies **(IGO3)** and IA is an n-dimensional interval-valued aggregation function, one has that

$$IGO_{IA}([1,1], \ldots, [1,1]) = IGO([1,1], \ldots, [1,1]) \cdot IA([1,1], \ldots, [1,1]) = [1,1] \cdot [1,1] = [1,1].$$

□

From Proposition 4.3, it is immediate that:

Corollary 4.2. *Given an n-dimensional iv-overlap function IOn*: $L([0,1])^n \to L([0,1])$ *and a commutative, Moore continuous n-dimensional interval-valued aggregation function IA*: $L([0,1])^n \to L([0,1])$, *one has that the function* IOn_{IA}: $L([0,1])^n \to L([0,1])$, *defined, for all* $X_1, \ldots, X_n \in L([0,1])$, *by*

$$IOn_{IA}(X_1, \ldots, X_n) = IOn(X_1, \ldots, X_n) \cdot IA(X_1, \ldots, X_n)$$

is a general iv-overlap function.

Example 4.6. Considering IOn_L as defined in Example 4.1, the function IGO_{PL}: $L([0,1])^n \to L([0,1])$ defined, for all $X_1, \ldots, X_n \in L([0,1])$, by

$$IGO_{PL}(X_1, \ldots, X_n) = \prod_{i=1}^{n} X_i \cdot IOn_L(X_1, \ldots, X_n),$$

is a general iv-overlap function, but not an n-dimensional iv-overlap function.

Theorem 4.3. *Let IA*: $L([0,1])^m \to L([0,1])$ *be a Moore continuous n-dimensional interval-valued aggregation function and* $\overrightarrow{IGO} = (IGO_1, \ldots, IGO_m)$ *a tuple of general iv-overlap functions. Then, the interval-valued function* $IA_{\overrightarrow{IGO}} : L([0,1])^n \to L([0,1])$, *defined, for all* $X_1, \ldots, X_n \in L([0,1])$, *by*

$$IA_{\overrightarrow{IGO}}(X_1, \ldots, X_n) = IA(IGO_1(X_1, \ldots, X_n), \ldots, IGO_m(X_1, \ldots, X_n))$$

is a general iv-overlap function.

Proof. It is immediate that $IA_{\overrightarrow{IGO}}$ is commutative **(IGO1)** and $\leq_{Pr}$-increasing in the first component **(IGO4)**. As it is also Moore continuous **(IGO5)**, let us prove that $IA_{\overrightarrow{IGO}}$ satisfies conditions **(IGO2)** and **(IGO3)**.

(IGO2) Consider $X_1, \ldots, X_n \in L([0,1])$ such that $\prod_{i=1}^{n} X_i = [0,0]$. Then, one has that

$$IGO_j(X_1, \ldots, X_n) = [0,0],$$

for all $j \in \{1, \ldots, m\}$, and, therefore,

$$IA(IGO_1(X_1, \ldots, X_n), \ldots, IGO_m(X_1, \ldots, X_n)) = IA([0,0], \ldots, [0,0]) = [0,0];$$

(IGO3) Consider $X_1, \ldots, X_n \in L([0,1])$ such that $\prod_{i=1}^{n} X_i = [1,1]$. Then, one has that $X_i = [1,1]$ for each $i \in \{1, \ldots, n\}$. Since IGO_j satisfies **(IGO3)** for all $j \in \{1, \ldots, m\}$, it holds that

$$IA(IGO_1([1,1], \ldots, [1,1]), \ldots, IGO_m([1,1], \ldots, [1,1])) = IA([1,1], \ldots, [1,1]) = [1,1].$$

□

It is immediate that:

Corollary 4.3. *Consider the tuple* $\overrightarrow{IGO} = (IGO_1, \ldots, IGO_m)$ *of general iv-overlap functions. Then, for* $w_1, \ldots, w_m \in [0,1]$ *such that* $w_1 + w_2 + \ldots + w_m = 1$, *the function given, for all* $X_1, \ldots, X_n \in L([0,1])$, *by*

$$SUM_{\overrightarrow{IGO}}(X_1, \ldots, X_n) = w_1 \cdot IGO_1(X_1, \ldots, X_n) + \ldots + w_m \cdot IGO_m(X_1, \ldots, X_n)$$

is a general iv-overlap function.

Proposition 4.1, it is immediate that:

Corollary 4.4. *Given a Moore continuous n-dimensional interval-valued aggregation function IA*: $L([0,1])^m \to L([0,1])$ *and the tuple of n-dimensional iv-overlap functions* $\overrightarrow{IOn} = (IOn_1, \ldots, IOn_m)$, *the interval-valued function* $IA_{\overrightarrow{IOn}} : L([0,1])^n \to L([0,1])$ *defined, for all* $X_1, \ldots, X_n \in L([0,1])$, *by*

$$IA_{\overrightarrow{IOn}}(X_1, \ldots, X_n) = IA(IOn_1(X_1, \ldots, X_n), \ldots, IOn_m(X_1, \ldots, X_n))$$

is a general iv-overlap function.

It is clear that Corollary 4.4 could be rewritten by swapping the tuple of n-dimensional iv-overlap functions by either a tuple of n-dimensional iv-0-overlap functions or a tuple of n-dimensional iv-1-overlap functions.

5. Interval-valued overlap index

The concept of overlap index has been used for measuring the degree of overlapping between two functions. To extend this approach to interval-valued functions, in this section we introduce the concept interval-valued overlap index. Furthermore, we present and analyze some construction methods for interval-valued overlap indices.

Definition 5.1. A mapping $\mathcal{IO} : IFS(U) \times IFS(U) \to L([0,1])$ is said to be an interval-valued (iv) overlap index if it respects the following conditions, for all $\mathcal{A}, \mathcal{B}, \mathcal{C} \in IFS(U)$:

($\mathcal{IO}$1) $\mathcal{IO}(\mathcal{A}, \mathcal{B}) = [0,0]$ if and only if for all $z \in U$, $\mathcal{A}(z) \cdot \mathcal{B}(z) = [0,0]$;
($\mathcal{IO}$2) $\mathcal{IO}(\mathcal{A}, \mathcal{B}) = \mathcal{IO}(\mathcal{B}, \mathcal{A})$;
($\mathcal{IO}$3) If $\mathcal{B} \leq \mathcal{C}$, meaning that $\mathcal{B}(z) \leq_{Pr} \mathcal{C}(z)$ for every $z \in U$, then $\mathcal{IO}(\mathcal{A}, \mathcal{B}) \leq_{Pr} \mathcal{IO}(\mathcal{A}, \mathcal{C})$,

An interval-valued overlap index is said to be normal, whenever it also satisfies the following condition:

($\mathcal{IO}$4) If there exists $z \in U$ such that $\mathcal{A}(z) \cdot \mathcal{B}(z) = [1,1]$, then $\mathcal{IO}(\mathcal{A}, \mathcal{B}) = [1,1]$.

Example 5.1. The function $\mathcal{IO}_\pi : IFS(U) \times IFS(U) \to L([0,1])$ given, for all $\mathcal{A}, \mathcal{B}, \mathcal{C} \in IFS(U)$, by:

$$\mathcal{IO}_\pi(\mathcal{A}, \mathcal{B}) = \left[\frac{1}{n} \sum_{i=1}^{n} \underline{A(z_i)} \cdot \underline{B(z_i)}, \frac{1}{n} \sum_{i=1}^{n} \overline{A(z_i)} \cdot \overline{B(z_i)} \right], \tag{2}$$

for $U = \{z_1, \ldots, z_n\}$, is an iv-overlap index.

Theorem 5.1. *Consider an n-dimensional interval-valued aggregation function IM: $L([0,\ 1])^n \to L([0,\ 1])$ such that, for all $X_1, \ldots, X_n \in L([0,1])$, it holds that:*

$$IM(X_1, \ldots, X_n) = [0,0] \Leftrightarrow X_1 = \ldots = X_n = [0,0],$$

and an iv-overlap function IO: $L([0,\ 1])^2 \to L([0,\ 1])$. Then, the function $\mathcal{IO}_{IM}^{IO} : IFS(U) \times IFS(U) \to L([0,1])$ given, for all $\mathcal{A}, \mathcal{B} \in IFS(U)$ and $U = \{z_1, \ldots, z_n\}$, by

$$\mathcal{IO}_{IM}^{IO}(\mathcal{A}, \mathcal{B}) = IM(IO(\mathcal{A}(z_1), \mathcal{B}(z_1)), \ldots, IO(\mathcal{A}(z_n), \mathcal{B}(z_n))),$$

is an iv-overlap index. Additionally, if $IM(X_1, \ldots, X_n) = [1,1]$ whenever $X_i = [1,1]$, for some $i \in \{1, \ldots, n\}$, meaning that IM has [1,1] as its nilpotent element, then $\mathcal{IO}_{IM}^{IO}$ is normal.

Proof.

($\mathcal{IO}$1) Suppose that for all $z \in U$, it holds that $\mathcal{A}(z) \cdot \mathcal{B}(z) = [0,0]$. Then, one has that $IO(\mathcal{A}(z_i), \mathcal{B}(z_i)) = [0,0]$, for $i \in \{1, \ldots, n\}$, meaning that $\mathcal{IO}_{IM}^{IO}(\mathcal{A}, \mathcal{B}) = IM([0,0], \ldots, [0,0]) = [0,0]$. Now, suppose that $\mathcal{IO}_{IM}^{IO}(\mathcal{A}, \mathcal{B}) = [0,0]$. Then, it follows that

$$IM(IO(\mathcal{A}(z_1), \mathcal{B}(z_1)), \ldots, IO(\mathcal{A}(z_n), \mathcal{B}(z_n))) = [0,0].$$

By the hypothesis, one has that

$$IM(IO(\mathcal{A}(z_1), \mathcal{B}(z_1)), \ldots, IO(\mathcal{A}(z_n), \mathcal{B}(z_n))) = [0,0] \Leftrightarrow IO(\mathcal{A}(z_i), \mathcal{B}(z_i)) = [0,0],$$

for every $i \in \{1, \ldots, n\}$. Thus, one has that $\mathcal{A}(z_i) \cdot \mathcal{B}(z_i) = [0,0]$, for every $i \in \{1, \ldots, n\}$.

($\mathcal{IO}$2) Immediate, since IO is commutative.

($\mathcal{IO}$3) Suppose that $\mathcal{B} \leq \mathcal{C}$. Since both IM and IO are $\leq_{Pr}$-increasing, it follows that

$$\mathcal{IO}_{IM}^{IO}(\mathcal{A}, \mathcal{B}) \leq_{Pr} \mathcal{IO}_{IM}^{IO}(\mathcal{A}, \mathcal{C});$$

($\mathcal{IO}$4) Suppose that IM has [1,1] as its nilpotent element and that there exists $z_i \in U$ such that $\mathcal{A}(z_i) \cdot \mathcal{B}(z_i) = [1,1]$, $i \in \{1, \ldots, n\}$. Then, we have that

$$IO(\mathcal{A}(z_i), \mathcal{B}(z_i)) = [1,1] \Rightarrow IM((IO(\mathcal{A}(z_1), \mathcal{B}(z_1)), \ldots, [1,1], \ldots, IO(\mathcal{A}(z_n), \mathcal{B}(z_n)))) = [1,1],$$

meaning that $\mathcal{IO}_{IM}^{IO}$ is normal. □

Proposition 5.1. *Let $\mathcal{O}_1, \mathcal{O}_2 : FS(U) \times FS(U) \to [0,1]$ be two overlap indices such that $\mathcal{O}_1 \leq \mathcal{O}_2$. Then, the mapping $\widetilde{\mathcal{O}_1, \mathcal{O}_2} : IFS(U) \times IFS(U) \to L([0,1])$ defined, for all $\mathcal{A}, \mathcal{B} \in IFS(U)$, by*

$$\widetilde{\mathcal{O}_1, \mathcal{O}_2}(\mathcal{A}, \mathcal{B}) = [\mathcal{O}_1(A_l, B_l), \mathcal{O}_2(A_u, B_u)]$$

is an iv-overlap index. If $\mathcal{O}_1$ and $\mathcal{O}_2$ are both normal overlap indices, then $\widetilde{\mathcal{O}_1, \mathcal{O}_2}$ is also normal.

Proof. Since $\mathcal{O}_1 \leq \mathcal{O}_2$, it is immediate that $\widetilde{\mathcal{O}_1, \mathcal{O}_2}$ is well defined. Now, let us see if all conditions from Definition 5.1 are met.

$$\begin{tikzcd} O \arrow[r, "Th.\ 2.3"] \arrow[d, "Th.\ 2.2"'] & \mathcal{O} \arrow[d, "Prop.\ 5.1"] \\ IO \arrow[r, "Th.\ 5.1"'] & \mathcal{IO} \end{tikzcd}$$

Fig. 2. Commutative diagram of the construction methods of an iv-overlap index $\mathcal{IO}$ based on an overlap function O.

($\mathcal{IO}$1) Suppose that for all $z \in U$, it holds that $\mathcal{A}(z) \cdot \mathcal{B}(z) = [0, 0]$. Then, for all $z \in U$, one has that:

$$\begin{aligned} [A_l(z) \cdot B_l(z), A_u(z) \cdot B_u(z)] = [0, 0] &\Leftrightarrow A_l(z) \cdot B_l(z) = 0 \ \wedge \ A_u(z) \cdot B_u(z) = 0 \\ &\Leftrightarrow \mathcal{O}_1(A_l(z), B_l(z)) = 0 \ \wedge \ \mathcal{O}_2(A_u(z), B_s(z)) = 0 \\ &\Leftrightarrow [\mathcal{O}_1(A_l, B_l), \mathcal{O}_2(A_u, B_u)] = [0, 0] \\ &\Leftrightarrow \widetilde{\mathcal{O}_1, \mathcal{O}_2}(\mathcal{A}, \mathcal{B}) = [0, 0]. \end{aligned}$$

($\mathcal{IO}$2) Immediate, as $\mathcal{O}_1$ and $\mathcal{O}_2$ are commutative.

($\mathcal{IO}$3) Suppose that $\mathcal{B} \leq \mathcal{C}$. Since both $\mathcal{O}_1$ and $\mathcal{O}_2$ are increasing, it follows that $\widetilde{\mathcal{O}_1, \mathcal{O}_2}(\mathcal{A}, \mathcal{B}) \leq_{Pr} \widetilde{\mathcal{O}_1, \mathcal{O}_2}(\mathcal{A}, \mathcal{C})$.

($\mathcal{IO}$4) Suppose that there exists $z \in U$ such that $\mathcal{A}(z) \cdot \mathcal{B}(z) = [1, 1]$ and that $\mathcal{O}_1$ and $\mathcal{O}_2$ are both normal overlap indices. Then, one has that:

$$\begin{aligned} A_l(z) \cdot B_l(z) = 1 \ \wedge \ A_u(z) \cdot B_u(z) = 1 &\Rightarrow [\mathcal{O}_1(A_l, B_l), \mathcal{O}_2(A_u, B_u)] = [1, 1] \\ &\Rightarrow \widetilde{\mathcal{O}_1, \mathcal{O}_2}(\mathcal{A}, \mathcal{B}) = [1, 1], \end{aligned}$$

meaning that $\widetilde{\mathcal{O}_1, \mathcal{O}_2}$ is normal. □

In what follows, denote $\widetilde{\mathcal{O}, \mathcal{O}}$ simply by $\widetilde{\mathcal{O}}$.

Theorem 5.2. *Let O: $[0, 1]^2 \to [0, 1]$ be an overlap function, $U = \{z_1, \ldots, z_n\}$ a finite set and M: $[0, 1]^n \to [0, 1]$ an aggregation function such that, for all $x_1, \ldots, x_n \in [0, 1]$, it holds that*

$$M(x_1, \ldots, x_n) = 0 \Leftrightarrow x_1 = \ldots = x_n = 0. \tag{3}$$

Then, one has that

$$\mathcal{IO}_{\widehat{M}}^{\widehat{O}}(\mathcal{A}, \mathcal{B}) = \widetilde{\mathcal{O}_M^O}(\mathcal{A}, \mathcal{B}),$$

for all $\mathcal{A}, \mathcal{B} \in IFS(U)$.

Proof. Consider $\mathcal{A}, \mathcal{B} \in IFS(U)$ such that $U = \{z_1, \ldots, z_n\}$, an overlap function O: $[0, 1]^2 \to [0, 1]$ and let M: $[0, 1]^n \to [0, 1]$ be an aggregation function satisfying the Condition (3). Since, by Definition 2.4, one has that $\underline{\widehat{O}(\mathcal{A}(z), \mathcal{B}(z))} = O(A_l(z), B_l(z))$ and $\overline{\widehat{O}(\mathcal{A}(z), \mathcal{B}(z))} = O(A_u(z), B_u(z))$, then it holds that:

$$\begin{aligned} &\mathcal{IO}_{\widehat{M}}^{\widehat{O}}(\mathcal{A}, \mathcal{B}) \\ &= \widehat{M}(\widehat{O}(\mathcal{A}(z_1), \mathcal{B}(z_1)), \ldots, \mathcal{A}(z_n), \mathcal{B}(z_n)) \\ &= \left[M(\underline{\widehat{O}(\mathcal{A}(z_1), \mathcal{B}(z_1))}, \ldots, \underline{\widehat{O}(\mathcal{A}(z_n), \mathcal{B}(z_n))}), M(\overline{\widehat{O}(\mathcal{A}(z_1), \mathcal{B}(z_1))}, \ldots, \overline{\widehat{O}(\mathcal{A}(z_n), \mathcal{B}(z_n))}) \right] \\ &= [M(O(A_l(z_1), B_l(z_1)), \ldots, O(A_l(z_n), B_l(z_n))), M(O(A_u(z_1), B_u(z_1)), \ldots, O(A_u(z_n), B_u(z_n)))] \\ &\qquad \text{by Definition 2.4} \\ &= \left[\mathcal{O}_M^O(A_l, B_l), \mathcal{O}_M^O(A_u, B_u) \right] \text{ by Theorem 2.3} \\ &= \widetilde{\mathcal{O}_M^O}(\mathcal{A}, \mathcal{B}) \text{ by Proposition 5.1,} \end{aligned}$$

which completes the proof. □

Observe that, based on Theorem 5.2, the diagram presented in Fig. 2 commutes.

Example 5.2. Let us consider the set $U = \{z_1, \ldots, z_n\}$ and the overlap function O_p: $[0, 1]^2 \to [0, 1]$ defined,for all $x, y \in [0, 1]$, by

$$O_p(x, y) = xy,$$

as our basis for constructing the iv-overlap $\mathcal{IO}_\pi$ presented in Example 5.1. The following construction methods are viable:

(1) First, we build an o-representable iv-overlap IO_p: $L([0, 1])^2 \to L([0, 1])$ by adopting O_p as the both representatives. Thus, for all $X, Y \in L([0, 1])$, one has that:

$$IO_p(X, Y) = \widehat{O_p}(X, Y).$$

Next, we define the n-dimensional interval-valued aggregation IM_A: $L([0,1])^n \to L([0,1])$, for all $X_1, \ldots, X_n \in L([0,1])$, as

$$IM_A(X_1, \ldots, X_n) = \left[\frac{1}{n} \sum_{i=1}^{n} \underline{X_i}, \frac{1}{n} \sum_{i=1}^{n} \overline{X_i} \right].$$

It is clear that

$$IM_A(X_1, \ldots, X_n) = [0,0] \Leftrightarrow X_1 = \ldots = X_n = [0,0].$$

So, we may obtain $\mathcal{IO}_\pi : IFS(U) \times IFS(U) \to L([0,1])$ by Theorem 5.1 as:

$$\mathcal{IO}_\pi(\mathcal{A}, \mathcal{B}) = \mathcal{IO}^{IO_p}_{IM_A}(\mathcal{A}, \mathcal{B}) = IM_A(IO_p(\mathcal{A}(z_1), \mathcal{B}(z_1)), \ldots, IO_p(\mathcal{A}(z_n), \mathcal{B}(z_n))), \tag{4}$$

for all $\mathcal{A}, \mathcal{B} \in IFS(U)$. By rewriting Eq. (4), we obtain the same iv-overlap index as defined in Eq. (2) in Example 5.1.

(2) In this method, first we build an overlap index $\mathcal{O}_\pi$ through Theorem 2.3. For that end, we consider the arithmetic mean as the aggregation function M_A, as it holds that

$$M_A(x_1, \ldots, x_n) = 0 \Leftrightarrow x_1 = \ldots = x_n = 0.$$

Thus, $\mathcal{O}_\pi : FS(U) \times FS(U) \to [0,1]$ is defined, for all $\mathcal{A}, \mathcal{B} \in IFS(U)$, as

$$\mathcal{O}_\pi(A, B) = M_A(O_p(A(z_1), B(z_1)), \ldots, O_p(A(z_n), B(z_n))).$$

Now, through Proposition 5.1, we construct an iv-overlap index $\mathcal{IO}_\pi : IFS(U) \times IFS(U) \to L([0,1])$, defined, for all $\mathcal{A}, \mathcal{B} \in IFS(U)$, by

$$\mathcal{IO}_\pi(\mathcal{A}, \mathcal{B}) = \widetilde{\mathcal{O}_\pi} = [\mathcal{O}_\pi(A_l, B_l), \mathcal{O}_\pi(A_s, B_u)], \tag{5}$$

which also coincides with Eq. (2) in Example 5.1.

6. An illustrative example

This section is aimed at presenting an illustrative example of the application of general iv-overlap functions and iv-overlap indices in classification problems using IV-FRBCSs, specifically IVTURS (Interval-Valued Fuzzy Reasoning Method with Tuning and Rule Selection) [46] as an state-of-the-art IV-FRBCS. Firstly, we recall the main concepts on FRBCSs and the interval-valued fuzzy reasoning method.

6.1. Fuzzy rule-based classification systems

A classification problem is composed by P training examples $\vec{x_p} = (x_{p1}, \ldots, x_{pn})$, $p \in \{1, \ldots, P\}$, where x_{pi} is the value of the i-th variable of the p-th training example. Each of these examples belongs to one of the M classes $y_p \in C = \{C_1, \ldots, C_M\}$. The goal of the learned classifier is to identify the class of new, unknown, testing examples.

FRBCSs are one of the most frequently adopted techniques to deal with classification problems. Through the usage of linguistic labels in their rules, they provide an interpretable model while still achieving accurate results [47]. The following structure is adopted for the fuzzy rules:

$$\text{Rule } R_j : \text{If } x_1 \text{ is } A_{j1} \text{ and } \ldots \text{ and } x_n \text{ is } A_{jn} \text{ then } Class = C'_j \text{ with } RW_j, \tag{6}$$

where R_j is the label of the j-th rule, $x = (x_1, \ldots, x_n)$ is an n-dimensional example vector, A_{ji} is the fuzzy set representing the linguistic term of the j-th rule in the i-th antecedent, $C'_j \in C$ is a class label, and $RW_j \in [0, 1]$ is the rule weight [31]. Specifically, we consider the computation of the rule weight using the fuzzy confidence value or certainty factor through overlap indices, as defined in [24] as

$$Cnf(R_j) = \frac{\mathcal{O}(\mathbb{C}_{js}, U)}{\mathcal{O}(\mathbb{C}_{jp}, U)}, \tag{7}$$

where $U = \{\vec{x_1}, \ldots, \vec{x_P}\}$ with $U(\vec{x_i}) = 1$, for all $i \in \{1, \ldots, P\}$, $\mathbb{C}_{jp}$ is the fuzzy set built with the matching degrees of all P examples, $\mathbb{C}_{js} \subseteq \mathbb{C}_{jp}$ is the fuzzy set built with the matching degrees of $s \leq P$ examples whose class is associated with the j-th rule and $\mathcal{O}$ is an overlap index.

Considering $\mathcal{O} = \mathcal{O}_\pi$, given in Eq. (1), in [24] it was obtained the classical confidence used for the rule weight:

$$RW_j = \frac{\sum_{x_p \in C'_j} A_j(x_p)}{\sum_{p=1}^{P} A_j(x_p)},$$

where $A_j(x_p)$ is the matching degree of the pattern x_p with the antecedent part of the fuzzy rule R_j, computed as

$$A_j(x_p) = T(A_{j1}(x_{p1}), \cdots, A_{jn}(x_{pn})),$$

with T being a conjunction operator (t-norm) and $j \in \{1, \ldots, L\}$.

In this paper, we apply the IVTURS algorithm to obtain the IV-FRBCS. We must point out that, in this case, the interval fuzzy rules follow a similar structure as in Eq. (6), with the linguistic labels A_{ji} being modeled using triangular shaped interval-valued membership functions and the rule weight is also an interval, now denoted by IRW_j. Furthermore, the interval-valued fuzzy reasoning method considers the ignorance degree represented by the IVFSs throughout the inference process. For an in-depth look at each step of the IVTURS algorithm, see [46].

6.2. Applying the new concepts on the inference process

After obtaining the interval-valued fuzzy rules, let us develop a method for classifying new examples. Thus, let $\vec{x_p} = (x_{p1}, \ldots, x_{pn})$ be a new example to be classified, L being the number of rules in the rule base and M being the number of classes of the problem. The new interval-valued fuzzy reasoning method can be defined by the following steps:

(1) Interval matching degree: First, we measure the similarity between the interval membership degrees (of each variable to the corresponding IVFS)and the ideal membership degree [1,1] through an IV-REF IR (see Definition 2.3). Then, we apply a general iv-overlap function IGO (instead of applying an interval-valued t-norm as in IVTURS), for $j \in \{1, \ldots, L\}$ as follows:

$$\left[\underline{\mathcal{A}_j(x_p)}, \overline{\mathcal{A}_j(x_p)}\right] = IGO\left(IR\left(\left[\underline{\mathcal{A}_{j1}(x_{p1})}, \overline{\mathcal{A}_{j1}(x_{p1})}\right], [1,1]\right), \ldots, IR\left(\left[\underline{\mathcal{A}_{jn}(x_{pn})}, \overline{\mathcal{A}_{jn}(x_{pn})}\right], [1,1]\right)\right).$$

The interval matching degree represents the strength of the activation of the if-part of the rules for each x_p.

(2) Interval association degree: For the class of each rule, the interval matching degree is weighted with the corresponding iv-rule weight $IRW_j^k \in L([0,1])$, resulting in the following expression:

$$\left[\underline{b_j^k}, \overline{b_j^k}\right] = \left[\underline{\mathcal{A}_j(x_p)}, \overline{\mathcal{A}_j(x_p)}\right] \cdot \left[\underline{IRW_j^k}, \overline{IRW_j^k}\right] \text{with } k = 1, \ldots, M \text{ and } j = 1, \ldots, L.$$

Here, we obtain the iv-rule weight for the j-rule R_j through an interval-valued fuzzy confidence, which we define by:

$$ICnf(R_j) = \mathcal{IO}(\mathcal{C}_{js}, \mathcal{U}) \div_H \mathcal{IO}(\mathcal{C}_{jP}, \mathcal{U}),$$

where $\mathcal{U} = \{\vec{x_1}, \ldots, \vec{x_P}\}$ with $\mathcal{U}(\vec{x_i}) = [1,1]$, for all $i \in \{1, \ldots, P\}$, $\mathcal{IO}$ is an iv-overlap index, $\mathcal{C}_{jP}$ is the interval fuzzy set built with the interval matching degrees of all P examples and $\mathcal{C}_{js} \subseteq \mathcal{C}_{jP}$ is the interval fuzzy set built with the interval-valued matching degrees of $s \leq P$ examples whose class is associated with the j-th rule.

The resulting expression for the iv-rule weight based on the interval-valued fuzzy confidence value when

$$\mathcal{IO} = \mathcal{IO}_\pi,$$

given in Eq. (2), is:

$$IRW_j = \sum_{x_p \in C_j'} \left[\underline{\mathcal{A}_j(x_p)}, \overline{\mathcal{A}_j(x_p)}\right] \div_H \sum_{p=1}^{P} \left[\underline{\mathcal{A}_j(x_p)}, \overline{\mathcal{A}_j(x_p)}\right].$$

The remaining steps of the IV-FRM are the same as those used in IVTURS.

6.3. Setup of the experiment

To analyze the behaviour of a classification system when applying general iv-overlap functions and iv-overlap indices, we have selected 31 real-world data-sets from the KEEL repository [1], which are publicly available on the webpage (http://www.keel.es/dataset.php) with all the relevant information about them. In Table 2, one can find some properties of the selected data-sets, such as number of attributes (Atts.), number of examples (Ex.), and the number of classes (Class.). It is noteworthy that the *magic, page-blocks, penbased, ring, satimage, shuttle*, and *twonorm* data-sets have been stratified sampled at 10% to improve the efficiency of the learning process. Missing values from *bands, cleveland* and *wisconsin* data-sets have been removed before the experimentation.

A *fivefold cross-validation model* has been applied in order to carry out the different experiments. This was done by splitting the data-set into five random partitions of data, employing a combination of four of them (80%) to train the system and the remaining one (20%) to test it. This process was executed 5 times, changing the testing partition in each iteration. The performance measure was done through the accuracy rate.

The configuration of the IVTURS classifier is the same as in [46], but we also apply the general iv-overlap functions shown in Table 3, as the conjunction operator (the definitions for GO_L, GO_{GM} and GO_{HM} can be seen in Table 1). When the chosen operation is the interval product (iv-Prod), then we have the original IVTURS algorithm.

Observe that the functions iv-Luk, iv-ProdLuk, iv-GmLuk and iv-HmLuk are also n-dimensional 0-iv-overlap functions, and when this type of aggregation is applied as the conjunction operator, examples with a low matching degree with the antecedent part of a fuzzy rule are not taken into account in the system, which is why those functions were selected for this example. The remaining ones (iv-Prod, iv-GM and iv-HM) were included just for comparison sake, as they were the

Table 2
Summary of the employed datasets.

id	Data-set	Atts.	Ex.	Class.
app	appendicitis	7	106	2
bal	balance	4	625	3
ban	banana	2	5300	2
bds	bands	19	365	2
bup	bupa	6	345	2
clv	cleveland	13	297	5
con	contraceptive	9	1473	3
eco	ecoli	7	336	8
gla	glass	9	214	7
hab	haberman	3	306	2
hay	hayes-hoth	4	160	3
ion	ionosphere	33	351	2
iri	iris	4	150	3
led	led7digit	7	500	10
mag	magic	10	19020	2
new	newthyroid	5	215	3
pag	pageblocks	10	5472	5
pen	penbased	16	10992	10
pho	phoneme	5	5404	2
pim	pima	8	768	2
rin	ring	20	7400	2
sah	saheart	9	462	2
sat	satimage	36	6435	7
shu	shuttle	9	58000	7
spe	spectfheart	44	267	2
tit	titanic	3	2201	2
two	twonorm	20	7400	2
veh	vehicle	18	846	4
win	wine	13	178	3
wis	wisconsin	9	683	2
yea	yeast	8	1484	10

Table 3
General iv-overlap functions used in the application.

General iv-overlap function identifier	Definition
iv-Prod	$IGO_p(X_1,\ldots,X_n) = \prod_{i=1}^n X_i$
iv-Luk	$IGO_L(X_1,\ldots,X_n) = \widehat{GO_L}(X_1,\ldots,X_n)$
iv-GM	$IGO_{GM}(X_1,\ldots,X_n) = \widehat{GO_{GM}}(X_1,\ldots,X_n)$
iv-HM	$IGO_{HM}(X_1,\ldots,X_n) = \widehat{GO_{HM}}(X_1,\ldots,X_n)$
iv-ProdLuk	$IGO_{pL}(X_1,\ldots,X_n) = IGO_p(X_1,\ldots,X_n) \cdot IGO_L(X_1,\ldots,X_n)$
iv-GmLuk	$IGO_{GmL}(X_1,\ldots,X_n) = IGO_{GM}(X_1,\ldots,X_n) \cdot IGO_L(X_1,\ldots,X_n)$
iv-HmLuk	$IGO_{HmL}(X_1,\ldots,X_n) = IGO_{HM}(X_1,\ldots,X_n) \cdot IGO_L(X_1,\ldots,X_n)$

functions that were combined through the construction method presented in Proposition 4.3 to obtain iv-ProdLuk, iv-GmLuk and iv-HmLuk.

In order to have some statistical support in our example, we use the aligned Friedman ranks test [29] to detect statistical differences among a group of results and report the obtained ranks of each algorithm (lower ranks are preferable). Next, we applied the Holm's post-hoc test [30] to compare the best ranking method with the other considered algorithms. We follow the suggestion to use these tests from [26], where it is shown that they are strongly recommended to be employed in machine learning.

6.4. Discussion of the results

The values presented in Table 4 are the average among the 5 testing results, where the best result for each data-set is highlighted in **bold-face**. We can observe that all the general iv-overlap functions obtain similar results to those achieved by the original IVTURS (iv-Prod). In order to give statistical support to the previous findings we have applied the aligned rank test, whose ranks as well as the adjusted p-values (APVs) provided by the Holm's post hoc test are shown in Table 5, with the best ranking method highlighted in **bold-face**.

From the statistical tests we see that the global results when applying general-iv overlap functions are comparable to those of the original IVTURS algorithm, as the ranks are close and every APV is equal to 1. The iv-HmLuk algorithm presented a higher global accuracy, although no operation has significantly improved the system globally, as none has statistically decreased the overall performance considering all data-sets.

Table 4
Results in testing for the different methods.

Data-set	iv-Prod	iv-Luk	iv-GM	iv-HM	iv-ProdLuk	iv-GmLuk	iv-HmLuk
app	84.89	83.94	83.98	84.89	83.98	**86.80**	83.98
bal	85.76	86.56	**86.72**	86.56	**86.72**	85.12	86.24
ban	81.45	81.79	80.68	**83.17**	81.89	82.21	82.49
bds	66.32	67.74	69.05	69.41	67.99	**70.8**	68.82
bup	64.06	**66.67**	63.77	62.32	63.77	64.06	64.06
clv	58.58	57.91	58.93	57.58	**59.25**	**59.25**	57.91
con	53.36	**53.91**	52.48	52.41	53.23	53.30	53.57
eco	80.96	79.74	79.78	78.57	**81.84**	80.07	80.37
gla	68.72	69.19	**71.97**	64.97	63.12	70.10	68.70
hab	72.85	75.47	72.19	71.89	75.47	74.16	**75.80**
hay	79.46	79.46	79.46	79.46	79.46	79.46	79.46
ion	92.04	91.74	92.03	91.76	**93.17**	91.18	92.61
iri	95.33	96.00	96.00	**96.67**	96.00	95.33	95.33
led	70.60	70.60	70.60	70.60	70.60	70.60	70.60
mag	79.91	80.23	80.39	79.97	80.02	**80.97**	79.44
new	97.21	**97.67**	95.81	96.74	96.28	96.74	95.81
pag	93.79	94.15	93.79	94.34	**94.70**	93.79	94.34
pen	92.18	91.36	91.18	**92.82**	91.46	91.27	91.73
pho	80.42	80.68	80.53	80.81	80.63	**80.92**	80.64
pim	74.61	73.95	**75.38**	73.70	73.95	74.48	74.86
rin	**90.81**	90.14	90.27	90.41	**90.81**	90.41	**90.81**
sah	70.13	**71.65**	70.33	70.77	69.91	70.15	69.70
sat	76.21	76.52	76.52	**77.30**	76.06	76.21	77.14
shu	90.30	91.54	93.29	**93.52**	91.13	91.13	91.82
son	**81.74**	79.37	79.36	81.25	80.31	78.40	80.29
spe	79.38	78.28	79.40	79.41	**80.51**	79.39	79.76
tit	78.87	78.87	78.87	78.87	78.87	78.87	78.87
two	**93.65**	93.51	93.38	93.24	92.57	92.43	93.24
veh	66.20	65.37	**67.61**	66.075	65.48	66.67	65.96
win	**97.19**	94.95	95.52	96.60	96.59	95.51	96.60
wis	96.34	96.63	**97.07**	96.49	96.63	96.34	96.05
yea	55.32	55.05	**57.34**	55.12	55.66	53.64	56.87
Mean	79.65	79.71	79.80	79.62	79.63	79.68	**79.81**

Table 5
Average Rankings of the algorithms (Aligned Friedman).

Algorithm	Rank	APV
iv-Prod (IVTURS)	113.66	1
iv-Luk	119.64	1
iv-GM	114.52	1
iv-HM	109.14	1
iv-ProdLuk	111.92	1
iv-GmLuk	119.94	1
iv-HmLuk	**98.69**	-

However, observe that IVTURS has the best performance in only 4 of the 32 employed data-sets (12.5%). In some datasets like *bands* or *haberman*, we have an enhancement of the results provided by IVTURS: 1) from 66.32% to 70.8% using the iv-GmLuk in *bands* and 2) from 72.85% to 75.8% applying iv-HmLuk in haberman. This shows that there is a good chance that some general iv-overlap function based method can be better suited for a given data-set, depending on the context of the problem at hand. This experimentation example illustrates the adaptability of general iv-overlap functions, as they can be constructed and applied in different ways, producing competitive results in classification systems, as they were comparable to the results of IVTURS.

7. Conclusion

In this paper, we introduced important concepts to overcome the limitations of the applicability of iv-overlap functions in n-dimensional problems with interval-valued data. First, we developed the concept and studied the representability of n-dimensional iv-overlap functions, also introducing the concept *o*-representable functions. Then, we presented some generalizations on the definition of n-dimensional iv-overlap functions, leading to the definition, characterization and construction methods for general iv-overlap functions.

The concept of iv-overlap index was also introduced, followed by the study of some construction methods, where we observed that from an overlap function we can obtain an iv-overlap index, either by an aggregation of representable iv-overlap functions or by a pair of overlap indices (Fig. 2).

We showed an illustrative example regarding an application in classification, through a new interval-valued fuzzy reasoning method in which we apply both general iv-overlap functions and iv-overlap indices.

A future research line could be to study whether datasets have properties of the data that make some general iv-overlap functions more suitable to be applied than others, which could lead to the development of a powerful classifier. Ongoing theoretical work includes the study of similar concepts presented in this paper in the context of admissible orders [13].

Declaration of Competing Interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

CRediT authorship contribution statement

Tiago da Cruz Asmus: Writing - original draft, Conceptualization, Methodology, Investigation, Software, Writing - review & editing. **Graçaliz Pereira Dimuro:** Conceptualization, Methodology, Writing - review & editing. **Benjamín Bedregal:** Conceptualization, Methodology, Writing - review & editing. **José Antonio Sanz:** Investigation, Software, Writing - review & editing. **Sidnei Pereira Jr.:** Methodology, Investigation. **Humberto Bustince:** Conceptualization, Writing - review & editing, Supervision, Project administration.

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5.1.2 N-dimensional admissibly ordered interval-valued overlap functions and its influence in interval-valued fuzzy rule-based classification systems

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N-Dimensional Admissibly Ordered Interval-valued Overlap Functions and its Influence in Interval-valued Fuzzy Rule-based Classification Systems

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Abstract*—Overlap functions are a type of aggregation functions that are not required to be associative, generally used to indicate the overlapping degree between two values. They have been successfully used as a conjunction operator in several practical problems, such as fuzzy rule-based classification systems (FRBCSs) and image processing. Some extensions of overlap functions were recently proposed, such as general overlap functions and, in the interval-valued context, n-dimensional interval-valued overlap functions. The latter allow them to be applied in n-dimensional problems with interval-valued inputs, like interval-valued classification problems, where one can apply interval-valued FRBCSs (IV-FRBCSs). In this case, the choice of an appropriate total order for intervals, like an admissible order, can play an important role. However, neither the relationship between the interval order and the n-dimensional interval-valued overlap function (which may or may not be increasing for that order) nor the impact of this relationship in the classification process have been studied in the literature. Moreover, there is not a clear preferred n-dimensional interval-valued overlap function to be applied in an IV-FRBCS. Hence, in this paper we: (i) present some new results on admissible orders, which allow us to introduce the concept of n-dimensional admissibly ordered interval-valued overlap functions, that is, n-dimensional interval-valued overlap functions that are increasing with respect to an admissible order; (ii) develop a width-preserving construction method for this kind of function, derived from an admissible order and an n-dimensional overlap function, discussing some of its features; (iii) analyze the behaviour of several combinations of admissible orders and n-dimensional (admissibly ordered) interval-valued overlap functions when applied in IV-FRBCSs. All in all, the contribution of this paper resides in pointing out the effect of admissible orders and n-dimensional admissibly ordered interval-valued overlap functions, both from a theoretical and applied points of view, the latter when considering classification problems.

***Index Terms*—n-dimensional overlap functions, interval-valued overlap functions, admissible orders, fuzzy rule-based classification systems**

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I. Introduction

In 2010, Bustince et al. [1] introduced the concept of overlap functions in order to deal with the overlap problem that usually appears in image processing. For that, overlap functions were conceived as continuous aggregation functions [2] that are not required to be associative. In fact, the associativity is not a relevant property for many applications besides image processing, such as decision making based on fuzzy preference relations, as properly discussed by Dimuro et al. in [3], [4], [5]. Observe that the continuity property of overlap functions was essential for the application in image processing, in the context where the concept was born.

Overlap functions are more general than the well known t-norms [6], although the required continuity may be more restrictive. In fact, there is an intersection between those two families: any continuous positive t-norm is an overlap function and any associative overlap function with 1 as neutral element is a t-norm. Nevertheless, the class of overlap functions is reacher than that of t-norms in many aspects, considering, e.g., the idempotency and homogeneity properties [7]. Moreover, overlap functions are closed to the convex sum and the aggregation by generalized composition of overlap functions, whereas neither the convex sum of t-norms nor the aggregation of t-norms by a t-norm results in t-norms, in general [8], [9].

Since the appearance of the concept of overlap functions, many authors have dedicated time to the theoretical research on their properties and related concepts, such as Qiao [10], Qiao and Hu [11], Dimuro et al. [5], [8], [12], [13], Zhou and Yan [14], Zhu et al. [15], Zhang et al. [16] and Cao et al. [17]. Moreover, the application of overlap function is getting attention mainly because the associativity is not required during the information aggregation process, like in image processing [18], decision making [19], [20], wavelet-fuzzy power quality diagnosis system [21], forest fire detection [22] and classification by generalizations of the Choquet integral [23], [24], [25], [26], [27]. Observe that, in some of the mentioned applications (e.g., decision making and classification), the continuity of overlap functions is not required.

However, overlap functions are bivariate functions, which implies that they can only be applied in problems involving just two classes or objects. This becomes a serious drawback when one faces n-dimensional problems (e.g., classification [28]), since overlap functions may be not associative. In order to overcome this limitation, Gómez et al. [29] introduced the concept of n-dimensional overlap functions. More recently, De Miguel et al. [30] defined general overlap functions by relaxing the boundary conditions of n-dimensional overlap functions, providing a more flexible definition.

Now, observe that in some applications there may be uncertainty in providing either the membership grades or the definition of membership functions [31]. To deal with this problem, one may adopt interval-valued fuzzy sets (IVFSs) [32], [33], [34], since it is capable to model both vagueness (soft class boundaries) and uncertainty (with respect to the membership function), as discussed in [35], [36], [37]. That is the reason why IVFSs have been successfully applied in several problems, such as game theory [38], decision making [39], pest control [40] and, specially, classification [37].

To address the problem of working in the interval-valued fuzzy context, Qiao and Hu [41] and Bedregal et al. [35] introduced independently the concept of interval-valued (iv) overlap functions. Latter, in [42], Asmus et al. introduced the concepts of n-dimensional iv-overlap functions and general iv-overlap functions, which were applied to compute the interval matching degree in Interval-Valued Fuzzy Rule Based Classification Systems (IV-FRBCSs) [43], [44].

IV-FRBCSs are Fuzzy Rule Based Classification Systems (FRBCSs) [45] whose linguistic labels are modeled by means of IVFSs, as in the work of Sanz et al. [37]. In IV-FRBCSs, the ignorance/uncertainty inherent to the definition of the membership functions, represented by IVFSs, is taken into account in the whole reasoning process, which implies that in the end of the classification process one needs to compare intervals instead of numbers. To carry out this comparison, a total order relation between intervals is needed, instead of the usual partial orders (e.g., the product order [46]). For that, one may use admissible orders introduced by Bustince et al. [47], which are total orders that may be constructed by means of aggregation functions, that is, different total orders can be obtained by varying the aggregation functions used in their construction. Since their definition, several works took into account admissible orders, such as [48], [49].

When defining IV-FRBCSs, both the aggregation function used to compute the interval matching degree and the adopted total order play a key role, as they can change the behaviour of the system. However, in the literature, there is not a consensus regarding which are the recommended n-dimensional iv-overlap functions to be applied to compute the interval matching degree in IV-FRBCSs. Moreover, there is no previous study concerning the relation between the chosen interval total order and n-dimensional iv-overlap function (which may or may not be increasing for that order), and the impact of such relation in the whole classification process.

Considering the discussion above, in this paper we have the following objectives:

1. To define n-dimensional admissibly ordered iv-overlap functions, that is, n-dimensional iv-overlap functions that are increasing with respect to an admissible order, studying their properties and showing examples;

2. To introduce a construction method of n-dimensional admissibly ordered iv-overlap functions based on n-dimensional overlap functions and a chosen admissible order, aiming at obtaining width-preserving iv-functions, that is, the resulting interval is never wider than any of the aggregated inputs, which is a desirable property in many applications;

3. To analyze the influence of both the admissible orders and the n-dimensional admissibly ordered iv-overlap functions in IV-FRBCSs.

The paper is organized as follows. Section II presents some preliminary concepts that are necessary for the development of the paper. In Section III, we present new results on admissible orders and introduce the concept of n-dimensional admissibly ordered iv-overlap functions, studying properties and showing examples. In section IV we develop a width-preserving construction method for n-dimensional admissibly ordered iv-overlap functions. In Section V, we analyze the influence of the combination of admissible orders and n-dimensional (admissibly ordered) interval-valued overlap functions, in classification problems. Section VI is the Conclusion.

II. Preliminaries

A. Interval Representation

Let us denote as $L([0,1])$ the set of all closed subintervals of the unit interval $[0,1]$. Denote $\vec{x} = (x_1, \ldots, x_n) \in [0,1]^n$ and $\vec{X} = (X_1, \ldots, X_n) \in L([0,1])^n$. Given any $X = [x_1, x_2] \in L([0,1])$, $\underline{X} = x_1$ and $\overline{X} = x_2$ denote, respectively, the left and right projections of X. The product and inclusion partial orders are defined for all $X, Y \in L([0,1])$, respectively, by:

$$X \leq_{Pr} Y \quad \Leftrightarrow \quad \underline{X} \leq \underline{Y} \wedge \overline{X} \leq \overline{Y};$$
$$X \subseteq Y \quad \Leftrightarrow \quad \underline{X} \geq \underline{Y} \wedge \overline{X} \leq \overline{Y}.$$

We call as $\leq_{Pr}$-increasing a function that is increasing with respect to the product order $\leq_{Pr}$. The projections $F^-, F^+ : [0,1]^n \to [0,1]$ of $F : L([0,1])^n \to L([0,1])$ are defined, respectively, by:

$$F^-(x_1, \ldots, x_n) = \underline{F([x_1, x_1], \ldots, [x_n, x_n])}; \quad (1)$$
$$F^+(x_1, \ldots, x_n) = \overline{F([x_1, x_1], \ldots, [x_n, x_n])}. \quad (2)$$

Given two functions $f, g : [0,1]^n \to [0,1]$ such that $f \leq g$, we define the function $\widehat{f,g} : L([0,1])^n \to L([0,1])$ as

$$\widehat{f,g}(X_1, \ldots, X_n) = [f(\underline{X_1}, \ldots, \underline{X_n}), g(\overline{X_1}, \ldots, \overline{X_n})]. \quad (3)$$

Definition 1. *[36] Let $F : L([0,1])^n \to L([0,1])$ be an $\leq_{Pr}$-increasing interval function. F is said to be representable if there exist increasing functions $f, g : [0,1]^n \to [0,1]$ such that $f \leq g$ and $F = \widehat{f,g}$.*

The functions f and g are the *representatives* of the interval function F. When $F = \widehat{f,f}$, we denote simply as $\widehat{f}$.

Proposition 1. *[42] For each $\leq_{Pr}$-increasing interval function $F : L([0,1])^n \to [0,1]$, F is representable if and only if F is inclusion monotonic.*

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Proposition 2. *[42] If an $\leq_{Pr}$-increasing interval function $F : L([0,1])^n \to L([0,1])$ is inclusion monotonic, then $\underline{F(X_1,\ldots,X_n)} = F^-(\underline{X_1},\ldots,\underline{X_n})$ and $\overline{F(X_1,\ldots,X_n)} = F^+(\overline{X_1},\ldots,\overline{X_n})$, for all $X_1,\ldots,X_n \in L([0,1])$.*

Definition 2. *[50] An interval-valued negation is a function $N : L([0,1]) \to L([0,1])$ that is $\leq_{Pr}$-decreasing and satisfies: **(N1)** $N([1,1]) = [0,0]$; **(N2)** $N([0,0]) = [1,1]$. If for all $X \in L([0,1]), N(N(X)) = X$, N is said to be involutive.*

Definition 3. *[51] An interval-valued restricted equivalence functions (IV-REF) is a function $IR : L([0,1])^2 \to L([0,1])$ satisfying: **(IR1)** IR is commutative; **(IR2)** $IR(X,Y) = [1,1] \Leftrightarrow X = Y$; **(IR3)** $IR(X,Y) = [0,0] \Leftrightarrow X = [0,0]$ and $Y = [1,1]$, or $X = [1,1]$ and $Y = [0,0]$; **(IR4)** $IR(X,Y) = IR(N(X),N(Y))$; **(IR5)** $\forall X,Y,Z \in L([0,1]), X \leq_{Pr} Y \leq_{Pr} Z \Rightarrow IR(X,Y) \geq_{Pr} IR(X,Z)$ and $IR(Y,Z) \geq_{Pr} IR(X,Z)$.*

Some interval operations that are used in this paper are defined, for all $X,Y \in L([0,1])$ as: [46], [52]

Sum: $X + Y = [\underline{X} + \underline{Y}, \overline{X} + \overline{Y}]$;

Product: $X \cdot Y = [\underline{X} \cdot \underline{Y}, \overline{X} \cdot \overline{Y}]$;

Generalized Hukuhara Division: with $\underline{Y} \neq 0$,

$X \div_H Y = [\min\{\underline{X}/\underline{Y}, \overline{X}/\overline{Y}\}, \max\{\underline{X}/\underline{Y}, \overline{X}/\overline{Y}\}]$.

B. Admissible orders

The notion of admissible orders for intervals came from the interest in extending the product order $\leq_{Pr}$ to a total order.

Definition 4. *[47] Let $(L([0,1]), \leq_{AD})$ be a partially ordered set. The order $\leq_{AD}$ is called an admissible order if*

(i) *$\leq_{AD}$ is a total order on $(L([0,1]), \leq_{AD})$;*

(ii) *For all $X,Y \in L([0,1])$, $X \leq_{AD} Y$ whenever $X \leq_{Pr} Y$.*

In other words, an order $\leq_{AD}$ on $L([0,1])$ is admissible, if it is total and refines the order $\leq_{Pr}$ [47].

Example 1. *The following relations on $L([0,1])$ are examples of admissible orders:*

(i) The lexicographical orders with respect to the first and second coordinate, defined, respectively, by:

$$X \leq_{Lex1} Y \Leftrightarrow \underline{X} < \underline{Y} \vee (\underline{X} = \underline{Y} \wedge \overline{X} \leq \overline{Y});$$

$$X \leq_{Lex2} Y \Leftrightarrow \overline{X} < \overline{Y} \vee (\overline{X} = \overline{Y} \wedge \underline{X} \leq \underline{Y}).$$

(ii) The order $\leq_{XY}$ introduced by Xu and Yager in [53], defined by:

$$X \leq_{XY} Y \Leftrightarrow \underline{X} + \overline{X} < \underline{Y} + \overline{Y} \text{ or } (\underline{X} + \overline{X} = \underline{Y} + \overline{Y} \text{ and } \overline{X} - \underline{X} \leq \overline{Y} - \underline{Y}).$$

(iii) Whenever one considers the comparison of the information quality [54] provided by the intervals X and Y in the order of Xu and Yager, it is possible to define, as in [43]:

$$X \leq_{IQ} Y \Leftrightarrow \underline{X} + \overline{X} < \underline{Y} + \overline{Y} \text{ or } (\underline{X} + \overline{X} = \underline{Y} + \overline{Y} \text{ and } \overline{Y} - \underline{Y} \leq \overline{X} - \underline{X}).$$

Proposition 3. *[47] Let $A, B : [0,1]^2 \to [0,1]$ be aggregation functions (see Def. 6), such that, for all $X,Y \in L([0,1])$, the equalities $A(\underline{X},\overline{X}) = A(\underline{Y},\overline{Y})$ and $B(\underline{X},\overline{X}) = B(\underline{Y},\overline{Y})$ can hold only if $X = Y$. Define the relation $\leq_{A,B}$ on $L([0,1])$ by*

$$X \leq_{A,B} Y \Leftrightarrow A(\underline{X},\overline{X}) < A(\underline{Y},\overline{Y}) \text{ or } (A(\underline{X},\overline{X}) = A(\underline{Y},\overline{Y}) \text{ and } B(\underline{X},\overline{X}) \leq B(\underline{Y},\overline{Y})).$$

Then $\leq_{A,B}$ is an admissible order on $L([0,1])$.

The pair (A,B) of aggregation functions that generates the order $\leq_{A,B}$ in Prop. 3 is called an admissible pair of aggregation functions [47]. Of particular interest is when the admissible order is generated by K_α mappings [47]. For $\alpha \in [0,1]$, the mapping $K_\alpha : [0,1]^2 \to [0,1]$ is defined by:

$$K_\alpha(x,y) = x + \alpha \cdot (y - x). \tag{4}$$

Definition 5. *[47] For $\alpha, \beta \in [0,1]$ such that $\alpha \neq \beta$, the relation $\leq_{\alpha,\beta}$ is defined by*

$$X \leq_{\alpha,\beta} Y \Leftrightarrow K_\alpha(\underline{X},\overline{X}) < K_\alpha(\underline{Y},\overline{Y}) \text{ or } (K_\alpha(\underline{X},\overline{X}) = K_\alpha(\underline{Y},\overline{Y}) \text{ and } K_\beta(\underline{X},\overline{X}) \leq K_\beta(\underline{Y},\overline{Y})).$$

Then, the relation $\leq_{\alpha,\beta}$ is an admissible order generated by an admissible pair of aggregation functions (K_α, K_β) [47].

Remark 1. *By varying the values of α and β one can recover some of the defined admissible orders, e.g., the lexicographical orders $\leq_{Lex1}$ and $\leq_{Lex2}$, and the orders $\leq_{XY}$ and $\leq_{IQ}$ are recovered, respectively, by $\leq_{0,1}$, $\leq_{1,0}$, $\leq_{0.5,1}$ and $\leq_{0.5,0}$.*

Lemma 1. *[47] For any $\alpha, \beta \in [0,1]$, $\alpha \neq \beta$, it holds that: (i)$\beta > \alpha \Rightarrow \leq_{\alpha,\beta} = \leq_{\alpha,1}$; (ii) $\beta < \alpha \Rightarrow \leq_{\alpha,\beta} = \leq_{\alpha,0}$.*

C. n-dimensional Overlap Functions

Definition 6. *[2] An aggregation function is a mapping $A : [0,1]^n \to [0,1]$ that is increasing in each argument and satisfying: **(A1)** $A(0,\ldots,0) = 0$; **(A2)** $A(1,\ldots,1) = 1$.*

Definition 7. *[28], [29] A function $On : [0,1]^n \to [0,1]$ is said to be an n-dimensional overlap function if the following conditions hold, for all $\vec{x} \in [0,1]^n$: **(On1)** On is commutative; **(On2)** $On(\vec{x}) = 0 \Leftrightarrow \prod_{i=1}^n x_i = 0$; **(On3)** $On(\vec{x}) = 1 \Leftrightarrow \prod_{i=1}^n x_i = 1$; **(On4)** On is increasing; **(On5)** On is continuous.*

If for all $x,y,z \in (0,1]$ one has that $x < y$ implies that $On(x,z,\ldots,z) < On(y,z,\ldots,z)$, then On is called a *strict n-dimensional overlap function*. In Table I we show some examples of n-dimensional overlap functions.

A 2-dimensional overlap function is just called overlap function. For properties on (n-dimensional) overlap functions and related concepts, see also: [3], [4], [9], [10], [11], [29].

D. n-dimensional Interval-valued Overlap Functions

Recently, the concepts of n-dimensional interval-valued aggregation/overlap functions and general interval-valued overlap functions were introduced by Asmus et al. in [42]:

Definition 8. *[42] A function $IA : L([0,1])^n \to L([0,1])$ is an n-dimensional interval-valued aggregation function*

TABLE I: Examples of n-dimensional overlap functions

Name	Definition
Product	$On_P(\vec{x}) = \prod_{i=1}^n x_i$
Minimum	$On_M(\vec{x}) = \min\{x_1, \ldots, x_n\}$
Hamacher	$On_{Hp}(\vec{x}) = \begin{cases} 0, & \text{if } x_1 = \ldots = x_n = 0; \\ \dfrac{\prod_{i=1}^n x_i}{\left(\sum_{i=1}^n \prod_{j \in N_i^n} x_j\right) - (n-1)\prod_{i=1}^n x_i}, & \text{otherwise} \\ \text{where } N_i^n = \{1, \ldots, n\} - \{i\} \end{cases}$
OB Overlap	$On_{OB}(\vec{x}) = \sqrt{\min\{x_1, \ldots, x_n\} \cdot \prod_{i=1}^n x_i}$
Geom. Mean	$On_{Gm}(\vec{x}) = \sqrt[n]{\prod_{i=1}^n x_i}$
Harm. Mean	$On_{Hm}(\vec{x}) = \begin{cases} \dfrac{n}{\frac{1}{x_i} + \ldots + \frac{1}{x_n}}, & \text{if } x_i \neq 0, \forall i \in \{1, \ldots, n\}; \\ 0, & \text{otherwise} \end{cases}$

whenever the following conditions hold: ***(IA1)*** IA *is* $\leq_{Pr}$*-increasing in each argument;* ***(IA2)*** IA *satisfies the boundary conditions:* ***(i)*** $IA([0,0], \ldots, [0,0]) = [0,0]$ *and* ***(ii)*** $IA([1,1], \ldots, [1,1]) = [1,1]$.

Definition 9. *[42] A function* $IOn : L([0,1])^n \to L([0,1])$ *is an n-dimensional interval-valued (iv) overlap function if, for all* $\vec{X} \in L([0,1])^n$ *and* $Y \in L([0,1])$*, it satisfies:* ***(IOn1)*** IOn *is commutative;* ***(IOn2)*** $IOn(\vec{X}) = [0,0] \Leftrightarrow \prod_{i=1}^n X_i = [0,0]$*;* ***(IOn3)*** $IOn(\vec{X}) = [1,1] \Leftrightarrow \prod_{i=1}^n X_i = [1,1]$*;* ***(IOn4)*** IOn *is* $\leq_{Pr}$*-increasing in the first component:* $X_1 \leq_{Pr} Y \Rightarrow IOn(X_1, X_2, \ldots, X_n) \leq_{Pr} IOn(Y, X_2, \ldots, X_n)$*;* ***(IOn5)*** IOn *is Moore continuous [46].*

Example 2. *Some examples of n-dimensional iv-overlap functions, for* $\vec{X} = \in L([0,1])^n$ *are:*

1. $IOn_M(\vec{X}) = \left[\min\{\underline{X_1}, \ldots, \underline{X_n}\}, \min\{\overline{X_1}, \ldots, \overline{X_n}\}\right]$*;*
2. $IOn_{Pp}(\vec{X}) = \left[\prod_{i=1}^n \underline{X_i}^p, \prod_{i=1}^n \overline{X_i}^p\right]$*, for* $p > 0$*;*
3. $IOn_{Mp}(\vec{X}) = IOn_M(\vec{X}) \cdot IOn_{Pp}(\vec{X})$.

Theorem 1. *[42] Let* $On_1, On_2 : [0,1]^n \to [0,1]$ *be n-dimensional overlap functions such that* $On_1 \leq On_2$*. Then, the function* $\widehat{On_1, On_2} : L([0,1])^n \to L([0,1])$*, as defined in Eq. (3), is an n-dimensional iv-overlap function.*

An n-dimensional iv-overlap function $IOn : L([0,1])^n \to L([0,1])$ is said to be *o*-representable if there exist n-dimensional overlap functions $On_1, On_2 : [0,1]^n \to [0,1]$ such that $On_1 \leq On_2$ and $IOn = \widehat{On_1, On_2}$.

Theorem 2. *[42] Let* $IOn : L([0,1])^n \to L([0,1])$ *be an n-dimensional iv-overlap function. Then,* IOn *is o-representable if and only if* IOn *is inclusion monotonic and satisfies, for all* $\vec{X} \in L([0,1])^n$*: (i)* $\underline{IOn(\vec{X})} = 0 \Leftrightarrow \prod_{i=1}^n \underline{X_i} = 0$*; (ii)* $\overline{IOn(\vec{X})} = 1 \Leftrightarrow \prod_{i=1}^n \overline{X_i} = 1$.

Corollary 1. *Let* $IOn : L([0,1])^n \to L([0,1])$ *be an n-dimensional iv-overlap function such that* $IOn^+ : L([0,1])^n \to L([0,1])$ *(Eq. (2)) is a strict n-dimensional overlap function. Then,* IOn *is o-representable if and only if it is inclusion monotonic and, for all* $\vec{X} \in L([0,1])^n$*:* $\underline{IOn(\vec{X})} = 0 \Leftrightarrow \prod_{i=1}^n \underline{X_i} = 0$.

Proof. It is immediate from Prop. 2 and Theorem 2. □

A 2-dimensional iv-overlap function is called iv-overlap function. For more properties on such functions, see [35], [41].

III. N-DIMENSIONAL ADMISSIBLY ORDERED INTERVAL-VALUED OVERLAP FUNCTIONS

In this section, we define the concept of n-dimensional admissibly ordered interval-valued overlap function, following by some properties and examples. But first, we introduce some new results regarding admissible orders.

Proposition 4. *For all* $\alpha_1, \alpha_2, \beta_1, \beta_2 \in [0,1]$ *such that* $\alpha_1 \neq \alpha_2, \alpha_1 \neq \beta_1$ *and* $\alpha_2 \neq \beta_2$*, one has that* $\leq_{\alpha_1,\beta_1} \neq \leq_{\alpha_2,\beta_2}$.

Proof. Consider $Y = [0,1]$ and $\alpha_1, \alpha_2 \in [0,1]$ such that $\alpha_1 < \alpha_2$. For all $X = [x,x]$ such that $\alpha_1 < x < \alpha_2$ one has that $Y \leq_{\alpha_1,\beta_1} X$ and $X \leq_{\alpha_2,\beta_2} Y$, for any $\beta_1 \neq \alpha_1$ and $\beta_2 \neq \alpha_2$. The proof for the case in which $\alpha_2 < \alpha_1$ is analogous. □

Proposition 5. *For all* $\alpha \in (0,1)$ *one has that* $\leq_{\alpha,0} \neq \leq_{\alpha,1}$.

Proof. For all $\alpha \in (0,1)$, it is possible to find $X, Y \in L([0,1])$, namely, $X = [\alpha, \alpha]$ and $Y = [0,1]$, such that $Y <_{\alpha,0} X$ and $X <_{\alpha,1} Y$. □

Corollary 2. *For all* $\alpha, \beta_1, \beta_2 \in [0,1]$ *such that* $\beta_1 < \alpha < \beta_2$*, one has that* $\leq_{\alpha,\beta_1} \neq \leq_{\alpha,\beta_2}$.

Proof. It is immediate from Lemma 1 and Prop. 5. □

From Prop. 4 and 5, and Lemma 1, it is clear that $\leq_{\alpha_1,\beta_1} \neq \leq_{\alpha_2,\beta_2}$, for all $\alpha_1, \alpha_2, \beta_1, \beta_2 \in [0,1]$ such that $\alpha_1 \neq \beta_1$ and $\alpha_2 \neq \beta_2$, except when $\alpha_1 = \alpha_2 = \alpha$, $\beta_1 < \alpha$ and $\beta_2 < \alpha$ or when $\alpha_1 = \alpha_2 = \alpha$, $\alpha < \beta_1$ and $\alpha < \beta_2$.

Here, we introduce the definition of n-dimensional admissibly ordered interval-valued overlap function.

Definition 10. *A function* $AOn : L([0,1])^n \to L([0,1])$ *is an n-dimensional admissibly ordered interval-valued overlap function for an admissible order* $\leq_{AD}$ *(n-dimensional* $\leq_{AD}$*-overlap function) if it satisfies the conditions* ***(IOn1)***, ***(IOn2)*** *and* ***(IOn3)*** *of Def. 9 and, for all* $Y, X_1, \ldots, X_n \in L([0,1])$*:*

(AOn4) AOn *is increasing for* $\leq_{AD}$ *in the first component:* $X_1 \leq_{AD} Y \Rightarrow AOn(X_1, \ldots, X_n) \leq_{AD} AOn(Y, X_2, \ldots, X_n)$.

Remark 2. *Condition* ***(IOn5)*** *from Def. 9 is not needed as the continuity was only a requirement in the original definition of overlap functions in order to enable them to be applied in image processing [1], which is not the case here. Besides that, the notion of continuity for admissible orders is still an open problem, and it is not the focus of this work.*

A 2-dimensional $\leq_{AD}$-overlap function is called $\leq_{AD}$-overlap function.

Example 3. *Some examples of* $\leq_{AD}$*-overlap functions are: (1) The interval minimum with respect to the order* $\leq_{IQ}$*, defined by*

$$\min_{\leq_{0.5,0}}(X,Y) = \begin{cases} X, & \text{if } X \leq_{0.5,0} Y \\ Y, & \text{otherwise} \end{cases}$$

is an $\leq_{0.5,0}$*-overlap function (see Prop. 6);*

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(2) The interval product is an n-dimensional $\leq_{1,0}$-overlap function, that is, it is increasing with respect to the order $\leq_{Lex2}$ (see Theorem 2);
(3) For a given $\alpha \in [0,1]$ and On_{OB} defined in Table I, the function $AOn_{OB}^{0.5}$ defined by

$$\begin{aligned} AOn_{OB}^{0.5}(X_1,\ldots,X_n) = \\ [On_{OB}(K_{0.5}(\underline{X_1},\overline{X_1}),\ldots,K_{0.5}(\underline{X_n},\overline{X_n})) - 0.5\cdot m, \\ On_{OB}(K_{0.5}(\underline{X_1},\overline{X_1}),\ldots,K_{0.5}(\underline{X_n},\overline{X_n})) + 0.5\cdot m], \end{aligned}$$

with

$$\begin{aligned} m = \min\{\overline{X_1}-\underline{X_1},\ldots,\overline{X_n}-\underline{X_n}, \\ On_{OB}(K_{0.5}(\underline{X_1},\overline{X_1}),\ldots,K_{0.5}(\underline{X_n},\overline{X_n})), \\ 1-On_{OB}(K_{0.5}(\underline{X_1},\overline{X_1}),\ldots,K_{0.5}(\underline{X_n},\overline{X_n}))\}, \end{aligned}$$

is an n-dimensional $\leq_{0.5,1}$-overlap function (see Theorem 4).

It is immediate that:

Proposition 6. *The interval minimum defined, for all $X,Y \in L([0,1])$, by*

$$\min_{\leq_{AD}}(X,Y) = \begin{cases} X, & \text{if } X \leq_{AD} Y \\ Y, & \text{otherwise} \end{cases}$$

is an $\leq_{AD}$-overlap function for any admissible order $\leq_{AD}$.

The result in Prop. 6 holds for a similarly defined n-dimensional interval minimum, that is, the function that returns the least interval from n interval-valued inputs accordingly to an admissible order $\leq_{AD}$.

Lemma 2. *Let $On : [0,1]^n \to [0,1]$ be an n-dimensional overlap function. Then, there exists $b \in (0,1)$ such that, for all $a \in (0,b)$, it holds that $On(a,1,\ldots,1) < On(b,1,\ldots,1)$.*

Proof. By condition $(\boldsymbol{On3})$ of Def. 7, one has that $On(x,1,\ldots,1) < 1$, for each $x \in (0,1)$, and by $(\boldsymbol{On5})$, we have that there exists $x_0 \in (0,1)$ such that, for each $y \in (x_0,1)$, it holds that $O_n(x_0,1\ldots,1) < O_n(y,1,\ldots,1) < 1$. So, taking $b = \frac{x_0+1}{2}$ we have that, for each $a \in (0,b)$, it holds that $On(a,1,\ldots,1) < On(b,1,\ldots,1)$. ☐

Remark 3. *Consider $X,Y \in L([0,1])$. Observe that whenever $X <_{\alpha,\beta} Y$, with $\overline{X} > \overline{Y}$ and $\underline{X}+\alpha(\overline{X}-\underline{X}) < \underline{Y}+\alpha(\overline{Y}-\underline{Y})$, then it is immediate that*

$$\alpha < \frac{\underline{Y}-\underline{X}}{(\underline{Y}-\underline{X})-(\overline{Y}-\overline{X})}. \tag{5}$$

The following theorem presents an important result regarding o-representable n-dimensional iv-overlap functions and the conditions for them to be increasing with respect to an admissible order $\leq_{\alpha,\beta}$.

Theorem 3. *Let $IOn : L([0,1])^n \to L([0,1])$ be an o-representable n-dimensional iv-overlap function and $\alpha,\beta \in [0,1]$, $\alpha \neq \beta$. Then, IOn is $\leq_{\alpha,\beta}$-increasing if and only if $\alpha = 1$ and IOn^+ is a strict n-dimensional overlap function.*

Proof. $(\Rightarrow)$ Based on Lemma 2, there exists $b \in (0,1)$ such that for all $a \in (0,b)$, $IOn^+(a,1,\ldots,1) < IOn^+(b,1,\ldots,1)$ holds. Consider $\alpha < 1$. It is possible to find $X,Y \in L([0,1])$ such that $X <_{\alpha,\beta} Y$, with $\underline{X} < \underline{Y} < \overline{Y} < \overline{X}$ and $\underline{X}+\alpha(\overline{X}-\underline{X}) < \underline{Y}+\alpha(\overline{Y}-\underline{Y})$. In fact, that is the case when $X = [\frac{b}{4}, b]$ and $Y = [\frac{b}{2}, \frac{b}{2-0.9\underbrace{9\ldots9}_{n-times}}]$, for n sufficiently great.

Next, suppose that $Z = [0,1]$. Then, it follows that $IOn(X,Z,\ldots,Z) = [0, IOn^+(\overline{X},1,\ldots,1)] >_{Pr} [0, IOn^+(\overline{Y},1,\ldots,1)] = IOn(Y,Z,\ldots,Z)$. As $\leq_{\alpha,\beta}$ is an admissible order, then, one has that $IOn(X,Z,\ldots,Z) >_{\alpha,\beta} IOn(Y,Z,\ldots,Z)$, showing that IOn is not $\leq_{\alpha,\beta}$-increasing. By the contrapositive, if IOn is $\leq_{\alpha,\beta}$-increasing then $\alpha = 1$.

Now, let us suppose that IOn^+ is not strict. Then, there exist $x_1,\ldots,x_n,y,z \in (0,1]$ such that $y < z$ and $IOn^+(x_1,\ldots,x_{n-1},y) = IO^+(x_1,\ldots,x_{n-1},z)$. As IOn is $\leq_{\alpha,\beta}$-increasing, one has that $\alpha = 1$, and thus, by Lemma 1, IOn is $\leq_{1,0}$-increasing. Since $[y,y] \leq_{1,0} [0,z]$, then:

$$\begin{aligned} IOn([x_1,x_1],\ldots,[x_{n-1},x_{n-1}],[y,y]) \leq_{1,0} \\ IOn([x_1,x_1],\ldots,[x_{n-1},x_{n-1}],[0,z]). \end{aligned}$$

As IOn is o-representable, one has that

$$\begin{aligned} IOn([x_1,x_1],\ldots,[x_{n-1},x_{n-1}],[y,y]) = \\ [IOn^-(x_1,\ldots,x_{n-1},y), IOn^+(x_1,\ldots,x_{n-1},y)], \end{aligned}$$

$$\begin{aligned} IOn([x_1,x_1],\ldots,[x_{n-1},x_{n-1}],[0,z]) = \\ [IOn^-(x_1,\ldots,x_{n-1},0), IOn^+(x_1,\ldots,x_{n-1},z)]. \end{aligned}$$

Since $IOn^+(x_1,\ldots,x_{n-1},y) = IOn^+(x_1,\ldots,x_{n-1},z)$, it follows that:

$$\begin{aligned} IOn([x_1,x_1],\ldots,[x_{n-1},x_{n-1}],[y,y]) \leq_{1,0} \\ IOn([x_1,x_1],\ldots,[x_{n-1},x_{n-1}],[0,z]) \Leftrightarrow \\ IOn^-(x_1,\ldots,x_{n-1},y) \leq IOn^-(x_1,\ldots,x_{n-1},0), \end{aligned}$$

which is a contradiction as $IOn^-(x_1,\ldots,x_{n-1},0) = 0$ and $IOn^-(x_1,\ldots,x_{n-1},y) > 0$, showing that IOn is not $\leq_{\alpha,\beta}$-increasing. Then, if IOn is $\leq_{\alpha,\beta}$-increasing then IOn^+ must be a strict n-dimensional overlap function.
$(\Leftarrow)$ Consider $X_1,\ldots,X_{n-1},[a,b],[c,d] \in L([0,1])$ such that $[a,b] \leq_{1,0} [c,d]$. Then, one has the following cases:
(i) $a > c$ and $b < d$: In this case, one has that $[a,b] \subset [c,d]$, and, thus, by Theorem 2,

$$IOn(X_1,\ldots,X_{n-1},[a,b]) \subseteq IOn(X_1,\ldots,X_n,[c,d]).$$

First, consider $X_i \neq [0,0]$ for all $i \in \{1,\ldots,n\}$. By Prop. 2, since IOn^+ is a strict overlap function, one has that:

$$\begin{aligned} \overline{IOn(X_1,\ldots,X_{n-1},[a,b])} = IOn^+(\overline{X_1},\ldots,\overline{X_{n-1}},b) \\ < IOn^+(\overline{X_1},\ldots,\overline{X_{n-1}},d) = \overline{IOn(X_1,\ldots,X_{n-1},[c,d])}. \end{aligned}$$

If X_i=$[0,0]$ for some $i \in \{1,\ldots,n\}$, then

$$\begin{aligned} IOn(X_1,\ldots,X_{n-1},[a,b]) = \\ IOn(X_1,\ldots,X_{n-1},[c,d]) = [0,0]. \end{aligned}$$

Thus, for any $X \in L([0,1])$, one concludes that

$$IOn(X_1,\ldots,X_{n-1},[a,b]) \leq_{1,0} IOn(X_1,\ldots,X_{n-1},[c,d]).$$

(ii) $b \leq d$ and $a \leq c$: In this case, $[a,b] \leq_{Pr} [c,d]$, and, thus,

$$IOn(X_1,\ldots,X_{n-1},[a,b]) \leq_{Pr} IOn(X_1,\ldots,X_{n-1},[c,d]).$$

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Since $\leq_{1,0}$ is an admissible order, one concludes that

$$IOn(X_1,\ldots,X_{n-1},[a,b]) \leq_{1,0} IOn(X_1,\ldots,X_{n-1},[c,d]).$$

Since IOn is o-representable and IOn^+ is a strict overlap function, then the result follows Lemma 1. ☐

Now, let us show an example to illustrate Theorem 3 with a specific o-representable n-dimensional $\leq_{AD}$-overlap function.

Example 4. *Let IOn_{Pp} be an n-dimensional iv-overlap function as defined in Example 2, and $\alpha,\beta \in [0,1]$, $\alpha \neq \beta$. As*

$$\begin{aligned} IOn_{Pp}(\vec{X}) &= \left[IOn^-_{Pp}(\vec{X}), IOn^+_{Pp}(\vec{X})\right] \\ &= \left[\underline{X_1}^p \cdot \ldots \cdot \underline{X_n}^p, \overline{X_1}^p \cdot \ldots \cdot \overline{X_n}^p\right], \end{aligned}$$

it is clear that IOn^+_{Pp} is a strict n-dimensional overlap function. Furthermore, suppose $\alpha < 1$ and consider $Z = [0,1]$, $X = [0.1, 0.5], Y = [0.4, 0.4\underbrace{9\ldots9}_{n-times}]$, for $n > 0$, where, clearly, $\underline{X} < \underline{Y}$ and $\overline{X} > \overline{Y}$. Then, there exists a sufficiently great n such that $X <_{\alpha,\beta} Y$, and, by Remark 3, Eq. (5) holds. However, one has that

$$\begin{aligned} IOn_{Pp}(X,Z) &= [0, 0.5^p] >_{Pr} [0, (0.4\underbrace{9\ldots9}_{n-times})^p] \\ &= IOn_{Pp}(Y,Z). \end{aligned}$$

As $\leq_{\alpha,\beta}$ is an admissible order, then, it follows that $IOn_{Pp}(X,Z) >_{\alpha,\beta} IOn_{Pp}(Y,Z)$, showing that IOn_{Pp} is not $\leq_{\alpha,\beta}$-increasing. Then, if IOn_{Pp} is $\leq_{\alpha,\beta}$-increasing then $\alpha = 1$. Since IOn_{Pp} is o-representable and IOn^+_{Pp} is a strict n-dimensional overlap function, from Theorem 3 and Lemma 1 one has that IOn_{Pp} is $\leq_{1,\beta}$-increasing. Thus, IOn_{Pp} is $\leq_{\alpha,\beta}$-increasing if and only if $\alpha = 1$.

IV. A Construction Method

In this section, we present a construction method to obtain n-dimensional $\leq_{\alpha,\beta}$-overlap functions, with $\alpha \neq \beta$, for a given α and a strict n-dimensional overlap function. For the sake of simplicity, let us denote $K_\alpha(\underline{X}, \overline{X})$ simply as $K_\alpha(X)$.

Theorem 4. *Let On be a strict n-dimensional overlap function, $\alpha \in (0,1)$ and $\beta \in [0,1]$ such that $\alpha \neq \beta$. Then $AOn^\alpha : L([0,1])^n \to L([0,1])$ defined, for all $\vec{X} \in L([0,1])^n$, by*

$$\begin{aligned} AOn^\alpha(\vec{X}) = [&On(K_\alpha(X_1),\ldots,K_\alpha(X_n)) - \alpha m, \\ &On(K_\alpha(X_1),\ldots,K_\alpha(X_n)) + (1-\alpha)m], \end{aligned}$$

where

$$\begin{aligned} m = &\min\{\overline{X_1} - \underline{X_1},\ldots,\overline{X_n} - \underline{X_n}, On(K_\alpha(X_1),\ldots,K_\alpha(X_n)), \\ &1 - On(K_\alpha(X_1),\ldots,K_\alpha(X_n))\}, \end{aligned}$$

is an n-dimensional $\leq_{\alpha,\beta}$-overlap function.

Proof. Consider $\alpha \in (0,1)$ and $\beta \in [0,1]$ such that $\alpha \neq \beta$. By Lemma 1 it is sufficient to consider the case $\beta = 0$ and $\beta = 1$. Clearly, AOn^α is well defined and commutative. Also:

$$\begin{aligned} &K_\alpha(AOn^\alpha(\vec{X})) \\ &\quad= On(K_\alpha(X_1),\ldots,K_\alpha(X_n)) - \alpha m \\ &\qquad+\alpha(On(K_\alpha(X_1),\ldots,K_\alpha(X_n)) + (1-\alpha)m \\ &\qquad-On(K_\alpha(X_1),\ldots,K_\alpha(X_n)) + \alpha m) \\ &\quad= On(K_\alpha(X_1),\ldots,K_\alpha(X_n)). \end{aligned}$$

Furthermore,

$$\begin{aligned} K_0(AOn^\alpha(\vec{X})) &= On(K_\alpha(X_1),\ldots,K_\alpha(X_n)) - \alpha m, \\ K_1(AOn^\alpha(\vec{X})) &= On(K_\alpha(X_1),\ldots,K_\alpha(X_n)) + (1-\alpha)m. \end{aligned}$$

Now, consider $\vec{X} \in L([0,1])^n$. Then, since $\alpha \neq 0$,

$$\begin{aligned} &AOn^\alpha(\vec{X}) = [0,0] \\ &\Leftrightarrow\ K_\alpha(AOn^\alpha(\vec{X})) = 0 \\ &\Leftrightarrow\ On(K_\alpha(X_1),\ldots,K_\alpha(X_n)) = 0 \\ &\Leftrightarrow\ K_\alpha(X_i) = 0 \text{ for some } i \in \{1,\ldots,n\} \\ &\Leftrightarrow\ X_i = [0,0] \text{ for some } i \in \{1,\ldots,n\}. \end{aligned}$$

Therefore, AOn^α satisfies **(IOn2)**.

Consider $\vec{X} \in L([0,1])^n$. Then, since $\alpha \neq 1$,

$$\begin{aligned} &AOn^\alpha(\vec{X}) = [1,1] \\ &\Leftrightarrow\ K_\alpha(AOn^\alpha(\vec{X})) = 1 \\ &\Leftrightarrow\ On(K_\alpha(X_1),\ldots,K_\alpha(X_n)) = 1 \\ &\Leftrightarrow\ K_\alpha(X_i) = 1 \text{ for each } i \in \{1,\ldots,n\} \\ &\Leftrightarrow\ X_i = [1,1] \text{ for each } i \in \{1,\ldots,n\}. \end{aligned}$$

Thus, AOn^α satisfies **(IOn3)**. In order to prove that AOn^α satisfies **(AOn4)** for $\leq_{\alpha,0}$, consider $Y <_{\alpha,0} Z$ and $X \in L([0,1])$ such that $K_\alpha(X) = 0$. Then,

$$\begin{aligned} &On(K_\alpha(Y), K_\alpha(X),\ldots,K_\alpha(X)) = 0 \\ &\qquad= On(K_\alpha(Z), K_\alpha(X),\ldots,K_\alpha(X)) \end{aligned}$$

and, therefore,

$$\begin{aligned} &\min\{\overline{Y} - \underline{Y}, \overline{X} - \underline{X},\ldots,\overline{X} - \underline{X}, \\ &\quad On(K_\alpha(Y), K_\alpha(X),\ldots,K_\alpha(X)), \\ &\quad 1 - On(K_\alpha(Y), K_\alpha(X),\ldots,K_\alpha(X))\} = 0 \\ &= \min\{\overline{Z} - \underline{Z}, \overline{X} - \underline{X},\ldots,\overline{X} - \underline{X}, \\ &\quad On(K_\alpha(Z), K_\alpha(X),\ldots,K_\alpha(X)), \\ &\quad 1 - On(K_\alpha(Z), K_\alpha(X),\ldots,K_\alpha(X))\}. \end{aligned}$$

Hence, $AOn^\alpha(Y, X,\ldots,X) = [0,0] = AOn^\alpha(Z, X,\ldots,X)$.

Now take $X \in L([0,1])$ such that $K_\alpha(X) > 0$. By definition, we have the following two cases:

1) $K_\alpha(Y) < K_\alpha(Z)$. Since On is strict, one has that

$$\begin{aligned} &On(K_\alpha(Y), K_\alpha(X),\ldots,K_\alpha(X)) < \\ &\qquad On(K_\alpha(Z), K_\alpha(X),\ldots,K_\alpha(X)) \end{aligned}$$

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and, therefore,

$$\begin{aligned}
&K_\alpha(AOn^\alpha(Y, X, \ldots, X)) \\
&\quad = \; On(K_\alpha(Y), K_\alpha(X), \ldots, K_\alpha(X)) < \\
&\qquad On(K_\alpha(Z), K_\alpha(X), \ldots, K_\alpha(X)) \\
&\quad = \; K_\alpha(AOn^\alpha(Z, X, \ldots, X)).
\end{aligned}$$

So, $AOn^\alpha(Y, X, \ldots, X) <_{\alpha,0} AOn^\alpha(Z, X, \ldots, X)$.

