General Admissibly Ordered Interval-valued Overlap Functions

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Abstract

Overlap functions are a class of aggregation functions that measure the overlapping degree between two values. They have been successfully applied in several problems in which associativity is not required, such as classification and image processing. Some generalizations of overlap functions were proposed for them to be applied in problems with more than two classes, such as n-dimensional and general overlap functions. To measure the overlapping of interval data, interval-valued overlap functions were defined, and, later, they were also generalized in the form of n-dimensional and general interval-valued overlap functions. In order to apply some of those concepts in problems with interval data considering the use of admissible orders, which are total orders that refine the most used partial order for intervals, n-dimensional admissibly ordered interval-valued overlap functions were recently introduced, proving to be suitable to be applied in classification problems. However, the sole construction method presented for this kind of function do not allow the use of the well known lexicographical orders. So, in this work we combine previous developments to introduce general admissibly ordered interval-valued overlap functions, while also presenting different construction methods and the possibility to combine such methods, showcasing the flexibility and adaptability of this approach, while also being compatible with the lexicographical orders.

1. Introduction

Overlap functions are aggregation functions, initially introduced in the context of image processing problems, to measure the overlapping between classes [1, 2, 3]. Since then, they have been studied in the literature by many authors, mainly because of either the advantages they present over the popular t-norms [4, 5] or their great applicability, as in: fuzzy rule-based

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classification [6, 7, 8, 9, 10], decision making [11, 12], wavelet -fuzzy power quality diagnosis system [13], forest fire detection [14], among others.

The concept of n-dimensional overlap functions was introduced [15] in order to allow the application of overlap functions, which were originally defined as bivariate functions that do not need to be associative, in problems with multiple classes. By relaxing the boundary conditions of n-dimensional overlap functions, general overlap functions were defined, also showing good behaviour when applied in classification problems [16].

Now observe that, when working with fuzzy systems, one may face the problem regarding the uncertainty in assigning the values of the membership degrees or defining the membership functions that are adopted in the system. In the literature, a proposed solution is given by the use of interval-valued fuzzy sets (IVFSs) [17, 18, 19], where the membership degrees are represented by intervals, whose widths represent such uncertainty [20, 21, 22]. IVFSs have been successfully applied in many different fields, such as classification [23, 24, 25], image processing [26, 27], game theory [28], multicriteria decision making [29], pest control [30], irrigation systems [31] and collaborative clustering [32].

To avoid a stalemate when comparing interval data, Bustince et al. [33] introduced the concept of admissible orders for intervals, that is, total order relations that refine the usual product order [34], which is a partial order. Since their introduction, several works were developed taking admissible orders into account, such as [35, 36, 37, 38].

Qiao and Hu [39] and Bedregal et al. [40] defined, independently, the concept of interval-valued overlap functions. By extending and generalizing interval-valued overlap functions, Asmus et al. [23] introduced the concepts of n-dimensional interval-valued overlap functions and general interval-valued overlap functions, both concepts taking into account the usual increasingness with respect to the product order.

Allowing for a broader practical application of (n-dimensional) interval-valued overlap functions, Asmus et al. [35] introduced the concept of n-dimensional admissibly ordered interval-valued overlap functions, which are n-dimensional interval-valued overlap functions that are increasing with respect to an admissible order. They also presented a construction method, which, however, cannot generate n-dimensional interval-valued overlap functions that are increasing with respect to the well known lexicographical orders [33]. Although this is not a serious problem, with the initial motivation to overcome this drawback, in this present work we combine the recent developed concepts on (n-dimensional, general) interval-valued overlap functions and admissible orders to introduce general admissibly ordered interval-valued overlap function. However, the resulting definition proved to be much more flexible and adaptable, allowing for the development of different construction methods, and even the composition of functions constructed through those methods.

The paper is organized as follows. Section 2 presents some preliminary concepts. In Section 3, we introduce the concept of general admissibly ordered iv-overlap functions, studying its representation and relation with n-dimensional admissibly ordered iv-overlap function. In section 4, we present some construction methods for general admissibly ordered iv-overlap functions. Section 5 is the Conclusion.

2. Preliminaries

In this section, we recall some concepts on general overlap functions, interval mathematics, admissible orders and (admissibly ordered) interval-valued overlap functions.