2) $K_\alpha(Y) = K_\alpha(Z)$ and $K_0(Y) < K_0(Z)$. Then, $\underline{Y} < \underline{Z} \leq \overline{Z} < \overline{Y}$ and, therefore $\overline{Y} - \underline{Y} > \overline{Z} - \underline{Z}$. Thus, since

$$On(K_\alpha(Y), K_\alpha(X), \ldots, K_\alpha(X)) = On(K_\alpha(Z), K_\alpha(X), \ldots, K_\alpha(X)),$$

it holds that

$$\begin{aligned}
m_1 = \min\{&\overline{Y} - \underline{Y}, \overline{X} - \underline{X}, \ldots, \overline{X} - \underline{X}, \\
&On(K_\alpha(Y), K_\alpha(X), \ldots, K_\alpha(X)), \\
&1 - On(K_\alpha(Y), K_\alpha(X), \ldots, K_\alpha(X))\} \geq \\
\min\{&\overline{Z} - \underline{Z}, \overline{X} - \underline{X}, \ldots \overline{X} - \underline{X}, \\
&On(K_\alpha(Z), K_\alpha(X), \ldots, K_\alpha(X)), \\
&1 - On(K_\alpha(Z), K_\alpha(X), \ldots, K_\alpha(X))\} \; = \; m_2.
\end{aligned}$$

Hence,

$$\begin{aligned}
&K_\alpha(AOn^\alpha(Y, X, \ldots, X)) \\
&\quad = \; On(K_\alpha(Y), K_\alpha(X), \ldots, K_\alpha(X)) \\
&\quad = \; On(K_\alpha(Z), K_\alpha(X), \ldots, K_\alpha(X)) \\
&\quad = \; K_\alpha(AOn^\alpha(Z, X, \ldots, X)),
\end{aligned}$$

and

$$\begin{aligned}
&K_0(AOn^\alpha(Y, X, \ldots, X)) \\
&\quad = \; On(K_\alpha(Y), K_\alpha(X), \ldots, K_\alpha(X)) - \alpha m_1 \\
&\quad \leq \; On(K_\alpha(Y), K_\alpha(X), \ldots, K_\alpha(X)) - \alpha m_2 \\
&\quad = \; On(K_\alpha(Z), K_\alpha(X), \ldots, K_\alpha(X)) - \alpha m_2 \\
&\quad = \; K_0(AOn^\alpha(Z, X, \ldots, X)).
\end{aligned}$$

So, $AOn^\alpha(Y, X, \ldots, X) \leq_{\alpha,0} AOn^\alpha(Z, X, \ldots, X)$.

Thus, for each $\alpha \in (0, 1)$, AOn^α is an n-dimensional $\leq_{\alpha,0}$-overlap function. The proof that AOn^α satisfies **(AOn4)** for $\leq_{\alpha,1}$ and $\alpha \in (0, 1)$ is obtained analogously. □

Now, let us see an example of a $\leq_{\alpha,\beta}$ order and overlap function that do not allow for the construction of an $\leq_{\alpha,\beta}$-increasing o-representable iv-overlap function IO, but in which one can obtain an $\leq_{\alpha,\beta}$-overlap function AO^α via the method presented in Theorem 4.

Example 5. *Consider the admissible order $\leq_{0.4,0}$ and be the overlap function $Op : [0, 1]^2 \to [0, 1]$ defined, for all $x, y \in [0, 1]$, by $Op(x, y) = x \cdot y$. From Theorem 4, for $On = Op$:*

$$AOp^{0.4}(X, Y) = [K_{0.4}(X) \cdot K_{0.4}(Y) - 0.4 \cdot m, \; K_{0.4}(X) \cdot K_{0.4}(Y) + (0.6)m],$$

where

$$m = \min\{\overline{X} - \underline{X}, \overline{Y} - \underline{Y}, K_{0.4}(X) \cdot K_{0.4}(Y), \; 1 - K_{0.4}(X) \cdot K_{0.4}(Y)\}.$$

Now, for $X = [0, 1], Y = [0.2, 0.2]$ and $Z = [0, 0.4]$ one has that $Z \leq_{0.4,0} Y$, $AOp^{0.4}(X, Z) = [0.0384, 0.1024]$ and $AOp^{0.4}(X, Z) = [0.08, 0.08]$, meaning that

$$Z \leq_{0.4,0} y \Leftrightarrow AOp^{0.4}(X, Z) \leq_{0.4,0} AOp^{0.4}(X, Y),$$

which is expected for an $\leq_{0.4,0}$-overlap function.

However, if we try to construct an (admissibly ordered) o-representable interval-valued overlap function IOp in which $IOp^- = IOp^+ = Op$, one can observe that $IOp(X, Z) = [0, 0.4]$ and $IOp(X, Y) = [0, 0.2]$, meaning that $Y \leq_{0.4,0} Z$ and $IOp(X, Z) >_{0.4,0} IOp(X, Y)$, proving that IOp is not an $\leq_{0.4,0}$-overlap function. This happens because $\alpha \neq 1$, which fails to follow the conditions stated in Theorem 3.

Remark 4. *The construction method introduced in Theorem 4 allows us to obtain different n-dimensional $\leq_{\alpha,\beta}$-overlap functions with respect to any $\leq_{\alpha,\beta}$ order. Thus, its adaptability allows for it to be employed in various applications with different approaches to the ranking of intervals, determined by the choice of different α and β.*

Remark 5. *As stated in Theorem 4, the n-dimensional overlap function that is the core of the construction method must be strict. Yet, this requirement does not present itself as a hindrance, as most n-dimensional overlap functions are, in fact, strict. One notable exception is the minimum operator. However, the interval minimum as show in Prop. 6 is an $\leq_{AD}$-overlap function for any admissible order $\leq_{AD}$, and turns out to be a more suitable interval representation of the minimum.*

Remark 6. *It is noteworthy that the width of the resulting interval when applying AOn^α is given by*

$$\begin{aligned}
m \;\; = \;\; \min\{&\overline{X_1} - \underline{X_1}, \ldots, \overline{X_n} - \underline{X_n}, \\
&On(K_\alpha(X_1), \ldots, K_\alpha(X_n)), \\
&1 - On(K_\alpha(X_1), \ldots, K_\alpha(X_n))\}.
\end{aligned}$$

Thus, AOn^α is a width-preserving operation, as the resulting interval will never be wider than any of the aggregated inputs, which is a desirable property in many applications. On the other hand, by the way m is defined, if at least one of the aggregated intervals is degenerate, than the resulting interval when applying AOn^α will also be degenerate.

V. Analysis of the Influence of the Studied Concepts in IV-FRBCSs

The objective of this section is to analyze the behaviour of different admissible orders and n-dimensional (admissibly ordered) iv-overlap functions applied on the interval-valued fuzzy reasoning method (IV-FRM) of an IV-FRBCS. In order to do that, first we are going to review the main points of FRBCSs and IV-FRBCSs, highlighting the steps where we apply our new theoretical results.

A. Interval-Valued Fuzzy Rule-based Classification Systems

A classification problem is composed by P training examples $\vec{x_p} = (x_{p1}, \ldots, x_{pn}), p \in \{1, \ldots, P\}$ where x_{pi} is the value of the i-th variable of the p-th example. Each example

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belongs to one of M classes in $C=\{C_1, \ldots, C_M\}$. The learned classifier aims to identify the class of new testing examples.

FRBCSs are one most frequently adopted technique to deal with classification problems. They provide a good balance between accuracy and interpretability, since the antecedents of their rules are composed of linguistic labels, while still providing accurate results [44]. We adopt the following structure for the fuzzy rules:

$$\text{Rule } R_j : \text{If } x_1 \text{ is } A_{j1} \text{ and } \ldots \text{ and } x_n \text{ is } A_{jn} \quad \text{then } Class = C'_j \text{ with } RW_j, \tag{6}$$

where R_j is the label of the j-th rule, $x = (x_1, \ldots, x_n)$ is an n-dimensional example vector, A_{ji} is the fuzzy set representing the linguistic term of the j-th rule in the i-th antecedent, $C'_j \in C$ is a class label, and $RW_j \in [0, 1]$ is the rule weight [55]. Specifically, we consider the computation of the rule weight using the fuzzy confidence value or certainty factor, given by:

$$RW_j = \frac{\sum_{x_p \in C'_j} A_j(x_p)}{\sum_{p=1}^{P} A_j(x_p)}, \tag{7}$$

where $A_j(x_p)$ is the matching degree of the pattern x_p with the antecedent part of the fuzzy rule R_j, computed as

$$A_j(x_p) = \mathfrak{c}(A_{j1}(x_{p1}), \cdots, A_{jn}(x_{pn})), \tag{8}$$

where $\mathfrak{c}$ is an n-dimensional conjunction operator and $j \in \{1, \ldots, L\}$.

IV-FRBCSs are FRBCSs where some of the linguistic labels (or all of them) are modelled using IVFSs. Furthermore, the FRM must work with intervals instead of numbers to take into account the degree of uncertainty throughout the whole inference process (see Section V-B).

B. New Interval-valued Fuzzy Reasoning Method

In this paper, we apply our new theoretical results in the IVTURS algorithm[1], which is a state of the art IV-FRBCS. Its learning process is composed of three steps:

1) To build an IV-FRBCS. This step involves the following tasks:

• The generation of an initial FRBCS by applying FARC-HD [56], whose first learning stage is based on the Apriori algorithm [57] that builds fuzzy rules using the support and confidence (Eq. (7)). In this process, the product t-norm is usually used as the conjunction operator $\mathfrak{c}$ in Eq. (8). In this present paper, we propose to replace the product t-norm by different n-dimensional overlap functions On. Those functions are considered in the construction of the n-dimensional (admissibly ordered) iv-overlap functions used in the IV-FRM (described in the sequence). This change is important because in this manner we can learn different fuzzy rules (resulting in different IV-FRBCSs) depending on the function On.

• Modelling the linguistic labels of the learned FRBCS by means of IVFSs;

• The generation of an initial IV-REF for each variable of the problem.

2) To apply an optimization approach with a double purpose:

[1]For an in-depth look at each step of the IVTURS algorithm, see [43].

• To learn the best values of the IV-REFs' parameters;

• To apply a rule selection process in order to decrease the system's complexity.

Once the interval-valued fuzzy rules composing the system have been created, let us modify the mechanism for classifying new examples. Thus, let $\vec{x_p} = (x_{p1}, \ldots, x_{pn})$ be a new example to be classified, L being the number of rules in the rule base and M being the number of classes of the problem. The steps of the new IV-FRM are the following:

(1) *Interval matching degree*: It represents the strength of the activation of the if-part of the rules for each x_p. We use an IV-REF IR to compute the similarity between the interval membership degrees (of each variable of the pattern to the corresponding IVFS) and the ideal membership degree $[1, 1]$, and then, we apply an interval-valued function $F_O : L([0, 1])^n \to L([0, 1])$, for $j \in \{1, ..., L\}$ as follows:

$$\left[\underline{\mathcal{A}_j(x_p)}, \overline{\mathcal{A}_j(x_p)}\right] = F_O\left(IR\left(\left[\underline{\mathcal{A}_{j1}(x_{p1})}, \overline{\mathcal{A}_{j1}(x_{p1})}\right], [1,1]\right), \ldots, IR\left(\left[\underline{\mathcal{A}_{jn}(x_{pn})}, \overline{\mathcal{A}_{jn}(x_{pn})}\right], [1,1]\right)\right),$$

with F_O being an interval conjunction operator that can be defined in two different ways:

a) IOn, an o-representable n-dimensional iv-overlap function;

b) AOn^{α}, an n-dimensional $\leq_{\alpha,\beta}$-overlap function, with $\leq_{\alpha,\beta}$ being the same order applied in Step (4) of the IV-FRM.

As we have mentioned previously, F_O is defined based on the n-dimensional overlap function On applied as the conjunction operator when generating the initial FRBCS.

(2) *Interval association degree*: For the class of each rule, the interval matching degree is weighted with the corresponding iv-rule weight $IRW_j^k \in L([0, 1])$, through an interval-valued function $F_P : L([0, 1])^n \to L([0, 1])$, resulting in:

$$\left[\underline{b_j^k}, \overline{b_j^k}\right] = F_P\left(\left[\underline{\mathcal{A}_j(x_p)}, \overline{\mathcal{A}_j(x_p)}\right], \left[\underline{IRW_j^k}, \overline{IRW_j^k}\right]\right), \tag{9}$$

with $k = 1, \ldots, M$, $j = 1, \ldots, L$ and F_P being defined according to the function F_O applied to obtain the interval matching degree in Step (1), resulting in two possibilities:

a) If $F_O = IOn$, then $F_P = IOn_P = \widehat{On_P}$ (representable interval product overlap), with On_p shown in Table I;

b) If $F_O = AOn^{\alpha}$, then $F_P = AOn_P^{\alpha}$ (admissibly ordered interval product overlap), with AOn_P^{α} being an n-dimensional $\leq_{\alpha,\beta}$-overlap function defined through the construction method presented in Theorem. 4 considering On_p as shown in Table I and the same α as the one in the chosen $\leq_{\alpha,\beta}$ order to be applied in Step (4) of the IV-FRM.

For the rule weight, we utilize the interval-valued confidence value as in [58]. The resulting equation is shown as follows:

$$IRW_j = \sum_{x_p \in C'_j} \left[\underline{\mathcal{A}_j(x_p)}, \overline{\mathcal{A}_j(x_p)}\right] \div_H \sum_{p=1}^{P} \left[\underline{\mathcal{A}_j(x_p)}, \overline{\mathcal{A}_j(x_p)}\right].$$

(3) *Interval pattern classification soundness degree for all classes*: We aggregate the interval association degrees of each

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class (obtained in Step (2)) in which the upper bound is greater than 0 by applying an interval-valued aggregation function IA:

$$\left[\underline{Y_k}, \overline{Y_k}\right] = IA\left(\left[\underline{b_j^k}, \overline{b_j^k}\right], j = 1, \ldots, L \text{ and } \overline{b_j^k} > 0\right),$$

with $k = 1, \ldots, M$.

(4) *Classification*: A decision function F is applied over the interval soundness degree of the system for the pattern classification for all classes, given by:

$$F\left(\left[\underline{Y_1}, \overline{Y_1}\right], \ldots, \left[\underline{Y_M}, \overline{Y_M}\right]\right) = \arg\max_{k=1,\ldots,M}\left(\left[\underline{Y_k}, \overline{Y_k}\right]\right).$$

The last step of the IV-FRM consists of selecting the maximum interval soundness degree. To avoid a stalemate, the usage of a total order for intervals is preferred in this step. So, we use an admissible order $\leq_{\alpha,\beta}$ as defined in Def. 5.

One can observe that there are many possibilities of configuration of this new IV-FRM, based on the chosen functions F_O, F_P and the admissible order $\leq_{\alpha,\beta}$. However, as some of those choices are interconnected, we first decide on the admissible order $\leq_{\alpha,\beta}$ to be used in Step (4), as it determines the α applied in the construction of the n-dimensional $\leq_{\alpha,\beta}$-overlap functions when $F_O = AOn^{\alpha}$ and $F_P = AOn^{\alpha}_P$.

Notice that the interval-valued function F_O plays a key role because it determines the choice of: 1) the n-dimensional overlap function On used in the rule learning process; 2) the interval-valued function F_P used in Step (2) of the IV-FRM.

C. Experimental Framework

To analyze the behaviour of a classification system when applying different n-dimensional (admissibly ordered) iv-overlap functions and different admissible orders, we have selected 31 real-world data-sets from the KEEL repository [59], which are publicly available on the webpage (http://www.keel.es/dataset.php). Table II summarizes the properties of the selected data-sets, showing for each data-set the number of attributes (Atts.), the number of examples (Ex.), and the number of classes (Class.). We must point out that the *magic, page-blocks, penbased, ring, satimage, shuttle*, and *twonorm* data-sets have been stratified sampled at 10% in order to improve the learning process efficiency. Missing values from *bands*, *cleaveland* and *wisconsin* data-sets have been removed before the experimentation.

A *fivefold cross-validation model* has been applied in order to carry out the different experiments. This was done by splitting the data-set into five random partitions of data, employing a combination of four of them (80%) to train the system and the remaining one (20%) to test it. This process is carried out 5 times, changing the testing partition in each iteration. The performance measure was done through the accuracy rate.

The set-up of the IVTURS classifier is as recommended in [43], but we apply our new theoretical developments described in Sections V-A and V-B. We study the behaviour of the classifier using several combinations of the new theoretical concepts, as shown in Table III. Looking at Table III, we can clearly observe that the interval-valued conjunction operator (F_O) used in Step (1) the IV-FRM determines the overlap function (On) used when generating the initial fuzzy rules as well as the interval product (F_P) used in the Step (2) of the IV-FRM. We must point out in the case of ADM-Min, $\min_{\leq_{\alpha,\beta}}$ is simply the n-dimensional interval minimum with respect to the $\leq_{\alpha,\beta}$ order at hand, as in Def. 6. Finally, for each combination we check the influence of the admissible order used in Step (4) of the IV-FRM. Specifically, we test three linear orders for intervals: $\leq_{Lex1}$ ($\alpha = 0, \beta = 1$), $\leq_{IQ}$ ($\alpha = 0.5, 0$) and $\leq_{Lex2}$ ($\alpha = 1, \beta = 0$)[2].

To give statistical support to our analysis, we use the aligned Friedman ranks test [60] to detect statistical differences among a group of results and report the obtained ranks of each method (with lower ranks being preferable). Next, we apply the Holm's post-hoc test [61] to compare the best ranking method with the other considered methods. Finally, we apply a Wilcoxon

TABLE II: Summary of the employed datasets

id	Data-set	Atts.	Ex.	Class.
app	appendicitis	7	106	2
bal	balance	4	625	3
ban	banana	2	5300	2
bds	bands	19	365	2
bup	bupa	6	345	2
clv	cleveland	13	297	5
con	contraceptive	9	1473	3
eco	ecoli	7	336	8
gla	glass	9	214	7
hab	haberman	3	306	2
hay	hayes-hoth	4	160	3
ion	ionosphere	33	351	2
iri	iris	4	150	3
led	led7digit	7	500	10
mag	magic	10	19020	2
new	newthyroid	5	215	3
pag	pageblocks	10	5472	5
pen	penbased	16	10992	10
pho	phoneme	5	5404	2
pim	pima	8	768	2
rin	ring	20	7400	2
sah	saheart	9	462	2
sat	satimage	36	6435	7
shu	shuttle	9	58000	7
spe	spectfheart	44	267	2
tit	titanic	3	2201	2
two	twonorm	20	7400	2
veh	vehicle	18	846	4
win	wine	13	178	3
wis	wisconsin	9	683	2
yea	yeast	8	1484	10

TABLE III: Configuration schemes for the used classifiers

Classifier identifier	On	F_O	F_P
REP-Prod	On_P	$IOn_P = \widehat{On_P}$	$IOn_P = \widehat{On_P}$
REP-Min	On_M	$IOn_M = \widehat{On_M}$	$IOn_P = \widehat{On_P}$
REP-Hp	On_{Hp}	$IOn_{Hp} = \widehat{On_{Hp}}$	$IOn_P = \widehat{On_P}$
REP-OB	On_{OB}	$IOn_{OB} = \widehat{On_O B}$	$IOn_P = \widehat{On_P}$
REP-Gm	On_{Gm}	$IOn_{Gm} = \widehat{On_{Gm}}$	$IOn_P = \widehat{On_P}$
REP-Hm	On_{Hm}	$IOn_{Hm} = \widehat{On_{Hm}}$	$IOn_P = \widehat{On_P}$
ADM-Prod	On_P	AOn^{α}_P	$F_P = AOn^{\alpha}_P$
ADM-Min	On_M	$\min_{\leq_{\alpha,\beta}}$	AOn^{α}_P
ADM-Hp	On_{Hp}	AOn^{α}_{Hp}	AOn^{α}_P
ADM-OB	On_{OB}	AOn^{α}_{OB}	AOn^{α}_P
ADM-Gm	On_{Gm}	AOn^{α}_{Gm}	AOn^{α}_P
ADM-Hm	On_{Hm}	AOn^{α}_{Hm}	AOn^{α}_P

[2]To respect Theorem 4, in all experiments with ADM classifiers we consider $\alpha = 0 + 1^{-10}$ and $\alpha = 1 - 1^{-10}$, for $\leq_{Lex1}$ and $\leq_{Lex2}$, respectively.

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TABLE IV: Results in testing for the different methods

Method	$\leq_{Lex1}$	$\leq_{IQ}$	$\leq_{Lex2}$
REP-Prod	78.96	79.67	79.17
REP-Min	78.92	79.52	79.51
REP-Hp	79.08	79.34	79.35
REP-OB	79.00	79.83	79.33
REP-Gm	79.14	79.48	79.57
REP-Hm	79.19	79.39	79.41
ADM-Prod	**79.49**	79.14	79.19
ADM-Min	79.22	79.39	79.40
ADM-Hp	79.28	79.47	79.54
ADM-OB	79.25	79.57	**79.64**
ADM-Gm	79.17	**79.93**	79.49
ADM-Hm	78.93	79.23	79.16

Signed-Ranks test [62] in order to do pairwise comparisons. This selection of tests is suggested in [63], where it is shown that its use in machine learning is highly recommended.

D. Discussion of the Results

In Table IV we show the averaged results in testing for all the possible combinations among the three orders (by columns) and the configurations shown in Table III (by rows). The result we show is the averaged behaviour of the system in the 31 datasets considered in the study. For each admissible order, we highlight in bold face the best result, that is, the best n-dimensional (admissibly ordered) overlap function. The detailed results, that is, the results in all the datasets (in all the partitions) for all the combinations can be queried on the webpage (https://github.com/tiagoasmus/TestingResults-Adm-Overlaps/find/master?q=).

By looking at the highlighted results in Table IV, we see that, for each admissible order, the best performing configuration of the algorithm (regarding the global mean) was based on an $\leq_{AD}$-overlap function (ADM-Prod, ADM-Gm and ADM-OB). Furthermore, it appears that both the admissible order and the (interval-valued) conjunction operators have an impact on the accuracy obtained by each classifier.

In first place we studied if there are differences in the accuracy for a given method when we vary the chosen admissible order. In order to do so, we applied the aligned test to compare the three total orders for each configuration. The obtained ranks, as well as the Adjusted P-Values (APVs, presented in brackets) provided by the Holm's post hoc test are shown in Table V, where we have highlighted in **bold-face** the best rank (the least one) and we have stressed with an asterisk (*) those cases in which there are statistical differences (using $\alpha = 0.05$) between the control method (the one associated with the best rank) and the method in the corresponding total order.

From the results in Table V, one can observe:

1) The order $\leq_{Lex1}$ is the control method for only one configuration (ADM-Prod), being the worst ranking method in most cases, with statistical differences in several comparisons;
2) Although the order $\leq_{Lex2}$ is considered the control method in six configurations, in all those cases there are no significant differences with respect to the order $\leq_{IQ}$, with both orders presenting similar ranks;
3) The order $\leq_{IQ}$ is the control method in five configurations, and in two of those cases, it presents statistical differences versus $\leq_{Lex2}$ (and a low APV for ADM-Gm). Furthermore, it produces comparable results with the other orders even when it is not the control method.

TABLE V: Average Rankings of the algorithms (Aligned Friedman) - Comparing $\leq_{\alpha,\beta}$ orders

Method	$\leq_{Lex1}$	$\leq_{IQ}$	$\leq_{Lex2}$
REP-Prod	55.16 (0.011)*	**36.11** (-)	49.73 (0.047)*
REP-Min	57.94 (0.033)*	41.55 (0.996)	**41.52** (-)
REP-Hp	52.29 (0.466)	44.60 (0.944)	**44.11** (-)
REP-OB	57.81 (0.001)*	**33.73** (-)	49.47 (0.022)*
REP-Gm	56.24 (0.065)	**41.60** (-)	43.16 (0.819)
REP-Hm	51.32 (0.567)	45.71 (0.799)	**43.9677** (-)
ADM-Prod	**43.05** (-)	48.95 (0.771)	49.00 (0.771)
ADM-Min	47.11 (1.000)	48.61 (1.000)	**45.27** (-)
ADM-Hp	49.69 (0.9230)	46.66 (0.9230)	**44.65** (-)
ADM-OB	52.58 (0.4129)	44.50 (0.9325)	**43.92** (-)
ADM-Gm	57.95 (0.002)*	**35.13** (-)	47.92 (0.062)
ADM-Hm	45.00 (0.079)	**40.89** (-)	45.11 (0.538)

TABLE VI: Average Rankings of the algorithms (Aligned Friedman)

	Group REP			Group ADM	
Method	Rank	APV	Method	Rank	APV
REP-Prod	84.82	0.453	ADM-Prod	100.74	0.186
REP-Min	98.24	0.250	ADM-Min	99.02	0.187
REP-Hp	104.61	0.112	ADM-Hp	97.03	0.187
REP-OB	**74.57**	-	ADM-OB	87.65	0.302
REP-Gm	92.11	0.399	ADM-Gm	**73.52**	-
REP-Hm	106.65	0.095	ADM-Hm	103.05	0.154

In summary, we can conclude that $\leq_{Lex1}$ is not a suitable choice and $\leq_{IQ}$ is providing a robust behaviour regardless of the configuration. For these reasons, we decided to investigate the behaviour of our classifiers by varying the n-dimensional (admissibly ordered) iv-overlap functions used in the IV-FRM, taking in consideration the admissible order $\leq_{IQ}$.

To do it, we divided the methods into two groups, based on the interval conjunction operator (F_O) applied in Step (1) of the IV-FRM: representable (REP) and admissibly ordered (ADM) n-dimensional iv-overlap functions. We applied the Aligned Friedman and Holm's tests to compare the six n-dimensional (admissibly ordered) iv-overlap function belonging to each group. The results obtained for the functions of groups REP and ADM are shown in Tables VI, with the best ranking method in each group highlighted in **bold-face**.

From the results presented in Table VI, one can observe:

1) The behaviours of the representable n-dimensional iv-overlap functions are similar, but REP-OB seems to be the best option among its group;
2) Though there are not statistical differences among the n-dimensional admissibly ordered iv-overlap functions, ADM-

TABLE VII: Pairwise comparisons via Wilcoxon test

Comparison	R^+	R^-	p-value
REP-OB vs ADM-Gm	230	266	0.649
REP-Prod vs ADM-Gm	195	301	0.285

Gm stands out as it obtains low APVs versus the remained functions in its group, except for ADM-OB.

An interesting observation is that both control methods (REP-OB and ADM-Gm) and their respective interval conjunction operations (IOn_{OB} and $AOn^{0.5}_{Gm}$) are based on non-associative operations (OB overlap and geometric mean), pointing out that n-dimensional overlap functions and their interval extensions are suitable to be applied in IV-FRBCS.

Next, we carry out a pairwise comparison between the two representatives of each group (control methods), using the Wilcoxon test. We also compare the best performing method overall (ADM-Gm) with the original IVTURS (which is obtained using the REP-Prod configuration and the order $\leq_{IQ}$). The results for of these two last pairwise comparisons can be seen in Table VII.

As first indicated by the global means and afterwards confirmed by the statistical analysis, the combination of the admissible order $\leq_{IQ}$ and the n-dimensional $\leq_{IQ}$-overlap function $AOn^{0.5}_{Gm}$ in the ADM-Gm method produces the most accurate classification results. It does not statistically improve the performance over all other configurations, but in the light of the obtained results, we can recommend it as the best option for this type of IV-FRBCS.

VI. Conclusion

In this paper, we presented new results regarding admissible orders and defined the concept of n-dimensional admissibly ordered interval-valued overlap functions. A width-preserving construction method for this type of function for a given admissible order was also presented, which allowed us to define different n-dimensional $\leq_{AD}$-overlap functions to be applied in the IV-FRM of IVTURS.

On the application side, our experimentation made clear the impact of the chosen admissible order on IV-FRBCSs, with the order $\leq_{IQ}$ presenting itself as the most robust one. We also conclude that n-dimensional (admissibly ordered) interval-valued overlap functions, particularly the non-associative ones, are recommended to be applied on the IV-FRM of an IV-FRBCSs, with a special mention to the n-dimensional $\leq_{IQ}$-overlap function $AOn^{0.5}_{Gm}$.

All of the aforementioned contributions aimed to address: (i) the theoretical and applied gap in the literature regarding the configuration possibilities of IV-FRBCSs; (ii) the characteristics of the applied interval-valued functions and related interval orders.

As future work, we intend to further research on the effect of different n-dimensional interval-valued aggregation functions (such as the ones studied in this paper) in the interval pattern classification soundness degree for all classes (Step (3) of the IV-FRM). Particularly, the relation between such interval functions applied in this third stage and the admissible orders chosen for the decision making in the classification phase (last stage of the IV-FRM).

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5.1.3 Towards interval uncertainty propagation control in bivariate aggregation processes and the introduction of width-limited interval-valued overlap functions

Related publication:

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Towards interval uncertainty propagation control in bivariate aggregation processes and the introduction of width-limited interval-valued overlap functions

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Abstract

Overlap functions are a class of aggregation functions that measure the overlapping degree between two values. They have been successfully applied as a fuzzy conjunction operation in several problems in which associativity is not required, such as image processing and classification. Interval-valued overlap functions were defined as an extension to express the overlapping of interval-valued data, and they have been usually applied when there is uncertainty regarding the assignment of membership degrees, as in interval-valued fuzzy rule-based classification systems. In this context, the choice of a total order for intervals can be significant, which motivated the recent developments on interval-valued aggregation functions and interval-valued overlap functions that are increasing to a given admissible order, that is, a total order that refines the usual partial order for intervals. Also, width preservation has been considered on these recent works, in an intent to avoid the uncertainty increase and guarantee the information quality, but no deeper study was made regarding the relation between the widths of the input intervals and the output interval, when applying interval-valued functions, or how one can control such uncertainty propagation based on this relation. Thus, in this paper we: (i) introduce and develop the concepts of width-limited interval-valued functions and width limiting functions, presenting a theoretical approach to analyze the relation between the widths of the input and output intervals of bivariate interval-valued functions, with special attention to interval-valued aggregation functions; (ii) introduce the concept of (a, b)-ultramodular aggregation functions, a less restrictive extension of one-dimension convexity for bivariate aggregation functions, which have an important predictable

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behaviour with respect to the width when extended to the interval-valued context; (iii) define width-limited interval-valued overlap functions, taking into account a function that controls the width of the output interval and a new notion of increasingness with respect to a pair of partial orders $(\leq_1, \leq_2)$; (iv) present and compare three construction methods for these width-limited interval-valued overlap functions, considering a pair of orders $(\leq_1, \leq_2)$, which may be admissible or not, showcasing the adaptability of our developments.

Keywords: Aggregation functions; Overlap functions; Interval-valued aggregation functions; Interval-valued overlap functions; Admissible orders

1. Introduction

Aggregation functions are useful operators that combine (fuse) several numerical values into a single representative one, being especially suitable to model fuzzy logic operations and they have been widely employed in several theoretical and applied fields [1,2].

Overlap functions are a particular class of aggregation functions that do not need to be associative, and they were originally defined as continuous functions in order to deal with the overlapping between classes in image processing problems [3–5]. They have quickly risen in popularity due to some desirable properties that they present. In [6,7], one can find clear discussions on the advantages that overlap functions have over the popular t-norms. For example, overlap functions are closed to the convex sum and the aggregation by internal generalized composition, whereas t-norms are not. Also, overlap functions showed good results when applied in problems in which associativity of the employed aggregation operator is not required, as in fuzzy rule-based classification [8–10], decision making [11], wavelet-fuzzy power quality diagnosis system [12] or forest fire detection [13], among others.

In fuzzy modelling, there may be uncertainty regarding the values of the membership degrees or the definition of the membership functions to be employed in the system [14]. One possible solution is the adoption of interval-valued fuzzy sets (IVFSs) [15,16], where the membership degrees are represented by intervals. In this manner, the widths of the assigned intervals are intrinsically related with the uncertainty/ignorance with respect to the modelling of the fuzzy sets [17–19]. IVFSs have been successfully applied in many different fields, such as classification [20,21], image processing [22], game theory [23], multicriteria decision making [24], pest control [25], irrigation systems [26] and collaborative clustering [27].

Interval-valued aggregation functions were defined in [28], in order to model the aggregation of interval-valued data in the unit interval. Following a similar approach, interval-valued overlap functions were defined, independently, by Qiao and Hu [29] and Bedregal et al. [30], as an extension of overlap functions to the interval-valued context. By extending and generalizing the concept of interval-valued overlap functions, Asmus et al. [21] introduced the concepts of n-dimensional interval-valued overlap functions and general interval-valued overlap functions.

It is important to observe that not all interval-valued aggregation functions are interval extensions of some known aggregation function on the unit interval [31]. Nevertheless, it is noteworthy that most popular definitions of interval-valued aggregation functions were developed to properly encompass the result of interval extensions of well known aggregation functions following the optimality (the least possible interval width) and correctness (the unknown value of the extended operation is contained in the resulting interval) criteria for interval representation (also called, the best interval representation), as discussed in [17,31], and taking into account the usual product order when comparing intervals [32]. Although this approach is both intuitive and theoretically sound, it may present some drawbacks on the application side:

(i) Observe that, although it is natural that the uncertainty carried out by intervals leads to a partial order, as the product order (and also the Fishburn interval orders [33]), one may face data that is not comparable, which is a serious hindrance in problems such as decision making and classification [34], in which the system must always decide and rely in just one interval result when comparing all possible alternatives;

(ii) In many interval-valued processes, the output intervals' widths become larger than a desirable threshold, according to the widths of the input intervals, which may be imposed by applications constraints concerning the quality of the information required for the interval results. In those cases, the interval outputs, although correct, usually carry no meaningful information about the exact value they are actually approximating [35,36].

To solve the first problem **(i)**, avoiding a stalemate when comparing intervals, Bustince et al. [34] introduced the concept of admissible orders, that is, total orders that refine the product order, in the sense that they coincide with the product order whenever the intervals are comparable. In particular, they defined the $\leq_{\alpha,\beta}$ order, based on an operator K_α that corresponds to the Hurwicz's criterion [37] for balancing pessimism and optimism under uncertainty [38].

Since then, many works using admissible orders have appeared in the literature, for example, [34,38–40]. In particular, Bustince et al. [38] presented a construction method for interval-valued aggregation functions that are increasing with respect to a given admissible order. In a similar line of work, Asmus et al. [40] introduced the concept of n-dimensional admissibly ordered interval-valued overlap functions, which are n-dimensional interval-valued overlap functions that are increasing with respect to an admissible order.

In an initial attempt to deal with the second drawback **(ii)**, the construction method presented by Bustince et al. [38] produces interval-valued aggregation functions that can be width-preserving whenever some restrictive conditions are satisfied. Note that the width of the interval output of a width-preserving interval-valued function is equal to the width of the interval inputs, when they all have the same width. However, Bustince et al. [38] clearly state that, ideally, the definition of width preservation would have to take into account the width of the interval inputs in every case, not only when those inputs have the same length, which we call the drawback **(iii)** to be overcome.

Considering the second problem **(ii)**, Asmus et al. [40] presented a construction method for n-dimensional admissibly ordered interval-valued overlap functions in which the width of the output is always less or equal to the minimal width of the inputs (see Theorem 2.2 in Section 2), which also comes to avoid the problem **(iii)**. However, this type of minimal width limitation has two sides: on one hand, as desired, the functions produced by the method avoid an increasing width (uncertainty) propagation; on the other hand, unfortunately, just one degenerate input interval (that is, with width equal to zero) is sufficient to completely remove all uncertainty of the output interval, which is clearly counterintuitive, to say the least.

Thus, the study of the relation between the width of the inputs and the output of interval-valued fuzzy operations coupled with adaptable tools to limit the increasing uncertainty in the output of such operations is still a challenge to overcome in the literature, especially regarding interval-valued aggregation and interval-valued overlap functions, which are of our particular interest.

The development of models that help to avoid that the interval outputs' widths become larger than the expected/required in practical applications certainly will increase the applicability of interval-valued fuzzy-based tools to solve many problems, as in interval-valued fuzzy-rule based classification systems (see, e.g.: [19,40,41]) and decision making (see, e.g.: [42,43]), by providing interval outputs with better information quality. We point out that the information quality of interval-valued results is a strong requirement claimed by scientists and engineers interested in interval-based tools [32].

So, in order to present a contribution to solve the problems **(ii)** and **(iii)** in the context of interval-valued overlap functions, and even in a more general framework, without disregarding the problem **(i)**, this paper has the following general objective:

- To develop a theoretical approach to aid the analysis of bivariate interval-valued operations with respect to the width of the operated intervals in order to control the uncertainty propagation, with special attention to interval-valued overlap functions, admissibly ordered or not.

To accomplish this goal, we have the following specific objectives:

1) To introduce the concepts of width-limited interval-valued functions and width-limiting functions, which are theoretical tools to study the relation between the widths of the inputs with the width of the output of interval-valued functions, necessary for the construction of interval-valued functions with controlled uncertainty propagation;
2) To define (a, b)-ultramodular aggregation functions, a less restrictive extension of one-dimension convexity for bivariate aggregation functions, which shall have an important predictable behaviour with respect to their interval output widths when extended to the interval-valued context;
3) To study the relation between some width-limited interval-valued functions and their respective width-limiting functions, especially when dealing with (a, b)-ultramodular aggregation functions, giving some backdrop for future comparisons with similarly constructed interval-valued functions;

4) To define the notion of increasingness for a pair of partial orders, allowing for more flexible construction methods for width-limited interval-valued functions;
5) To introduce the concept of width-limited interval-valued overlap functions, taking into account a width-limiting function and a pair of partial orders, which allows the definition of interval-valued overlap operations that provide output intervals that do not exceed a desirable uncertainty (width) threshold;
6) To study the relation between width-limited interval-valued overlap functions and some of their width-limiting functions, particularly when considering the best interval representation of some overlap function;
7) To present and study three construction methods for width-limited interval-valued overlap functions, presenting examples and comparisons between them to showcase the versatility and applicability of our approach.

Regarding the paper organization, in Section 2 we present some preliminary concepts, followed by Section 3, where Specific Objectives 1-3 are addressed. In Section 4, we encompass Specific Objectives 4-7, with the final remarks being presented in Section 5.

2. Preliminaries

In this section, we recall some concepts on (ultramodular) aggregation functions, overlap functions, interval mathematics, admissible orders and (admissibly ordered) interval-valued overlap functions.

2.1. Fuzzy negations and aggregation functions

Definition 2.1. [44] A function $N : [0, 1] \to [0, 1]$ is a fuzzy negation if the following conditions hold:

(N1) $N(0) = 1$ and $N(1) = 0$;
(N2) If $x \leq y$ then $N(y) \leq N(x)$, for all $x, y \in [0, 1]$.

If N also satisfies the involutive property,

(N3) $N(N(x)) = x$, for all $x \in [0, 1]$,

then it is said to be a strong fuzzy negation.

Example 2.1. The Zadeh negation given by

$$N_Z(x) = 1 - x,$$

is a strong fuzzy negation [44].

Definition 2.2. [44] Given a fuzzy negation $N : [0, 1] \to [0, 1]$ and a function $F : [0, 1]^2 \to [0, 1]$, then the function $F^N : [0, 1]^2 \to [0, 1]$ defined, for all $x, y \in [0, 1]$, by

$$F^N(x, y) = N(F(N(x), N(y))), \tag{1}$$

is the N-dual of F.

When it is clear by the context, the N_Z-dual function (dual with respect to the Zadeh negation) of F will be just called dual of F, and will be denoted by F^d.

Definition 2.3. [2] An aggregation function is any function $A : [0, 1]^n \to [0, 1]$ that satisfies the following conditions:

(A1) A is increasing in each argument;
(A2) $A(0, \ldots, 0) = 0$ and $A(1, \ldots, 1) = 1$.

Table 1
Examples of overlap functions.

Name	Definition
Product	$O_P(x, y) = x \cdot y$
Minimum	$O_M(x, y) = \min\{x, y\}$
Geom. Mean	$O_{Gm}(x, y) = \sqrt{x \cdot y}$
OmM Overlap	$O_M(x, y) = \min\{x, y\} \cdot \max\{x^2, y^2\}$
OB Overlap	$O_{OB}(x, y) = \min\{x\sqrt{y}, y\sqrt{x}\}$
Ot Overlap	$O_t(x, y) = \frac{(2x-1)^3+1}{2} \cdot \frac{(2y-1)^3+1}{2}$

The Ot Overlap is an original definition introduced here.
The others can be found in the literature (e.g. [10]).

Example 2.2. For $\alpha \in [0, 1]$, the mapping $K_\alpha : [0, 1]^2 \to [0, 1]$, defined, for all $x, y \in [0, 1]$, by

$$K_\alpha(x, y) = x + \alpha \cdot (y - x), \tag{2}$$

is an aggregation function.

Observe that the operator K_α corresponds to Hurwicz's criterion [37] for adjusting pessimism and optimism under uncertainty, when working in contexts of imperfect information.

In [2], one may find the concepts of conjunctive and disjunctive aggregation function. In this paper, we need a more general definition:

Definition 2.4. Consider a function $F : [0, 1]^2 \to [0, 1]$. Then, F is said to be:

a) Conjunctive, if $F(x, y) \leq \min\{x, y\}$ for all $x, y \in [0, 1]$;
b) Disjunctive, if $F(x, y) \geq \max\{x, y\}$ for all $x, y \in [0, 1]$.

The definition of ultramodular aggregation functions is a key concept in this work:

Definition 2.5. [45] An aggregation function $A : [0, 1]^2 \to [0, 1]$ is called ultramodular if, for all $x_1, x_2, y_1, y_2, \epsilon, \delta \in [0, 1]$, such that $x_2 + \epsilon, y_2 + \delta \in [0, 1]$, $x_1 \leq x_2$ and $y_1 \leq y_2$, it holds that:

$$A(x_1 + \epsilon, y_1 + \delta) - A(x_1, y_1) \leq A(x_2 + \epsilon, y_2 + \delta) - A(x_2, y_2). \tag{3}$$

Proposition 2.1. *[45] Assume that all partial derivatives of order 2 of the aggregation function $A : [0, 1]^2 \to [0, 1]$ exist. Then A is ultramodular if and only if all partial derivatives of order 2 are non-negative.*

Theorem 2.1. *[45] Let $A_1, A_2, A_3 : [0, 1]^2 \to [0, 1]$ be ultramodular aggregation functions. Then, the composite function $A : [0, 1]^2 \to [0, 1]$ given, for all $x, y \in [0, 1]$, by $A(x, y) = A_3(A_1(x, y), A_2(x, y))$ is an ultramodular aggregation function.*

Corollary 2.1. *[45] Let $A_1, A_2 : [0, 1]^2 \to [0, 1]$ be ultramodular aggregation functions and $K_\alpha : [0, 1]^2 \to [0, 1]$ as defined in Equation (2). Then, we have that the function $A_\alpha : [0, 1]^2 \to [0, 1]$ given, for all $x, y, \alpha \in [0, 1]$, by $A_\alpha(x, y) = K_\alpha(A_1(x, y), A_2(x, y))$, is an ultramodular aggregation function.*

Example 2.3. The following are examples of ultramodular aggregation functions:

1) The weighted sum K_α, as defined in Equation (2);
2) The product overlap (see Table 1).

By Propositions 2.2 and 2.7 in [45], the following result is immediate.

Proposition 2.2. *Let* $A : [0,1]^2 \to [0,1]$ *be an ultramodular aggregation function. Then, it holds that:*

$$A(x^*, y) + A(x, y^*) \le A(x^*, y^*) + A(x, y),$$

for all $x^*, y^*, x, y \in [0,1]$ *such that* $x \le x^*$ *and* $y \le y^*$.

Observe that Proposition 2.2 also follows directly from Definition 2.5, by taking $x_1 = x_2 = x$, $y_1 = y$, $y_2 = y^*$, $\epsilon = x^* - x$ and $\delta = 0$.

Definition 2.6. [3,46] An overlap function is any bivariate function $O : [0,1]^2 \to [0,1]$ that satisfies the following conditions, for all $x, y \in [0,1]$:

(O1) O is commutative;
(O2) $O(x, y) = 0$ if and only if $xy = 0$;
(O3) $O(x, y) = 1$ if and only if $xy = 1$;
(O4) O is increasing;
(O5) O is continuous.

Note that an overlap function is, in particular, an aggregation function. If for all $x, y, z \in (0,1]$ one has that $x < y \Leftrightarrow O(x,z) < O(y,z)$, then O is called a *strict overlap function*.

By Theorem 4 in [3], one has that:

Proposition 2.3. *Let* $O_1, O_2, O_3 : [0,1]^2 \to [0,1]$ *be overlap functions. Then, the composite function* $O_C : [0,1]^2 \to [0,1]$ *given, for all* $x, y \in [0,1]$ *by* $O_C(x,y) = O_3(O_1(x,y), O_2(x,y))$ *is an overlap function.*

Proposition 2.4. *[3] Let* $O_1, O_2 : [0,1]^2 \to [0,1]$ *be overlap functions. Then, we have that function* $O_\alpha : [0,1]^2 \to [0,1]$ *given, for all* $x, y, \alpha \in [0,1]$*, by* $O_\alpha(x,y) = K_\alpha(O_1(x,y), O_2(x,y))$ *is an overlap function.*

The theoretical development of both overlap functions and their dual (grouping functions) are summarized by Bustince et al. in [5]. Additionally, some examples of studies on overlap and grouping functions are described in the sequence. The basic properties of overlap functions and grouping functions, like homogeneity, migrativity and idempotency, were studied by Bedregal et al. in [46]. Archimedean overlap functions were introduced by Dimuro et al. in [47]. Additive generators of overlap functions and grouping functions were introduced by Dimuro et al. in [48,49], and their multiplicative generators by Qiao et al. in [50]. Further studies on the migrativity property of overlap functions were presented in [51,52]. Dimuro et al. developed the concept of fuzzy implication functions derived overlap and grouping functions in [7,53,54]. The properties of such fuzzy implications were studied in [6,55]. Extensions of overlap and grouping functions to the n-dimensional context were studied in [56,57].

2.2. *Interval mathematics*

Let us denote as $L([0,1])$ the set of all closed subintervals of the unit interval $[0,1]$. Given any $X = [x_1, x_2] \in L([0,1])$, $\underline{X} = x_1$ and $\overline{X} = x_2$ denote, respectively, the left and right projections of X, and $w(X) = \overline{X} - \underline{X}$ denotes the width of X. When $\underline{X} = \overline{X}$, and consequently $w(X) = 0$, we call X a degenerate interval.

The interval product is defined, for all $X, Y \in L([0,1])$, by:

$$X \cdot Y = [\underline{X} \cdot \underline{Y}, \overline{X} \cdot \overline{Y}].$$

The product and inclusion partial orders are defined for all $X, Y \in L([0,1])$, respectively, by [32]:

$$X \le_{Pr} Y \Leftrightarrow \underline{X} \le \underline{Y} \wedge \overline{X} \le \overline{Y};$$
$$X \subseteq Y \Leftrightarrow \underline{X} \ge \underline{Y} \wedge \overline{X} \le \overline{Y}.$$

We call as $\le_{Pr}$-increasing a function that is increasing with respect to the product order $\le_{Pr}$. The projections $IF^-, IF^+ : [0,1]^2 \to [0,1]$ of $IF : L([0,1])^2 \to L([0,1])$ are defined, respectively, by:

$$IF^-(x,y) = \underline{IF([x,x],[y,y])}; \tag{4}$$
$$IF^+(x,y) = \overline{IF([x,x],[y,y])}. \tag{5}$$

Given two increasing functions $F, G : [0,1]^2 \to [0,1]$ such that $F \le G$, we define the function $\widehat{F,G} : L([0,1])^2 \to L([0,1])$ as

$$\widehat{F,G}(X,Y) = [F(\underline{X},\underline{Y}), G(\overline{X},\overline{Y})]. \tag{6}$$

An interval-valued function IF is said to be Moore-continuous if it is continuous with respect to the Moore metric [32] $d_M : L([0,1])^2 \to \mathbb{R}$, defined, for all $X, Y \in L([0,1])$, by:

$$d_M(X,Y) = \max(|\underline{X} - \underline{Y}|, |\overline{X} - \overline{Y}|).$$

Definition 2.7. [18] Let $IF : L([0,1])^2 \to L([0,1])$ be an $\le_{Pr}$-increasing interval function. IF is said to be representable if there exist increasing functions $F, G : [0,1]^2 \to [0,1]$ such that $F \le G$ and $F = \widehat{F,G}$.

The functions F and G are the *representatives* of the interval function IF. When $IF = \widehat{F,F}$, we denote simply as $\widehat{F}$. In this case, IF is said to be the best interval representation of F, as in [17,18].

Consider $\alpha \in [0,1]$ and the aggregation function K_α as defined in Equation (2). Then, given an interval $X \in L([0,1])$, we denote $K_\alpha(\underline{X}, \overline{X})$ simply as $K_\alpha(X)$. Also, it is immediate that

$$[K_\alpha(X) - \alpha \cdot w(X), K_\alpha(X) + (1-\alpha)\cdot w(X)] = X, \tag{7}$$

for all $\alpha \in [0,1]$.

2.3. Admissible orders

The notion of admissible orders for intervals came from the interest in extending the product order $\le_{Pr}$ to a total order.

Definition 2.8. [34] Let $(L([0,1]), \le_{AD})$ be a partially ordered set. The order $\le_{AD}$ is called an admissible order if

(i) $\le_{AD}$ is a total order on $L([0,1])$;
(ii) For all $X, Y \in L([0,1])$, $X \le_{AD} Y$ whenever $X \le_{Pr} Y$.

In other words, an order $\le_{AD}$ on $L([0,1])$ is admissible, if it is total and refines the order $\le_{Pr}$ [34].

Proposition 2.5. *[34] Let $A_1, A_2 : [0,1]^2 \to [0,1]$ be two continuous aggregation functions, such that, for all $X, Y \in L([0,1])$, the equalities $A_1(\underline{X},\overline{X}) = A_1(\underline{Y},\overline{Y})$ and $A_2(\underline{X},\overline{X}) = A_2(\underline{Y},\overline{Y})$ can hold only if $X = Y$. Define the relation $\le_{A_1,A_2}$ on $L([0,1])$ by*

$$X \le_{A_1,A_2} Y \Leftrightarrow A_1(\underline{X},\overline{X}) < A_1(\underline{Y},\overline{Y}) \text{ or }$$
$$(A_1(\underline{X},\overline{X}) = A_1(\underline{Y},\overline{Y}) \text{ and } A_2(\underline{X},\overline{X}) \le A_2(\underline{Y},\overline{Y})).$$

Then $\le_{A_1,A_2}$ is an admissible order on $L([0,1])$.

The pair (A_1, A_2) of aggregation functions that generates the order $\le_{A_1,A_2}$ in Proposition 2.5 is called an admissible pair of aggregation functions [34].

For $\alpha, \beta \in [0,1]$, such that $\alpha \ne \beta$, when $A_1 = K_\alpha$ and $A_2 = K_\beta$, with K_α and K_β given by Equation (2), we write $\le_{\alpha,\beta}$ for the order $\le_{K_\alpha,K_\beta}$, which is given by:

$$X \le_{\alpha,\beta} Y \Leftrightarrow K_\alpha(\underline{X},\overline{X}) < K_\alpha(\underline{Y},\overline{Y}) \text{ or } \tag{8}$$
$$(K_\alpha(\underline{X},\overline{X}) = K_\alpha(\underline{Y},\overline{Y}) \text{ and } K_\beta(\underline{X},\overline{X}) \le K_\beta(\underline{Y},\overline{Y})).$$

Lemma 2.1. *[34] For any $\alpha, \beta \in [0,1]$, $\alpha \ne \beta$, it holds that: (i) $\beta > \alpha \Rightarrow \le_{\alpha,\beta} = \le_{\alpha,1}$; (ii) $\beta < \alpha \Rightarrow \le_{\alpha,\beta} = \le_{\alpha,0}$.*

Remark 2.1. By varying the values of α and β one can recover some of the known admissible orders, e.g., the lexicographical orders $\leq_{Lex1}$ and $\leq_{Lex2}$ are recovered, respectively, by $\leq_{0,1}$ and $\leq_{1,0}$, and the Xu and Yager order [58] $\leq_{XY}$ is recovered by $\leq_{0.5,1}$.

2.4. Interval-valued overlap functions

Definition 2.9. [28] An interval-valued function $IA : L([0,1])^2 \to L([0,1])$ is said to be an interval-valued aggregation function if the following conditions hold:

(IA1) IA is $\leq_{Pr}$-increasing;
(IA2) $IA([0,0],[0,0]) = [0,0]$ and $IA([1,1],[1,1]) = [1,1]$.

Definition 2.10. [29,30] An interval-valued (iv) overlap function is a mapping $IO : L([0,1])^2 \to L([0,1])$ that respects the following conditions:

(IO1) IO is commutative;
(IO2) $IO(X,Y) = [0,0]$ if and only if $X \cdot Y = [0,0]$;
(IO3) $IO(X,Y) = [1,1]$ if and only if $X \cdot Y = [1,1]$;
(IO4) IO is $\leq_{Pr}$-increasing in the first component: $IO(Y,X) \leq_{Pr} IO(Z,X)$ when $Y \leq_{Pr} Z$.
(IO5) IO is Moore continuous.

Note that, by **(IO1)** and **(IO4)**, iv-overlap functions are also monotonic in the second component.

An iv-overlap function $IO : L([0,1])^2 \to L([0,1])$ is said to be o-representable [21] if there exist overlap functions $O_1, O_2 : [0,1]^2 \to [0,1]$ such that $O_1 \leq O_2$ and $IO = \widehat{O_1, O_2}$.

Definition 2.11. [40] A function $AO : L([0,1])^2 \to L([0,1])$ is an admissibly ordered interval-valued overlap function for an admissible order $\leq_{AD}$ ($\leq_{AD}$-overlap function) if it satisfies the conditions **(IO1)**, **(IO2)** and **(IO3)** of Definition 2.10 and, for all $X, Y, Z \in L([0,1])$:

(AO4) AO is $\leq_{AD}$-increasing: $X \leq_{AD} Y \Rightarrow AO(X,Z) \leq_{AD} AO(Y,Z)$.

The following construction method for admissibly ordered interval-valued overlap functions preserves the minimal width of the input intervals:

Theorem 2.2. *[40] Let O be a strict overlap function and $\alpha \in (0,1), \beta \in [0,1]$ such that $\alpha \neq \beta$. Then $AO^\alpha : L([0,1])^2 \to L([0,1])$ defined, for all $X, Y \in L([0,1])$, by*

$$AO^\alpha(X,Y) = [O(K_\alpha(X), K_\alpha(Y)) - \alpha m, O(K_\alpha(X), K_\alpha(Y)) + (1-\alpha)m], \tag{9}$$

where

$$m = \min\{w(X), w(Y), O(K_\alpha(X), K_\alpha(Y)), 1 - O(K_\alpha(X), K_\alpha(Y))\}$$

is an $\leq_{\alpha,\beta}$-overlap function.

3. Width-limited interval-valued functions

As a motivation to the developments presented in this section, we pose the following question: with respect to a given interval-valued function, how can the width of the output interval be affected by the widths of the input intervals? In order to aid on such discussion, concerning the uncertainty propagation control in aggregation processes, we introduce the following definition:

Definition 3.1. Consider an interval-valued function $IF : L([0,1])^2 \to L([0,1])$ and a mapping $B : [0,1]^2 \to [0,1]$. Then, IF is said to be width-limited by B if $w(IF(X,Y)) \leq B(w(X), w(Y))$, for all $X, Y \in L([0,1])$. B is called a width-limiting function of IF.

Remark 3.1. Every function $IF : L([0,1])^2 \to L([0,1])$ is width-limited by the function $B_1 : [0,1]^2 \to [0,1]$ defined by $B_1(x, y) = 1$, for all $x, y \in [0,1]$.

In the following, denote:

$$\mathcal{IF} = \{IF : L([0,1])^2 \to L([0,1]) \mid IF \text{ is a binary interval-valued function}\}$$

and

$$\mathcal{F} = \{F : [0,1]^2 \to [0,1] \mid F \text{ is binary function}\}.$$

First, we analyze how to obtain the least width-limiting function for a given interval-valued function:

Theorem 3.1. *The mapping $\mathfrak{L} : \mathcal{IF} \to \mathcal{F}$ defined for all $IF \in \mathcal{IF}$ and $\epsilon, \delta \in [0,1]$, by*

$$\mathfrak{L}(IF)(\epsilon, \delta) = \sup_{\substack{u \in [0, 1-\epsilon] \\ v \in [0, 1-\delta]}} \{w(IF([u, u+\epsilon], [v, v+\delta]))\}$$

provides the least width-limiting function $\mathfrak{L}(IF) : [0,1]^2 \to [0,1]$ for IF.

Proof. It is clear that $\mathfrak{L}(IF)$ is well defined, since

$$\sup_{\substack{u \in [0, 1-\epsilon] \\ v \in [0, 1-\delta]}} \{w(IF([u, u+\epsilon], [v, v+\delta]))\} \in [0,1],$$

for all $IF \in \mathcal{IF}$ and all $\epsilon, \delta \in [0,1]$.

Now, observe that

$$\begin{aligned} \mathfrak{L}(IF)(\epsilon, \delta) &= \sup_{\substack{u \in [0, 1-\epsilon] \\ v \in [0, 1-\delta]}} \{w(IF([u, u+\epsilon], [v, v+\delta]))\} \\ &\geq w(IF([u, u+\epsilon], [v, v+\delta])) \end{aligned}$$

for all $u \in [0, 1-\epsilon]$, $v \in [0, 1-\delta]$, showing that IF is width-limited by $\mathfrak{L}(IF)$, since $w([u, u+\epsilon]) = \epsilon$ and $w([v, v+\delta]) = \delta$.

Finally, suppose that there exists a function $B : [0,1]^2 \to [0,1]$ such that: (i) B is a width-limiting function for IF; (ii) there exist $\epsilon_0, \delta_0 \in [0,1]$ such that $B(\epsilon_0, \delta_0) < \mathfrak{L}(IF)(\epsilon_0, \delta_0)$. So, it follows that

$$B(\epsilon_0, \delta_0) < \sup_{\substack{u \in [0, 1-\epsilon_0] \\ v \in [0, 1-\delta_0]}} \{w(IF([u, u+\epsilon_0], [v, v+\delta_0]))\}.$$

Then, there exist $u_0 \in [u, u+\epsilon_0]$, $v_0 \in [v, v+\delta_0]$ such that

$$B(\epsilon_0, \delta_0) < w(IF([u_0, u_0+\epsilon_0], [v_0, v_0+\delta_0])),$$

meaning that IF cannot be width-limited by B, which is a contradiction. The conclusion is that $\mathfrak{L}(IF)$ is the least function that is width-limiting for IF. $\square$

In the following, denote:

$$\mathcal{A} = \{A : [0,1]^2 \to [0,1] \mid A \text{ is an aggregation function}\}$$

and

$$\mathcal{IA} = \{IA : L([0,1])^2 \to L([0,1]) \mid IA \text{ is the best interval representation of an aggregation function } A \in \mathcal{A}\}.$$

Then, a similar approach of Theorem 3.1 can be used to obtain the least width-liming aggregation function for a given representable interval-valued aggregation function.

Theorem 3.2. *The mapping $\mathfrak{L} : \mathcal{IA} \to \mathcal{F}$ defined for all $IA \in \mathcal{IA}$ and $\epsilon, \delta \in [0,1]$, by*

$$\mathfrak{L}(IA)(\epsilon,\delta) = \sup_{\substack{u \in [0,1-\epsilon] \\ v \in [0,1-\delta]}} \{w(IA([u,u+\epsilon],[v,v+\delta]))\} \tag{10}$$

provides the least width-limiting function $\mathfrak{L}(IA) : [0,1]^2 \to [0,1]$ for IA. Moreover, $\mathfrak{L}(IA)$ is an aggregation function.

Proof. From Theorem 3.1, it only remains to be shown that $\mathfrak{L}(IA)$ respects the conditions for it to be an aggregation function, for all $IA \in \mathcal{IA}$:

(A1) Consider $\epsilon_1, \epsilon_2, \delta_1, \delta_2 \in [0,1]$ such that $\epsilon_1 \leq \epsilon_2$ and $\delta_1 \leq \delta_2$. Thus, for all $u \in [0, 1-\epsilon_2]$ and $v \in [0, 1-\delta_2]$, it holds that

$$[u, u+\epsilon_1] \leq [u, u+\epsilon_2] \text{ and } [v, v+\delta_1] \leq [v, v+\delta_2].$$

Since IA is $\leq_{Pr}$-increasing, for all $u \in [0, 1-\epsilon_2]$ and $v \in [0, 1-\delta_2]$, it follows that

$$IA([u,u+\epsilon_1],[v,v+\delta_1]) \leq_{Pr} IA([u,u+\epsilon_2],[v,v+\delta_2]). \tag{11}$$

As $IA \in \mathcal{IA}$, then there exists an aggregation function $A : [0,1]^2 \to [0,1]$ such that

$$IA(X,Y) = [A(\underline{X},\underline{Y}), A(\overline{X},\overline{Y})],$$

for all $X, Y \in L([0,1])$. Thus, by Equation (11), one has that

$$\begin{aligned}
&[A(u,v), A(u+\epsilon_1, v+\delta_1)] \leq_{Pr} [A(u,v), A(u+\epsilon_2, v+\delta_2)] \\
&\quad \Rightarrow A(u+\epsilon_1, v+\delta_1) - A(u,v) \leq A(u+\epsilon_2, v+\delta_2) - A(u,v) \\
&\quad \Rightarrow w([A(u,v), A(u+\epsilon_1, v+\delta_1)]) \leq w([A(u,v), A(u+\epsilon_2, v+\delta_2)]) \\
&\quad \Rightarrow w(IA([u,u+\epsilon_1],[v,v+\delta_1])) \leq w(IA([u,u+\epsilon_2],[v,v+\delta_2])) \\
&\quad \Rightarrow \sup_{\substack{u \in [0,1-\epsilon_1] \\ v \in [0,1-\delta_1]}} \{w(IA([u,u+\epsilon_1],[v,v+\delta_1]))\} \leq \sup_{\substack{u \in [0,1-\epsilon_2] \\ v \in [0,1-\delta_2]}} \{w(IA([u,u+\epsilon_2],[v,v+\delta_2]))\} \\
&\quad \Rightarrow \mathfrak{L}(IA)(\epsilon_1,\delta_1) \leq \mathfrak{L}(IA)(\epsilon_2,\delta_2),
\end{aligned}$$

showing that $\mathfrak{L}(IA)$ is increasing.

(A2) As $IA \in \mathcal{IA}$, it follows that

$$\mathfrak{L}(IA)(0,0) = \sup_{u,v \in [0,1]} \{w(IA([u,u],[v,v]))\} = \sup_{u,v \in [0,1]} \{A(u,v) - A(u,v)\} = 0,$$

and

$$\mathfrak{L}(IA)(1,1) = w(IA([0,1],[0,1])) = A(1,1) - A(0,0) = 1. \quad \square$$

Example 3.1. Let $A : [0,1]^2 \to [0,1]$ be an aggregation function defined, for all $x, y \in [0,1]$, by $A(x,y) = \frac{x+y+x \cdot y}{3}$. Then, the mapping $\mathfrak{L}(\widehat{A}) : [0,1]^2 \to [0,1]$ defined, for all $\epsilon, \delta \in [0,1]$, by

$$\begin{aligned}
\mathfrak{L}(\widehat{A})(\epsilon,\delta) &= \sup_{\substack{u \in [0,1-\epsilon] \\ v \in [0,1-\delta]}} \{w(\widehat{A}([u,u+\epsilon],[v,v+\delta]))\} \\
&= \sup_{\substack{u \in [0,1-\epsilon] \\ v \in [0,1-\delta]}} \{A(u+\epsilon, v+\delta) - A(u,v)\} \\
&= \sup_{\substack{u \in [0,1-\epsilon] \\ v \in [0,1-\delta]}} \left\{ \frac{u+\epsilon+v+\delta+(u+\epsilon)\cdot(v+\delta)}{3} - \left(\frac{u+v+y\cdot v}{3}\right) \right\}
\end{aligned}$$

$$= \sup_{\substack{u \in [0,1-\epsilon] \\ v \in [0,1-\delta]}} \left\{ \frac{\epsilon + \delta + u \cdot \delta + \epsilon \cdot v + \epsilon \cdot \delta}{3} \right\}$$
$$= \frac{\epsilon + \delta + (1-\epsilon) \cdot \delta + \epsilon \cdot (1-\delta) + \epsilon \cdot \delta}{3}$$
$$= \frac{2\epsilon + 2\delta - \epsilon \cdot \delta}{3}$$

is the least width-limiting function for $\widehat{A}$. Observe that $\mathfrak{L}(\widehat{A})$ is an aggregation function.

Based on the concept of ultramodularity, let us define a less restrictive extension of one-dimension convexity for bivariate aggregation functions:

Definition 3.2. Consider $a, b \in [0, 1]$. An aggregation function $A : [0, 1]^2 \to [0, 1]$ is called (a, b)-ultramodular if, for all $x, y, \epsilon, \delta \in [0, 1]$ and $x + \epsilon, y + \delta, a - \epsilon, b - \delta \in [0, 1]$, it holds that:

$$A(x + \epsilon, y + \delta) - A(x, y) \leq A(a, b) - A(a - \epsilon, b - \delta). \tag{12}$$

Proposition 3.1. *Let $A : [0, 1]^2 \to [0, 1]$ be an ultramodular aggregation function. Then, A is an $(1, 1)$-ultramodular aggregation function.*

Proof. Immediate, since Equation (12), with $a = b = 1$, is a particular case of Equation (3) when $\epsilon + x_2 = 1$ and $\delta + y_2 = 1$. □

Remark 3.2. If an aggregation function $A : [0, 1]^2 \to [0, 1]$ is $(1, 1)$-ultramodular, then, for all $x, y, \epsilon, \delta \in [0, 1]$ such that $x + \epsilon, y + \delta, a - \epsilon, b - \delta \in [0, 1]$, it holds that:

$$A(x + \epsilon, y + \delta) - A(x, y) \leq A^d(\epsilon, \delta), \tag{13}$$

where A^d is the dual of A.

Remark 3.3. From Proposition 3.1, we have that every ultramodular function is also $(1, 1)$-ultramodular. However, the converse may not hold. For example, the Ot overlap (Table 1) given by $O_t(x, y) = \frac{(2x-1)^3+1}{2} \cdot \frac{(2y-1)^3+1}{2}$, for all $x, y \in [0, 1]$, is an $(1, 1)$-ultramodular function. However, by Proposition 2.1, O_t is clearly not an ultramodular aggregation function.

Now, let us present a characterization for the least width-limiting function of the best interval representation of an $(1, 1)$-ultramodular aggregation function, or the best interval representation of its dual:

Theorem 3.3. *Let $A : [0, 1]^2 \to [0, 1]$ be an aggregation function, $\mathfrak{L}(\widehat{A}), \mathfrak{L}(\widehat{A^d}) : [0, 1]^2 \to [0, 1]$ be the least width-limiting functions for $\widehat{A}$ and $\widehat{A^d}$, respectively. Then, $\mathfrak{L}(\widehat{A}) = \mathfrak{L}(\widehat{A^d}) = A^d$ if and only if A is an $(1, 1)$-ultramodular aggregation function.*

Proof. $(\Rightarrow)$ Suppose that $\mathfrak{L}(\widehat{A}) = \mathfrak{L}(\widehat{A^d}) = A^d$. Then, we have that:

$$\mathfrak{L}(\widehat{A}) = A^d$$
$$\Rightarrow \sup_{\substack{u \in [0,1-\epsilon] \\ v \in [0,1-\delta]}} \{w(\widehat{A}([u, u+\epsilon], [v, v+\delta]))\} = 1 - A(1-\epsilon, 1-\delta)$$
$$\Rightarrow A(u+\epsilon, v+\delta) - A(u, v) \leq A(1, 1) - A(1-\epsilon, 1-\delta), \text{ for all } u \in [0, 1-\epsilon], v \in [0, 1-\delta]. \tag{14}$$

From Equation (14), we conclude that A is $(1, 1)$-ultramodular.

$(\Leftarrow)$ Suppose that $A : [0, 1]^2 \to [0, 1]$ is an $(1, 1)$-ultramodular aggregation function. Then, for all $\epsilon, \delta \in [0, 1]$, it holds that:

$$
\begin{aligned}
\mathfrak{L}(\widehat{A})(\epsilon,\delta) &= \sup_{\substack{u\in[0,1-\epsilon]\\ v\in[0,1-\delta]}} \{w(\widehat{A}([u,u+\epsilon],[v,v+\delta]))\}\\
&= \sup_{\substack{u\in[0,1-\epsilon]\\ v\in[0,1-\delta]}} \{A(u+\epsilon,v+\delta)-A(u,v)\}\\
&= A(1-\epsilon+\epsilon,1-\delta+\delta)-A(1-\epsilon,1-\delta)\\
&= 1-A(1-\epsilon,1-\delta)\\
&= A^d(\epsilon,\delta),
\end{aligned}
$$

and

$$
\begin{aligned}
\mathfrak{L}(\widehat{A^d})(\epsilon,\delta) &= \sup_{\substack{u\in[0,1-\epsilon]\\ v\in[0,1-\delta]}} \{w(\widehat{A^d}([u,u+\epsilon],[v,v+\delta]))\}\\
&= \sup_{\substack{u\in[0,1-\epsilon]\\ v\in[0,1-\delta]}} \{A(1-u,1-v)-A(1-u-\epsilon,1-v-\delta)\}\\
&= A(1,1)-A(1-\epsilon,1-\delta)\\
&= A^d(\epsilon,\delta),
\end{aligned}
$$

since A is $(1,1)$-ultramodular. Thus, if A is an $(1,1)$-ultramodular aggregation function, then $\mathfrak{L}(\widehat{A}) = \mathfrak{L}(\widehat{A^d}) = A^d$. $\square$

Remark 3.4. In the context of Theorem 3.3, as $\widehat{A}$ and $\widehat{A^d}$ are representable iv-aggregation functions, then their least width-limiting function A^d is an aggregation function, as stated by Theorem 3.2. Also, observe that the function A does not need to be ultramodular.

Example 3.2. The least width-limiting function for either $\widehat{O_t}$ (the best interval representation of the overlap function O_t, shown in Table 1) or $\widehat{O_t^d}$ (the best interval representation of the dual of O_t) is O_t^d, as O_t is an $(1,1)$-ultramodular aggregation function.

Since every ultramodular aggregation function is also $(1,1)$-ultramodular, the following result is immediate.

Corollary 3.1. *Let $A:[0,1]^2\to[0,1]$ be an aggregation function, $\mathfrak{L}(\widehat{A}), \mathfrak{L}(\widehat{A^d}) : L([0,1])^2 \to L([0,1])$ be the least width-limiting functions for $\widehat{A}$ and $\widehat{A^d}$, respectively. Then, $\mathfrak{L}(\widehat{A}) = \mathfrak{L}(\widehat{A^d}) = A^d$ if and only if A is an ultramodular aggregation function.*

Example 3.3. Here we present some examples of width-limiting functions for the best interval representation of either an ultramodular aggregation function or its dual:

1) The least width-limiting function for either $\widehat{O_P}$ (the best interval representation of the product overlap) or $\widehat{O_P^d}$ (the best interval representation of the dual of O_P) is O_P^d;
2) The least width-limiting function for $\widehat{K_\alpha}$ (the best interval representation of the weighted sum), is $K_\alpha^d = K_\alpha$, with $\alpha\in[0,1]$;
3) Consider the aggregation function $AM:[0,1]^2\to[0,1]$ given by $AM(x,y)=\frac{x+y}{2}$ (arithmetic mean). So, the least width-limiting function for $\widehat{AM}$ (the best interval representation of the arithmetic mean), is $AM^d = AM$.

Proposition 3.2. *Let $IF_1, IF_2, IG, IH \in \mathcal{IA}$, such that $IH(X,Y) = IG(IF_1(X,Y), IF_2(X,Y))$, for all $X,Y\in L([0,1])$. Then, it holds that:*

$$
\mathfrak{L}(IH) \le \mathfrak{L}(IG)(\mathfrak{L}(IF_1), \mathfrak{L}(IF_2)).
$$

Proof. Consider $IF_1([x_1, x_1+\epsilon], [x_2, x_2+\delta]) = [y_1, y_1+\epsilon^*]$, $IF_2([x_1, x_1+\epsilon], [x_2, x_2+\delta]) = [y_2, y_2+\delta^*]$, with $\epsilon, \delta, \epsilon^*, \delta^*, \in [0,1]$ and $x_1+\epsilon, x_2+\delta, y_1+\epsilon^*, y_2+\delta^* \in [0,1]$. Then, it follows that:

$$\begin{aligned}
&w(IH([x_1, x_1+\epsilon], [x_2, x_2+\delta])) \\
&\quad = w(IG(IF_1([x_1, x_1+\epsilon], [x_2, x_2+\delta]), IF_2([x_1, x_1+\epsilon], [x_2, x_2+\delta]))) \\
&\quad = w(IG([y_1, y_1+\epsilon^*], [y_2, y_2+\delta^*]) \\
&\quad \leq \mathfrak{L}(IG)(\epsilon^*, \delta^*), \text{ by Theorem 3.2} \\
&\quad \leq \mathfrak{L}(IG)(\mathfrak{L}(IF_1)(\epsilon, \delta), \mathfrak{L}(IF_2)(\epsilon, \delta)),
\end{aligned}$$

which means that $\mathfrak{L}(IG)(\mathfrak{L}(IF_1), \mathfrak{L}(IF_2))$ is a width-limiting function for IH.

However, as $\mathfrak{L}(IH)$ is the least width-limiting function for IH (by Theorem 3.2), thus, one concludes that

$$\mathfrak{L}(IH) \leq \mathfrak{L}(IG)(\mathfrak{L}(IF_1), \mathfrak{L}(IF_2)). \quad \square$$

Example 3.4.

1) Take $IF_1 = \widehat{AM}$, $IF_2 = \widehat{O_P}$, $IG = \widehat{K_\alpha}$, as presented in Example 3.3. Then, let $IH : L([0,1])^2 \to L([0,1])$ be an iv-aggregation function defined, for all $X, Y \in L([0,1])$ with $\alpha \in [0,1]$, by

$$\begin{aligned}
IH(X,Y) &= \widehat{K_\alpha}(\widehat{AM}(X,Y), \widehat{O_P}(X,Y)) \\
&= \widehat{K_\alpha}([AM(\underline{X}, \underline{Y}), AM(\overline{X}, \overline{Y})], [O_P(\underline{X}, \underline{Y}), O_P(\overline{X}, \overline{Y})])) \\
&= \widehat{K_\alpha}\left(\left[\frac{\underline{X}+\underline{Y}}{2}, \frac{\overline{X}+\overline{Y}}{2}\right], [\underline{X}\cdot\underline{Y}, \overline{X}\cdot\overline{Y}]\right).
\end{aligned}$$

Since AM, O_P and K_α are ultramodular, it holds that:

$$\begin{aligned}
&\mathfrak{L}(IH)(\epsilon, \delta) \\
&= \sup_{\substack{u \in [0, 1-\epsilon] \\ v \in [0, 1-\delta]}} \left\{ K_\alpha\left(\frac{u+\epsilon+v+\delta}{2}, (u+\epsilon)\cdot(v+\delta)\right) - K_\alpha\left(\frac{u+v}{2}, u\cdot v\right)\right\} \\
&= K_\alpha\left(\frac{1-\epsilon+\epsilon+1-\delta+\delta}{2}, (1-\epsilon+\epsilon)\cdot(1-\delta+\delta)\right) - K_\alpha\left(\frac{1-\epsilon+1-\delta}{2}, (1-\epsilon)\cdot(1-\delta)\right) \\
&= K_\alpha(1,1) - K_\alpha\left(\frac{(1-\epsilon)+(1-\delta)}{2}, (1-\epsilon)\cdot(1-\delta)\right) \\
&= 1 - K_\alpha(AM(1-\epsilon, 1-\delta), O_P(1-\epsilon, 1-\delta)) \\
&= 1 - K_\alpha\left(1 - AM(\epsilon, \delta), 1 - O_P^d(\epsilon, \delta)\right) \\
&= K_\alpha\left(AM(\epsilon, \delta), O_P^d(\epsilon, \delta)\right).
\end{aligned}$$

From Theorem 3.3 we have that $\mathfrak{L}(\widehat{O_P}) = O_P^d$, $\mathfrak{L}(\widehat{AM}) = AM^d = AM$ and $\mathfrak{L}(\widehat{K_\alpha}) = K_\alpha^d = K_\alpha$, for all $\alpha \in [0,1]$. So, we conclude that

$$\mathfrak{L}(\widehat{K_\alpha})(\mathfrak{L}(\widehat{AM})(\epsilon, \delta), \mathfrak{L}(\widehat{O_P})(\epsilon, \delta)) = K_\alpha(AM(\epsilon, \delta), O_P^d(\epsilon, \delta)) = \mathfrak{L}(IH).$$

2) Now, take $IF_1 = \widehat{O_P}$, $IF_2 = \widehat{O_P^d}$, $IG = \widehat{K_{0.25}}$, with $\alpha = 0.25$. Then, let $IH : L([0,1])^2 \to L([0,1])$ be the iv-aggregation function defined, for all $X, Y \in L([0,1])$, by

$$\begin{aligned}
IH(X,Y) &= \widehat{K_{0.25}}(\widehat{O_P}(X,Y), \widehat{O_P^d}(X,Y)) \\
&= \widehat{K_{0.25}}([O_P(\underline{X}, \underline{Y}), O_P(\overline{X}, \overline{Y})], [O_P^d(\underline{X}, \underline{Y}), O_P^d(\overline{X}, \overline{Y})])) \\
&= \widehat{K_{0.25}}\left([\underline{X}\cdot\underline{Y}, \overline{X}\cdot\overline{Y}], [\underline{X}+\underline{Y}-\underline{X}\cdot\underline{Y}, \overline{X}+\overline{Y}-\overline{X}\cdot\overline{Y}]\right) \\
&= \left[\frac{\underline{X}+\underline{Y}+2\cdot\underline{X}\cdot\underline{Y}}{4}, \frac{\overline{X}+\overline{Y}+2\cdot\overline{X}\cdot\overline{Y}}{4}\right].
\end{aligned}$$

Thus,

$$
\begin{aligned}
&\mathfrak{L}(IH)(\epsilon,\delta)\\
&=\sup_{\substack{u\in[0,1-\epsilon]\\ v\in[0,1-\delta]}}\left\{\frac{u+\epsilon+v+\delta+2\cdot(u+\epsilon)\cdot(v+\delta)}{4}-\frac{u+v+2\cdot u\cdot v}{4}\right\}\\
&=\sup_{\substack{u\in[0,1-\epsilon]\\ v\in[0,1-\delta]}}\left\{\frac{\epsilon+\delta+2\cdot u\cdot\delta+2\cdot\epsilon\cdot v+2\cdot\epsilon\cdot\delta}{4}\right\}\\
&=\frac{\epsilon+\delta+2\cdot(1-\epsilon)\cdot\delta+2\cdot\epsilon\cdot(1-\delta)+2\cdot\epsilon\cdot\delta}{4}\\
&=\frac{3\epsilon+3\delta-2\epsilon\delta}{4}.
\end{aligned}
$$

From Theorem 3.3 we have that $\mathfrak{L}(\widehat{O_P})=\mathfrak{L}(\widehat{O_P^d})=O_P^d$ and $\mathfrak{L}(\widehat{K_{0.25}})=K_{0.25}^d=K_{0.25}$. So, we have that

$$
\begin{aligned}
&\mathfrak{L}(\widehat{K_{0.25}})(\mathfrak{L}(\widehat{O_P})(\epsilon,\delta),\mathfrak{L}(\widehat{O_P^d})(\epsilon,\delta))\\
&=K_{0.25}(O_P^d(\epsilon,\delta),O_P^d(\epsilon,\delta))\\
&=O_P^d(\epsilon,\delta)\\
&\geq\frac{3\epsilon+3\delta-2\epsilon\delta}{4}\\
&=\mathfrak{L}(IH).
\end{aligned}
$$

Remark 3.5. Consider an interval-valued function $IF : L([0,1])^2 \to L([0,1])$ and an aggregation function $A : [0,1]^2 \to [0,1]$. If IF is width-limited by A, we have that, for any $X, Y \in L([0,1])$:

1) If $A=\max$, then IF is limited by the maximal width of the input intervals X, Y;
2) If $A=\min$, then IF is limited by the minimal width of the input intervals X, Y;
3) If A is conjunctive and either X or Y is degenerate, then $IF(X,Y)$ is also degenerate;
4) If A is averaging, then $\min\{w(X),w(Y)\}\leq w(IF(X,Y))\leq\max\{w(X),w(Y)\}$.

4. Width-limited interval-valued overlap functions

The aim of this section is to apply the newly developed concepts of width-limited interval-valued functions and width limiting functions to obtain a new definition of width-limited interval-valued overlap functions, taking into consideration different partial orders. Also, we are going to present three construction methods for width-limited interval-valued overlap functions, followed by some examples and comparisons.

First, to enable a more flexible definition of interval-valued functions, let us define the concept of increasingness with respect to a pair of partial orders:

Definition 4.1. Let $IF : L([0,1])^2 \to L([0,1])$ be an interval-valued function and $\leq_1$, $\leq_2$ be two partial order relations on $L([0,1])$. Then, IF is said to be $(\leq_1,\leq_2)$-increasing if the following condition holds, for all $X_1, X_2, Y_1, Y_2 \in L([0,1])$:

$$X_1 \leq_1 X_2 \ \wedge\ Y_1 \leq_1 Y_2 \Rightarrow IF(X_1,Y_1) \leq_2 IF(X_2,Y_2).$$

When an interval-valued function $IF : L([0,1])^2 \to L([0,1])$ is $(\leq,\leq)$-increasing, we denote it simply as $\leq$-increasing, for any partial order relation $\leq$ on $L([0,1])$.

Proposition 4.1. *Let $\leq_{AD}$ be an admissible order on $L([0,1])$. Then, an $\leq_{Pr}$-increasing function $IF : L([0,1])^2 \to L([0,1])$ is also $(\leq_{Pr},\leq_{AD})$-increasing.*

Proof. Immediate, as $\leq_{AD}$ is an admissible order and, as such, refines $\leq_{Pr}$. $\square$

Example 4.1. Given an overlap function $O : [0,1]^2 \to [0,1]$, the $\leq_{\alpha,\beta}$-overlap function $AO^{\alpha} : L([0,1])^2 \to L([0,1])$ defined in Equation (9) (Theorem 2.2) is $(\leq_{Pr}, \leq_{\alpha,\beta})$-increasing for all $\alpha, \beta \in [0,1]$ such that $\alpha \neq \beta$.

Here, we present the definition of width-limited interval-valued overlap functions:

Definition 4.2. Let $B : [0,1]^2 \to [0,1]$ be a commutative and increasing function and $\leq_1$, $\leq_2$ be two partial order relations on $L([0,1])$. Then, the mapping $IOw : L([0,1])^2 \to L([0,1])$ is said to be a width-limited interval-valued overlap function (w-iv-overlap function) with respect to the tuple $(\leq_1, \leq_2, B)$, if the following conditions hold for all $X, Y \in L([0,1])$:

(IOw1) IOw is commutative;
(IOw2) $IOw(X,Y) = [0,0] \Leftrightarrow X \cdot Y = [0,0]$;
(IOw3) $IOw(X,Y) = [1,1] \Leftrightarrow X \cdot Y = [1,1]$;
(IOw4) IOw is $(\leq_1, \leq_2)$-increasing;
(IOw5) IOw is width-limited by B.

Remark 4.1. Taking a similar approach as in [40] when defining admissibly ordered interval-valued overlap functions, we do not require the continuity as a condition in Definition 4.2. The original definition of overlap functions (Definition 2.10) included the Moore continuity as a necessary condition as the goal was to be applied in image processing problems [3], which is not the case here.

Now, let us presents some results regarding width-limited interval-valued overlap functions obtained through the best interval representation of an overlap function:

Proposition 4.2. *Let $O : [0,1]^2 \to [0,1]$ be an $(1,1)$-ultramodular overlap function. Then, the function $IF : L([0,1])^2 \to L([0,1])$, such that $IF = \widehat{O}$ is an w-iv-overlap function for the tuple $(\leq_{Pr}, \leq_{Pr}, O^d)$, where O^d is the dual of O.*

Proof. Immediate from Theorem 3.3. $\square$

Example 4.2. Let $O_t : [0,1]^2 \to [0,1]$ be the Ot overlap, given in Table 1. Then, the function $IF : L([0,1])^2 \to L([0,1])$, such that $IF = \widehat{O_t}$ is an w-iv-overlap function for the tuple $(\leq_{Pr}, \leq_{Pr}, O_t^d)$, where O_t^d is the dual of O_t.

Proposition 4.3. *Let $O_1, O_2, O_3 : [0,1]^2 \to [0,1]$ be ultramodular overlap functions, and $O_C : [0,1]^2 \to [0,1]$ be an overlap function given, for all $x, y \in [0,1]$, by $O_C(x,y) = O_3(O_1(x,y), O_2(x,y))$. Then, the function $IF : L([0,1])^2 \to L([0,1])$, such that $IF = \widehat{O_C}$ is an w-iv-overlap function for the tuple $(\leq_{Pr}, \leq_{Pr}, O_C^d)$, where O_C^d is the dual of O_C.*

Proof. Immediate from Theorem 2.1, Proposition 2.3 and Theorem 3.3. $\square$

Example 4.3. Consider the overlap functions $O_1, O_2, O_3, O_C : [0,1]^2 \to [0,1]$ given, for all $x, y, \in [0,1]$, respectively, by $O_1(x,y) = x^{2p} \cdot y^{2p}$, $O_2(x,y) = x^{2q} \cdot y^{2q}$, $O_3(x,y) = x \cdot y$ and $O_C(x,y) = O_3(O_1(x,y), O_2(x,y))$, with $p, q \in \mathbb{N}^+$. Since O_1, O_2 and O_3 are ultramodular, it follows that the function $IF : L([0,1])^2 \to L([0,1])$, such that $IF = \widehat{O_C}$, is an w-iv-overlap function for the tuple $(\leq_{Pr}, \leq_{Pr}, O_C^d)$.

Proposition 4.4. *Let $O_1, O_2 : [0,1]^2 \to [0,1]$ be ultramodular overlap functions, and $O_\alpha : [0,1]^2 \to [0,1]$ be an overlap function given, for all $x, y, \alpha \in [0,1]$, by $O_\alpha(x,y) = K_\alpha(O_1(x,y), O_2(x,y))$. Then, the function $IF : L([0,1])^2 \to L([0,1])$ such that $IF = \widehat{O_\alpha}$, for all $\alpha \in [0,1]$, is an w-iv-overlap function for $(\leq_{Pr}, \leq_{Pr}, O_\alpha^d)$, where O_α^d is the dual of O_α.*

Proof. Immediate from Corollary 2.1, Proposition 2.4 and Theorem 3.3. □

Example 4.4. Consider the ultramodular overlap functions $O_1, O_2, O_\alpha : [0,1]^2 \to [0,1]$ given, for all $x, y, \alpha \in [0,1]$, respectively, by $O_1(x,y) = x^2y^2$, $O_2(x,y) = x^4y^4$ and $O_\alpha = K_\alpha(O_1(x,y), O_2(x,y))$. It follows that the function $IF : L([0,1])^2 \to L([0,1])$, such that $IF = \widehat{O_\alpha}$, for all $\alpha \in [0,1]$, is an w-iv-overlap function for the tuple $(\leq_{Pr}, \leq_{Pr}, O_\alpha^d)$.

As discussed in the Introduction, our aim is to construct interval-valued overlap functions in which the width of the interval output does not surpass a desirable threshold, according to the width-limiting function applied to the widths of the interval inputs. We point out that the desirable maximal threshold is determined by the application requirement, concerning the extent of the necessity to conserve the information quality of the results, with respect to the information quality of the inputs (see Remark 4.6).

In the following, we introduce the definition of such maximal width threshold, a key concept to be applied in two of the construction methods presented latter in the paper:

Definition 4.3. Consider a function $B : [0,1]^2 \to [0,1]$ and let $IF : L([0,1])^2 \to L([0,1])$ be an interval-valued function. Then, the function $m_{IF,B} : L([0,1])^2 \to [0,1]$, defined for all $X, Y \in L([0,1])$ by:

$$m_{IF,B}(X,Y) = \min\{w(IF(X,Y)), w(IF(Y,X)), B(w(X), w(Y)), B(w(Y), w(X))\}, \tag{15}$$

is called the maximal width threshold for the pair (IF, B). Whenever B and IF are both commutative, then Equation (15) can be reduced to:

$$m_{IF,B}(X,Y) = \min\{w(IF(X,Y)), B(w(X), w(Y))\}.$$

Proposition 4.5. *Let $m_{\widehat{F},B} : L([0,1])^2 \to [0,1]$ be the maximal width threshold for the pair $(\widehat{F}, B)$ with $\widehat{F} : L([0,1])^2 \to L([0,1])$ being an interval-valued function having an increasing function $F : [0,1]^2 \to [0,1]$ as both its representatives. Whenever it holds that: i) both X and Y are degenerate or ii) either X or Y is degenerate and B is a conjunctive function, then $m_{\widehat{F},B}(X,Y) = 0$.*

Proof. Consider an increasing function $F : [0,1]^2 \to [0,1]$, a conjunctive function $B : [0,1]^2 \to [0,1]$ and the maximal width threshold $m_{\widehat{F},B} : L([0,1])^2 \to [0,1]$ given by Definition 4.3. Then:

i) Take $X, Y \in L([0,1])$ such that $\underline{X} = \overline{X}$ and $\underline{Y} = \overline{Y}$, that is, both X and Y are degenerate. Then, we have that $w(\widehat{F}(X,Y)) = F(\overline{X}, \overline{Y}) - F(\underline{X}, \underline{Y}) = 0$ and, similarly, $w(\widehat{F}(Y,X)) = 0$. So, it holds that

$$m_{\widehat{F},B}(X,Y) = \min\{0, 0, B(w(X), w(Y)), B(w(Y), w(X))\} = 0;$$

ii) Take $X, Y \in L([0,1])$ such that $\underline{X} = \overline{X}$, meaning that $w(X) = 0$. Since B is conjunctive, it holds that $B(w(X), w(Y)) = B(0, w(Y)) = 0$ and, analogously, $B(w(Y), w(X)) = 0$. Then, we have that

$$m_{\widehat{F},B}(X,Y) = \min\{w(\widehat{F}(X,Y)), w(\widehat{F}(Y,X)), 0, 0\} = 0.$$

The same result applies when Y is degenerate. □

Lemma 4.1. *Consider a strict overlap function $O : [0,1]^2 \to [0,1]$ and $X, Y, Z \in L([0,1])$ such that $X \leq_{Pr} Y$ and $\underline{Z} > 0$. Then, one has that:*

a) *If $\underline{X} = \underline{Y}$ and $\overline{X} < \overline{Y}$, then $K_\alpha(\widehat{O}(X,Z)) < K_\alpha(\widehat{O}(Y,Z))$, for all $\alpha \in (0,1]$;*
b) *If $\underline{X} < \underline{Y}$ and $\overline{X} = \overline{Y}$, then $K_\alpha(\widehat{O}(X,Z)) < K_\alpha(\widehat{O}(Y,Z))$, for all $\alpha \in [0,1)$;*
c) *If $\underline{X} < \underline{Y}$ and $\overline{X} < \overline{Y}$, then $K_\alpha(\widehat{O}(X,Z)) < K_\alpha(\widehat{O}(Y,Z))$, for all $\alpha \in [0,1]$.*

Proof. Consider a strict overlap function $O : [0,1]^2 \to [0,1]$ and $X, Y, Z \in L([0,1])$ such that $X <_{Pr} Y$. Then, we have the following cases:

a) $\underline{X} = \underline{Y}$ and $\overline{X} < \overline{Y}$. As O is strict and $\overline{Z} > 0$, we have that $O(\underline{X}, \underline{Z}) = O(\underline{Y}, \underline{Z})$ and $O(\overline{Y}, \overline{Z}) < O(\overline{Y}, \overline{Z})$. Then, $K_\alpha(\widehat{O}(X, Z)) = (1-\alpha) \cdot O(\underline{X}, \underline{Z}) + \alpha \cdot O(\overline{X}, \overline{Z}) < (1-\alpha) \cdot O(\underline{Y}, \underline{Z}) + \alpha \cdot O(\overline{Y}, \overline{Z}) = K_\alpha(\widehat{O}(Y, Z))$, for all $\alpha \in (0, 1]$;

b) $\underline{X} < \underline{Y}$ and $\overline{X} = \overline{Y}$. Again, as O is strict and $\underline{Z} > 0$, we have that $O(\underline{X}, \underline{Z}) < O(\underline{Y}, \underline{Z})$ and $O(\overline{Y}, \overline{Z}) = O(\overline{Y}, \overline{Z})$. So, $K_\alpha(\widehat{O}(X, Z)) = (1-\alpha) \cdot O(\underline{X}, \underline{Z}) + \alpha \cdot O(\overline{X}, \overline{Z}) < (1-\alpha) \cdot O(\underline{Y}, \underline{Z}) + \alpha \cdot O(\overline{Y}, \overline{Z}) = K_\alpha(\widehat{O}(Y, Z))$, for all $\alpha \in [0, 1)$;

c) $\underline{X} < \underline{Y}$ and $\overline{X} < \overline{Y}$. Analogously to the other cases, we have that $O(\underline{X}, \underline{Z}) < O(\underline{Y}, \underline{Z})$ and $O(\overline{Y}, \overline{Z}) < O(\overline{Y}, \overline{Z})$. Thus, $K_\alpha(\widehat{O}(X, Z)) = (1-\alpha) \cdot O(\underline{X}, \underline{Z}) + \alpha \cdot O(\overline{X}, \overline{Z}) < (1-\alpha) \cdot O(\underline{Y}, \underline{Z}) + \alpha \cdot O(\overline{Y}, \overline{Z}) = K_\alpha(\widehat{O}(Y, Z))$, for all $\alpha \in [0, 1]$. □

Here, we present the first construction method for w-iv-overlap functions:

Theorem 4.1. *Consider a commutative and increasing function $B : [0, 1]^2 \to [0, 1]$, a strict overlap function $O : [0, 1]^2 \to [0, 1]$ and take $\alpha \in (0, 1]$ and $\beta \in [0, \alpha)$. Then, the interval-valued function $IOw_B^\alpha : L([0, 1])^2 \to L([0, 1])$ defined, for all $X, Y \in L([0, 1])$, by*

$$IOw_B^\alpha(X, Y) = [K_\alpha(\widehat{O}(X, Y)) - \alpha \cdot m_{\widehat{O},B}(X, Y), K_\alpha(\widehat{O}(X, Y)) + (1-\alpha) \cdot m_{\widehat{O},B}(X, Y)], \tag{16}$$

is a w-iv-overlap function for the tuple $(\leq_{Pr}, \leq_{\alpha,\beta}, B)$.

Proof. See Appendix A. □

Proposition 4.6. *Let $O : [0, 1]^2 \to [0, 1]$ be a strict overlap function, $B : [0, 1]^2 \to [0, 1]$ be an increasing and commutative function and $IOw_B^\alpha : L([0, 1])^2 \to L([0, 1])$ be an w-iv-overlap function for the tuple $(\leq_{Pr}, \leq_{\alpha,\beta}, B)$ obtained through Theorem 4.1 for any $\alpha, \beta \in [0, 1]$ such that $\alpha \neq \beta$. Then, for any $X, Y \in L([0, 1])$ one has that $IOw_B^\alpha(X, Y) \subseteq \widehat{O}(X, Y)$.*

Proof. It is immediate that $K_\alpha(IOw_B^\alpha(X, Y)) = K_\alpha(\widehat{O}(X, Y))$, for any $\alpha \in [0, 1]$. Then, either $IOw_B^\alpha(X, Y) \subseteq \widehat{O}(X, Y)$ or $\widehat{O}(X, Y) \subseteq IOw_B^\alpha(X, Y)$. On the other hand, as

$$w(IOw_B^\alpha(X, Y)) = m_{\widehat{O},B}(X, Y) = \min\{w(\widehat{O}(X, Y)), B(w(X), w(Y))\} \leq w(\widehat{O}(X, Y)),$$

then $IOw_B^\alpha(X, Y) \subseteq \widehat{O}(X, Y)$. □

The next result is immediate from Theorem 3.3.

Proposition 4.7. *Let $O : [0, 1]^2 \to [0, 1]$ be an $(1, 1)$-ultramodular overlap function, $A : [0, 1]^2 \to [0, 1]$ be an aggregation function such that $A \geq O^d$ and $IOw_A^\alpha : L([0, 1])^2 \to L([0, 1])$ be the w-iv-overlap function for the tuple $(\leq_{Pr}, \leq_{\alpha,\beta}, A)$, obtained by Theorem 4.1 with $\alpha, \beta \in [0, 1]$. Then, $IOw_A^\alpha(X, Y) = \widehat{O}(X, Y)$, for all $X, Y \in L([0, 1])$.*

Remark 4.2. From Proposition 4.7, when we apply construction method presented in Theorem 4.1 to obtain an w-iv-overlap function IOw_A^α based on an $(1, 1)$-ultramodular overlap function O with a width-limiting aggregation function A, such that $A < O^d$ and $\alpha, \beta \in [0, 1]$, the output interval is narrower (with greater quality of information) than the one obtained by $\widehat{O}$. Furthermore, from Proposition 4.6, it holds that this interval is contained in the one obtained by $\widehat{O}$, which is a desirable property, since $\widehat{O}$ is the best interval representation of O, in the sense of [17,31].

The following examples aim to illustrate how the construction method presented in Theorem 4.1 works, comparing the results with the ones obtained through o-representable iv-overlap functions.

Example 4.5. Consider an increasing and commutative function $B : [0, 1]^2 \to [0, 1]$, the product overlap function $Op : [0, 1]^2 \to [0, 1]$, $\alpha \in (0, 1]$ and $\beta \in [0, \alpha)$. Then, the interval-valued function $IOpw_B^\alpha : L([0, 1])^2 \to L([0, 1])$ defined, for all $X, Y \in L([0, 1])$, by

$$IOpw_B^\alpha(X, Y) = [K_\alpha(\widehat{Op}(X, Y)) - \alpha \cdot m_{\widehat{Op},B}(X, Y), K_\alpha(\widehat{Op}(X, Y)) + (1-\alpha) \cdot m_{\widehat{Op},B}(X, Y)], \tag{17}$$

is a w-iv-overlap function for the tuple $(\leq_{Pr}, \leq_{\alpha,\beta}, \max)$.

1) Take $B = \max$, $X = [0.2, 0.8]$ and $Y = [0.5, 1]$. So, we have that $\widehat{Op}([0.2, 0.8], [0.5, 1]) = [0.1, 0.8]$. It is clear that $\widehat{Op}$ is not width-limited by max, as $w(\widehat{Op}([0.2, 0.8], [0.5, 1])) = 0.7 > 0.6 = \max(w([0.2, 0.8]), w([0.5, 1]))$. Also, by Equation (7), observe that $\widehat{Op}([0.2, 0.8], [0.5, 1])$ can be obtained as:

$$\widehat{Op}([0.2, 0.8], [0.5, 1]) = [K_\alpha([0.1, 0.8]) - \alpha \cdot 0.7, K_\alpha([0.1, 0.8]) + (1-\alpha) \cdot 0.7], \tag{18}$$

which also results in $[0.1, 0.8]$, for all $\alpha \in (0, 1]$.
The maximal width threshold for the pair $(Op, \max)$ in this context is given by

$$m_{\widehat{Op},\max}([0.2, 0.8], [0.5, 1]) = \\ \min\{w(\widehat{Op}([0.2, 0.8], [0.5, 1])), \max(w([0.2, 0.8]), w([0.5, 1]))\} = \min\{0.7, \max\{0.6, 0.5\}\} = 0.6.$$

By Equation (17), we have that

$$IOpw^\alpha_{\max}([0.2, 0.8], [0.5, 1]) = [K_\alpha([0.1, 0.8]) - \alpha \cdot 0.6, K_\alpha([0.1, 0.8]) + (1-\alpha) \cdot 0.6], \tag{19}$$

and $w(IOpw^\alpha_{\max}([0.2, 0.8], [0.5, 1])) = 0.6 \leq \max(w([0.2, 0.8]), w([0.5, 1]))$, which is expected as $IOpw^\alpha_{\max}$ is width-limited by max.
Notice, from Equations (18) and (19), that $K_\alpha(\widehat{Op}([0.2, 0.8], [0.5, 1])) = K_\alpha(IOpw^\alpha_{\max}([0.2, 0.8], [0.5, 1]))$, and that $w(\widehat{Op}([0.2, 0.8], [0.5, 1])) = 0.7 > 0.6 = w(IOpw^\alpha_{\max}([0.2, 0.8], [0.5, 1]))$.
Let us assign some values for α to observe what is the resulting interval for $IOpw^\alpha_{\max}([0.2, 0.8], [0.5, 1])$.

a) If $\alpha = 0.01$, then

$$IOp^{0.01}_{\max}([0.2, 0.8], [0.5, 1]) = [K_{0.01}([0.1, 0.8]), K_{0.01}([0.1, 0.8]) + 0.6] = [0.107, 0.707];$$

b) If $\alpha = 0.5$, then

$$IOp^{0.5}_{\max}([0.2, 0.8], [0.5, 1]) = [K_{0.5}([0.1, 0.8]) - 0.5 \cdot 0.6, K_{0.5}([0.1, 0.8]) + 0.5 \cdot 0.6] = [0.15, 0.75];$$

c) If $\alpha = 1$, then

$$IOp^{1}_{\max}([0.2, 0.8], [0.5, 1]) = [K_1([0.1, 0.8]) - 0.6, K_1([0.1, 0.8])] = [0.2, 0.8].$$

2) Now, consider $B = \max$ and take $X = [0.6, 0.9]$ and $Y = [0.8, 0.8]$. So, we have that

$$\widehat{Op}([0.6, 0.9], [0.8, 0.8]) = [0.48, 0.72].$$

Although $\widehat{Op}$ is not width-limited by max, in this case it holds that as

$$w(\widehat{Op}([0.6, 0.9], [0.8, 0.8])) = 0.24 < 0.3 = \max(w([0.6, 0.9]), w([0.8, 0.8])).$$

Moreover, by Equation (7), $\widehat{Op}([0.6, 0.9], [0.8, 0.8])$ can be written as:

$$\widehat{Op}([0.6, 0.9], [0.8, 0.8]) = [K_\alpha([0.48, 0.72]) - \alpha \cdot 0.24, K_\alpha([0.48, 0.72]) + (1-\alpha) \cdot 0.24] = [0.48, 0.72].$$

The maximal width threshold for the pair $(Op, \max)$ in this context is given by

$$m_{\widehat{Op},\max}([0.6, 0.9], [0.8, 0.8]) = \\ \min\{w(\widehat{Op}([0.6, 0.9], [0.8, 0.8])), \max(w([0.6, 0.9]), w([0.8, 0.8]))\} = \min\{0.24, \max\{0.3, 0\}\} = 0.24.$$

By Equation (17), we have that

$$IOpw^\alpha_{\max}([0.6, 0.9], [0.8, 0.8]) = [K_\alpha([0.48, 0.72]) - \alpha \cdot 0.24, K_\alpha([0.48, 0.72]) + (1-\alpha) \cdot 0.24] = [0.48, 0.72].$$

Thus, $\widehat{Op}([0.6, 0.9], [0.8, 0.8]) = IOpw^\alpha_{\max}([0.6, 0.9], [0.8, 0.8]) = [0.48, 0.72]$, for all $\alpha \in (0, 1]$.

3) Next, take the same $X = [0.6, 0.9]$, and $Y = [0.8, 0.8]$, but now with $B = \min$. So,

$$m_{\widehat{Op},\min}([0.6, 0.9], [0.8, 0.8]) = \\ \min\{w(\widehat{Op}([0.6, 0.9], [0.8, 0.8])), \min\{w([0.6, 0.9]), w([0.8, 0.8])\}\} = \min\{0.24, \min\{0.3, 0\}\} = 0,$$

and, therefore,

$$IOp^{\alpha}_{\min}([0.6, 0.9], [0.8, 0.8]) = [K_\alpha([0.48, 0.72]), K_\alpha([0.48, 0.72])],$$

for any $\alpha \in (0, 1]$. One can observe that $w(IOp^{\alpha}_{\min}([0.6, 0.9], [0.8, 0.8])) = 0$, which is expected from Remark 3.5 as $Y = [0.8, 0.8]$ is degenerate and min is a conjunctive function.

4) Finally, take $X = [0.2, 0.8]$ and $Y = [0.5, 1]$, and let $B = Op^d$. Then, the maximal width threshold for the pair $(\widehat{Op}, Op^d)$ is given by

$$m_{\widehat{Op},Op^d}([0.2, 0.8], [0.5, 1]) =$$
$$\min\{w(\widehat{Op}([0.2, 0.8], [0.5, 1])), Op^d(w([0.2, 0.8]), w([0.5, 1]))\} = \min\{0.7, Op^d(0.6, 0.5)\} = 0.7.$$

By Equation (17), we have that

$$\begin{aligned} &IOpw^{\alpha}_{Op^d}([0.2, 0.8], [0.5, 1]) \\ &= [K_\alpha([0.1, 0.8]) - \alpha \cdot 0.7, K_\alpha([0.1, 0.8]) + (1 - \alpha) \cdot 0.7] \\ &= \widehat{Op}([0.2, 0.8], [0.5, 1]) \\ &= [0.1, 0.8], \end{aligned}$$

which is expected, by Proposition 4.7, since Op is an $(1, 1)$-ultramodular overlap function.

Next, we present the second construction method for w-iv-overlap functions:

Theorem 4.2. *Let $O : [0, 1]^2 \to [0, 1]$ be a strict overlap function, $B : [0, 1]^2 \to [0, 1]$ be a commutative, increasing and conjunctive function and $\alpha \in (0, 1)$, $\beta \in [0, 1]$ such that $\alpha \neq \beta$. Then $IOw^{\alpha}_{B} : L([0, 1])^2 \to L([0, 1])$ defined, for all $X, Y \in L([0, 1])$, by*

$$IOw^{\alpha}_{B}(X, Y) = [O(K_\alpha(X), K_\alpha(Y)) - \alpha\theta, O(K_\alpha(X), K_\alpha(Y)) + (1 - \alpha)\theta],$$

where

$$\theta = B(B(w(X), w(Y)), B(O(K_\alpha(X), K_\alpha(Y)), 1 - O(K_\alpha(X), K_\alpha(Y))))$$

is a w-iv-overlap function for the tuple $(\leq_{\alpha,\beta}, \leq_{\alpha,\beta}, B)$.

Proof. See Appendix B. □

The following result is immediate as a w-iv-overlap function for the tuple $(\leq_{\alpha,\beta}, \leq_{\alpha,\beta}, B)$ is also a $\leq_{\alpha,\beta}$-overlap function (Definition 2.11), in the sense of [40].

Corollary 4.1. *Let $O : [0, 1]^2 \to [0, 1]$ be a strict overlap function, $B : [0, 1]^2 \to [0, 1]$ be a commutative, increasing and conjunctive function and $\alpha \in (0, 1)$, $\beta \in [0, 1]$ such that $\alpha \neq \beta$. Then $IOw^{\alpha}_{B} : L([0, 1])^2 \to L([0, 1])$ defined, for all $X, Y \in L([0, 1])$, by*

$$IOw^{\alpha}_{B}(X, Y) = [O(K_\alpha(X), K_\alpha(Y)) - \alpha\theta, O(K_\alpha(X), K_\alpha(Y)) + (1 - \alpha)\theta],$$

where

$$\theta = B(B(w(X), w(Y)), B(O(K_\alpha(X), K_\alpha(Y)), 1 - O(K_\alpha(X), K_\alpha(Y))))$$

is a $\leq_{\alpha,\beta}$-overlap function.

Example 4.6. Consider a function $B : [0, 1]^2 \to [0, 1]$ such that $B = \min$ and the product overlap function $Op : [0, 1]^2 \to [0, 1]$. Then, the interval-valued function $IOpw^{\alpha}_{\min} : L([0, 1])^2 \to L([0, 1])$ defined, for all $X, Y \in L([0, 1])$, by

$$IOpw^{\alpha}_{\min}(X, Y) = [Op(K_\alpha(X), K_\alpha(Y)) - \alpha\theta, Op(K_\alpha(X), K_\alpha(Y)) + (1 - \alpha)\theta], \tag{20}$$

where

$$\theta = \min(\min(w(X), w(Y)), \min(Op(K_\alpha(X), K_\alpha(Y)), 1 - Op(K_\alpha(X), K_\alpha(Y))))$$

is a w-iv-overlap function for the tuple $(\leq_{\alpha,\beta}, \leq_{\alpha,\beta}, \min)$, for all $\alpha \in (0, 1)$, $\beta \in [0, 1]$ with $\alpha \neq \beta$.

1) Take $X = [0.2, 0.8]$, and $Y = [0.5, 1]$. By Equation (20), we have that

$$\begin{aligned} &IOpw^\alpha_{\min}([0.2, 0.8], [0.5, 1]) \\ &\quad = [Op(K_\alpha([0.2, 0.8]), K_\alpha([0.5, 1])) - \alpha\theta, Op(K_\alpha([0.2, 0.8]), K_\alpha([0.5, 1])) + (1 - \alpha)\theta], \end{aligned}$$

where

$$\begin{aligned} \theta = {} & \min(\min(w([0.2, 0.8]), w([0.5, 1])), \\ & \min(Op(K_\alpha([0.2, 0.8]), K_\alpha([0.5, 1])), 1 - Op(K_\alpha([0.2, 0.8]), K_\alpha([0.5, 1])))) \end{aligned}$$

Let us assign some values for α to observe what is the resulting interval for $IOpw^\alpha_{\min}([0.2, 0.8], [0.5, 1])$.

a) If $\alpha = 0.01$ then

$$\begin{aligned} \theta = {} & \min(\min(w([0.2, 0.8]), w([0.5, 1])), \\ & \quad \min(Op(K_{0.01}([0.2, 0.8]), K_{0.01}([0.5, 1])), 1 - Op(K_{0.01}([0.2, 0.8]), K_{0.01}([0.5, 1])))) \\ & = \min(\min(0.6, 0.5), \min(0.104, 0.896)) = 0.104 \end{aligned}$$

and

$$IOpw^0_{\min}([0.2, 0.8], [0.5, 1]) = [0.104 - 0.01 \cdot 0.104, 0.104 + 0.99 \cdot 0.104] = [0.103, 0.207];$$

b) If $\alpha = 0.5$ then

$$\begin{aligned} \theta = {} & \min(\min(w([0.2, 0.8]), w([0.5, 1])), \\ & \quad \min(Op(K_{0.5}([0.2, 0.8]), K_{0.5}([0.5, 1])), 1 - Op(K_{0.5}([0.2, 0.8]), K_{0.5}([0.5, 1])))) \\ & = \min(\min(0.6, 0.5), \min(0.375, 0.625)) = 0.375 \end{aligned}$$

and

$$IOpw^{0.5}_{\min}([0.2, 0.8], [0.5, 1]) = [0.375 - 0.5 \cdot 0.375, 0.375 + 0.5 \cdot 0.375] = [0.1875, 0.5625];$$

c) If $\alpha = 0.99$ then

$$\begin{aligned} \theta = {} & \min(\min(w([0.2, 0.8]), w([0.5, 1])), \\ & \quad \min(Op(K_{0.99}([0.2, 0.8]), K_{0.99}([0.5, 1])), 1 - Op(K_{0.99}([0.2, 0.8]), K_{0.99}([0.5, 1])))) \\ & = \min(\min(0.6, 0.5), \min(0.79, 0.2099)) = 0.2099 \end{aligned}$$

and

$$IOpw^{0.99}_{\min}([0.2, 0.8], [0.5, 1]) = [0.79 - 0.99 * 0.2099, 0.79 + 0.01 * 2099] = [0.5822, 0.7921].$$

2) Now, take $X = [0.6, 0.9]$, and $Y = [0.8, 0.8]$. By Equation (20), we have that

$$\begin{aligned} &IOpw^\alpha_{\min}([0.6, 0.9], [0.8, 0.8]) \\ &\quad = [Op(K_\alpha([0.6, 0.9]), K_\alpha([0.8, 0.8])) - \alpha\theta, Op(K_\alpha([0.6, 0.9]), K_\alpha([0.8, 0.8])) + (1 - \alpha)\theta], \end{aligned}$$

where

$$\begin{aligned} & \theta = \min(\min(w([0.6, 0.9], w([0.8, 0.8])), \min(Op(K_\alpha([0.6, 0.9]), \\ & K_\alpha([0.8, 0.8])), 1 - Op(K_\alpha([0.6, 0.9]), K_\alpha([0.8, 0.8])))) = 0. \end{aligned}$$

Thus, $IOpw^\alpha_{\min}([0.6, 0.9], [0.8, 0.8]) = [K_\alpha([0.6, 0.9]) \cdot K_\alpha([0.8, 0.8])]$, for any $\alpha \in (0, 1)$. For example:

a) If $\alpha = 0.01$ then $IOpw_{\min}^{0.01}([0.6, 0.9], [0.8, 0.8]) = [0.603 \cdot 0.8, 0.603 \cdot 0.8] = [0.4824, 0.4824]$;
b) If $\alpha = 0.5$ then $IOpw_{\min}^{0.5}([0.6, 0.9], [0.8, 0.8]) = [0.7 \cdot 0.8, 0.7 \cdot 0.8] = [0.56, 0.56]$;
c) If $\alpha = 0.99$ then $IOpw_{\min}^{0.99}([0.6, 0.9], [0.8, 0.8]) = [0.897 \cdot 0.8, 0.897 \cdot 0.8] = [0.7176, 0.7176]$.

Remark 4.3. Considering Theorem 4.2, when $B = \min$ we recover the construction method presented in Theorem 2.2, meaning that Theorem 4.2 is more general. Also, it is noteworthy that the reason for $\alpha \in (0, 1)$ is to assure that the construction method produces an w-iv-overlap function. For example, if $\alpha = 0$, then $IOw_B^0([0, 1], [0.2, 0.2]) = [0, 0]$, which would contradict **(IOw2)**. Also, one can observe that IOw_B^α falls into the conditions of Remark 3.5, meaning that if either X or Y is degenerate, then $IOw_B^\alpha(X, Y)$ is also degenerate, as shown in Example 4.6, for $X = [0.6, 0.9]$ and $Y = [0.8, 0.8]$. Finally, although the w-iv-overlap constructed by the method presented in Theorem 4.2 is width-limited by the chosen function B, the output interval may not be contained in the best interval representation of the chosen overlap function O, as shown in the next example.

Example 4.7. Consider an w-iv-overlap function $IOpw_{\min}^{0.99}$ for the tuple $(\leq_{0.99,\beta}, \leq_{0.99,\beta}, \min)$ obtained via the construction method presented in Theorem 4.2 by taking $B = \min$, $O = O_P$ (the product overlap) and $\beta \in [0, 1]$ such that $\beta \neq 0.99$. In the case when $X = Y = [0.1, 0.4]$, we have that

$$\widehat{Op}([0.1, 0.4], [0.1, 0.4]) = [0.1 \cdot 0.1, 0.4 \cdot 0.4] = [0.01, 0.16].$$

From Theorem 4.2, it holds that

$$\begin{aligned}
\theta = \min(&\min(w([0.1, 0.4]), w([0.1, 0.4])),\\
&\min(Op(K_{0.99}([0.1, 0.4]), K_{0.99}([0.1, 0.4])), 1 - Op(K_{0.99}([0.1, 0.4]), K_{0.99}([0.1, 0.4]))))\\
&= \min(\min(0.3, 0.3), \min(Op(0.397, 0.397), 1 - Op(0.397, 0.397))))\\
&= \min(0.3, \min(0.1576, 0.8424)))\\
&= 0.1576.
\end{aligned}$$

So,

$$\begin{aligned}
IOpw_{\min}^{0.99}([0.1, 0.4], [0.1, 0.4]) = [&Op(K_{0.99}([0.1, 0.4]), K_{0.99}([0.1, 0.4])) - 0.99 \cdot 0.1576,\\
&Op(K_{0.99}([0.1, 0.4]), K_{0.99}([0.1, 0.4])) + 0.01 \cdot 0.1576]\\
&= [0.0016, 0.1502],
\end{aligned}$$

showing that $IOpw_{\min}^{0.99}([0.1, 0.4], [0.1, 0.4]) \nsubseteq \widehat{Op}([0.1, 0.4], [0.1, 0.4])$.

Before presenting the third construction method for w-iv-overlaps, let us recall some important concepts presented in [38]:

Definition 4.4. Let $c \in [0, 1]$ and $\alpha \in [0, 1]$. We denote by $d_\alpha(c)$ the maximal possible width of an interval $Z \in L([0, 1])$ such that $K_\alpha(Z) = c$. Moreover, for any $X \in L([0, 1])$, define

$$\lambda_\alpha(X) = \frac{w(X)}{d_\alpha(K_\alpha(X))},$$

where we set $\frac{0}{0} = 1$.

Proposition 4.8. *For all* $\alpha \in [0, 1]$ *and* $X \in L([0, 1])$ *it holds that*

$$d_\alpha(K_\alpha(X)) = \min\left\{\frac{K_\alpha(X)}{\alpha}, \frac{1 - K_\alpha(X)}{1 - \alpha}\right\},$$

where we set $\frac{r}{0} = 1$*, for all* $r \in [0, 1]$.

Now, we present a version of Theorem 3.16 in [38] in the context of 2-dimensional functions.

Theorem 4.3. *Let $\alpha, \beta \in [0, 1]$, such that, $\alpha \neq \beta$. Let $A_1, A_2 : [0, 1]^2 \to [0, 1]$ be two aggregation functions where A_1 is strictly increasing. Then $IF^\alpha : L([0, 1])^2 \to L([0, 1])$ defined by:*

$$IF^\alpha_{A1,A2}(X, Y) = R, \text{ where, } \begin{cases} K_\alpha(R) = A_1(K_\alpha(X), K_\alpha(Y)), \\ \lambda_\alpha(R) = A_2(\lambda_\alpha(X), \lambda_\alpha(Y)), \end{cases}$$

for all $X, Y \in L([0, 1])$, is an $\leq_{\alpha,\beta}$-increasing iv-aggregation function.

Proof. It follows from Theorem 3.16 in [38]. □

As overlap functions are a class of aggregation functions, the following result is immediate.

Corollary 4.2. *Let $\alpha, \beta \in [0, 1]$, such that, $\alpha \neq \beta$. Let $O : [0, 1]^2 \to [0, 1]$ be a strict overlap function and $A : [0, 1]^2 \to [0, 1]$ be an aggregation function. Then $IF^\alpha_{O,A} : L([0, 1])^2 \to L([0, 1])$ defined by:*

$$IF^\alpha_{O,A}(X, Y) = R, \text{ where, } \begin{cases} K_\alpha(R) = O(K_\alpha(X), K_\alpha(Y)), \\ \lambda_\alpha(R) = A(\lambda_\alpha(X), \lambda_\alpha(Y)), \end{cases}$$

for all $X, Y \in L([0, 1])$, is an $\leq_{\alpha,\beta}$-increasing iv-aggregation function.

The following result is immediate from Definition 4.4 and Corollary 4.2.

Corollary 4.3. *Let $\alpha, \beta \in [0, 1]$ be such that, $\alpha \neq \beta$. Let $O : [0, 1]^2 \to [0, 1]$ be a strict overlap function, $A : [0, 1]^2 \to [0, 1]$ be an aggregation function and $IF^\alpha_{O,A} : L([0, 1])^2 \to L([0, 1])$ be an iv-aggregation function constructed as in Corollary 4.2. Then, for all $X, Y \in L([0, 1])$, we have that*

$$w(IF^\alpha_{O,A}(X, Y)) = A(\lambda_\alpha(X), \lambda_\alpha(Y)) \cdot d_\alpha(K_\alpha(IF^\alpha_{O,A}(X, Y))).$$

Finally, the third construction method for w-iv-overlaps is obtained as follows:

Theorem 4.4. *Consider a strict overlap function $O : [0, 1]^2 \to [0, 1]$, a commutative aggregation function $B : [0, 1]^2 \to [0, 1]$, an interval-valued aggregation function $IF^\alpha_{O,B} : L([0, 1])^2 \to L([0, 1])$ defined as in Corollary 4.2, the maximal width threshold $m_{IF^\alpha_{O,B},B} : L([0, 1])^2 \to L([0, 1])$ for the pair $(IF^\alpha_{O,B}, B)$, $\alpha \in (0, 1)$ and $\beta \in [0, 1]$ with $\alpha \neq \beta$. Then, the interval-valued function $IOw^\alpha_B : L([0, 1])^2 \to L([0, 1])$ defined by*

$$IOw^\alpha_B(X, Y) = R,$$

where:

(i) $K_\alpha(R) = O(K_\alpha(X), K_\alpha(Y))$;
(ii) $w(R) = m_{IF^\alpha_{O,B},B}(X, Y)$,

is a w-iv-overlap function for the tuple $(\leq_{\alpha,\beta}, \leq_{\alpha,\beta}, B)$.

Proof. See Appendix C. □

The following result is immediate as a w-iv-overlap function for the tuple $(\leq_{\alpha,\beta}, \leq_{\alpha,\beta}, B)$ is also a $\leq_{\alpha,\beta}$-overlap function (Definition 2.11), in the sense of [40].

Corollary 4.4. *Consider a strict overlap function $O : [0, 1]^2 \to [0, 1]$, a commutative aggregation function $B : [0, 1]^2 \to [0, 1]$, an interval-valued aggregation function $IF^\alpha_{O,B} : L([0, 1])^2 \to L([0, 1])$ defined as in Corollary 4.2, the maximal width threshold $m_{IF^\alpha_{O,B},B} : L([0, 1])^2 \to L([0, 1])$ for the pair $(IF^\alpha_{O,B}, B)$, $\alpha \in (0, 1)$ and $\beta \in [0, 1]$ with $\alpha \neq \beta$. Then, the interval-valued function $IOw^\alpha_B : L([0, 1])^2 \to L([0, 1])$ defined by*

$$IOw^\alpha_B(X, Y) = R,$$

where:

(i) $K_\alpha(R) = O(K_\alpha(X), K_\alpha(Y))$;
(ii) $w(R) = m_{IF^\alpha_{O,B},B}(X, Y)$.

is a $\leq_{\alpha,\beta}$-overlap function.

Example 4.8. Consider a commutative aggregation function $B : [0, 1]^2 \to [0, 1]$ and the product overlap function $Op : [0, 1]^2 \to [0, 1]$. Then, the interval-valued function $IOpw^\alpha_B : L([0, 1])^2 \to L([0, 1])$ defined, for all $X, Y \in L([0, 1])$, by

$$IOpw^\alpha_B(X, Y) = R,$$

where:

(i) $K_\alpha(R) = Op(K_\alpha(X), K_\alpha(Y))$;
(ii) $w(R) = m_{IF^\alpha_{Op,B},B}(X, Y)$,

is a w-iv-overlap function for the tuple $(\leq_{\alpha,\beta}, \leq_{\alpha,\beta}, B)$, for all $\alpha, \beta \in [0, 1]$ such that $\alpha \neq \beta$.

1) Take $B = \max$, $X = [0.2, 0.8]$ and $Y = [0.5, 1]$. By **(i)**, we have that

$$K_\alpha(R) = Op(K_\alpha([0.2, 0.8]), K_\alpha([0.5, 1]))$$

and

$$\begin{aligned} w(R) &= m_{IF^\alpha_{Op,\max},\max}([0.2, 0.8], [0.5, 1]) \text{ by } \mathbf{(ii)} \\ &= \min\{w(IF^\alpha_{O,\max}([0.2, 0.8], [0.5, 1])), \max(w([0.2, 0.8]), w([0.5, 1]))\} \text{ by Definition 4.3} \\ &= \min\{\max(\lambda_\alpha([0.2, 0.8]), \lambda_\alpha([0.5, 1])) \cdot d_\alpha(K_\alpha(IF^\alpha_{O,\max}([0.2, 0.8], [0.5, 1]))), \max(0.6, 0.5)\} \\ &\qquad \text{by Corollary 4.3} \\ &= \min\{\max(\lambda_\alpha([0.2, 0.8]), \lambda_\alpha([0.5, 1])) \cdot d_\alpha(Op(K_\alpha([0.2, 0.8]), K_\alpha([0.5, 1]))), 0.6\}. \\ &\qquad \text{by Corollary 4.2} \end{aligned}$$

Let us assign some values for α to observe what is the resulting interval for $IOpw^\alpha_{\max}([0.2, 0.8], [0.5, 1])$.
a) If $\alpha = 0.01$ then

$$K_{0.01}(R) = Op(K_{0.01}([0.2, 0.8]), K_{0.01}([0.5, 1])) = 0.206 \cdot 0.505 = 0.104,$$

and

$$\begin{aligned} &w(R) \\ &= \min\{\max(\lambda_{0.01}([0.2, 0.8]), \lambda_{0.01}([0.5, 1])) \cdot d_{0.01}(Op(K_\alpha([0.2, 0.8]), K_{0.01}([0.5, 1]))), 0.6\} \\ &= \min\left\{\max\left(\frac{w([0.2, 0.8])}{d_{0.01}(K_{0.01}([0.2, 0.8]))}, \frac{w([0.5, 1])}{d_{0.01}(K_{0.01}([0.5, 1]))}\right) \cdot d_{0.01}(0.104), 0.6\right\} \\ &\qquad \text{by Definition 4.8} \\ &= \min\left\{\max\left(\frac{0.6}{\min\left\{\frac{0.206}{0.01}, \frac{0.794}{0.99}\right\}}, \frac{0.5}{\min\left\{\frac{0.505}{0.01}, \frac{0.495}{0.99}\right\}}\right) \cdot \min\left\{\frac{0.104}{0.01}, \frac{0.896}{0.99}\right\}, 0.6\right\} \\ &\qquad \text{by Proposition 4.4} \\ &= \min\left\{\max\left(\frac{0.6}{0.802}, \frac{0.5}{0.5}\right) \cdot 0.905, 0.6\right\} = \min\{0.905, 0.6\} = 0.6. \end{aligned}$$

So, by Equation (7), $IOpw^{0.01}_{\max}([0.2, 0.8], [0.5, 1]) = [0.104 - 0.01 \cdot 0.6, 0.104 + 0.99 \cdot 0.6] = [0.098, 0.698]$.
In the next cases, we will just present the final results.

b) If $\alpha = 0.5$ then

$$K_{0.5}(R) = Op(K_{0.5}([0.2, 0.8]), K_{0.5}([0.5, 1])) = 0.5 \cdot 0.75 = 0.375,$$

and

$$w(R) = \min\left\{\max\left(\frac{0.6}{1}, \frac{0.5}{0.5}\right) \cdot 0.625, 0.6\right\} = \min\{0.625, 0.6\} = 0.6.$$

Thus, $IOpw^{0.5}_{\max}([0.2, 0.8], [0.5, 1]) = [0.375 - 0.5 \cdot 0.6, 0.375 + 0.5 \cdot 0.6] = [0.075, 0.675]$.

c) If $\alpha = 0.99$ then

$$K_{0.99}(R) = Op(K_{0.99}([0.2, 0.8]), K_{0.99}([0.5, 1])) = 0.794 \cdot 0.995 = 0.79,$$

and

$$w(R) = \min\left\{\max\left(\frac{0.6}{0.802}, \frac{0.5}{0.5}\right) \cdot 0.798, 0.6\right\} = \min\{0.798, 0.6\} = 0.6.$$

Therefore, $IOpw^{0.99}_{\max}([0.2, 0.8], [0.5, 1]) = [0.79 - 0.99 \cdot 0.6, 0.79 + 0.01 \cdot 0.6] = [0.196, 0.796]$.

2) Now, take $X = [0.6, 0.9]$, and $Y = [0.8, 0.8]$. Then, we have that

$$K_\alpha(R) = Op(K_\alpha([0.6, 0.9]), K_\alpha([0.8, 0.8]))$$

and, by **(ii)**,

$$w(R) = \min\{\max(\lambda_\alpha([0.6, 0.9]), \lambda_\alpha([0.8, 0.8])) \cdot d_\alpha(Op(K_\alpha([0.6, 0.9]), K_\alpha([0.8, 0.8]))), 0.3\}.$$

by Corollary 4.2

Once again, let us observe the value of $IOpw^{\alpha}_{\max}([0.6, 0.9], [0.8, 0.8])$ by varying the value of α:

a) If $\alpha = 0.01$ then

$$K_{0.01}(R) = Op(K_{0.01}([0.6, 0.9]), K_{0.01}([0.8, 0.8])) = 0.603 \cdot 0.8 = 0.4824,$$

and

$$w(R) = \min\left\{\max\left(\frac{0.3}{0.401}, \frac{0}{0.202}\right) \cdot 0.5228, 0.3\right\} = \min\{0.3911, 0.3\} = 0.3.$$

So, $IOpw^{0.01}_{\max}([0.6, 0.9], [0.8, 0.8]) = [0.4824 - 0.01 \cdot 0.3, 0.4824 + 0.99 \cdot 0.3] = [0.4794, 0.7794]$.

b) If $\alpha = 0.5$ then

$$K_{0.5}(R) = Op(K_{0.5}([0.6, 0.9]), K_{0.5}([0.8, 0.8])) = 0.75 \cdot 0.8 = 0.6,$$

and

$$w(R) = \min\left\{\max\left(\frac{0.3}{0.5}, \frac{0}{0.4}\right) \cdot 0.8, 0.3\right\} = \min\{0.48, 0.3\} = 0.3.$$

Thus, $IOpw^{0.5}_{\max}([0.6, 0.9], [0.8, 0.8]) = [0.6 - 0.5 \cdot 0.3, 0.6 + 0.5 \cdot 0.3] = [0.45, 0.75]$.

c) If $\alpha = 0.99$ then

$$K_{0.99}(R) = Op(K_{0.99}([0.6, 0.9]), K_{0.99}([0.8, 0.8])) = 0.897 \cdot 0.8 = 0.7176,$$

and

$$w(R) = \min\left\{\max\left(\frac{0.3}{0.906}, \frac{0}{0.808}\right) \cdot 0.7248, 0.3\right\} = \min\{0.24, 0.3\} = 0.24.$$

Therefore, $IOpw^{0.99}_{\max}([0.6, 0.9], [0.8, 0.8]) = [0.7176 - 0.99 \cdot 0.24, 0.7176 + 0.01 \cdot 0.24] = [0.48, 0.72]$.

Table 2
Comparison between construction methods of a w-iv-overlap IOw_B^α, based on an overlap function O and a width-limiting function B.

	Construction 1	Construction 2	Construction 3
Advantages			
IOw_B^α is $(\leq_{Pr}, \leq_{\alpha,\beta})$-increasing	✓	✓	✓
IOw_B^α is $\leq_{\alpha,\beta}$-increasing		✓	✓
For all $X, Y \in L([0,1])$: $IOw_B^\alpha(X,Y) \subseteq \widehat{O}(X,Y)$	✓		
Drawbacks			
α must be different than 1		✗	✗
$\beta < \alpha$ must hold	✗		
B needs to be conjunctive		✗	
For all B: $(w(X)=0$ or $w(Y)=0) \Rightarrow IOw_B^\alpha(X,Y)=0$		✗	

3) Finally, take $X = [0.6, 0.9]$, and $Y = [0.8, 0.8]$, but consider $B = \min$. Then, we have that

$$K_\alpha(R) = Op(K_\alpha([0.6, 0.9]), K_\alpha([0.8, 0.8]))$$

and, by **(ii)**,

$$w(R) = \min\{\min(\lambda_\alpha([0.6, 0.9]), \lambda_\alpha([0.8, 0.8])) \cdot d_\alpha(Op(K_\alpha([0.6, 0.9]), K_\alpha([0.8, 0.8]))), 0\} = 0.$$

So, let us see the different values of $IOpw_{\min}^\alpha([0.6, 0.9], [0.8, 0.8])$ in this case by varying the value of α:

a) If $\alpha = 0.01$ then $IOpw_{\min}^{0.01}([0.6, 0.9], [0.8, 0.8]) = [0.4824, 0.4824]$;

b) If $\alpha = 0.5$ then $IOpw_{\min}^{0.5}([0.6, 0.9], [0.8, 0.8]) = [0.56, 0.56]$;

c) If $\alpha = 0.99$ then $IOpw_{\min}^{0.99}([0.6, 0.9], [0.8, 0.8]) = [0.7176, 0.7176]$.

Remark 4.4. The reason why $\alpha \in (0, 1)$ is to assure that the construction method results in an w-iv-overlap function, so that conditions **(IOw2)** and **(IOw3)** are respected. Moreover, one may observe that the construction method presented in Theorem 4.4, for a given overlap O, may not produce intervals contained in the best interval representation of O. However, it generates an interval-valued function which is $\leq_{\alpha,\beta}$-increasing and the chosen width-limiting aggregation function B does not need to be conjunctive. In the case when B is conjunctive, as Remark 3.5 states, when either X or Y is degenerate, then $IOw_B^\alpha(X,Y)$ is also degenerate.

Table 2 shows a comparison between the three construction methods for w-iv-overlap functions presented in Theorems 4.1 (Construction 1), 4.2 (Construction 2) and 4.4 (Construction 3), regarding some desirable properties (marked with ✓) and some possible drawbacks (marked with ✗).

On Table 3, we review the results obtained from Examples 4.5 and 4.8, to further compare the constructions methods presented on Theorems 4.1 (Construction 1) and 4.4 (Construction 3), all based on the product overlap O_P, but with different choices of the width-limiting function B and different values of α. As the construction method provided by Theorem 4.4 (Construction 3) does not allow for $\alpha = 1$, we present the values obtained by this method for $\alpha = 0.99$, instead. We omitted the results from Example 4.6 on Table 3, as the construction method based on Theorem 4.2 (Construction 2) presented itself as the most restrictive one, by a simple analysis of Table 2.

Remark 4.5. It is noteworthy that our construction methods of w-iv-overlap functions are all based on a core overlap function O, but do not necessarily provide a proper interval extension of O. However, such constructed w-iv-overlap functions satisfy interval counterparts (**(IOw1)-(IOw4)** of Definition 4.2) of most of the defining properties of O (**(O1)-(O4)** of Definition 2.6), being well fitted to measure the overlap of interval data in a similar manner as O measures overlap of real data.

Remark 4.6. Concerning the application of the presented construction methods of width-limited iv-overlap functions in practical problems, a number of choices need to be made by the domain expert:

Table 3
Comparison of the results obtained in Examples 4.5 and 4.8.

	Construction 1	Construction 3	Best interval representation
$X=[0.2,0.8]$ $Y=[0.5,1]$ $A=\max$ $\alpha=0.01$	$IOwp^{0.01}_{\max}=[0.107,0.707]$	$IOwp^{0.01}_{\max}=[0.098,0.698]$	$\widehat{O_P}(X,Y)=[0.1,0.8]$
$X=[0.2,0.8]$ $Y=[0.5,1]$ $A=\max$ $\alpha=0.5$	$IOwp^{0.5}_{\max}=[0.15,0.75]$	$IOwp^{0.5}_{\max}=[0.075,0.675]$	$\widehat{O_P}(X,Y)=[0.1,0.8]$
$X=[0.2,0.8]$ $Y=[0.5,1]$ $A=\max$ $\alpha=1$	$IOwp^{1}_{\max}=[0.2,0.8]$	$IOwp^{0.99}_{\max}=[0.196,0.796]$	$\widehat{O_P}(X,Y)=[0.1,0.8]$
$X=[0.6,0.9]$ $Y=[0.8,0.8]$ $A=\max$ $\alpha=0.01$	$IOwp^{0.01}_{\max}=[0.48,0.72]$	$IOwp^{0.01}_{\max}=[0.4794,0.7794]$	$\widehat{O_P}(X,Y)=[0.48,0.72]$
$X=[0.6,0.9]$ $Y=[0.8,0.8]$ $A=\max$ $\alpha=0.5$	$IOwp^{0.5}_{\max}=[0.48,0.72]$	$IOwp^{0.5}_{\max}=[0.45,0.75]$	$\widehat{O_P}(X,Y)=[0.48,0.72]$
$X=[0.6,0.9]$ $Y=[0.8,0.8]$ $A=\max$ $\alpha=1$	$IOwp^{1}_{\max}=[0.48,0.72]$	$IOwp^{0.99}_{\max}=[0.48,0.72]$	$\widehat{O_P}(X,Y)=[0.48,0.72]$
$X=[0.6,0.9]$ $Y=[0.8,0.8]$ $A=\min$ $\alpha=0.01$	$IOwp^{0}_{\min}=[0.4824,0.4824]$	$IOwp^{0.01}_{\min}=[0.4824,0.4824]$	$\widehat{O_P}(X,Y)=[0.48,0.72]$
$X=[0.6,0.9]$ $Y=[0.8,0.8]$ $A=\min$ $\alpha=0.5$	$IOwp^{0.5}_{\min}=[0.6,0.6]$	$IOwp^{0.5}_{\min}=[0.56,0.56]$	$\widehat{O_P}(X,Y)=[0.48,0.72]$
$X=[0.6,0.9]$ $Y=[0.8,0.8]$ $A=\min$ $\alpha=1$	$IOwp^{1}_{\min}=[0.72,0.72]$	$IOwp^{0.99}_{\min}=[0.7176,0.7176]$	$\widehat{O_P}(X,Y)=[0.48,0.72]$

1. **The choice of overlap function O:** According to the considered application, some overlap functions produce better results than others. For example, in the literature, it is possible to verify that some overlap functions are more suitable to be applied in image processing [4] while others present good behaviour in classification problems [59,60,21,40].
2. **The choice of α and β:** It is completely determined by the admissible order $\leq_{\alpha,\beta}$ that is suitable for the application, reflecting the adopted attitude of the expert in front of uncertainty [37]. A pessimist/caution attitude towards the uncertainty is considered when one relies that the exact real value that an interval is approximating is much closer to its left endpoint than to its right endpoint [38]. The optimist/audacious attitude is defined analogously.
That is, the more pessimist/cautious attitude is needed in the decision process, the closer to zero should be α stated. On the contrary, the more optimist/audacious attitude is expected in the decision process, the closer to one should be α defined. The value of β is only used when the compared K_α points have the same value. In this case, β determines the ordering according to the interval widths, that is, the information quality required by the

application. For $\beta = 0$, the ordering respects the information quality ordering. On the other hand, for $\beta = 1$, the ordering respects the inclusion set order.

3. The choice of the width-limiting function B: Different applications may require that the aggregation process produces interval-valued outputs with more or less uncertainty tolerance, which will inform the definition of B. This will be determined by the information quality required by the application. For example, when using $B = \min$ one has a more rigid control of the information quality of the interval result than when $B = \max$. That is, the higher the output's width, the lesser will be the information quality [35].

5. Conclusion

We introduced and developed the concepts of width-limited interval-valued functions and their respective width-limiting functions, as a way to analyze the effect of the width of the input intervals on the width of the output interval, accordingly to the interval-valued function at hand. Furthermore, it was shown a way to obtain the least width-limiting function for a given interval-valued function, which informs how much width-propagation one can expect for such interval-valued operation. A relaxation of the concept of ultramodularity was presented, in the form of (a, b)-ultramodular functions, allowing us to analyze the width-limiting functions of the best interval representation of some aggregation functions. Also, we introduced the notion of an interval-valued function that is increasing with respect to a pair of partial orders, a more flexible approach for increasingness of interval-valued functions.

These new developed concepts could aid the definition of different interval-valued functions with controlled width propagation. As our primary interest was to apply such notions on interval-valued overlap operations, width-limited interval-valued overlap functions were defined and studied. Following that, three construction methods for w-iv-overlap functions were presented, analyzed and compared. As these construction methods are all based on choices of overlap functions, width-limiting functions and admissible orders, it was made clear the adaptability of the developed concepts, as one can obtain an interval-valued overlap operations that best satisfy the restrictions of the context regarding the acceptable amount of width propagation and/or the ordering of intervals to be applied.

Thus, the contributions of this work aimed to address the gap in the literature regarding the analysis of the width of interval-valued functions, especially interval-valued overlap functions, while providing the initial theoretical tools to allow the application of similarly defined width-limited interval-valued functions in practical problems, where the increasing uncertainty associated with the widths of the operated intervals may be an obstacle to overcome, in order to maintain the information quality. On the near future, we intend to generalize adequately the presented theoretical approach to allow for applications in the context of interval-valued fuzzy rule-based classification systems.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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Appendix A. Proof of Theorem 4.1

Proof. Consider a commutative and increasing function $B : [0, 1]^2 \to [0, 1]$, a strict overlap function $O : [0, 1]^2 \to [0, 1]$ and take $\alpha \in (0, 1]$, $\beta \in [0, \alpha)$. Observe that, for all $X, Y \in L([0, 1])$:

(i) $K_\alpha(IOw_B^\alpha(X, Y)) = K_\alpha(\widehat{O}(X, Y))$;
(ii) $w(IOw_B^\alpha(X, Y)) = m_{\widehat{O},B}(X, Y) = \min\{w(\widehat{O}(X, Y)), B(w(X), w(Y))\}$.

So, it is immediate that IOw_B^α is well defined. Now, let us verify if IOw_B^α respects conditions **(IOw1)-(IOw5)** of Definition 4.2.

(IOw1) Immediate, as O and B are both commutative;

(IOw2) ($\Rightarrow$) Suppose that there are $X, Y \in L([0,1])$ such that $IOw_B^\alpha(X,Y) = [0,0]$. Then, we have the following cases:

1) $m_{\widehat{O},B}(X,Y) = w(\widehat{O}(X,Y))$
From Equations (2) and (16), it follows that:

$$\begin{aligned}
&[K_\alpha(\widehat{O}(X,Y)) - \alpha \cdot w(\widehat{O}(X,Y)), K_\alpha(\widehat{O}(X,Y)) + (1-\alpha)\cdot w(\widehat{O}(X,Y))] = [0,0]\\
&\Rightarrow [O(\underline{X},\underline{Y}) + \alpha \cdot w(\widehat{O}(X,Y)) - \alpha \cdot w(\widehat{O}(X,Y)),\\
&\quad O(\underline{X},\underline{Y}) + \alpha \cdot w(\widehat{O}(X,Y)) + w(\widehat{O}(X,Y)) - \alpha \cdot w(\widehat{O}(X,Y))] = [0,0]\\
&\Rightarrow [O(\underline{X},\underline{Y}), O(\underline{X},\underline{Y}) + w(\widehat{O}(X,Y))] = [0,0] \Rightarrow [O(\underline{X},\underline{Y}), O(\overline{X},\overline{Y})] = [0,0]\\
&\Rightarrow \widehat{O}(X,Y) = [0,0] \Leftrightarrow X \cdot Y = [0,0].
\end{aligned}$$

2) $m_{\widehat{O},B}(X,Y) = B(w(X), w(Y))$
From Equations (2) and (16), it holds that:

$$\begin{aligned}
&[K_\alpha(\widehat{O}(X,Y)) - \alpha \cdot B(w(X),w(Y)), K_\alpha(\widehat{O}(X,Y)) + (1-\alpha)\cdot B(w(X),w(Y))] = [0,0]\\
&\Rightarrow -\alpha \cdot B(w(X),w(Y)) = (1-\alpha)\cdot B(w(X),w(Y)) \Rightarrow B(w(X),w(Y)) = 0\\
&\Rightarrow [K_\alpha(\widehat{O}(X,Y)), K_\alpha(\widehat{O}(X,Y))] = [0,0] \Rightarrow K_\alpha(\widehat{O}(X,Y)) = 0\\
&\Rightarrow \widehat{O}(X,Y) = [0,0] \Leftrightarrow X \cdot Y = [0,0].
\end{aligned}$$

($\Leftarrow$) Consider $X, Y \in L([0,1])$ such that $X \cdot Y = [0,0]$. Then, it is immediate that $\widehat{O}(X,Y) = [0,0]$ and $m_{\widehat{O},B}(X,Y) = 0$. Furthermore, from Equation (16):

$$IOw_B^\alpha(X,Y) = [K_\alpha([0,0]) - \alpha \cdot 0, K_\alpha([0,0]) + (1-\alpha)\cdot 0] = [0,0].$$

(IOw3) ($\Rightarrow$) Consider $X, Y \in L([0,1])$ such that $IOw_B^\alpha(X,Y) = [1,1]$. Then, we have the following cases:

1) $m_{\widehat{O},B}(X,Y) = w(\widehat{O}(X,Y))$
From Equations (2) and (16), it follows that:

$$\begin{aligned}
&[K_\alpha(\widehat{O}(X,Y)) - \alpha \cdot w(\widehat{O}(X,Y)), K_\alpha(\widehat{O}(X,Y)) + (1-\alpha)\cdot w(\widehat{O}(X,Y))] = [1,1]\\
&\Rightarrow [O(\underline{X},\underline{Y}) + \alpha \cdot w(\widehat{O}(X,Y)) - \alpha \cdot w(\widehat{O}(X,Y)),\\
&\quad O(\underline{X},\underline{Y}) + \alpha \cdot w(\widehat{O}(X,Y)) + w(\widehat{O}(X,Y)) - \alpha \cdot w(\widehat{O}(X,Y))] = [1,1]\\
&\Rightarrow [O(\underline{X},\underline{Y}), O(\underline{X},\underline{Y}) + w(\widehat{O}(X,Y))] = [1,1] \Rightarrow [O(\underline{X},\underline{Y}), O(\overline{X},\overline{Y})] = [1,1]\\
&\Rightarrow \widehat{O}(X,Y) = [1,1] \Leftrightarrow X \cdot Y = [1,1].
\end{aligned}$$

2) $m_{\widehat{O},B}(X,Y) = B(w(X), w(Y))$
From Equations (2) and (16), it holds that:

$$\begin{aligned}
&[K_\alpha(\widehat{O}(X,Y)) - \alpha \cdot B(w(X),w(Y)), K_\alpha(\widehat{O}(X,Y)) + (1-\alpha)\cdot B(w(X),w(Y))] = [1,1]\\
&\Rightarrow -\alpha \cdot B(w(X),w(Y)) = (1-\alpha)\cdot B(w(X),w(Y)) \Rightarrow B(w(X),w(Y)) = 0\\
&\Rightarrow [K_\alpha(\widehat{O}(X,Y)), K_\alpha(\widehat{O}(X,Y))] = [1,1] \Rightarrow K_\alpha(\widehat{O}(X,Y)) = 1\\
&\Rightarrow \widehat{O}(X,Y) = [1,1] \Leftrightarrow X \cdot Y = [1,1].
\end{aligned}$$

($\Leftarrow$) Consider $X, Y \in L([0,1])$ such that $X \cdot Y = [1,1]$. Then, it is immediate that $\widehat{O}(X,Y) = [1,1]$ and $m_{\widehat{O},B}(X,Y) = 0$. Furthermore, from Equation (16):

$$IOw_B^\alpha(X,Y) = [K_\alpha([1,1]) - \alpha \cdot 0, K_\alpha([1,1]) + (1-\alpha)\cdot 0] = [1,1].$$

(IOw4) Consider $X, Y, Z \in L([0,1])$ such that $X \leq_{Pr} Y$. Then:

$$IOw_B^\alpha(X,Z) = [K_\alpha(\widehat{O}(X,Z)) - \alpha \cdot m_{\widehat{O},B}(X,Z), K_\alpha(\widehat{O}(X,Z)) + (1-\alpha)\cdot m_{\widehat{O},B}(X,Z)], \tag{A.1}$$

and

$$IOw_B^\alpha(Y,Z) = [K_\alpha(\widehat{O}(Y,Z)) - \alpha \cdot m_{\widehat{O},B}(Y,Z), K_\alpha(\widehat{O}(Y,Z)) + (1-\alpha)\cdot m_{\widehat{O},B}(Y,Z)]. \tag{A.2}$$

Observe that $IOw_B^\alpha(X,Z)$ is obtained by constructing an interval around the value of $K_\alpha(\widehat{O}(X,Z))$, and that $\widehat{O}(X,Z)$ is an o-representable iv-overlap function with O as both its representatives. Then, from Equations (2) and (A.1), it follows that:

$$K_\alpha(IOw_B^\alpha(X,Z)) = K_\alpha(\widehat{O}(X,Z)), \tag{A.3}$$
$$K_\beta(IOw_B^\alpha(X,Z)) = K_\alpha(\widehat{O}(X,Z)) - \alpha \cdot m_{\widehat{O},B}(X,Z) + \beta \cdot m_{\widehat{O},B}(X,Z).$$

As $\beta < \alpha$, by Lemma 2.1, one can consider $\beta = 0$. Thus, we have that:

$$K_\beta(IOw_B^\alpha(X,Z)) = K_\alpha(\widehat{O}(X,Z)) - \alpha \cdot m_{\widehat{O},B}(X,Z). \tag{A.4}$$

Analogously, from Equations (2) and (A.2), it follows that:

$$K_\alpha(IOw_B^\alpha(Y,Z)) = K_\alpha(\widehat{O}(Y,Z)), \tag{A.5}$$
$$K_\beta(IOw_B^\alpha(Y,Z)) = K_\alpha(\widehat{O}(Y,Z)) - \alpha \cdot m_{\widehat{O},B}(Y,Z). \tag{A.6}$$

Now, we have the following possibilities regarding $m_{\widehat{O},B}(X,Z)$ and $m_{\widehat{O},B}(Y,Z)$ that affects the values of $IOw_B^\alpha(X,Z)$ and $IOw_B^\alpha(Y,Z)$, respectively:

1) $m_{\widehat{O},B}(X,Z) = w(\widehat{O}(X,Z))$ and $m_{\widehat{O},B}(Y,Z) = w(\widehat{O}(Y,Z))$
In this case, we have

$$IOw_B^\alpha(X,Z) = \widehat{O}(X,Z) \leq_{Pr} \widehat{O}(Y,Z) = IOw_B^\alpha(Y,Z),$$

meaning that $IOw_B^\alpha(X,Z) \leq_{\alpha,\beta} IOw_B^\alpha(Y,Z)$.

2) $m_{\widehat{O},B}(X,Z) = B(w(X), w(Z))$ and $m_{\widehat{O},B}(Y,Z) = B(w(Y), w(Z))$
It follows that

$$IOw_B^\alpha(X,Z) = [K_\alpha(\widehat{O}(X,Z)) - \alpha \cdot B(w(X),w(Z)), K_\alpha(\widehat{O}(X,Z)) + (1-\alpha)\cdot B(w(X),w(Z))],$$

and

$$IOw_B^\alpha(Y,Z) = [K_\alpha(\widehat{O}(Y,Z)) - \alpha \cdot B(w(Y),w(Z)), K_\alpha(\widehat{O}(Y,Z)) + (1-\alpha)\cdot B(w(Y),w(Z))].$$

Now, let us verify all the cases in which $X \leq_{Pr} Y$ holds:

a) $\underline{X} = \underline{Y}$ and $\overline{X} = \overline{Y}$:
We have that $X = Y$, meaning that

$$IOw_B^\alpha(X,Z) = IOw_B^\alpha(Y,Z) \Rightarrow IOw_B^\alpha(X,Z) \leq_{\alpha,\beta} IOw_B^\alpha(Y,Z).$$

b) $\underline{X} = \underline{Y}$ and $\overline{X} < \overline{Y}$:
When $\underline{Z} \neq 0$, from Lemma 4.1, it holds that $K_\alpha(\widehat{O}(X,Z)) < K_\alpha(\widehat{O}(Y,Z))$, since O is a strict overlap function and $\alpha \in (0,1]$. As $K_\alpha(IOw_B^\alpha(X,Z)) = K_\alpha(\widehat{O}(X,Z))$ and $K_\alpha(IOw_B^\alpha(Y,Z)) = K_\alpha(\widehat{O}(Y,Z))$, we have that

$$K_\alpha(IOw_B^\alpha(X,Z)) < K_\alpha(IOw_B^\alpha(Y,Z)) \Rightarrow IOw_B^\alpha(X,Z) \leq_{\alpha,\beta} IOw_B^\alpha(Y,Z).$$

If $\underline{Z} = 0$ and $\overline{Z} \neq 0$, by **(O2)**, one has that

$$\widehat{O}(X, Z) = [0, O(\overline{X}, \overline{Z})],$$

and

$$\widehat{O}(Y, Z) = [0, O(\overline{Y}, \overline{Z})].$$

Since $\overline{X} < \overline{Y}$ and O is strict, then

$$K_\alpha(IOw_B^\alpha(X, Z)) = K_\alpha(\widehat{O}(X, Z)) < K_\alpha(\widehat{O}(Y, Z)) = K_\alpha(IOw_B^\alpha(Y, Z))$$
$$\Rightarrow IOw_B^\alpha(X, Z) \leq_{\alpha,\beta} IOw_B^\alpha(Y, Z).$$

If $\underline{Z} = 0$ and $\overline{Z} = 0$, then

$$\widehat{O}(X, Z) = IOw_B^\alpha(X, Z) = [0, 0] = IOw_B^\alpha(Y, Z) = \widehat{O}(X, Z).$$

So, we have that $IOw_B^\alpha(X, Z) \leq_{\alpha,\beta} IOw_B^\alpha(Y, Z)$, for all $X, Y, Z \in L([0, 1])$, such that $\underline{X} = \underline{Y}$ and $\overline{X} < \overline{Y}$.

c) $\underline{X} < \underline{Y}$ and $\overline{X} = \overline{Y}$:
When $\underline{Z} \neq 0$ and $\alpha \neq 1$, from Lemma 4.1, we have that $K_\alpha(\widehat{O}(X, Z)) < K_\alpha(\widehat{O}(Y, Z))$. So, it holds that

$$K_\alpha(IOw_B^\alpha(X, Z)) < K_\alpha(IOw_B^\alpha(Y, Z)) \Rightarrow IOw_B^\alpha(X, Z) \leq_{\alpha,\beta} IOw_B^\alpha(Y, Z).$$

When taking $\underline{Z} \neq 0$ and $\alpha = 1$, we have that $K_\alpha(IOw_B^\alpha(X, Z)) = K_\alpha(IOw_B^\alpha(Y, Z))$. Moreover, from Equations (A.4) and (A.6):

$$K_\beta(IOw_B^1(X, Z)) = O(\overline{X}, \overline{Z}) - B(w(X), w(Z))$$

and

$$K_\beta(IOw_B^1(Y, Z)) = O(\overline{Y}, \overline{Z}) - B(w(Y), w(Z)).$$

As $\underline{X} < \underline{Y}$ and $\overline{X} = \overline{Y}$, we have that $w(Y) < w(X)$, and thus, $B(w(Y), w(Z)) \leq B(w(X), w(Z))$, as B is increasing. So,

$$K_\beta(IOw_B^1(X, Z)) = O(\overline{X}, \overline{Z}) - B(w(X), w(Z)) \leq O(\overline{Y}, \overline{Z}) - B(w(Y), w(Z)) = K_\beta(IOw_B^1(Y, Z)).$$

Then,

$$K_\alpha(IOw_B^\alpha(X, Z)) = K_\alpha(IOw_B^\alpha(Y, Z)) \text{ and } K_\beta(IOw_B^\alpha(X, Z)) \leq K_\beta(IOw_B^\alpha(Y, Z))$$
$$\Rightarrow IOw_B^\alpha(X, Z) \leq_{\alpha,\beta} IOw_B^\alpha(Y, Z).$$

If $\underline{Z} = 0$, by **(O2)**, one has that

$$\widehat{O}(X, Z) = [0, O(\overline{X}, \overline{Z})],$$

and

$$\widehat{O}(Y, Z) = [0, O(\overline{Y}, \overline{Z})].$$

Since $\overline{X} = \overline{Y}$, then $K_\alpha(IOw_B^\alpha(X, Z)) = K_\alpha(IOw_B^\alpha(Y, Z))$ and, analogous to the previous case when $\underline{Z} \neq 0$ and $\alpha = 1$, we have that

$$K_\alpha(IOw_B^\alpha(X, Z)) = K_\alpha(IOw_B^\alpha(Y, Z)) \text{ and } K_\beta(IOw_B^\alpha(X, Z)) \leq K_\beta(IOw_B^\alpha(Y, Z))$$
$$\Rightarrow IOw_B^\alpha(X, Z) \leq_{\alpha,\beta} IOw_B^\alpha(Y, Z).$$

So, we have that $IOw_B^\alpha(X, Z) \leq_{\alpha,\beta} IOw_B^\alpha(Y, Z)$, for all $X, Y, Z \in L([0, 1])$, such that $\underline{X} < \underline{Y}$ and $\overline{X} = \overline{Y}$.

d) $\underline{X} < \underline{Y}$ and $\overline{X} < \overline{Y}$:
When $\underline{Z} \neq 0$, from Lemma 4.1, it holds that $K_\alpha(\widehat{O}(X, Z)) < K_\alpha(\widehat{O}(Y, Z))$. So, we have that

$$K_\alpha(IOw_B^\alpha(X, Z)) < K_\alpha(IOw_B^\alpha(Y, Z)) \Rightarrow IOw_B^\alpha(X, Z) \leq_{\alpha,\beta} IOw_B^\alpha(Y, Z).$$

If $\underline{Z} = 0$ and $\overline{Z} \neq 0$, by **(O2)**, one has that

$$\widehat{O}(X, Z) = [0, O(\overline{X}, \overline{Z})],$$

and

$$\widehat{O}(Y, Z) = [0, O(\overline{Y}, \overline{Z})].$$

Since $\overline{X} < \overline{Y}$ and O is strict, then

$$K_\alpha(IOw_B^\alpha(X, Z)) = K_\alpha(\widehat{O}(X, Z)) < K_\alpha(\widehat{O}(Y, Z)) = K_\alpha(IOw_B^\alpha(Y, Z))$$
$$\Rightarrow IOw_B^\alpha(X, Z) \leq_{\alpha,\beta} IOw_B^\alpha(Y, Z).$$

If $\underline{Z} = 0$ and $\overline{Z} = 0$, then

$$\widehat{O}(X, Z) = IOw_B^\alpha(X, Z) = [0, 0] = IOw_B^\alpha(Y, Z) = \widehat{O}(X, Z).$$

So, we have that $IOw_B^\alpha(X, Z) \leq_{\alpha,\beta} IOw_B^\alpha(Y, Z)$, for all $X, Y, Z \in L([0, 1])$, such that $\underline{X} < \underline{Y}$ and $\overline{X} < \overline{Y}$. Thus, one can conclude that, for all $X, Y, Z \in L([0, 1])$, when $m_{\widehat{O},B}(X, Z) = B(w(X), w(Z))$ and $m_{\widehat{O},B}(Y, Z) = B(w(Y), w(Z))$, then

$$X \leq_{Pr} Y \Rightarrow IOw_B^\alpha(X, Z) \leq_{\alpha,\beta} IOw_B^\alpha(Y, Z).$$

3) $m_{\widehat{O},B}(X, Z) = w(\widehat{O}(X, Z))$ and $m_{\widehat{O},B}(Y, Z) = B(w(Y), w(Z))$
It follows that

$$IOw_B^\alpha(X, Z) = \widehat{O}(X, Z),$$

and

$$IOw_B^\alpha(Y, Z) = [K_\alpha(\widehat{O}(Y, Z)) - \alpha \cdot B(w(Y), w(Z)), K_\alpha(\widehat{O}(Y, Z)) + (1 - \alpha) \cdot B(w(Y), w(Z))].$$

Now, let us verify all the cases in which $X \leq_{Pr} Y$ holds:

a) $\underline{X} = \underline{Y}$ and $\overline{X} = \overline{Y}$:
We have that $X = Y$ and

$$IOw_B^\alpha(X, Z) = IOw_B^\alpha(Y, Z) \Rightarrow IOw_B^\alpha(X, Z) \leq_{\alpha,\beta} IOw_B^\alpha(Y, Z).$$

b) $\underline{X} = \underline{Y}$ and $\overline{X} < \overline{Y}$:
When $\underline{Z} \neq 0$, from Lemma 4.1, it holds that $K_\alpha(\widehat{O}(X, Z)) < K_\alpha(\widehat{O}(Y, Z))$, since O is a strict overlap function and $\alpha \in (0, 1]$. So, as $K_\alpha(IOw_B^\alpha(X, Z)) = K_\alpha(\widehat{O}(X, Z))$ and $K_\alpha(IOw_B^\alpha(Y, Z)) = K_\alpha(\widehat{O}(Y, Z))$, we have that

$$K_\alpha(IOw_B^\alpha(X, Z)) < K_\alpha(IOw_B^\alpha(Y, Z)) \Rightarrow IOw_B^\alpha(X, Z) \leq_{\alpha,\beta} IOw_B^\alpha(Y, Z).$$

If $\underline{Z} = 0$ and $\overline{Z} \neq 0$, by **(O2)**, one has that

$$\widehat{O}(X, Z) = [0, O(\overline{X}, \overline{Z})],$$

and

$$\widehat{O}(Y, Z) = [0, O(\overline{Y}, \overline{Z})].$$

Since $\overline{X} < \overline{Y}$ and O is strict, then

$$K_\alpha(IOw_B^\alpha(X, Z)) = K_\alpha(\widehat{O}(X, Z)) < K_\alpha(\widehat{O}(Y, Z)) = K_\alpha(IOw_B^\alpha(Y, Z))$$
$$\Rightarrow IOw_B^\alpha(X, Z) \leq_{\alpha,\beta} IOw_B^\alpha(Y, Z).$$

If $\underline{Z}=0$ and $\overline{Z}=0$, then

$$\widehat{O}(X,Z)=IOw_B^\alpha(X,Z)=[0,0]=IOw_B^\alpha(Y,Z)=\widehat{O}(X,Z).$$

So, we have that $IOw_B^\alpha(X,Z)\leq_{\alpha,\beta} IOw_B^\alpha(Y,Z)$, for all $X,Y,Z\in L([0,1])$, such that $\underline{X}=\underline{Y}$ and $\overline{X}<\overline{Y}$.

c) $\underline{X}<\underline{Y}$ and $\overline{X}=\overline{Y}$:

When $\underline{Z}\neq 0$ and $\alpha\neq 1$, from Lemma 4.1, we have that $K_\alpha(\widehat{O}(X,Z))<K_\alpha(\widehat{O}(Y,Z))$. So, it holds that

$$K_\alpha(IOw_B^\alpha(X,Z))<K_\alpha(IOw_B^\alpha(Y,Z))\Rightarrow IOw_B^\alpha(X,Z)\leq_{\alpha,\beta} IOw_B^\alpha(Y,Z).$$

If $\underline{Z}\neq 0$ and $\alpha=1$, we have that $K_\alpha(IOw_B^\alpha(X,Z))=K_\alpha(IOw_B^\alpha(Y,Z))$. Moreover, from Equations (A.4) and (A.6):

$$K_\beta(IOw_B^1(X,Z))=O(\overline{X},\overline{Z})-w(\widehat{O}(X,Z))$$

and

$$K_\beta(IOw_B^1(Y,Z))=O(\overline{Y},\overline{Z})-B(w(Y),w(Z)).$$

As $\underline{X}<\underline{Y}$ and $\overline{X}=\overline{Y}$, we have that

$$B(w(Y),w(Z))\leq w(\widehat{O}(Y,Z))=O(\overline{Y},\overline{Z})-O(\underline{Y},\underline{Z})\leq O(\overline{X},\overline{Z})-O(\underline{X},\underline{Z})=w(\widehat{O}(X,Z)),$$

as O is increasing. So,

$$K_\beta(IOw_B^1(X,Z))=O(\overline{X},\overline{Z})-w(\widehat{O}(X,Z))\leq O(\overline{Y},\overline{Z})-B(w(Y),w(Z))=K_\beta(IOw_B^1(Y,Z)).$$

Then,

$$K_\alpha(IOw_B^\alpha(X,Z))=K_\alpha(IOw_B^\alpha(Y,Z)) \text{ and } K_\beta(IOw_B^\alpha(X,Z))\leq K_\beta(IOw_B^\alpha(Y,Z))$$
$$\Rightarrow IOw_B^\alpha(X,Z)\leq_{\alpha,\beta} IOw_B^\alpha(Y,Z).$$

When $\underline{Z}=0$, by **(O2)** we have that

$$\widehat{O}(X,Z)=[0,O(\overline{X},\overline{Z})],$$

and

$$\widehat{O}(Y,Z)=[0,O(\overline{Y},\overline{Z})].$$

Since $\overline{X}=\overline{Y}$, then $K_\alpha(IOw_B^\alpha(X,Z))=K_\alpha(IOw_B^\alpha(Y,Z))$ and, analogous to the previous case when $\underline{Z}\neq 0$ and $\alpha=1$, we have that

$$K_\alpha(IOw_B^\alpha(X,Z))=K_\alpha(IOw_B^\alpha(Y,Z)) \text{ and } K_\beta(IOw_B^\alpha(X,Z))\leq K_\beta(IOw_B^\alpha(Y,Z))$$
$$\Rightarrow IOw_B^\alpha(X,Z)\leq_{\alpha,\beta} IOw_B^\alpha(Y,Z).$$

So, we have that $IOw_B^\alpha(X,Z)\leq_{\alpha,\beta} IOw_B^\alpha(Y,Z)$, for all $X,Y,Z\in L([0,1])$, such that $\underline{X}<\underline{Y}$ and $\overline{X}=\overline{Y}$

d) $\underline{X}<\underline{Y}$ and $\overline{X}<\overline{Y}$:

If $\underline{Z}\neq 0$, from Lemma 4.1, it holds that $K_\alpha(\widehat{O}(X,Z))<K_\alpha(\widehat{O}(Y,Z))$. So, we have that

$$K_\alpha(IOw_B^\alpha(X,Z))<K_\alpha(IOw_B^\alpha(Y,Z))\Rightarrow IOw_B^\alpha(X,Z)\leq_{\alpha,\beta} IOw_B^\alpha(Y,Z).$$

If $\underline{Z}=0$ and $\overline{Z}\neq 0$, by **(O2)**, one has that

$$\widehat{O}(X,Z)=[0,O(\overline{X},\overline{Z})],$$

and

$$\widehat{O}(Y,Z)=[0,O(\overline{Y},\overline{Z})].$$

Since $\overline{X}<\overline{Y}$ and O is strict, then

$$K_\alpha(IOw_B^\alpha(X,Z)) = K_\alpha(\widehat{O}(X,Z)) < K_\alpha(\widehat{O}(Y,Z)) = K_\alpha(IOw_B^\alpha(Y,Z))$$
$$\Rightarrow IOw_B^\alpha(X,Z) \leq_{\alpha,\beta} IOw_B^\alpha(Y,Z).$$

If $\underline{Z} = 0$ and $\overline{Z} = 0$, then

$$\widehat{O}(X,Z) = IOw_B^\alpha(X,Z) = [0,0] = IOw_B^\alpha(Y,Z) = \widehat{O}(X,Z).$$

So, we have that $IOw_B^\alpha(X,Z) \leq_{\alpha,\beta} IOw_B^\alpha(Y,Z)$, for all $X,Y,Z \in L([0,1])$, such that $\underline{X} < \underline{Y}$ and $\overline{X} < \overline{Y}$. Thus, one can conclude that, for all $X,Y,Z \in L([0,1])$, when $m_{\widehat{O},B}(X,Z) = w(\widehat{O}(X,Z))$ and $m_{\widehat{O},B}(Y,Z) = B(w(Y),w(Z))$, then

$$X \leq_{Pr} Y \Rightarrow IOw_B^\alpha(X,Z) \leq_{\alpha,\beta} IOw_B^\alpha(Y,Z).$$

4) $m_{\widehat{O},B}(X,Z) = B(w(X),w(Z))$ and $m_{\widehat{O},B}(Y,Z) = w(\widehat{O}(Y,Z))$
It follows that

$$IOw_B^\alpha(X,Z) = [K_\alpha(\widehat{O}(X,Z)) - \alpha \cdot B(w(X),w(Z)), K_\alpha(\widehat{O}(X,Z)) + (1-\alpha)\cdot B(w(X),w(Z))],$$

and

$$IOw_B^\alpha(Y,Z) = \widehat{O}(Y,Z).$$

Now, let us verify all the cases in which $X \leq_{Pr} Y$ holds:

a) $\underline{X} = \underline{Y}$ and $\overline{X} = \overline{Y}$:
We have that $X = Y$ and $IOw_B^\alpha(X,Z) = IOw_B^\alpha(Y,Z) \Rightarrow IOw_B^\alpha(X,Z) \leq_{\alpha,\beta} IOw_B^\alpha(Y,Z)$.

b) $\underline{X} = \underline{Y}$ and $\overline{X} < \overline{Y}$:
When $\underline{Z} \neq 0$, from Lemma 4.1, it holds that $K_\alpha(\widehat{O}(X,Z)) < K_\alpha(\widehat{O}(Y,Z))$, since O is a strict overlap function and $\alpha \in (0,1]$. So, as $K_\alpha(IOw_B^\alpha(X,Z)) = K_\alpha(\widehat{O}(X,Z))$ and $K_\alpha(IOw_B^\alpha(Y,Z)) = K_\alpha(\widehat{O}(Y,Z))$, we have that

$$K_\alpha(IOw_B^\alpha(X,Z)) < K_\alpha(IOw_B^\alpha(Y,Z)) \Rightarrow IOw_B^\alpha(X,Z) \leq_{\alpha,\beta} IOw_B^\alpha(Y,Z).$$

If $\underline{Z} = 0$ and $\overline{Z} \neq 0$, by **(O2)**, one has that

$$\widehat{O}(X,Z) = [0, O(\overline{X},\overline{Z})],$$

and

$$\widehat{O}(Y,Z) = [0, O(\overline{Y},\overline{Z})].$$

Since $\overline{X} < \overline{Y}$ and O is strict, then

$$K_\alpha(IOw_B^\alpha(X,Z)) = K_\alpha(\widehat{O}(X,Z)) < K_\alpha(\widehat{O}(Y,Z)) = K_\alpha(IOw_B^\alpha(Y,Z))$$
$$\Rightarrow IOw_B^\alpha(X,Z) \leq_{\alpha,\beta} IOw_B^\alpha(Y,Z).$$

If $\underline{Z} = 0$ and $\overline{Z} = 0$, then

$$\widehat{O}(X,Z) = IOw_B^\alpha(X,Z) = [0,0] = IOw_B^\alpha(Y,Z) = \widehat{O}(X,Z).$$

So, we have that $IOw_B^\alpha(X,Z) \leq_{\alpha,\beta} IOw_B^\alpha(Y,Z)$, for all $X,Y,Z \in L([0,1])$, such that $\underline{X} = \underline{Y}$ and $\overline{X} < \overline{Y}$.

c) $\underline{X} < \underline{Y}$ and $\overline{X} = \overline{Y}$:
When $\underline{Z} \neq 0$ and $\alpha \neq 1$, from Lemma 4.1, we have that $K_\alpha(\widehat{O}(X,Z)) < K_\alpha(\widehat{O}(Y,Z))$. So, it holds that

$$K_\alpha(IOw_B^\alpha(X,Z)) < K_\alpha(IOw_B^\alpha(Y,Z)) \Rightarrow IOw_B^\alpha(X,Z) \leq_{\alpha,\beta} IOw_B^\alpha(Y,Z).$$

If $\underline{Z} \neq 0$ and $\alpha = 1$, we have that $K_\alpha(IOw_B^\alpha(X,Z)) = K_\alpha(IOw_B^\alpha(Y,Z))$. Moreover, from Equations (A.4) and (A.6):

$$K_\beta(IOw_B^1(X,Z)) = O(\overline{X},\overline{Z}) - B(w(X),w(Z))$$

and

$$K_\beta(IOw_B^1(Y,Z)) = O(\overline{Y},\overline{Z}) - w(\widehat{O}(Y,Z)).$$

As $\underline{X} < \underline{Y}$ and $\overline{X} = \overline{Y}$, we have that $w(Y) < w(X)$, and thus,

$$w(\widehat{O}(Y,Z)) \leq B(w(Y),w(Z)) \leq B(w(X),w(Z)),$$

as B is increasing. So,

$$K_\beta(IOw_B^1(X,Z)) = O(\overline{X},\overline{Z}) - w(\widehat{O}(X,Z)) \leq O(\overline{Y},\overline{Z}) - B(w(Y),w(Z)) = K_\beta(IOw_B^1(Y,Z)).$$

Then,

$$K_\alpha(IOw_B^\alpha(X,Z)) = K_\alpha(IOw_B^\alpha(Y,Z)) \text{ and } K_\beta(IOw_B^\alpha(X,Z)) \leq K_\beta(IOw_B^\alpha(Y,Z))$$
$$\Rightarrow IOw_B^\alpha(X,Z) \leq_{\alpha,\beta} IOw_B^\alpha(Y,Z).$$

So, we have that $IOw_B^\alpha(X,Z) \leq_{\alpha,\beta} IOw_B^\alpha(Y,Z)$, for all $X,Y,Z \in L([0,1])$, such that $\underline{X} < \underline{Y}$ and $\overline{X} = \overline{Y}$.

d) $\underline{X} < \underline{Y}$ and $\overline{X} < \overline{Y}$:
When $\underline{Z} \neq 0$, from Lemma 4.1, it holds that $K_\alpha(\widehat{O}(X,Z)) < K_\alpha(\widehat{O}(Y,Z))$. So, we have that

$$K_\alpha(IOw_B^\alpha(X,Z)) < K_\alpha(IOw_B^\alpha(Y,Z)) \Rightarrow IOw_B^\alpha(X,Z) \leq_{\alpha,\beta} IOw_B^\alpha(Y,Z).$$

If $\underline{Z} = 0$ and $\overline{Z} \neq 0$, by **(O2)**, one has that

$$\widehat{O}(X,Z) = [0, O(\overline{X},\overline{Z})],$$

and

$$\widehat{O}(Y,Z) = [0, O(\overline{Y},\overline{Z})].$$

Since $\overline{X} < \overline{Y}$ and O is strict, then

$$K_\alpha(IOw_B^\alpha(X,Z)) = K_\alpha(\widehat{O}(X,Z)) < K_\alpha(\widehat{O}(Y,Z)) = K_\alpha(IOw_B^\alpha(Y,Z))$$
$$\Rightarrow IOw_B^\alpha(X,Z) \leq_{\alpha,\beta} IOw_B^\alpha(Y,Z).$$

If $\underline{Z} = 0$ and $\overline{Z} = 0$, then

$$\widehat{O}(X,Z) = IOw_B^\alpha(X,Z) = [0,0] = IOw_B^\alpha(Y,Z) = \widehat{O}(X,Z).$$

So, we have that $IOw_B^\alpha(X,Z) \leq_{\alpha,\beta} IOw_B^\alpha(Y,Z)$, for all $X,Y,Z \in L([0,1])$, such that $\underline{X} < \underline{Y}$ and $\overline{X} < \overline{Y}$.
Thus, one can conclude that, for all $X,Y,Z \in L([0,1])$, when $m_{\widehat{O},B}(X,Z) = B(w(X),w(Z))$ and $m_{\widehat{O},B}(Y,Z) = w(\widehat{O}(Y,Z))$, then

$$X \leq_{Pr} Y \Rightarrow IOw_B^\alpha(X,Z) \leq_{\alpha,\beta} IOw_B^\alpha(Y,Z).$$

As verified for all possible scenarios, it holds that IOw_B^α is $(\leq_{Pr}, \leq_{\alpha,\beta})$-increasing, for all $\alpha,\beta \in [0,1]$ such that $\alpha \neq \beta$.