2.1. General Overlap Functions

Definition 1. *[41] An aggregation function is a mapping $A : [0,1]^n \rightarrow [0,1]$ that is increasing in each argument and satisfying:* ***(A1)*** *$A(0,\ldots,0) = 0$;* ***(A2)*** *$A(1,\ldots,1) = 1$.*

Definition 2. *[42, 15] A function $On : [0,1]^n \rightarrow [0,1]$ is said to be an n-dimensional overlap function if, for all $\vec{x} \in [0,1]^n$:* ***(On1)*** *On is commutative;* ***(On2)*** *$On(\vec{x}) = 0 \Leftrightarrow \prod_{i=1}^{n} x_i = 0$;* ***(On3)*** *$On(\vec{x}) = 1 \Leftrightarrow \prod_{i=1}^{n} x_i = 1$;* ***(On4)*** *$On$ is increasing;* ***(On5)*** *On is continuous.*

When On is strictly increasing in $(0,1]$, it is called a *strict n-dimensional overlap function.* A 2-dimensional overlap function is just called overlap function [43, 1].

By changing the boundaries conditions **(On2)** and **(On3)** to obtain a less restrictive definition, *general overlap functions* were introduced as follows:

Definition 3. *[16] A function $GO : [0,1]^n \rightarrow [0,1]$ is said to be a general overlap function if, for all $\vec{x} \in [0,1]^n$:* ***(GO1)*** *On is commutative;* ***(GO2)*** *$\prod_{i=1}^{n} x_i = 0 \Rightarrow GO(\vec{x}) = 0$* ***(GO3)*** *$\prod_{i=1}^{n} x_i = 1 \Rightarrow GO(\vec{x}) = 1$;* ***(GO4)*** *$GO$ is increasing;* ***(GO5)*** *GO is continuous.*

Proposition 1. *[16] If $On : L([0,1])^n \rightarrow [0,1]$ is an n-dimensional overlap function, then On is also a general overlap function, but the converse may not hold.*

Example 1. *The following are all examples of general overlap functions, defined for all $\vec{x} \in [0,1]^n$:*

a) *The product GO_P, given by $GO_P(\vec{x}) = \prod_{i=1}^{n} x_i$, which is a strict n-dimensional overlap function, and, whenever $n = 2$, it is the product t-norm [44].*

b) *The function GO_L, given by $GO_L(\vec{x}) = \max\{(\sum_{i=1}^{n} x_i) - (n-1), 0\}$, which is not neither an n-dimensional overlap function nor strictly increasing. For $n = 2$, it is the Lukasiewicz t-norm [44].*

c) *The geometric mean GO_{Gm}, given by $GO_{Gm}(\vec{x}) = \sqrt[n]{\prod_{i=1}^{n} x_i}$, which is a strict n-dimensional overlap function, but it is not a t-norm, when $n = 2$ [1].*

For properties on (n-dimensional) overlap functions, general overlap functions and related concepts, see also: [16, 45, 4, 46, 47, 15, 48, 49, 50].

2.2. Interval Mathematics and Admissible Orders

Let us denote as $L([0,1])$ the set of all closed subintervals of the unit interval $[0,1]$. Denote $\vec{x} = (x_1,\ldots,x_n) \in [0,1]^n$ and $\vec{X} = (X_1,\ldots,X_n) \in L([0,1])^n$. Given any $X = [x_1, x_2] \in L([0,1])$, $\underline{X} = x_1$ and $\overline{X} = x_2$ denote, respectively, the left and right projections of X, and

$w(X) = \overline{X} - \underline{X}$ denotes the width of X. The interval product is defined for all $X, Y \in L([0,1])$ by:

$$X \leq_{Pr} Y \quad \Leftrightarrow \quad \underline{X} \leq \underline{Y} \wedge \overline{X} \leq \overline{Y}.$$

We call as $\leq_{Pr}$-increasing a function that is increasing with respect to the product order $\leq_{Pr}$. The projections $F^-, F^+ : [0,1]^n \to [0,1]$ of $F : L([0,1])^n \to L([0,1])$ are defined, respectively, by:

$$F^-(x_1, \ldots, x_n) \quad = \quad \underline{F([x_1, x_1], \ldots, [x_n, x_n])}; \tag{1}$$
$$F^+(x_1, \ldots, x_n) \quad = \quad \overline{F([x_1, x_1], \ldots, [x_n, x_n])}. \tag{2}$$

Given two functions $f, g : [0,1]^n \to [0,1]$ such that $f \leq g$, we define the function $\widehat{f,g} : L([0,1])^n \to L([0,1])$ as

$$\widehat{f,g}(\vec{X}) = [f(\underline{X_1}, \ldots, \underline{X_n}), g(\overline{X_1}, \ldots, \overline{X_n})]. \tag{3}$$

Definition 4. *[21] Let $IF : L([0,1])^n \to L([0,1])$ be an $\leq_{Pr}$-increasing interval function. IF is said to be representable if there exist increasing functions $f, g : [0,1]^n \to [0,1]$ such that $f \leq g$ and $F = \widehat{f,g}$.*

The functions f and g are the *representatives* of the interval function F. When $F = \widehat{f,f}$, we denote simply as $\widehat{f}$.