(IOw5)

$$\begin{aligned} w(IOw_B^\alpha(X,Y)) &= K_\alpha(\widehat{O}(X,Y)) + (1-\alpha)\cdot m_{\widehat{O},B}(X,Y) - (K_\alpha(\widehat{O}(X,Y)) - \alpha\cdot m_{\widehat{O},B}(X,Y)) \\ &= m_{\widehat{O},B}(X,Y) \\ &= \min\{w(\widehat{O}(X,Y)), B(w(X),w(Y))\} \\ &\leq B(w(X),w(Y)). \end{aligned}$$

Then, it holds that IOw_B^α is width-limited by B for all $\alpha \in [0,1]$. □

Appendix B. Proof of Theorem 4.2

Proof. Consider a commutative, increasing and conjunctive function $B : [0, 1]^2 \to [0, 1]$, a strict overlap function $O : [0, 1]^2 \to [0, 1]$ and let $\alpha \in (0, 1)$, $\beta \in [0, 1]$ such that $\alpha \neq \beta$. Observe that, for all $X, Y \in L([0, 1])$:

(i) $K_\alpha(IOw_B^\alpha(X, Y)) = O(K_\alpha(X), K_\alpha(Y))$;
(ii) $w(IOw_B^\alpha(X, Y)) = \theta = B(B(w(X), w(Y)), B(O(K_\alpha(X), K_\alpha(Y)), 1 - O(K_\alpha(X), K_\alpha(Y))))$.

So, it is clear that IOw_B^α is well defined. Now, let us verify if IOw_B^α respects conditions **(IOw1)-(IOw5)** from Definition 4.2.

(IOw1) Immediate, as O and B are commutative;

(IOw2) ($\Rightarrow$) Take $X, Y \in L([0, 1])$ and suppose that $IOw_B^\alpha(X, Y) = [0, 0]$. Then, by **(i)**, we have that

$$K_\alpha(IOw_B^\alpha(X, Y)) = K_\alpha([0, 0]) = 0 = O(K_\alpha(X), K_\alpha(Y)),$$

since $\alpha \in (0, 1)$. Thus, by condition **(O2)**, either $K_\alpha(X) = 0$ or $K_\alpha(Y) = 0$, and, therefore, $X \cdot Y = [0, 0]$;
($\Leftarrow$) Consider $X, Y \in L([0, 1])$ such that $X \cdot Y = [0, 0]$. So, $K_\alpha(X) \cdot K_\alpha(Y) = 0$, since $\alpha \in (0, 1)$. Then, by **(i)** and **(O2)**, one has that $K_\alpha(IOw_B^\alpha(X, Y)) = O(K_\alpha(X), K_\alpha(Y)) = 0$, meaning that $IOw_B^\alpha(X, Y) = [0, 0]$;

(IOw3) ($\Rightarrow$) Take $X, Y \in L([0, 1])$ such that $IOw_B^\alpha(X, Y) = [1, 1]$. Then, by **(i)**, one has that

$$K_\alpha(IOw_B^\alpha(X, Y)) = K_\alpha([1, 1]) = 1 = O(K_\alpha(X), K_\alpha(Y)).$$

By **(O3)**, $K_\alpha(X) \cdot K_\alpha(Y) = 1$, since $\alpha \in (0, 1)$, meaning that $X \cdot Y = [1, 1]$;
($\Leftarrow$) Consider $X, Y \in L([0, 1])$ such that $X \cdot Y = [1, 1]$. So, $K_\alpha(X) \cdot K_\alpha(Y) = 1$, since $\alpha \in (0, 1)$. Then, by **(i)** and **(O3)**, one has that $K_\alpha(IOw_B^\alpha(X, Y)) = O(K_\alpha(X), K_\alpha(Y)) = 1$, meaning that $IOw_B^\alpha(X, Y) = [1, 1]$;

(IOw4) Consider $X, Y, Z \in L([0, 1])$ such that $X \leq_{\alpha,\beta} Y$ with $\alpha \in (0, 1)$, $\beta \in [0, 1]$, $\alpha \neq \beta$. By Lemma 2.1, it is sufficient to consider the cases $\beta = 0$ and $\beta = 1$. First, for $X <_{\alpha,\beta} Y$ and $\beta = 0$ we have the following possibilities:

1) $X <_{\alpha,0} Y$ and $K_\alpha(Z) = 0$. Then, $O(K_\alpha(X), K_\alpha(Z)) = 0 = O(K_\alpha(Y), K_\alpha(Z))$, and, therefore, since $\alpha \neq 0$, by **(i)** it holds that $IOw_B^\alpha(X, Z) = IOw_B^\alpha(Y, Z) = [0, 0]$;
2) $X <_{\alpha,0} Y$ and $K_\alpha(Z) > 0$. Here, we have the following possibilities:
 a) $K_\alpha(X) < K_\alpha(Y)$. Since O is strict, by **(O4)**, one has that $O(K_\alpha(X), K_\alpha(Z)) < O(K_\alpha(Y), K_\alpha(Z))$, and, thus, by **(i)** it follows that $IOw_B^\alpha(X, Z) <_{\alpha,0} IOw_B^\alpha(Y, Z)$;
 b) $K_\alpha(X) = K_\alpha(Y)$ and $K_{\beta=0}(X) < K_{\beta=0}(Y)$. Then, $\underline{X} < \underline{Y} \leq \overline{Y} < \overline{X}$, meaning that $w(X) > w(Y)$. So, by **(i)**,

$$K_\alpha(IOw_B^\alpha(X, Z)) = O(K_\alpha(X), K_\alpha(Z)) = O(K_\alpha(Y), K_\alpha(Z)) = K_\alpha(IOw_B^\alpha(Y, Z)),$$

and

$$\begin{aligned} K_{\beta=0}(IOw_B^\alpha(X, Z)) &= K_\alpha(IOw_B^\alpha(X, Z)) - \alpha \cdot w(IOw_B^\alpha(X, Z)) \text{ by Equation (8)} \\ &= K_\alpha(IOw_B^\alpha(X, Z)) - \alpha \cdot B(B(w(X), w(Z)), B(O(K_\alpha(X), K_\alpha(Z)), 1 - O(K_\alpha(X), K_\alpha(Z)))) \\ &\qquad \text{by (ii)} \\ &\leq K_\alpha(IOw_B^\alpha(Y, Z)) - \alpha \cdot B(B(w(Y), w(Z)), B(O(K_\alpha(Y), K_\alpha(Z)), 1 - O(K_\alpha(Y), K_\alpha(Z)))) \\ &= K_\alpha(IOw_B^\alpha(Y, Z)) - \alpha \cdot w(IOw_B^\alpha(Y, Z)) \\ &= K_{\beta=0}(IOw_B^\alpha(Y, Z)), \end{aligned}$$

as B is increasing. Therefore, $IOw_B^\alpha(X, Z) \leq_{\alpha,0} IOw_B^\alpha(Y, Z)$.

When $X = Y$, it is immediate that $IOw_B^\alpha(X, Z) = IOw_B^\alpha(Y, Z)$. Then, for $\beta = 0$ it holds that

$$IOw_B^\alpha(X, Z) \leq_{\alpha,0} IOw_B^\alpha(Y, Z).$$

The proof for $\beta = 1$ can be obtained analogously.

(IOw5) By **(ii)**, since B is conjunctive, it holds that

$$w(IOw_B^\alpha(X, Y)) = \theta$$
$$= B(B(w(Y), w(Z)), B(O(K_\alpha(Y), K_\alpha(Z)), 1 - O(K_\alpha(Y) \leq B(w(X), w(Y)).$$

Then, it holds that IOw_B^α is width-limited by B for all $\alpha \in (0, 1)$. □

Appendix C. Proof of Theorem 4.4

Proof. Consider a commutative aggregation function $B : [0, 1]^2 \to [0, 1]$, a strict overlap function $O : [0, 1]^2 \to [0, 1]$ and let $\alpha \in (0, 1)$ and $\beta \in [0, 1]$ such that $\alpha \neq \beta$. Observe that it is immediate that IOw_B^α is well defined. In fact, considering that $IOw_B^\alpha(X, Y) = R$, one has that $w(R) = m_{IF^\alpha_{O,B}, B}(X, Y)$ which, by Definition 4.3, is uniquely defined for the pair $(IF^\alpha_{O,B}, B)$. As $K_\alpha(R) = O(K_\alpha(X), K_\alpha(Y))$, then, it follows that $\underline{R} = K_\alpha(R) - \alpha \cdot w(R)$ and $\overline{R} = K_\alpha(R) + (1 - \alpha) \cdot w(R)$.

Now, let us verify if IOw_B^α respects conditions **(IOw1)-(IOw5)** from Definition 4.2.

(IOw1) Observe that, since O and B are commutative, then $IF^\alpha_{O,B}$ is commutative, as well as $m_{IF^\alpha_{O,B}, B}$. Then, it is immediate that IOw_B^α is commutative;

(IOw2) ($\Rightarrow$) Take $X, Y \in L([0, 1])$ and suppose that $IOw_B^\alpha(X, Y) = R = [0, 0]$. Then, by **(i)**, we have that

$$K_\alpha(R) = K_\alpha([0, 0]) = 0 = O(K_\alpha(X), K_\alpha(Y)),$$

since $\alpha \in (0, 1)$. Thus, by condition **(O2)**, either $K_\alpha(X) = 0$ or $K_\alpha(Y) = 0$, and, therefore, $X \cdot Y = [0, 0]$;

($\Leftarrow$) Consider $X, Y \in L([0, 1])$ such that $X \cdot Y = [0, 0]$. So, $K_\alpha(X) \cdot K_\alpha(Y) = 0$, since $\alpha \in (0, 1)$. Then, by **(i)** and **(O2)**, one has that

$$K_\alpha(R) = O(K_\alpha(X), K_\alpha(Y)) = 0,$$

meaning that $IOw_B^\alpha(X, Y) = R = [0, 0]$;

(IOw3) ($\Rightarrow$) Take $X, Y \in L([0, 1])$ such that $IOw_B^\alpha(X, Y) = R = [1, 1]$. Then, by **(i)**, one has that

$$K_\alpha(R) = K_\alpha([1, 1]) = 1 = O(K_\alpha(X), K_\alpha(Y)).$$

By **(O3)**, $K_\alpha(X) \cdot K_\alpha(Y) = 1$, since $\alpha \in (0, 1)$, meaning that $X \cdot Y = [1, 1]$;

($\Leftarrow$) Consider $X, Y \in L([0, 1])$ such that $X \cdot Y = [1, 1]$. So, $K_\alpha(X) \cdot K_\alpha(Y) = 1$, since $\alpha \in (0, 1)$. Then, by **(i)** and **(O3)**, one has that

$$K_\alpha(R) = O(K_\alpha(X), K_\alpha(Y)) = 1,$$

meaning that $IOw_B^\alpha(X, Y) = R = [1, 1]$;

(IOw4) Consider $X, Y, Z \in L([0, 1])$ such that $X \leq_{\alpha,\beta} Y$ with $\alpha \in (0, 1)$, $\beta \in [0, 1]$, such that $\alpha \neq \beta$. By Lemma 2.1, it is sufficient to consider the cases $\beta = 0$ and $\beta = 1$. First, for $X <_{\alpha,\beta} Y$ and $\beta = 0$ we have the following possibilities:

1) $X <_{\alpha,0} Y$ and $K_\alpha(Z) = 0$. Then, $O(K_\alpha(X), K_\alpha(Z)) = 0 = O(K_\alpha(Y), K_\alpha(Z))$, and, therefore, since $\alpha \neq 0$, by **(i)** it holds that $IOw_B^\alpha(X, Z) = [0, 0] = IOw_B^\alpha(Y, Z)$;

2) $X <_{\alpha,0} Y$ and $K_\alpha(Z) > 0$. Here, we have the following possibilities:

a) $K_\alpha(X) < K_\alpha(Y)$. Since O is strict, by **(O4)**, one has that $O(K_\alpha(X), K_\alpha(Z)) < O(K_\alpha(Y), K_\alpha(Z))$, and, thus, by **(i)** it follows that $IOw_B^\alpha(X,Z) <_{\alpha,0} IOw_B^\alpha(Y,Z)$;

b) $K_\alpha(X) = K_\alpha(Y)$ and $K_{\beta=0}(X) < K_{\beta=0}(Y)$. Then, $\underline{X} < \underline{Y} \le \overline{Y} < \overline{X}$, meaning that $w(X) > w(Y)$ and, therefore, by Definition 4.4, $\lambda_\alpha(X) > \lambda_\alpha(Y)$. So, by **(i)**,

$$K_\alpha(IOw_B^\alpha(X,Z)) = O(K_\alpha(X), K_\alpha(Z)) = O(K_\alpha(Y), K_\alpha(Z)) = K_\alpha(IOw_B^\alpha(Y,Z)),$$

and

$$\begin{aligned}
K_{\beta=0}(IOw_B^\alpha(X,Z)) &= K_\alpha(IOw_B^\alpha(X,Z)) - \alpha \cdot w(IOw_B^\alpha(X,Z)) \text{ by Equation (8)}\\
&= K_\alpha(IOw_B^\alpha(X,Z)) - \alpha \cdot m_{IF^\alpha_{O,B},B}(X,Z) \text{ by } \textbf{(ii)}\\
&= K_\alpha(IOw_B^\alpha(X,Z)) - \alpha \cdot \min\{B(w(X), w(Z)), B(\lambda_\alpha(X), \lambda_\alpha(Z)) \cdot d_\alpha(K_\alpha(IOw_B^\alpha(X,Z)))\}\\
&\qquad \text{by Definition 4.3}\\
&\le K_\alpha(IOw_B^\alpha(Y,Z)) - \alpha \cdot \min\{B(w(Y), w(Z)), B(\lambda_\alpha(Y), \lambda_\alpha(Z)) \cdot d_\alpha(K_\alpha(IOw_B^\alpha(Y,Z)))\}\\
&= K_\alpha(IOw_B^\alpha(Y,Z)) - \alpha \cdot m_{IF^\alpha_{O,B},B}(Y,Z) \text{ by Definition 4.3}\\
&= K_{\beta=0}(IOw_B^\alpha(Y,Z)),
\end{aligned}$$

as B is increasing. Therefore, $IOw_B^\alpha(X,Z) \le_{\alpha,0} IOw_B^\alpha(Y,Z)$.

When $X = Y$, it is immediate that $IOw_B^\alpha(X,Z) = IOw_B^\alpha(Y,Z)$. Then, for $\beta = 0$ it holds that

$$IOw_B^\alpha(X,Z) \le_{\alpha,0} IOw_B^\alpha(Y,Z).$$

The proof for $\beta = 1$ can be obtained analogously.

(IOw5) By **(ii)** and Definition 4.3, it holds that

$$\begin{aligned}
w(IOw_B^\alpha(X,Y)) &= m_{IF^\alpha_{O,B},B}(X,Y)\\
&= \min\{B(w(X), w(Y)), B(\lambda_\alpha(X), \lambda_\alpha(Y)) \cdot d_\alpha(K_\alpha(R))\}\\
&\le B(w(X), w(Y)).
\end{aligned}$$

Then, it holds that IOw_B^α is width-limited by B for all $\alpha \in (0,1)$. □

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5.1.4 A methodology for controlling the information quality in interval-valued fusion processes: theory and application

Related publication:

- T. Asmus, G. Dimuro, B. Bedregal, J. Sanz, R. Mesiar and H. Bustince, “A methodology for controlling the information quality in interval-valued fusion processes: theory and application.”, Knowledge Based Systems (submitted).
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A methodology for controlling the information quality in interval-valued fusion processes: theory and application

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Abstract

An important problem faced when dealing with imperfect information in fusion processes is that of the uncertainty regarding the values of the membership degrees to be employed in fuzzy modeling. In this scenario, one can apply interval-valued (iv) fuzzy sets, where the membership degrees are represented by intervals. A recurrent issue is the situation where the quality of information carried by the intervals, expressed by their widths, suffers degradation during the fusion process. So, the main objective of this paper is to develop a general framework to construct iv-fusion functions whose outputs conserve the information quality of the operated intervals. To achieve that, we first extend important concepts such as width-limiting functions and width-limited iv-functions to the n-dimensional context. Then, we present a characterization for any subclass of increasing fusion function by their set of properties, followed by the interval extension of such characterization to obtain classes of width-limited iv-fusion functions. We show that our methodology is general enough to retrieve several classes of iv-aggregation functions from the literature. Two approaches for constructing width-limited iv-fusion functions are also presented, which enables the application of different subclasses of width-limited iv-fusion functions in fusion/aggregation processes with imperfect information. Finally, we present a case study in a classification problem. Specifically, we use IVTURS, a state-of-the-art iv-fuzzy rule-based classification system, and a particular subclass of width-limited iv-fusion functions (n-dimensional width-limited iv-overlap functions), showing that the control of the information quality through width limitation significantly enhances the accuracy of the classifier.

Keywords: Fusion functions, interval-valued aggregation functions, interval information quality, n-dimensional interval-valued overlap functions, fuzzy rule-based classification systems

1. Introduction

Fusion functions are useful operators that combine several numerical values into a single representative one [1]. The most important class of fusion functions is that of aggregation functions [2] (or, more generally, pre-aggregation functions [3]), which are especially suitable to model fuzzy logic operations. For example, t-norms [4] and overlap functions [5] can be applied as fuzzy conjunction operators, while t-conorms [4] and grouping functions [6] can be applied as fuzzy disjunction operators. For that reason, aggregation functions have been widely used in several theoretical and applied fields [2, 7]. In particular, we have worked with n-dimensional overlap functions [8], which constitute a subclass of aggregation functions that do not require associativity and have been successfully applied in the reasoning method of fuzzy rule-based classification systems (FRBCSs) [9, 10].

When facing problems with imperfect information [11, 12], there may be uncertainty regarding the values of the membership degrees or even in the definition of the membership functions to be used in a fuzzy modeling [13, 14]. A viable and popular solution is the adoption of interval-valued fuzzy sets (IVFSs) [15, 16], where the membership degrees are represented by intervals. In this context, the width of the assigned intervals are intrinsically related with the uncertainty/ignorance with respect to the modeling of the fuzzy sets [17, 18]. IVFSs have been successfully applied in many different fields, such as image processing [19], game theory [20], multicriteria decision making [21], pest control [22], irrigation systems [23] and collaborative clustering [24]. The modeling of linguistic labels via IVFSs in FRBCSs gave birth to interval-valued rule-based classification systems (IV-FRBCSs) [25, 26, 27], where the aggregation process is a key component for the success of the classifier.

To accomplish aggregation processes with interval data, dif-

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ferent aggregation functions had to be extended to the interval context [28]. Since then, several classes of iv-aggregation were introduced, such as interval-valued t-norms and t-conorms [18], iv-overlap and grouping functions [29, 30] and general iv-overlap and grouping functions [26, 31]. In most cases, a particular class of iv-aggregation function is defined by extending the definition of a given class of aggregation function based on the concept of best interval representation [17]. That is, the interval output of the iv-aggregation function is defined by the application of the original aggregation function to the endpoints of the input intervals. Besides being intuitive and theoretically sound, extending aggregation functions to the interval context through the best interval representation has other benefits: the computation is generally easy, as one only deals with the endpoints of the input intervals, and correctness is guaranteed, since the output interval contains the exact unknown aggregated value [32].

Nevertheless, there are some drawbacks when applying iv-aggregation functions defined in this manner in practical problems. First, monotonicity is usually evaluated with respect to the product order [33], which also considers only the endpoints of the intervals when comparing them. However, the product order is not a total order, meaning that one may have intervals that are not comparable, a hindrance that has to be avoided in problems such as decision making and classification [34]. To tackle this drawback, Bustince et al. [34] introduced the concept of admissible orders, that is, total orders that refine the product order, and that can be constructed by a pair of aggregation functions. Since then, many works using admissible orders have appeared in the literature [35, 36, 37]. In the context of aggregation of interval data, Bustince et al. [38] presented a construction method for iv-aggregation functions that are increasing with respect to a given admissible order. In the same context, Asmus et al. [27] introduced the concept of n-dimensional admissibly ordered iv-overlap functions, which are n-dimensional iv-overlap functions that are increasing with respect to an admissible order, showing good results when applied in IV-FRBCSs.

Another drawback in a practical sense is that, due to some applications constraints concerning the quality of the information [39, 40] required for the interval result, the interval output of iv-aggregation functions based on the best interval representation may be larger than a desirable threshold. In this case, the interval result is guaranteed to be correct, however, it may carry no meaningful information about the real value it is approximating.

In an initial study to address this problem, Bustince et al. [38] introduced the concept of width-preserving functions, that is, iv-functions that, under some conditions, can provide outputs with the same width of all the inputs. However, the concept of width preservation only takes into account the very specific case where all the interval inputs have the same width. Then, more recently, Asmus et al. [41] introduced the concept of interval width limitation, where the width of the output of a bivariate iv-function is limited by a function applied to the widths of its inputs. Nevertheless, such theoretical approach for conserving the interval information quality in fusion processes was not considered in any applied problem, which means that there is a challenge yet to be addressed in a practical sense.

Motivated by the discussion above, this paper brings a novel and general methodology to deal with the problem of guaranteeing the information quality by controlling the width of interval outputs that are generated when applying the so called iv-fusion functions (in particular, iv-aggregation functions). Then, the main theoretical objective of this paper is to provide a general framework for n-dimensional width-limited interval-valued (w-iv) fusion functions, which enables the definition and construction methods for different subclasses of w-iv-aggregation functions, capable of retrieving known definitions of iv-aggregation functions from the literature and suitable to be applied on different practical problems where the information quality has to be controlled. To accomplish this goal, we have the following specific objectives:

1. To extend the concepts of width-limited w-iv-fusion functions and width-limiting fusion functions to the n-dimensional context (Sect. 3);

2. To present a characterization of any class of increasing fusion function through a set of properties (Sect. 4);

3. To define classes of w-iv-fusion functions based on an increasing fusion function, the interval extension of its set of properties and a pair of partial orders (Sect, 4);

4. To present two general approaches to provide construction methods for w-iv-fusion functions, one based on representable interval functions and other on admissibly ordered interval functions, discussing examples (Sect. 5).

On the application side, we show the beneficial effects of this type of information quality control in classification problems (Sect. 6). Specifically, we apply the new framework in IVTURS, which is a state-of-the-art IV-FRBCS. For that, we develop a new interval-valued fuzzy reasoning method, in which the information quality is controlled by w-iv-fusion functions, in particular, n-dimensional w-iv-overlap functions. We analyze the effect of the interval width control on the performance of the classifier, since the construction methods allow one to determine the control level by means of a hyper-parameter. Finally, we conduct an experimental study where we compare the results of the original IVTURS classifier versus the best performing configurations of our new approach in order to clearly observe the obtained improvement, regardless of the chosen construction method to obtain w-iv-fusion functions. Additionally, Section 2 presents some necessary preliminary concepts, and the main conclusions are drawn in Section 7, which completes the organization of the paper.

2. Preliminaries

In this section, we recall some basic concepts on aggregation functions, interval mathematics and iv-aggregation functions.

2.1. Aggregation Functions

Denote $\vec{x} = (x_1, \ldots, x_n) \in [0,1]^n$. Any $F : [0,1]^n \to [0,1]$ is called a *fusion function* [1].

Definition 1. *[42] A function $N : [0,1] \to [0,1]$ is a fuzzy negation if, for all $x, y \in [0,1]$: **(N1)** $N(0) = 1$ and $N(1) = 0$; **(N2)** If $x \leq y$ then $N(y) \leq N(x)$. If (involutive property) **(N3)** $N(N(x)) = x$, then N is a strong fuzzy negation.*

Example 1. *The Zadeh negation given, for all $x \in [0,1]$, by $N_Z(x) = 1 - x$, is a strong fuzzy negation.*

Definition 2. *Let H be the set of annihilator elements of a fusion function $F : [0,1]^n \to [0,1]$. F is said to be a strict fusion function is if it is strictly increasing on $([0,1] - H)^n$.*

Definition 3. *[42] Given a strong fuzzy negation $N : [0,1] \to [0,1]$ and a fusion function $F : [0,1]^n \to [0,1]$, then the fusion function $F^N : [0,1]^n \to [0,1]$ defined, for all $\vec{x} \in [0,1]^n$, by $F^N(\vec{x}) = N(F(N(x_1), \ldots, N(x_n)))$, it the N-dual of F.*

When it is clear by the context, the N_Z-dual function (dual with respect to the Zadeh negation) of F will be just called dual of F, and will be denoted by F^d.

A particularly important class of fusion function is that of aggregation functions [2], defined as follows.

Definition 4. *[2] An aggregation function is any fusion function $A : [0,1]^n \to [0,1]$ respecting: **(A1)** A is increasing; **(A2)** $A(0, \ldots, 0) = 0$ and $A(1, \ldots, 1) = 1$.*

An aggregation function A that is strictly increasing in $([0,1] - H)^n$, with H being the set of annihilator elements of F, is said to be a *strict aggregation function*.

Definition 5. *[43] An aggregation function $A : [0,1]^n \to [0,1]$ is called ultramodular if, for all $\vec{x}, \vec{y}, \vec{\epsilon} \in [0,1]^n$, such that $\vec{y} + \vec{\epsilon} \in [0,1]^n$ and $\vec{x} \leq \vec{y}$: $A(\vec{x} + \vec{\epsilon}) - A(\vec{x}) \leq A(\vec{y} + \vec{\epsilon}) - A(\vec{y})$.*

Here we extend the concept of (a, b)-ultramodular binary function [41] for the n-dimensional context:

Definition 6. *Consider $\vec{a} \in [0,1]^n$. An aggregation function $A : [0,1]^n \to [0,1]$ is called $\vec{a}$-ultramodular if, for all $\vec{x}, \vec{\epsilon} \in [0,1]^n$ and $\vec{x} + \vec{\epsilon}, \vec{a} - \vec{\epsilon} \in [0,1]^n$, it holds that:*

$$A(\vec{x} + \vec{\epsilon}) - A(\vec{x}) \leq A(\vec{a}) - A(\vec{a} - \vec{\epsilon}). \tag{1}$$

When $\vec{a} = (1, \ldots, 1)$, from Eq. (1) and condition **(A2)** from Def. 4, we have that $A(\vec{x} + \vec{\epsilon}) - A(\vec{x}) \leq A^d(\vec{\epsilon})$, where A^d is the dual of A. In this case, A is said to be $\vec{1}$-ultramodular.

The next result is an extension to the n-dimensional context of Prop. 3.1 from the work of Asmus et al. [41]:

Proposition 1. *Let $A : [0,1]^n \to [0,1]$ be an ultramodular aggregation function. Then, A is an $(\vec{1})$-ultramodular aggregation function, but the converse may not hold.*

There are many classes of aggregation functions defined in the literature. Here we highlight some of them that are going to be of importance on this work.

Definition 7. *[10] A fusion function $On : [0,1]^n \to [0,1]$ is an n-dimensional overlap function if, for all $\vec{x} \in [0,1]^n$: **(On1)** On is symmetric; **(On2)** $On(\vec{x}) = 0 \Leftrightarrow \prod_{i=1}^n x_i = 0$; **(On3)** $On(\vec{x}) = 1 \Leftrightarrow \prod_{i=1}^n x_i = 1$; **(On4)** On is increasing; **(On5)** On is continuous.*

A 2-dimensional overlap function is just called *overlap function* [5].

Definition 8. *[8] A fusion function $Gn : [0,1]^n \to [0,1]$ is said to be an n-dimensional grouping function if, for all $\vec{x} \in [0,1]^n$: **(Gn1)** Gn is symmetric; **(Gn2)** $Gn(\vec{x}) = 0 \Leftrightarrow x_i = 0$ for all $i \in \{1, \ldots, n\}$; **(Gn3)** $Gn(\vec{x}) = 1 \Leftrightarrow$ there exists $i \in \{1, \ldots, n\}$ such that $x_i = 1$; **(Gn4)** Gn is increasing; **(Gn5)** Gn is continuous.*

By duality, one can obtain n-dimensional grouping functions from n-dimensional overlap functions, and vice-versa.

Example 2. ***a)** The arithmetic mean $AM : [0,1]^n \to [0,1]$, defined, for all $\vec{x} \in [0,1]^n$, by $AM(\vec{x}) = \frac{\sum_{i=1}^n x_i}{n}$, is an aggregation function that is strict (with $H = \emptyset$) and $\vec{1}$-ultramodular;*
***b)** The geometric mean: $GM : [0,1]^n \to [0,1]$, given, for all $\vec{x} \in [0,1]^n$, by*

$$GM(\vec{x}) = \sqrt{\prod_{i=1}^{n} x_i}, \tag{2}$$

is a strict (with $H = \{0\}$) n-dimensional overlap function;
***c)** $OnB : [0,1]^n \to [0,1]$, given, for all $\vec{x} \in [0,1]^n$, by*

$$OnB(\vec{x}) = \sqrt{\left(\prod_{i=1}^{n} x_i\right) \cdot (\min\{x_1, \ldots, x_n\})}, \tag{3}$$

is a strict (with $H = \{0\}$) n-dimensional overlap function;
***d)** $OnT : [0,1]^n \to [0,1]$, given, for all $\vec{x} \in [0,1]^n$, by*

$$OnT(\vec{x}) = \prod_{i=1}^{n} \frac{(2x_i - 1)^3 + 1}{2}, \tag{4}$$

is an n-dimensional overlap function that is also an $\vec{1}$-ultramodular aggregation function, but it is not an ultramodular aggregation function.
***e)** The functions $GnB, GnT : [0,1]^n \to [0,1]$ such that $GnB = OnB^d$ and $GnT = OnT^d$, are n-dimensional grouping functions.*

2.2. Interval Mathematics and Admissible Orders

Denote by $L([0,1])$ the set of closed subintervals of the unit interval $[0,1]$ and $\vec{X} = (X_1, \ldots, X_n) \in L([0,1])^n$. For any $X = [x_1, x_2] \in L([0,1])$, the left and right projections of X are denoted, respectively, by $\underline{X} = x_1$ and $\overline{X} = x_2$. The width of X is denoted $w(X)$, which is given by $w(X) = \overline{X} - \underline{X}$.

We call by *interval-valued (iv) fusion function* any interval-valued function $IF : L([0,1])^n \to L([0,1])$ that merges n intervals from $L([0,1])$ into a single interval in $L([0,1])$.

Definition 9. *[38] An iv-fusion function $IF : L([0,1])^n \to L([0,1])$ is called width-preserving (or w-preserving, for simplicity) if, for any $\vec{X} \in L([0,1])^n$ such that $w(X_1) = \ldots = w(X_n)$, it holds that $w(IF(\vec{X})) = w(X_1)$.*

An iv-fusion function $IF : L([0,1])^n \to L([0,1])$ is said to be increasing with respect to a partial order $\leq$ on $L([0,1])$ (or, simply, $\leq$-increasing) if, for all $\vec{X}, \vec{Y} \in L([0,1])^n$, the following condition holds:

$$X_i \leq Y_i \text{ for all } i \in \{1, \ldots, n\} \Rightarrow IF(\vec{X}) \leq IF(\vec{Y}).$$

Definition 10. *[41] Let $IF : L([0,1])^n \to L([0,1])$ be an iv-fusion function and $\leq_1$, $\leq_2$ be two partial order relations on $L([0,1])$. Then, IF is said to be $(\leq_1, \leq_2)$-increasing if the following condition holds, for all $\vec{X}, \vec{Y} \in L([0,1])^n$:*

$$X_i \leq_1 Y_i \text{ for all } i \in \{1, \ldots, n\} \Rightarrow IF(\vec{X}) \leq_2 IF(\vec{Y}).$$

When an iv-fusion function $IF : L([0,1])^n \to L([0,1])$ is $(\leq, \leq)$-increasing, we denote it simply as $\leq$-increasing, for any partial order relation $\leq$ on $L([0,1])$.

The *product order* [33], denoted by $\leq_{Pr}$, is a partial order relation, defined, for all $X, Y \in L([0,1])$, by:

$$X \leq_{Pr} Y \quad \Leftrightarrow \quad \underline{X} \leq \underline{Y} \wedge \overline{X} \leq \overline{Y}.$$

Let $f, g : [0,1]^n \to [0,1]$ be two fusion functions such that $f \leq g$. Then, the iv-fusion function $\widehat{f,g} : L([0,1])^n \to L([0,1])$ is given by: $\widehat{f,g}(\vec{X}) = [f(\underline{X_1}, \ldots, \underline{X_n}), g(\overline{X_1}, \ldots, \overline{X_n})]$.

Definition 11. *[18] Let $IF : L([0,1])^n \to L([0,1])$ be a $\leq_{Pr}$-increasing iv-fusion function. Then, IF is said to be representable if there exist increasing fusion functions $f, g : [0,1]^n \to [0,1]$ such that $f \leq g$ and $IF = \widehat{f,g}$.*

f and g are called the *representatives* of IF. When $IF = \widehat{f,f}$, we denote simply as $\widehat{f}$. In this case, IF is said to be *the best interval representation (BIR)* of f [18].

The next interval operations, defined for all $X, Y \in L([0,1])$, are used in this paper: [33, 44]

Sum: $X + Y = [\underline{X} + \underline{Y}, \overline{X} + \overline{Y}]$, with $\overline{X} + \overline{Y} \leq 1$;

Product: $X \cdot Y = [\underline{X} \cdot \underline{Y}, \overline{X} \cdot \overline{Y}]$;

Generalized Hukuhara Division: for $\underline{Y} \neq 0, X \leq_{Pr} Y$:

$$X \div_H Y = [\min\{\underline{X}/\underline{Y}, \overline{X}/\overline{Y}\}, \max\{\underline{X}/\underline{Y}, \overline{X}/\overline{Y}\}]. \quad (5)$$

Here, we recall the concept of admissible orders.

Definition 12. *[34] Let $(L([0,1]), \leq_{AD})$ be a partially ordered set. The order $\leq_{AD}$ is an admissible order if, for all $X, Y \in L([0,1])$: **(i)** $\leq_{AD}$ is a total order on $(L([0,1]), \leq_{AD})$; **(ii)** $X \leq_{Pr} Y \Rightarrow X \leq_{AD} Y$.*

Thus, an order $\leq_{AD}$ on $L([0,1])$ is said to be admissible if it is a total order that refines the product order $\leq_{Pr}$ [34]. Since every admissible order $\leq_{AD}$ refines $\leq_{Pr}$, it is immediate that every $\leq_{AD}$-increasing function is also $\leq_{Pr}$-increasing.

Example 3. *Here are some examples of admissible orders: **(i)** The lexicographical orders $\leq_{Lex1}$ and $\leq_{Lex2}$, corresponding, respectively, to the first and second coordinates, are given by:*

$$X \leq_{Lex1} Y \quad \Leftrightarrow \quad \underline{X} < \underline{Y} \vee (\underline{X} = \underline{Y} \wedge \overline{X} \leq \overline{Y});$$
$$X \leq_{Lex2} Y \quad \Leftrightarrow \quad \overline{X} < \overline{Y} \vee (\overline{X} = \overline{Y} \wedge \underline{X} \leq \underline{Y}).$$

***(ii)** The order of Xu and Yager $\leq_{XY}$ [45], given by:*

$$X \leq_{XY} Y \Leftrightarrow \underline{X} + \overline{X} < \underline{Y} + \overline{Y} \text{ or } (\underline{X} + \overline{X} = \underline{Y} + \overline{Y} \text{ and } \overline{X} - \underline{X} \leq \overline{Y} - \underline{Y}).$$

***(iii)** The order $\leq_{IQ}$ [27], given by:*

$$X \leq_{IQ} Y \Leftrightarrow \underline{X} + \overline{X} < \underline{Y} + \overline{Y} \text{ or } (\underline{X} + \overline{X} = \underline{Y} + \overline{Y} \text{ and } \overline{Y} - \underline{Y} \leq \overline{X} - \underline{X}). \quad (6)$$

Observe that the order $\leq_{IQ}$ is based on the order of Xu and Yager, but takes into consideration the information quality [39] when comparing the intervals.

Next, we recall the definition of the admissible order $\leq_{\alpha,\beta}$:

Definition 13. *[34] For $\alpha, \beta \in [0,1]$ such that $\alpha \neq \beta$, the relation $\leq_{\alpha,\beta}$ is defined, for all $X, Y \in L([0,1])$, by*

$$X \leq_{\alpha,\beta} Y \Leftrightarrow K_\alpha(\underline{X}, \overline{X}) < K_\alpha(\underline{Y}, \overline{Y}) \text{ or } (K_\alpha(\underline{X}, \overline{X}) = K_\alpha(\underline{Y}, \overline{Y}) \text{ and } K_\beta(\underline{X}, \overline{X}) \leq K_\beta(\underline{Y}, \overline{Y})),$$

where $K_\alpha, K_\beta : [0,1]^2 \to [0,1]$ are aggregation functions defined, for all $x, y \in [0,1]$, respectively, by

$$K_\alpha(x, y) = x + \alpha \cdot (y - x); \; K_\beta(x, y) = x + \beta \cdot (y - x). \quad (7)$$

Remark 1. *The order $\leq_{\alpha,\beta}$ can recover other known admissible orders, by an appropriate choice of α and β. For example, **i)** The lexicographical orders $\leq_{Lex1}$ and $\leq_{Lex2}$ are recovered, respectively, by $\leq_{0,1}$ and $\leq_{1,0}$; **ii)** The orders $\leq_{XY}$ and $\leq_{IQ}$ are recovered, respectively, by $\leq_{0.5,1}$ and $\leq_{0.5,0}$.*

Whenever we apply the mapping K_α on the endpoints of an interval $X \in [0,1]$, we denote $K_\alpha(\underline{X}, \overline{X})$ simply as $K_\alpha(X)$.

Lemma 1. *[34] For any $\alpha, \beta \in [0,1]$, $\alpha \neq \beta$, it holds that: (i) $\beta > \alpha \Rightarrow \leq_{\alpha,\beta} = \leq_{\alpha,1}$; (ii) $\beta < \alpha \Rightarrow \leq_{\alpha,\beta} = \leq_{\alpha,0}$.*

2.3. Interval-valued Fusion Functions

Definition 14. *[46] $IN : L([0,1]) \to L([0,1])$ is called an iv-fuzzy negation if it is $\leq_{Pr}$-decreasing, **(IN1)** $IN([1,1]) = [0,0]$ and **(IN2)** $IN([0,0]) = [1,1]$. If $IN(IN(X)) = X$, for all $X \in L([0,1])$, then IN is said to be involutive.*

Definition 15. *[47] $IR : L([0,1])^2 \to L([0,1])$ is an iv-restricted equivalence function (IV-REF) with respect to an iv-fuzzy negation IN, if, for all $X, Y, Z \in L([0,1])$: **(IR1)** IR is commutative; **(IR2)** $IR(X,Y) = [1,1] \Leftrightarrow X = Y$; **(IR3)** $IR(X,Y) = [0,0] \Leftrightarrow X = [0,0]$ and $Y = [1,1]$, or $X = [1,1]$ and $Y = [0,0]$; **(IR4)** $IR(X,Y) = IR(IN(X), IN(Y))$; **(IR5)** $X \leq_{Pr} Y \leq_{Pr} Z \Rightarrow IR(X,Y) \geq_{Pr} IR(X,Z)$, $IR(Y,Z) \geq_{Pr} IR(X,Z)$.*

Definition 16. *[26] An iv-fusion function $IA : L([0,1])^n \to L([0,1])$ is called an iv-aggregation function if: **(IA1)** IA is $\leq_{Pr}$-increasing; **(IA2)** $IA([0,0], \ldots, [0,0]) = [0,0]$ and $IA([1,1], \ldots, [1,1]) = [1,1]$.*

Definition 17. *[38] Consider $c \in [0,1]$ and $\alpha \in [0,1]$. Then, the maximal possible width of an interval $Z \in L([0,1])$ is denoted by $d_\alpha(c)$, such that $K_\alpha(Z) = c$. Also, define, for any $X \in L([0,1])$,*

$$\lambda_\alpha(X) = \frac{w(X)}{d_\alpha(K_\alpha(X))}, \tag{8}$$

where we set $\frac{0}{0} = 1$.

Proposition 2. *[38] For all $\alpha \in [0,1]$ and $X \in L([0,1])$, one has that*

$$d_\alpha(K_\alpha(X)) = \min\left\{\frac{K_\alpha(X)}{\alpha}, \frac{1-K_\alpha(X)}{1-\alpha}\right\}, \tag{9}$$

where we set $\frac{r}{0} = 1$, for all $r \in [0,1]$.

Theorem 1. *[38] Let $\alpha, \beta \in [0,1]$ be such that $\alpha \neq \beta$. Let $A_1, A_2 : [0,1]^n \to [0,1]$ be two aggregation functions where A_1 is strictly increasing. Then $IF^\alpha : L([0,1])^n \to L([0,1])$ defined, for all $\vec{X} \in L([0,1])^n$, by:*

$$IF^\alpha_{A1,A2}(\vec{X}) = R$$
$$\text{where}\begin{cases} K_\alpha(R) = A_1(K_\alpha(X_1), \ldots, K_\alpha(X_n)), \\ \lambda_\alpha(R) = A_2(\lambda_\alpha(X_1), \ldots, \lambda_\alpha(X_n)), \end{cases}$$

is an $\leq_{\alpha,\beta}$-increasing iv-aggregation function.

Corollary 1. *[41] Let $\alpha \in (0,1], \beta \in [0,1]$ be such that $\alpha \neq \beta$. Let $On : [0,1]^n \to [0,1]$ be a strict n-dimensional overlap function and A be an aggregation function. Then $IF^\alpha_{On,A} : L([0,1])^n \to L([0,1])$ defined, for all $\vec{X} \in L([0,1])^n$, by:*

$$IF^\alpha_{On,A}(\vec{X}) = R$$
$$\text{where}\begin{cases} K_\alpha(R) = On(K_\alpha(X_1), \ldots, K_\alpha(X_n)), \\ \lambda_\alpha(R) = A(\lambda_\alpha(X_1), \ldots, \lambda_\alpha(X_n)), \end{cases}$$

is an $\leq_{\alpha,\beta}$-increasing iv-aggregation function.

Definition 18. *[26] An iv-fusion function $IOn : L([0,1])^n \to L([0,1])$ is called an n-dimensional iv-overlap function if, for all $\vec{X} \in L([0,1])^n$: **(IOn1)** IOn is symmetric; **(IOn2)** $IOn(\vec{X}) = [0,0] \Leftrightarrow \prod_{i=1}^n X_i = [0,0]$; **(IOn3)** $IOn(\vec{X}) = [1,1] \Leftrightarrow \prod_{i=1}^n X_i = [1,1]$; **(IOn4)** IOn is $\leq_{Pr}$-increasing; **(IOn5)** IOn is Moore continuous [33].*

For $n = 2$, IOn is just called iv-overlap function [29, 30].

Definition 19. *[27] An iv-fusion function $AOn : L([0,1])^n \to L([0,1])$ is called an n-dimensional admissibly ordered iv-overlap function for an admissible order $\leq_{AD}$ ($\leq_{AD}$-overlap function) if it respects the conditions **(IOn1)**, **(IOn2)** and **(IOn3)** of Def. 18, and **(AOn4)** AOn is $\leq_{AD}$-increasing.*

Although the construction method presented in Corollary 1 is based on an n-dimensional overlap function, the constructed function is not necessarily an $\leq_{\alpha,\beta}$-overlap function. It is not trivial to obtain such type of function, so we present here a new result, which is an adaptation of Corollary 1 with that purpose, as $\leq_{\alpha,\beta}$-overlap functions are featured throughout our theoretical and practical developments:

Theorem 2. *Consider a strict n-dimensional overlap function $On : [0,1]^n \to [0,1]$, an increasing and symmetric fusion function $B : [0,1]^n \to [0,1]$ and $\alpha \in (0,1), \beta \in [0,1]$, such that, $\alpha \neq \beta$. $AOn^\alpha_B : L([0,1])^n \to L([0,1])$ defined by:*

$$AOn^\alpha_B(\vec{X}) = R$$
$$\text{where}\begin{cases} K_\alpha(R) = On(K_\alpha(X_1), \ldots, K_\alpha(X_n)), \\ \lambda_\alpha(R) = B(\lambda_\alpha(X_1), \ldots, \lambda_\alpha(X_n)), \end{cases}$$

for all $\vec{X} \in L([0,1])^n$, is an $\leq_{\alpha,\beta}$-overlap function.

Proof. See Appendix Appendix A. □

The following result is immediate from Def. 17 and Theorem 2.

Corollary 2. *Let $\alpha \in (0,1), \beta \in [0,1]$ be such that, $\alpha \neq \beta$. Let $On : [0,1]^n \to [0,1]$ be a strict n-dimensional overlap function, $B : [0,1]^n \to [0,1]$ be an increasing and symmetric fusion function and $AOn^\alpha_B : L([0,1])^n \to L([0,1])$ be an iv-aggregation function constructed as in Theorem 2. Then, for all $\vec{X} \in L([0,1])^n$, we have that*

$$w(AOn^\alpha_B(\vec{X})) = B(\lambda_\alpha(X_1), \ldots, \lambda_\alpha(X_n)) \cdot d_\alpha(K_\alpha(AOn^\alpha_B(\vec{X}))). \tag{10}$$

3. Width-Limited iv-Fusion Functions

Here, we extend for the n-dimensional context the main results on width-limitation presented by Asmus et al. [41], as these concepts are going to be thoroughly featured on our present theoretical developments and are also required in the practical application presented in Section 6.

Definition 20. *[41] Consider an iv-fusion function $IF : L([0,1])^n \to L([0,1])$ and $B : [0,1]^n \to [0,1]$. IF is said to be width-limited by B if $w(IF(\vec{X})) \leq B(w(X_1), \ldots, w(X_n))$, for all $\vec{X} \in L([0,1])^n$. B is called a width-limiting function of IF.*

Denote $\mathcal{F}=\{F{:}[0,1]^n \to [0,1] | F \text{ is a fusion function}\}$ and $\mathcal{IF}=\{IF{:}L([0,1])^n {\to} L([0,1]) | IF \text{ is an iv-fusion function}\}$. Next theorem shows how to obtain the least width-limiting function for a given iv-fusion function:

Theorem 3. *[41] The mapping $\mathfrak{L} : \mathcal{IF} \to \mathcal{F}$ defined, for all $IF \in \mathcal{IF}$ and $\vec{\epsilon} \in [0,1]^n$, by*

$$\mathfrak{L}(IF)(\vec{\epsilon}) = \sup_{\substack{u_1 \in [0, 1-\epsilon_1] \\ \ldots \\ u_n \in [0, 1-\epsilon_n]}} \{w(IF([u_1, u_1+\epsilon_1], \ldots, [u_n, u_n+\epsilon_n]))\}$$

provides the least width-limiting function $\mathfrak{L}(IF) : [0,1]^n \to [0,1]$ for IF.

Denote $\mathcal{A}=\{A{:}[0,1]^n {\to} [0,1] | A \text{ is an aggregation function}\}$,

$$\mathcal{IA} = \{IA : L([0,1])^n \to L([0,1]) \,|\, IA \text{ is the best interval representation of an aggregation function } A \in \mathcal{A}\}.$$

Then, an approach similar to that one considered in Theorem 3 can be used to obtain the least width-liming aggregation function for a given representable iv-aggregation function.

Theorem 4. *[41] The mapping $\mathfrak{L} : \mathcal{IA} \rightarrow \mathcal{F}$ defined, for all $IA \in \mathcal{IA}$ and $\vec{\epsilon} \in [0,1]^n$, by*

$$\mathfrak{L}(IA)(\vec{\epsilon})= \sup_{\substack{u_1 \in [0,1-\epsilon_1] \\ \ldots \\ u_n \in [0,1-\epsilon_n]}} \{w(IA([u_1, u_1+\epsilon_1], \ldots, [u_n, u_n+\epsilon_n]))\} \tag{11}$$

provides the least width-limiting function $\mathfrak{L}(IA) : [0,1]^n \rightarrow [0,1]$ for IA. Moreover, $\mathfrak{L}(IA)$ is an aggregation function.

Now, let us present a characterization for the least width-limiting function of the best interval representation (BIR) of an $\vec{1}$-ultramodular aggregation function, or the BIR of its dual:

Theorem 5. *[41] Let $A : [0,1]^n \rightarrow [0,1]$ be an aggregation function, $A^d : [0,1]^n \rightarrow [0,1]$ be the dual of A, $\mathfrak{L}(\widehat{A}), \mathfrak{L}(\widehat{A^d}) : [0,1]^n \rightarrow [0,1]$ be the least width-limiting functions for $\widehat{A}$ and $\widehat{A^d}$, respectively. Then, $\mathfrak{L}(\widehat{A}) = \mathfrak{L}(\widehat{A^d}) = A^d$ if and only if A is an $\vec{1}$-ultramodular aggregation function.*

In the context of Theorem 5, as $\widehat{A}$ and $\widehat{A^d}$ are representable iv-aggregation functions, then their least width-limiting function A^d is an aggregation function, as stated by Theorem 4. Observe that the function A does not need to be ultramodular.

Example 4. ***a)*** *Every iv-fusion function $IF : L([0,1])^n \rightarrow L([0,1])$ is width-limited by the function $B : [0,1]^n \rightarrow [0,1]$ given by $B(\vec{x}) = 1$, for all $\vec{x} \in [0,1]^n$.*
b) *The n-dimensional iv-overlap function IOnP, defined, for all $\vec{X} \in L([0,1])^n$, by $IOnP(\vec{X}) = \prod_{i=1}^n X_i$, is width-limited by the probabilistic sum, given by $B(\vec{x}) = 1 - \prod_{i=1}^n 1 - x_i$, for all $\vec{x} \in [0,1]^n$. This holds because the IOn_p is the BIR of the product, which is an ultramodular aggregation function, and the probabilistic sum is the dual of the product.*
c) *Similar to the last case, the BIR of the n-dimensional overlap function OnT, defined in Eq. (4), denoted by $\widehat{OnT}$, is width-limited by $B = OnT^d$, since OnT is an $\vec{1}$-ultramodular aggregation function.*

Remark 2. *Observe that a width-limited function (Def. 20) differs from a width-preserving function (Def. 9), since one can only guarantee the uncertainty control in width-preserving functions when all the inputs have the same width. On the other hand, some width-limited functions can guarantee the uncertainty control accordingly to some width-limiting function B and still not be considered width-preserving as by Def. 9. So, width-limitation is a more suitable (and flexible) concept to indicate the potential uncertainty on the outputs of a given interval-valued function than width-preservation.*

4. A Framework for Width-Limited iv-Fusion Functions

The goal of this section is to present a way to obtain different classes of aggregation functions that have their outputs' widths limited by some arbitrary width-limiting function. In the work by Asmus et al. [41], for instance, width-limited interval-valued overlap functions were introduced based on such notion. Here, we recall their definition:

Definition 21. *[41] Let $B : [0,1]^2 \rightarrow [0,1]$ be a commutative and increasing function and $\leq_1$, $\leq_2$ be two partial order relations on $L([0,1])$. Then, the mapping $IOw : L([0,1])^2 \rightarrow L([0,1])$ is said to be a width-limited interval-valued overlap function (w-iv-overlap function) with respect to the tuple ($\leq_1$, $\leq_2$, B), if the following conditions hold for all $X, Y \in L([0,1])$:* ***(IOw1)*** *IOw is commutative;* ***(IOw2)*** *$IOw(X,Y) = [0,0] \Leftrightarrow X \cdot Y = [0,0]$;* ***(IOw3)*** *$IOw(X,Y) = [1,1] \Leftrightarrow X \cdot Y = [1,1]$;* ***(IOw4)*** *IOw is $(\leq_1, \leq_2)$-increasing;* ***(IOw5)*** *IOw is width-limited by B.*

Notice that Def. 21 is quite similar to Def. 18 for $n = 2$. In fact, conditions **(IOn1)**, **(IOn2)** and **(IOn3)**, for $n = 2$, are the same as **(IOw1)**, **(IOw2)** and **(IOw3)**. That is, both classes of iv-fusion functions share the same properties except for (i) monotonicity, (ii) continuity and (iii) width-limitation. Furthermore, it is easy to observe that those properties that they share, commutativity and boundaries conditions, are interval counterparts of properties **(On1)**, **(On2)** and **(On3)** of n-dimensional overlap functions (Def. 7). In a sense, one can say that n-dimensional overlap functions are the core functions in which both Def.s 18 and 21 derive from. So, one could obtain analogous definitions for width-limited interval-valued aggregation functions based on other core aggregation functions, if the properties of those core functions can be extended to the interval context.

For that, inspired by the approach of directional increasing fusion functions developed by Bustince et al. [48], first we present a characterization of any subclass $\mathcal{F}$ of increasing n-dimensional fusion functions through a set of properties $P_{\mathcal{F}}$ such that: (i) includes boundary conditions for any $F \in \mathcal{F}$ and (ii) possibly includes some other constraints not related to the monotonicity. Such subclass of fusion functions is given by:

$$\mathcal{F} = \{F : [0,1]^n \rightarrow [0,1] \mid F \text{ is increasing and satisfies all the properties in } P_{\mathcal{F}}\}. \tag{12}$$

Remark 3. *In this paper, we work with the usual definition of monotonicity, however, our characterization is not restricted by this specific definition, meaning that other kinds of monotonicity could be considered, such as weak monotonicity [49], directional monotonicity [50], ordered directionally monotonicity [51] or strengthened ordered directionally monotonicity [52].*

Example 5. *Based on Eq. (12), the class of aggregation functions $\mathcal{A}$ is given by $\mathcal{A} = \{A : [0,1]^n \rightarrow [0,1] \mid A$ is increasing and satisfies all the properties in $P_{\mathcal{A}}\}$, where $P_{\mathcal{A}}=\{A(0,\ldots,0)= 0, A(1,\ldots,1) = 1\}$. Analogously, any subclass of aggregation functions can be denoted in this manner. For example, the class of n-dimensional overlap functions On can be defined by: $On = \{On : [0,1]^n \rightarrow [0,1] \mid On$ is increasing and satisfies all the properties in $P_{On}\}$, where $P_{On} = \{$**(On1)**, **(On2)**, **(On3)**, **(On5)**$\}$.*

Now, given the set $P_{\mathcal{F}}$ of properties of a fusion function $F \in \mathcal{F}$, denote by $IP_{\mathcal{F}}$ the set of interval extensions of the properties in $P_{\mathcal{F}}$. Usually, there are more than one way to extend a given property of a function to the interval context, so $IP_{\mathcal{F}}$ varies accordingly to how one extends such properties.

Finally, consider the function $B \in \mathcal{B}$, where a $\mathcal{B}$ is a subclass of increasing fusion functions (with its corresponding set of properties $P_{\mathcal{B}}$) and let $\leq_1, \leq_2$ be partial orders on $L([0,1])$. Then, denote the class of width-limited interval-valued fusion functions (w-iv-fusion functions) for the tuple $(\leq_1, \leq_2, B)$ by $\mathcal{IFW}^B_{\leq_1,\leq_2}$, which is then given by:

$$\begin{aligned}\mathcal{IF}w^B_{\leq_1,\leq_2} = \\ \{IF : L([0,1])^n \to L([0,1]) | IF \text{ is } (\leq_1, \leq_2)\text{-increasing}, \\ \text{width-limited by } B \text{ and satisfies the properties in } IP_{\mathcal{F}}\}.\end{aligned} \tag{13}$$

Example 6. *Consider the class of aggregation functions $\mathcal{A}$, with its respective set of properties $P_{\mathcal{A}}$ (as shown in Ex. 5). Also, consider an increasing fusion function $B : [0,1]^n \to [0,1]$ and let $\leq_1, \leq_2$ be partial orders on $L([0,1])$. Then, based on Eq. (13), $\mathcal{IA}w^B_{\leq_1,\leq_2}$ is the class of width-limited interval-valued aggregation functions (w-iv-aggregation functions) for the tuple $(\leq_1, \leq_2, B)$, given by:*

$$\begin{aligned}\mathcal{IA}w^B_{\leq_1,\leq_2} = \\ \{IA : L([0,1])^n \to L([0,1]) | IA \textit{ is } (\leq_1, \leq_2)\textit{-increasing}, \\ \textit{width-limited by } B \textit{ and satisfies the properties in } IP_{\mathcal{A}}\}\end{aligned} \tag{14}$$

*where $IP_{\mathcal{A}}$ is an interval extension of $P_{\mathcal{A}}$, given by $IP_{\mathcal{A}} = \{IA([0,0],\ldots,[0,0]) = [0,0], IA([1,1],\ldots,[1,1]) = [1,1]\}$. Observe that if $\leq_1 = \leq_2 = \leq_{Pr}$ and $B(\vec{x}) = 1$, for all $\vec{x} \in [0,1]^n$, then $\mathcal{IA}w^B_{\leq_1,\leq_2}$ is the class of iv-aggregation functions defined in Def. 16. Similarly, other subclasses of iv-aggregation functions can be retrieved by Eq. (13), depending on the set of properties $IP_{\mathcal{F}}$. For example, take $IP_{\mathcal{F}} = IP_{On}$, where IP_{On} is the interval extension of the set P_{On} (see Ex. 5): $IP_{On} = \{$**(IOn1)**, **(IOn2)**, **(IOn3)**, **(IOn5)**$\}$, and $B(\vec{x}) = 1$, for all $\vec{x} \in [0,1]^n$. Then, $\mathcal{IO}nw^B_{\leq_{Pr}}$, given by*

$$\begin{aligned}\mathcal{IO}nw^B_{\leq_{Pr}} = \\ \{IOn : L([0,1])^n \to L([0,1]) | IOn \textit{ is } \leq_{Pr}\textit{-increasing}, \\ \textit{width-limited by } B \textit{ and satisfies the properties in } IP_{On}\},\end{aligned} \tag{15}$$

is the class of n-dimensional iv-overlap functions (Def. 18).

Remark 4. *The representation of w-iv-fusion functions by Eq. (13) is general enough so that different iv-aggregation functions defined in the literature may be retrieved, such as interval-valued t-norms and t-conorms [18], general interval-valued overlap functions [26], general interval-valued grouping functions [31], among others, by restricting to the case where $\leq_1 = \leq_2 = \leq_{Pr}$ and $B(\vec{x}) = 1$, for all $\vec{x} \in [0,1]^n$. However, those functions clearly have no limitation regarding their output widths and may not be applicable in problems where admissible orders must be considered.*

Example 7. *Consider a function $B \in \mathcal{B}$, where $\mathcal{B}$ is the subclass of increasing fusion functions, such that $P_{\mathcal{B}} = \{simmetry\}$, and two partial orders $\leq_1, \leq_2$ on $L([0,1])$. Then, $\mathcal{IO}nw^B_{\leq_1,\leq_2}$ is the class of width-limited n-dimensional interval-valued overlap functions (w-iv-overlap functions) for the tuple $(\leq_1, \leq_2, B)$, given by:*

$$\begin{aligned}\mathcal{IO}nw^B_{\leq_1,\leq_2} = \{IOnw : L([0,1])^n \to L([0,1]) | \\ IOnw \textit{ is } (\leq_1, \leq_2)\textit{-increasing, width-limited by } B \\ \textit{and satisfies the properties in } IP_{On'}\},\end{aligned} \tag{16}$$

*where $IP_{On'} = \{$**(IOn1)**, **(IOn2)**, **(IOn3)**$\}$.*

Observe that when $n = 2$, $\mathcal{IO}nw^B_{\leq_1,\leq_2}$ is the class of w-iv-overlap functions as shown in Def. 21. In other case, when $\leq_1 = \leq_2 = \leq_{\alpha,\beta}$, with $\alpha, \beta \in [0,1]$, such that $\alpha \neq \beta$, and $B(\vec{x}) = 1$, for all $\vec{x} \in [0,1]^n$, then, also by Eq. (16), $\mathcal{IO}nw^B_{\leq_{\alpha,\beta}}$ is the class of n-dimensional admissibly ordered interval-valued overlap function as presented in Def. 19.

Then, one can see that classes of iv-fusion functions that may not be $\leq_{Pr}$-increasing can also be retrieved by Eq. (13).

Remark 5. *It is noteworthy that the continuity (**On5**) was not extended to the interval context, so was not considered in IP'_{On}. Neither Def. 19 nor Def. 21 has continuity as a condition, as its interval extension it is not fully developed in the context of admissible orders. This is not a drawback, since the continuity requirement was added to the definition of overlap functions just because it was introduced firstly to be applied in image processing [5]. Actually, it is well known that the continuity can be disregarded in several applications, which is the case, for example, when overlap functions are applied in classification problems [26, 27, 53, 54]. Nevertheless, if one is defining a class of width-limited $\leq_{Pr}$-increasing fusion functions, than the continuity can be extended to the interval-context by the Moore-continuity [33], as in Ex. 6, when defining the class of n-dimensional iv-overlap functions.*

Remark 6. *The associativity property was proved to be difficult to maintain in iv-fusion functions that have controlled width-limitation (see the construction methods in Sect. 5), so we do not include it when defining a set $IP_{\mathcal{F}}$. We point out that this is not a drawback of our approach, in the sense that we explain below. Observe that, for several applications (e.g., classification), it is necessary to consider n-dimensional inputs for the aggregation process. That is why the associativity property has been considered an important requirement for extending bi-variate aggregation functions to the n-dimensional context in a very direct way, which is the case, for example, of t-norms and t-conorms [42]. However, it is well known that possibly non-associative bi-variate functions (e.g., overlap/grouping functions) can be extended to the n-dimensional context in many ways (e.g., n-dimensional overlap/grouping functions and general overlap/grouping functions). Also, one can find in the literature several possibly non-associative aggregation functions which can be used as alternatives to t-norms/t-conorms, such as t-seminorms or semi-copulas [55], weak t-norms [56], pseudo-t-norms [57], semi-uninorms [58], MICA operators [59], and micanorms [60].*

Remark 7. *In general, there are no restrictions regarding the width-limiting function B, or its set of properties $P_{\mathcal{B}}$, when defining a class $\mathcal{IFW}^B_{\leq_1,\leq_2}$ by Eq. (13). But, to construct some examples of w-iv-fusion functions of the class $\mathcal{IFW}^B_{\leq_1,\leq_2}$ respecting the properties of $IP_{\mathcal{F}}$, it may be necessary that $P_{\mathcal{B}}$ shares some properties with $P_{\mathcal{F}}$. That is the reason for which we required that B to be symmetric in Ex. 7, a shared property with On. This relation between the core function F and* *the width-limiting function B becomes clear in the construction methods presented in Section 5.*

5. Construction Methods

With the concepts of width-limited functions and least width-limiting functions, by Theorem 3, one can expect the maximum amount of uncertainty on the outputs of a given iv-fusion function. However, in order to control such uncertainty to an arbitrary degree (given by a chosen width-limiting function B), one can apply the developed theory to obtain some construction methods for width-limited iv-fusion functions. This is the main goal of this section.

In the following, we present a key concept to be applied in the construction methods for w-iv-fusion functions:

Definition 22. *Consider a fusion function $B : [0,1]^n \to [0,1]$ and let $IF : L([0,1])^n \to L([0,1])$ be an iv-fusion function. Then, the function $m_{IF,B} : L([0,1])^n \to [0,1]$, defined for all $\vec{X} \in L([0,1])^n$ by:*

$$m_{IF,B}(\vec{X}) = \min\{w(IF(\vec{X})), B(w(X_1), \ldots, w(X_n))\},$$

is called the maximal width threshold for the pair (IF, B).

5.1. Construction Method Based on representable iv-fusion functions (CMR)

The main idea of the Construction Method based on Representable iv-fusion functions (CMR) is to reduce the output's width of a representable iv-fusion function when it surpasses the limit imposed by a width-limiting fusion function B. The outputs of the constructed function are given by the maximal threshold for the pair $(\widehat{F}, B)$, where $\widehat{F}$ is the best interval representation (BIR) of a strict fusion function F. The reduction of the output occurs in the direction of a point of the interval, accordingly to a chosen value of $\alpha \in [0,1]$. For example, if $\alpha = 0.5$, then the output is "narrowed" towards the medium point of the interval obtained through the BIR.

The formalization of this concept is presented in the following three theorems, each one with some specificity regarding the chosen strict fusion function F and its respective restriction on the choice of admissible order that is suitable for the construction method.

Theorem 6. *Consider an increasing fusion function $B : [0,1]^n \to [0,1]$, a strict fusion function $F : [0,1]^n \to [0,1]$ with $h = 0$ as its annihilator element, $\alpha \in (0,1]$ and $\beta \in [0,\alpha)$. Then, the iv-fusion function $IFw_B^\alpha : L([0,1])^n \to L([0,1])$ defined, for all $\vec{X} \in L([0,1])^n$, by*

$$\begin{aligned} IFw_B^\alpha(\vec{X}) = [&K_\alpha(\widehat{F}(\vec{X})) - \alpha \cdot m_{\widehat{F},B}(\vec{X}), \\ &K_\alpha(\widehat{F}(\vec{X})) + (1-\alpha) \cdot m_{\widehat{F},B}(\vec{X})], \end{aligned} \tag{17}$$

is a width-limited fusion function for the tuple $(\leq_{Pr}, \leq_{\alpha,\beta}, B)$.

Proof. See Appendix Appendix B. □

Theorem 7. *Consider an increasing fusion function $B : [0,1]^n \to [0,1]$, a strict fusion function $F : [0,1]^n \to [0,1]$ with $h = 1$ as its annihilator element $\alpha \in [0,1)$ and $\beta \in (\alpha,1]$. Then, the iv-fusion function $IFw_B^\alpha : L([0,1])^n \to L([0,1])$ defined, for all $\vec{X} \in L([0,1])^n$, by*

$$\begin{aligned} IFw_B^\alpha(\vec{X}) = [&K_\alpha(\widehat{F}(\vec{X})) - \alpha \cdot m_{\widehat{F},B}(\vec{X}), \\ &K_\alpha(\widehat{F}(\vec{X})) + (1-\alpha) \cdot m_{\widehat{F},B}(\vec{X})], \end{aligned}$$

is a width-limited fusion function for the tuple $(\leq_{Pr}, \leq_{\alpha,\beta}, B)$.

Proof. Analogous to the proof of Theorem 6. □

The following theorem follows from Theorems 6 and 7.

Theorem 8. *Consider an increasing fusion function $B : [0,1]^n \to [0,1]$, a strictly increasing fusion function $F : [0,1]^n \to [0,1]$ and $\alpha, \beta \in [0,1]$ with $\alpha \neq \beta$. Then, the iv-fusion function $IFw_B^\alpha : L([0,1])^n \to L([0,1])$ defined, for all $\vec{X} \in L([0,1])^n$, by*

$$\begin{aligned} IFw_B^\alpha(\vec{X}) = [&K_\alpha(\widehat{F}(\vec{X})) - \alpha \cdot m_{\widehat{F},B}(\vec{X}), \\ &K_\alpha(\widehat{F}(\vec{X})) + (1-\alpha) \cdot m_{\widehat{F},B}(\vec{X})], \end{aligned}$$

is a width-limited fusion function for the tuple $(\leq_{Pr}, \leq_{\alpha,\beta}, B)$.

One can impose certain conditions on the functions B and F, reflected on the sets $P_{\mathcal{B}}$ and $P_{\mathcal{F}}$, respectively, in order to obtain specific subclasses of fusion functions. In the following, consider the class of w-iv-aggregation functions $\mathcal{IA}w^B_{\leq_{Pr},\leq_{\alpha,\beta}}$ given by Eq. (14), in Ex. 6.

Theorem 9. *Consider an increasing function $B : [0,1]^n \to [0,1]$, a strict aggregation function $A : [0,1]^n \to [0,1]$ with $h = 0$ as its annihilator element, $\alpha \in (0,1]$ and $\beta \in [0,\alpha)$. Then, the iv-fusion function $IAw_B^\alpha : L([0,1])^n \to L([0,1])$ defined, for all $\vec{X} \in L([0,1])^n$, by*

$$\begin{aligned} IAw_B^\alpha(\vec{X}) = [&K_\alpha(\widehat{A}(\vec{X})) - \alpha \cdot m_{\widehat{A},B}(\vec{X}), \\ &K_\alpha(\widehat{A}(\vec{X})) + (1-\alpha) \cdot m_{\widehat{A},B}(\vec{X})], \end{aligned}$$

is a width-limited aggregation function for $(\leq_{Pr}, \leq_{\alpha,\beta}, B)$.

Proof. From the proof of Theorem 6, IAw_B^α is well defined, $(\leq_{Pr}, \leq_{\alpha,\beta})$-increasing and width-limited by B. We prove that: **(i)** $IAw_B^\alpha([0,0],\ldots,[0,0])=[0,0]$, **(ii)** $IAw_B^\alpha([1,1],\ldots,[1,1])=[1,1]$.
(i) From **(A2)**, one has that $\widehat{A}([0,0],\ldots,[0,0]) = [0,0]$. Then: $w(\widehat{A}([0,0],\ldots,[0,0])) = 0 = m_{\widehat{A},B}([0,0],\ldots,[0,0])$.

So, $IAw_B^\alpha([0,0],\ldots,[0,0]) = \widehat{A}([0,0],\ldots,[0,0]) = [0,0]$.
(ii) Also, from **(A2)**, one has that $\widehat{A}([1,1],\ldots,[1,1]) = [1,1]$. Then: $w(\widehat{A}([1,1],\ldots,[1,1])) = 0 = m_{\widehat{A},B}([1,1],\ldots,[1,1])$. So, $IAw_B^\alpha([1,1],\ldots,[1,1])=\widehat{A}([1,1],\ldots,[1,1])=[1,1]$. □

Theorem 10. *Consider an increasing function $B : [0,1]^n \to [0,1]$, a strict aggregation function $A : [0,1]^n \to [0,1]$ with $h = 1$ as its annihilator element, $\alpha \in [0,1)$ and $\beta \in (\alpha,1]$. Then, the iv-fusion function $IAw_B^\alpha : L([0,1])^n \to L([0,1])$ defined, for all $\vec{X} \in L([0,1])^n$, by*

$$IAw_B^\alpha(\vec{X}) = [K_\alpha(\widehat{A}(\vec{X})) - \alpha \cdot m_{\widehat{A},B}(\vec{X}),$$

$$K_\alpha(\widehat{A}(\vec{X})) + (1-\alpha)\cdot m_{\widehat{A},B}(\vec{X})],$$

is a width-limited aggregation function for $(\leq_{Pr}, \leq_{\alpha,\beta}, B)$.

Proof. Analogous to the proof of Theorem 9. □

The next theorem follows from Theorems 9 and 10:

Theorem 11. *Consider an increasing function* $B : [0,1]^n \to [0,1]$*, an aggregation function* $A : [0,1]^n \to [0,1]$ *that is strictly increasing on* $[0,1]^n$*, and* $\alpha, \beta \in [0,1]$ *with* $\alpha \neq \beta$*. Then, the iv-fusion function* $IAw_B^\alpha : L([0,1])^n \to L([0,1])$ *defined, for all* $\vec{X} \in L([0,1])^n$*, by*

$$IAw_B^\alpha(\vec{X}) = [K_\alpha(\widehat{A}(\vec{X})) - \alpha\cdot m_{\widehat{A},B}(\vec{X}),$$
$$K_\alpha(\widehat{A}(\vec{X})) + (1-\alpha)\cdot m_{\widehat{A},B}(\vec{X})],$$

is a w-iv-aggregation function for the tuple $(\leq_{Pr}, \leq_{\alpha,\beta}, B)$.

Example 8. *Consider* $B = \min$*,* $A = AM$ *(arithmetic mean),* $\alpha = 0.5$ *and* $\beta = 0$ *(admissible order* $\leq_{IQ}$*). Then, the iv-fusion function* $IAMw_{\min}^\alpha : L([0,1])^n \to L([0,1])$ *defined, for all* $\vec{X} \in L([0,1])^n$*, by*

$$IAMw_{\min}^\alpha(\vec{X}) = [K_\alpha(\widehat{AM}(\vec{X})) - \alpha\cdot m_{\widehat{AM},\min}(\vec{X}),$$
$$K_\alpha(\widehat{AM}(\vec{X})) + (1-\alpha)\cdot m_{\widehat{AM},\min}(\vec{X})],$$

is a w-iv-aggregation function for the tuple $(\leq_{Pr}, \leq_{IQ}, \min)$.

Remark 8. *It is noteworthy that, by Theorem 5,* $\widehat{AM}$ *(BIR of the arithmetic mean) is width-limited by AM, as AM is an* $\vec{1}$*-ultramodular aggregation function and* $AM = AM^d$*. However, if one must control the output's width by a width-limiting function B such that* $B \leq AM$ *(which is the case for* $B = \min$*), then the construction method shown in Theorem 11 may be employed, with the result presented in Ex. 8.*

Analogously, one can construct w-iv-fusion functions that are interval extensions of specific types of aggregation functions. For instance, consider the class of w-iv-overlap functions $\mathcal{IO}nw^B_{\leq_{Pr},\leq_{\alpha\beta}}$ given by Eq. (15), in Ex. 7.

Theorem 12. *Consider a symmetric and increasing function* $B : [0,1]^n \to [0,1]$*, a strict n-dimensional overlap function* $On : [0,1]^n \to [0,1]$*,* $\alpha \in (0,1)$ *and* $\beta \in [0,\alpha)$*. Then, the iv-fusion function* $IOnw_B^\alpha : L([0,1])^n \to L([0,1])$ *defined, for all* $\vec{X} \in L([0,1])^n$*, by*

$$IOnw_B^\alpha(\vec{X}) = [K_\alpha(\widehat{On}(\vec{X})) - \alpha\cdot m_{\widehat{On},B}(\vec{X}), \qquad (18)$$
$$K_\alpha(\widehat{On}(\vec{X})) + (1-\alpha)\cdot m_{\widehat{On},B}(\vec{X})],$$

is a width-limited overlap function for $(\leq_{Pr}, \leq_{\alpha,\beta}, B)$.