The interval-product is defined, for all $X, Y \in L([0,1])$, by $X \cdot Y = [\underline{X} \cdot \underline{Y}, \overline{X} \cdot \overline{Y}]$.

The notion of admissible orders for intervals came from the interest in refining the product order $\leq_{Pr}$ to a total order.

Definition 5. *[33] Let $(L([0,1]), \leq_{AD})$ be a partially ordered set. The order $\leq_{AD}$ is called an admissible order if*

(i) *$\leq_{AD}$ is a total order on $(L([0,1]), \leq_{AD})$;*

(ii) *For all $X, Y \in L([0,1])$, $X \leq_{AD} Y$ whenever $X \leq_{Pr} Y$.*

In other words, an order $\leq_{AD}$ on $L([0,1])$ is admissible, if it is total and refines the order $\leq_{Pr}$ [33].

Example 2. *Examples of admissible orders on $L([0,1])$ are the lexicographical orders with respect to the first and second coordinate, defined, respectively, by:*

$$X \leq_{Lex1} Y \Leftrightarrow \underline{X} < \underline{Y} \vee (\underline{X} = \underline{Y} \wedge \overline{X} \leq \overline{Y});$$
$$X \leq_{Lex2} Y \Leftrightarrow \overline{X} < \overline{Y} \vee (\overline{X} = \overline{Y} \wedge \underline{X} \leq \underline{Y}).$$

Definition 6. *[33] For $\alpha, \beta \in [0,1]$ such that $\alpha \neq \beta$, the relation $\leq_{\alpha,\beta}$ is defined by*

$$X \leq_{\alpha,\beta} Y \Leftrightarrow K_\alpha(\underline{X}, \overline{X}) < K_\alpha(\underline{Y}, \overline{Y}) \text{ or}$$
$$(K_\alpha(\underline{X}, \overline{X}) = K_\alpha(\underline{Y}, \overline{Y}) \text{ and } K_\beta(\underline{X}, \overline{X}) \leq K_\beta(\underline{Y}, \overline{Y})),$$

where $K_\alpha, K_\beta : [0,1]^2 \to [0,1]$ are aggregation functions defined, respectively, by

$$K_\alpha(x, y) = x + \alpha \cdot (y - x),$$
$$K_\beta(x, y) = x + \beta \cdot (y - x). \tag{4}$$

Then, the relation $\leq_{\alpha,\beta}$ is an admissible order.

Remark 1. *By varying the values of α and β one can recover some of the known admissible orders, e.g., the lexicographical orders $\leq_{Lex1}$ and $\leq_{Lex2}$ can be recovered by $\leq_{0,1}$ and $\leq_{1,0}$, respectively.*

Whenever we apply the mapping K_α on the endpoints of an interval $X \in [0,1]$, we denote $K_\alpha(\underline{X}, \overline{X})$ simply as $K_\alpha(X)$.

We denote an interval-valued function that is increasing with respect to an admissible order $\leq_{AD}$ as $\leq_{AD}$-increasing. Obviously, every $\leq_{AD}$-increasing function is also $\leq_{Pr}$-increasing, since every admissible order $\leq_{AD}$ refines $\leq_{Pr}$.

2.3. General Interval-valued Overlap Functions

Definition 7. *[51] A function $IA : L([0,1])^n \to L([0,1])$ is an interval-valued (iv) aggregation function whenever: **(IA1)** IA is $\leq_{Pr}$-increasing; **(IA2)** IA satisfies: $IA([0,0],\ldots,[0,0]) = [0,0]$ and $IA([1,1],\ldots,[1,1]) = [1,1]$.*

Definition 8. *[23] A function $IOn : L([0,1])^n \to L([0,1])$ is an n-dimensional interval-valued (iv) overlap function if, for all $\vec{X} \in L([0,1])^n$, it satisfies: **(IOn1)** IOn is commutative; **(IOn2)** $IOn(\vec{X}) = [0,0] \Leftrightarrow \prod_{i=1}^n X_i = [0,0]$; **(IOn3)** $IOn(\vec{X}) = [1,1] \Leftrightarrow \prod_{i=1}^n X_i = [1,1]$; **(IOn4)** IOn is $\leq_{Pr}$-increasing; **(IOn5)** IOn is Moore continuous [34].*

For $n = 2$, IOn is just called iv-overlap function [40, 39].