Proof. From the proof of Theorem 6, it is immediate that $IOnw_B^\alpha$ is well defined, $(\leq_{Pr}, \leq_{\alpha,\beta})$-increasing and width-limited by B. It remains to be proven that $IOnw_B^\alpha$ has the properties of the set $IP_{On'} = \{$**(IOn1)**, **(IOn2)** and **(IOn3)**$\}$:
(IOn1) It is immediate, since On and B are both symmetric.
(IOn2) $(\Rightarrow)$ Take $\vec{X} \in L([0,1])^n$, such that $IOnw_B^\alpha(\vec{X}) = [0,0]$. Then, we have the following cases:

1) $m_{\widehat{On},B}(\vec{X}) = w(\widehat{On}(\vec{X}))$:
From Eqs. (7) and (18), it follows that:

$$[K_\alpha(\widehat{On}(\vec{X})) - \alpha w(\widehat{On}(\vec{X})),$$
$$K_\alpha(\widehat{On}(\vec{X})) + (1-\alpha)w(\widehat{On}(\vec{X}))] = [0,0]$$
$$\Rightarrow [On(\underline{X_1},\ldots,\underline{X_n}) + \alpha w(\widehat{On}(\vec{X})) - \alpha w(\widehat{On}(\vec{X})),$$
$$On(\underline{X_1},\ldots,\underline{X_n}) + \alpha w(\widehat{On}(\vec{X})) + w(\widehat{On}(\vec{X})) -$$
$$\alpha w(\widehat{On}(\vec{X}))] = [0,0]$$
$$\Rightarrow [On(\underline{X_1},\ldots,\underline{X_n}),$$
$$On(\underline{X_1},\ldots,\underline{X_n}) + w(\widehat{On}(\vec{X}))] = [0,0]$$
$$\Rightarrow [On(\underline{X_1},\ldots,\underline{X_n}), On(\overline{X_1},\ldots,\overline{X_n})] = [0,0]$$
$$\Rightarrow \widehat{On}(\vec{X}) = [0,0] \Leftrightarrow \prod_{i=1}^n X_i = [0,0], \text{ by } \textbf{(On2)}.$$

2) $m_{\widehat{On},B}(\vec{X}) = B(w(X_1),\ldots,w(X_n))$:
From Eqs. (7) and (18), it holds that:

$$[K_\alpha(\widehat{On}(\vec{X})) - \alpha B(w(X_1),\ldots,w(X_n)), \qquad (19)$$
$$K_\alpha(\widehat{On}(\vec{X})) + (1-\alpha)B(w(X_1),\ldots,w(X_n))]=[0,0]$$
$$\Rightarrow -\alpha B(w(X_1),\ldots,w(X_n)) =$$
$$(1-\alpha)B(w(X_1),\ldots,w(X_n))$$
$$\Rightarrow B(w(X_1),\ldots,w(X_n)) = 0$$
$$\Rightarrow [K_\alpha(\widehat{On}(\vec{X})), K_\alpha(\widehat{On}(\vec{X}))] = [0,0], \text{ by Eq. (19)}$$
$$\Rightarrow K_\alpha(\widehat{On}(\vec{X})) = 0$$
$$\Rightarrow \widehat{On}(\vec{X}) = [0,0], \text{ since } \alpha \neq 0$$
$$\Leftrightarrow \prod_{i=1}^n X_i = [0,0], \text{ by } \textbf{(On2)}.$$

$(\Leftarrow)$ Consider $\vec{X} \in L([0,1])^n$, such that $\prod_{i=1}^n X_i = [0,0]$. Then, it is immediate that $\widehat{On}(\vec{X}) = [0,0]$ and $m_{\widehat{On},B}(\vec{X}) = 0$. Furthermore, from Eq. (18):

$$IOnw_B^\alpha(\vec{X}) = [K_\alpha([0,0]) - \alpha\cdot 0, K_\alpha([0,0]) + (1-\alpha)\cdot 0] = [0,0].$$

(IOn3) $(\Rightarrow)$ Take $\vec{X} \in L([0,1])^n$, such that $IOnw_B^\alpha(X,Y) = [1,1]$. We have the following cases:
1) $m_{\widehat{On},B}(\vec{X}) = w(\widehat{On}(\vec{X}))$
From Eqs. (7) and (18), it follows that:

$$[K_\alpha(\widehat{On}(\vec{X})) - \alpha w(\widehat{On}(\vec{X})),$$
$$K_\alpha(\widehat{On}(\vec{X})) + (1-\alpha)tw(\widehat{On}(\vec{X}))] = [1,1]$$
$$\Rightarrow [On(\underline{X_1},\ldots,\underline{X_n}) + \alpha w(\widehat{On}(\vec{X})) - \alpha w(\widehat{On}(\vec{X})),$$
$$O(\underline{X},\underline{Y}) + \alpha w(\widehat{O}(X,Y)) + w(\widehat{O}(X,Y))$$
$$-\alpha w(\widehat{On}(\vec{X}))] = [1,1]$$
$$\Rightarrow [On(\underline{X_1},\ldots,\underline{X_n}), On(\underline{X_1},\ldots,\underline{X_n}) + w(\widehat{On}(\vec{X}))]=[1,1]$$
$$\Rightarrow [On(\underline{X_1},\ldots,\underline{X_n}), On(\overline{X_1},\ldots,\overline{X_n})] = [1,1]$$
$$\Rightarrow \widehat{On}(\vec{X}) = [1,1] \Leftrightarrow \prod_{i=1}^n X_i = [1,1], \text{ by } \textbf{(On3)}.$$

2) $m_{\widehat{On},B}(\vec{X}) = B(w(X_1),\ldots,w(X_n))$

From Eqs. (7) and (18), it holds that:

$$
\begin{aligned}
&[K_\alpha(\widehat{On}(\vec{X})) - \alpha B(w(X_1),\ldots,w(X_n)), K_\alpha(\widehat{On}(\vec{X})) \qquad (20)\\
&\quad +(1-\alpha)B(w(X_1),\ldots,w(X_n))]=[1,1]\\
\Rightarrow\ & -\alpha B(w(X_1),\ldots,w(X_n)) =\\
&\quad (1-\alpha)B(w(X_1),\ldots,w(X_n))\\
\Rightarrow\ & B(w(X_1),\ldots,w(X_n)) = 0 \qquad (21)\\
\Rightarrow\ & [K_\alpha(\widehat{On}(\vec{X})), K_\alpha(\widehat{On}(\vec{X}))] = [1,1], \text{ by Eq. (20)}\\
\Rightarrow\ & K_\alpha(\widehat{On}(\vec{X})) = 1\\
\Rightarrow\ & \widehat{On}(\vec{X}) = [1,1], \text{ since } \alpha \neq 1\\
\Leftrightarrow\ & \prod_{i=1}^{n} X_i = [1,1], \text{ by } \textbf{(On3)}.
\end{aligned}
$$

($\Leftarrow$) Consider $\vec{X} \in L([0,1])^n$ such that $\prod_{i=1}^n X_i = [1,1]$. Then, it is immediate that $\widehat{On}(\vec{X}) = [1,1]$ and $m_{\widehat{On},B}(\vec{X}) = 0$. Furthermore, from Eq. (18): $IOnw^{\alpha}_{B}(\vec{X}) = [K_\alpha([1,1]) - \alpha \cdot 0, K_\alpha([1,1])(1-\alpha)\cdot 0] = [1,1]$. □

Remark 9. *In Theorem 12, we have that $\alpha \neq 1$, which is not necessary in Theorems 6 and 9. This is to ensure that $IOnw^{\alpha}_{B}$ respects condition **(IOn3)**. Furthermore, B must be symmetric for $IOnw^{\alpha}_{B}$ to respect condition **(IOn1)**. These restrictions on B and α may vary accordingly to the class of aggregation function on which the construction method is based.*

Here, we present an example of applying CMR based on the n-dimensional overlap function OnB, given in Eq. (3), which is a function that produces good results when applied in classification problems, as shown in Section 6.

Example 9. *Consider $B = \max$, $On = OnB$, given in Eq. (3), $\alpha = 0.5$ and $\beta = 0$ (admissible order $\leq_{IQ}$). Then, by Theorem 12, the iv-fusion function $IOnw^{\alpha}_{OnB,\max} : L([0,1])^n \to L([0,1])$ defined, for all $\vec{X} \in L([0,1])^n$, by*

$$
\begin{aligned}
&IOnw^{\alpha=0.5}_{OnB,\max}(\vec{X}) = \qquad (22)\\
&\quad [K_{\alpha=0.5}(\widehat{OnB}(\vec{X})) - 0.5 \cdot m_{\widehat{OnB},\max}(\vec{X}),\\
&\quad K_{\alpha=0.5}(\widehat{OnB}(\vec{X})) + 0.5 \cdot m_{\widehat{OnB},\max}(\vec{X})],
\end{aligned}
$$

is a w-iv-overlap function for the tuple $(\leq_{Pr}, \leq_{IQ}, \max)$.
a) *Let $n = 2$, $X_1 = [0.2, 0.8]$ and $X_2 = [0.5, 1]$. So, we have that $w(X_1) = 0.6$, $w(X_2) = 0.5$ and $\max\{w(X_1), w(X_2)\} = 0.6$. Also, it holds that:*

$$\widehat{OnB}(\vec{X}) = [OnB(0.2, 0.5), OnB(0.8, 1)] = [0.1414, 0.8].$$

Observe that

$$w(\widehat{OnB}(\vec{X})) = 0.6586 > 0.6 = \max\{w(X_1), w(X_2)\},$$

meaning that $\widehat{OnB}$ is not width-limited by max.
However, from Eq. (22), one has that:

$$
\begin{aligned}
&IOnw^{\alpha=0.5}_{OnB,\max}(\vec{X})\\
&= [K_{\alpha=0.5}(\widehat{OnB}(\vec{X})) - 0.5 \cdot m_{\widehat{OnB},\max}(\vec{X}),\\
&\quad K_{\alpha=0.5}(\widehat{OnB}(\vec{X})) + 0.5 \cdot m_{\widehat{OnB},\max}(\vec{X})]\\
&= [K_{\alpha=0.5}([0.1414, 0.8]) - 0.5 \cdot \min\{w(\widehat{OnB}(\vec{X})),\\
&\quad \max\{w(X_1), w(X_2)\}\}, K_{\alpha=0.5}([0.1414, 0.8])\\
&\quad +0.5 \cdot \min\{w(\widehat{OnB}(\vec{X})), \max\{w(X_1), w(X_2)\}\}]\\
&= [0.4707 - 0.5 \cdot \min\{0.6586, 0.6\}, 0.4707\\
&\quad +0.5 \cdot \min\{0.6586, 0.6\}] = [0.1707, 0.7707].
\end{aligned}
$$

So, $w(IOnw^{\alpha=0.5}_{OnB,\max}(\vec{X})) = 0.6 \leq \max\{w(X_1), w(X_2)\}$, which is expected from a function that is width-limited by max.

One can visualise the way that the method works by taking the interval $[0.1414, 0.8]$ (output of the BIR) and "narrowing" it in the direction of its K_α point. In this case, as $\alpha = 0.5$, its the midpoint (0.4707). This can be verified, since:

$$
\begin{aligned}
&K_{\alpha=0.5}(\widehat{OnB}(\vec{X})) = K_{\alpha=0.5}([0.1414, 0.8]) = 0.4707\\
&\quad = K_{\alpha=0.5}([0.1707, 0.7707]) = K_{\alpha=0.5}(IOnw^{\alpha=0.5}_{OnB,\max}(\vec{X})).
\end{aligned}
$$

If we consider $\alpha = 0.99$, then the narrowing of the interval $[0.1414, 0.8]$ would occur towards the value 0.7934, near its right endpoint. In this case, $IOnw^{\alpha=0.99}_{OnB,\max}(\vec{X}) = [0.1994, 0.7994]$.
b) *Take $X_1 = [0.6, 0.9]$ and $X_2 = [0.8, 0.8]$, then, $w(X_1) = 0.3$, $w(X_2) = 0$ and $\max\{w(X_1), w(X_2)\} = 0.3$. In this case,*

$$\widehat{OnB}(\vec{X}) = [OnB(0.6, 0.8), OnB(0.9, 0.8)] = [0.5367, 0.759].$$

So, $w(\widehat{OnB}(\vec{X})) = 0.2223 < 0.3 = \max\{w(X_1), w(X_2)\}$. Also, from Eq. (22):

$$
\begin{aligned}
&IOnw^{\alpha=0.5}_{OnB,\max}(\vec{X})\\
&= [K_{\alpha=0.5}(\widehat{OnB}(\vec{X})) - 0.5 \cdot m_{\widehat{OnB},\max}(\vec{X}),\\
&\quad K_{\alpha=0.5}(\widehat{OnB}(\vec{X})) + 0.5 \cdot m_{\widehat{OnB},\max}(\vec{X})]\\
&= [K_{\alpha=0.5}([0.5367, 0.759]) - 0.5 \cdot \min\{w(\widehat{OnB}),\\
&\quad \max\{w(X_1), w(X_2)\}\}, K_{\alpha=0.5}([0.5367, 0.759])\\
&\quad +0.5 \cdot \min\{w(\widehat{OnB}), \max\{w(X_1), w(X_2)\}\}]\\
&= [0.6479 - 0.5 \cdot 0.2223, 0.6479 + 0.5 \cdot 0.2223]\\
&= [0.5367, 0.759].
\end{aligned}
$$

Observe that, although $\widehat{OnB}$ is not width-limited by max*, in this case, the width of the output does not exceed the limit imposed by the chosen width-limiting function* (max)*. That is why, by applying the construction method as by Eq. (22), it follows that $IOnw^{\alpha=0.5}_{OnB,\max}(\vec{X}) = [0.5367, 0.759] = \widehat{OnB}(\vec{X})$.*

In Table 1, we show the results obtained for $IOnw^{\alpha}_{OnB,B}(\vec{X})$, given in Ex. 9, by different choices of α and width-limiting function B. In this table, it is possible to compare the results obtained by CMR with the ones given by the BIR. Also, one can observe that, in every case shown in Table 1, $IOnw^{\alpha}_{OnB,B}(\vec{X}) \subseteq \widehat{OnB}(\vec{X})$.

By extending n-dimensional grouping functions to the interval context in a similar manner as done in Ex. 7 with n-dimensional overlap functions, one can obtain the class of n-dimensional w-iv-grouping functions, denoted by $\mathcal{IG}nw^{B}_{\leq_{Pr},\leq_{\alpha,\beta}}$. As n-dimensional grouping functions are the dual of n-dimensional overlap functions, the following result is immediate from Theorems 10 and 12.

Table 1: Ex. of CMR, comparing with the BIR

	CMR	Best Interval Representation
$X_1 = [0.2, 0.8]$ $X_2 = [0.5, 1]$ $B = \max$ $\alpha = 0.5$	$IOnw^{0.5}_{OnB,\max}(\vec{X}) = [0.1707, 0.7707]$	$\widehat{OnB}(\vec{X}) = [0.1414, 0.8]$
$X_1 = [0.2, 0.8]$ $X_2 = [0.5, 1]$ $B = \max$ $\alpha = 0.99$	$IOnwp^{0.99}_{OnB,\max}(\vec{X}) = [0.1994, 0.7994]$	$\widehat{OnB}(\vec{X}) = [0.1414, 0.8]$
$X_1 = [0.2, 0.8]$ $X_2 = [0.5, 1]$ $B = \min$ $\alpha = 0.5$	$IOnw^{0.5}_{OnB,\min}(\vec{X}) = [0.2207, 0.7207]$	$\widehat{OnB}(\vec{X}) = [0.1414, 0.8]$
$X_1 = [0.2, 0.8]$ $X_2 = [0.5, 1]$ $B = \min$ $\alpha = 0.99$	$IOnwp^{0.99}_{OnB,\min}(\vec{X}) = [0.2984, 0.7984]$	$\widehat{OnB}(\vec{X}) = [0.1414, 0.8]$

Theorem 13. *Consider a symmetric and increasing function $B : [0,1]^n \to [0,1]$, a strict n-dimensional grouping function $Gn : [0,1]^n \to [0,1]$, $\alpha \in (0,1)$ and $\beta \in (\alpha, 1]$. Then, the iv-fusion function $IOnw^{\alpha}_B : L([0,1])^n \to L([0,1])$ defined, for all $\vec{X} \in L([0,1])^n$, by*

$$IOnw^{\alpha}_B(\vec{X}) = [K_\alpha(\widehat{On}(\vec{X})) - \alpha \cdot m_{\widehat{On},B}(\vec{X}), \quad K_\alpha(\widehat{On}(\vec{X})) + (1-\alpha) \cdot m_{\widehat{On},B}(\vec{X})], \tag{23}$$

is n-dimensional w-iv-grouping function for $(\leq_{Pr}, \leq_{\alpha,\beta}, B)$.

Example 10. *Consider $B = AM$ (arithmetic mean), $Gn = Gn_p$ (probabilistic sum), $\alpha = 0.5$ and $\beta = 1$ (admissible order $\leq_{XY}$). Then, the iv-fusion function $IGPw^{\alpha}_{\min} : L([0,1])^n \to L([0,1])$ defined, for all $\vec{X} \in L([0,1])^n$, by*

$$IGPw^{\alpha}_{AM}(\vec{X}) = [K_\alpha(\widehat{Gn_p}(\vec{X})) - \alpha m_{\widehat{Gn_p},AM}(\vec{X}), \quad K_\alpha(\widehat{Gn_p}(\vec{X})) + (1-\alpha) m_{\widehat{Gn_p},AM}(\vec{X})], \tag{24}$$

is a w-iv-grouping function for the tuple $(\leq_{Pr}, \leq_{XY}, AM)$.

Regardless of the core fusion function F employed on CMR, the following result holds:

Proposition 3. *Let $IFw^{\alpha}_B : L([0,1])^n \to L([0,1])$ be a w-iv-fusion function for the tuple $(\leq_{Pr}, \leq_{\alpha,\beta}, B)$ obtained through Theorem 6, Theorem 7 or Theorem 8, with the respective choices of $\alpha, \beta \in [0,1]$ such that $\alpha \neq \beta$. Then, for any $\vec{X} \in L([0,1])^n$ one has that $IFw^{\alpha}_B(\vec{X}) \subseteq \widehat{F}(\vec{X})$.*

Remark 10. *Prop. 3 ensures that any w-iv-fusion function obtained through the CMR never generates outputs outside of the interval output of the BIR of the base fusion function F.*

5.2. A Construction Method based on admissibly ordered iv-fusion functions (CMA)

This method follows a similar approach as CMR, by applying the maximal threshold to limit the outputs' widths of the constructed function around a K_α point. The main difference is that, instead of being based on representable iv-fusion functions, it is based on $\leq_{\alpha,\beta}$-increasing fusion functions.

Theorem 14. *Consider an increasing fusion function $B : [0,1]^n \to [0,1]$, a strict fusion function $F : [0,1]^n \to [0,1]$ with $h = 0$ as its annihilator element, $\alpha \in (0,1], \beta \in [0,1]$ such that $\alpha \neq \beta$, and an $\leq_{\alpha,\beta}$-increasing iv-fusion function $IF^\alpha : L([0,1])^n \to L([0,1])$, such that $K_\alpha(IF^\alpha)(\vec{X}) = F(K_\alpha(X_1), \ldots, K_\alpha(X_n))$, for all $\vec{X} \in L([0,1])^n$. Then, the iv-fusion function $IFw^{\alpha}_B : L([0,1])^n \to L([0,1])$ defined, for all $\vec{X} \in L([0,1])^n$, by*

$$IFw^{\alpha}_B(\vec{X}) = [K_\alpha(IF^\alpha(\vec{X})) - \alpha \cdot m_{IF^\alpha,B}(\vec{X}), \quad K_\alpha(IF^\alpha(\vec{X})) + (1-\alpha) \cdot m_{IF^\alpha,B}(\vec{X})],$$

is a width-limited fusion function for $(\leq_{\alpha,\beta}, \leq_{\alpha,\beta}, B)$.

Proof. See Appendix Appendix C. □

Theorem 15. *Consider an increasing fusion function $B : [0,1]^n \to [0,1]$, a strict fusion function $F : [0,1]^n \to [0,1]$ with $h = 1$ as its annihilator element, $\alpha \in [0,1), \beta \in [0,1]$ such that $\alpha \neq \beta$, and an $\leq_{\alpha,\beta}$-increasing iv-fusion function $IF^\alpha : L([0,1])^n \to L([0,1])$, such that $K_\alpha(IF^\alpha)(\vec{X}) = F(K_\alpha(X_1), \ldots, K_\alpha(X_n))$, for all $\vec{X} \in L([0,1])^n$. Then, the iv-fusion function $IFw^{\alpha}_B : L([0,1])^n \to L([0,1])$ defined, for all $\vec{X} \in L([0,1])^n$, by*

$$IFw^{\alpha}_B(\vec{X}) = [K_\alpha(IF^\alpha(\vec{X})) - \alpha \cdot m_{IF^\alpha,B}(\vec{X}), \quad K_\alpha(IF^\alpha(\vec{X})) + (1-\alpha) \cdot m_{IF^\alpha,B}(\vec{X})],$$

is a width-limited fusion function for $(\leq_{\alpha,\beta}, \leq_{\alpha,\beta}, B)$.

Proof. Analogous to the proof of Theorem 14. □

The next theorem follows from Theorems 14 and 15.

Theorem 16. *Consider an increasing fusion function $B : [0,1]^n \to [0,1]$, a strictly increasing fusion function $F : [0,1]^n \to [0,1]$, $\alpha, \beta \in [0,1]$ such that $\alpha \neq \beta$, and an $\leq_{\alpha,\beta}$-increasing iv-fusion function $IF^\alpha : L([0,1])^n \to L([0,1])$, such that $K_\alpha(IF^\alpha)(\vec{X}) = F(K_\alpha(X_1), \ldots, K_\alpha(X_n))$, for all $\vec{X} \in L([0,1])^n$. Then, the iv-fusion function $IFw^{\alpha}_B : L([0,1])^n \to L([0,1])$ defined, for all $\vec{X} \in L([0,1])^n$, by*

$$IFw^{\alpha}_B(\vec{X}) = [K_\alpha(IF^\alpha(\vec{X})) - \alpha \cdot m_{IF^\alpha,B}(\vec{X}), \quad K_\alpha(IF^\alpha(\vec{X})) + (1-\alpha) \cdot m_{IF^\alpha,B}(\vec{X})],$$

is a width-limited fusion function for $(\leq_{\alpha,\beta}, \leq_{\alpha,\beta}, B)$.

Similarly as done with CMR, one can obtain w-iv-aggregation functions as follows:

Theorem 17. *Consider an increasing fusion function $B : [0,1]^n \to [0,1]$, a strict aggregation function $A : [0,1]^n \to [0,1]$ with $h = 0$ as its annihilator element, $\alpha \in (0,1], \beta \in [0,1]$ such that $\alpha \neq \beta$, and an $\leq_{\alpha,\beta}$-increasing iv-aggregation function IA^α : $L([0,1])^n \to L([0,1])$, such that $K_\alpha(IA)(\vec{X}) = A(K_\alpha(X_1), \ldots, K_\alpha(X_n))$, for all $\vec{X} \in L([0,1])^n$. Then, the iv-fusion function IAw^{α}_B : $L([0,1])^n \to L([0,1])$ defined, for all $\vec{X} \in L([0,1])^n$, by*

$$IAw^{\alpha}_B(\vec{X}) = [K_\alpha(IA^\alpha(\vec{X})) - \alpha \cdot m_{IA^\alpha,B}(\vec{X}), \quad K_\alpha(IA^\alpha(\vec{X})) + (1-\alpha) \cdot m_{IA,B}(\vec{X})],$$

is a w-iv-aggregation function for the tuple $(\leq_{\alpha,\beta}, \leq_{\alpha,\beta}, B)$.

Proof. From the proof of Theorem 16, it is immediate that IAw_B^α is well defined, $\leq_{\alpha,\beta}$-increasing and width-limited by B. It remains to be proven that: **(i)** $IAw_B^\alpha([0,0],\ldots,[0,0]) = [0,0]$ and **(ii)** $IAw_B^\alpha([1,1],\ldots,[1,1]) = [1,1]$.
(i) By **(IA2)**, one has that $IA([0,0],\ldots,[0,0]) = [0,0]$. Then $w(IA([0,0],\ldots,[0,0])) = 0 = m_{IA^\alpha,B}([0,0],\ldots,[0,0])$. So, $IAw_B^\alpha([0,0],\ldots,[0,0]) = IA^\alpha([0,0],\ldots,[0,0]) = [0,0]$.
(ii) By **(IA2)**, $IA^\alpha([1,1],\ldots,[1,1]) = [1,1]$. So, we have that $w(IA^\alpha([1,1],\ldots,[1,1])) = 0 = m_{IA^\alpha,B}([1,1],\ldots,[1,1])$, and $IAw_B^\alpha([1,1],\ldots,[1,1]){=}IA^\alpha([1,1],\ldots,[1,1]) = [1,1]$. □

Theorem 18. *Consider an increasing fusion function $B : [0,1]^n \to [0,1]$, a strict aggregation function $A : [0,1]^n \to [0,1]$ with $h = 1$ as its annihilator element, $\alpha \in [0,1), \beta \in [0,1]$ such that $\alpha \neq \beta$, and an $\leq_{\alpha,\beta}$-increasing iv-aggregation function $IA^\alpha : L([0,1])^n \to L([0,1])$, such that*

$$K_\alpha(IA^\alpha)(\vec{X}) = A(K_\alpha(X_1),\ldots,K_\alpha(X_n)),$$

for all $\vec{X} \in L([0,1])^n$. Then, the iv-fusion function $IAw_B^\alpha : L([0,1])^n \to L([0,1])$ defined, for all $\vec{X} \in L([0,1])^n$, by

$$\begin{aligned} IAw_B^\alpha(\vec{X}) = [&K_\alpha(IA^\alpha(\vec{X})) - \alpha \cdot m_{IA^\alpha,B}(\vec{X}),\\ &K_\alpha(IA^\alpha(\vec{X})) + (1-\alpha) \cdot m_{IA^\alpha,B}(\vec{X})], \end{aligned}$$

is a w-iv-aggregation function for the tuple $(\leq_{\alpha,\beta}, \leq_{\alpha,\beta}, B)$.

Proof. Analogous to the proof of Theorem 17. □

The next theorem follows from Theorems 17 and 18:

Theorem 19. *Consider an increasing fusion function $B : [0,1]^n \to [0,1]$, a strictly increasing aggregation function $A : [0,1]^n \to [0,1]$, $\alpha,\beta \in [0,1]$ such that $\alpha \neq \beta$, and an $\leq_{\alpha,\beta}$-increasing iv-aggregation function $IA^\alpha : L([0,1])^n \to L([0,1])$, such that $K_\alpha(IA^\alpha)(\vec{X}) = A(K_\alpha(X_1),\ldots,K_\alpha(X_n))$, for all $\vec{X} \in L([0,1])^n$. Then, the iv-fusion function $IAw_B^\alpha : L([0,1])^n \to L([0,1])$ defined, for all $\vec{X} \in L([0,1])^n$, by*

$$\begin{aligned} IAw_B^\alpha(\vec{X}) = [&K_\alpha(IA^\alpha(\vec{X})) - \alpha \cdot m_{IA^\alpha,B}(\vec{X}),\\ &K_\alpha(IA^\alpha(\vec{X})) + (1-\alpha) \cdot m_{IA^\alpha,B}(\vec{X})], \end{aligned}$$

is a w-iv-aggregation function for the tuple $(\leq_{\alpha,\beta}, \leq_{\alpha,\beta}, B)$.

Width-limited functions based on a specific class of aggregation functions can also be constructed. In the following, we present the construction method for w-iv-overlap functions:

Theorem 20. *Consider an increasing and symmetric fusion function $B : [0,1]^n \to [0,1]$, a strict n-dimensional overlap function $On : [0,1]^n \to [0,1]$, $\alpha \in (0,1)$, $\beta \in [0,1]$ with $\alpha \neq \beta$, and an $\leq_{\alpha,\beta}$-overlap function $IOn^\alpha : L([0,1])^n \to L([0,1])$, such that $K_\alpha(IOn)(\vec{X}) = On(K_\alpha(X_1),\ldots,K_\alpha(X_n))$, for all $\vec{X} \in L([0,1])^n$. Then, the iv-fusion function $IOnw_B^\alpha : L([0,1])^n \to L([0,1])$ defined, for all $\vec{X} \in L([0,1])^n$, by*

$$\begin{aligned} IOnw_B^\alpha(\vec{X}) = [&K_\alpha(IOn^\alpha(\vec{X})) - \alpha \cdot m_{IOn^\alpha,B}(\vec{X}),\\ &K_\alpha(IOn^\alpha(\vec{X})) + (1-\alpha) \cdot m_{IOn^\alpha,B}(\vec{X})], \end{aligned}$$

is a w-iv-overlap function for the tuple $(\leq_{\alpha,\beta}, \leq_{\alpha,\beta}, B)$.

Proof. From the proof of Theorem 19, it is immediate that $IOnw_B^\alpha$ is well defined, $(\leq_{Pr}, \leq_{\alpha,\beta})$-increasing and width-limited by B. It remains to be proven that $IOnw_B^\alpha$ has the properties of the set $IP_{On'} = \{$**(IOn1)**, **(IOn2)**, **(IOn3)**$\}$:
(IOn1) It is immediate, since On and B are both symmetric.
(IOn2) ($\Rightarrow$) Take $\vec{X} \in L([0,1])^n$ and suppose that $IOnw_B^\alpha(\vec{X}) = [0,0]$. Then, we have that

$$K_\alpha(IOnw_B^\alpha(\vec{X})){=}K_\alpha([0,0]){=}0{=}On(K_\alpha(X_1),\ldots,K_\alpha(X_n)),$$

since $\alpha \in (0,1)$. Thus, by condition **(On2)**, $K_\alpha(X_i) = 0$ for some $i \in \{0,\ldots,n\}$, and, therefore, $\prod_{i=1}^n X_i = [0,0]$;

($\Leftarrow$) Consider $\vec{X} \in L([0,1])^n$ such that $\prod_{i=1}^n X_i = [0,0]$. So, $K_\alpha(X_i)\cdot\ldots\cdot K_\alpha(X_n) = 0$, since $\alpha \in (0,1)$. Then, by **(On2)**, one has that

$$K_\alpha(IOnw_B^\alpha(\vec{X})) = On(K_\alpha(X_1),\ldots,K_\alpha(X_n)) = 0,$$

meaning that $IOnw_B^\alpha(\vec{X}) = [0,0]$;

(IOn3) ($\Rightarrow$) Take $\vec{X} \in L([0,1])^n$ such that $IOnw_B^\alpha(\vec{X}) = [1,1]$. Then, one has that

$$K_\alpha(IOnw_B^\alpha(\vec{X})){=}K_\alpha([1,1]){=}1{=}On(K_\alpha(X_1),\ldots,K_\alpha(X_n)).$$

By **(On3)**, $K_\alpha(X_1)\cdot\ldots\cdot K_\alpha(X_n) = 1$, since $\alpha \in (0,1)$, meaning that $\prod_{i=1}^n X_i = [1,1]$;

($\Leftarrow$) Consider $\vec{X} \in L([0,1])^n$ such that $\prod_{i=1}^n X_i = [1,1]$. So, $K_\alpha(X_1)\cdot\ldots\cdot K_\alpha(X_n) = 1$, since $\alpha \in (0,1)$. Then, by **(On3)**, one has that

$$K_\alpha(IOnw_B^\alpha(\vec{X})){=}On(K_\alpha(X_1),\ldots,K_\alpha(X_n)){=}1,$$

meaning that $IOnw_B^\alpha(\vec{X}) = [1,1]$. □

Example 11. *Consider the $\leq_{IQ}$-overlap function $AOn_{OnB,\max}^\alpha$ given by Theorem 2 for $B = \max$, $On = OnB$ given by Eq. (3), $\alpha = 0.5$ and $\beta = 0$ (admissible order $\leq_{IQ}$). Then, the iv-fusion function $IOnw_{OnB,\max}^\alpha : L([0,1])^n \to L([0,1])$ defined, for all $\vec{X} \in L([0,1])^n$, by*

$$\begin{aligned} IOnw_{OnB,\max}^\alpha(\vec{X}) = &\qquad (25)\\ [&K_\alpha(AOn_{OnB,\max}^\alpha(\vec{X})) - \alpha m_{AOn_{OnB,\max}^\alpha,\max}(\vec{X}),\\ &K_\alpha(AOn_{OnB,\max}^\alpha(\vec{X})) + (1-\alpha) m_{AOn_{OnB,\max}^\alpha,\max}(\vec{X})], \end{aligned}$$

is a w-iv-overlap function for the tuple $(\leq_{IQ}, \leq_{IQ}, \max)$.

Let us consider the same cases as in Ex. 9, but now applying CMA. So, take $n = 2$, $X_1 = [0.2, 0.8]$ and $X_2 = [0.5, 1]$, meaning that $\max\{w(X_1), w(X_2)\} = 0.6$. Also, from Theorem 2, it holds that:

$$AOn_{OnB,\max}^\alpha(\vec{X}) = R \qquad (26)$$
$$\text{where} \begin{cases} K_\alpha(R) = OnB(K_\alpha(X_1), K_\alpha(X_2)), \\ \lambda_\alpha(R) = \max\{\lambda_\alpha(X_1), \lambda_\alpha(X_2)\}, \end{cases}$$

with $\lambda_\alpha(X_1)$ and $\lambda_\alpha(X_2)$ given by Def. 17. So, we have that:

$$\begin{aligned} &w(AOn_{OnB,\max}^\alpha(\vec{X}))\\ &= \lambda_\alpha(AOn_{OnB,\max}^\alpha) \cdot \min\left\{\frac{K_\alpha(AOn_{OnB,\max}^\alpha)}{\alpha},\right. \end{aligned}$$

$$
\begin{aligned}
& \qquad\qquad \left.\frac{1-K_\alpha(AOn^\alpha_{OnB,\max})}{1-\alpha}\right\} \text{ by Eq. (10)} \\
= \; & \max\{\lambda_\alpha([0.2,0.8]),\lambda_\alpha([0.5,1])\}\cdot \\
& \min\left\{\frac{OnB(K_\alpha([0.2,0.8]),K_\alpha([0.5,1]))}{0.5},\right. \\
& \qquad \left.\frac{1-OnB(K_\alpha([0.2,0.8]),K_\alpha([0.5,1]))}{1-0.5}\right\} \text{ by Eq. (26)} \\
= \; & \max\left\{\frac{w([0.2,0.8])}{d_\alpha(K_\alpha([0.2,0.8]))},\frac{w([0.5,1])}{d_\alpha(K_\alpha([0.5,1]))}\right\}\cdot \\
& \min\left\{\frac{OnB(0.5,0.75)}{0.5},\frac{1-OnB(0.5,0.75)}{0.5}\right\} \text{ by Eq. (8)} \\
= \; & \max\left\{\frac{0.6}{\min\left\{\frac{0.5}{0.5},\frac{0.5}{0.5}\right\}},\frac{0.5}{\min\left\{\frac{0.75}{0.5},\frac{0.25}{0.5}\right\}}\right\}\cdot \\
& \qquad\qquad \min\left\{\frac{0.433}{0.5},\frac{1-0.433}{0.5}\right\} \text{ by Eq. (9)} \\
= \; & \max\left\{\frac{0.6}{1},\frac{0.5}{0.5}\right\}\cdot\min\{0.866,1.134\} = 0.866.
\end{aligned}
$$

So, it follows that:

$$w(AOn^\alpha_{OnB,\max}(\vec{X})) = 0.866 > 0.6 = \max\{w(X_1), w(X_2)\},$$

meaning that $AOn^\alpha_{OnB,\max}$ *is not width-limited by* max. *By applying CMA, from Eq. (25), one has that:*

$$
\begin{aligned}
& IOnw^{\alpha=0.5}_{OnB,\max}(\vec{X}) \\
= \; & [K_{\alpha=0.5}(AOn^\alpha_{OnB,\max}(\vec{X})) - 0.5\cdot m_{AOn^\alpha_{OnB,\max},\max}(\vec{X}), \\
& K_{\alpha=0.5}(AOn^\alpha_{OnB,\max}(\vec{X})) + 0.5\cdot m_{AOn^\alpha_{OnB,\max},\max}(\vec{X})] \\
= \; & [OnB(0.5,0.75) - 0.5\cdot\min\{0.866,0.6\}, \\
& \qquad OnB(0.5,0.75) + 0.5\cdot\min\{0.866,0.6\}] \\
= \; & [0.433 - 0.5\cdot 0.6, 0.433 + 0.5\cdot 0.6] \\
= \; & [0.133, 0.733].
\end{aligned}
$$

So, $w(IOnw^{\alpha=0.5}_{OnB,\max}(\vec{X}) = 0.6 \le \max\{w(X_1), w(X_2)\}$, *which is expected from a function that is width-limited by* max.

For a similar reason as observed in Ex. 9, whenever one has that $w(AOn^\alpha_{OnB,\max}(\vec{X})) \le \max\{w(X_1), w(X_2)\}$, *for some* $\vec{X} \in L([0,1])^n$ *and* $\alpha \in (0,1]$, *then the output already respects the width control dictated by* max, *and, in this case,* $IOnw^\alpha_{OnB,\max}(\vec{X}) = AOn^\alpha_{OnB,\max}(\vec{X})$.

Some values obtained by both $IOnw^\alpha_{OnB,B}(\vec{X})$ and $AOn^\alpha_{OnB,B}(\vec{X})$ (from Ex. 11) can be seen on Table 2, by varying the value of α and the chosen width-limiting function B. One can observe that, in Table 2, every result obtained by CMA ($IOnw^\alpha_{OnB,B}(\vec{X})$) is contained on the interval given by the corresponding $\le_{\alpha,\beta}$-overlap function without width-limitation ($AOn^\alpha_{OnB,B}(\vec{X})$). Also, different from the BIR, $AOn^\alpha_{OnB,B}$ varies accordingly to the chosen α. Finally, it is noteworthy that, in both Ex. 11 and Table 2, we applied the same function B for both $IOnw^\alpha_{OnB,B}(\vec{X})$ and $AOn^\alpha_{OnB,B}(\vec{X})$, but this is not a requirement. We decided to keep both iv-functions based on the same B for simplicity.

At this point, it is clear that other classes of w-iv-aggregation functions can be obtained through CMA, by the appropriate choices of a strict aggregation function A, a width-limiting function B, α,β and an $\le_{\alpha,\beta}$-increasing iv-aggregation function IA^α.

Table 2: CMA, comparing with $\le_{\alpha,\beta}$-overlap functions

	CMA	$\le_{\alpha,\beta}$-overlap function
$X_1 = [0.2, 0.8]$ $X_2 = [0.5, 1]$ $B = \max$ $\alpha = 0.5$	$IOnw^{0.5}_{OnB,\max}(\vec{X}) = [0.13, 0.73]$	$AOn^{0.5}_{OnB,\max}(\vec{X}) = [0, 0.87]$
$X_1 = [0.2, 0.8]$ $X_2 = [0.5, 1]$ $B = \max$ $\alpha = 0.99$	$IOnw^{0.99}_{OnB,\max}(\vec{X}) = [0.19, 0.79]$	$AOn^{0.99}_{OnB,\max}(\vec{X}) = [0, 0.80]$
$X_1 = [0.2, 0.8]$ $X_2 = [0.5, 1]$ $B = \min$ $\alpha = 0.5$	$IOnw^{0.5}_{OnB,\min}(\vec{X}) = [0.18, 0.68]$	$AOn^{0.5}_{OnB,\min}(\vec{X}) = [0.17, 0.69]$
$X_1 = [0.2, 0.8]$ $X_2 = [0.5, 1]$ $B = \min$ $\alpha = 1$	$IOnwp^{0.99}_{OnB,\min}(\vec{X}) = [0.29, 0.79]$	$AOn^{0.99}_{OnB,\min}(\vec{X}) = [0.19, 0.80]$

5.3. Applications to practical problems

When applying either of the construction methods of width-limited iv-fusion functions in practical problems, the domain expert has to make some key choices, as explained bellow:

1. *The choice of fusion function F*: usually, when extending a fusion/aggregation process modelled by a fusion function F to the interval context, one can maintain the same fusion function F as the core of the construction method to be employed. For example, it was shown by Asmus et al. [27] that the n-dimensional overlap functions GM and OnB, defined in Eqs. (2) and (3), respectively, are well fitted to be employed in classification problems. Thus, it is natural that those functions are chosen to be the core of some w-iv-fusion functions to be applied in IV-FRBCSs with width-limitation (see Sect. 6);

2. *The choice of* α: It is entirely determined by the admissible order $\le_{\alpha,\beta}$ that is indicated for the application. The choice of the interval order depends on how the intervals are obtained or interpreted [34, 36, 38]. To keep the same example, Asmus et al. [27] showed that the admissible order $\le_{IQ}$ ($\alpha = 0.5, \beta = 0$), defined in Eq. (6), is a suitable choice for IV-FRBCSs.

3. *The choice of the width-limiting function B*: it depends on how much the length of the output interval's width has to be controlled to conserve the information quality of the interval inputs, since the larger the width of the interval output, the lesser is the information quality carried by it [39]. This level of control to be determined may not be obvious, but there are ways to test/compare different configurations of the same construction method by taking in to account different width-limiting functions, as we show in the application in a classification problem, presented in Sect. 6.

6. Application to Classification Problems

To showcase the applicability of our developments in practical problems, in this section we apply interval operators of specific subclasses of w-iv-fusion functions in the IVTURS IV-FRBCS. In the work by Asmus et al. [27], it was shown that n-dimensional overlap functions (and interval-valued functions based on them) are recommended to be applied in this type of problem. Furthermore, the best performing methods on that paper are based on n-dimensional $\leq_{\alpha,\beta}$-increasing iv-overlap functions that are width-limited by the minimum, that is, the outputs' widths are lesser or equal than the widths of the inputs. So, the class of n-dimensional w-iv-overlap functions as defined in Ex. 5 seems a natural choice to provide functions for this application.

In the following, first we recall some key concepts regarding IV-FRBCSs. After that, we present the experimental framework, followed by the analysis of the results.

6.1. IV-FRBCSs

A classification problem is composed by P training examples $\vec{x_p} = (x_{p1}, \ldots, x_{pn})$, $p \in \{1, \ldots, P\}$ where x_{pi} is the value of the i-th variable of the p-th example. Each example belongs to one of M classes in $C=\{C_1, \ldots, C_M\}$. The goal of the learned classifier is to identify the class of new testing examples.

One of the most frequently applied techniques to deal with classification problems are the FRBCSs. They can achieve accurate results by using highly interpretable models, since the fuzzy rules are expressed by linguistic labels [61]. The structure of the fuzzy rules is given by:

$$\text{Rule } R_j : \text{If } x_1 \text{ is } A_{j1} \text{ and } \ldots \text{ and } x_n \text{ is } A_{jn} \quad \text{then } Class = C'_j \text{ with } RW_j, \tag{27}$$

where R_j is the label of the j-th rule, $x = (x_1, \ldots, x_n)$ is an n-dimensional example vector, A_{ji} is the fuzzy set modeling the linguistic term of the j-th rule in the i-th antecedent, $C'_j \in C$ is a class label, and $RW_j \in [0, 1]$ is the rule weight [62]. In particular, the rule weight is computed by the fuzzy confidence value (or certainty factor), as follows:

$$RW_j = \frac{\sum_{x_p \in C'_j} A_j(x_p)}{\sum_{p=1}^{P} A_j(x_p)}, \tag{28}$$

where $A_j(x_p)$ is the matching degree of the pattern x_p with the antecedent part of the fuzzy rule R_j, given by

$$A_j(x_p) = \mathfrak{c}(A_{j1}(x_{p1}), \cdots, A_{jn}(x_{pn})), \tag{29}$$

where $\mathfrak{c}$ is an n-dimensional conjunction operator and $j \in \{1, \ldots, L\}$.

A FRBCS becomes an IV-FRBCS when some of the linguistic labels (or all of them) are modelled using IVFSs. This means that the fuzzy reasoning method must work with intervals instead of numbers, being called an iv-fuzzy reasoning method (IV-FRM), to take into account the interval widths (uncertainty) throughout the whole inference process [63] (see Sect. 6.2). As a novelty for this kind of classifier, we apply width-limited functions to control the information quality of the interval operations that occur on the IV-FRM, and analyze if such width control improves the performance of the system.

6.2. New iv-Fuzzy Reasoning Method

For the following experimentation, we apply our new theoretical concepts in the IVTURS algorithm, which is a state of the art IV-FRBCS (an in-depth look at each step of the IVTURS algorithm can be seen in the work by Sanz et al. [64]). Here, we recall the steps of its learning process:

1) *The building of an IV-FRBCS*, by the following procedures:

a) To generate an initial FRBCS by applying the two first stages of FARC-HD [65], a technique that is based on the Apriori algorithm [66] to build fuzzy rules (Eq. (28)) in its first learning stage. In those fuzzy rules, the product t-norm is usually applied as the conjunction operator $\mathfrak{c}$ in Eq. (29). However, as shown by Asmus et at. [27], one can benefit from replacing the product t-norm by other n-dimensional overlap functions On, such as the Geometric Mean and the n-dimensional OnB-overlap, given by Eq.s (2) and (3), respectively. This allows for the chosen n-dimensional overlap functions to be considered as the core functions for the construction of the n-dimensional w-iv-overlap functions to be used in the IV-FRM (described in the sequence). Thus, the function On impacts the learning process of the fuzzy rules, which means that if On is not the product, then the obtained fuzzy rules would not necessarily be the same than those obtained by the original IVTURS.

b) To define IVFSs to model the linguistic labels of the learned FRBCS;

c) To generate initial IV-REFs for each variable of the problem.

2) T*he application of an optimization approach*, so that:

a) It learns the best values of the IV-REFs' parameters;

b) It applies a rule selection process to decrease the system's complexity.

After creating the interval-valued fuzzy rules that compose the system, we define the new mechanism for classifying examples, as follows.

Let $\vec{x_p} = (x_{p1}, \ldots, x_{pn})$ be a new example to be classified, L be the number of rules in the rule base and M be the number of classes of the problem. The new IV-FRM is defined by the following steps:

(1) *Interval matching degree*: It expresses activation strength the rules' antecedents for each x_p. The similarity between the interval membership degrees (of each variable of the pattern to the corresponding IVFS) and the ideal membership degree $[1, 1]$ is computed by an IV-REF IR and, then, we use an interval-valued fusion function $F_O : L([0, 1])^n \to L([0, 1])$, for $j \in \{1, ..., L\}$ to combine their results as follows:

$$\mathcal{A}_j(x_p)=F_O\left(IR\left(\mathcal{A}_{j1}(x_{p1}), [1, 1]\right), \ldots, IR\left(\mathcal{A}_{jn}(x_{pn}), [1, 1]\right)\right),$$

with F_O being an n-dimensional w-iv-overlap function based on the n-dimensional overlap function On (applied as the conjunction operator when generating the initial FRBCS) and an increasing and symmetric width-limiting function B, which will determine how much information quality control we desire on the IV-FRM.

(2) *Interval association degree*: For the respective class of each rule, the interval matching degree is weighted with the corresponding iv-rule weight $IRW_j^k \in L([0, 1])$, through an iv-fusion function $F_P : L([0, 1])^n \to L([0, 1])$, as follows:

$$b_j^k = F_P\left(\mathcal{A}_j(x_p), IRW_j^k\right), \tag{30}$$

with $k = 1, \ldots, M$, $j = 1, \ldots, L$ and F_P being an interval-valued product operation, applied with the same criteria for width-limitation as the one for F_O.

The rule weight is defined by the interval-valued confidence value [67], given by:

$$IRW_j = \sum_{x_p \in C'_j} \mathcal{A}_j(x_p) \div_H \sum_{p=1}^{P} \mathcal{A}_j(x_p),$$

with $\div_H$ being defined as in Eq. (5).

(3) *Interval pattern classification soundness degree for all classes*: All interval association degrees of each class (obtained in Step (2)) with upper bounds that are greater than 0 are aggregated by an interval-valued aggregation function IA:

$$Y_k = IA\left(b_j^k, j = 1, \ldots, L \text{ and } \overline{b_j^k} > 0\right),$$

with $k = 1, \ldots, M$.

(4) *Classification*: A decision function F is applied over the interval pattern classification soundness degrees for all classes, as follows:

$$F(Y_1, \ldots, Y_M) = \arg \max_{k=1,\ldots,M} (Y_k).$$

In this final step of the IV-FRM, the system selects the greatest interval soundness degree, which is done by comparing the resulting intervals through an admissible order (in order to avoid a stalemate). As discussed by Asmus et al. [27], the order $\leq_{IQ}$ (order of Xu-Yager based on the quality of information, or $\leq_{\alpha,\beta}$ with $\alpha = 0.5$ and $\beta = 0$) is a suitable option for this type of classification problem, so we opt for this admissible order in our experiment.

6.3. Experimental Framework

The general goal of our experiment is to analyze the classification performance of the system when applying different n-dimensional w-iv-overlap functions obtained by either the construction method based on representable fusion functions (CMR) or the construction method based on $\leq_{\alpha,\beta}$-increasing fusion functions (CMA). To conduct our experiment, we have selected 31 real-world data-sets from the KEEL repository [68], which are publicly available on the webpage (http://www.keel.es/dataset.php). Table 3 summarizes the properties of the considered data-sets, presenting, for each data-set, the numbers of attributes (Atts.), examples (Ex.), and classes (Class.). To improve the learning process efficiency, the *magic, page-blocks, penbased, ring, satimage, shuttle*, and *twonorm* data-sets have been stratified sampled at 10%. Also, missing values from *bands*, *cleaveland* and *wisconsin* data-sets have been removed before our experiments.

We apply a *5-fold cross-validation model*, by dividing each data-set in 5 random partitions. Four of them (80%) are combined to train the system and the remaining one (20%) is reserved for testing. This process is executed 5 times, changing the testing partition in each iteration. The accuracy rate is used to measure the system's performance.

Table 3: Summary of the employed datasets

id	Data-set	Atts.	Ex.	Class.
app	appendicitis	7	106	2
bal	balance	4	625	3
ban	banana	2	5300	2
bds	bands	19	365	2
bup	bupa	6	345	2
clv	cleveland	13	297	5
con	contraceptive	9	1473	3
eco	ecoli	7	336	8
gla	glass	9	214	7
hab	haberman	3	306	2
hay	hayes-hoth	4	160	3
ion	ionosphere	33	351	2
iri	iris	4	150	3
led	led7digit	7	500	10
mag	magic	10	19020	2
new	newthyroid	5	215	3
pag	pageblocks	10	5472	5
pen	penbased	16	10992	10
pho	phoneme	5	5404	2
pim	pima	8	768	2
rin	ring	20	7400	2
sah	saheart	9	462	2
sat	satimage	36	6435	7
shu	shuttle	9	58000	7
spe	spectfheart	44	267	2
tit	titanic	3	2201	2
two	twonorm	20	7400	2
veh	vehicle	18	846	4
win	wine	13	178	3
wis	wisconsin	9	683	2
yea	yeast	8	1484	10

We follow the recommendation provided by Sanz et al. [64] for the set-up of the IVTURS classifier, with the modifications explained in Section 6.2. Then, we analyze the classification performance by comparing different configurations based on the function F_O applied on Step (1) of the IV-FRM, which is determined by both the corresponding On used on the learning process of the fuzzy rules and the weighting operation (F_P) used in the Step (2) of the IV-FRM, as shown in Table 4. The selected n-dimensional overlap functions (On) to be used as the core of F_O were based on the best performing operations for this kind of classifier [27], namely, GM and OnB (Eq. (2) and (3), respectively), as well as the product, since it is the operation used on the original IVTURS.

From Table 4, it can be seen that there are nine methods belonging to three groups: (i) REP: F_O is obtained by the BIR of On ($\widehat{On}$), (ii) CONR: F_O is obtained by CMR, via Theorem 12 ($IOnw^{\alpha}_{On,B}$), and (iii) CONA: F_O is obtained by CMA, via Theorem 20 ($AOnw^{\alpha}_{On,B}$). In each group we test the three n-dimensional overlap functions selected in this study, for instance, for the representable group, we have REP-Prod, REP-GM and REP-OnB.

When F_O is given by an n-dimensional w-iv-overlap function (all the approaches derived for the CONR and CONA groups), we check the effect of the width-limitation by comparing the results of the classifier when varying the width-limiting function B, given by:

$$B^{\rho}(w(x_1), \ldots, w(x_n)) = \min\{w(x_1), \ldots, w(x_n)\} + \rho(\max\{w(x_1), \ldots, w(x_n)\} - \min\{w(x_1), \ldots, w(x_n)\}), \tag{31}$$

with $\rho \in [0, 1]$. Specifically, we test each configuration with five possible values for ρ: $\rho = 0$ ($B = \min$); $\rho = 0.25$; $\rho = 0.5$; $\rho = 0.75$; $\rho = 1$ ($B = \max$). In this manner, the parameter

Table 4: Configuration schemes for the used classifiers

Classifier identifier	On	F_O	F_P
REP-Prod	OnP	$IOn_P = \widehat{OnP}$	$IOn_P = \widehat{OnP}$
REP-OnB	OnB	$IOnB = \widehat{OnB}$	$IOn_P = \widehat{OnP}$
REP-Gm	GM	$IGM = \widehat{GM}$	$IOn_P = \widehat{OnP}$
CONR-Prod	OnP	$IOnw^{\alpha}_{OnP,B}$	$IOnw^{\alpha}_{OnP,B}$
CONR-OnB	OnB	$IOnw^{\alpha}_{OnB,B}$	$IOnw^{\alpha}_{OnP,B}$
CONR-Gm	GM	$IOnw^{\alpha}_{GM,B}$	$IOnw^{\alpha}_{OnP,B}$
CONA-Prod	OnP	$AOnw^{\alpha}_{OnP,B}$	$AOnw^{\alpha}_{OnP,B}$
CONA-OnB	OnB	$AOnw^{\alpha}_{OnB,B}$	$AOnw^{\alpha}_{OnP,B}$
CONA-Gm	GM	$AOnw^{\alpha}_{GM,B}$	$AOnw^{\alpha}_{OnP,B}$

Table 5: Results in testing for the different methods

Method					
REP-Prod			79.56		
REP-OnB			79.83		
REP-Gm			79.75		
	$\rho = 0$	$\rho = 0.25$	$\rho = 0.5$	$\rho = 0.75$	$\rho = 1$
CONR-Prod	79.82	79.62	79.58	79.61	79.20
CONR-OnB	**80.11**	79.76	79.79	79.87	79.76
CONR-Gm	79.65	79.90	79.71	79.71	79.95
CONA-Prod	79.54	79.48	79.46	79.43	79.60
CONA-OnB	80.06	**80.00**	**80.02**	**80.09**	79.87
CONA-Gm	79.91	79.94	79.94	79.91	**80.04**

ρ indicates the amount of width control that we are imposing on the system. When $\rho = 0$, the output's width is limited by the minimum of the inputs' widths, representing the most strict width limitation. Conversely, when $\rho = 1$, the output's width is limited by the maximum of the inputs' widths, representing the less width control.

To detect if there are statistical differences in performance among the methods in a selected group, first, we use the aligned Friedman ranks test [69], reporting the obtained ranks of each method (the less the rank, the better). The best ranking method of such group is compared with the others through a Holm's post-hoc test [70]. When the goal is to provide a pairwise comparison, we apply a Wilcoxon Signed-Ranks test [71]. García et al. [72] showed that this selection of statistical tests is highly recommended to be used in machine learning.

6.4. Discussion of the Results

In Table 5 we present results in testing (average in the 31 datasets) for all the possible configurations of the new method, one in each row (the same ones as shown in Table 4). All approaches based on the construction methods CONR and CONA allow the control of the interval widths by means of the hyper-parameter ρ, whose results are shown in columns. On the other hand, approaches belonging to the REP group do not allow such control and, therefore, we present their results in a single column as they are not affected by the hyper-parameter ρ. For each ρ, we highlight in bold face the best result, that is, the configuration of the classifier that produced the greatest global mean. The detailed results for all the datasets (in all the partitions), with every possible combination, can be queried on https://github.com/tiagoasmus/testingResults-w-iv-overlaps.

Table 6: Average Rankings of the algorithms (Aligned Friedman) - Comparing levels of width control ρ

Method	$\rho = 0$	$\rho = 0.25$	$\rho = 0.5$	$\rho = 0.75$	$\rho = 1$
CONR-Prod	**60.61 (-)**	77.87 (0.390)	72 (0.390)	77.76 (0.390)	101.76 (0.001)*
CONR-OnB	**61.31 (-)**	84.81 (0.118)	81.40 (0.156)	75.34 (0.218)	87.15 (0.094)*
CONR-GM	86.58 (0.383)	70.47 (0.800)	80.18 (0.539)	85.19 (0.383)	**67.58 (-)**
CONA-Prod	73.02 (0.583)	80.19 (0.477)	83.97 (0.394)	86.07 (0.362)	**66.76 (-)**
CONA-OnB	80.16 (1.000)	78.42 (1.000)	76.79 (1.000)	**72.24 (-)**	82.39 (1.000)
CONA-GM	81.65 (0.529)	78.36 (0.529)	79.69 (0.529)	84.08 (0.469)	**66.23 (-)**

By observing Table 5, we see that the methods from group REP are not able to obtain better averaged behaviours than those highlighted in the second part of the table, that is, the best configurations of the methods that allow one to control de output's interval width. Moreover, most of the highlighted results come from the group CONA, with the exception of one method that is from group CONR (CONR-OnB with $\rho = 0$), which is also the method with the best global mean. Therefore, we can affirm that methods from groups CONR and CONA produce good results, possibly due to the control of the interval widths in the interval operations. In particular, the method CONA-OnB achieves a very robust performance for every considered level of width limitation.

Next, we study the effect of the level of width control performed in each method (the different values of ρ). To do it, we compare the five possible values of ρ for each method, by applying the aligned Friedman's test. The obtained ranks, as well as the Adjusted P-Values (APVs, presented in brackets) obtained by the Holm's post hoc test are shown in Table 6, where we have highlighted in **bold-face** the best rank (the less one) and we have stressed with an asterisk (*) those cases where there are statistical differences (using $\alpha = 0.1$) in favour to the ρ associated to the less rank. Looking at Table 6, one can observe that the benefit from a more rigid width control depends on the applied interval-valued function:

a) For the group CONR, two algorithms with a more strict width control produced better results (CONR-Prod and CONR-OnB, both with $\rho = 0$ as the control method). In both cases, there are significant differences from the control method ($\rho = 0$, strong width-limitation), and the algorithm with $\rho = 1$ (least considered width-limitation);

b) Considering the group CONA, the method CONA-Prod appear to perform better with a less aggressive width limitation (control method has $\rho = 1$). Confirming the previous observation, CONA-OnB achieved good results for every considered level of width control, as indicated by the same APV = 1 obtained for each of its configurations when compared to the control method ($\rho = 0.75$);

c) Independently of the employed Construction method, the algorithms that are based on the geometric mean also seem more

Table 7: Average Rankings of the algorithms (Aligned Friedman) - Comparison by groups

Method	Rank (APV)	Method	Rank (APV)	Method	Rank (APV)
REP-Prod	50.61 (0.500)	CONR-Prod^0	49.44 (0.628)	CONA-Prod^1	57.31 (0.029*)
REP-OnB	**42.73 (-)**	CONR-OnB^0	**42.53 (-)**	CONA-OnB$^{0.75}$	**40.55 (-)**
REP-GM	47.66 (0.500)	CONR-GM^1	49.03 (0.628)	CONA-GM^1	43.15 (0.705)

Table 8: Average Rankings of the algorithms (Aligned Friedman) - Comparing construction methods

Method	Rank (APV)
REP-OnB	55.87 (0.069*)
CONR-OnB^0	43.77 (0.724)
CONA-OnB$^{0.75}$	**41.36 (-)**

accurate with less strict width control, as the control methods have $\rho = 1$ for both CONR-GM and CONA-GM.

Next, we apply three Aligned Friedman and Holm's tests, one for each group (REP, CONR and CONA), to compare the best performing algorithms from each group. In the case of the group REP, we test the three considered methods as they do not depend on the values of ρ. For the groups CONR and CONA, we compare the control methods we obtained from Table 6. We indicate the value of ρ of each method as a superindex, when necessary. For example, the method CONR-Prod with $\rho = 0$ is denoted by CON-PROD^0. The results of these tests are shown in Table 7, with the best ranking method in each group highlighted in **bold-face** and methods that present significant difference from the control method are marked with an asterisk (*).

Observing Table 7, it is clear that the methods based on the n-dimensional overlap function OnB have good performance, as they are the control methods in each of the groups (REP, CONR and CONA). Next, we statistically compare those three best performing methods in another aligned test, whose obtained results are presented on Table 8. From Table 8, we see that the method CONA-OnB$^{0.75}$ is the best option, although CONR-OnB^0 also performs well. The method REP-OnB, does not achieve the same level of performance of the other two compared methods, being significantly less accurate than the control method. As those three methods are all based on the same core n-dimensional overlap function (OnB), which is used throughout all the components of those algorithms, the main difference between them lies on the construction of the interval-valued operations that take place in the IV-FRM, which may or may not control the widths of the outputs of such operations. Thus, we can conclude that controlling the width of the intervals, which implies having intervals with better information quality, is beneficial for the system's performance.

Finally, to further analyze the benefits of the new proposed methods, we carry out three pairwise comparisons between the best performing method from each group with the original configuration of the IVTURS algorithm (REP-Prod), through the Wilcoxon test. These results can be seen on Table 9, with

Table 9: Pairwise comparisons via Wilcoxon test

Comparison	R^+	R^-	p-value
IVTURS vs REP-OnB	181.5	314.5	0.214
IVTURS vs CONR-OnB^0	131.5	364.5	0.024*
IVTURS vs CONA-OnB$^{0.75}$	125.5	370.5	0.018*

the results with significant differences marked with an asterisk (*). Analyzing Table 9, it is clear that the configurations of both CONR-OnB^0 and CONA-OnB$^{0.75}$ improve significantly the performance of the IVTURS algorithm, whereas the method REP-OnB does not improve the accuracy of IVTURS in the same manner. Therefore, we can conclude that the exchange from the product to the n-dimensional overlap OnB is not the sole reason for the better performance of CONR-OnB^0 and CONA-OnB$^{0.75}$, indicating that these new methods benefit from a certain amount of width limitation.

7. Conclusion

When aggregating interval data through iv-aggregation functions, usually by means of the BIR, one may face the problem of dealing with interval outputs with large widths and, thus, low information quality. With the motivation to tackle this sort of challenge, in this paper, we presented a general framework to define and construct different subclasses of w-iv-fusion functions, allowing for the control of the interval output widths provided by interval aggregation operations that occur in practical problems, such as in IV-FRBCSs. From that, we have the following contributions:

1. The development of the concept of width-limitation, with the extension to the n-dimensional context of width-limited iv-fusion functions and width-limiting fusion functions;

2. The characterization of increasing fusion functions through a set of properties, a form of representation that facilitates the definition of interval-valued counterparts of such functions;

3. The definition of classes of w-iv-fusion functions based on an increasing fusion function, the interval extension of its set of properties and a pair of partial orders. This general methodology is capable of retrieving known definitions of iv-aggregation functions from the literature while also providing a flexible way to obtain iv-fusion functions with a desirable amount of width-limitation;

4. Two approaches to provide construction methods for w-iv-fusion functions, one based on the representable interval functions (CMR) and one based on admissibly ordered interval functions (CMA), where the interval outputs are "narrowed" in the direction of a K_α point and whose widths to not surpass a given threshold.

One of the key aspects of the developed framework and the presented construction methods is their flexibility, derived from the different possible choices of fusion functions, interval orders and width-limiting functions. This flexibility translates into potential applicability, as one can define a particular class of w-iv-fusion function accordingly to the constraints/requirements of a given practical problem. This aspect was highlighted in our case study, where we developed and applied a new IV-FRM for IV-FRBCSs, in which the information quality is controlled by n-dimensional w-iv-overlap functions, whose class is defined through our general framework. From our experimentation and subsequent statistical analysis, we can draw the following conclusions:

1. Configurations of the classifier based on n-dimensional w-iv-overlap functions constructed via either CMR or CMA have good classification accuracy, in general;

2. Although the amount of width control that benefits the performance of the system varies for each choice of w-iv-fusion function, this information can be retrieved by defining the width-limiting function through a parameter (ρ). In this manner, we can compare different values of ρ for each algorithm and possibly determine how much the widths of the outputs have to be constrained;

3. Configurations of the classifier based on the n-dimensional overlap function OnB produce the best results. Among those configurations, the one based on the CMA method have a significantly higher classification accuracy than the one based on the BIR of OnB. Particularly, CMA-OnB produces good results for every considered level of width limitation, presenting itself as a very stable method;

4. The two best performing methods, one based on CMR and other based on CMA, significantly enhances the classification accuracy of the state-of-the-art IVTURS algorithm, showing that both construction method approaches are suitable to provide w-iv-fusion functions to be applied in classification problems, which can benefit from the width control provided by such methods.

In future works we intend to apply our general framework to define and construct w-iv-fusion functions to be employed in aggregation processes with uncertainty (e.g., sensor data fusion), as in techniques for multicriteria decision making [21] and image processing [19].

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Appendix A. Proof of Theorem 2

Proof. Consider a symmetric aggregation function $B : [0,1]^n \to [0,1]$, a strict n-dimensional overlap function $On : [0,1]^n \to [0,1]$ and let $\alpha \in (0,1)$ and $\beta \in [0,1]$ such that $\alpha \neq \beta$. Observe that AOn^{α}_B is well defined. In fact, considering that $AOn^{\alpha}_B(\vec{X}) = R$, one has that $w(R) = On(\lambda_\alpha(X_1), \ldots, \lambda_\alpha(X_n)) \cdot d_\alpha(K_\alpha(R))$, $\underline{R} = K_\alpha(R) - \alpha \cdot w(R)$ and $\overline{R} = K_\alpha(R) + (1-\alpha) \cdot w(R)$. Now, let us verify if $IAOn^{\alpha}_B$ respects all conditions from Def. 19.

(IOn1) Immediate, since On and B are both symmetric;

(IOn2) ($\Rightarrow$) Take $\vec{X} \in L([0,1])^n$ and suppose that $AOn^{\alpha}_B(\vec{X}) = R = [0,0]$. Then, we have that

$$K_\alpha(R) = K_\alpha([0,0]) = 0 = On(K_\alpha(X_1), \ldots, K_\alpha(X_n)),$$

since $\alpha \in (0,1)$. Thus, by condition **(On2)**, $K_\alpha(X_i) = 0$ for some $i \in \{1, \ldots, n\}$, and, therefore, $\prod_{i=1}^n X_i = [0,0]$;

($\Leftarrow$) Consider $\vec{X_i} \in L([0,1])^n$ such that $\prod_{i=1}^n X_i = [0,0]$. So, $K_\alpha(X_1) \cdot \ldots \cdot K_\alpha(X_n) = 0$, since $\alpha \in (0,1)$. Then, by **(O2)**, one has that $K_\alpha(R) = On(K_\alpha(X_1), \ldots, K_\alpha(X_n)) = 0$, meaning that $AOn^{\alpha}_B(\vec{X}) = R = [0,0]$;

(IOn3) ($\Rightarrow$) Take $\vec{X} \in L([0,1])^n$ such that $AOn^{\alpha}_B(\vec{X}) = R = [1,1]$. Then, one has that

$$K_\alpha(R) = K_\alpha([1,1]) = 1 = On(K_\alpha(X_1), \ldots, K_\alpha(X_n)).$$

By **(On3)**, $K_\alpha(X_1) \cdot \ldots \cdot K_\alpha(X_n) = 1$, since $\alpha \in (0,1)$, meaning that $\prod_{i=1}^n X_i = [1,1]$;

($\Leftarrow$) Consider $\vec{X} \in L([0,1])^n$ such that $\prod_{i=1}^n X_i = [1,1]$. So, $K_\alpha(X_1) \cdot \ldots \cdot K_\alpha(X_n) = 1$, since $\alpha \in (0,1)$. Then, by **(i)** and **(O3)**, one has that $K_\alpha(R) = On(K_\alpha(X_1), \ldots, K_\alpha(X_n)) = 1$, meaning that $AOn^{\alpha}_B(\vec{X}) = R = [1,1]$;

(AOn4) Consider $Z \in L([0,1])$, $\vec{X}, \vec{Y} \in L([0,1])^n$, such that there exist $k \in \{1, \ldots, n\}$ for which $X_k \leq_{\alpha,\beta} Y_k$ and $X_i = Y_i = Z$ for all $i \in \{1, \ldots, n\} - \{k\}$. So, it holds that $X_i \leq_{\alpha,\beta} Y_i$ for all $i \in \{1, \ldots, n\}$. By Lemma 1, one can consider $\beta = 0$ or $\beta = 1$. Here, we present the proof for $\beta = 0$.

If $K_\alpha(Z) = 0$, then we have that $K_\alpha(AOn^{\alpha}_B(\vec{X})) = 0 = K_\alpha(AOn^{\alpha}_B(\vec{Y}))$, which means that $AOn^{\alpha}_B(\vec{X}) = [0,0] = AOn^{\alpha}_B(\vec{Y})$, since $\alpha \neq 0$. Then, $AOn^{\alpha}_B(\vec{X}) \leq_{\alpha,\beta} AOn^{\alpha}_B(\vec{Y})$.

If $K_\alpha(Z) \neq 0$, then we have the following cases:

a) $K_\alpha(X_k) < K_\alpha(Y_k)$: Since On is strict, one has that $K_\alpha(AOn^{\alpha}_B(\vec{X})) < K_\alpha(AOn^{\alpha}_B(\vec{Y}))$. Therefore,

$$\begin{aligned} &K_\alpha(AOn^{\alpha}_B(\vec{X})) < K_\alpha(AOn^{\alpha}_B(\vec{Y})) \\ &\quad \Rightarrow \quad AOn^{\alpha}_B(\vec{X}) \leq_{\alpha,\beta} AOn^{\alpha}_B(\vec{Y}). \end{aligned}$$

b) $K_\alpha(X_k) = K_\alpha(Y_k)$ and $K_\beta(X_k) < K_\beta(Y_k)$: Then, $\underline{X_k} < \underline{Y_k} \leq \overline{Y_k} < \overline{X_k}$, meaning that $w(X_k) > w(Y_k)$ and, therefore, by Def. 17, $\lambda_\alpha(X_k) > \lambda_\alpha(Y_k)$. So,

$$\begin{aligned} K_\alpha(AOn^{\alpha}_B(\vec{X})) &= On(K_\alpha(Z), \ldots, K_\alpha(X_k), \ldots, K_\alpha(Z)) \\ &= On(K_\alpha(Z), \ldots, K_\alpha(Y_k), \ldots, K_\alpha(Z)) \\ &= K_\alpha(AOn^{\alpha}_B(\vec{Y})), \text{ and} \end{aligned}$$

$$\begin{aligned} &K_{\beta=0}(AOn^{\alpha}_B(\vec{X})) \\ &\quad = K_\alpha(AOn^{\alpha}_B(\vec{X})) - \alpha w(AOn^{\alpha}_B(\vec{X})) \text{ by Def. 13} \\ &\quad = K_\alpha(AOn^{\alpha}_B(\vec{X})) - \\ &\qquad \alpha B(\lambda_\alpha(Z), \ldots, \lambda_\alpha(X_k), \ldots, \lambda_\alpha(Z)) d_\alpha(K_\alpha(AOn^{\alpha}_B(\vec{X}))) \\ &\quad \leq K_\alpha(AOn^{\alpha}_B(\vec{Y})) - \\ &\qquad \alpha B(\lambda_\alpha(Z), \ldots, \lambda_\alpha(Y_k), \ldots, \lambda_\alpha(Z)) d_\alpha(K_\alpha(AOn^{\alpha}_B(\vec{Y}))) \\ &\quad = K_{\beta=0}(AOn^{\alpha}_B(\vec{Y})), \end{aligned}$$

since B is increasing. Therefore, $AOn^{\alpha}_B(\vec{X}) \leq_{\alpha,\beta} AOn^{\alpha}_B(\vec{Y})$.

c) $K_\alpha(X_k) = K_\alpha(Y_k)$ and $K_\beta(X_k) = K_\beta(Y_k)$: In this case, $\vec{X} = \vec{Y}$, so, it is immediate that $AOn^\alpha_B(\vec{X}) \leq_{\alpha,\beta} AOn^\alpha_B(\vec{Y})$. So, for every scenario when $\beta = 0$, it holds that if $X_i \leq_{\alpha,\beta} Y_i$, for all $i \in \{1,\ldots,n\}$, then $AOn^\alpha_B(\vec{X}) \leq_{\alpha,\beta} AOn^\alpha_B(\vec{Y})$. The proof for $\beta = 1$ is obtained analogously. □

Appendix B. Proof of Theorem 6

Proof. Consider an increasing fusion function $B : [0,1]^n \to [0,1]$, a strict fusion function $F : [0,1]^n \to [0,1]$ with $h = 0$ as its annihilator element and take $\alpha \in (0,1]$, $\beta \in [0,\alpha)$. Observe that, for all $\vec{X} \in L([0,1])^n$:
(i) $K_\alpha(IFw^\alpha_B(\vec{X})) = K_\alpha(\widehat{F}(\vec{X}))$;
(ii) $K_\beta(IFw^\alpha_B(\vec{X})) = K_\alpha(\widehat{F}(\vec{X})) - \alpha m_{\widehat{F},B}(\vec{X}) + \beta m_{\widehat{F},B}(\vec{X})$;
(iii) $w(IFw^\alpha_B(\vec{X})) = m_{\widehat{F},B}(\vec{X})$
$= \min\{w(\widehat{F}(\vec{X})), B(w(X_1),\ldots,w(X_n))\}$.

So, it is immediate that IFw^α_B is well defined and, by **(iii)**, that is width-limited by B. Now, consider $Z \in L([0,1])$, $\vec{X},\vec{Y} \in L([0,1])^n$, such that there exists $k \in \{1,\ldots,n\}$ for which $X_k \leq_{Pr} Y_k$ and $X_i = Y_i = Z$ for all $i \in \{1,\ldots,n\} - \{k\}$. So, it holds that $X_i \leq_{Pr} Y_i$ for all $i \in \{1,\ldots,n\}$. As $\beta < \alpha$, by Lemma 1, one can consider $\beta = 0$. Thus:

$$K_{\beta=0}(IFw^\alpha_B(\vec{X})) = K_\alpha(\widehat{F}(\vec{X})) - \alpha \cdot m_{\widehat{F},B}(\vec{X}) \qquad \text{(B.1)}$$

$$K_{\beta=0}(IFw^\alpha_B(\vec{Y})) = K_\alpha(\widehat{F}(\vec{Y})) - \alpha \cdot m_{\widehat{F},B}(\vec{Y}). \qquad \text{(B.2)}$$

Now, we have the following possibilities regarding $m_{\widehat{F},B}(\vec{X})$ and $m_{\widehat{F},B}(\vec{Y})$ that affects the values of $IFw^\alpha_B(\vec{X})$ and $IFw^\alpha_B(\vec{Y})$, respectively:
1) $m_{\widehat{F},B}(\vec{X}) = w(\widehat{F}(\vec{X}))$ and $m_{\widehat{F},B}(\vec{Y}) = w(\widehat{F}(\vec{Y}))$: In this case, we have $IFw^\alpha_B(\vec{X}) = \widehat{F}(\vec{X}) \leq_{Pr} \widehat{F}(\vec{Y}) = IFw^\alpha_B(\vec{Y})$, meaning that $IFw^\alpha_B(\vec{X}) \leq_{\alpha,\beta} IFw^\alpha_B(\vec{Y})$.
2) $m_{\widehat{F},B}(\vec{X}) = B(w(Z),\ldots,w(X_k),\ldots,w(Z))$ and
$m_{\widehat{F},B}(\vec{Y}) = B(w(Z),\ldots,w(Y_k),\ldots,w(Z))$: It follows that

$$IFw^\alpha_B(\vec{X}) = [K_\alpha(\widehat{F}(\vec{X})) - \alpha B(w(Z),\ldots,w(X_k),\ldots,w(Z)), \; K_\alpha(\widehat{F}(\vec{X})) + (1-\alpha)B(w(Z),\ldots,w(X_k),\ldots,w(Z))], \text{ and}$$

$$IFw^\alpha_B(\vec{Y}) = [K_\alpha(\widehat{F}(\vec{Y})) - \alpha B(w(Z),\ldots,w(Y_k),\ldots,w(Z)), \; K_\alpha(\widehat{F}(\vec{Y})) + (1-\alpha)B(w(Z),\ldots,w(Y_k),\ldots,w(Z))].$$

Now, let us verify all the cases in which $X_k \leq_{Pr} Y_k$ holds:
a) $\underline{X_k} = \underline{Y_k}$ and $\overline{X_k} = \overline{Y_k}$: We have that $X_k = Y_k$, meaning that $IFw^\alpha_B(\vec{X}) = IFw^\alpha_B(\vec{Y}) \Rightarrow IFw^\alpha_B(\vec{X}) \leq_{\alpha,\beta} IFw^\alpha_B(\vec{Y})$.
b) $\underline{X_k} = \underline{Y_k}$ and $\overline{X_k} < \overline{Y_k}$: When $\underline{Z} \neq h = 0$, it holds that $K_\alpha(\widehat{F}(\vec{X})) < K_\alpha(\widehat{F}(\vec{Y}))$, since F is strictly increasing on $(0,1]^n$ and $\alpha \in (0,1]$. So, it follows that

$$K_\alpha(IFw^\alpha_B(\vec{X})) < K_\alpha(IFw^\alpha_B(\vec{Y})) \Rightarrow IFw^\alpha_B(\vec{X}) \leq_{\alpha,\beta} IFw^\alpha_B(\vec{Y}).$$

If $\underline{Z} = h = 0$ and $\overline{Z} \neq h = 0$, one has that

$$\begin{aligned} \widehat{F}(\vec{X}) &= [0, F(\overline{Z},\ldots,\overline{X_k},\ldots,\overline{Z})], \\ \widehat{F}(\vec{Y}) &= [0, F(\overline{Z},\ldots,\overline{Y_k},\ldots,\overline{Z})]. \end{aligned}$$

Since $\overline{X_k} < \overline{Y_k}$ and F is strict, then

$$\begin{aligned} K_\alpha(IFw^\alpha_B(\vec{X})) &= K_\alpha(\widehat{F}(\vec{X})) \\ &< K_\alpha(\widehat{F}(\vec{Y})) \\ &= K_\alpha(IFw^\alpha_B(\vec{Y})) \\ &\Rightarrow IFw^\alpha_B(\vec{X}) \leq_{\alpha,\beta} IFw^\alpha_B(\vec{Y}). \end{aligned}$$

If $\underline{Z} = h = 0$ and $\overline{Z} = h = 0$, then

$$\widehat{F}(\vec{X}) = IFw^\alpha_B(\vec{X}) = [0,0] = IFw^\alpha_B(\vec{Y}) = \widehat{F}(\vec{Y}).$$

So, we have that $IFw^\alpha_B(\vec{X}) \leq_{\alpha,\beta} IFw^\alpha_B(Y,Z)$, for all $X_k, Y_k, Z \in L([0,1])$, such that $\underline{X_k} = \underline{Y_k}$ and $\overline{X_k} < \overline{Y_k}$.
c) $\underline{X_k} < \underline{Y_k}$ and $\overline{X_k} = \overline{Y_k}$: When $\underline{Z} \neq h = 0$ and $\alpha \neq 1$, we have that $K_\alpha(\widehat{F}(\vec{X})) < K_\alpha(\widehat{F}(\vec{Y}))$. So, it holds that

$$K_\alpha(IFw^\alpha_B(\vec{X})) < K_\alpha(IFw^\alpha_B(\vec{Y})) \Rightarrow IFw^\alpha_B(\vec{X}) \leq_{\alpha,\beta} IFw^\alpha_B(\vec{Y}).$$

When taking $\underline{Z} \neq h = 0$ and $\alpha = 1$, we have that $K_\alpha(IFw^\alpha_B(\vec{X})) = K_\alpha(IFw^\alpha_B(\vec{Y})) = K$. Moreover, from Eqs. (C.1) and (C.2):

$$\begin{aligned} K_\beta(IFw^\alpha_B(\vec{X})) &= K - B(w(Z),\ldots,w(X_k),\ldots,w(Z)) \\ K_\beta(IFw^\alpha_B(\vec{Y})) &= K - B(w(Z),\ldots,w(Y_k),\ldots,w(Z)). \end{aligned}$$

As $\underline{X_k} < \underline{Y_k}$ and $\overline{X_k} = \overline{Y_k}$, we have that $w(Y_k) < w(X_k)$, and, since B is increasing,

$$B(w(Z),\ldots,w(Y_k),\ldots,w(Z)) \leq B(w(Z),\ldots,w(X_k),\ldots,w(Z)).$$

So:

$$\begin{aligned} K_\beta(IFw^\alpha_B(\vec{X})) &= K - B(w(Z),\ldots,w(X_k),\ldots,w(Z)) \\ &\leq K - B(w(Z),\ldots,w(Y_k),\ldots,w(Z)) \\ &= K_\beta(IFw^\alpha_B(\vec{Y})). \text{ Then:} \end{aligned}$$

$$\begin{aligned} & K_\alpha(IFw^\alpha_B(\vec{X})) = K_\alpha(Ifw^\alpha_B(\vec{Y})) \\ & \text{and} \quad K_\beta(IFw^\alpha_B(\vec{X})) \leq K_\beta(IFw^\alpha_B(\vec{Y})) \\ & \Rightarrow \quad IFw^\alpha_B(\vec{X}) \leq_{\alpha,\beta} IFw^\alpha_B(\vec{Y}). \end{aligned}$$

If $\underline{Z} = h = 0$, one has that

$$\begin{aligned} \widehat{F}(\vec{X}) &= [0, F(\overline{Z},\ldots,\overline{X_k},\ldots,\overline{Z})], \\ \widehat{F}(\vec{Y}) &= [0, F(\overline{Z},\ldots,\overline{Y_k},\ldots,\overline{Z})]. \end{aligned}$$

Since $\overline{X_k} = \overline{Y_k}$, then $K_\alpha(IFw^\alpha_B(\vec{X})) = K_\alpha(IFw^\alpha_B(\vec{Y}))$ and, analogously to the previous case, when $\underline{Z} \neq h = 0$ and $\alpha = 1$, we have that

$$\begin{aligned} & K_\alpha(IFw^\alpha_B(\vec{X})) = K_\alpha(IFw^\alpha_B(\vec{Y})) \\ & \text{and} \quad K_\beta(IFw^\alpha_B(\vec{X})) \leq K_\beta(IFw^\alpha_B(\vec{Y})) \\ & \Rightarrow \quad IFw^\alpha_B(\vec{X}) \leq_{\alpha,\beta} IFw^\alpha_B(\vec{Y}). \end{aligned}$$

So, we have that $IFw^\alpha_B(\vec{X}) \leq_{\alpha,\beta} IFw^\alpha_B(\vec{Y})$, for all $X_k, Y_k, Z \in L([0,1])$, such that $\underline{X_k} < \underline{Y_k}$ and $\overline{X_k} = \overline{Y_k}$.
d) $\underline{X} < \underline{Y}$ and $\overline{X} < \overline{Y}$: When $\underline{Z} \neq h = 0$, it holds that $K_\alpha(\widehat{F}(\vec{X})) < K_\alpha(\widehat{F}(\vec{Y}))$. So, we have that

$$K_\alpha(IFw^\alpha_B(\vec{X})) < K_\alpha(IFw^\alpha_B(\vec{Y})) \Rightarrow IFw^\alpha_B(\vec{X}) \leq_{\alpha,\beta} IFw^\alpha_B(\vec{X}).$$

If $\underline{Z} = h = 0$ and $\overline{Z} \neq h = 0$, one has that

$$\widehat{F}(\vec{X}) = [0, F(\overline{Z}, \ldots, \overline{X_k}, \ldots, \overline{Z})],$$
$$\widehat{F}(\vec{X}) = [0, F(\overline{Z}, \ldots, \overline{Y_k}, \ldots, \overline{Z})].$$

Since $\overline{X_k} < \overline{Y_k}$ and F is strict, then

$$\begin{aligned} K_\alpha(IFw_B^\alpha(\vec{X})) &= K_\alpha(\widehat{F}(\vec{X})) \\ &< K_\alpha(\widehat{F}(\vec{X})) \\ &= K_\alpha(IFw_B^\alpha(\vec{Y})) \\ &\Rightarrow IFw_B^\alpha(\vec{X}) \leq_{\alpha,\beta} IFw_B^\alpha(\vec{Y}). \end{aligned}$$

If $\underline{Z} = \overline{Z} = h = 0$, then

$$\widehat{F}(\vec{X}) = IFw_B^\alpha(\vec{X}) = [0, 0] = IFw_B^\alpha(\vec{Y}) = \widehat{F}(\vec{Y}).$$

So, we have that $IFw_B^\alpha(\vec{X}) \leq_{\alpha,\beta} IFw_B^\alpha(\vec{Y})$, for all $X_k, Y_k, Z \in L([0, 1])$, such that $\underline{X_k} < \underline{Y_k}$ and $\overline{X_k} < \overline{Y_k}$. Thus, we conclude that, for all $X_k, Y_k, Z \in L([0, 1])$, when

$$\begin{aligned} m_{\widehat{F},B}(X, Z) &= B(w(Z), \ldots, w(X_k), \ldots, w(Z)) \\ m_{\widehat{F},B}(Y, Z) &= B(w(Z), \ldots, w(Y_k), \ldots, w(Z)), \text{ then} \end{aligned}$$

$$X_i \leq_{Pr} Y_i \text{ for all } i \in \{1, \ldots, n\} \Rightarrow IOw_B^\alpha(\vec{X}) \leq_{\alpha,\beta} IOw_B^\alpha(\vec{Y}).$$

3) $m_{\widehat{F},B}(\vec{X}) = w(\widehat{F}(\vec{X}))$ and $m_{\widehat{F},B}(\vec{X}) = B(w(Z), \ldots, w(Y_k), \ldots, w(Z))$: It follows that

$$IFw_B^\alpha(\vec{X}) = \widehat{F}(\vec{X}) \text{ and}$$

$$\begin{aligned} IFw_B^\alpha(\vec{Y}) = [&K_\alpha(\widehat{F}(\vec{Y})) - \alpha B(w(Z), \ldots, w(Y_k), \ldots, w(Z)), \\ &K_\alpha(\widehat{F}(\vec{Y})) + (1 - \alpha) B(w(Z), \ldots, w(Y_k), \ldots, w(Z))]. \end{aligned}$$

Again, we analyze all the possibilities in which $X_k \leq_{Pr} Y_k$ holds. The results are exactly the same as the ones presented on the proof when $m_{\widehat{F},B}(\vec{X}) = B(w(Z), \ldots, w(X_k), \ldots, w(Z))$ and $m_{\widehat{F},B}(\vec{X}) = B(w(Z), \ldots, w(Y_k), \ldots, w(Z))$, with the exception of the particular case when $\underline{X_k} < \underline{Y_k}$, $\overline{X_k} = \overline{Y_k}$, $\underline{Z} \neq h = 0$ and $\alpha = 1$. In this case, we have that $K_\alpha(IFw_B^\alpha(\vec{X})) = K_\alpha(IFw_B^\alpha(\vec{Y})) = K$. Moreover, from Eqs. (C.1) and (C.2): $K_\beta(IFw_B^\alpha(\vec{X})) = K - w(\widehat{F}(\vec{X}))$ and $K_\beta(IFw_B^\alpha(\vec{Y})) = K - B(w(Z), \ldots, w(Y_k), \ldots, w(Z))$.