Theorem 1. *[23] Let $On_1, On_2 : [0,1]^n \to [0,1]$ be n-dimensional overlap functions such that $On_1 \leq On_2$. Then, the function $IOn : L([0,1])^n \to L([0,1])$ given, for all $\vec{X} \in L([0,1])^n$, by $IOn(\vec{X}) = \widehat{On_1, On_2}(\vec{X})$, as defined in Eq. (3), is an n-dimensional iv-overlap function.*

Regarding Theo. 1, IOn is a representable interval-valued function. As both its representatives are n-dimensional overlap functions, it is said to be o-representable [23].

By changing **(IOn2)** and **(IOn3)** in Def. 8, general interval-valued overlap functions were defined as follows:

Definition 9. *[23] A function $IGO : L([0,1])^n \to L([0,1])$ is said to be a general interval-valued (iv) overlap function if, for all $\vec{X} \in L([0,1])^n$: **(IGO1)** IGO is commutative; **(IGO2)** If $\prod_{i=1}^n X_i = [0,0]$ then $IGO(\vec{X}) = [0,0]$; **(IGO3)** If $\prod_{i=1}^n X_i = [1,1]$ then $IGO(\vec{X}) = [1,1]$; **(IGO4)** IGO is $\leq_{Pr}$-increasing; **(IGO5)** IGO is Moore continuous.*

Proposition 2. *[23] If $IOn : L([0,1])^n \to L([0,1])$ is an n-dimensional iv-overlap function, then it is also a general iv-overlap function, but the converse may not hold.*

Theorem 2. *[23] Let $GO_1, GO_2 : [0,1]^n \to [0,1]$ be two general overlap functions such that $GO_1 \leq GO_2$. Then, the function $IGO : L([0,1])^n \to L([0,1])$ given, for all $\vec{X} \in L([0,1])^n$, by $IGO(\vec{X}) = \widehat{GO_1, GO_2}(\vec{X})$, is a (representable) general iv-overlap function.*

In order to apply n-dimensional iv-overlap functions in problems where admissible orders must be considered, the following definition was introduced:

Definition 10. *[35] A function $AOn : L([0,1])^n \to L([0,1])$ is an n-dimensional admissibly ordered interval-valued overlap function for an admissible order $\leq_{AD}$ (n-dimensional $\leq_{AD}$-overlap function) if it satisfies **(IOn1)**, **(IOn2)** and **(IOn3)** from Def. 8, and the following condition holds:*

(AOn4) *AOn is $\leq_{AD}$-increasing.*

Remark 2. *Observe that condition **(IOn5)** was not considered in Def. 10, as the continuity condition of overlap functions was only a requirement in order for them to be applied in image processing problems, which was not the case in [35].*

Theorem 3. *[35] Let $IOn : L([0,1])^n \rightarrow L([0,1])$ be an o-representable n-dimensional iv-overlap function and $\alpha, \beta \in [0,1]$, $\alpha \neq \beta$. Then, IOn is $\leq_{\alpha,\beta}$-increasing if and only if $\alpha = 1$ and IOn^+ is a strict n-dimensional overlap function.*

The following Theorem presents a construction method for n-dimensional $\leq_{\alpha,\beta}$-overlap functions:

Theorem 4. *[35] Let On be a strict n-dimensional overlap function, $\alpha \in (0,1)$ and $\beta \in [0,1]$ such that $\alpha \neq \beta$. Then $AOn^\alpha : L([0,1])^n \rightarrow L([0,1])$ defined, for all $\vec{X} \in L([0,1])^n$, by*

$$AOn^\alpha(\vec{X}) = [On(K_\alpha(X_1), \ldots, K_\alpha(X_n)) - \alpha m, \\ On(K_\alpha(X_1), \ldots, K_\alpha(X_n)) + (1-\alpha)m],$$

where

$$m = \min\{\overline{X_1} - \underline{X_1}, \ldots, \overline{X_n} - \underline{X_n}, On(K_\alpha(X_1), \ldots, K_\alpha(X_n)), \\ 1 - On(K_\alpha(X_1), \ldots, K_\alpha(X_n))\},$$

is an n-dimensional $\leq_{\alpha,\beta}$-overlap function.