As $\underline{X_k} < \underline{Y_k}$ and $\overline{X_k} = \overline{Y_k}$, we have that:

$$\begin{aligned} &B(w(Z), \ldots, w(Y_k), \ldots, w(Z)) \\ &\leq w(\widehat{F}(\vec{Y})) \\ &= F(\overline{Z}, \ldots, \overline{Y_k}, \ldots, \overline{Z}) - F(\underline{Z}, \ldots, \underline{Y_k}, \ldots, \underline{Z}) \\ &\leq F(\overline{Z}, \ldots, \overline{X_k}, \ldots, \overline{Z}) - F(\underline{Z}, \ldots, \underline{X_k}, \ldots, \underline{Z}) \\ &= w(\widehat{F}(\vec{X})), \end{aligned}$$

since F is strictly increasing. So,

$$\begin{aligned} &K_\beta(IFw_B^\alpha(\vec{X})) \\ &= F(\overline{Z}, \ldots, \overline{X}, \ldots, \overline{Z}) - w(\widehat{F}(\vec{X})) \\ &\leq F(\overline{Z}, \ldots, \overline{Y_k}, \ldots, \overline{Z}) - B(w(Z), \ldots, w(Y_k), \ldots, w(Z)) \\ &= K_\beta(IFw_B^\alpha(\vec{Y})). \text{ Then:} \end{aligned}$$

$$\begin{aligned} &K_\alpha(IFw_B^\alpha(\vec{X})) = K_\alpha(IFw_B^\alpha(\vec{Y})) \\ &\text{and } , K_\beta(IFw_B^\alpha(\vec{X})) \leq K_\beta(IFw_B^\alpha(\vec{Y})) \\ &\Rightarrow IFw_B^\alpha(\vec{X}) \leq_{\alpha,\beta} IOw_B^\alpha(Y, Z). \end{aligned}$$

Thus, one can conclude that, for all $X_k, Y_k, Z \in L([0, 1])$, when $m_{\widehat{F},B}(\vec{X}) = w(\widehat{F}(\vec{X}))$ and $m_{\widehat{F},B}(\vec{Y}) = B(w(Z), \ldots, w(Y_k), \ldots, w(Z))$, then

$$X_i \leq_{Pr} Y_i \text{ for all } i \in \{1, \ldots, n\} \Rightarrow IOw_B^\alpha(\vec{X}) \leq_{\alpha,\beta} IOw_B^\alpha(\vec{Y}).$$

4) $m_{\widehat{F},B}(\vec{X}) = B(w(Z), \ldots, w(X_k), \ldots, w(Z))$ and $m_{\widehat{F},B}(\vec{Y}) = w(\widehat{F}(\vec{Y}))$: It follows that

$$\begin{aligned} IFw_B^\alpha(\vec{X}) = [&K_\alpha(\widehat{F}(\vec{X})) - \alpha B(w(Z), \ldots, w(X_k), \ldots, w(Z)), \\ &K_\alpha(\widehat{F}(\vec{X})) + (1 - \alpha) B(w(Z), \ldots, w(X_k), \ldots, w(Z))] \text{ and} \\ &IFw_B^\alpha(\vec{Y}) = \widehat{F}(\vec{Y}). \end{aligned}$$

Once more, by analyzing every possibility in which $X_k \leq_{Pr} Y_k$ holds, one may observe that the results are exactly the same as the ones presented previously, with the exception when $\underline{X_k} < \underline{Y_k}$, $\overline{X_k} = \overline{Y_k}$, $\underline{Z} \neq h = 0$ and $\alpha = 1$. In this case, we have that $K_\alpha(IFw_B^\alpha(\vec{X})) = K_\alpha(IFw_B^\alpha(\vec{Y}))$. Moreover, from Eqs. (C.1) and (C.2):

$$\begin{aligned} &K_\beta(IFw_B^\alpha(\vec{X})) \\ &= F(\overline{Z}, \ldots, \overline{X_k}, \ldots, \overline{Z}) - B(w(Z), \ldots, w(X_k), \ldots, w(Z)) \text{ and} \\ &K_\beta(IFw_B^\alpha(\vec{Y})) = F(\overline{Z}, \ldots, \overline{Y_k}, \ldots, \overline{Z}) - w(\widehat{F}(\vec{Y})). \end{aligned}$$

As $\underline{X_k} < \underline{Y_k}$ and $\overline{X_k} = \overline{Y_k}$, we have that $w(Y_k) < w(X_k)$, and, since B is increasing:

$$\begin{aligned} &w(\widehat{F}(\vec{Y})) \leq B(w(Z), \ldots, w(Y_k), \ldots, w(Z)) \\ &\leq B(w(Z), \ldots, w(X_k), \ldots, w(Z)), \text{ So:} \end{aligned}$$

$$\begin{aligned} &K_\beta(IFw_B^\alpha(\vec{X})) \\ &= F(\overline{Z}, \ldots, \overline{X_k}, \ldots, \overline{Z}) - w(\widehat{F}(\vec{X})) \\ &\leq F(\overline{Z}, \ldots, \overline{Y_k}, \ldots, \overline{Z}) - B(w(Z), \ldots, w(Y_k), \ldots, w(Z)) \\ &= K_\beta(IFw_B^\alpha(\vec{Y})). \text{ Then:} \end{aligned}$$

$$\begin{aligned} &K_\alpha(IFw_B^\alpha(\vec{X})) = K_\alpha(IFw_B^\alpha(\vec{Y})) \\ &\text{and } K_\beta(IFw_B^\alpha(\vec{X})) \leq K_\beta(IFw_B^\alpha(\vec{Y})) \\ &\Rightarrow IFw_B^\alpha(\vec{X}) \leq_{\alpha,\beta} IFw_B^\alpha(\vec{Y}). \end{aligned}$$

Then, we have that $IFw_B^\alpha(\vec{X}) \leq_{\alpha,\beta} IFw_B^\alpha(\vec{Y})$, for all $X_k, Y_k, Z \in L([0, 1])$, such that $\underline{X_k} < \underline{Y_k}$ and $\overline{X_k} = \overline{Y_k}$.

Thus, one conclude that, for all $X_k, Y_k, Z \in L([0, 1])$, when

$$\begin{aligned} m_{\widehat{F},B}(\vec{X}) &= B(w(Z), \ldots, w(X), \ldots, w(Z)) \\ m_{\widehat{F},B}(\vec{Y}) &= w(\widehat{F}(\vec{Y})), \text{ then} \\ \forall i \in \{1, \ldots, n\} : \quad & X_i \leq_{Pr} Y_i \Rightarrow IFw_B^\alpha(\vec{X}) \leq_{\alpha,\beta} IFw_B^\alpha(\vec{Y}). \end{aligned}$$

As verified for all possible scenarios, it holds that IFw_B^α is $(\leq_{Pr}, \leq_{\alpha,\beta})$-increasing, for all $\alpha \in (0, 1]$ and $\beta \in [0, \alpha)$, which completes the proof that IFw_B^α is a w-iv-fusion function for the tuple $(\leq_{Pr}, \leq_{\alpha,\beta}, B)$. □

Appendix C. Proof of Theorem 14

Proof. Consider an increasing fusion function $B : [0,1]^n \to [0,1]$, a strict fusion function $F : [0,1]^n \to [0,1]$ with $h = 0$ as its annihilator element, $\alpha \in (0,1]$, $\beta \in [0,1]$ such that $\alpha \neq \beta$, and an $\leq_{\alpha,\beta}$ increasing fusion function $IF^\alpha : L([0,1])^n \to L([0,1])$ such that $K_\alpha(IF)(\vec{X}) = F(K_\alpha(X_1),\ldots,K_\alpha(X_n))$, for all $\vec{X} \in L([0,1])^n$. Observe that, for all $\vec{X} \in L([0,1])^n$: **(i)** $K_\alpha(IFw_B^\alpha(\vec{X})) = K_\alpha(IF^\alpha(\vec{X}))$; **(ii)** $K_\beta(IFw_B^\alpha(\vec{X})) = K_\alpha(IF^\alpha(\vec{X})) - \alpha \cdot m_{IF^\alpha,B}(\vec{X}) + \beta \cdot m_{IF^\alpha,B}(\vec{X})$; **(iii)** $w(IFw_B^\alpha(\vec{X})) = m_{IF^\alpha,B}(\vec{X}) = \min\{w(IF^\alpha(\vec{X})), B(w(X_1),\ldots,w(X_n))\}$. So, it is immediate that IFw_B^α is well defined and, by **(iii)**, that is width-limited by B.

Now, consider $Z \in L([0,1])$, $\vec{X},\vec{Y} \in L([0,1])^n$, such that there exist $k \in \{1,\ldots,n\}$ for which $X_k \leq_{\alpha,\beta} Y_k$ and $X_i = Y_i = Z$ for all $i \in \{1,\ldots,n\} - \{k\}$. So, it holds that $X_i \leq_{\alpha,\beta} Y_i$ for all $i \in \{1,\ldots,n\}$. By Lemma 1, one can consider $\beta = 0$ or $\beta = 1$. First, we present the proof for $\beta = 0$. Thus:

$$K_{\beta=0}(IFw_B^\alpha(\vec{X})) = K_\alpha(IF^\alpha(\vec{X})) - \alpha m_{IF^\alpha,B}(\vec{X}) \qquad \text{(C.1)}$$

$$K_{\beta=0}(IFw_B^\alpha(\vec{Y})) = K_\alpha(IF^\alpha(\vec{Y})) - \alpha m_{IF^\alpha,B}(\vec{Y}). \qquad \text{(C.2)}$$

Next, if $K_\alpha(Z) = 0$, then $K_\alpha(IFw_B^\alpha(\vec{X})) = K_\alpha(IF^\alpha(\vec{X})) = 0 = K_\alpha(IF^\alpha(\vec{Y})) = K_\alpha(IFw_B^\alpha(\vec{Y}))$, which means that $IFw_B^\alpha(\vec{X}) = IF^\alpha(\vec{X}) = [0,0] = IF^\alpha(\vec{Y}) = IFw_B^\alpha(\vec{Y})$, since $\alpha \neq 0$. Then, $IFw_B^\alpha(\vec{X}) \leq_{\alpha,\beta} IFw_B^\alpha(\vec{Y})$.

If $K_\alpha(Z) \neq 0$, then we have the following cases:

a) $K_\alpha(X_k) < K_\alpha(Y_k)$: Since F is strict, one has that:

$$\begin{aligned} &K_\alpha(IFw_B^\alpha(\vec{X})) \\ &\quad = K_\alpha(IF^\alpha(\vec{X})) < K_\alpha(IF^\alpha(\vec{Y})) = K_\alpha(IFw_B^\alpha(\vec{Y})). \text{ Thus:} \end{aligned}$$

$$\begin{aligned} &K_\alpha(IFw_B^\alpha(\vec{X})) < K_\alpha(IFw_B^\alpha(\vec{Y})) \\ &\quad \Rightarrow IFw_B^\alpha(\vec{X}) \leq_{\alpha,\beta} IFw_B^\alpha(\vec{Y}). \end{aligned}$$

b) $K_\alpha(X_k) = K_\alpha(Y_k)$ and $K_\beta(X_k) < K_\beta(Y_k)$: Since $K_\alpha(X_k) = K_\alpha(Y_k)$, then $K_\alpha(IFw_B^\alpha(\vec{X})) = K_\alpha(IF^\alpha(\vec{X})) = K_\alpha(IF^\alpha(\vec{Y})) = K_\alpha(IFw_B^\alpha(\vec{Y})) = K$ and, since IF^α is $\leq_{\alpha,\beta}$-increasing, we have that $K_\beta(IF^\alpha(\vec{X})) \leq K_\beta(IF^\alpha(\vec{Y}))$. As $\beta = 0$, then $w(X_k) > w(Y_k)$ and $w(IF^\alpha(\vec{X})) \geq w(IF^\alpha(\vec{Y}))$. From Eqs (C.1) and (C.2), the values of $K_\beta(IFw_B^\alpha(\vec{X}))$ and $K_\beta(IFw_B^\alpha(\vec{Y}))$ depend on the values of $m_{IF^\alpha,B}(\vec{X})$ and $m_{IF^\alpha,B}(\vec{Y})$, respectively. So, let us analyze each possibility regarding those maximal thresholds.

1) $m_{IF^\alpha,B}(\vec{X}) = w(IF^\alpha(\vec{X}))$ and $m_{IF^\alpha,B}(\vec{Y}) = w(IF^\alpha(\vec{Y}))$: In this case, $IFw_B^\alpha(\vec{X}) = IF^\alpha(\vec{X}) = IF^\alpha(\vec{Y}) = IFw_B^\alpha(\vec{Y})$.

2) $m_{IF^\alpha,B}(\vec{X}) = B(w(Z),\ldots,w(X_k),\ldots,w(Z))$ and $m_{IF^\alpha,B}(\vec{Y}) = B(w(Z),\ldots,w(Y_k),\ldots,w(Z))$: From Eqs. (C.1) and (C.2):

$$\begin{aligned} K_\beta(IFw_B^\alpha(\vec{X})) &= K - B(w(Z),\ldots,w(X_k),\ldots,w(Z)) \\ &\leq K - B(w(Z),\ldots,w(Y_k),\ldots,w(Z)) \\ &= K_\beta(IFw_B^\alpha(\vec{Y})), \end{aligned}$$

since B is increasing and $w(X_k) > w(Y_k)$. Thus,

$$IFw_B^\alpha(\vec{X}) \leq_{\alpha,\beta} IFw_B^\alpha(\vec{Y}).$$

3) $m_{IF^\alpha,B}(\vec{X}) = w(IF^\alpha(\vec{X}))$ and $m_{IF^\alpha,B}(\vec{Y}) = B(w(Z),\ldots,w(Y_k),\ldots,w(Z))$: From Eqs. (C.1) and (C.2):

$$\begin{aligned} K_\beta(IFw_B^\alpha(\vec{X})) &= K - w(IF^\alpha(\vec{X})) \\ &\leq K - B(w(Z),\ldots,w(Y_k),\ldots,w(Z)) \\ &= K_\beta(IFw_B^\alpha(\vec{Y})), \end{aligned}$$

since $B(w(Z),\ldots,w(Y_k),\ldots,w(Z)) \leq w(IF^\alpha(\vec{Y})) \leq w(IF^\alpha(\vec{X}))$. Thus,

$$IFw_B^\alpha(\vec{X}) \leq_{\alpha,\beta} IFw_B^\alpha(\vec{Y}).$$

4) $m_{IF^\alpha,B}(\vec{X}) = B(w(Z),\ldots,w(X_k),\ldots,w(Z))$ and $m_{IF^\alpha,B}(\vec{Y}) = w(IF^\alpha(\vec{Y}))$: Since $w(Y_k) \leq w(X_k)$, then

$$\begin{aligned} w(IF^\alpha(\vec{Y})) &\leq B(w(Z),\ldots,w(Y_k),\ldots,w(Z)) \\ &\leq B(w(Z),\ldots,w(X_k),\ldots,w(Z)). \end{aligned}$$

From Eqs. (C.1) and (C.2):

$$\begin{aligned} &K_\beta(IFw_B^\alpha(\vec{X})) \\ &\quad = K - B(w(Z),\ldots,w(X_k),\ldots,w(Z)) \leq K - w(IF^\alpha(\vec{Y})), \end{aligned}$$

meaning that $IFw_B^\alpha(\vec{X}) \leq_{\alpha,\beta} IFw_B^\alpha(\vec{Y})$. So, when $K_\alpha(X_k) = K_\alpha(Y_k)$ and $K_\beta(X_k) < K_\beta(Y_k)$, we have that $IFw_B^\alpha(\vec{X}) \leq_{\alpha,\beta} IFw_B^\alpha(\vec{Y})$.

c)$K_\alpha(X_k) = K_\alpha(Y_k)$ and $K_\beta(X_k) = K_\beta(Y_k)$: In this case, $\vec{X} = \vec{Y}$, so, it is immediate that $IFw_B^\alpha(\vec{X}) \leq_{\alpha,\beta} IFw_B^\alpha(\vec{Y})$. So, for every scenario when $\beta = 0$, it holds that, if $X_i \leq_{\alpha,\beta} Y_i$ for all $i \in \{1,\ldots,n\}$, then $IFw_B^\alpha(\vec{X}) \leq_{\alpha,\beta} IFw_B^\alpha(\vec{Y})$.

The proof for $\beta = 1$ is obtained analogously.

Thus, as verified for all possible scenarios, it holds that IFw_B^α is $(\leq_{\alpha,\beta}, \leq_{\alpha,\beta})$-increasing, for all $\alpha \in (0,1]$ and $\beta \in [0,1]$, which completes the proof that IFw_B^α is a w-iv-fusion function for the tuple $(\leq_{\alpha,\beta}, \leq_{\alpha,\beta}, B)$. □

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5.1.5 A constructive framework to define fusion functions with floating domains in arbitrary closed real intervals

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A constructive framework to define fusion functions with floating domains in arbitrary closed real intervals

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Abstract

Fusion functions and their most important subclass, aggregation functions, have been successfully applied in fuzzy modeling. However, there are practical problems, such as classification via neural networks, where the data to be aggregated are not modeling membership degrees in the unit interval. In this scenario, systems could benefit from the application of operators defined in domains different from $[0, 1]$, although, presenting similar behavior of some aggregation functions whose subclasses are currently defined only in the fuzzy context (e.g., overlap functions and t-norms). So, the main objective of this paper is to present a general framework to characterize classes of fusion functions with floating domains, called (a, b)-fusion functions, defined on any closed real interval $[a, b]$, based on classes of core fusion functions defined on $[0, 1]$. The fundamental aspect of this framework is that the properties of a core fusion function are preserved in the context of the analogous (a, b)-fusion function. Construction methods for (a, b)-aggregation functions are presented and some properties are studied. Finally, we introduce a similar framework to define fusion functions in which the inputs come from an interval $[a, b]$ but the output is mapped on a possibly different interval $[c, d]$, called (a, b, c, d)-fusion functions, along with some construction methods.

Keywords: (a, b)-fusion functions; (a, b)-aggregation functions; n-dimensional overlap functions; t-norms; uninorms

1. Introduction

Fusion functions are operators defined to combine/fuse several numerical values from the unit interval $[0, 1]$ into a single representative one, also from this same interval [42]. The most known and studied class of fusion functions is that of aggregation functions [8, 29], which are increasing fusion functions with some boundary conditions. Aggregation functions, in fact, can be defined on any interval $[a, b]$, with $a, b \in \mathbb{R}$ and $a < b$, such as the ordered weighted

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averaging (OWA) [57] operator and the Choquet integral [13, 20]. However, most of its subclasses (e.g., that of overlap functions [7, 11, 12], t-norms [36] and uninorms [58]) were defined specifically on $[0, 1]$, as they are mostly used to model fuzzy logic operations over membership degrees or truth-values.

For that reason, aggregation functions and their subclasses have been successfully employed in a plethora of theoretical and applied fields that involves some sort of fuzzy modeling. For instance, overlap functions and their generalizations (such as general overlap functions [15]) show good results when applied as a fuzzy conjunction operator in problems where associativity of the applied aggregation function is not required, such as in image processing [12, 34], fuzzy rule-based classification [38, 39, 40], decision making [23], wavelet-fuzzy power quality diagnosis system [45] and forest fire detection [26].

Some problems, though, may have imperfect information [60, 61], meaning that there may be uncertainty in the process of assigning the membership degrees or defining the membership functions to be applied in the fuzzy modeling [41, 44]. Several works tackled this challenge in different ways accordingly to their perspective on uncertainty [9], by using, for example, intervals [43], interval-valued fuzzy sets [30, 59], intuitionistic fuzzy sets [6] soft sets [1] or rought sets [48] . Naturally, aggregation functions (and many of its subclasses) were extended to be applied in each one of those contexts (e.g., interval-valued aggregation functions [2, 3, 4, 19] and intuitionistic aggregations [56]). Such generalizations can also be studied through the lens of lattice theory[1]. Recently, it was observed in the literature the development of many classes of aggregation functions on lattices, such as t-norms and t-conorms [24, 51, 53], uninorms [14, 35], overlap and grouping functions [46, 49, 54]. Although some of those defined functions could operate with inputs that are not from the unit interval, there has not been an interest in applying such generalizations of aggregation functions in applications that are not fuzzy in nature.

We point out that the necessity of defining aggregation functions in intervals that are not the unit interval may be observed in the literature, even in the fuzzy context. For example, the ordinal sums of t-norms (t-conorms) [36, 51] and overlap (grouping) functions [18, 55] acting on $[0, 1]$ are defined on the basis of t-norms (t-conorms) and overlap (grouping) functions acting on a family of non-empty, pairwise disjoint open subintervals (x, y), which, although included in $[0, 1]$, are not equal to $[0, 1]$.

Still, there are practical problems where the data to be aggregated are not modeling membership degrees, truth values or some extension of them considering uncertainty modeling, which could benefit from the application of functions with similar behaviour of aggregation functions that are currently defined only to operate in the fuzzy context. That is the case, for example, of the pooling process in convolutional neural networks [37], which are widely applied in image processing [17, 47], and recurrent neural networks [31], such as Long Short-Term Memory [33], which are used in several machine learning problems with sequential information [32].

Then, the main objective of this paper is to present a framework to characterize extended classes of fusion functions on a floating domain $[a, b]^n$, which we call (a, b)-fusion functions, based on core classes of fusion functions defined on $[0, 1]^n$. The fundamental aspect of this framework is that the properties of the core fusion function, defined in the context of the unit interval, are preserved in the context of an arbitrary interval $[a, b]$ when defining an analogous (a, b)-fusion function. We point out that this property preservation is not trivial, since there are a multitude of ways of characterizing properties that are equivalent in the context of the unit interval, but that can lead to different concepts when defined in another interval $[a, b]$.

Since the motivation comes from an application standpoint, we present some construction methods for these newly defined (a, b)-aggregation functions, based on some core known aggregation functions (e.g., n-dimensional overlap functions [28], t-norms [36] and uninorms [58]), guaranteeing that the constructed function behaves in $[a, b]$ in a similar manner as the core function does in $[0, 1]$. Furthermore, the presented construction methods are based on the choice of core aggregation function and an increasing bijective function, both able to be defined with parameters that can be manipulated/adapted/learned, accordingly to the application at hand, without sacrificing the main properties of the desired constructed function. Then, we proceed to study some interesting properties of aggregation functions, namely, idempotency, a kind of generalized migrativity (introduced here) and abstract homogeneity [52], and how such properties are preserved when our construction methods for (a, b)-aggregation functions are applied.

Finally, we present the main concepts to develop a similar framework to define fusion functions whose the inputs come from an interval $[a, b]$ but the output is mapped on a possibly different interval $[c, d]$. We call them (a, b, c, d)-

[1]For more on lattice theory, see [27].

fusion functions. Then, based on this framework, subclasses of (a, b, c, d)-fusion functions are defined and construction methods for them are presented. We show that, under some constraints, when a constructed (a, b, c, d)-aggregation function is based on an (a, b)-aggregation function, which, in turn, is based on a core aggregation function defined on $[0, 1]^n$, then, it is equivalent to the (a, b, c, d)-aggregation function obtained directly from the same core aggregation function defined in $[0, 1]$.

The paper is organized as follows:

Section 2 Important preliminary concepts are presented;

Section 3 We introduce and discuss the notion of property shifting, which is how we denominate the action of properly transpose a given property from one domain to another, and develop a general framework for defining classes of (a, b)-fusion functions based on classes of fusion functions, showing examples;

Section 4 The construction methods for different classes of (a, b)-aggregation functions are presented and discussed;

Section 5 We study some properties of aggregation functions and their counterparts in the context of (a, b)-aggregation functions, with particular interest in the study of (a, b)-aggregation functions obtained by our construction methods;

Section 6 The main concepts of (a, b, c, d)-fusion concepts are developed, focusing on different ways to construct them;

Section 7 In our concluding remarks, we review the main contributions of the paper and propose some possible future lines of work.

2. Preliminary concepts

In this section, we recall some preliminary concepts that are relevant for the development of the paper.
Let us denote $\vec{x} = (x_1, \ldots, x_n) \in [0, 1]^n$, where $n > 1$.

Definition 2.1. *[36] A function $N : [0, 1] \to [0, 1]$ is a fuzzy negation if the following conditions hold:*

(N1) $N(0) = 1$ *and* $N(1) = 0$*;*

(N2) *If* $x \leq y$ *then* $N(y) \leq N(x)$*, for all* $x, y \in [0, 1]$.

If N also satisfies the involutive property,

(N3) $N(N(x)) = x$*, for all* $x \in [0, 1]$*,*

then it is said to be a strong fuzzy negation.

Example 2.1. *The Zadeh negation given, for all $x \in [0, 1]$, by*

$$N_Z(x) = 1 - x,$$

is a strong fuzzy negation.

The concept of fusion function [42] was originally defined in the context of the unit interval as an arbitrary function $F : [0, 1]^n \to [0, 1]$.

Definition 2.2. *[36] Given a strong fuzzy negation $N : [0, 1] \to [0, 1]$ and a fusion function $F : [0, 1]^n \to [0, 1]$, then the fusion function $F^N : [0, 1]^n \to [0, 1]$ defined, for all $\vec{x} \in [0, 1]^n$, by*

$$F^N(\vec{x}) = N(F(N(x_1), \ldots, N(x_n))), \tag{1}$$

is the N-dual of F.

When it is clear by the context, the N_Z-dual function (dual with respect to the Zadeh negation) of F is just called dual of F, and is denoted by F^d. Observe that $(F^N)^N = F$, since N is a strong negation.

Here we recall the representation, introduced by Asmus et al. [5], of a class of fusion functions through its set of sufficient and necessary properties, which we denominate as *constitutive properties*. Let $\mathcal{F}$ be a subclass of fusion functions $F : [0,1]^n \to [0,1]$ and $P_{\mathcal{F}}$ be a set of constitutive properties of the functions from $\mathcal{F}$, such that it includes: (i) boundary conditions for any $F \in \mathcal{F}$, (ii) some kind of monotonicity and (iii) possibly other constraints not related to neither (i) nor (ii). Such subclass of functions is given by:

$$\mathcal{F} = \{F : [0,1]^n \to [0,1] |\ F \text{ satisfies all the properties in } P_{\mathcal{F}}\}. \tag{2}$$

We present the same style of representation for the definition of aggregation functions, which is the most important subclass of fusion functions, as follows:

Definition 2.3. *[8] An aggregation function is any function $A \in \mathcal{A}$, where:*

$$\mathcal{A} = \{A : [0,1]^n \to [0,1] |\ A \textit{ satisfies all the properties in } P_{\mathcal{A}}\}$$

with

$$P_{\mathcal{A}} = \{\textbf{(A1)}, \textbf{(A2)}\},$$

and

(A1) *A is increasing;*

(A2) *$A(0, \dots, 0) = 0$ and $A(1, \dots, 1) = 1$.*

Example 2.2. **i)** *The function $AM : [0,1]^n \to [0,1]$ (arithmetic mean), given, for all $\vec{x} \in [0,1]^n$, by*

$$AM(\vec{x}) = \frac{\sum_{i=1}^n x_i}{n}, \tag{3}$$

is an aggregation function.

ii) *The function $AW : [0,1]^n \to [0,1]$ (weighted arithmetic mean), given, for all $\vec{x}, \vec{w} \in [0,1]^n$, by*

$$AW(\vec{x}) = \sum_{i=1}^n x_i \cdot w_i, \tag{4}$$

such that $\sum_{i=1}^n w_i = 1$, is an aggregation function.

There are many subclasses of aggregation functions defined in the literature. Here we highlight some of them that are going to be of importance on this work.

Definition 2.4. *[22, 28] An n-dimensional overlap function is any fusion function $O \in \mathcal{O}$, such that:*

$$\mathcal{O} = \{O : [0,1]^n \to [0,1] |\ O \textit{ satisfies all the properties in } P_{\mathcal{O}}\}$$

where

$$P_{\mathcal{O}} = \{\textbf{(O1)}, \textbf{(O2)}, \textbf{(O3)}, \textbf{(O4)}, \textbf{(O5)}\},$$

and

(O1) *O is symmetric;*

(O2) *$O(\vec{x}) = 0 \Leftrightarrow \prod_{i=1}^n x_i = 0$;*

(O3) $O(\vec{x}) = 1 \Leftrightarrow \prod_{i=1}^{n} x_i = 1$*;*

(O4) *O is increasing;*

(O5) *O is continuous.*

A 2-dimensional overlap function is just called overlap function [7, 11].

Remark 2.1. *Taking into consideration Definitions 2.3 and 2.4, one can observe that conditions **(A1)** and **(O4)** are the same one (increasingness). However, we decide to label them differently so that each condition is associated with one respective class of functions, to aid the readability of the mathematical proofs in this paper.*

Example 2.3. **i)** *The function $O_P : [0,1]^n \to [0,1]$ (product overlap), given, for all $\vec{x} \in [0,1]^n$, by*

$$GM(\vec{x}) = \prod_{i=1}^{n} x_i, \tag{5}$$

is an n-dimensional overlap function.

ii) *The function $GM : [0,1]^n \to [0,1]$ (geometric mean), given, for all $\vec{x} \in [0,1]^n$, by*

$$GM(\vec{x}) = \sqrt[n]{\prod_{i=1}^{n} x_i}, \tag{6}$$

is an n-dimensional overlap function.

Theorem 2.1. *[28] Consider a continuous aggregation function $A : [0,1]^m \to [0,1]$, such that*

(PA) *$A(\vec{x}) = 0$ if and only if $x_i = 0$, for some $i \in \{1, \ldots, m\}$;*

(PB) *$A(\vec{x}) = 1$ if and only if $x_i = 1$, for all $i \in \{1, \ldots, m\}$;*

and a tuple of n-dimensional overlap functions $\overrightarrow{O} = (O_1, \ldots, O_m)$. Then, the mapping $A_{\overrightarrow{O}} : [0,1]^n \to [0,1]$, defined, for all $\vec{x} \in [0,1]^n$, by

$$A_{\overrightarrow{O}}(\vec{x}) = A(O_1(\vec{x}), \ldots, O_m(\vec{x})), \tag{7}$$

is an n-dimensional overlap function.

Corollary 2.1. *[28] Consider an m-dimensional overlap function $OC : [0,1]^m \to [0,1]$ and a tuple of n-dimensional overlap functions $\overrightarrow{O} = (O_1, \ldots, O_m)$. Then, the mapping $OC_{\overrightarrow{O}} : [0,1]^n \to [0,1]$, defined for all $\vec{x} \in [0,1]^n$, by*

$$OC_{\overrightarrow{O}}(\vec{x}) = OC(O_1(\vec{x}), \ldots, O_m(\vec{x})), \tag{8}$$

is an n-dimensional overlap function.

By Corollary 2.1, one can observe that the class of n-dimensional overlap functions is self closed with respect to the generalized composition.

Definition 2.5. *[36] A t-norm is any bivariate fusion function $T \in \mathcal{T}$, such that:*

$$\mathcal{T} = \{T : [0,1]^2 \to [0,1] |\ T \text{ satisfies all the properties in } P_{\mathcal{T}}\}$$

where

$$P_{\mathcal{T}} = \{\textbf{(T1)}, \textbf{(T2)}, \textbf{(T3)}, \textbf{(T4)}\},$$

and

(T1) *T is symmetric;*

(T2) *T is associative;*

(T3) *T has 1 as its neutral element;*

(T4) *T is increasing.*

Example 2.4. **i)** *The function $T_L : [0,1]^2 \to [0,1]$ (Łukasiewicz t-norm), given, for all $x, y \in [0,1]$, by*

$$T_L(x,y) = \max\{x+y-1, 0\}, \tag{9}$$

is a t-norm.

ii) *The function $T_H : [0,1]^2 \to [0,1]$ (Hamacher product), given, for all $x, y \in [0,1]$, by*

$$T_H(x,y) = \begin{cases} 0 & \textit{if } x = y = 0, \\ \frac{xy}{x+y-xy} & \textit{otherwise}, \end{cases} \tag{10}$$

is a t-norm.

Definition 2.6. *[58] An uninorm is any bivariate fusion function $U \in \mathcal{U}$, such that:*

$$\mathcal{U} = \{U : [0,1]^2 \to [0,1] |\ U \textit{ satisfies all the properties in } P_{\mathcal{U}}\}$$

where

$$P_{\mathcal{U}} = \{\textbf{\textit{(U1)}}, \textbf{\textit{(U2)}}, \textbf{\textit{(U3)}}, \textbf{\textit{(U4)}}\},$$

and

(U1) *U is symmetric;*

(U2) *U is associative;*

(U3) *U has a neutral element;*

(U4) *U is increasing.*

Example 2.5. **i)** *Consider $e \in [0,1]$. Then, the function $U_C : [0,1]^2 \to [0,1]$, given, for all $x, y \in [0,1]$, by*

$$U_C(x,y) = \begin{cases} \max\{x,y\} & \textit{if } (x,y) \in [e,1]^2, \\ \min\{x,y\} & \textit{otherwise}, \end{cases} \tag{11}$$

is an uninorm with e as its neutral element;

ii) *The function $U_P : [0,1]^2 \to [0,1]$, given, for all $x, y \in [0,1]$, by*

$$U_P(x,y) = \begin{cases} 0 & \textit{if } (x,y) \in \{(1,0),(0,1)\}, \\ \frac{xy}{(1-x)+(1-y)+xy} & \textit{otherwise}, \end{cases} \tag{12}$$

is an uninorm with $\frac{1}{2}$ as its neutral element.

3. $\mathcal{F}$-shifted (a,b)-fusion functions

The main goal of this section is to introduce a general framework to define new classes of functions with similar behaviour as some known subclasses of fusion/aggregation functions, but that are not limited to the unit interval. The idea is to define those new classes of functions (acting on an interval $[a,b]$) through a sets of properties that mirrors the ones from the known functions (acting on $[0,1]$).

From the remainder of this paper, consider $a, b \in \mathbb{R}$, such that $a < b$.

Definition 3.1. *An (a,b)-fusion function is an arbitrary function $F^{a,b} : [a,b]^n \to [a,b]$.*

It is clear that every fusion function is an (a,b)-fusion function for $a = 0$ and $b = 1$. Then, henceforward, every $(0,1)$-fusion function is called here just as fusion function.

We denote by $\mathcal{F}^{a,b}$ a subclass of (a,b)-fusion functions determined by a set of constitutive properties $P_{\mathcal{F}^{a,b}}$.

The action of *shifting* a property **(P1)** of a function $F_1 : [a_1,b_1]^n \to [a_1,b_1]$ from $[a_1,b_1]$ to $[a_2,b_2]$ is to "rewrite" **(P1)** so that it conveys the same concept in the context of $[a_2,b_2]$, resulting in a property **(P2)** of a function $F_2 : [a_2,b_2]^n \to [a_2,b_2]$. In other words, **(P2)** is the counterpart in $[a_2,b_2]$ for the property **(P1)** (see Example 3.1). Some properties can be shifted without any rewriting (e.g., monotonicity, continuity, associativity and idempotency). However, boundary conditions, in general, have to be rewritten when shifted.

Example 3.1. *Suppose that we intend to define a property **(A2')** that conveys the boundary conditions of a function $F : [-10,10]^n \to [-10,10]$ by shifting the property **(A2)** of aggregation functions (Definition 2.3). It is clear that **(A2)** is written taking into consideration the boundaries of $[0,1]$, since aggregation functions are defined on the unit interval. So, a natural way to shift **(A2)** from $[0,1]$ to $[-10,10]$ is to rewrite it by changing the lower and upper boundaries accordingly, resulting in **(A2')** as follows:*

(A2') $A(-10,\ldots,-10) = -10$ *and* $A(10,\ldots,10) = 10$.

Remark 3.1. *A given property in the context of the interval $[0,1]$ can be defined for a general interval $[a,b]$ in different ways, so that it coincides with the original definition when $a = 0$ and $b = 1$. This is the case of the 1-Lipschitz property [29]. A bivariate fusion function $F : [0,1]^2 \to [0,1]$ has this property if, for all $x_1, x_2, y_1, y_2 \in [0,1]$, one has that:*

$$|F(x_1,y_1) - F(x_2,y_2)| \leq |x_1 - x_2| + |y_1 - y_2|. \tag{13}$$

Observe that this property, expressed by Inequality (13), can be defined without modifications for (a,b)-fusion functions. Now, consider the following expression for a property of a bivariate (a,b)-fusion function $F^{a,b} : [a,b]^2 \to [a,b]$:

$$|F^{a,b}(x_1,y_1) - F^{a,b}(x_2,y_2)| \leq \frac{|x_1 - x_2| + |y_1 - y_2|}{(b-a)^k}, k \in [0,+\infty), \tag{14}$$

for all $x_1, x_2, y_1, y_2 \in [a,b]$. The property expressed by Equation (14) coincides with the 1-Lipschitz property in the particular case when $k = 0$, or when $b - a = 1$. However, it is clear that the properties expressed by Equations (13) and (14) are not equivalent, that is, they do not convey the same concept. That is why, when shifting the 1-Lipschitz property from $[0,1]$ to $[a,b]$, one should express it by Equation (13), without rewriting it, in order to avoid introducing a different concept.

Definition 3.2. *Let $\mathcal{F}$ be the subclass of fusion functions $F : [0,1]^n \to [0,1]$ determined by the set of constitutive properties $P_{\mathcal{F}}$, defined in Equation (2). Then, a set of constitutive properties $P_{\mathcal{F}^{a,b}}$ of a class of (a,b)-fusion functions $\mathcal{F}^{a,b}$ is said to be $\mathcal{F}$-shiftable if $P_{\mathcal{F}}$ coincides with the set composed of all the properties obtained by shifting each property of $P_{\mathcal{F}^{a,b}}$ from $[a,b]$ to $[0,1]$.*

Definition 3.3. *Let $P_{\mathcal{F}}$ be the set of constitutive properties of a class of fusion functions $\mathcal{F}$. Then, $\mathcal{F}^{a,b}$, given by*

$$\mathcal{F}^{a,b} = \{F^{a,b} : [a,b]^n \to [a,b] |\ F^{a,b} \text{ satisfies all the properties in } P_{\mathcal{F}}^{a,b}\}, \tag{15}$$

is said to be $\mathcal{F}$-shifted if $P_{\mathcal{F}}^{a,b}$ is $\mathcal{F}$-shiftable.

A $\mathcal{F}$-shifted class of (a,b)-fusion functions $\mathcal{F}^{a,b}$ is a counterpart (in $[a,b]$) of a class of fusion function $\mathcal{F}$ (in $[0,1]$).

Example 3.2. **i)** *Consider a subclass of* $(-10,10)$*-fusion functions* $\mathcal{FA}^{-10,10}$*, with its set of constitutive properties* $P_{\mathcal{FA}^{-10,10}}$ *given by:*

$$P_{\mathcal{FA}^{-10,10}} = \{\textbf{(A1')}, \textbf{(A2')}\},$$

where, for all $FA^{-10,10} \in \mathcal{FA}^{-10,10}$*, it holds that:*

(A1') $FA^{-10,10}$ *is increasing;*

(A2') $FA^{-10,10}(-10,\dots,-10) = -10$ *and* $FA^{-10,10}(10,\dots,10) = 10.$

Then, $P_{\mathcal{FA}^{-10,10}}$ *is* $\mathcal{A}$*-shiftable, since we obtain* ***(A1)*** *and* ***(A2)*** *(Definition 2.3), which are the defining properties of* $\mathcal{A}$*, by shifting* ***(A1')*** *and* ***(A2')*** *from* $[-10,10]$ *to* $[0,1]$*. Thus,* $\mathcal{FA}^{-10,10}$ *is an* $\mathcal{A}$*-shifted class of* (a,b)*-fusion functions.*

ii) *Consider the class of* n*-dimensional overlap functions* $\mathcal{O}$ *and a subclass of* (a,b)*-fusion functions* $\mathcal{H}^{a,b}$ *with its set of constitutive properties* $P_{\mathcal{H}^{a,b}}$*, given by:*

$$P_{\mathcal{H}^{a,b}} = \{\textbf{(H1)}, \textbf{(H2)}\},$$

where, for all for all $H^{a,b} \in \mathcal{H}^{a,b}$*, it holds that:*

(H1) $H^{a,b}$ *is symmetric;*

(H2) $H^{a,b}$ *is associative.*

Clearly, $P_{\mathcal{H}^{a,b}}$ *is not* $\mathcal{O}$*-shiftable, since we cannot transpose their properties to the context of the unit interval so that they coincide with the properties from* $P_{\mathcal{O}}$ *(Definition 2.4). Thus,* $\mathcal{H}^{a,b}$ *is not an* $\mathcal{O}$*-shifted class of* (a,b)*-fusion functions. However, if we consider the class* $\mathcal{H}$ *of symmetric and associative fusion functions, then it is immediate that* $\mathcal{H}^{a,b}$ *is* $\mathcal{H}$*-shifted.*

In [8], aggregation functions were already defined in the context of a domain $[a,b]^n$. But here, to avoid confusion, we call them aggregation functions only when $a = 0$ and $b = 1$ (Definition 2.3). Otherwise, we call them (a,b)-aggregation functions, just to standardize the notation. The definition of the class of (a,b)-aggregation function is given as follows:

Definition 3.4. *[8] An* (a,b)*-aggregation function is any function* $A^{a,b} \in \mathcal{A}^{a,b}$*, such that:*

$$\mathcal{A}^{a,b} = \{A^{a,b} : [a,b]^n \to [a,b] \mid A^{a,b} \text{ satisfies all the properties in } P_{\mathcal{A}}^{a,b}\}$$

where

$$P_{\mathcal{A}}^{a,b} = \{\textbf{(A1*)}, \textbf{(A2*)}\},$$

and

(A1*) $A^{a,b}$ *is increasing;*

(A2*) $A^{a,b}(a,\dots,a) = a$ *and* $A^{a,b}(b,\dots,b) = b.$

Example 3.3. **i)** *The arithmetic mean* $AM : [a,b]^n \to [a,b]$*, given by Equation (3), is an* (a,b)*-aggregation function for any arbitrary* $a,b \in \mathbb{R}$*, such that* $a < b$*;*

ii) *The product operation is a* $(0,b)$*-fusion function with* $b \leq 1$ *and an* (a,b)*-fusion function when* $a < 0, b \leq 1$ *and* $a^2 \leq b$ *(e.g.,* $[-1,1]$*). It is only considered an* (a,b)*-aggregation function in the particular case where* $a = 0$ *and* $b = 1$*. However, in Section 4 we present a construction method in which one can obtain an* (a,b)*-aggregation function based on the product (or any other aggregation function, for that matter) for any arbitrary* $a,b \in \mathbb{R}$*, such that* $a < b$*.*

The following results are immediate:

Proposition 3.1. *Consider the class of aggregation functions $\mathcal{A}$ and its set of constitutive properties $P_{\mathcal{A}}$ (from Definition 2.3). Then the set of properties $P_{\mathcal{A}}^{a,b}$ (from Definition 3.4) is $\mathcal{A}$-shiftable.*

Corollary 3.1. *The class $\mathcal{A}^{a,b}$ of (a,b)-aggregation functions (Definition 3.4) is $\mathcal{A}$-shifted.*

3.1. Some $\mathcal{A}$-shifted subclasses of (a,b)-aggregation functions

Here we study some $\mathcal{A}$-shifted subclasses of (a,b)-aggregation functions.

Analogous to Definition 3.3 of $\mathcal{F}$-shifted subclasses of (a,b)-fusion functions, one can define $\mathcal{A}$-shifted subclasses of (a,b)-aggregation functions, as follows:

Definition 3.5. *Let $P_{\mathcal{A}'}$ be the set of constitutive properties of a subclass of aggregation functions $\mathcal{A}'$. Then, $\mathcal{A}'^{a,b}$, given by*

$$\mathcal{A}'^{a,b} = \{A'^{a,b} : [a,b]^n \to [a,b] \mid A'^{a,b} \text{ satisfies all the properties in } P_{\mathcal{A}'}^{a,b}\}, \tag{16}$$

is said to be $\mathcal{A}'$-shifted if $P_{\mathcal{A}'}^{a,b}$ is $\mathcal{A}'$-shiftable.

Observe that any $\mathcal{A}$-shifted subclass of (a,b)-aggregation functions is also an $\mathcal{A}$-shifted subclass of (a,b)-fusion functions.

Now, let us define different $\mathcal{A}'$-shifted subclasses of (a,b)-aggregation functions $\mathcal{A}'^{a,b} \subseteq \mathcal{A}^{a,b}$, based on a subclass of aggregation functions $\mathcal{A}' \subseteq \mathcal{A}$. First, for a given subclass $\mathcal{A}'^{a,b}$, one must define its set of constitutive properties $P_{\mathcal{A}'}^{a,b}$ in a way for it to be $\mathcal{A}'$-shiftable.

Example 3.4. *Suppose that we intend to define an $\mathcal{O}$-shifted subclass $\mathcal{O}^{a,b}$ of (a,b)-aggregation functions as the counterpart in $[a,b]$ for the class of n-dimensional overlap functions $\mathcal{O}$ (Definition 2.4). For that, we have to define the set of constitutive properties $P_{\mathcal{O}^{a,b}}$ in a way for it to be $\mathcal{O}$-shiftable, that is, so that $P_{\mathcal{O}^{a,b}} = P_{\mathcal{O}}$ when shifting the properties of $P_{\mathcal{O}^{a,b}}$ from $[a,b]$ to $[0,1]$.*

*From Definition 2.4, we see that the set $P_{\mathcal{O}}$ has three properties that can be shifted without rewriting them: **(O1)**, **(O4)** and **(O5)**. So, these three properties can be part of the set $P_{\mathcal{O}^{a,b}}$. However, properties **(O2)** and **(O3)** are the lower and upper boundary conditions, respectively, and, thus, they depend on the values of such boundaries (0 and 1). Also, they are defined by means of the product operation which, in the context of the interval $[0,1]$, has the lower boundary as its annihilator element and the upper boundary as its neutral element. This characteristic is not carried when defining such boundary conditions on a different interval $[a,b]$.*

*So, it is clear that we cannot simply exchange 0 for the left endpoint (a) on condition **(O2)** and 1 for right endpoint (b) on condition **(O3)** to obtain the analogous boundary conditions for $P_{\mathcal{O}^{a,b}}$. There are more than one way to define such boundary conditions so that they are equivalent to **(O2)** and **(O3)** when $a = 0$ and $b = 1$. Here we present a viable alternative. Considering an (a,b)-fusion function $O^{a,b} : [a,b]^n \to [a,b]$, the following properties complete the set $P_{\mathcal{O}^{a,b}}$:*

(OAB1) *$O^{a,b}$ is symmetric;*

(OAB2) *$O^{a,b}(x_1, \ldots, x_n) = a$ if and only if $\prod_{i=1}^{n}(x_i - a) = 0$;*

(OAB3) *$O^{a,b}(x_1, \ldots, x_n) = b$ if and only if $\prod_{i=1}^{n}(\frac{x_i - a}{b-a}) = 0$;*

(OAB4) *$O^{a,b}$ is increasing;*

(OAB5) *$O^{a,b}$ is continuous.*

*One can observe that **(OAB2)** and **(OAB3)** are equivalent to **(O2)** and **(O3)**, respectively, when $a = 0$ and $b = 1$, since the relevant properties of the product operation are respected in $[0,1]$. The other three properties were just relabelled to not mix the notation. Thus, the set of properties $P_{\mathcal{O}^{a,b}} = \{$**(OAB1)**, **(OAB2)**, **(OAB3)**, **(OAB4)**, **(OAB5)**$\}$ is $\mathcal{O}$-shiftable.*

Based on the set of properties $P_{\mathcal{O}^{a,b}}$ defined in Example 3.4, one can define the class of n-dimensional (a,b)-overlap functions.

Definition 3.6. *The class $\mathcal{O}^{a,b}$ of n-dimensional (a,b)-overlap functions $O^{a,b}$ is given by:*

$$\mathcal{O}^{a,b} = \{O^{a,b} : [a,b]^n \to [a,b] |\ O^{a,b} \text{ satisfies all the properties in } P_{\mathcal{O}^{a,b}}\} \tag{17}$$

*where $P_{\mathcal{O}^{a,b}} = \{$**(OAB1), (OAB2),(OAB3), (OAB4), (OAB5)**$\}$.*

Proposition 3.2. *Consider the class of n-dimensional (a,b)-overlap functions $\mathcal{O}^{a,b}$ (Definition 3.6). Then, $\mathcal{O}^{a,b}$ is $\mathcal{O}$-shifted.*

PROOF. Immediate, since $\mathcal{O} \subseteq \mathcal{A}$ and, as shown in Example 3.4, $P_{\mathcal{O}^{a,b}}$ is $\mathcal{O}$-shiftable. □

Example 3.5. i) *The function $MIN : [a,b]^n \to [a,b]$, given, for all $\vec{x} \in [a,b]^n$, by*

$$MIN(\vec{x}) = \min\{x_1, \ldots, x_n\}, \tag{18}$$

is an n-dimensional (a,b)-overlap function;

ii) *The geometric mean, given by Equation (6), is only an n-dimensional (a,b)-overlap function when $a = 0$ and $b > 0$. In Section 4, we present a construction method to obtain an n-dimensional (a,b)-overlap function $O^{a,b}$ based on a given n-dimensional overlap function O (e.g., the geometric mean), for any arbitrary $a, b \in \mathbb{R}$, such that $a < b$.*

Since n-dimensional (a,b)-overlap functions are defined by shifting the properties of Definition 2.4 from $[0,1]$ to $[a,b]$, then some other properties of n-dimensional overlap functions that are not explicitly stated on their definition can also be shifted in a similar manner. The next two results exemplify that the properties expressed by Theorem 2.1 and Corollary 2.1 can be shifted from $[0,1]$ to $[a,b]$:

Theorem 3.1. *Consider a continuous (a,b)-aggregation function $A^{a,b} : [a,b]^m \to [a,b]$, such that*

(PA*) *$A^{a,b}(\vec{x}) = a$ if and only if $x_i = a$, for some $i \in \{1, \ldots, m\}$;*

(PB*) *$A^{a,b}(\vec{x}) = b$ if and only if $x_i = b$, for all $i \in \{1, \ldots, m\}$;*

and a tuple $\overrightarrow{O^{a,b}} = (O_1^{a,b}, \ldots, O_m^{a,b})$ of n-dimensional (a,b)-overlap functions. Then, the mapping $A^{a,b}_{\overrightarrow{O^{a,b}}} : [a,b]^n \to [a,b]$, defined for all $\vec{x} \in [a,b]^n$, by

$$A^{a,b}_{\overrightarrow{O^{a,b}}}(\vec{x}) = A^{a,b}(O_1^{a,b}(\vec{x}), \ldots, O_m^{a,b}(\vec{x})), \tag{19}$$

is an n-dimensional (a,b)-overlap function.

PROOF. It is immediate that $A^{a,b}_{\overrightarrow{O^{a,b}}}$ is well defined. Then, by **(O1)**, **(O4)** and **(O5)**, we have that $A^{a,b}_{\overrightarrow{O^{a,b}}}$ respects conditions **(OAB1)**, **(OAB4)** and **(OAB5)**. Now, let us prove that $A^{a,b}_{\overrightarrow{O^{a,b}}}$ respects the remaining conditions of Definition 3.6:

(OAB2) Suppose that $A^{a,b}_{\overrightarrow{O^{a,b}}}(\vec{x}) = a$, for some $\vec{x} \in [a,b]^n$. Then, by Equation (19) and **(PA*)**, we have that:

$$O_j^{a,b}(\vec{x}) = a \text{ for some } j \in \{1, \ldots, m\} \Leftrightarrow x_i = a \text{ for some } i \in \{1, \ldots, n\} \quad \text{by } \textbf{(OAB2)}.$$

On the other hand, if we take $\vec{x} \in [a,b]^n$, such that $\vec{x} = (x_1, \ldots, x_i, \ldots, x_n)$ with $x_i = a$ for some $i \in \{1, \ldots, n\}$, then, by **(OAB2)**, **(PA*)** and Equation (19), we have that $A^{a,b}_{\overrightarrow{O^{a,b}}}(\vec{x}) = a$.

(OAB3) Suppose that $A_{\overrightarrow{O^{a,b}}}^{a,b}(\vec{x}) = b$, for all $\vec{x} \in [a,b]^n$. Then, by Equation (19) and **(PB*)**, it follows that:

$$O_j^{a,b}(\vec{x}) = b \text{ for all } j \in \{1, \ldots, m\} \Leftrightarrow \vec{x} = (b, \ldots, b) \quad \text{by } \textbf{(OAB3)}.$$

Conversely, if $\vec{x} = (b, \ldots, b)$, then, by **(OAB3)**, **(A2*)** and Equation (19), we have that $A_{\overrightarrow{O^{a,b}}}^{a,b}(\vec{x}) = b$.

□

Corollary 3.2. *Consider an m-dimensional (a,b)-overlap function $OC^{a,b} : [a,b]^m \to [a,b]$ and a tuple $\overrightarrow{O^{a,b}} = (O_1^{a,b}, \ldots, O_m^{a,b})$ of n-dimensional (a,b)-overlap functions. Then, the mapping $OC_{\overrightarrow{O^{a,b}}} : [a,b]^n \to [a,b]$, defined for all $\vec{x} \in [a,b]^n$, by*

$$OC_{\overrightarrow{O^{a,b}}}(\vec{x}) = OC^{a,b}(O_1^{a,b}(\vec{x}), \ldots, O_m^{a,b}(\vec{x})), \tag{20}$$

is an n-dimensional (a,b)-overlap function.

Proof. Immediate, since $OC^{a,b}$ is a continuous (a,b)-aggregation function that respects **(PA*)** and **(PB*)**.

Corollary 3.3. *Consider the weighted arithmetic mean $AW^{a,b} : [a,b]^m \to [a,b]$ given, for all $\vec{x} \in [a,b]^n$, by Equation (4), with $\vec{w} \in [0,1]^m$, such that $\sum_{i=1}^m w_i = 1$, and a tuple $\overrightarrow{O^{a,b}} = (O_1^{a,b}, \ldots, O_m^{a,b})$ of n-dimensional (a,b)-overlap functions. Then, the mapping $AW_{\overrightarrow{O^{a,b}}} : [a,b]^n \to [a,b]$, defined, for all $\vec{x} \in [a,b]^n$, by*

$$\begin{aligned} AW_{\overrightarrow{O^{a,b}}}(\vec{x}) &= AW^{a,b}(O_1^{a,b}(\vec{x}), \ldots, O_m^{a,b}(\vec{x})) \\ &= O_1^{a,b}(\vec{x}) \cdot w_1 + \ldots + O_m^{a,b}(\vec{x}) \cdot w_m, \end{aligned} \tag{21}$$

is an n-dimensional (a,b)-overlap function.

Proof. Immediate, since $AW^{a,b}$ is a continuous (a,b)-aggregation function that respects **(PA*)** and **(PB*)**.

Remark 3.2. *Notice that, by Corollary 3.2, one can state that the class of (a,b)-overlap functions is self closed with respect to the generalized composition, and, by Corollary 3.3, one can observe that the convex sum of n-dimensional (a,b)-overlap functions is also an n-dimensional (a,b)-overlap function. These properties are especially useful in practical applications, since one can combine different (a,b)-overlap functions to obtain new functions with the same behaviour.*

Remark 3.3. *In a similar manner in which n-dimensional (a,b)-overlap functions were defined as a counterpart for n-dimensional overlap functions, one could define n-dimensional (a,b)-grouping functions as a counterpart for n-dimensional grouping functions. Since n-dimensional grouping functions are the dual notion of n-dimensional overlap functions, properties such as the one expressed in Corollary 3.2 can also be obtained in the context of n-dimensional (a,b)-grouping functions.*

Other $\mathcal{A}'$-shifted class of (a,b)-aggregation functions can be defined in a similar manner as presented in Example 3.4 and Definition 3.6. To exemplify that, in the following we define (a,b)-t-norms and (a,b)-uninorms.

Definition 3.7. *Consider a bivariate (a,b)-fusion function $T^{a,b} : [a,b]^2 \to [a,b]$ and the following properties:*

(TAB1) *$T^{a,b}$ is symmetric;*

(TAB2) *$T^{a,b}$ is associative;*

(TAB3) *$T^{a,b}$ has b as its neutral element;*

(TAB4) *$T^{a,b}$ is increasing.*

Then, the class $\mathcal{T}^{a,b}$ of (a,b)-t-norms $T^{a,b}$ is given by:

$$\mathcal{T}^{a,b} = \{T^{a,b} : [a,b]^2 \to [a,b] |\ T^{a,b} \text{ satisfies all the properties in } P_{\mathcal{T}^{a,b}}\} \tag{22}$$

where $P_{\mathcal{T}^{a,b}} = \{$***(TAB1), (TAB2),(TAB3),(TAB4)***$\}$.

The following result is immediate:

Proposition 3.3. *Consider the class of t-norms $\mathcal{T}$ (Definition 2.5) and the class of (a,b)-t-norms $\mathcal{T}^{a,b}$ (Definition 3.7). Then, the class $\mathcal{T}^{a,b}$ is $\mathcal{T}$-shifted.*

Example 3.6. i) *The (a,b)-fusion function $T_L^{a,b} : [a,b]^2 \to [a,b]$, given, for all $x, y \in [a,b]$, by*

$$T_L^{a,b}(x,y) = \max\{x + y - b, a\}, \tag{23}$$

is an (a,b)-t-norm. When $a = 0$ and $b = 1$, $T_L^{a,b} = T_L$, which is the Łukasiewicz t-norm, given in Equation (9);

ii) *The function $T_H : [a,b]^2 \to [a,b]$, such that $b > 1$, given by*

$$T_H(x,y) = \begin{cases} a & \text{if } x = y = a, \\ \frac{xy}{x+y-xy} & \text{otherwise,} \end{cases}$$

was inspired by the Hamacher product t-norm, defined in Equation (46), but cannot be an (a,b)-t-norm, since it is not well defined. It is not trivial to define an "Hamacher product-like" (a,b)-t-norm, so we show in Section 4 a construction method to obtain an (a,b)-t-norm $T^{a,b}$ based on any given core t-norm T.

Remark 3.4. *Observe that there is not an analogous result for (a,b)-t-norms as the ones stated in Theorem 3.1 and Corollary 3.2 for n-dimensional (a,b)-overlap functions. Those results derive from the fact that the generalized composition of n-dimensional overlap functions provides an n-dimensional overlap function (Theorem 2.1 and Corollary 2.1), but the same property does not necessarily hold for t-norms.*

Remark 3.5. *In a similar discussion from the one in Remark 3.3, by the duality between t-norms and t-conorms, one can define (a,b)-t-conorms in an analogous way as done with (a,b)-t-norms. In an attempt to keep this paper concise, we will reserve such developments for future works.*

Definition 3.8. *Consider a bivariate (a,b)-fusion function $U^{a,b} : [a,b]^2 \to [a,b]$ and the following properties:*

(UAB1) *$U^{a,b}$ is symmetric;*

(UAB2) *$U^{a,b}$ is associative;*

(UAB3) *$U^{a,b}$ has a neutral element;*

(UAB4) *$U^{a,b}$ is increasing.*

Then, the class $\mathcal{U}^{a,b}$ of (a,b)-uninorms $U^{a,b}$ is given by:

$$\mathcal{U}^{a,b} = \{U^{a,b} : [a,b]^2 \to [a,b] |\ U^{a,b} \text{ satisfies all the properties in } P_{\mathcal{U}^{a,b}}\} \tag{24}$$

where $P_{\mathcal{U}^{a,b}} = \{$***(UAB1), (UAB2),(UAB3),(UAB4)***$\}$.

The following result is immediate:

Proposition 3.4. *Consider the class of uninorms $\mathcal{U}$ (Definition 2.6) and the class of (a,b)-uninorms $\mathcal{U}^{a,b}$ (Definition 3.8). Then, the class $\mathcal{U}^{a,b}$ is $\mathcal{U}$-shifted.*

Example 3.7. **i)** *Consider $q \in [a,b]$. Then, the function $U_C^{a,b} : [a,b]^2 \to [a,b]$, given, for all $x, y \in [a,b]$, by*

$$U_C(x,y) = \begin{cases} \max\{x,y\} & \text{if } x,y \in [q,b], \\ \min\{x,y\} & \text{otherwise}, \end{cases} \tag{25}$$

is an (a,b)-uninorm with q as its neutral element. One may observe that $U_C^{a,b}$ is a counterpart on $[a,b]$ for the uninorm U_C (Equation (11));

ii) *As discussed on Examples 3.5 and 3.6, some aggregation functions are not trivially transposed to obtain an analogous definition on $[a,b]$. That is the case of the U_P uninorm, given by Equation (12). So, in Section 4, we present a construction method to obtain (a,b)-unimorms, based on a choice of any core unimorm, such as U_P.*

Remark 3.6. *In the same manner that uninorms can be seen as a generalization of t-norms and t-conorms, it is immediate that (a,b)-uninorms are a generalization of (a,b)-t-norms and (a,b)-t-conorms.*

4. Construction methods for $\mathcal{F}$-shifted (a,b)-fusion functions

In [54], Wang et al. introduced a construction method for overlap functions on a lattice L based on a "generator triple" composed of an overlap function (which is bivariate) on some lattice M and two complete homomorphisms from L to M, under several constraints. Here, we develop construction methods for any n-dimensional (a,b)-fusion function, with focus on (a,b)-aggregation functions and their subclasses, based on a core fusion function and an increasing bijective function, without imposing any additional constraints.

Consider a fusion function $F : [0,1]^n \to [0,1]$ and an increasing and bijective function $\phi : [a,b] \to [0,1]$ and the (a,b)-fusion function $F_\phi^{a,b} : [a,b]^n \to [a,b]$ given, for all $x_1, \ldots, x_n \in [a,b]$, by

$$F_\phi^{a,b}(x_1, \ldots, x_n) = \phi^{-1}\left(F\left(\phi(x_1), \ldots, \phi(x_n)\right)\right). \tag{26}$$

Then, F is said to be the core function of $F_\phi^{a,b}$. Equation (26) place an important role in the following construction methods. In the remainder of the paper, we denote $F_\phi^{a,b}$ simply by $F^{a,b}$.

Theorem 4.1. *Consider a fusion function $A : [0,1]^n \to [0,1]$, an increasing and bijective function $\phi : [a,b] \to [0,1]$ and an (a,b)-fusion function $A^{a,b} : [a,b]^n \to [a,b]$ given, for all $x_1, \ldots, x_n \in [a,b]$, by*

$$A^{a,b}(x_1, \ldots, x_n) = \phi^{-1}\left(A\left(\phi(x_1), \ldots, \phi(x_n)\right)\right). \tag{27}$$

Then, $A^{a,b}$ is an (a,b)-aggregation function if and only if A is an aggregation function.

PROOF. It is immediate that $A^{a,b}$ is well defined.

($\Leftarrow$) Suppose that A is an aggregation function. Then, let us prove that $A^{a,b}$ has all properties from $P_{\mathcal{A}}^{a,b}$:

(A1*) Let $\vec{x}, \vec{y} \in [a,b]^n$ be such that $\vec{x} \leq \vec{y}$. Since ϕ and A are increasing, then it follows that

$$\vec{x} \leq \vec{y} \Rightarrow A^{a,b}(\vec{x}) \leq A^{a,b}(\vec{y});$$

(A2*) Consider $\vec{a} = (a, \ldots, a)$ and $\vec{b} = (b, \ldots, b)$. Then:

$$\begin{aligned} A^{a,b}(\vec{a}) &= \phi^{-1}\left(A\left(\phi(a), \ldots, \phi(a)\right)\right) \\ &= \phi^{-1}\left(A(0, \ldots, 0)\right) && \text{since } \phi \text{ is bijective and increasing} \\ &= \phi^{-1}(0) && \text{by } \mathbf{(A2)} \\ &= a, \end{aligned}$$

and

$$\begin{aligned} A^{a,b}(\vec{b}) &= \phi^{-1}\left(A\left(\phi(b),\ldots,\phi(b)\right)\right) \\ &= \phi^{-1}\left(A\left(1,\ldots,1\right)\right), && \text{since } \phi \text{ is bijective and increasing} \\ &= \phi^{-1}(1) && \text{by } \textbf{(A2)} \\ &= b. \end{aligned}$$

($\Rightarrow$) Suppose that $A^{a,b}$ is an (a,b)-aggregation function. Now, let us prove that A respects all conditions from Definition 2.3:

(A1) Let $\vec{x},\vec{y} \in [0,1]^n$ be such that $\vec{x} \le \vec{y}$. Then, it holds that $\phi^{-1}(x_i) \le \phi^{-1}(y_i)$, for all $i \in \{1,\ldots,n\}$, since ϕ^{-1} is increasing. From **(A1*)**, one has that:

$$\begin{aligned} & A^{a,b}(\phi^{-1}(x_1),\ldots,\phi^{-1}(x_n)) \le A^{a,b}(\phi^{-1}(y_1),\ldots,\phi^{-1}(y_n)) \\ \Rightarrow\ & \phi^{-1}(A(\phi(\phi^{-1}(x_1)),\ldots,\phi(\phi^{-1}(y_1)))) \le \phi^{-1}(A(\phi(\phi^{-1}(x_1)),\ldots,\phi(\phi^{-1}(y_1)))), && \text{by Equation (27)} \\ \Rightarrow\ & A(x_1,\ldots,x_n) \le A(y_1,\ldots,y_n), && \text{since } \phi^{-1} \text{ is bijective and increasing.} \end{aligned}$$

(A2) From **(A2*)**, one has that:

$$\begin{aligned} & A^{a,b}(a,\ldots,a) = a \\ \Rightarrow\ & \phi^{-1}\left(A\left(\phi(a),\ldots,\phi(a)\right)\right) = a \\ \Rightarrow\ & \phi^{-1}\left(A\left(0,\ldots,0\right)\right) = a \\ \Rightarrow\ & A\left(0,\ldots,0\right) = 0, \end{aligned}$$

and

$$\begin{aligned} & A^{a,b}(b,\ldots,b) = b \\ \Rightarrow\ & \phi^{-1}\left(A\left(\phi(b),\ldots,\phi(b)\right)\right) = b \\ \Rightarrow\ & \phi^{-1}\left(A\left(1,\ldots,1\right)\right) = b \\ \Rightarrow\ & A\left(1,\ldots,1\right) = 1. \end{aligned}$$

□

Example 4.1. *A basic increasing bijection $\phi_A : [b,a] \to [0,1]$ is the only affine transform between $[a,b]$ and $[0,1]$, defined, for all $x \in [a,b]$, by*

$$\phi_A(x) = \left(\frac{x-a}{b-a}\right). \tag{28}$$

More generally, one may consider $\phi_A^p : [a,b] \to [0,1]$, defined, for all $x \in [a,b]$, by

$$\phi_A^p(x) = \left(\frac{x-a}{b-a}\right)^p. \tag{29}$$

Then, let $GM : [0,1]^n \to [0,1]$ be the geometric mean, given by Equation (6). Thus, the (a,b)-fusion function $GM^{a,b} : [a,b]^n \to [a,b]$, given, for all $x_1,\ldots,x_n \in [a,b]$, by

$$GM^{a,b}(x_1,\ldots,x_n) = (\phi_A^p)^{-1}\left(GM\left(\phi_A^p(x_1),\ldots,\phi_A^p(x_n)\right)\right) = {\phi_A}^{-1}\left(GM\left(\phi_A(x_1),\ldots,\phi_A(x_n)\right)\right), \tag{30}$$

is an (a,b)-aggregation function. We can rewrite Equation (30) as follows:

$$GM^{a,b}(x_1,\ldots,x_n) = GM\left(\frac{x_1-a}{b-a},\ldots,\frac{x_n-a}{b-a}\right)\cdot(b-a)+a.$$

Remark 4.1. *It is immediate that any aggregation function $A : [0,1]^n \to [0,1]$ can be the core function of the construction method presented in Theorem 4.1, as it was the case with the geometric mean in Example 4.1. By applying the construction method, one can obtain an analogous (a,b)-aggregation function for any given aggregation function.*

Remark 4.2. *In the context of Theorem 4.1, when considering the basic increasing bijection ϕ_A, shown in Example 4.1, some (a,b)-aggregation functions and their respective core aggregation functions share the same formula. This is the case for positively homogeneous and shift invariant aggregation functions [29], like the Choquet integral [13]. In fact, the (a,b)-Choquet integral, constructed by this method, corresponds to the asymmetric Choquet integral introduced by Denneberg [16]. Hence, all the special instances of this function, such as the minimum, maximum, arithmetic mean, weighted mean and OWA [57], preserve their formulas when applied as the core of the construction method for defining analogous (a,b)-aggregation functions.*

Remark 4.3. *In Theorem 4.1, one could also obtain an (a,b)-aggregation function by considering ϕ as a decreasing bijection. However, for this and the following construction methods, we focus only on applying increasing bijections to facilitate the shifting of properties of the core aggregation function from $[0,1]$ to $[a,b]$.*

Remark 4.4. *More complex ways could be considered for constructing (a,b)-fusion functions based on increasing (or decreasing) bijections, instead of just ϕ and ϕ^{-1}, as defined on Equation (26). For instance, one could consider the monotonic bijections $\eta, \phi_1, \ldots, \phi_n : [a,b] \to [0,1]$ and a fusion function $F : [0,1]^n \to [0,1]$ to construct an (a,b)-fusion $F^{a,b} : [a,b]^n \to [a,b]$, defined, for all $\vec{x} \in [a,b]^n$, by*

$$F^{a,b}(\vec{x}) = \eta^{-1} F(\phi_1(x_1), \ldots, \phi_n(x_n)).$$

With this approach, some shifted properties from F are preserved for $F^{a,b}$ (e.g., $F^{a,b}$ is an (a,b)-aggregation function if and only if F is an aggregation function) but others properties may not be preserved (e.g., symmetry and associativity).

Similar construction methods from the one in Theorem 4.1 can be obtained for different subclasses of (a,b)-aggregation functions.

Theorem 4.2. *Consider a fusion function $O : [0,1]^n \to [0,1]$, an increasing and bijective function $\phi : [a,b] \to [0,1]$ and an (a,b)-fusion function $O^{a,b} : [a,b]^n \to [a,b]$ given, for all $x_1, \ldots, x_n \in [a,b]$, by*

$$O^{a,b}(x_1, \ldots, x_n) = \phi^{-1}\left(O\left(\phi(x_1), \ldots, \phi(x_n)\right)\right), \tag{31}$$

Then, $O^{a,b}$ is an n-dimensional (a,b)-overlap function if and only if O is an n-dimensional overlap function.

PROOF. ($\Rightarrow$) Suppose that $O^{a,b}$ is an n-dimensional (a,b)-overlap function. Then, it is immediate that O is increasing, symmetric and continuous. Let us prove that O respects the remaining conditions of Definition 2.4:

(O2)

$$\begin{aligned}
&O(x_1, \ldots, x_n) = 0 \\
&\Leftrightarrow \quad O(\phi(\phi^{-1}(x_1)), \ldots, \phi(\phi^{-1}(x_n))) = 0, && \text{since } \phi \text{ is bijective} \\
&\Leftrightarrow \quad \phi^{-1}(O(\phi(\phi^{-1}(x_1)), \ldots, \phi(\phi^{-1}(x_n)))) = \phi^{-1}(0) \\
&\Leftrightarrow \quad O^{a,b}(\phi^{-1}(x_1), \ldots, \phi^{-1}(x_n)) = a, && \text{by Equation (31)} \\
&\Leftrightarrow \quad \phi^{-1}(x_i) = a, \text{ for some } i \in \{1, \ldots, n\}, && \text{by } \textbf{(OAB2)} \\
&\Leftrightarrow \quad x_i = 0, \text{ for some } i \in \{1, \ldots, n\}.
\end{aligned}$$

(O3)

$$
\begin{aligned}
& O(x_1,\ldots,x_n)=1 \\
& \Leftrightarrow \quad O(\phi(\phi^{-1}(x_1)),\ldots,\phi(\phi^{-1}(x_n)))=1, && \text{since } \phi \text{ is bijective} \\
& \Leftrightarrow \quad \phi^{-1}(O(\phi(\phi^{-1}(x_1)),\ldots,\phi(\phi^{-1}(x_n))))=\phi^{-1}(1) \\
& \Leftrightarrow \quad O^{a,b}(\phi^{-1}(x_1),\ldots,\phi^{-1}(x_n))=b, && \text{by Equation (31)} \\
& \Leftrightarrow \quad \phi^{-1}(x_i)=a, \ \text{for all } i\in\{1,\ldots,n\}, && \text{by } \textbf{(OAB3)} \\
& \Leftrightarrow \quad x_i=1, \ \text{for all } i\in\{1,\ldots,n\}.
\end{aligned}
$$

($\Leftarrow$) Suppose that O is an n-dimensional overlap function. From **(O1)**, **(O4)** and **(O5)**, we also have that $O^{a,b}$ is symmetric, increasing and continuous. Now, let us prove that it respects the remaining conditions of Definition 3.6:

(OAB2) Suppose that $O^{a,b}(\vec{x})=a$, for some $\vec{x}\in[a,b]^n$. Then, from Equation (31), we have that:

$$a=\phi^{-1}\left(O\left(\phi(x_1),\ldots,\phi(x_n)\right)\right)\Leftrightarrow 0=O\left(\phi(x_1),\ldots,\phi(x_n)\right),$$

since ϕ is increasing and bijective. From **(O2)**, it follows that:

$$\phi(x_i)=0 \text{ for some } i\in\{1,\ldots,n\}\Leftrightarrow x_i=a \text{ for some } i\in\{1,\ldots,n\}.$$

(OAB3) Suppose that $O^{a,b}(\vec{x})=b$, for all $\vec{x}\in[a,b]^n$. Then, from Equation (31), we have that:

$$b=\phi^{-1}\left(O\left(\phi(x_1),\ldots,\phi(x_n)\right)\right)\Leftrightarrow 1=O\left(\phi(x_1),\ldots,\phi(x_n)\right),$$

since ϕ is increasing and bijective. From **(O3)**, it follows that:

$$\phi(x_i)=1 \text{ for all } i\in\{1,\ldots,n\}\Leftrightarrow x_i=b \text{ for all } i\in\{1,\ldots,n\}.$$

$\square$

Example 4.2. *The (a,b)-aggregation function $GM^{a,b}:[a,b]^n\to[a,b]$ defined in Example 4.1 is an n-dimensional (a,b)-overlap function.*

In the next theorem, we show that one can obtain the same n-dimensional (a,b)-overlap function from two distinct methods, both based on a tuple of core n-dimensional overlap functions $\overrightarrow{O}=(O_1,\ldots,O_m)$. One method consists in first obtaining the n-dimensional overlap function $A_{\overrightarrow{O}}$ by the generalized composition of the core n-dimensional overlap functions by an aggregation function A (as in Theorem 2.1), followed by the application of the construction method of Theorem 4.2 taking $A_{\overrightarrow{O}}$ as the core function. The other method consists in first applying both the construction method of Theorem 4.2 m times, one for each core overlap function from $\overrightarrow{O}$, as well as the construction method of Theorem 4.1 with an aggregation function A as the core function, followed by the generalized composition of the m resulting n-dimensional (a,b)-overlap functions $(O_1^{a,b},\ldots,O_m^{a,b})$ by the resulting (a,b)-aggregation function $A^{a,b}$.

Theorem 4.3. *Consider a continuous aggregation function $A:[0,1]^m\to[0,1]$, such that*

(PA) *$A(\vec{x})=0$ if and only if $x_i=0$, for some $i\in\{1,\ldots,m\}$;*

(PB) *$A(\vec{x})=1$ if and only if $x_i=1$, for all $i\in\{1,\ldots,m\}$;*

a tuple $\overrightarrow{O}=(O_1,\ldots,O_m)$ of n-dimensional overlap functions and the n-dimensional overlap function $A_{\overrightarrow{O}}:[0,1]^n\to[0,1]$, defined, for all $\vec{x}\in[0,1]^n$, by

$$A_{\overrightarrow{O}}(\vec{x})=A(O_1(\vec{x}),\ldots,O_m(\vec{x})). \tag{32}$$

$$
\begin{tikzcd}
\overrightarrow{O} \arrow[r, "Th.\ 4.2"] \arrow[d, "Th.\ 2.1"'] & \overrightarrow{O^{a,b}} \arrow[d, "Th.\ 3.1"] \\
A_{\overrightarrow{O}} \arrow[r, "Th.\ 4.1"'] & A^{a,b}_{\overrightarrow{O}} = OC^{a,b}
\end{tikzcd}
$$

Figure 1: Commutative diagram of the construction methods of an n-dimensional (a, b)-overlap function based on a tuple of n-dimensional overlap functions.

Also, consider an increasing and bijective function $\phi : [a, b] \to [0, 1]$, the n-dimensional (a, b)-overlap function $A^{a,b}_{\overrightarrow{O}} : [a, b]^n \to [a, b]$ given, for all $\vec{y} \in [a, b]^n$, by

$$A^{a,b}_{\overrightarrow{O}}(\vec{y}) = \phi^{-1}\left(A_{\overrightarrow{O}}(\phi(y_1), \ldots, \phi(y_n))\right), \tag{33}$$

the (a, b)-aggregation function $A^{a,b} : [a, b]^m \to [a, b]$, given, for all $\vec{z} \in [a, b]^m$, by

$$A^{a,b}(\vec{z}) = \phi^{-1}\left(A(\phi(z_1), \ldots, \phi(z_m))\right), \tag{34}$$

the n-dimensional (a, b)-overlap functions $O^{a,b}_1, \ldots, O^{a,b}_m : [a, b]^n \to [a, b]$, given, for all $\vec{y} \in [a, b]^n$, by

$$O^{a,b}_i(\vec{y}) = \phi^{-1}\left(O_i(\phi(y_1), \ldots, \phi(y_n))\right), i \in \{1, \ldots, m\}, \tag{35}$$

and the n-dimensional (a, b)-overlap function $OC^{a,b} : [a, b]^n \to [a, b]$, defined, for all $\vec{y} \in [a, b]^n$ by

$$OC^{a,b}(\vec{y}) = A^{a,b}(O^{a,b}_1(\vec{y}), \ldots, O^{a,b}_m(\vec{y})). \tag{36}$$

Then, it holds that $A^{a,b}_{\overrightarrow{O}} = OC^{a,b}$.

PROOF. Consider $\vec{x} \in [0, 1]^n$ and $\vec{y} \in [a, b]^n$ such that $x_i = \phi(y_i)$ for all $i \in \{1, \ldots, n\}$. As ϕ is bijective, it is immediate that $y_i = \phi^{-1}(x_i)$ for all $i \in \{1, \ldots, n\}$. Then, it follows that:

$$
\begin{aligned}
A^{a,b}_{\overrightarrow{O}}(\vec{y}) &= \phi^{-1}\left(A_{\overrightarrow{O}}(\phi(y_1), \ldots, \phi(y_n))\right), && \text{by Equation (33)} \\
&= \phi^{-1}\left(A_{\overrightarrow{O}}(x_1, \ldots, x_n)\right) \\
&= \phi^{-1}\left(A(O_1(\vec{x}), \ldots, O_m(\vec{x}))\right), && \text{by Equation (32)} \\
&= \phi^{-1}\left(A(\phi(\phi^{-1}(O_1(\vec{x}))), \ldots, \phi(\phi^{-1}(O_m(\vec{x}))))\right), && \text{since } \phi \text{ is bijective} \\
&= A^{a,b}\left(\phi^{-1}(O_1(\vec{x})), \ldots, \phi^{-1}(O_m(\vec{x}))\right), && \text{by Equation (34)} \\
&= A^{a,b}\left(\phi^{-1}(O_1(\phi(y_1), \ldots, \phi(y_n))), \ldots, \phi^{-1}(O_m(\phi(y_1), \ldots, \phi(y_n)))\right) \\
&= A^{a,b}\left(O^{a,b}_1(y_1, \ldots, y_n), \ldots, O^{a,b}_m(y_1, \ldots, y_n)\right), && \text{by Equation (35)} \\
&= OC^{a,b}(\vec{y}), && \text{by Equation (36).}
\end{aligned}
$$

□

Theorem 4.3 shows that the diagram of Figure 1 commutes, where $\overrightarrow{O} = (O_1, \ldots, O_m)$ and $\overrightarrow{O^{a,b}} = (O^{a,b}_1, \ldots, O^{a,b}_m)$.

Theorem 4.4. *Consider a bivariate fusion function $T : [0, 1]^2 \to [0, 1]$, an increasing and bijective function $\phi : [a, b] \to [0, 1]$ and a bivariate (a, b)-fusion function $T^{a,b} : [a, b]^2 \to [a, b]$ given, for all $x, y \in [a, b]$, by*

$$T^{a,b}(x, y) = \phi^{-1}\left(T\left(\phi(x), \phi(y)\right)\right), \tag{37}$$

Then, $T^{a,b}$ is an (a, b)-t-norm if and only if T is a t-norm.

PROOF. ($\Rightarrow$) Suppose that $T^{a,b}$ is an (a,b)-t-norm. Then, it is immediate that T is symmetric **(T1)** and increasing **(T4)**. Let us prove the remaining conditions:

(T2) From **(TAB2)**, one has that, for all $x, y, z \in [0,1]$:

$$\begin{aligned}
&T^{a,b}(T^{a,b}(\phi^{-1}(x), \phi^{-1}(y)), \phi^{-1}(z)) = T^{a,b}(\phi^{-1}(x), T^{a,b}(\phi^{-1}(y), \phi^{-1}(z))) \\
&\quad \Rightarrow \quad T^{a,b}(\phi^{-1}(T(\phi(\phi^{-1}(x)), \phi(\phi^{-1}(y)))), \phi^{-1}(z)) = T^{a,b}(\phi^{-1}(x), \phi^{-1}(T(\phi(\phi^{-1}(y)), \phi(\phi^{-1}(z))))), \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{by Equation (37)} \\
&\quad \Rightarrow \quad \phi^{-1}(T(T(x,y),z)) = \phi^{-1}(T(x, T(y,z))), \qquad \text{since } \phi \text{ is bijective} \\
&\quad \Rightarrow \quad T(T(x,y),z) = T(x, T(y,z)),
\end{aligned}$$

which means that T is associative.

(T3) From **(TAB3)**, one has that, for all $x \in [0,1]$:

$$\begin{aligned}
&T^{a,b}(\phi^{-1}(x), b) = T^{a,b}(b, \phi^{-1}(x)) = \phi^{-1}(x) \\
&\quad \Rightarrow \quad \phi^{-1}(T(\phi(\phi^{-1}(x)), \phi(b))) = \phi^{-1}(x), \qquad \text{by Equation (37)} \\
&\quad \Rightarrow \quad \phi^{-1}(T(x,1)) = \phi^{-1}(x), \qquad \text{since } \phi \text{ is bijective} \\
&\quad \Rightarrow \quad T(x,1) = x,
\end{aligned}$$

which implies that T has 1 as its neutral element. Since T is symmetric and increasing, the result follows.

Thus, T is a t-norm.

($\Leftarrow$) Suppose that T is a t-norm. From **(T1)** and **(T4)**, we also have that $T^{a,b}$ is symmetric and increasing. Now, let us prove the remaining conditions:

(TAB2) For all $x, y, z \in [a,b]$, one has that:

$$\begin{aligned}
T^{a,b}(T^{a,b}(x,y),z) &= \phi^{-1}(T(\phi(T^{a,b}(x,y)), \phi(z))) && \text{by Equation (37)} \\
&= \phi^{-1}(T(T(\phi(x), \phi(y)), \phi(z))) && \text{since } \phi \text{ is bijective} \\
&= \phi^{-1}(T(\phi(x), T(\phi(y), \phi(z)))) && \text{by } \textbf{(T2)} \\
&= \phi^{-1}(T(\phi(x), \phi(T^{a,b}(y,z)))) \\
&= T^{a,b}(x, T^{a,b}(y,z)),
\end{aligned}$$

showing that $T^{a,b}$ is associative.

(TAB3) For all $x \in [a,b]$, it holds that:

$$\begin{aligned}
T^{a,b}(x,b) &= \phi^{-1}(T(\phi(x), \phi(b))), && \text{by Equation (37)} \\
&= \phi^{-1}(T(\phi(x), 1)), && \text{since } \phi \text{ is bijective} \\
&= \phi^{-1}(\phi(x)) && \text{by } \textbf{(T3)} \\
&= x.
\end{aligned}$$

Since $T^{a,b}$ is symmetric, it follows that b is its neutral element. □

Example 4.3. *Consider the Hamacher product $T_H : [0,1]^2 \to [0,1]$, given by Equation (46), and $\phi_A^p : [a,b] \to [0,1]$, defined in Equation (29). Then, the (a,b)-fusion function $T_H^{a,b} : [a,b]^2 \to [a,b]$, given, for all $x, y \in [a,b]$, by*

$$T_H^{a,b}(x,y) = (\phi_A^p)^{-1} \left(T_H \left(\phi_A^p(x), \phi_A^p(y) \right) \right), \tag{38}$$

is an (a,b)-t-norm. By taking $p = 1$, we can rewrite Equation (38) as follows:

$$T_H^{a,b}(x,y) = T_H \left(\frac{x-a}{b-a}, \frac{y-a}{b-a} \right) \cdot (b-a) + a.$$

Remark 4.5. *It is clear that, in the context of Theorem 4.4, when $a = 0$ and $b = 1$, Equation (37) provides a t-norm. In this case, if $T = T_P$ (the product t-norm), then the constructed t-norm $T^{0,1}$ is a continuous strict t-norm (strictly increasing in $(0, 1]$). If $T = T_L$ (Łukasiewicz t-norm, given in Equation (9)), then the constructed $T^{0,1}$ is a continuous nilpotent t-norm.*

Theorem 4.5. *Consider $e \in [0, 1]$, $q \in [a, b]$, a bivariate fusion function $U : [0, 1]^2 \to [0, 1]$, an increasing and bijective function $\phi : [a, b] \to [0, 1]$, such that $\phi(q) = e$, and a bivariate (a, b)-fusion function $U^{a,b} : [a, b]^2 \to [a, b]$ given, for all $x, y \in [a, b]$, by*

$$U^{a,b}(x, y) = \phi^{-1} (U (\phi(x), \phi(y))) , \tag{39}$$

Then, $U^{a,b}$ is an (a, b)-uninorm with q as its neutral element if and only if U is an uninorm with e as its neutral element.

PROOF. Analogous to the proof of Theorem 4.4.

Example 4.4. *Consider the unimorm $U_P : [0, 1]^2 \to [0, 1]$, given by Equation (12), and $\phi^p_A : [a, b] \to [0, 1]$, defined in Equation (29). Then, the (a, b)-fusion function $U^{a,b}_P : [a, b]^2 \to [a, b]$, given, for all $x, y \in [a, b]$, by*

$$U^{a,b}_P(x, y) = (\phi^p_A)^{-1} (U_P (\phi^p_A(x), \phi^p_A(y))) , \tag{40}$$

is an (a, b)-uninorm. By taking $p = 1$, we can rewrite Equation (40) as follows:

$$U^{a,b}_P(x, y) = U_P \left(\frac{x - a}{b - a}, \frac{y - a}{b - a} \right) \cdot (b - a) + a.$$

Remark 4.6. *In the same manner that uninorms were defined as a generalization of t-norms and t-conorms [58], it is immediate that (a, b)-uninorms are a generalization of $(a, b$-t-norms and $(a, b$-t-conorms.*

5. Study of some properties of (a, b)-aggregation functions

In this section, we analyze some properties of (a, b)-aggregation functions, in particular, the cases in which the properties of the core aggregation functions are preserved/shifted when constructing an analogous (a, b)-aggregation functions via the previously introduced construction methods.

5.1. Idempotency and averaging properties

A fusion function $F : [0, 1]^n \to [0, 1]$ is idempotent [29] if, for all $x \in [0, 1]$, it holds that:

$$F(x, \dots, x) = x. \tag{41}$$

Clearly, idempotency can be analogously defined for (a, b)-fusion functions.

Proposition 5.1. *Let $A^{a,b} \in \mathcal{A}^{a,b}$ be an (a, b)-aggregation function. Then, $A^{a,b} |_{[c,d]}$ is a (c, d)-aggregation function for all $[c, d] \subseteq [a, b]$ if and only if $A^{a,b}$ is idempotent.*

PROOF. ($\Rightarrow$) Suppose that $A^{a,b} |_{[c,d]}$ is a (c, d)-aggregation function for all $[c, d] \subseteq [a, b]$. Then, for all $[c, d] \subseteq [a, b]$, it holds that $A^{a,b}(c, \dots, c) = c$ and $A^{a,b}(d, \dots, d) = d$, meaning that $A^{a,b}(x, \dots, x) = x$, for all $x \in [a, b]$;

($\Leftarrow$) Now, suppose that $A^{a,b}$ is idempotent. Then, it is immediate that $A^{a,b} |_{[c,d]}$ is increasing and idempotent. Moreover, for all $[c, d] \subseteq [a, b]$, it follows that $A^{a,b} |_{[c,d]} (c, \dots, c) = c$ and $A^{a,b} |_{[c,d]} (d, \dots, d) = d$, meaning that $A^{a,b} |_{[c,d]}$ is a (c, d)-aggregation function. □

Theorem 5.1. *Let $A : [0, 1]^n \to [0, 1]$ be an aggregation function, $\phi : [a, b] \to [0, 1]$ an increasing bijective function and $A^{a,b} : [a, b]^n \to [a, b]$ an (a, b)-aggregation function defined, for all $\vec{x} \in [a, b]^n$, by $A^{a,b}(\vec{x}) = \phi^{-1}(A(\phi(x_1), \dots, \phi(x_n)))$. Then, $A^{a,b}$ is idempotent if and only if A is idempotent.*

PROOF. ($\Rightarrow$) Suppose that $A^{a,b}$ is idempotent. So, for all $x \in [0,1]$, it follows that:

$$\begin{aligned}
&A^{a,b}(\phi^{-1}(x),\ldots,\phi^{-1}(x)) = \phi^{-1}(x) \\
&\quad\Rightarrow \quad \phi^{-1}(A(\phi(\phi^{-1}(x)),\ldots,\phi(\phi^{-1}(x)))) = \phi^{-1}(x), \qquad \text{by Equation (27)} \\
&\quad\Rightarrow \quad A(x,\ldots,x) = x, \qquad \text{since } \phi \text{ is bijective,}
\end{aligned}$$

showing that A is idempotent.

($\Leftarrow$) Suppose that A is idempotent. Thus, for all $x \in [a,b]^n$, it holds that:

$$\begin{aligned}
A^{a,b}(x,\ldots,x) &= \phi^{-1}(A(\phi(x),\ldots,\phi(x))), && \text{by Equation (27)} \\
&= \phi^{-1}(\phi(x)), && \text{since } A \text{ is idempotent,} \\
&= x, && \text{since } \phi \text{ is bijective,}
\end{aligned}$$

which means that $A^{a,b}$ is idempotent. □

A fusion function $F : [0,1]^n \to [0,1]$ is averaging when, for all $\vec{x} \in [0,1]^n$, it holds that:

$$\min\{\vec{x}\} \leq F(\vec{x}) \leq \max\{\vec{x}\}.$$

In the context of aggregation functions, since they are increasing, the idempotency and averaging properties are equivalent [29]. The same holds for (a,b)-aggregation functions, since they are also increasing, and the averaging property can be naturally shifted from $[0,1]$ to $[a,b]$ (the same holds for idempotency). Therefore, the following result is immediate.

Corollary 5.1. *Let $A : [0,1]^n \to [0,1]$ be an aggregation function, $\phi : [a,b] \to [0,1]$ an increasing bijective function and $A^{a,b} : [a,b]^n \to [a,b]$ the (a,b)-aggregation function defined, for all $\vec{x} \in [a,b]^n$, by $A^{a,b}(\vec{x}) = \phi^{-1}(A(\phi(x_1),\ldots,\phi(x_n)))$. Then, $A^{a,b}$ is averaging if and only if A is averaging.*

Example 5.1. i) *The arithmetic mean is an idempotent and averaging (a,b)-aggregation function;*

ii) *The n-dimensional (a,b)-overlap function $GM^{a,b}$, given by Equation (30), is also idempotent and averaging.*

5.2. Generalized migrativity

Consider $\alpha \in [0,1]$. A bivariate fusion function $F : [0,1]^2 \to [0,1]$ is said to be α-migrative [21] if, for all $x, y \in [0,1]$, it holds that:

$$F(\alpha \cdot x, y) = F(x, \alpha \cdot y). \tag{42}$$

In [25], α-migrativity was generalized by replacing both product operations on Equation (42) by a t-norm T, obtaining the concept of (α, T)-migrativity. Humberto et al. [10] generalized this concept by considering an aggregation function B, instead of a t-norm, introducing the (α, B)-migrativity. Qiao and Hu [50] studied the migrativity property for an overlap function O, rewriting Equation (42), with $F = O$ and replacing the first product operation by an overlap function O_1 and the second product operation by an overlap function O_2, resulting in the concept of (α, O_1, O_2)-migrativity for overlap functions. More recently, Qiao [49] introduced a similar definition of migrativity for overlap functions on lattices, where O_1 and O_2 are replaced, respectively, by binary operators A, B on a lattice L, with $\alpha \in L$, named (α, A, B)-migrativity of overlap functions. Inspired by such developments, here we introduce the concept of (α, F_1, F_2)-migrativity of a fusion function F, as follows:

Definition 5.1. *Consider $\alpha \in [0,1]$ and two fusion functions $F_1, F_2 : [0,1]^n \to [0,1]$. A fusion function $F : [0,1]^n \to [0,1]$ is said to be (k, α, F_1, F_2)-migrative if, for all $\vec{x} \in [0,1]^n$, it holds that:*

$$F(F_1(\alpha, x_1), x_2, \ldots, x_n) = F(x_1, \ldots, F_2(\alpha, x_k), \ldots, x_n), \tag{43}$$

for some $k \in \{2,\ldots,n\}$. Whenever, F is (k, α, F_1, F_2)-migrative for all $k \in \{2,\ldots,n\}$, then it is said to be (α, F_1, F_2)-migrative.

However, when constructing an (a,b)-aggregation function as a counterpart of a (generalized) migrative aggregation function, the constructed function, most likely, does not respect any definitions of migrativity that are made in the context of the unit interval. So, here we shift the property of (α, F_1, F_2)-migrativity (Definition 5.1) from $[0,1]$ to $[a,b]$, which results in the following definition:

Definition 5.2. *Consider $\delta \in [a,b]$ and two (a,b)-fusion functions $F_1^{a,b}, F_2^{a,b} : [a,b]^n \to [a,b]$. An (a,b)-fusion function $F^{a,b} : [a,b]^n \to [0,1]$ is said to be $(k, \delta, F_1^{a,b}, F_2^{a,b})$-migrative if, for all $\vec{x} \in [a,b]^n$, it holds that:*

$$F^{a,b}(F_1^{a,b}(\delta, x_1), x_2, \dots, x_n) = F^{a,b}(x_1, \dots, F_2^{a,b}(\delta, x_k), \dots, x_n), \tag{44}$$

for some $k \in \{2, \dots, n\}$. Whenever, $F^{a,b}$ is $(k, \delta, F_1^{a,b}, F_2^{a,b})$-migrative for all $k \in \{2, \dots, n\}$, then it is said to be $(\delta, F_1^{a,b}, F_2^{a,b})$-migrative.

Theorem 5.2. *Let $\phi : [a,b] \to [0,1]$ be an increasing bijective function, $F_1^{a,b}, F_2^{a,b} : [a,b]^2 \to [a,b]$ be two bivariate (a,b)-fusion functions defined, for all $\vec{x} \in [a,b]^2$, by Equation (26), with $F_1, F_2 : [0,1]^2 \to [0,1]$ as their respective core fusion functions, and $A^{a,b} : [a,b]^n \to [a,b]$ be an (a,b)-aggregation function defined, for all $\vec{y} \in [a,b]^n$, by Equation (27), with $A : [0,1]^n \to [0,1]$ as its core aggregation function. Then, for $\delta \in [a,b]$, $A^{a,b}$ is $(\delta, F_1^{a,b}, F_2^{a,b})$-migrative if and only if A is $(\phi(\delta), F_1, F_2)$-migrative.*

PROOF. ($\Rightarrow$) Suppose that $A^{a,b}$ is $(\delta, F_1^{a,b}, F_2^{a,b})$-migrative. So, for all $\delta \in [a,b]$, $\vec{x} \in [0,1]^n$ and $i \in \{2, \dots, n\}$, by Definition 5.2, it follows that:

$$\begin{aligned}
& A^{a,b}(F_1^{a,b}(\delta, \phi^{-1}(x_1)), \phi^{-1}(x_2), \dots, \phi^{-1}(x_n)) = A^{a,b}(\phi^{-1}(x_1), \dots, F_2^{a,b}(\delta, \phi^{-1}(x_i)), \dots, \phi^{-1}(x_n)) \\
\Rightarrow\ & A^{a,b}(\phi^{-1}(F_1(\phi(\delta), \phi(\phi^{-1}(x_1)))), \phi^{-1}(x_2), \dots, \phi^{-1}(x_n)) = \\
& A^{a,b}(\phi^{-1}(x_1), \dots, \phi^{-1}(F_2(\phi(\delta), \phi(\phi^{-1}(x_i)))), \dots, \phi^{-1}(x_n)), && \text{by Equation (26)} \\
\Rightarrow\ & \phi^{-1}(A(\phi(\phi^{-1}(F_1(\phi(\delta), \phi(\phi^{-1}(x_1))))), \phi(\phi^{-1}(x_2)), \dots, \phi(\phi^{-1}(x_n)))) = \\
& \phi^{-1}(A(\phi(\phi^{-1}(x_1)), \dots, \phi(\phi^{-1}(F_2(\phi(\delta), \phi(\phi^{-1}(x_i))))), \dots, \phi(\phi^{-1}(x_n)))), && \text{by Equation (27)} \\
\Rightarrow\ & A(F_1(\phi(\delta), x_1), x_2, \dots, x_n) = A(x_1, \dots, F_2(\phi(\delta), x_i), \dots, x_n), && \text{since } \phi^{-1} \text{ is bijective,}
\end{aligned}$$

showing that A is $(\phi(\delta), F_1, F_2)$-migrative.

($\Leftarrow$) Suppose that A is $(\phi(\delta), F_1, F_2)$-migrative. Thus, for all $\delta, x, \dots, x_n \in [a,b]$ and $i \in \{2, \dots, n\}$, it holds that:

$$\begin{aligned}
A^{a,b}(F_1^{a,b}(\delta, x_1), x_2, \dots, x_n) &= A^{a,b}(\phi^{-1}(F_1(\phi(\delta), \phi(x_1))), x_2, \dots, x_n), && \text{by Equation (26)} \\
&= \phi^{-1}(A(\phi(\phi^{-1}(F_1(\phi(\delta), \phi(x_1)))), \phi(x_2), \dots, \phi(x_n))), \\
& && \text{by Equation (27)} \\
&= \phi^{-1}(A(F_1(\phi(\delta), \phi(x_1)), \phi(x_2), \dots, \phi(x_n))), && \text{since } \phi \text{ is bijective,} \\
&= \phi^{-1}(A(\phi(x_1), \dots, F_2(\phi(\delta), \phi(x_i)), \dots, \phi(x_n))), \\
& && \text{since } A \text{ is } (\phi(\delta), F_1, F_2)\text{-migrative,} \\
&= \phi^{-1}(A(\phi(x_1), \dots, \phi(\phi^{-1}(F_2(\phi(\delta), \phi(x_i)))), \dots, \phi(x_n))), \\
& && \text{since } \phi \text{ is bijective,} \\
&= A^{a,b}(x_1, \dots, \phi^{-1}(F_2(\phi(\delta), \phi(x_i))), \dots, x_n), && \text{by Equation (27)} \\
&= A^{a,b}(x_1, \dots, F_2^{a,b}(\delta, x_i), \dots, x_n), && \text{by Equation (26)}
\end{aligned}$$

which means that $A^{a,b}$ is $(\delta, F_1^{a,b}, F_2^{a,b})$-migrative. □

Corollary 5.2. *Let $\phi : [a,b] \to [0,1]$ be an increasing bijective function, $F_1^{a,b}, F_2^{a,b} : [a,b]^2 \to [a,b]$ be two bivariate (a,b)-fusion functions defined, for all $\vec{x} \in [a,b]^2$, by Equation (26), with $F_1, F_2 : [0,1]^2 \to [0,1]$ as their respective core fusion functions, and $A^{a,b} : [a,b]^n \to [a,b]$ be an (a,b)-aggregation function defined, for all $\vec{y} \in [a,b]^n$, by Equation (27), with $A : [0,1]^n \to [0,1]$ as its core aggregation function. Then, for $\delta \in [a,b]$ and $k \in \{2, \dots, n\}$, $A^{a,b}$ is $(k, \delta, F_1^{a,b}, F_2^{a,b})$-migrative if and only if A is $(k, \phi(\delta), F_1, F_2)$-migrative.*

Example 5.2. **i)** *Let $\mathfrak{A} = \{A_k : [0,1]^n \to [0,1] \,|k \in \{2,\dots,n\}\}$, where A_k is defined, for all $\vec{x} \in [0,1]^n$, by*

$$A_k(\vec{x}) = \begin{cases} 0, & \text{if } x_k = 0, \\ \frac{\prod_{i=1}^n x_i^2}{x_k}, & \text{otherwise}, \end{cases} \tag{45}$$

be a family of aggregation functions and $F_1, F_2 : [0,1]^2 \to [0,1]$ be two bivariate fusion functions, defined, for all $x, y \in [0,1]$, respectively, by

$$F_1(x,y) = x \cdot y,$$

and

$$F_2(x,y) = x^2 \cdot y,$$

It is immediate that each aggregation function $A_k \in \mathfrak{A}$ is (k, α, F_1, F_2)-migrative, with $\alpha \in [0,1]$. Now, considering an increasing and bijective function $\phi : [a,b] \to [0,1]$, define the (a,b)-fusion functions $F_1^{a,b}, F_2^{a,b} : [a,b]^2 \to [a,b]$, through Equation (26), with F_1 and F_2 as their respective core aggregation functions. Also, define the (a,b)-aggregation functions $A_k : [a,b]^n \to [a,b]$, through Equation (27), with A_k as their core aggregation functions and $k \in \{2,\dots,n\}$. Thus, for $\delta = \phi^{-1}(\alpha)$, one has that every $A_k^{a,b}$ is a $(k, \delta, F_1^{a,b}, F_2^{a,b})$-migrative function. Observe that this result does not imply that, for some specific $k \in \{2,\dots,n\}$, $A_k^{a,b}$ is $(\delta, F_1^{a,b}, F_2^{a,b})$-migrative.

ii) *Consider $\delta \in [a,b]$, the product overlap O_P, given by Equation (5), and let $O_P^{a,b} : [a,b]^n \to [a,b]$ be the n-dimensional (a,b)-overlap function obtained by Theorem 4.2, based on O_P as its core n-dimensional overlap function and an increasing and bijective function $\phi : [a,b] \to [0,1]$. Then, one has that $O_P^{a,b}$ is $(\delta, O_P^{a,b}, O_P^{a,b})$-migrative. If $n = 2$, $(\delta, O_P^{a,b}, O_P^{a,b})$-migrativity is the result of shifting the traditional α-migrativity property from $[0,1]$ to $[a,b]$;*

iii) *Consider $\alpha \in (0,1)$, the overlap function $O_q : [0,1]^2 \to [0,1]$, defined, for all $x, y \in [0,1]$, by $O_q(x,y) = x^q \cdot y^q$, with $q > 0$, and the aggregation function $A : [0,1]^n \to [0,1]$, given, for $\vec{x} \in [0,1]^n$, by*

$$A(\vec{x}) = \begin{cases} \prod_{i=1}^n x_i, & \text{if } x_j \in [0,\alpha] \text{ for some } j \in \{1,\dots,n\} \\ 1, & \text{otherwise}. \end{cases} \tag{46}$$

Then, A is (α, O_q, O_q)-migrative. Observe that, if $n = 2$, A coincides with the function $O^{(\alpha)}$, presented in [50] (Example 3.1), which is an example of a function that is (α, O_1, O_2) migrative, with $O_1 = O_2 = O_q$. Considering an increasing and bijective function $\phi : [a,b] \to [0,1]$, define the (a,b)-overlap function $O_q^{a,b} : [a,b]^2 \to [a,b]$, through Equation (31), with O_q as its core overlap function. Also, define the (a,b)-aggregation function $A^{a,b} : [a,b]^n \to [a,b]$, through Equation (27), with A as its core aggregation function. Then, for $\delta = \phi^{-1}(\alpha)$, $A^{a,b}$ is a $(\delta, O_q^{a,b}, O_q^{a,b})$-migrative function;

iv) *Consider $\alpha \in [0,1]$, the projection function $PROJ_2 : [0,1]^2 \to [a,b]$, given, for all $x, y \in [0,1]$, by $F_2(x,y) = y$, the bivariate arithmetic mean $BAM : [0,1]^2 \to [0,1]$ given, for all $x, y \in [0,1]$, by $BAM^{a,b}(x,y) = \frac{x+y}{2}$ and the projection function $PROJ_1 : [0,1]^n \to [0,1]$, given, for all $\vec{x} \in [0,1]^n$, by $PROJ_1(\vec{x}) = x_1$. Then, one has that $PROJ_1$ is $(\alpha, PROJ_2, BAM)$-migrative. Considering $\delta \in [a,b]$, if we define the functions $PROJ_1^{a,b} : [a,b]^n \to [a,b]$ and $PROJ_2^{a,b}, BAM^{a,b} : [a,b]^2 \to [a,b]$ analogously, then, it is immediate that $PROJ_1^{a,b}$ is $(\delta, PROJ_2^{a,b}, BAM^{a,b})$-migrative.*

5.3. Generalized homogeneity

A fusion function $F : [0,1]^n \to [0,1]$ is said to be homogeneous of order $\gamma \in [0, +\infty)$ if, for any $\alpha, x_1, \dots, x_n \in [0,1]$, it holds that:

$$F(\alpha \cdot x_1, \dots, \alpha \cdot x_n) = \alpha^\gamma \cdot F(x_1, \dots, x_n), \tag{47}$$

considering $0^0 = 0$. This property was generalized in [52], in the form of abstract homogeneity of order 1, by replacing the product operations in Equation (47) by another bivariate fusion function g and applying an automorphism on the parameter α, with $\gamma = 1$. When this automorphism is the identity function, we obtain the g-homogeneity property, defined as follows:

Definition 5.3. *[52] Consider a bivariate fusion function $g : [0,1]^2 \to [0,1]$. A fusion function $F : [0,1]^n \to [0,1]$ is said to be g-homogeneous if, for any $\alpha, x_1, \ldots, x_n \in [0,1]$, it holds that:*

$$F(g(\alpha, x_1), \ldots, g(\alpha, x_n)) = g(\alpha, F(x_1, \ldots, x_n)). \tag{48}$$

As discussed for the generalized migrativity property, (a,b)-aggregation functions constructed based on g-homogeneous aggregation functions are not expected to be g-homogeneous, since g is not an (a,b)-fusion function. So, let us shift the g-homogeneity property from $[0,1]$ to $[a,b]$, as follows:

Definition 5.4. *Consider a bivariate (a,b)-fusion function $g^{a,b} : [a,b]^2 \to [a,b]$. An (a,b)-fusion function $F^{a,b} : [a,b]^n \to [a,b]$ is said to be $g^{a,b}$-homogeneous if, for any $\delta, x_1, \ldots, x_n \in [a,b]$, it holds that:*

$$F^{a,b}(g^{a,b}(\delta, x_1), \ldots, g^{a,b}(\delta, x_n)) = g^{a,b}(\delta, F^{a,b}(x_1, \ldots, x_n)). \tag{49}$$

Theorem 5.3. *Let $\phi : [a,b] \to [0,1]$ be an increasing bijective function, $g^{a,b} : [a,b]^n \to [a,b]$ be an (a,b)-fusion function defined, for all $\vec{x} \in [a,b]^n$, by Equation (26), with $g : [0,1]^2 \to [0,1]$ as its core fusion function, and $A^{a,b} : [a,b]^n \to [a,b]$ be an (a,b)-aggregation function defined, for all $\vec{x} \in [a,b]^n$, by Equation (27), with $A : [0,1]^n \to [0,1]$ as its core aggregation function. Then, $A^{a,b}$ is $g^{a,b}$-homogeneous if and only if A is g-homogeneous.*

PROOF. ($\Rightarrow$) Suppose that $A^{a,b}$ is $g^{a,b}$-homogeneous. So, for all $\delta \in [a,b]$ and $\vec{x} \in [0,1]^n$, it follows that:

$$\begin{aligned}
&A^{a,b}(g^{a,b}(\delta, \phi^{-1}(x_1)), \ldots, g^{a,b}(\delta, \phi^{-1}(x_n))) = g^{a,b}(\delta, A^{a,b}(\phi^{-1}(x_1), \ldots, \phi^{-1}(x_n))) \\
&\Rightarrow \quad A^{a,b}(\phi^{-1}(g(\phi(\delta), \phi(\phi^{-1}(x_1)))), \ldots, \phi^{-1}(g(\phi(\delta), \phi(\phi^{-1}(x_1))))) \\
&\qquad = \phi^{-1}(g(\phi(\delta), \phi(A^{a,b}(\phi^{-1}(x_1), \ldots, \phi^{-1}(x_n))))) && \text{by Equation (26)} \\
&\Rightarrow \quad \phi^{-1}(A(\phi(\phi^{-1}(g(\phi(\delta), \phi(\phi^{-1}(x_1))))), \ldots, \phi(\phi^{-1}(g(\phi(\delta), \phi(\phi^{-1}(x_1))))))) \\
&\qquad = \phi^{-1}(g(\phi(\delta), \phi(\phi^{-1}(A(\phi(\phi^{-1}(x_1)), \ldots, \phi(\phi^{-1}(x_n))))))), && \text{by Equation (27)} \\
&\Rightarrow \quad A(g(\phi(\delta), x_1), \ldots, g(\phi(\delta), x_n)) = g(\phi(\delta), A(x_1, \ldots, x_n)), && \text{since } \phi \text{ is bijective,}
\end{aligned}$$

showing that A is g-homogeneous.

($\Leftarrow$) Suppose that A is g-homogeneous. Thus, for all $\delta, x_1, \ldots, x_n \in [a,b]$, it holds that:

$$\begin{aligned}
A^{a,b}(g^{a,b}(\delta, x_1), \ldots, g^{a,b}(\delta, x_n)) &= A^{a,b}(\phi^{-1}(g(\phi(\delta), \phi(x_1))), \ldots, \phi^{-1}(g(\phi(\delta), \phi(x_n)))), \\
& \qquad \text{by Equation (26)} \\
&= \phi^{-1}(A(\phi(\phi^{-1}(g(\phi(\delta), \phi(x_1)))), \ldots, \phi(\phi^{-1}(g(\phi(\delta), \phi(x_n)))))), \\
& \qquad \text{by Equation (27)} \\
&= \phi^{-1}(A(g(\phi(\delta), \phi(x_1)), \ldots, g(\phi(\delta), \phi(x_n)))) && \text{since } \phi \text{ is bijective,} \\
&= \phi^{-1}(g(\phi(\delta), A(\phi(x_1), \ldots, \phi(x_n)))), && \text{since } A \text{ is } g\text{-homogeneous,} \\
&= \phi^{-1}(g(\phi(\delta), \phi(\phi^{-1}(A(\phi(x_1), \ldots, \phi(x_n)))))), && \text{since } \phi \text{ is bijective,} \\
&= \phi^{-1}(g(\phi(\delta), \phi(A^{a,b}(x_1, \ldots, x_n)))) && \text{by Equation (27)} \\
&= g^{a,b}(\delta, A^{a,b}(x_1, \ldots, x_n)), && \text{by Equation (26)}
\end{aligned}$$

which means that $A^{a,b}$ is $g^{a,b}$-homogeneous. □

Example 5.3. **i)** *Consider the bivariate arithmetic mean $BAM^{a,b} : [a,b]^2 \to [a,b]$ given, for all $x, y \in [a,b]$ by $BAM^{a,b}(x,y) = \frac{x+y}{2}$. Then, the (n-ary) arithmetic mean, given by Equation (3), is a $BAM^{a,b}$-homogeneous (a,b)-aggregation function;*

ii) *Consider the (a,b)-overlap function $BGM^{a,b}$, constructed via Theorem 4.2 with the overlap function $BGM : [0,1]^2 \to [0,1]$, given, for all $x, y \in [0,1]$, by $BGM(x,y) = \sqrt{x \cdot y}$, as its core function. Also, consider the (a,b)-aggregation functions $MIN : [a,b]^n \to [a,b]$, given by Equation (18) (minimum operator), and $MAX : [a,b]^n \to [a,b]$, given, for all $\vec{x} \in [a,b]^n$, by*

$$MAX(\vec{x}) = \max\{x_1, \ldots, x_n\}.$$

Then, MIN, MAX are $BGM^{a,b}$-homogeneous (a,b)-aggregation functions.

6. Towards $\mathcal{F}$-shifted (a,b,c,d)-fusion functions

In Section 3, we presented a framework to define new classes of functions with domain $[a,b]^n$ and codomain $[a,b]$ based on functions with domain $[0,1]^n$ and codomain $[0,1]$. That is, we showed how to define (a,b)-fusion functions based on fusion functions, by shifting their defining properties. Here, we discuss the concepts necessary to develop a similar framework to define classes of functions with domain $[a,b]^n$ and codomain $[c,d]$, such that $c, d \in \mathbb{R}$ and $c < d$. We call those functions as (a,b,c,d)-fusion functions.

Definition 6.1. *An (a,b,c,d)-fusion function is an arbitrary function $F_{a,b}^{c,d} : [a,b]^n \to [c,d]$.*

It is immediate that every fusion function is an (a,b,c,d)-fusion function for $a = c = 0$ and $b = d = 1$. Also, every (a,b)-fusion function is an (a,b,c,d)-fusion function when $a = c$ and $b = d$. So, every $(0,1,0,1)$-fusion function is called just as fusion function and every (a,b,a,b)-fusion function is called just as (a,b)-fusion function.

Properties from either fusion functions or (a,b)-fusion functions can be shifted to the context of (a,b,c,d)-fusion functions, by taking into consideration the domain $[a,b]^n$ and codomain $[c,d]$.

Example 6.1. *Suppose that we intend to shift the property **(A2')** (see Example 3.1) that conveys the boundary conditions of an $(-10,10)$-aggregation function $F : [-10,10]^n \to [-10,10]$ to obtain an analogous property for a $(-10,10,0,10)$-fusion function $H : [-10,10]^n \to [0,10]$. The shifted property **(A2†)** is defined as follows:*

(A2†) $A(-10, \ldots, -10) = 0$ *and* $A(10, \ldots, 10) = 10$.

Based on Definition 3.4, one we define (a,b,c,d)-aggregation functions in the following.

Definition 6.2. *An (a,b,c,d)-aggregation function is any function $A_{a,b}^{c,d} \in \mathcal{A}_{a,b}^{c,d}$, such that:*

$$\mathcal{A}_{a,b}^{c,d} = \{A_{a,b}^{c,d} : [a,b]^n \to [c,d] |\ A_{a,b}^{c,d} \text{ satisfies all the properties in } P_{\mathcal{A}}^\dagger\}$$

where

$$P_{\mathcal{A}}^\dagger = \{\textbf{(A1}^\dagger\textbf{)}, \textbf{(A2}^\dagger\textbf{)}\},$$

and

(A1†) $A_{a,b}^{c,d}$ *is increasing;*

(A2†) $A_{a,b}^{c,d}(a, \ldots, a) = c$ *and* $A_{a,b}^{c,d}(b, \ldots, b) = d$.

Example 6.2. *The bivariate $(-10,10,0,10)$-fusion function $H : [-10,10]^2 \to [0,10]$, given, for all $x, y \in [-10,10]$, by*

$$H(x,y) = \frac{x + y + 20}{4}$$

is a bivariate $(-10,10,0,10)$-aggregation function.

The construction method for (a, b)-aggregation functions presented in Theorem 4.1 can be adapted to obtain a construction method for (a, b, c, d)-aggregation functions based on a core aggregation function.

Theorem 6.1. *Consider a fusion function $A : [0, 1]^n \to [0, 1]$, an increasing and bijective function $\phi : [a, b] \to [0, 1]$, an increasing and bijective function $\psi : [0, 1] \to [c, d]$ and an (a, b, c, d)-fusion function $A^{c,d}_{a,b} : [a, b]^n \to [c, d]$ given, for all $x_1, \ldots, x_n \in [a, b]$, by*

$$A^{c,d}_{a,b}(x_1, \ldots, x_n) = \psi\left(A\left(\phi(x_1), \ldots, \phi(x_n)\right)\right). \tag{50}$$

Then, $A^{c,d}_{a,b}$ is an (a, b, c, d)-aggregation function if and only if A is an aggregation function.

PROOF. Analogous to the proof of Theorem 4.1. □

Remark 6.1. *Observe that Equation (50) is more general than Equation (27), even in the particular case when $[a, b] = [c, d]$, since ψ does not need to be the inverse of ϕ.*

Example 6.3. *Consider the geometric mean $GM : [0, 1]^n \to [0, 1]$, given by Equation (6), an increasing and bijective function $\phi : [a, b] \to [0, 1]$, defined, for all $x \in [a, b]$, by*

$$\phi(x) = \left(\frac{x - a}{b - a}\right)^p, \ p > 0,$$

and an increasing and bijective function $\psi : [0, 1] \to [c, d]$, defined, for all $y \in [0, 1]$, by

$$\psi(y) = y^{\frac{1}{q}} \cdot (d - c) + c, \ q > 0.$$

Then, the (a, b, c, d)-fusion function $GM^{c,d}_{a,b} : [a, b]^n \to [c, d]$, given, for all $x_1, \ldots, x_n \in [a, b]$, by

$$GM^{c,d}_{a,b}(x_1, \ldots, x_n) = \psi\left(GM\left(\phi(x_1), \ldots, \phi(x_n)\right)\right), \tag{51}$$

is an (a, b, c, d)-aggregation function. By taking $p = q = 1$, we can rewrite Equation (51) as follows:

$$GM^{c,d}_{a,b}(x_1, \ldots, x_n) = GM\left(\frac{x_1 - a}{b - a}, \ldots, \frac{x_n - a}{b - a}\right) \cdot (d - c) + c. \tag{52}$$

In the following, we present a construction method for (a, b, c, d)-aggregation function based on a core (a, b)-aggregation function.

Theorem 6.2. *Consider an (a, b)-fusion function $A^{a,b} : [a, b]^n \to [a, b]$, an increasing and bijective function $\theta : [a, b] \to [c, d]$ and an (a, b, c, d)-fusion function $A^{c,d}_{a,b} : [a, b]^n \to [c, d]$ given, for all $x_1, \ldots, x_n \in [a, b]$, by*

$$A^{c,d}_{a,b}(x_1, \ldots, x_n) = \theta\left(A^{a,b}\left(x_1, \ldots, x_n\right)\right). \tag{53}$$

Then, $A^{c,d}_{a,b}$ is an (a, b, c, d)-aggregation function if and only if $A^{a,b}$ is an (a, b)-aggregation function.

PROOF. Analogous to the proof of Theorem 4.1. □

Example 6.4. *Consider the aggregation function $GM : [0, 1]^n \to [0, 1]$, given by Equation (6), the (a, b)-aggregation function $GM^{a,b} : [a, b]^n \to [a, b]$, given by Equation (30), and increasing and bijective function $\theta : [a, b] \to [c, d]$, defined, for all $x \in [a, b]$, by*

$$\theta(x) = \left(\frac{x - a}{b - a}\right) \cdot (d - c) + c. \tag{54}$$

Then, the (a, b, c, d)-fusion function $GM^{c,d}_{a,b} : [a, b]^n \to [c, d]$, given, for all $x_1, \ldots, x_n \in [a, b]$, by

$$GM^{c,d}_{a,b}(x_1, \ldots, x_n) = \theta\left(GM^{a,b}\left(x_1, \ldots, x_n\right)\right), \tag{55}$$

is an (a, b, c, d)-aggregation function. From Equations (30), (54) and (55), one has that:

$$GM^{c,d}_{a,b}(x_1, \ldots, x_n) = GM\left(\frac{x_1 - a}{b - a}, \ldots, \frac{x_n - a}{b - a}\right) \cdot (d - c) + c. \tag{56}$$

$$
\begin{tikzcd}
A : [0,1]^n \to [0,1] \arrow[rr, "Th.\ 4.1"] \arrow[dr, "Th.\ 6.1"'] & & A^{a,b} : [a,b]^n \to [a,b] \arrow[dl, "Th.\ 6.2"] \\
& AR^{c,d}_{a,b} = AS^{c,d}_{a,b} : [a,b]^n \to [c,d] &
\end{tikzcd}
$$

Figure 2: Commutative diagram of the construction methods of an n-dimensional (a,b,c,d)-aggregation function based on a core aggregation function A.

One can observe that Equations (52) and (56) coincide. This fact is derived from the following theorem.

Theorem 6.3. *Let $AR^{c,d}_{a,b} : [a,b]^n \to [c,d]$ be an (a,b,c,d)-aggregation function constructed via Theorem 6.1 using increasing and bijective functions $\phi : [a,b] \to [0,1]$ and $\psi : [0,1] \to [c,d]$, and a core aggregation function $A : [0,1]^n \to [0,1]$. Let $AS^{c,d}_{a,b} : [a,b]^n \to [c,d]$ be an (a,b,c,d)-aggregation function constructed via Theorem 6.2 using an increasing and bijective function $\theta : [a,b] \to [c,d]$ and the core (a,b)-aggregation function $A^{a,b} : [a,b]^n \to [a,b]$, which, in turn, is constructed via Theorem 4.1 using ϕ and the core aggregation function A. Thus, if $\psi = \theta \circ \phi^{-1}$ then $AR^{c,d}_{a,b} = AS^{c,d}_{a,b}$.*

PROOF. For all $\vec{x} \in [a,b]^n$, one has that:

$$
\begin{aligned}
&\psi = \theta \circ \phi^{-1} \\
&\quad \Rightarrow \quad \psi\left(A\left(\phi(x_1), \ldots, \phi(x_n)\right)\right) = (\theta \circ \phi^{-1})(A(\phi(x_1), \ldots, \phi(x_n))) \\
&\quad \Rightarrow \quad \psi\left(A\left(\phi(x_1), \ldots, \phi(x_n)\right)\right) = \theta\left(\phi^{-1}(A(\phi(x_1), \ldots, \phi(x_n)))\right) \\
&\quad \Rightarrow \quad \psi\left(A\left(\phi(x_1), \ldots, \phi(x_n)\right)\right) = \theta\left(A^{a,b}\left(x_1, \ldots, x_n\right)\right), \qquad \text{by Equation (27)} \\
&\quad \Rightarrow \quad AR^{c,d}_{a,b} = AS^{c,d}_{a,b}, \qquad \text{by Equations (50) and (53).}
\end{aligned}
$$

□

Theorem 6.3 shows that the diagram presented in Figure 2 commutes, whenever $\psi = \theta \circ \phi^{-1}$.

7. Conclusion

In this paper, we sought to provide a theoretical tool set to support the definition of new classes of fusion operators that can aggregate data from any real closed interval, based on analogous known classes of such operators that are defined, specifically, on the unit interval. There are many practical applications that can benefit from the developed concepts, in particular with the assurance that the advantageous properties of known aggregation functions can be preserved (shifted) when applying the newly developed functions, even on problems that do not necessarily involve fuzzy modeling.

Here, we review our main contributions:

- The introduction of the concept property shifting, which is a novel denomination for the action of properly transposing properties from one domain to another without sacrificing their fundamental characteristics;
- The development of a general framework for defining (a,b)-fusion functions, possibly in intervals other then $[0,1]$, by shifting the defining properties of known fusion functions;
- The introduction of construction methods for different subclasses of (a,b)-fusion functions, based on choices of a core fusion function and an increasing bijective function, which makes them highly adaptable and prone to be applied in different practical problems;
- The study of both known and newly defined properties of aggregation functions, along with their shifted counterparts in $[a,b]$, and how they are related when we construct (a,b)-aggregation functions via our construction methods;

- The development of a general framework for defining (a, b, c, d)-fusion functions, which is designed in an analogous manner as the one for (a, b)-fusion functions;
- The introduction of construction methods for (a, b, c, d)-aggregation functions, highlighting the different ways one can obtain a given (a, b, c, d)-aggregation function.

Backed by the developed concepts, in an ongoing work, we intend to use (a, b)-aggregation functions in the pooling process of a convolutional neural network, since the aggregated values do not come from the unit interval, to be applied in classification and image processing problems. Also, we see promise in applying (a, b)-fusion functions to generalize the discrete Choquet Integral, defined in any interval $[a, b]$, with possible applications in recurrent neural networks. Future works, regarding the theoretical standpoint, may include a deeper study of particular classes of (a, b)-fusion functions, defined through our framework, with special interest in cases in which the shifting of properties may not be trivial.

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5.2 Complementary publications

In this section, we present the four complementary papers discussed in Section 3.2. Since they are all conferences papers, we provide a brief description on their respective publications and conferences.

5.2.1 General grouping functions

- H. Santos, G. Dimuro, T. Asmus, G. Lucca, E. Bueno, B. Bedregal, J. Sanz and H. Bustince, "General grouping functions", Information Processing and Management of Uncertainty in Knowledge-Based Systems 1238 (2020) 481–495.
  - Conference: Information Processing and Management of Uncertainty in Knowledge-Based Systems IPMU 2020
  - Status: Part of the Communications in Computer and Information Science book series (CCIS, volume 1238)
    - DOI: https://doi.org/10.1007/978-3-030-50143-3_38

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General grouping functions

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Abstract. Some aggregation functions that are not necessarily associative, namely overlap and grouping functions, have called the attention of many researchers in the recent past. This is probably due to the fact that they are a richer class of operators whenever one compares with other classes of aggregation functions, such as t-norms and t-conorms, respectively. In the present work we introduce a more general proposal for disjunctive n-ary aggregation functions entitled general grouping functions, in order to be used in problems that admit n dimensional inputs in a more flexible manner, allowing their application in different contexts. We present some new interesting results, like the characterization of that operator and also provide different construction methods.

Keywords: Grouping functions · n-dimensional grouping functions · General grouping functions · General overlap functions.

1 Introduction

Overlap functions are a kind of aggregation functions [3] that are not required to be associative, and they were introduced by Bustince et al. in [4] to measure the degree of overlapping between two classes or objects. Grouping functions are the dual notion of overlap function. They were introduced by Bustince et al. [5] in order to express the measure of the amount of evidence in favor of either of two alternatives when performing pairwise comparisons [1] in decision making based on fuzzy preference relations [6]. They have also been used as the disjunction operator in some important problems, such as image thresholding [17] and the construction of a class of implication functions for the generation of fuzzy subsethood and entropy measures [8].

Overlap and grouping have been largely studied since they are richer than t-norms and t-conorms [18], respectively, in many aspects, considering, for example, some properties like idempotency, homogeneity, and, mainly, the self-closeness feature with respect to the convex sum and the aggregation by generalized composition of overlap/grouping functions [9,10,12,7]. For example, there is just one idempotent t-conorm (namely, the maximum t-conorm) and two homogeneous t-conorms (namely, the maximum and the probabilistic sum of t-conorms). On the contrary, there are uncountable numbers of idempotent, as well as homogenous, grouping functions [2,13]. For comparisons among properties of grouping functions and t-conorms, see [2,5,17]

However, grouping functions are bivariate functions. Since they may be non associative, they can only be applied in bi-dimensional problems (that is, when just two classes or objects are considered). In order to solve this drawback, Gómez et al. [16] introduced the concept of n-dimensional grouping functions, with an application to fuzzy community detection.

Recently, De Miguel et al. [20] introduced general overlap functions, by relaxing some boundary conditions, in order to apply to an n-ary problem, namely, fuzzy rule based classification systems, more specifically, in the determination of the matching degree in the fuzzy reasoning method. Then, inspired on the paper by De Miguel et al. [20], the objective of this present paper is to introduce the concept of general grouping functions, providing their characterization and different construction methods. The aim is to define the theoretical basis of a tool that can be used to express the measure of the amount of evidence in favor of one of multiple alternatives when performing n-ary comparisons in multi-criteria decision making based on n-ary fuzzy heterogeneous, incomplete preference relations [14,19,26], which we let for future work.

The paper is organized as follows. Section 2 presents some preliminary concepts. In Sect. 3, we define general grouping functions, studying some properties. In Sect. 4, we study the characterization of general grouping functions, providing some construction methods. Section 5 is the Conclusion.

2 Preliminary concepts

In this section, we highlight some relevant concepts used in this work.

Definition 1. *A function $N\colon [0,1] \to [0,1]$ is a fuzzy negation if it holds: $(N1)$ N is antitonic, i.e. $N(x) \leq N(y)$ whenever $y \leq x$ and $(N2)$ $N(0) = 1$ and $N(1) = 0$.*

Definition 2. *[3] An n-ary aggregation function is a mapping $A\colon [0,1]^n \to [0,1]$ satisfying: $(A1)$ $A(0,\ldots,0) = 0$ and $A(1,\ldots,1) = 1$; $(A2)$ increasingness: for each $i \in \{1,\ldots,n\}$, if $x_i \leq y$ then $A(x_1,\ldots,x_n) \leq A(x_1,\ldots,x_{i-1},y,x_{i+1},\ldots,x_n)$.*

Definition 3. *An n-ary aggregation function $A\colon [0,1]^n \to [0,1]$ is called conjunctive if, for any $\overrightarrow{x} = (x_1,\ldots,x_n) \in [0,1]^n$, it holds that $A(\overrightarrow{x}) \leq \min(\overrightarrow{x}) = \min\{x_1,\ldots,x_n\}$. And A is called disjunctive if, for any $\overrightarrow{x} = (x_1,\ldots,x_n) \in [0,1]^n$, it holds that $A(\overrightarrow{x}) \geq \max(\overrightarrow{x}) = \max\{x_1,\ldots,x_n\}$.*

Definition 4. *[4] A binary function $O\colon [0,1]^2 \to [0,1]$ is said to be an overlap function if it satisfies the following conditions, for all $x,y,z \in [0,1]$:*

(O1) $O(x, y) = O(y, x)$;
(O2) $O(x, y) = 0$ *if and only if* $x = 0$ *or* $y = 0$;
(O3) $O(x, y) = 1$ *if and only if* $x = y = 1$;
(O4) *if* $x \leq y$ *then* $O(x, z) \leq O(y, z)$;
(O5) O *is continuous;*

Definition 5. *[5] A binary function* $G\colon [0, 1]^2 \to [0, 1]$ *is said to be a grouping function if it satisfies the following conditions, for all* $x, y, z \in [0, 1]$:

(G1) $G(x, y) = G(y, x)$;
(G2) $G(x, y) = 0$ *if and only if* $x = y = 0$;
(G3) $G(x, y) = 1$ *if and only if* $x = 1$ *or* $y = 1$;
(G4) *If* $x \leq y$ *then* $G(x, z) \leq G(y, z)$;
(G5) G *is continuous;*

For all properties and related concepts on overlap functions and grouping functions, see [2,5,9,11,12,21,23,24,25].

Definition 6. *[22] A function* $G\colon [0, 1]^2 \to [0, 1]$ *is a* 0*-grouping function if the second condition in Def. 5 is replaced by: (G2′) If* $x = y = 0$ *then* $G(x, y) = 0$. *Analogously, a function* $G\colon [0, 1]^2 \to [0, 1]$ *is a* 1*-grouping function if the third condition in Def. 5 is replaced by: (G3′) If* $x = 1$ *or* $y = 1$ *then* $G(x, y) = 1$.

Both notions were extended in several ways and some of them are presented in the following definitions.

Definition 7. *[15] An* n*-ary function* $\mathcal{G}\colon [0, 1]^n \to [0, 1]$ *is called an* n*-dimensional grouping function if for all* $\vec{x} = (x_1, ..., x_n) \in [0, 1]^n$:

1. $\mathcal{G}$ *is commutative;*
2. $\mathcal{G}(\vec{x}) = 0$ *if and only if* $x_i = 0$, *for all* $i = 1, \ldots, n$;
3. $\mathcal{G}(\vec{x}) = 1$ *if and only if there exists* $i \in \{1, \ldots, n\}$ *with* $x_i = 1$;
4. $\mathcal{G}$ *is increasing;*
5. $\mathcal{G}$ *is continuous.*

Definition 8. *[20] A function* $\mathcal{GO}\colon [0, 1]^n \to [0, 1]$ *is said to be a general overlap function if it satisfies the following conditions, for all* $\vec{x} = (x_1, \ldots, x_n) \in [0, 1]^n$:

($\mathcal{GO}$1) $\mathcal{GO}$ *is commutative;*
($\mathcal{GO}$2) *If* $\prod_{i=1}^{n} x_i = 0$ *then* $\mathcal{GO}(\vec{x}) = 0$;
($\mathcal{GO}$3) *If* $\prod_{i=1}^{n} x_i = 1$ *then* $\mathcal{GO}(\vec{x}) = 1$;
($\mathcal{GO}$4) $\mathcal{GO}$ *is increasing;*
($\mathcal{GO}$5) $\mathcal{GO}$ *is continuous.*

3 General grouping functions

Following the ideas given in [20], we can also generalize the idea of general grouping functions as follows.

Definition 9. *A function $\mathcal{GG}\colon [0,1]^n \to [0,1]$ is called a general grouping function if the following conditions hold, for all $\vec{x} = (x_1, \ldots, x_n) \in [0,1]^n$:*

($\mathcal{GG}1$) *$\mathcal{GG}$ is commutative;*
($\mathcal{GG}2$) *If $\sum_{i=1}^{n} x_i = 0$ then $\mathcal{GG}(\vec{x}) = 0$;*
($\mathcal{GG}3$) *If there exists $i \in \{1, \ldots, n\}$ such that $x_i = 1$ then $\mathcal{GG}(\vec{x}) = 1$;*
($\mathcal{GG}4$) *$\mathcal{GG}$ is increasing;*
($\mathcal{GG}5$) *$\mathcal{GG}$ is continuous.*

Note that ($\mathcal{GG}2$) is the same of saying that 0 is an anhilator of the general grouping function $\mathcal{GG}$.

Proposition 1. *If $\mathcal{G}\colon [0,1]^n \to [0,1]$ is an n-dimensional grouping function, then $\mathcal{G}$ is also a general grouping function.*

Proof. Straighforward. □

From this proposition, we can conclude that the concept of general grouping functions is a generalization of n-dimensional grouping functions, which on its turn is a generalization of the concepts of 0-grouping functions and 1-grouping functions.

Example 1.
1. Every grouping function $G\colon [0,1]^2 \to [0,1]$ is a general grouping function, but the converse does not hold.
2. The function $\mathcal{GG}(x,y) = \min\{1, 2-(1-x)^2-(1-y)^2\}$ is a general grouping function, but it is not a bidimensional grouping function, since $\mathcal{GG}(0.5, 0.5) = 1$.
3. Consider $G(x,y) = \max\{1-(1-x)^p, 1-(1-y)^p\}$, for $p > 0$ and $S_{\mathfrak{L}}(x,y) = \min\{1, x+y\}$. Then, the function $\mathcal{GG}^{S_{\mathfrak{L}}}(x,y) = G(x,y)S_{\mathfrak{L}}(x,y)$ is a general grouping function.
4. Take any grouping function G, and a continuous t-conorm S. Then, the generalization of the previous item is the binary general grouping function given by: $\mathcal{GG}(x,y) = G(x,y)S(x,y)$
5. Other examples are:

$$Prod\_S\_Luk(x_1, \ldots, x_n) = \left(1 - \prod_{i=1}^{n}(1-x_i)\right) * \left(\min\left\{\sum_{i=1}^{n} x_i, 1\right\}\right)$$

$$GM\_S\_Luk(x_1, \ldots, x_n) = \left(1 - \sqrt[n]{\prod_{i=1}^{n}(1-x_i)}\right) * \left(\min\left\{\sum_{i=1}^{n} x_i, 1\right\}\right).$$

The generalization of the third item of Example 1 can be seen as follows.

Proposition 2. *Take any grouping function G, and any t-conorm S. Then, the binary general grouping function given by: $\mathcal{GG}(x,y) = G(x,y)S(x,y)$.*

Proposition 3. *Let $F\colon [0,1]^n \to [0,1]$ be a commutative and continuous aggregation function. Then the following statements hold:*

(i) *If F is disjunctive, then F is a general grouping function.*
(ii) *If F is conjunctive, then F is neither a general grouping function nor an n-dimensional grouping function.*

Proof. Consider a commutative and continuous aggregation function $F\colon [0,1]^n \to [0,1]$. It follows that:
(i) Since F is commutative ($\mathcal{GG}1$), continuous ($\mathcal{GG}5$) and clearly increasing ($\mathcal{GG}4$), then it remains to prove the following:
($\mathcal{GG}2$) Suppose that $\sum_{i=1}^{n} x_i = 0$. Then, since F is an aggregation function, it holds that $F(0,\ldots,0) = 0$.
($\mathcal{GG}3$) Suppose that, for some $\vec{x} = (x_1,\ldots,x_n) \in [0,1]^n$, there exists $i \in \{1,\ldots,n\}$ such that $x_i = 1$. Then, since F is disjunctive, then $F(\vec{x}) \geq \max\{x_1,\ldots,1,\ldots,x_n\} = 1$, which means that $F(\vec{x}) = 1$.
(ii) Suppose that F is a conjunctive aggregation function and it is either a general grouping function or an n-dimensional grouping function. Then, by either **(**$\mathcal{GG}3$**)** or **(**$\mathcal{G}3$**)**, if for some $\vec{x} = (x_1 \ldots, x_n) \in [0,1]^n$, there exists $i \in \{1,\ldots,n\}$ such that $x_i = 1$, then $F(\vec{x}) = 1$. Take $\vec{x} = (1, 0 \ldots, 0)$, it follows that $F(1, 0 \ldots, 0) = 1 = \max\{1, 0 \ldots, 0\} \not\leq 0 = \min\{1, 0 \ldots, 0\}$, which is a contradiction with the conjunctive property. Thus, one concludes that F is neither a general grouping function nor an n-dimensional grouping function. □

We say that an element $a \in [0,1]$ is a neutral element of $\mathcal{GG}$ if for each $x \in [0,1]$, $\mathcal{GG}(x, \underbrace{a,\ldots,a}_{(n-1)}) = x$.

Proposition 4. *Let $\mathcal{GG}\colon [0,1]^n \to [0,1]$ be a general grouping function with a neutral element $a \in [0,1]$. Then, $a = 0$ if and only if $\mathcal{GG}$ satisfies, for all $\vec{x} = (x_1,\ldots,x_n) \in [0,1]^n$, the following condition:*

($\mathcal{GG}2'$) *If $\mathcal{GG}(\vec{x}) = 0$, then $\sum_{i=1}^{n} x_i = 0$.*

Proof. ($\Rightarrow$) Suppose that (i) the neutral element of $\mathcal{GG}$ is $a = 0$ and (ii) $\mathcal{GG}(x_1,\ldots,x_n) = 0$. Then, by (i), one has that, for each $x_1 \in [0,1]$, it holds that $x_1 = \mathcal{GG}(x_1, 0 \ldots, 0)$. By (ii) and since $\mathcal{GG}$ is increasing, it follows that

$$x_1 = \mathcal{GG}(x_1, 0 \ldots, 0) \leq \mathcal{GG}(x_1,\ldots,x_n) = 0.$$

Similarly, one shows that $x_2,\ldots,x_n = 0$, that is $\sum_{i=1}^{n} x_i = 0$.
($\Leftarrow$) Suppose that $\mathcal{GG}$ satisfies ($\mathcal{GG}2'$) and that $\mathcal{GG}(x_1,\ldots,x_n) = 0$, for $(x_1,\ldots,x_n) \in [0,1]^n$. Then, by ($\mathcal{GG}2'$), it holds that $\sum_{i=1}^{n} x_i = 0$. Since a is the neutral element of $\mathcal{GG}$, one has that $\mathcal{GG}(0, a, \ldots, a) = 0$, which means that $a = 0$, by ($\mathcal{GG}2'$). □

Remark 1. Observe that the result stated by Proposition 4 does not mean that when a general general grouping function has a neutral element, then it is necessarily equal to 0. In fact, for each $a \in (0,1)$, the function $\mathcal{GG}\colon [0,1]^n \to [0,1]$, for all $\vec{x} = (x_1 \dots, x_n) \in [0,1]^n$, defined by:

$$\mathcal{GG}(\vec{x}) = \begin{cases} \min\{\vec{x}\}, & \text{if } \max\{\vec{x}\} \le a \\ \max\{\vec{x}\}, & \text{if } \min\{\vec{x}\} \ge a \\ \frac{\min\{\vec{x}\}+\max\{\vec{x}\}\left(1-\min\{\vec{x}\}\right)-a}{1-a}, & \text{if } \min\{\vec{x}\} < a < \max\{\vec{x}\} \end{cases}$$

is a general grouping function with a as neutral element.

Proposition 5. *If* 0 *is the neutral element of a general grouping function* $\mathcal{GG}\colon [0,1]^n \to [0,1]$ *and* $\mathcal{GG}$ *is idempotent, then* $\mathcal{GG}$ *is the maximum.*

Proof. Since $\mathcal{GG}$ is idempotent and increasing in each argument, then one has that for all $\vec{x} = (x_1, \dots, x_n) \in [0,1]^n$: **(1)** $\mathcal{GG}(x_1, \dots, x_n) \le \mathcal{GG}(\max(\vec{x}), \dots, \max(\vec{x})) = \max\{\vec{x}\}$. Then we have that $x_k = \max\{\vec{x}\}$ for some $k = 1, \dots, n$; so we have $x_k = \mathcal{GG}(0, \dots, x_k, \dots, 0) \le \mathcal{GG}(x_1, \dots, x_k, \dots, x_n)$ and then **(2)** $\mathcal{GG}(x_1, \dots, x_n) \ge x_k = \max\{\vec{x}\}$. So, from **(1)** and **(2)** one has that $\mathcal{GG}(x_1, \dots, x_n) = \max\{\vec{x}\}$, for each $\vec{x} \in [0,1]^n$. □

3.1 General grouping functions on lattices

Following a similar procedure described in [20] for general overlap functions on lattices, it is possible to characterize general grouping functions. In order to do that, first we introduce some properties and notations.

Let us denote by $\mathfrak{G}^n$ the set of all general grouping functions. Define the ordering relation $\le_{\mathfrak{G}^n} \in \mathfrak{G}^n \times \mathfrak{G}^n$, for all $\mathcal{GG}_1, \mathcal{GG}_2 \in \mathfrak{G}^n$ by:

$$\mathcal{GG}_1 \le_{\mathfrak{G}^n} \mathcal{GG}_2 \Leftrightarrow \mathcal{GG}_1(\vec{x}) \le \mathcal{GG}_2(\vec{x}), \text{ for all } \vec{x} = (x_1, \dots, x_n) \in [0,1]^n.$$

The supremum and infimum of two arbitrary general grouping functions $\mathcal{GG}_1, \mathcal{GG}_2 \in \mathfrak{G}^n$ are, respectively, the general grouping functions $\mathcal{GG}_1 \vee \mathcal{GG}_2, \mathcal{GG}_1 \wedge \mathcal{GG}_2 \in \mathfrak{G}^n$, defined, for all $\vec{x} = (x_1, \dots, x_n) \in [0,1]^n$ by: $\mathcal{GG}_1 \vee \mathcal{GG}_2(\vec{x}) = \max\{\mathcal{GG}_1(\vec{x}), \mathcal{GG}_2(\vec{x})\}$ and $\mathcal{GG}_1 \wedge \mathcal{GG}_2(\vec{x}) = \min\{\mathcal{GG}_1(\vec{x}), \mathcal{GG}_2(\vec{x})\}$.

The following result is immediate:

Theorem 1. *The ordered set* $(\mathfrak{G}^n, \le_{\mathfrak{G}^n})$ *is a lattice.*

Remark 2. Note that the supremum of the lattice $(\mathfrak{G}^n, \le_{\mathfrak{G}^n})$ is given, for all $\vec{x} = (x_1, \dots, x_n) \in [0,1]^n$, by:

$$GG_{\sup}(\vec{x}) = \begin{cases} 0, & \text{if } \sum_{i=1}^{n} x_i = 0 \\ 1, & \text{otherwise.} \end{cases}$$

On the other hand, the infimum of $(\mathfrak{G}^n, \le_{\mathfrak{G}^n})$ is given, for all $\vec{x} = (x_1, \dots, x_n) \in [0,1]^n$, by:

$$GG_{\inf}(\vec{x}) = \begin{cases} 1, & \text{if } \exists i \in \{1,\dots,n\} : x_i = 1 \\ 0, & \text{otherwise.} \end{cases}.$$

However, neither $GG_{\sup}$ nor $GG_{\inf}$ are general grouping functions, since they are not continuous. Thus, in the lattice $(\mathfrak{G}^n, \leq_{\mathfrak{G}^n})$ there is no bottom neither top elements. Then, similarly to general overlap functions, the lattice $(\mathfrak{G}^n, \leq_{\mathfrak{G}^n})$ is not complete.

4 Characterization of General Grouping Functions and Construction Methods

In this section we provide a characterization and some constructions methods for general grouping functions.

Theorem 2. *The mapping $\mathcal{GG}\colon [0,1]^n \to [0,1]$ is a general grouping function if and only if*

$$\mathcal{GG}(\vec{x}) = \frac{f(\vec{x})}{f(\vec{x}) + h(\vec{x})} \tag{1}$$

for some $f, h\colon [0,1]^n \to [0,1]$ the following properties hold, for all $\vec{x} \in [0,1]^n$:

(i) *f and h are commutative;*
(ii) *f is increasing and h is decreasing.*
(iii) *If $\sum_{i=1}^{n} x_i = 0$, then $f(\vec{x}) = 0$.*
(iv) *If there exists $i \in \{1,\dots,n\}$ such that $x_i = 1$, then $h(\vec{x}) = 0$.*
(v) *f and h are continuous.*
(vi) *$f(\vec{x}) + h(\vec{x}) \neq 0$ for any $\vec{x} \in [0,1]^n$.*

Proof. It follows that:
($\Rightarrow$) Suppose that $\mathcal{GG}$ is a general grouping function, and take $f(\vec{x}) = \mathcal{GG}(\vec{x})$ and $h(\vec{x}) = 1 - f(\vec{x})$. Then one always have $f(\vec{x}) + h(\vec{x}) \neq 0$, and so Equation (1) is well defined. Also, conditions **(i)**-**(v)** trivially hold.
($\Leftarrow$) Consider $f, h\colon [0,1]^n \to [0,1]$ satisfying conditions **(i)**-**(v)**. We will show that $\mathcal{GG}$ defined according to Equation (1) is a general grouping function. It is immediate that $\mathcal{GG}$ is commutative ($\mathcal{GG}1$) and continuous ($\mathcal{GG}5$). To prove ($\mathcal{GG}2$), notice that if $\sum_{i=1}^{n} x_i = 0$ then $f(\vec{x}) = 0$ and thus $\mathcal{GG}(\vec{x}) = 0$. Now, let us prove that ($\mathcal{GG}3$) holds. For that, observe that if there exists $i \in \{1,\dots,n\}$ such that $x_i = 1$, then $h(\vec{x}) = 0$, and, thus, it is immediate that $\mathcal{GG}(\vec{x}) = 1$. The proof of ($\mathcal{GG}4$) is similar to [20, Theorem 3]. $\square$

Example 2. Observe that Theorem 2 provides a method for constructing general grouping functions. For example, take the maximum powered by p, defined by:

$$\max^p(\vec{x}) = \max_{1\leq i \leq n}\{x_i^p\},$$

with $p > 0$. So, if we take the function $T\max_\alpha^p : [0,1]^n \to [0,1]$, called α-truncated maximum powered by p, given, for all $\vec{x} \in [0,1]^n$ and $\alpha \in (0,1)$, by:

$$T\max_\alpha^p(\vec{x}) = \begin{cases} 0, & \text{if } \max^p(\vec{x}) \leq \alpha \\ \max^p(\vec{x}), & \text{if } \max^p(\vec{x}) > \alpha \end{cases} \tag{2}$$

then it is clear that $T\max^p_\alpha$ is not continuous. However, one can consider the function $CT\max^p_{\alpha,\epsilon}\colon [0,1]^n \to [0,1]$, called the continuous truncated maximum powered by p, for all $\vec{x} \in [0,1]^n$, $\alpha \in [0,1]$ and $\epsilon \in (0,\alpha]$, which is defined by:

$$CT\max^p_{\alpha,\epsilon}(\vec{x}) = \begin{cases} 0, & \text{if } \max^p(\vec{x}) \leq \alpha - \epsilon \\ \frac{\alpha}{\epsilon}\left(\max^p(\vec{x}) - (\alpha - \epsilon)\right), & \text{if } \alpha - \epsilon < \max^p(\vec{x}) < \alpha \\ \max^p(\vec{x}), & \text{if } \max^p(\vec{x}) \geq \alpha. \end{cases} \tag{3}$$

Observe that taking $f = CT\max^p_{\alpha,\epsilon}$, then f satisfies conditions **(i)-(iii)** and **(v)** in Theorem 2. Now, take $h(\vec{x}) = \min_{1\leq i\leq n}\{1 - x_i\}$, which satisfies conditions **(i)-(ii)** and **(iv)-(v)** required in Theorem 2. Thus, this assures that

$$\mathcal{GG}(\vec{x}) = \frac{CT\max^p_{\alpha,\epsilon}(\vec{x})}{CT\max^p_{\alpha,\epsilon}(\vec{x}) + \min_{1\leq i\leq n}\{1 - x_i\}}$$

is a general grouping function.

Remark 3. Observe that the maximum powered by p is an n-dimensional grouping function [15] and that $CT\max^p_{\alpha,\epsilon}$ is a general grouping function. However, $CT\max^p_{\alpha,\epsilon}$ is not an n-dimensional grouping function, for $\alpha - \epsilon > 0$, since $CT\max^p_{\alpha,\epsilon}(\alpha - \epsilon, \ldots, \alpha - \epsilon) = 0$.

Corollary 1. *Consider the functions $f, h\colon [0,1]^n \to [0,1]$ and let $\mathcal{GG}\colon [0,1]^n \to [0,1]$ be a general grouping function constructed according to Theorem 2, and taking into account functions f and h. Then $\mathcal{GG}$ is idempotent if and only if, for all $x \in [0,1)$, it holds that:*

$$f(x,\ldots,x) = \frac{x}{1-x}h(x,\ldots,x).$$

Proof. ($\Rightarrow$) If $\mathcal{GG}$ is idempotent, then by Theorem 2 it holds that:

$$\mathcal{GG}(x,\ldots,x) = \frac{f(x,\ldots,x)}{f(x,\ldots,x) + h(x,\ldots,x)} = x.$$

It follows that: $f(x,\ldots,x) = x(f(x,\ldots,x) + h(x,\ldots,x))$

$$(1-x)f(x,\ldots,x) = x\, h(x,\ldots,x)$$
$$f(x,\ldots,x) = \frac{x}{1-x}\, h(x,\ldots,x).$$

($\Leftarrow$) It is immediate. □

Example 3. Take the function $\alpha\beta$-truncated maximum powered by p, $T\max^p_{\alpha\beta}\colon [0,1]^n \to [0,1]$, for all $\vec{x} \in [0,1]^n$; $\alpha, \beta \in (0,1)$ and $\alpha < \beta$, defined by:

$$T\max^p_{\alpha\beta}(\vec{x}) = \begin{cases} 0, & \max^p(\vec{x}) \leq \alpha \\ \max^p(\vec{x}), & \alpha < \max^p(\vec{x}) < \beta \\ 1, & \max^p(\vec{x}) \geq \beta \end{cases}$$

It is clear that $T\max^p_{\alpha\beta}$ is not continuous. However, we can define its continuous version, $CT\max^p_{\alpha\beta,\epsilon\delta}\colon [0,1]^n \to [0,1]$, for all $\vec{x} \in [0,1]^n$; $\alpha \in [0,1)$; $\beta, \epsilon, \delta \in (0,1]$; $\alpha + \epsilon, \beta - \delta \in (0,1)$ and $\alpha + \epsilon < \beta - \delta$, as follows:

$$CT\max^p_{\alpha\beta,\epsilon\delta}(\vec{x}) = \begin{cases} 0, & \max^p(\vec{x}) \leq \alpha \\ \frac{1-(\alpha+\epsilon)}{\epsilon}(\alpha - \max^p(\vec{x})), & \alpha < \max^p(\vec{x}) < \alpha + \epsilon \\ 1 - \max^p(\vec{x}), & \alpha + \epsilon \leq \max^p(\vec{x}) \leq \beta - \delta \\ 1 - (\beta - \delta) - \frac{\beta-\delta}{\delta}(\beta - \delta - \max^p(\vec{x})), & \beta - \delta < \max^p(\vec{x}) < \beta \\ 1, & \max^p(\vec{x}) \geq \beta \end{cases}$$

Observe that $CT\max^p_{\alpha\beta,\epsilon\delta}$ satisfies conditions $(\mathcal{GG}1)$-$(\mathcal{GG}5)$ from Def. 9, and then it is a general grouping function. But, whenever $\alpha \neq 0$ or $\beta \neq 1$, then $CT\max^p_{\alpha\beta,\epsilon\delta}$ is not an n-dimensional grouping function, once $CT\max^p_{\alpha\beta,\epsilon\delta}(\alpha - \epsilon, \ldots, \alpha - \epsilon) = 0$, for $\alpha - \epsilon > 0$, because $\max^p(\alpha - \epsilon, \ldots, \alpha - \epsilon) = \alpha - \epsilon < \alpha$.

The following Theorem generalizes Example 3 providing a construction method for general grouping functions from truncated n-dimensional grouping functions.

Theorem 3. *Consider $\alpha \in [0,1)$; $\beta, \epsilon, \delta \in (0,1]$; $\alpha + \epsilon, \beta - \delta \in (0,1)$ and $\alpha < \beta$, $\alpha + \epsilon < \beta - \delta$. Let $\mathcal{G}$ be an n-dimensional grouping function, whose $\alpha\beta$-truncated version is defined, for all $\vec{x} = (x_1, \ldots, x_n) \in [0,1]^n$, by:*

$$T\mathcal{G}_{\alpha\beta}(\vec{x}) = \begin{cases} 0, & \mathcal{G}(\vec{x}) \leq \alpha \\ \mathcal{G}(\vec{x}), & \alpha < \mathcal{G}(\vec{x}) < \beta \\ 1, & \mathcal{G}(\vec{x}) \geq \beta \end{cases}$$

Then, the continuous version of $T\mathcal{G}_{\alpha\beta}$, for all $\vec{x} = (x_1, \ldots, x_n) \in [0,1]^n$, is given by:

$$CT\mathcal{G}_{\alpha\beta,\epsilon\delta}(\vec{x}) = \begin{cases} 0, & \mathcal{G}(\vec{x}) \leq \alpha \\ \frac{1-(\alpha+\epsilon)}{\epsilon}(\alpha - \mathcal{G}(\vec{x})), & \alpha < \mathcal{G}(\vec{x}) < \alpha + \epsilon \\ 1 - \mathcal{G}(\vec{x}), & \alpha + \epsilon \leq \mathcal{G}(\vec{x}) \leq \beta - \delta \\ 1 - (\beta - \delta) - \frac{\beta-\delta}{\delta}(\beta - \delta - \mathcal{G}(\vec{x})), & \beta - \delta < \mathcal{G}(\vec{x}) < \beta \\ 1, & \mathcal{G}(\vec{x}) \geq \beta \end{cases}$$

and it is a general grouping function. Besides that, whenever $\alpha = 0$ and $\beta = 1$, then $CT\mathcal{G}_{\alpha\beta,\epsilon\delta}$ is an n-dimensional grouping function.

The following proposition shows a construction method of general grouping functions that generalizes Example 1(4).

Proposition 6. *Let $\mathcal{G}\colon [0,1]^n \to [0,1]$ be an n-dimensional grouping function and let $F\colon [0,1]^n \to [0,1]$ be a commutative and continuous aggregation function such that, for all $\vec{x} = (x_1, \ldots, x_n) \in [0,1]^n$, if there exists $i \in \{1, \ldots, n\}$ such that $x_i = 1$, then $F(\vec{x}) = 1$. Then $\mathcal{GG}_{\mathcal{G}F}(\vec{x}) = \mathcal{G}(\vec{x})F(\vec{x})$ is a general grouping function.*

Proof. It is immediate that $\mathcal{GG}_{\mathcal{G}F}$ is well defined, ($\mathcal{GG}1$) commutative, ($\mathcal{GG}4$) increasing and ($\mathcal{GG}5$) continuous, since $\mathcal{G}$, F and the product operation are commutative, increasing and continuous. To prove ($\mathcal{GG}2$), whenever $\sum_{i=1}^{n} x_i = 0$, then by ($\mathcal{G}2$), it holds that $\mathcal{G}(\vec{x}) = 0$, and, thus, $\mathcal{GG}_{\mathcal{G}F}(\vec{x}) = \mathcal{G}(\vec{x})F(\vec{x}) = 0$. For ($\mathcal{GG}3$), whenever there exists $i \in \{1, \ldots, n\}$ such that $x_i = 1$, then, by ($\mathcal{G}3$), one has that $\mathcal{G}(\vec{x}) = 1$, and, by the property of F, it holds that $F(\vec{x}) = 1$. It follows that: $\mathcal{GG}_{\mathcal{G}F}(\vec{x}) = \mathcal{G}(\vec{x})F(\vec{x}) = 1$. □

The following result is immediate.

Corollary 2. *Let $\mathcal{GH}\colon [0,1]^n \to [0,1]$ be a general grouping function and let $F\colon [0,1] \to [0,1]$ be a commutative and continuous aggregation function such that, for all $\vec{x} = (x_1, \ldots, x_n) \in [0,1]^n$, if there exists $i \in \{1, \ldots, n\}$ such that $x_i = 1$, then $F(\vec{x}) = 1$. Then $\mathcal{GG}_{\mathcal{GH},F}(\vec{x}) = \mathcal{GH}(\vec{x})F(\vec{x})$ is a general grouping function.*

Note that $\mathfrak{G}^n$ is closed with respect to some aggregation functions, as stated by the following results, which provide a construction methods of general grouping functions.

Theorem 4. *Consider $M\colon\ [0,1]^2 \to [0,1]$. For any $\mathcal{GG}_1, \mathcal{GG}_2 \in \mathfrak{G}^n$, define the mapping $M_{\mathcal{GG}_1,\mathcal{GG}_2}\colon [0,1]^n \to [0,1]$, for all $\vec{x} \in [0,1]^n$, by:*

$$M_{\mathcal{GG}_1,\mathcal{GG}_2}(\vec{x}) = M(\mathcal{GG}_1(\vec{x}), \mathcal{GG}_2(\vec{x})).$$

Then, $M_{\mathcal{GG}_1,\mathcal{GG}_2} \in \mathfrak{G}^n$ if and only if M is a continuous aggregation function.

Proof. It follows that:
($\Rightarrow$) Suppose that $M_{\mathcal{GG}_1,\mathcal{GG}_2} \in \mathfrak{G}^n$. Then it is immediate that M is continuous and increasing (A2). Now consider $\vec{x} = (x_1, \ldots, x_n) \in [0,1]^n$ and suppose that $\sum_{i=1}^{n} x_i = 0$. Then, by ($\mathcal{GG}2$), one has that: $M_{\mathcal{GG}_1,\mathcal{GG}_2}(\vec{x}) = M(\mathcal{GG}_1(\vec{x}), \mathcal{GG}_2(\vec{x})) = 0$ and $\mathcal{GG}_1(\vec{x}) = \mathcal{GG}_2(\vec{x}) = 0$. Thus, it holds that $M(0,0) = 0$. Now, consider $\vec{x} = (x_1, \ldots, x_n) \in [0,1]^n$, such that there exists $i \in \{1, \ldots, n\}$ such that $x_i = 1$. Then, by ($\mathcal{GG}3$), one has that: $M_{\mathcal{GG}_1,\mathcal{GG}_2}(\vec{x}) = M(\mathcal{GG}_1(\vec{x}), \mathcal{GG}_2(\vec{x})) = 1$ and $\mathcal{GG}_1(\vec{x}) = \mathcal{GG}_2(\vec{x}) = 1$. Therefore, it holds that $M(1,1) = 1$. This proves that M also satisfies (A1), and, thus, M is a continuous aggregation function.
($\Leftarrow$) Suppose that M is a continuous aggregation function. Then it is immediate that $M_{\mathcal{GG}_1,\mathcal{GG}_2}$ is ($\mathcal{GG}1$) commutative, ($\mathcal{GG}4$) increasing and ($\mathcal{GG}5$) continuous. For ($\mathcal{GG}2$), consider $\vec{x} = (x_1, \ldots, x_n) \in [0,1]^n$ such that $\sum_{i=1}^{n} x_i = 0$. Then, by ($\mathcal{GG}2$), one has that $\mathcal{GG}_1(\vec{x}) = \mathcal{GG}_2(\vec{x}) = 0$. It follows that: $M_{\mathcal{GG}_1,\mathcal{GG}_2}(\vec{x}) = M(\mathcal{GG}_1(\vec{x}), \mathcal{GG}_2(\vec{x}))= M(0,0) = 0$, by (A1), since M is an aggregation function. Finally, for ($\mathcal{GG}3$) consider that there exists $i \in \{1, \ldots, n\}$ such that $x_i = 1$ for some $\vec{x} = (x_1, \ldots, x_n) \in [0,1]^n$. Then, it holds that $\mathcal{GG}_1(\vec{x}) = \mathcal{GG}_2(\vec{x}) = 1$. It follows that: $M_{\mathcal{GG}_1,\mathcal{GG}_2}(\vec{x})= M(\mathcal{GG}_1(\vec{x}), \mathcal{GG}_2(\vec{x})) = M(1,1) = 1$, by (A1), since M is an aggregation function. This proves that $M_{\mathcal{GG}_1,\mathcal{GG}_2} \in \mathfrak{G}^n$. □

Example 4. In the sense of Theorem 4, $\mathfrak{G}^n$ is closed under any bidimensional overlap functions, grouping functions and continuous t-norms and t-conorms [18].

Corollary 3. *Consider* $M: [0,1]^2 \to [0,1]$. *For any* n*-dimensional grouping functions* $\mathcal{G}_1, \mathcal{G}_2: [0,1]^n \to [0,1]$, *define the mapping* $M_{\mathcal{G}_1,\mathcal{G}_2}: [0,1]^n \to [0,1]$, *for all* $\vec{x} = (x_1, \ldots, x_n) \in [0,1]^n$, *by:*

$$M_{\mathcal{G}_1,\mathcal{G}_2}(\vec{x}) = M(\mathcal{G}_1(\vec{x}), \mathcal{G}_2(\vec{x})).$$

Then, $M_{\mathcal{G}_1,\mathcal{G}_2} \in \mathfrak{G}^n$ *if and only if* M *is a continuous aggregation function.*

Proof. It follows from Theorem 4, since any n-dimensional grouping function is a general grouping function. □

Theorem 4 can be easily extended for n-ary functions $M^n: [0,1]^n \to [0,1]$:

Theorem 5. *Consider* $M^n: [0,1]^n \to [0,1]$. *For any* $\mathcal{GG}_1, \ldots, \mathcal{GG}_n \in \mathfrak{G}^n$, *define the mapping* $M_{\mathcal{GG}_1,\ldots,\mathcal{GG}_n}: [0,1]^n \to [0,1]$, *for all* $\vec{x} \in [0,1]^n$, *by:*

$$M_{\mathcal{GG}_1,\ldots,\mathcal{GG}_n}(\vec{x}) = M^n(\mathcal{GG}_1(\vec{x}), \ldots, \mathcal{GG}_n(\vec{x})).$$

Then, $M_{\mathcal{GG}_1,\ldots,\mathcal{GG}_n} \in \mathfrak{G}^n$ *if and only if* $M^n : [0,1]^n \to [0,1]$ *is a continuous* n*-ary aggregation function.*

Proof. Analogous to the proof of Theorem 4. □

This result can be extended for n-dimensional grouping functions.

Corollary 4. *Consider* $M^n: [0,1]^n \to [0,1]$ *and fr any* n*-dimensional grouping functions* $\mathcal{G}_1, \ldots, \mathcal{G}_n$ *define the mapping* $M_{\mathcal{G}_1,\ldots,\mathcal{G}_n}: [0,1]^n \to [0,1]$, *for all* $\vec{x} = (x_1, \ldots, x_n) \in [0,1]^n$, *by:*

$$M_{\mathcal{G}_1,\ldots,\mathcal{G}_n}(\vec{x}) = M^n(\mathcal{G}_1(\vec{x}), \ldots, \mathcal{G}_n(\vec{x})).$$

Then, $M_{\mathcal{G}_1,\ldots,\mathcal{G}_n}$ *is a general grouping function if and only if* $M^n: [0,1]^n \to [0,1]$ *is a continuous* n*-ary aggregation function.*

Corollary 5. *Let* $\mathcal{GG}_1, \ldots, \mathcal{GG}_m: [0,1]^n \to [0,1]$ *be general grouping functions and consider weights* $w_1, \ldots, w_m \in [0,1]$ *such that* $\sum_{i=1}^{m} w_i = 1$. *Then the convex sum* $\mathcal{GG} = \sum_{i=1}^{m} w_i \mathcal{GG}_i$ *is also a general grouping function.*

Proof. Since the weighted sum is a continuous commutative n-ary aggregation function, the result follows from Theorem 5. □

It is possible to obtain general grouping functions from the generalized composition of general grouping functions and aggregation functions satisfying especial conditions:

Theorem 6. *Let* $\mathcal{GG}_2: [0,1]^n \to [0,1]$ *be a general grouping function and let the* n*-ary aggregation functions* $A_1, \ldots, A_n: [0,1]^n \to [0,1]$ *be continuous, commutative and disjunctive. Then, the function* $\mathcal{GG}_1: [0,1]^n \to [0,1]$ *defined, for all* $\vec{x} = (x_1, \ldots, x_n) \in [0,1]^n$, *by:* $\mathcal{GG}_1(\vec{x}) = \mathcal{GG}_2(A_1(\vec{x}), \ldots, A_n(\vec{x}))$ *is a general grouping function.*

Proof. Since $\mathcal{GG}_2$, $A_1, \ldots, A_n$ are commutative, increasing and continuous functions, then $\mathcal{GG}_1$ satisfies conditions $(\mathcal{GG}1)$, $(\mathcal{GG}4)$ and $(\mathcal{GG}5)$. So, it remains to prove:

$(\mathcal{GG}2)$ Let $\vec{x} = (x_1, \ldots, x_n) \in [0,1]^n$ be such that $\sum_{i=1}^{n} x_i = 0$. Then, since A_1 is disjunctive, we have that $A_1(\vec{x}) \geq \max(\vec{x}) = 0$, that is $A_1(\vec{x}) = 0$. Equivalently, one obtains $A_2(\vec{x}), \ldots, A_n(\vec{x}) = 0$. Thus, since $\mathcal{GG}_2$ is a general grouping function, one has that $\mathcal{GG}_1(\vec{x}) = \mathcal{GG}_2(A_1(\vec{x}), \ldots, A_n(\vec{x})) = \mathcal{GG}_2(0, \ldots, 0) = 0$.

$(\mathcal{GG}3)$ Suppose that, for some $\vec{x} = (x_1, \ldots, x_n) \in [0,1]^n$, there exists $i \in \{1, \ldots, n\}$ such that $x_i = 1$. So, since A_1 is disjunctive then $A_1(\vec{x}) \geq \max(\vec{x}) = 1$, that is $A_1(\vec{x}) = 1$. Since $\mathcal{GG}_2$ is a general grouping function, it follows that $\mathcal{GG}_1(\vec{x}) = \mathcal{GG}_2(A_1(\vec{x}), \ldots, A_n(\vec{x})) = \mathcal{GG}_2(1, A_2(\vec{x}), \ldots, A_n(\vec{x})) = 1$. □

Next proposition uses the n-duality property.

Proposition 7. *Consider a continuous fuzzy negation* $N\colon [0,1] \to [0,1]$ *and a general overlap function* $\mathcal{GO}\colon [0,1]^n \to [0,1]$*, then for all* $\vec{x} = (x_1, \ldots, x_n) \in [0,1]^n$*:*

$$\mathcal{GG}(\vec{x}) = N(\mathcal{GO}(N(x_1), \ldots, N(x_n))) \tag{4}$$

is a general grouping function. Reciprocally, if $\mathcal{GG}\colon [0,1]^n \to [0,1]$ *is a general grouping function, then for all* $\vec{x} = (x_1, \ldots, x_n) \in [0,1]^n$*:*

$$\mathcal{GO}(\vec{x}) = N(\mathcal{GG}(N(x_1), \ldots, N(x_n))) \tag{5}$$

is a general overlap function.

Proof. Since we have a continuous fuzzy negation and bearing in mind that general overlap functions and general grouping functions are commutative, increasing and continuous functions according to Def. 8 and Def. 9, respectively, then $\mathcal{GO}$ and $\mathcal{GG}$ satisfy conditions $(\mathcal{GO}1), (\mathcal{GG}1)$; $(\mathcal{GO}4), (\mathcal{GG}4)$ and $(\mathcal{GO}5), (\mathcal{GG}5)$. So, it remains to prove:

$(\mathcal{GG}2)$ For Eq. (4), take $x_i = 0$, for all $i \in \{1, \ldots, n\}$. Therefore,

$$\mathcal{GG}(\vec{x}) = N(\mathcal{GO}(N(0), \ldots, N(0))) \overset{N2}{=} N(\mathcal{GO}(1, \ldots, 1)) \overset{\mathcal{GO}3}{=} N(1) \overset{N2}{=} 0.$$

$(\mathcal{GG}3)$ If there exists a $x_i = 1$, for some $i \in \{1, \ldots, n\}$, then

$$\begin{aligned} \mathcal{GG}(\vec{x}) &= N(\mathcal{GO}(N(x_1), \ldots, N(1), \ldots, N(x_n))) \\ &\overset{N2}{=} N(\mathcal{GO}(N(x_1), \ldots, 0, \ldots, N(x_n))) \\ &\overset{\mathcal{GO}2}{=} N(0) \overset{N2}{=} 1. \end{aligned}$$

$(\mathcal{GO}2)$ Similarly, for Eq. (5), take a $x_i = 0$ for some $i \in \{1, \ldots, n\}$. So,

$$\begin{aligned} \mathcal{GO}(\vec{x}) &= N(\mathcal{GG}(N(x_1), \ldots, N(0), \ldots, N(x_n))) \\ &\overset{N2}{=} N(\mathcal{GG}(N(x_1), \ldots, 1, \ldots, N(x_n))) \\ &\overset{\mathcal{GG}3}{=} N(1) \overset{N2}{=} 0. \end{aligned}$$

$(\mathcal{GO}3)$ Now, consider that $x_i = 1$, for all $i \in \{1, \ldots, n\}$. Then,

$$\mathcal{GO}(\vec{x}) = N(\mathcal{GG}(N(1), \ldots, N(1))) \overset{N2}{=} N(\mathcal{GG}(0, \ldots, 0)) \overset{\mathcal{GG}2}{=} N(0) \overset{N2}{=} 1.$$

□

5 Conclusions

In this paper, we first introduced the concept of general grouping functions and studied some of their properties. Then we provide a characterization of general grouping functions and some construction methods.

The theoretical developments presented here allow for a more flexible approach when dealing with decision making problems with multiple alternatives. Immediate future work is concerned with the development of an application in multi-criteria decision making based on n-ary fuzzy heterogeneous, incomplete preference relations.

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5.2.2 General interval-valued grouping functions

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General Admissibly Ordered Interval-valued Overlap Functions

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Abstract

Overlap functions are a class of aggregation functions that measure the overlapping degree between two values. They have been successfully applied in several problems in which associativity is not required, such as classification and image processing. Some generalizations of overlap functions were proposed for them to be applied in problems with more than two classes, such as n-dimensional and general overlap functions. To measure the overlapping of interval data, interval-valued overlap functions were defined, and, later, they were also generalized in the form of n-dimensional and general interval-valued overlap functions. In order to apply some of those concepts in problems with interval data considering the use of admissible orders, which are total orders that refine the most used partial order for intervals, n-dimensional admissibly ordered interval-valued overlap functions were recently introduced, proving to be suitable to be applied in classification problems. However, the sole construction method presented for this kind of function do not allow the use of the well known lexicographical orders. So, in this work we combine previous developments to introduce general admissibly ordered interval-valued overlap functions, while also presenting different construction methods and the possibility to combine such methods, showcasing the flexibility and adaptability of this approach, while also being compatible with the lexicographical orders.

1. Introduction

Overlap functions are aggregation functions, initially introduced in the context of image processing problems, to measure the overlapping between classes [1, 2, 3]. Since then, they have been studied in the literature by many authors, mainly because of either the advantages they present over the popular t-norms [4, 5] or their great applicability, as in: fuzzy rule-based

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classification [6, 7, 8, 9, 10], decision making [11, 12], wavelet -fuzzy power quality diagnosis system [13], forest fire detection [14], among others.

The concept of n-dimensional overlap functions was introduced [15] in order to allow the application of overlap functions, which were originally defined as bivariate functions that do not need to be associative, in problems with multiple classes. By relaxing the boundary conditions of n-dimensional overlap functions, general overlap functions were defined, also showing good behaviour when applied in classification problems [16].

Now observe that, when working with fuzzy systems, one may face the problem regarding the uncertainty in assigning the values of the membership degrees or defining the membership functions that are adopted in the system. In the literature, a proposed solution is given by the use of interval-valued fuzzy sets (IVFSs) [17, 18, 19], where the membership degrees are represented by intervals, whose widths represent such uncertainty [20, 21, 22]. IVFSs have been successfully applied in many different fields, such as classification [23, 24, 25], image processing [26, 27], game theory [28], multicriteria decision making [29], pest control [30], irrigation systems [31] and collaborative clustering [32].

To avoid a stalemate when comparing interval data, Bustince et al. [33] introduced the concept of admissible orders for intervals, that is, total order relations that refine the usual product order [34], which is a partial order. Since their introduction, several works were developed taking admissible orders into account, such as [35, 36, 37, 38].

Qiao and Hu [39] and Bedregal et al. [40] defined, independently, the concept of interval-valued overlap functions. By extending and generalizing interval-valued overlap functions, Asmus et al. [23] introduced the concepts of n-dimensional interval-valued overlap functions and general interval-valued overlap functions, both concepts taking into account the usual increasingness with respect to the product order.

Allowing for a broader practical application of (n-dimensional) interval-valued overlap functions, Asmus et al. [35] introduced the concept of n-dimensional admissibly ordered interval-valued overlap functions, which are n-dimensional interval-valued overlap functions that are increasing with respect to an admissible order. They also presented a construction method, which, however, cannot generate n-dimensional interval-valued overlap functions that are increasing with respect to the well known lexicographical orders [33]. Although this is not a serious problem, with the initial motivation to overcome this drawback, in this present work we combine the recent developed concepts on (n-dimensional, general) interval-valued overlap functions and admissible orders to introduce general admissibly ordered interval-valued overlap function. However, the resulting definition proved to be much more flexible and adaptable, allowing for the development of different construction methods, and even the composition of functions constructed through those methods.

The paper is organized as follows. Section 2 presents some preliminary concepts. In Section 3, we introduce the concept of general admissibly ordered iv-overlap functions, studying its representation and relation with n-dimensional admissibly ordered iv-overlap function. In section 4, we present some construction methods for general admissibly ordered iv-overlap functions. Section 5 is the Conclusion.

2. Preliminaries

In this section, we recall some concepts on general overlap functions, interval mathematics, admissible orders and (admissibly ordered) interval-valued overlap functions.

2.1. General Overlap Functions

Definition 1. *[41] An aggregation function is a mapping $A : [0,1]^n \to [0,1]$ that is increasing in each argument and satisfying:* ***(A1)*** *$A(0,\ldots,0) = 0$;* ***(A2)*** *$A(1,\ldots,1) = 1$.*

Definition 2. *[42, 15] A function $On : [0,1]^n \to [0,1]$ is said to be an n-dimensional overlap function if, for all $\vec{x} \in [0,1]^n$:* ***(On1)*** *On is commutative;* ***(On2)*** *$On(\vec{x}) = 0 \Leftrightarrow \prod_{i=1}^{n} x_i = 0$;* ***(On3)*** *$On(\vec{x}) = 1 \Leftrightarrow \prod_{i=1}^{n} x_i = 1$;* ***(On4)*** *$On$ is increasing;* ***(On5)*** *On is continuous.*

When On is strictly increasing in $(0,1]$, it is called a *strict n-dimensional overlap function*. A 2-dimensional overlap function is just called overlap function [43, 1].

By changing the boundaries conditions **(On2)** and **(On3)** to obtain a less restrictive definition, *general overlap functions* were introduced as follows:

Definition 3. *[16] A function $GO : [0,1]^n \to [0,1]$ is said to be a general overlap function if, for all $\vec{x} \in [0,1]^n$:* ***(GO1)*** *On is commutative;* ***(GO2)*** *$\prod_{i=1}^{n} x_i = 0 \Rightarrow GO(\vec{x}) = 0$* ***(GO3)*** *$\prod_{i=1}^{n} x_i = 1 \Rightarrow GO(\vec{x}) = 1$;* ***(GO4)*** *$GO$ is increasing;* ***(GO5)*** *GO is continuous.*

Proposition 1. *[16] If $On : L([0,1])^n \to [0,1]$ is an n-dimensional overlap function, then On is also a general overlap function, but the converse may not hold.*

Example 1. *The following are all examples of general overlap functions, defined for all $\vec{x} \in [0,1]^n$:*

a) *The product GO_P, given by $GO_P(\vec{x}) = \prod_{i=1}^{n} x_i$, which is a strict n-dimensional overlap function, and, whenever $n = 2$, it is the product t-norm [44].*

b) *The function GO_L, given by $GO_L(\vec{x}) = \max\{(\sum_{i=1}^{n} x_i) - (n-1), 0\}$, which is not neither an n-dimensional overlap function nor strictly increasing. For $n = 2$, it is the Lukasiewicz t-norm [44].*

c) *The geometric mean GO_{Gm}, given by $GO_{Gm}(\vec{x}) = \sqrt[n]{\prod_{i=1}^{n} x_i}$, which is a strict n-dimensional overlap function, but it is not a t-norm, when $n = 2$ [1].*

For properties on (n-dimensional) overlap functions, general overlap functions and related concepts, see also: [16, 45, 4, 46, 47, 15, 48, 49, 50].

2.2. Interval Mathematics and Admissible Orders

Let us denote as $L([0,1])$ the set of all closed subintervals of the unit interval $[0,1]$. Denote $\vec{x} = (x_1,\ldots,x_n) \in [0,1]^n$ and $\vec{X} = (X_1,\ldots,X_n) \in L([0,1])^n$. Given any $X = [x_1,x_2] \in L([0,1])$, $\underline{X} = x_1$ and $\overline{X} = x_2$ denote, respectively, the left and right projections of X, and

$w(X) = \overline{X} - \underline{X}$ denotes the width of X. The interval product is defined for all $X, Y \in L([0,1])$ by:

$$X \leq_{Pr} Y \quad \Leftrightarrow \quad \underline{X} \leq \underline{Y} \wedge \overline{X} \leq \overline{Y}.$$

We call as $\leq_{Pr}$-increasing a function that is increasing with respect to the product order $\leq_{Pr}$. The projections $F^-, F^+ : [0,1]^n \to [0,1]$ of $F : L([0,1])^n \to L([0,1])$ are defined, respectively, by:

$$F^-(x_1, \ldots, x_n) = \underline{F([x_1, x_1], \ldots, [x_n, x_n])}; \tag{1}$$

$$F^+(x_1, \ldots, x_n) = \overline{F([x_1, x_1], \ldots, [x_n, x_n])}. \tag{2}$$

Given two functions $f, g : [0,1]^n \to [0,1]$ such that $f \leq g$, we define the function $\widehat{f,g} : L([0,1])^n \to L([0,1])$ as

$$\widehat{f,g}(\vec{X}) = [f(\underline{X_1}, \ldots, \underline{X_n}), g(\overline{X_1}, \ldots, \overline{X_n})]. \tag{3}$$

Definition 4. *[21] Let $IF : L([0,1])^n \to L([0,1])$ be an $\leq_{Pr}$-increasing interval function. IF is said to be representable if there exist increasing functions $f, g : [0,1]^n \to [0,1]$ such that $f \leq g$ and $F = \widehat{f,g}$.*

The functions f and g are the *representatives* of the interval function F. When $F = \widehat{f,f}$, we denote simply as $\widehat{f}$.

The interval-product is defined, for all $X, Y \in L([0,1])$, by $X \cdot Y = [\underline{X} \cdot \underline{Y}, \overline{X} \cdot \overline{Y}]$.

The notion of admissible orders for intervals came from the interest in refining the product order $\leq_{Pr}$ to a total order.

Definition 5. *[33] Let $(L([0,1]), \leq_{AD})$ be a partially ordered set. The order $\leq_{AD}$ is called an admissible order if*

(i) *$\leq_{AD}$ is a total order on $(L([0,1]), \leq_{AD})$;*

(ii) *For all $X, Y \in L([0,1])$, $X \leq_{AD} Y$ whenever $X \leq_{Pr} Y$.*

In other words, an order $\leq_{AD}$ on $L([0,1])$ is admissible, if it is total and refines the order $\leq_{Pr}$ [33].

Example 2. *Examples of admissible orders on $L([0,1])$ are the lexicographical orders with respect to the first and second coordinate, defined, respectively, by:*

$$X \leq_{Lex1} Y \Leftrightarrow \underline{X} < \underline{Y} \vee (\underline{X} = \underline{Y} \wedge \overline{X} \leq \overline{Y});$$
$$X \leq_{Lex2} Y \Leftrightarrow \overline{X} < \overline{Y} \vee (\overline{X} = \overline{Y} \wedge \underline{X} \leq \underline{Y}).$$

Definition 6. *[33] For $\alpha, \beta \in [0,1]$ such that $\alpha \neq \beta$, the relation $\leq_{\alpha,\beta}$ is defined by*

$$X \leq_{\alpha,\beta} Y \Leftrightarrow K_\alpha(\underline{X}, \overline{X}) < K_\alpha(\underline{Y}, \overline{Y}) \text{ or}$$
$$(K_\alpha(\underline{X}, \overline{X}) = K_\alpha(\underline{Y}, \overline{Y}) \text{ and } K_\beta(\underline{X}, \overline{X}) \leq K_\beta(\underline{Y}, \overline{Y})),$$

where $K_\alpha, K_\beta : [0,1]^2 \to [0,1]$ are aggregation functions defined, respectively, by

$$K_\alpha(x, y) = x + \alpha \cdot (y - x), \tag{4}$$
$$K_\beta(x, y) = x + \beta \cdot (y - x).$$

Then, the relation $\leq_{\alpha,\beta}$ is an admissible order.

Remark 1. *By varying the values of α and β one can recover some of the known admissible orders, e.g., the lexicographical orders $\leq_{Lex1}$ and $\leq_{Lex2}$ can be recovered by $\leq_{0,1}$ and $\leq_{1,0}$, respectively.*

Whenever we apply the mapping K_α on the endpoints of an interval $X \in [0,1]$, we denote $K_\alpha(\underline{X}, \overline{X})$ simply as $K_\alpha(X)$.

We denote an interval-valued function that is increasing with respect to an admissible order $\leq_{AD}$ as $\leq_{AD}$-increasing. Obviously, every $\leq_{AD}$-increasing function is also $\leq_{Pr}$-increasing, since every admissible order $\leq_{AD}$ refines $\leq_{Pr}$.

2.3. General Interval-valued Overlap Functions

Definition 7. *[51] A function $IA : L([0,1])^n \to L([0,1])$ is an interval-valued (iv) aggregation function whenever: **(IA1)** IA is $\leq_{Pr}$-increasing; **(IA2)** IA satisfies: $IA([0,0],\ldots,[0,0]) = [0,0]$ and $IA([1,1],\ldots,[1,1]) = [1,1]$.*

Definition 8. *[23] A function $IOn : L([0,1])^n \to L([0,1])$ is an n-dimensional interval-valued (iv) overlap function if, for all $\vec{X} \in L([0,1])^n$, it satisfies: **(IOn1)** IOn is commutative; **(IOn2)** $IOn(\vec{X}) = [0,0] \Leftrightarrow \prod_{i=1}^n X_i = [0,0]$; **(IOn3)** $IOn(\vec{X}) = [1,1] \Leftrightarrow \prod_{i=1}^n X_i = [1,1]$; **(IOn4)** IOn is $\leq_{Pr}$-increasing; **(IOn5)** IOn is Moore continuous [34].*

For $n = 2$, IOn is just called iv-overlap function [40, 39].

Theorem 1. *[23] Let $On_1, On_2 : [0,1]^n \to [0,1]$ be n-dimensional overlap functions such that $On_1 \leq On_2$. Then, the function $IOn : L([0,1])^n \to L([0,1])$ given, for all $\vec{X} \in L([0,1])^n$, by $IOn(\vec{X}) = \widehat{On_1, On_2}(\vec{X})$, as defined in Eq. (3), is an n-dimensional iv-overlap function.*

Regarding Theo. 1, IOn is a representable interval-valued function. As both its representatives are n-dimensional overlap functions, it is said to be o-representable [23].

By changing **(IOn2)** and **(IOn3)** in Def. 8, general interval-valued overlap functions were defined as follows:

Definition 9. *[23] A function $IGO : L([0,1])^n \to L([0,1])$ is said to be a general interval-valued (iv) overlap function if, for all $\vec{X} \in L([0,1])^n$: **(IGO1)** IGO is commutative; **(IGO2)** If $\prod_{i=1}^n X_i = [0,0]$ then $IGO(\vec{X}) = [0,0]$; **(IGO3)** If $\prod_{i=1}^n X_i = [1,1]$ then $IGO(\vec{X}) = [1,1]$; **(IGO4)** IGO is $\leq_{Pr}$-increasing; **(IGO5)** IGO is Moore continuous.*

Proposition 2. *[23] If $IOn : L([0,1])^n \to L([0,1])$ is an n-dimensional iv-overlap function, then it is also a general iv-overlap function, but the converse may not hold.*

Theorem 2. *[23] Let $GO_1, GO_2 : [0,1]^n \to [0,1]$ be two general overlap functions such that $GO_1 \leq GO_2$. Then, the function $IGO : L([0,1])^n \to L([0,1])$ given, for all $\vec{X} \in L([0,1])^n$, by $IGO(\vec{X}) = \widehat{GO_1, GO_2}(\vec{X})$, is a (representable) general iv-overlap function.*

In order to apply n-dimensional iv-overlap functions in problems where admissible orders must be considered, the following definition was introduced:

Definition 10. *[35] A function $AOn : L([0,1])^n \to L([0,1])$ is an n-dimensional admissibly ordered interval-valued overlap function for an admissible order $\leq_{AD}$ (n-dimensional $\leq_{AD}$-overlap function) if it satisfies **(IOn1)**, **(IOn2)** and **(IOn3)** from Def. 8, and the following condition holds:*

(AOn4) *AOn is $\leq_{AD}$-increasing.*

Remark 2. *Observe that condition **(IOn5)** was not considered in Def. 10, as the continuity condition of overlap functions was only a requirement in order for them to be applied in image processing problems, which was not the case in [35].*

Theorem 3. *[35] Let $IOn : L([0,1])^n \to L([0,1])$ be an o-representable n-dimensional iv-overlap function and $\alpha, \beta \in [0,1]$, $\alpha \neq \beta$. Then, IOn is $\leq_{\alpha,\beta}$-increasing if and only if $\alpha = 1$ and IOn^+ is a strict n-dimensional overlap function.*

The following Theorem presents a construction method for n-dimensional $\leq_{\alpha,\beta}$-overlap functions:

Theorem 4. *[35] Let On be a strict n-dimensional overlap function, $\alpha \in (0,1)$ and $\beta \in [0,1]$ such that $\alpha \neq \beta$. Then $AOn^\alpha : L([0,1])^n \to L([0,1])$ defined, for all $\vec{X} \in L([0,1])^n$, by*

$$\begin{aligned} AOn^\alpha(\vec{X}) = [&On(K_\alpha(X_1), \ldots, K_\alpha(X_n)) - \alpha m, \\ &On(K_\alpha(X_1), \ldots, K_\alpha(X_n)) + (1-\alpha)m], \end{aligned}$$

where

$$\begin{aligned} m = \min\{&\overline{X_1} - \underline{X_1}, \ldots, \overline{X_n} - \underline{X_n}, On(K_\alpha(X_1), \ldots, K_\alpha(X_n)), \\ &1 - On(K_\alpha(X_1), \ldots, K_\alpha(X_n))\}, \end{aligned}$$

is an n-dimensional $\leq_{\alpha,\beta}$-overlap function.

Remark 3. *Notice that **(IOn2)** and **(IOn3)** are both necessary and sufficient conditions. For that reason, the construction method presented in Theo. 4 must consider $\alpha \in (0,1)$ and, consequently, cannot be applied to obtain neither an n-dimensional $\leq_{0,1}$-overlap function nor an n-dimensional $\leq_{1,0}$-overlap function, that is, n-dimensional admissibly ordered iv-overlap functions that are increasing with respect to the lexicographical orders $\leq_{Lex1}$ and $\leq_{Lex2}$, respectively. This drawback is going to be addressed in our developments in this work. Furthermore, the chosen n-dimensional overlap function On must be strict, to ensure that the constructed function is $\leq_{\alpha,\beta}$-increasing.*

Here, we recall some concepts presented in [37] that were used to introduce a construction method for iv-aggregation functions that are $\leq_{\alpha,\beta}$-increasing.

Definition 11. *[37] Let $c \in [0,1]$ and $\alpha \in [0,1]$. We denote by $d_\alpha(c)$ the maximal possible width of an interval $Z \in L([0,1])$ such that $K_\alpha(Z) = c$. Moreover, for any $X \in L([0,1])$, define*

$$\lambda_\alpha(X) = \frac{w(X)}{d_\alpha(K_\alpha(X))},$$

where we set $\frac{0}{0} = 1$.

Proposition 3. *[37] For all $\alpha \in [0,1]$ and $X \in L([0,1])$ it holds that*

$$d_\alpha(K_\alpha(X)) = \min\left\{\frac{K_\alpha(X)}{\alpha}, \frac{1-K_\alpha(X)}{1-\alpha}\right\},$$

where we set $\frac{r}{0} = 1$, for all $r \in [0,1]$.

Theorem 5. *[37] Let $\alpha, \beta \in [0,1]$, such that, $\alpha \neq \beta$. Let $A_1, A_2 : [0,1]^n \to [0,1]$ be two aggregation functions where A_1 is strictly increasing. Then $IF^\alpha : L([0,1])^n \to L([0,1])$ defined by:*

$$IF^{\alpha}_{A1,A2}(\vec{X}) = R, \ \textit{where,} \begin{cases} K_\alpha(R) = A_1(K_\alpha(X_1), \ldots, K_\alpha(X_n)), \\ \lambda_\alpha(R) = A_2(\lambda_\alpha(X_1), \ldots, \lambda_\alpha(X_n)), \end{cases}$$

for all $\vec{X} \in L([0,1])^n$, is an $\leq_{\alpha,\beta}$-increasing iv-aggregation function.

As n-dimensional overlap functions are a class of aggregation functions, the following result is immediate.

Corollary 1. *Let $\alpha, \beta \in [0,1]$, such that, $\alpha \neq \beta$. Let $On : [0,1]^n \to [0,1]$ be a strict n-dimensional overlap function and $A : [0,1]^n \to [0,1]$ be an aggregation function. Then $IF^{\alpha}_{O,A} : L([0,1])^n \to L([0,1])$ defined by:*

$$IF^{\alpha}_{On,A}(\vec{X}) = R, \ \textit{where,} \begin{cases} K_\alpha(R) = On(K_\alpha(X_1), \ldots, K_\alpha(X_n)), \\ \lambda_\alpha(R) = A(\lambda_\alpha(X_1), \ldots, \lambda_\alpha(X_n)), \end{cases}$$

for all $\vec{X} \in L([0,1])^n$, is an $\leq_{\alpha,\beta}$-increasing iv-aggregation function.

Remark 4. *Concerning Coro. 1, observe that although we apply an n-dimensional overlap function as part of the construction method, the resulting iv-aggregation function $IF^{\alpha}_{O,A}$ may not be an $\leq_{\alpha,\beta}$-overlap function, as one can only guarantee that condition **(AOn4)** is satisfied.*

3. General admissibly ordered interval-valued overlap functions

By combining the concepts of general iv-overlap functions and n-dimensional admissibly ordered iv-overlap functions, we introduce the following definition:

Definition 12. *A function $AGO : L([0,1])^n \to L([0,1])$ is a general admissibly ordered interval-valued overlap function for an admissible order $\leq_{AD}$ (general $\leq_{AD}$-overlap function) if it satisfies the conditions **(IGO1)**, **(IGO2)** and **(IGO3)** of Def. 3, and the following condition holds:*

(AGO4) *AGO is $\leq_{AD}$-increasing.*

The following result is immediate:

Proposition 4. *If $AOn : L([0,1])^n \to L([0,1])$ is an n-dimensional $\leq_{AD}$-overlap function, then it is also a general $\leq_{AD}$-overlap function, but the converse may not hold.*

Here we present some results regarding representable general iv-overlap functions and their increasingness with respect to a particular admissible order. In the following result, consider that a strict general overlap function is a general overlap function that is strictly increasing in $(0,1]$.

Lemma 1. *Let $GO : [0,1]^n \to [0,1]$ be a strict general overlap function. Then, GO is an n-dimensional overlap function.*

*Proof. It is immediate that GO respects conditions **(On1)**, **(On4)** and **(On5)** and, by **(GO2)** and **(GO3)**, it respects the necessary conditions ($\Leftarrow$) of **(On2)** and **(On3)**. It remains to prove the sufficient conditions ($\Rightarrow$) of **(On2)** and **(On3)**:*

(On2) *($\Rightarrow$) Suppose that GO is strict and does not respect **(On2)** ($\Rightarrow$). Take $\vec{y} \in (0,1]^n$ such that $GO(\vec{y}) = 0$. Then, there exist $\vec{x} \in (0,1]^n$ such that $\vec{x} < \vec{y}$ and, by **(GO4)**, $GO(\vec{x}) = GO(\vec{y}) = 0$, which is a contradiction since GO is strict. Thus, GO respects **(On2)**.*

(On3) *($\Rightarrow$) Suppose that GO is strict and does not respect **(On3)** ($\Rightarrow$). By **(GO2)**, one has that $\vec{x} = (1,\ldots,1) \Rightarrow GO(\vec{x}) = 1$. Now, take $\vec{y} \in [0,1]^n$ such that $y_i \neq 1$ for some $i \in \{1,\ldots,n\}$ and $GO(\vec{y}) = 1$. Then, one has that $\vec{y} < \vec{x}$ and $GO(\vec{y}) = GO(\vec{x}) = 1$, which is a contradiction since GO is strict. Thus, GO respects **(On3)**.*

Theorem 6. *Let $IGO : L([0,1])^n \to L([0,1])$ be a representable general iv-overlap function and $\alpha, \beta \in [0,1]$, $\alpha \neq \beta$. Then, IGO is $\leq_{\alpha,\beta}$-increasing if and only if $\alpha = 1$ and IGO^+ is a strict n-dimensional overlap function.*

Proof. Analogous to the proof of Theo. 3 in [35], taking into account Lem. 1.

Then, the following result is immediate:

Corollary 2. *Let $On : [0,1]^n \to [0,1]$ be an n-dimensional overlap function and $IGO : L([0,1])^n \to L([0,1])$ be a general iv-overlap function such that $IGO = \widehat{On}$, and $\alpha, \beta \in [0,1]$, $\alpha \neq \beta$. Then, IGO is a general $\leq_{\alpha,\beta}$-overlap if and only if $\alpha = 1$ and On is a strict n-dimensional overlap function.*

Example 3. *Consider the general overlap function GO_P as defined in Ex. 1 for $n = 2$. As it is a strict general overlap function, then, by Lem. 1, it is also a strict overlap function. Then, the interval-valued function $AGO_P : L([0,1])^2 \to L([0,1])$ defined, for all $\vec{X} \in L([0,1])^2$, by*

$$AGO_P(\vec{X}) = \widehat{GO_P}(\vec{X})$$

is a general $\leq_{1,0}$-overlap function, and also an 2-dimensional $\leq_{1,0}$-overlap function.

4. Construction methods

The first construction method for general $\leq_{AD}$-overlap functions is an adaptation of Theo. 4, by taking $\alpha \in [0,1]$, obtaining a general $\leq_{\alpha,\beta}$-overlap function.

Theorem 7. *Let On be a strict n-dimensional overlap function, $\alpha, \beta \in [0,1]$ such that $\alpha \neq \beta$. Then $AGO^\alpha : L([0,1])^n \to L([0,1])$ defined, for all $\vec{X} \in L([0,1])^n$, by*

$$\begin{aligned} AGO^\alpha(\vec{X}) = [&On(K_\alpha(X_1),\ldots,K_\alpha(X_n)) - \alpha m, \\ &On(K_\alpha(X_1),\ldots,K_\alpha(X_n)) + (1-\alpha)m], \end{aligned}$$

where

$$\begin{aligned} m = \min\{&\overline{X_1} - \underline{X_1},\ldots,\overline{X_n} - \underline{X_n}, On(K_\alpha(X_1),\ldots,K_\alpha(X_n)), \\ &1 - On(K_\alpha(X_1),\ldots,K_\alpha(X_n))\}, \end{aligned}$$

is a general $\leq_{\alpha,\beta}$-overlap function.

Proof. Analogous to the proof of Theo. 4 in [35].

Remark 5. *Observe that **(IGO2)** and **(IGO3)** are only sufficient conditions, allowing for $\alpha \in [0,1]$ in the construction method presented in Theo. 7, differently than in Theo. 4, in which $\alpha \in (0,1)$. This means that, through Theo. 7, one can obtain general $\leq_{AD}$-overlap functions that are increasing with respect to either one of the lexicographical orders.*

Remark 6. *Regarding Theo. 7, one could think that it could be based on a general overlap function GO instead of a n-dimensional overlap function On, for it to be even more broad of a method. However, as the base function needs to be strictly increasing in order to the constructed interval-valued function AGO^{α} to be $\leq_{\alpha,\beta}$-increasing, by Lem. 1, one has that every strict general overlap function is also an n-dimensional overlap function, and that is why we chose to maintain On in Theo. 7 to reinforce this fact.*

Example 4. *Consider the general overlap function GO_P as defined in Ex. 1. Then, for $\alpha = 1$ and $\beta = 0$, the interval-valued function $AGO_P^1 : L([0,1])^n \to L([0,1])$ defined for all $\vec{X} \in L([0,1])^n$, by*

$$AGO_P^1(\vec{X}) = [GO_P(\overline{X_1}, \ldots, \overline{X_n}) - m, , GO_P(\overline{X_1}, \ldots, \overline{X_n})],$$

where

$$m = \min\{\overline{X_1} - \underline{X_1}, \ldots, \overline{X_n} - \underline{X_n}, GO_P(\overline{X_1}, \ldots, \overline{X_n}), \\ 1 - GO_P(\overline{X_1}, \ldots, \overline{X_n})\},$$

is a general $\leq_{1,0}$-overlap function, or in other words, a general $\leq_{Lex2}$-overlap function. It is noteworthy that AGO_P^1 is not an n-dimensional $\leq_{1,0}$-overlap function.

The next construction methods are inspired on Theo. 5. First, we will present a more restrictive construction method for n-dimensional $\leq_{\alpha,\beta}$-overlap functions:

Theorem 8. *Let $\alpha, \beta \in (0,1)$, such that, $\alpha \neq \beta$. Let $On : [0,1]^n \to [0,1]$ be a strict n-dimensional overlap function and $A : [0,1]^n \to [0,1]$ be a commutative aggregation function. Then $AOn_A^{\alpha} : L([0,1])^n \to L([0,1])$ defined by:*

$$AOn_A^{\alpha}(\vec{X}) = R, \ \ where, \begin{cases} K_{\alpha}(R) = On(K_{\alpha}(X_1), \ldots, K_{\alpha}(X_n)), \\ \lambda_{\alpha}(R) = A(\lambda_{\alpha}(X_1), \ldots, \lambda_{\alpha}(X_n)), \end{cases}$$

for all $\vec{X} \in L([0,1])^n$, is an n-dimensional $\leq_{\alpha,\beta}$-overlap function.

Proof. *From Theo. 5, it is immediate that AOn_A^{α} is well defined and $\leq_{\alpha,\beta}$-increasing, thus, respecting condition **(AOn4)**. Now, let us verify if AOn_A^{α} respects the remainder conditions from Def. 10:*
(IOn1) *Immediate, since On and A are commutative.*
(IOn2) *($\Rightarrow$) Take $\vec{X} \in L([0,1])^n$ and suppose that $AOn_A^{\alpha}(\vec{X}) = R = [0,0]$. Then, we have that*

$$K_{\alpha}(R) = K_{\alpha}([0,0]) = 0 = On(K_{\alpha}(X_1), \ldots, K_{\alpha}(X_n)),$$

*for all $\alpha \in (0,1)$. Thus, by condition **(On2)**, $K_{\alpha}(X_i) = 0$ for some $i \in \{1, \ldots, n\}$, for all $\alpha \in (0,1)$, and, therefore, $\prod_{i=1}^{n} X_i = [0,0]$;*
*($\Leftarrow$) Consider $\vec{X} \in L([0,1])^n$ such that $\prod_{i=1}^{n} = [0,0]$. So, $K_{\alpha}(X_1) \cdot \ldots \cdot K_{\alpha}(X_n) = 0$, for all $\alpha \in (0,1)$. Then, by **(On2)**, one has that*

$$K_{\alpha}(R) = On(K_{\alpha}(X_1), \ldots, K_{\alpha}(X_n)) = 0,$$

for all $\alpha \in (0,1)$, meaning that $AOn_A^{\alpha}(\vec{X}) = R = [0,0]$;

(IOw3) *($\Rightarrow$) Take $\vec{X} \in L([0,1])^n$ such that $AOn_A^{\alpha}(\vec{X}) = R = [1,1]$. Then, one has that*

$$K_{\alpha}(R) = K_{\alpha}([1,1]) = 1 = On(K_{\alpha}(X_1), \ldots, K_{\alpha}(X_n)).$$

By **(On3)**, $K_\alpha(X_1)\cdot\ldots\cdot K_\alpha(X_n)=1$, for all $\alpha\in(0,1)$, meaning that $\prod_{i=1}^n X_i=[1,1]$;

($\Leftarrow$) Consider $\vec{X}\in L([0,1])^n$ such that $\prod_{i=1}^n X_i=[1,1]$. So, $K_\alpha(X_1)\cdot\ldots\cdot K_\alpha(X_n)=1$, for all $\alpha\in(0,1)$. Then, by **(i)** and **(O3)**, one has that

$$K_\alpha(R)=On(K_\alpha(X_1),\ldots,K_\alpha(X_n))=1,$$

for all $\alpha\in(0,1)$, meaning that $AOn_A^\alpha(\vec{X})=R=[1,1]$.

The following result is immediate, as it derives from a similar situation as discussed in Remarks 5 and 6.

Theorem 9. *Let $\alpha,\beta\in[0,1]$, such that, $\alpha\neq\beta$. Let $On:[0,1]^n\to[0,1]$ be a strict n-dimensional overlap function and $A:[0,1]^n\to[0,1]$ be a commutative aggregation function. Then $AGO_A^\alpha:L([0,1])^n\to L([0,1])$ defined by:*

$$AGO_A^\alpha(\vec{X})=R,\ \text{ where, }\begin{cases}K_\alpha(R)=On(K_\alpha(X_1),\ldots,K_\alpha(X_n)),\\ \lambda_\alpha(R)=A(\lambda_\alpha(X_1),\ldots,\lambda_\alpha(X_n)),\end{cases}$$

for all $\vec{X}\in L([0,1])^n$, is an general $\leq_{\alpha,\beta}$-overlap function.

Example 5. *Consider the general overlap functions GO_L and GO_{Gm} as defined in Ex. 1. Then, for $\alpha=1$ and $\beta=0$, the interval-valued function $AGm_{GO_L}^1:L([0,1])^n\to L([0,1])$ defined for all $\vec{X}\in L([0,1])^n$, by*

$$AGm_{GO_L}^1(\vec{X}){=}R,\ \text{ where, }\begin{cases}K_1(R){=}GO_{Gm}(\overline{X_1},\ldots,\overline{X_n}),\\ \lambda_1(R){=}GO_L(\lambda_1(X_1),\ldots,\lambda_1(X_n)),\end{cases}$$

is a general $\leq_{1,0}$-overlap function, but not an n-dimensional $\leq_{1,0}$-overlap function.

The following method allow the construction of general $\leq_{AD}$-overlap functions by the generalized composition of general $\leq_{AD}$-overlap functions by an $\leq_{AD}$-increasing iv-aggregation function.

Theorem 10. *Consider $IM:L([0,1])^m\to L([0,1])$. For a tuple $\overrightarrow{AGO}=(AGO_1,\ldots,AGO_m)$ of general $\leq_{AD}$-overlap functions, define the mapping $IM_{\overrightarrow{AGO}}:L([0,1])^n\to L([0,1])$, for all $\vec{X}\in L([0,1])^n$, by:*

$$IM_{\overrightarrow{AGO}}(\vec{X})=IM(AGO_1(\vec{X}),\ldots,AGO_m(\vec{X})).$$

Then, $IM_{\overrightarrow{AGO}}$ is a general $\leq_{AD}$-overlap function if and only if IM is an $\leq_{AD}$-increasing iv-aggregation function.

Proof. It follows that:

($\Rightarrow$) Suppose that $IM_{\overrightarrow{AGO}}$ is a general $\leq_{AD}$-overlap function. Then it is immediate that IM $\leq_{AD}$-increasing, and, also, $\leq_{Pr}$-increasing **(IA2)**. Now consider $\vec{X}\in L([0,1])^n$ such that $\prod_{i=1}^n X_i=[0,0]$. Then, by **(IGO2)**, one has that: $IM_{\overrightarrow{AGO}}(\vec{X})=IM(AGO_1(\vec{X}),\ldots,AGO_m(\vec{X}))=[0,0]$ and $AGO_1(\vec{X})=\ldots=AGO_m(\vec{X})=[0,0]$. Thus, it holds that $IM([0,0],\ldots,[0,0])=[0,0]$. Now, consider $\vec{X}\in L([0,1])^n$, such that $X_i=[1,1]$ for all $i\in\{1,\ldots,n\}$. Then, by **(IGO3)**, one has that: $IM_{\overrightarrow{AGO}}(\vec{X})=IM(AGO_1(\vec{X}),\ldots,AGO_m(\vec{X}))=[1,1]$ and $AGO_1(\vec{X})=\ldots=AGO_m(\vec{X})=[1,1]$. Therefore, it holds that $IM([1,1],\ldots,[1,1])=[1,1]$. This proves that IM also satisfies condition **(IA1)**, and, thus, an $\leq_{AD}$-increasing iv-aggregation function.

($\Leftarrow$) Suppose that IM is an $\leq_{AD}$-increasing iv-aggregation function. Then it is immediate that $IM_{\overrightarrow{AGO}}$ is commutative (by **(IGO1)**), and respects **(AGO4)**. It remains to prove:

(IGO2) *Consider $\vec{X} \in L([0,1])^n$ such that $\prod_{i=1}^n X_i = [0,0]$. Then, by **(IGO2)**, one has that $AGO_1(\vec{X}) = \ldots = AGO_m(\vec{X}) = [0,0]$. It follows that: $IM_{\overrightarrow{AGO}}(\vec{X}) = IM(AGO_1(\vec{X}), \ldots, AGO_m(\vec{X})) = IM([0,0], \ldots, [0,0]) = [0,0]$, by condition **(IA1)**, since IM is an iv-aggregation function.*

(IGO3) *Take $\vec{X} \in L([0,1])^n$ such that $X_i = [1,1]$ for all $i \in \{1, \ldots, n\}$. Then, **(IGO3)**, it holds that $AGO_1(\vec{X}) = \ldots = AGO_m(\vec{X}) = [1,1]$. It follows that: $IM_{\overrightarrow{AGO}}(\vec{X}) = IM(AGO_1(\vec{X}), \ldots, AGO_m(\vec{X})) = IM([1,1], \ldots, [1,1]) = [1,1]$, by condition **(IA1)**. This proves that $IM_{\overrightarrow{AGO}}(\vec{X})$ is a general $\leq_{AD}$-overlap function.*

Example 6. *Consider the general $\leq_{1,0}$-overlap functions AGO_P, AGO_P^1 and $AGm^1_{GO_L}$, from Ex.s 3, 4 and 5. Then, the interval-valued function $AGO : L([0,1])^n \to L([0,1])$ defined, for all $\vec{X} \in L([0,1])^n$, by*

$$AGO(\vec{X}) = AGO_P(AGO_P^1(\vec{X}), AGm^1_{GO_L}(\vec{X})),$$

is a general $\leq_{1,0}$-overlap function.

5. Conclusion

In this paper we presented the concept of general admissibly ordered interval-valued overlap functions, a more flexible definition of n-dimensional interval-valued overlap functions that are increasing with respect to an admissible order. This new definition allowed us to construct several interval-valued overlap operations taking into account different admissible orders, in particular, $\leq_{\alpha,\beta}$ orders with any $\alpha, \beta \in [0,1]$ such that $\alpha \neq \beta$. Finally, those constructed functions can be combined by generalized composition to obtain new general admissibly ordered interval-valued overlap functions, showcasing their adaptability.

Most construction methods for $\leq_{\alpha,\beta}$-increasing functions are based on the aggregation of the K_α values of the inputs by strictly increasing aggregation functions, which is a restriction that could be interesting to overcome in our future work. We also intend to apply the developed functions (with different combination of construction methods) in classification problems with interval-valued data.

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5.2.4 Enhancing the efficiency of the interval-valued fuzzy rule-based classifier with tuning and rule selection

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Enhancing the efficiency of the interval-valued fuzzy rule-based classifier with tuning and rule selection

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Abstract. Interval-Valued fuzzy rule-based classifier with TUning and Rule Selection, IVTURS, is a state-of-the-art fuzzy classifier. One of the key point of this method is the usage of interval-valued restricted equivalence functions because their parametrization allows one to tune them to each problem, which leads to obtaining accurate results. However, they require the application of the exponentiation several times to obtain a result, which is a time demanding operation implying an extra charge to the computational burden of the method.
In this contribution, we propose to reduce the number of exponentiation operations executed by the system, so that the efficiency of the method is enhanced with no alteration of the obtained results. Moreover, the new approach also allows for a reduction on the search space of the evolutionary method carried out in IVTURS. Consequently, we also propose four different approaches to take advantage of this reduction on the search space to study if it can imply an enhancement of the accuracy of the classifier. The experimental results prove: 1) the enhancement of the efficiency of IVTURS and 2) the accuracy of IVTURS is competitive versus that of the approaches using the reduced search space.

Keywords: Interval-Valued Fuzzy Rule-based Classification Systems · Interval-Valued Fuzzy Sets · Interval Type-2 Fuzzy Sets · Evolutionary Fuzzy Systems.

1 Introduction

Classification problems [10], which consist of assigning objects into predefined groups or classes based on the observed variables related to the objects, have been widely studied in machine learning. To tackle them, a mapping function from the input to the output space, called classifier, needs to be induced applying a learning algorithm. That is, a classifier is a model encoding a set of criteria that allows a data instance to be assigned to a particular class depending on the value of certain variables.

Fuzzy Rule-Based Classification Systems (FRBCSs) [16] are applied to deal with classification problems, since they obtain accurate results while providing the user with a model composed of a set of rules formed of linguistic labels easily understood by humans. Interval-Valued FRBCSs (IVFRBCSs) [21], are an extension of FRBCSs where some (or all) linguistic labels are modelled by means of Interval-Valued Fuzzy Sets (IVFSs) [19].

IVTURS [22] is a state-of-the-art IVFRBCS built upon the basis of FARC-HD [1]. First, the two first steps of FARC-HD are applied to learn an initial fuzzy rule base, which is augmented with IVFSs to represent the inherent ignorance in the definition of the membership functions [20]. One of the key components of IVTURS is its Fuzzy Reasoning Method (FRM) [6], where all the steps consider intervals instead of numbers. When the matching degree between an example and the antecedent of a rule has to be computed, IVTURS makes usage of Interval-Valued Restricted Equivalence Functions (IV-REFs) [18]. These functions are introduced to measure the closeness between the interval membership degrees and the ideal ones, $[1,1]$. Their interest resides in their parametric construction method, which allows them to be optimized for each specific problem. In fact, the last step of IVTURS applies an evolutionary algorithm to find the most appropriate values for the parameters used in their construction.

However, the accurate results obtained when using IV-REFs comes at the price of the computational cost. To use an IV-REF it is necessary to apply several exponentiation operations, which are very time demanding. Consequently, the aim of this contribution is to reduce the run-time of IVTURS by decreasing the number of exponentiation operations required to obtain the same results. To do so, we propose two modifications:

- A mathematical simplification of the construction method of IV-REFs, which allows one to reduce to half the number of exponentiation operations.
- Add a verification step to avoid making computations both with incompatible interval-valued fuzzy rules as well as with do not care labels.

Moreover, the mathematical simplification also offers the possibility of reducing the search space of the evolutionary process carried out in IVTURS. This reduction may imply a different behaviour of the classifier, which may derive to an enhancement of the results. In this contribution, we propose four different approaches to explore the reduced search space for the sake of studying whether they allow one to improve the system's performance or not.

We use the same experimental framework that was used in the paper where IVTURS was defined [22], which consist of twenty seven datasets selected from the KEEL dataset repository [2]. We will test whether our two modifications reduce the run-time of IVTURS and the reduction rate achieved as well as the performance of the four different approaches considered to explore the reduced search space. To support our conclusions, we conduct an appropriate statistical study as suggested in the literature [7,13].

The rest of the contribution is arranged as follows: in Section 2 we recall some preliminary concepts on IVFSs, IV-REFs and IVTURS. The proposals for speeding IVTURS up and those to explore the reduced search space are described in Section 3. Next, the experimental framework and the analysis of the results are presented in Sections 4 and 5, respectively. Finally, the conclusions are drawn in Section 6.

2 Preliminaries

In this section, we review several preliminary concepts on IVFSs (Section 2.1), IV-REFs (Section 2.2) and IVFRBCSs (Section 2.3).

2.1 Interval-Valued Fuzzy Sets

This section is aimed at recalling the theoretical concepts related to IVFSs. We start showing the definition of IVFSs, whose history and relationship with other type of FSs as interval type-2 FSs can be found in [4].

Let $L([0,1])$ be the set of all closed subintervals in $[0,1]$:

$$L([0,1]) = \{\mathbf{x} = [\underline{x}, \overline{x}] | (\underline{x}, \overline{x}) \in [0,1]^2 \text{ and } \underline{x} \leq \overline{x}\}.$$

Definition 1. *[19] An interval-valued fuzzy set A on the universe $U \neq \emptyset$ is a mapping $A_{IV} : U \to L([0,1])$, so that*

$$A_{IV}(u_i) = [\underline{A}(u_i), \overline{A}(u_i)] \in L([0,1]), \textit{ for all } u_i \in U.$$

It is immediate that $[\underline{A}(u_i), \overline{A}(u_i)]$ is the interval membership degree of the element u_i to the IVFS A.

In order to model the conjunction among IVFSs we apply t-representable interval-valued t-norms [9] without zero divisors, that is, they verify that $\mathbf{T}(\mathbf{x}, \mathbf{y}) = 0_L$ if and only if $\mathbf{x} = 0_L$ or $\mathbf{y} = 0_L$. We denote them $\mathbf{T}_{T_a,T_b}$, since they are represented by T_a and T_b, which are the t-norms applied over the lower and the upper bounds, respectively. That is, $\mathbf{T}_{T_a,T_b}(\mathbf{x}, \mathbf{y}) = [\mathbf{T_a}(\underline{\mathbf{x}}, \underline{\mathbf{y}}), \mathbf{T_b}(\overline{\mathbf{x}}, \overline{\mathbf{y}})]$.

Furthermore, we need to use interval arithmetical operations [8] to make some computations. Specifically, the interval arithmetic operations we need in the work are:

- Addition: $[\underline{x}, \overline{x}] + [\underline{y}, \overline{y}] = [\underline{x} + \underline{y}, \overline{x} + \overline{y}]$.
- Multiplication: $[\underline{x}, \overline{x}] * [\underline{y}, \overline{y}] = [\underline{x} * \underline{y}, \overline{x} * \overline{y}]$.
- Division: $\frac{[\underline{x}, \overline{x}]}{[\underline{y}, \overline{y}]} = [\min(\min(\frac{\underline{x}}{\underline{y}}, \frac{\overline{x}}{\overline{y}}), 1), \min(\max(\frac{\underline{x}}{\underline{y}}, \frac{\overline{x}}{\overline{y}}), 1)]$ with $\underline{y} \neq 0$.

where $[\underline{x}, \overline{x}]$, $[\underline{y}, \overline{y}]$ are two intervals in $\mathbb{R}^+$ so that **x** is larger than **y**.

Finally, when a comparison between interval membership degrees is necessary, we use the total order relationship for intervals defined by Xu and Yager [23] (see Eq.(1)), which is also an admissible order [5].

$$[\underline{x}, \overline{x}] \leq [\underline{y}, \overline{y}] \text{ if and only if } \underline{x} + \overline{x} < \underline{y} + \overline{y} \text{ or } \underline{x} + \overline{x} = \underline{y} + \overline{y} \text{ and } \overline{x} - \underline{x} \geq \overline{y} - \underline{y} \quad (1)$$

Using Eq.(1) it is easy to observe that $0_L = [0,0]$ and $1_L = [1,1]$ are the smallest and largest elements in $L([0,1])$, respectively.

2.2 Interval-Valued Restricted Equivalence Functions

In IVTURS [22], one of the key components are the IV-REFs [11,18], whose aim is to quantify the equivalence degree between two intervals. They are the extension on IVFSs of REFs [3] and their definition is as follows:

Definition 2. *[11,18] An Interval-Valued Restricted Equivalence Function (IV-REF) associated with a interval-valued negation N is a function*

$$IV\text{-}REF : L([0,1])^2 \to L([0,1])$$

so that:

(IR1) $IV\text{-}REF(\boldsymbol{x},\boldsymbol{y}) = IV\text{-}REF(\boldsymbol{y},\boldsymbol{x})$ *for all* $\boldsymbol{x},\boldsymbol{y} \in L([0,1])$*;*
(IR2) $IV\text{-}REF(\boldsymbol{x},\boldsymbol{y}) = 1_L$ *if and only if* $\boldsymbol{x} = \boldsymbol{y}$*;*
(IR3) $IV\text{-}REF(\boldsymbol{x},\boldsymbol{y}) = 0_L$ *if and only if* $\boldsymbol{x} = 1_L$ *and* $\boldsymbol{y} = 0_L$ *or* $\boldsymbol{x} = 0_L$ *and* $\boldsymbol{y} = 1_L$*;*
(IR4) $IV\text{-}REF(\boldsymbol{x},\boldsymbol{y}) = IV\text{-}REF(N(\boldsymbol{x}),N(\boldsymbol{y}))$ *with N an involutive interval-valued negation;*
(IR5) For all $\boldsymbol{x},\boldsymbol{y},\boldsymbol{z} \in L([0,1])$*, if* $\boldsymbol{x} \leq_L \boldsymbol{y} \leq_L \boldsymbol{z}$*, then* $IV\text{-}REF(\boldsymbol{x},\boldsymbol{y}) \geq_L IV\text{-}REF(\boldsymbol{x},\boldsymbol{z})$ *and* $IV\text{-}REF(\boldsymbol{y},\boldsymbol{z}) \geq_L IV\text{-}REF(\boldsymbol{x},\boldsymbol{z})$.

In this work we use the standard negation, that is, $N(x) = 1 - x$.

An interesting feature of IV-REFs is the possibility of parametrize them by means of automorphisms as follows.

Definition 3. *An automorphism of the unit interval is any continuous and strictly increasing function* $\phi : [0,1] \to [0,1]$ *so that* $\phi(0) = 0$ *and* $\phi(1) = 1$.

An easy way of constructing automorphisms is by means of a parameter $\lambda \in (0,\infty)$: $\varphi(x) = x^{\lambda}$, and hence, $\varphi^{-1}(x) = x^{1/\lambda}$. Some automorphims constructed using different values of the parameter λ are shown in Figure 1.

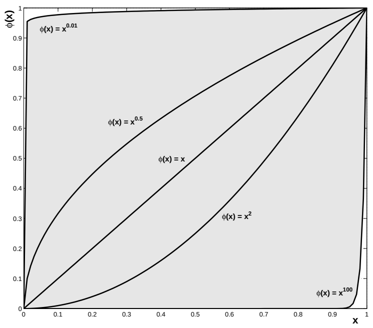

Fig. 1: Example of different automorphisms generated by different values of λ.

Then, the construction method of IV-REFs used in IVTURS can be seen in Eq.(2):

$$
\begin{aligned}
IV\text{-}REF(\mathbf{x},\mathbf{y}) = [&T(\phi_1^{-1}(1 - |\phi_2(\underline{x}) - \phi_2(\underline{y})|), \phi_1^{-1}(1 - |\phi_2(\overline{x}) - \phi_2(\overline{y})|)),\\
&S(\phi_1^{-1}(1 - |\phi_2(\underline{x}) - \phi_2(\underline{y})|), \phi_1^{-1}(1 - |\phi_2(\overline{x}) - \phi_2(\overline{y})|))]
\end{aligned} \tag{2}
$$

where T is the minimum t-norm, S is the maximum t-conorm and φ_1, φ_2 are two automorphisms of the interval $[0,1]$ parametrized by λ_1 and λ_2, respectively. Therefore, the IV-REFs used in IVTURS are as follows:

$$
\begin{aligned}
IV\text{-}REF(\mathbf{x},\mathbf{y}) = [&\min((1 - |\underline{x}^{\lambda_2} - \underline{y}^{\lambda_2}|)^{1/\lambda_1}, (1 - |\overline{x}^{\lambda_2} - \overline{y}^{\lambda_2}|)^{1/\lambda_1}),\\
&\max((1 - |\underline{x}^{\lambda_2} - \underline{y}^{\lambda_2}|)^{1/\lambda_1}, (1 - |\overline{x}^{\lambda_2} - \overline{y}^{\lambda_2}|)^{1/\lambda_1})]
\end{aligned} \tag{3}
$$

2.3 Interval-Valued Fuzzy Rule-Based Classification Systems

Solving a classification problem consists in learning a mapping function called *classifier* from a set of training examples, named *training set*, that allows new examples to be classified. The training set is composed of P examples, $x_p = (x_{p1}, \ldots, x_{pn}, y_p)$, where x_{pi} is the value of the i-th attribute $(i = 1, 2, \ldots, n)$ of the p-th training example. Each example belongs to a class $y_p \in \mathbb{C} = \{C_1, C_2, ..., C_m\}$, where m is the number of classes of the problem.

IVFRBCSs are a technique to deal with classification problems [20], where each of the n attributes is described by a set of linguistic terms modeled by their corresponding IVFSs. Consequently, they provide an interpretable model as the antecedent part of the fuzzy rules is composed of a subset of these linguistic terms as shown in Eq. (4).

$$\text{Rule } R_j : \text{ If } x_1 \text{ is } A_{j1} \text{ and } \ldots \text{ and } x_n \text{ is } A_{jn} \text{ then Class} = C_j \text{ with } RW_j \tag{4}$$

where R_j is the label of the jth rule, $x = (x_1, \ldots, x_n)$ is an n-dimensional pattern vector, A_{ji} is an antecedent IVFS representing a linguistic term, C_j is the class label, and RW_j is the rule weight [17].

IVTURS [22] is an state-of-the-art IVFRBCSs, whose learning process is composed of two steps:

1. To build an IV-FRBCS. This step involves the following tasks:
   - The generation of an initial FRBCS by applying FARC-HD [1].
   - Modelling the linguistic labels of the learned FRBCS by means of IVFSs.
   - The generation of an initial IV-REF for each variable of the problem.
2. To apply an optimization approach with a double purpose:
   - To learn the best values of the IV-REFs' parameters, that is, the values of the exponents of the automorphisms (λ_1 and λ_2).
   - To apply a rule selection process in order to decrease the system's complexity.

In order to be able to classify new examples, $x_p = (x_{p1}, \ldots, x_{pn})$, IVTURS considers an Interval-Valued Fuzzy Reasoning Method [22] (IV-FRM), which uses the L interval-valued fuzzy rules composing the model as follows:

1. *Interval matching degree*: It quantifies the strength of activation of the if-part for all rules (L) in the system with the example x_p:

$$\begin{aligned} [\underline{A_j}(x_p), \overline{A_j}(x_p)] = \mathbf{T}_{T_a,T_b}(&IV\text{-}REF([\underline{A_{j1}}(x_{p1}), \overline{A_{j1}}(x_{p1})], [1,1]), \ldots, \\ &IV\text{-}REF([\underline{A_{jn}}(x_{pn}), \overline{A_{jn}}(x_{pn})], [1,1])), \qquad j = 1, \ldots, L. \end{aligned} \tag{5}$$

2. *Interval association degree*: for each rule, R_j, the interval matching degree is weighted by its rule weight $RW_j = [\underline{RW_j}, \overline{RW_j}]$::

$$[\underline{b_j}(x_p), \overline{b_j}(x_p)] = [\underline{\mu_{A_j}}(x_p), \overline{\mu_{A_j}}(x_p)] * [\underline{RW_j}, \overline{RW_j}] \quad j = 1, \ldots, L. \tag{6}$$

3. *Interval pattern classification soundness degree for all classes*. The positive interval association degrees are aggregated by class applying an aggregation function *f*.

$$[\underline{Y_k}, \overline{Y_k}] = f_{R_j \in RB;\ C_j = k}([\underline{b_j}(x_p), \overline{b_j}(x_p)] | [\underline{b_j}(x_p), \overline{b_j}(x_p)] > 0_L), \qquad k = 1, \ldots, m. \tag{7}$$

4. *Classification*. A decision function F is applied over the interval soundness degrees:

$$F([\underline{Y_1}, \overline{Y_1}], ..., [\underline{Y_m}, \overline{Y_m}]) = \arg \max_{k=1,\ldots,m}([\underline{Y_k}, \overline{Y_k}]) \tag{8}$$

3 Enhancing the efficiency of IVTURS

IVTURS provides accurate results when tackling classification problems. However, we are concerned about its computational burden as it may be an obstacle to use it in real-world problems. The most computationally expensive operation in IVTURS is the exponentiation operation required when computing the IV-REFs, which are constantly used in the IV-FRM (Eq. (5)). Though there are twelve exponentiation operations in Eq. (3), only four of them need to be computed because: 1) $\underline{y} = \overline{y} = 1$, implying that the computation of $\underline{y}^{\lambda_2}$ and $\overline{y}^{\lambda_2}$ can be avoided as one raised to any number is one; 2) the lower and the upper bound of the resulting IV-REF are based on the minimum and maximum of the same operations, which reduces the number of operations to the half.

The aim of this contribution is to reduce the number of exponentiation operations needed to execute IVTURS, which will imply an enhancement of the system's efficiency. To do so, we propose two modifications to the original IVTURS: 1) to apply a mathematical simplification of the IV-REFs that reduces to half the number of exponentiation operations (Section 3.1) and 2) to avoid applying IV-REFs with both do not care labels and incompatible interval-valued fuzzy rules (Section 3.2).

Furthermore, the mathematical simplification of IV-REFs, besides reducing the number of exponentiation operations while obtaining the same results, would also allow us to also reduce the search space of the evolutionary algorithm, possibly implying in a different behavior in the system. We will study whether this reduction of the search space could result in a better performance of the system by using four different approaches to explore it (Section 3.3).

3.1 IV-REFs simplification

IV-REFs are used to measure the degree of closeness (equivalence) between two intervals. In IVTURS, they are used to compute the equivalence between the interval membership degrees and the ideal membership degree, that is, IV-REF($[\underline{x}, \overline{x}], [1, 1]$). Precisely, because one of the input intervals is $[1, 1]$, we can apply the following mathematical simplification.

$$IV\!-\!REF([\underline{x}, \overline{x}], [1, 1]) = [(1 - |\underline{x}^{\lambda_2} - 1^{\lambda_2}|)^{1/\lambda_1}, (1 - |\overline{x}^{\lambda_2} - 1^{\lambda_2}|)^{1/\lambda_1}] = [\underline{x}^{\lambda_2/\lambda_1}, \overline{x}^{\lambda_2/\lambda_1}] \tag{9}$$

Therefore, we can obtain the same result by just raising the value of the interval membership degree to the division of both exponents (λ_2/λ_1), which imply reducing to half the number of operations.

3.2 Avoiding incompatible rules and do not care labels

When the inference process is applied to classify a new example, the interval matching degree has to be obtained for each rule of the system. The maximum number of antecedents of the interval-valued fuzzy rules used in IVTURS is limited to a certain hyper-parameter of the algorithm, k_t, whose default value is 3. This fact implies that in almost all the classification problems the usage of do not care labels is necessary, since the number of input attributes is greater than that of k_t. In order to program this feature of IVTURS, a do not care label is considered as an extra membership function that returns the neutral element for the t-representable interval-valued t-norm used ($[1,1]$ in this case as the product is applied). In this manner, when performing the conjunction of the antecedents the usage of do not care labels do not change the obtained result. However, this fact implies that when having a do not care label, it returns $[1,1]$ as interval membership degree and IV-REF($[1,1],[1,1]$) needs to be computed ($[\underline{x},\overline{x}] = [1,1]$). Consequently, a large number of exponentiation operations can be saved if we avoid computing IV-REF in this situations as the result is always $[1,1]$.

On the other hand, we also propose to avoid obtaining the interval matching degree and thus computing the associated IV-REFs when the example is not compatible with the antecedent of the interval-valued fuzzy rule. To do so, we need to perform an initial iteration where we check whether the example is compatible with the rule. Then, the interval matching degree is only computed when they are compatible. This may see to be an extra charge for the run-time but we take advantage of this first iteration to obtain the interval matching degrees, avoiding the do not care labels, and we send them to the function that computes the interval matching degree.

These two modifications could have a huge impact on the run-time of IVTURS because do not care labels are very common in the interval-valued fuzzy rules of the system and the proportion of compatible rules with an example is usually small.

3.3 Reducing the search space in the evolutionary process of IVTURS

In Section 3.1 we have presented a mathematical simplification of the IV-REFs that allows one to reduce the number of exponentiation operations obtaining exactly the same results than that obtained in the original formulation of IVTURS. However, according to Eq. (9) we can observe that both parameters of the simplified IV-REF (λ_1, λ_2) can be collapsed into a unique one (λ) as shown in Eq. (10).

$$IV-REF([\underline{x},\overline{x}],[1,1]) = [\underline{x}^{\lambda_2/\lambda_1}, \overline{x}^{\lambda_2/\lambda_1}] = [\underline{x}^{\lambda}, \overline{x}^{\lambda}] \tag{10}$$

In this manner, the search space of the evolutionary process carried out in IVTURS, where the values of λ_1 and λ_2 are tuned to each problem, can be also reduced to half because only the value of λ needs to be tuned. Consequently, the behaviour of the algorithm can change and we aim at studying whether this reduction is beneficial or not. Specifically, the structure of the chromosome is: $C_i = (g_{\lambda_1}, g_{\lambda_2}, \ldots, g_{\lambda_n})$, where g_{λ_i}, $i = 1, \ldots, n$, are the genes representing the value of λ_i and n is the number of input variables of the classification problem.

The parameter λ can vary theoretically between zero and infinity. However, in IV-TURS, λ_1 and λ_2 are limited to the interval $[0.01, 100]$. On the other hand, in the evolutionary process, those genes used to encode them are codified in $[0.01, 1.99]$, $g_{\lambda_i} \in [0.01, 1.99]$, in such a way that the chances of learning values in $[0.01, 1]$ and in $(1, 100]$ are the same. Consequently, these genes have to be decoded so that they are in the range $[0.01, 100]$ when used in the corresponding IV-REF. The decoding process is driven by the following equation:

$$g_{\lambda_i} = \begin{cases} g_{\lambda_i}, & \text{if } 0 < g_{\lambda_i} \leq 1 \\ \frac{1}{2 - g_{\lambda_i}}, & \text{if } 1 < g_{\lambda_i} < 2 \end{cases} \tag{11}$$

In [12], Galar et. al use REFs (the numerical counterpart of IV-REFs) to deal with the problem of difficult classes applying the OVO decomposition strategy. In this method, on the one hand, those genes used for representing the parameter λ are coded in the range $(0, 1)$. On the other hand, the decoding process of the genes is driven by Eq.12.

$$\lambda_i = \begin{cases} (2 \cdot g_{\lambda_i})^2 & \text{if } g_{\lambda_i} \leq 0.5 \\ \dfrac{1}{(1 - 2 \cdot (g_{\lambda_i} - 0.5))^2} & \text{otherwise.} \end{cases} \tag{12}$$

There are two main differences between these two methods: 1) the decoded value by Eq. (11) is in the range $[0.01, 100]$, whereas when using Eq. (12) the values are in $(0, \infty)$ and 2) the search space is explored in a different way as can be seen in the two first rows of Figure 2, where the left and the right columns show how the final values when $\lambda_i \leq 1.0$ and $\lambda_i > 1.0$ are obtained, respectively.

Looking at these two methods, we propose another two new ones:

– Linear exploration of the search space: we encode all the genes in the range $[0.01, 1.99]$ and we decode them using a linear normalization in the ranges $[0.0001, 1.0]$ and $(1.0, 10000]$ for the genes in $[0.01, 1.0]$ and $(1.0, 1.99]$, respectively.
– Mixture of Eq. (11) and Eq. (12). Genes are encoded in $(0, 1)$ and they are decoded using Eq. (11) for genes in $(0, 0.5]$ (linear decoding: $2 \cdot g_{\lambda_i}$)and Eq. (12) for genes in $(0.5, 1.0]$.

4 Experimental Framework

We have considered the same datasets which were used in the paper where IVTURS was proposed. That is, we select twenty-seven real world data-sets from the KEEL data-set repository [2]. Table 1 summarizes their properties: number of examples (#Ex.), at-

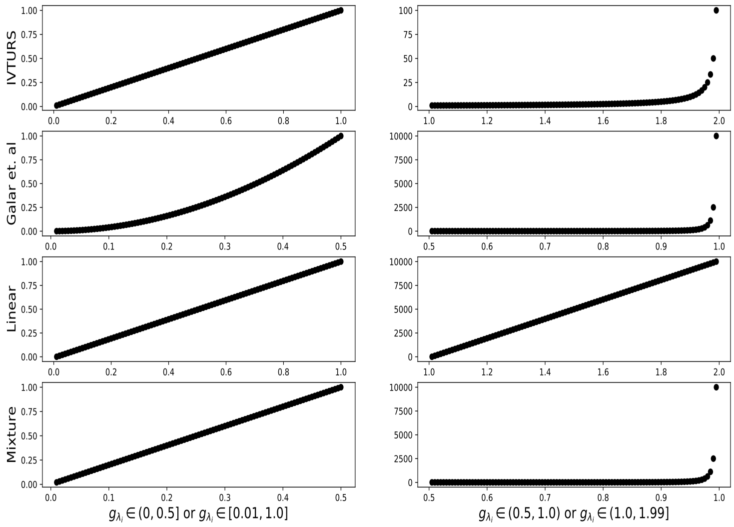

Fig. 2: Effect of the decoding method of the parameter g_λ on the way how the search space is explored.

Table 1: Description of the selected data-sets.

Id.	Data-set	#Ex.	#Atts.	#Class.
aus	Australian	690	14	2
bal	Balance	625	4	3
cle	Cleveland	297	13	5
con	Contraceptive	1,473	9	3
crx	Crx	653	15	2
der	Dermatology	358	34	6
eco	Ecoli	336	7	8
ger	German	1,000	20	2
hab	Haberman	306	3	2
hay	Hayes-Roth	160	4	3
hea	Heart	270	13	2
ion	Ionosphere	351	33	10
iri	Iris	150	4	3
mag	Magic	1,902	10	2
new	New-Thyroid	215	5	3
pag	Page-blocks	548	10	5
pen	Penbased	1,992	16	10
pim	Pima	768	8	2
sah	Saheart	462	9	2
spe	Spectfheart	267	44	2
tae	Tae	151	5	3
tit	Titanic	2,201	3	2
two	Twonorm	740	20	2
veh	Vehicle	846	18	4
win	Wine	178	13	3
wiR	Winequality-Red	1,599	11	11
wis	Wisconsin	683	9	2

tributes (#Atts.) and classes (#Class.)[4]. We apply a *5-fold cross-validation model* using the standard accuracy rate to measure the performance of the classifiers.

[4] We must recall that, as in the IVTURS' paper, the *magic, page-blocks, penbased, ring, satimage* and *shuttle* data-sets have been stratified sampled at 10% in order to reduce their size for training. In the case of missing values (*crx, dermatology* and *wisconsin*), those instances have been removed from the data-set

In this contribution we use the configuration of IVTURS that was used in the paper were it was defined:

- Fuzzy rule learning:
  - $Minsup$: 0.05.
  - $Maxconf$: 0.8.
  - $Depth_{max}$: 3.
  - k_t: 2.
- Evolutionary process
  - Population Size: 50 individuals.
  - Number of evaluations: 20,000.
  - Bits per gene for the Gray codification (for incest prevention): 30 bits.
- IVFSs construction:
  - Number of linguistic labels per variable: 5 labels.
  - Shape: Triangular membership functions.
  - Upper bound: 50% greater than the lower bound ($W = 0.25$).
- Configuration of the initial IV-REFs:
  - T-norm: minimum.
  - T-conorm: maximum.
  - First automorphism: $\phi_1(x) = x^1$ (a = 1).
  - Second automorphism: $\phi_2(x) = x^1$ (b = 1).
- Rule weight: fuzzy confidence (certainty factor) [17].
- Fuzzy reasoning method: additive combination [6].
- Conjunction operator: product interval-valued t-norm.
- Combination operator: product interval-valued t-norm.

5 Analysis of the obtained results

This section is aimed at showing the obtained results having a double aim:

1. To check whether the two modifications proposed for enhancing the run-time of IVTURS allow one to speed it up or not.
2. To study if the reduction of the search space made possible by the mathematical simplification of the IV-REFs allows one to improve the results of IVTURS.

In first place we show in Table 2 the run-time in seconds of the three versions of IVTURS[5], namely, the original IVTURS, IVTURS using the mathematical simplification of the IV-REFs ($IVTURS_{v1}$) and IVTURS avoiding the usage of incompatible interval-valued fuzzy rules and do not care labels ($IVTURS_{v2}$). For $IVTURS_{v1}$, the number in parentheses is the reduction rate achieved versus the original IVTURS, whereas in the case of $IVTURS_{v2}$ it is the reduction rate achieved with respect to $IVTURS_{v1}$.

Looking at the obtained results we can conclude that the two versions allow IVTURS to be more efficient. In fact, $IVTURS_{v1}$ allows one to reduce to half the run-time of IVTURS as expected, since the number of exponentiation operations is also

[5] We do not show the accuracy of the methods because they obtain the same results.

Table 2: Run-time in seconds of IVTURS besides the two versions developed for speeding it up. The number in parentheses shows the reduction rate of the method in the column versus the method in its respective left column.

Dataset	IVTURS	$IVTURS_{v1}$	$IVTURS_{v2}$
aus	2032.22	1019.17 (x1.99)	288.22 (x3.54)
bal	909.98	463.19 (x1.96)	193.64 (x2.39)
cle	1460.90	720.98 (x2.03)	133.24 (x5.41)
con	5572.14	2896.43 (x1.92)	971.67 (x2.98)
crx	1781.01	885.32 (x2.01)	227.00 (x3.90)
der	2167.93	1087.80 (x1.99)	107.32 (x10.14)
eco	466.07	244.90 (x1.90)	87.32 (x2.80)
ger	9462.10	4894.81 (x1.93)	838.09 (x5.84)
hab	97.28	53.69 (x1.81)	31.78 (x1.69)
hay	99.16	52.54 (x1.89)	32.75 (x1.60)
hea	764.16	388.49 (x1.97)	102.06 (x3.81)
ion	1314.86	663.30 (x1.98)	81.15 (x8.17)
iri	35.10	19.61 (x1.79)	7.40 (x2.65)
mag	3840.45	1976.30 (x1.94)	581.34 (x3.40)
new	96.94	51.98 (x1.87)	22.77 (x2.28)
pag	571.19	293.44 (x1.95)	66.80 (x4.39)
pen	6417.29	3273.78 (x1.96)	511.53 (x6.40)
pim	1433.95	700.68 (x2.05)	306.43 (x2.29)
sah	963.71	485.20 (x1.99)	181.29 (x2.68)
spe	1319.11	647.94 (x2.04)	117.44 (x5.52)
tae	141.29	76.32 (x1.85)	40.33 (x1.89)
tit	490.51	261.57 (x1.88)	141.57 (x1.85)
two	2426.83	1203.81 (x2.02)	225.71 (x5.33)
veh	4264.57	2162.36 (x1.97)	416.60 (x5.19)
win	231.12	117.65 (x1.96)	23.41 (x5.02)
wiR	5491.86	2807.14 (x1.96)	842.27 (x3.33)
wis	726.39	360.51 (x2.01)	91.74 (x3.93)
Mean	2021.41	1029.96 (x1.95)	247.07 (x4.02)

reduced to half. On the other hand, $IVTURS_{v2}$ exhibits a huge reduction on the run-time with respect to that of the original IVTURS as it is 7.839 times faster (1.95*4.02). These modifications allow IVTURS to be applied in a wider range of classification problems as its efficiency has been notably enhanced. The code of the IVTURS method using the two modification for speeding it up can be found at: `https://github.com/JoseanSanz/IVTURS`.

The second part of the study is to analyze whether the reduction of the search space enabled by the mathematical simplification if the IV-REFs allows one to improve the accuracy of the system or not. As we have explained in Section 3.3, we propose four approaches to codify and explore the reduced search space: 1) the same approach than that used in the original IVTURS but using the reduced search space (IVTURS$_{Red.}$); 2) the approach defined by Galar et al. [12] but extended on IVFSs (IVTURS$_{Galar}$); 3) the mixture of the two previous approaches (IVTURS$_{Mix.}$) and 4) the linear exploration of the search space (IVTURS$_{Linear}$).

In Table 3 we show the results obtained in testing by these four approach besides those obtained by the original IVTURS. We stress in **bold-face** the best result for each dataset. Furthermore, we also show the averaged performance in the 27 datasets (Mean).

According to the results shown in Table 3, we can observe that both methods using the approach defined by Galar et. al (IVTURS$_{Galar}$ and IVTURS$_{Mix.}$) allows one to

Table 3: Testing results provided by IVTURS and the four approaches using the reduced search space.

Dataset	IVTURS	IVTURS$_{Red.}$	IVTURS$_{Galar}$	IVTURS$_{Linear}$	IVTURS$_{Mix.}$
aus	**85.80**	85.07	84.20	84.64	85.36
bal	**85.76**	85.28	85.28	85.60	85.12
cle	**59.60**	58.24	58.58	56.22	58.93
con	53.36	53.02	53.16	**53.97**	53.57
crx	**87.14**	85.91	86.68	85.92	85.15
der	**94.42**	93.58	94.13	93.02	94.14
eco	78.58	78.28	80.96	**82.13**	80.07
ger	73.10	72.00	72.90	73.10	**73.30**
hab	72.85	73.17	72.19	71.55	**73.50**
hay	80.23	75.70	81.00	78.66	**81.77**
hea	**88.15**	85.93	87.41	86.67	**88.15**
ion	89.75	90.60	**92.60**	91.46	92.04
iri	96.00	**97.33**	96.00	96.00	96.00
mag	79.76	79.07	80.28	**80.91**	80.49
new	95.35	95.81	95.35	96.74	**97.21**
pag	95.07	94.16	94.70	**95.43**	94.89
pen	92.18	89.91	**92.64**	91.73	91.64
pim	75.90	74.48	74.87	**76.04**	74.61
sah	70.99	70.13	69.05	**71.20**	70.56
spe	80.52	79.39	**81.26**	80.15	80.16
tae	50.34	**58.30**	57.66	53.68	57.01
tit	**78.87**	**78.87**	**78.87**	**78.87**	**78.87**
two	92.30	90.95	92.43	91.22	**92.70**
veh	**67.38**	64.54	66.43	64.43	67.26
win	**97.19**	94.37	95.48	94.94	96.06
wiR	58.28	59.47	59.04	**59.66**	59.16
wis	96.49	**96.63**	**96.63**	96.34	96.34
Mean	80.57	80.01	80.73	80.38	**80.89**

improve the averaged accuracy of IVTURS. The reduction of the search space using the original approach defined in IVTURS, IVTURS$_{Red.}$, does not provide competitive results whereas the approach using a linear exploration of the search space also obtains worse results than those of the original IVTURS.

In order to give statistical support to our analysis we have carried out the Aligned Friedman's ranks test [14] to compare these five methods, whose obtained p-value is 1.21E-4 that implies the existence of statistical differences among them. For this reason, we have applied the Holm's post hoc test [15] to compare the control method (the one associated with the less rank) versus the remainder ones. In Table 4, we show both the ranks of the methods computed by the Aligned Friedman's test as well as the Adjusted P-Value (APV) obtained when applying the Holm's test.

Table 4: Results obtained by the Aligned Friedman's rank test and the Holm's test.

Method	Rank	APV
IVTURS$_{Mix.}$	52.48	
IVTURS	58.00	0.76
IVTURS$_{Galar}$	61.85	0.76
IVTURS$_{Linear}$	74.91	0.11
IVTURS$_{Red.}$	92.76	6.19E-4

Looking at the results of the statistical study we can conclude that IVTURS, IVTURS$_{Mix.}$ and IVTURS$_{Galar.}$ are statistically similar. However, there are statistical differences with respect to IVTURS$_{Red.}$ and a trend in favour to the three former methods when compared versus IVTURS$_{Linear}$. All in all, we can conclude that the approach defined in the original IVTURS provides competitive results even when compared against methods whose search space is reduced to half.

6 Conclusion

In this contribution we have proposed two modifications over IVTURS aimed at enhancing its efficiency. On the one hand, we have used a mathematical simplification of the IV-REFs used in the inference process. On the other hand, we avoid making computations with both incompatible interval-valued fuzzy rules and do not care labels, since they do not affect the obtained results and they entail a charge to the computational burden of the method. Moreover, we have proposed a reduction of the search space of the evolutionary process carried out in IVTURS using four different approaches.

The experimental results have proven the improvement of the run-time of the method, since it is almost eight times faster that the original IVTURS when applying the two modifications. Regarding the reduction of the seach space we have learned the following lessons: 1) the new methods based on the approach defined by Galar et. all allow one to improve the results without statistical differences versus IVTURS; 2) the simplification of the search space using the same setting defined in IVTURS does not provide competitive results, possibly due to the limited range where the genes are decoded when compared with respect the remainder approaches and 3) the linear exploration of the search space does not provide good results neither, which led us think that the most proper values are closer to one than to ∞.

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