Remark 3. *Notice that **(IOn2)** and **(IOn3)** are both necessary and sufficient conditions. For that reason, the construction method presented in Theo. 4 must consider $\alpha \in (0,1)$ and, consequently, cannot be applied to obtain neither an n-dimensional $\leq_{0,1}$-overlap function nor an n-dimensional $\leq_{1,0}$-overlap function, that is, n-dimensional admissibly ordered iv-overlap functions that are increasing with respect to the lexicographical orders $\leq_{Lex1}$ and $\leq_{Lex2}$, respectively. This drawback is going to be addressed in our developments in this work. Furthermore, the chosen n-dimensional overlap function On must be strict, to ensure that the constructed function is $\leq_{\alpha,\beta}$-increasing.*

Here, we recall some concepts presented in [37] that were used to introduce a construction method for iv-aggregation functions that are $\leq_{\alpha,\beta}$-increasing.

Definition 11. *[37] Let $c \in [0,1]$ and $\alpha \in [0,1]$. We denote by $d_\alpha(c)$ the maximal possible width of an interval $Z \in L([0,1])$ such that $K_\alpha(Z) = c$. Moreover, for any $X \in L([0,1])$, define*

$$\lambda_\alpha(X) = \frac{w(X)}{d_\alpha(K_\alpha(X))},$$

where we set $\frac{0}{0} = 1$.

Proposition 3. *[37] For all $\alpha \in [0,1]$ and $X \in L([0,1])$ it holds that*

$$d_\alpha(K_\alpha(X)) = \min\left\{\frac{K_\alpha(X)}{\alpha}, \frac{1 - K_\alpha(X)}{1-\alpha}\right\},$$

where we set $\frac{r}{0} = 1$, for all $r \in [0,1]$.

Theorem 5. *[37] Let* $\alpha, \beta \in [0,1]$*, such that,* $\alpha \neq \beta$*. Let* $A_1, A_2 : [0,1]^n \to [0,1]$ *be two aggregation functions where* A_1 *is strictly increasing. Then* $IF^\alpha : L([0,1])^n \to L([0,1])$ *defined by:*

$$IF^\alpha_{A1,A2}(\vec{X}) = R, \text{ where, } \begin{cases} K_\alpha(R) = A_1(K_\alpha(X_1), \ldots, K_\alpha(X_n)), \\ \lambda_\alpha(R) = A_2(\lambda_\alpha(X_1), \ldots, \lambda_\alpha(X_n)), \end{cases}$$

for all $\vec{X} \in L([0,1])^n$*, is an* $\leq_{\alpha,\beta}$*-increasing iv-aggregation function.*

As n-dimensional overlap functions are a class of aggregation functions, the following result is immediate.

Corollary 1. *Let* $\alpha, \beta \in [0,1]$*, such that,* $\alpha \neq \beta$*. Let* $On : [0,1]^n \to [0,1]$ *be a strict* n*-dimensional overlap function and* $A : [0,1]^n \to [0,1]$ *be an aggregation function. Then* $IF^\alpha_{O,A} : L([0,1])^n \to L([0,1])$ *defined by:*

$$IF^\alpha_{On,A}(\vec{X}) = R, \text{ where, } \begin{cases} K_\alpha(R) = On(K_\alpha(X_1), \ldots, K_\alpha(X_n)), \\ \lambda_\alpha(R) = A(\lambda_\alpha(X_1), \ldots, \lambda_\alpha(X_n)), \end{cases}$$

for all $\vec{X} \in L([0,1])^n$*, is an* $\leq_{\alpha,\beta}$*-increasing iv-aggregation function.*

Remark 4. *Concerning Coro. 1, observe that although we apply an n-dimensional overlap function as part of the construction method, the resulting iv-aggregation function* $IF^\alpha_{O,A}$ *may not be an* $\leq_{\alpha,\beta}$*-overlap function, as one can only guarantee that condition* ***(AOn4)*** *is satisfied.*

3. General admissibly ordered interval-valued overlap functions

By combining the concepts of general iv-overlap functions and n-dimensional admissibly ordered iv-overlap functions, we introduce the following definition:

Definition 12. *A function* $AGO : L([0,1])^n \to L([0,1])$ *is a general admissibly ordered interval-valued overlap function for an admissible order* $\leq_{AD}$ *(general* $\leq_{AD}$*-overlap function) if it satisfies the conditions* ***(IGO1)****,* ***(IGO2)*** *and* ***(IGO3)*** *of Def. 3, and the following condition holds:*

(AGO4) AGO *is* $\leq_{AD}$*-increasing.*

The following result is immediate:

Proposition 4. *If* $AOn : L([0,1])^n \to L([0,1])$ *is an* n*-dimensional* $\leq_{AD}$*-overlap function, then it is also a general* $\leq_{AD}$*-overlap function, but the converse may not hold.*

Here we present some results regarding representable general iv-overlap functions and their increasingness with respect to a particular admissible order. In the following result, consider that a strict general overlap function is a general overlap function that is strictly increasing in $(0,1]$.

Lemma 1. *Let* $GO : [0,1]^n \to [0,1]$ *be a strict general overlap function. Then,* GO *is an* n*-dimensional overlap function.*

Proof. It is immediate that GO *respects conditions* ***(On1)****,* ***(On4)*** *and* ***(On5)*** *and, by* ***(GO2)*** *and* ***(GO3)****, it respects the necessary conditions (*$\Leftarrow$*) of* ***(On2)*** *and* ***(On3)****. It remains to prove the sufficient conditions (*$\Rightarrow$*) of* ***(On2)*** *and* ***(On3)****:*

(On2) *($\Rightarrow$) Suppose that GO is strict and does not respect* **(On2)** *($\Rightarrow$). Take $\vec{y} \in (0,1]^n$ such that $GO(\vec{y}) = 0$. Then, there exist $\vec{x} \in (0,1]^n$ such that $\vec{x} < \vec{y}$ and, by* **(GO4)**, *$GO(\vec{x}) = GO(\vec{y}) = 0$, which is a contradiction since GO is strict. Thus, GO respects* **(On2)**.

(On3) *($\Rightarrow$) Suppose that GO is strict and does not respect* **(On3)** *($\Rightarrow$). By* **(GO2)**, *one has that $\vec{x} = (1,\ldots,1) \Rightarrow GO(\vec{x}) = 1$. Now, take $\vec{y} \in [0,1]^n$ such that $y_i \neq 1$ for some $i \in \{1,\ldots,n\}$ and $GO(\vec{y}) = 1$. Then, one has that $\vec{y} < \vec{x}$ and $GO(\vec{y}) = GO(\vec{x}) = 1$, which is a contradiction since GO is strict. Thus, GO respects* **(On3)**.

Theorem 6. *Let $IGO : L([0,1])^n \to L([0,1])$ be a representable general iv-overlap function and $\alpha, \beta \in [0,1]$, $\alpha \neq \beta$. Then, IGO is $\leq_{\alpha,\beta}$-increasing if and only if $\alpha = 1$ and IGO^+ is a strict n-dimensional overlap function.*

Proof. Analogous to the proof of Theo. 3 in [35], taking into account Lem. 1.

Then, the following result is immediate:

Corollary 2. *Let $On : [0,1]^n \to [0,1]$ be an n-dimensional overlap function and $IGO : L([0,1])^n \to L([0,1])$ be a general iv-overlap function such that $IGO = \widehat{On}$, and $\alpha, \beta \in [0,1]$, $\alpha \neq \beta$. Then, IGO is a general $\leq_{\alpha,\beta}$-overlap if and only if $\alpha = 1$ and On is a strict n-dimensional overlap function.*

Example 3. *Consider the general overlap function GO_P as defined in Ex. 1 for $n = 2$. As it is a strict general overlap function, then, by Lem. 1, it is also a strict overlap function. Then, the interval-valued function $AGO_P : L([0,1])^2 \to L([0,1])$ defined, for all $\vec{X} \in L([0,1])^2$, by*

$$AGO_P(\vec{X}) = \widehat{GO_P}(\vec{X})$$

is a general $\leq_{1,0}$-overlap function, and also an 2-dimensional $\leq_{1,0}$-overlap function.

4. Construction methods

The first construction method for general $\leq_{AD}$-overlap functions is an adaptation of Theo. 4, by taking $\alpha \in [0,1]$, obtaining a general $\leq_{\alpha,\beta}$-overlap function.

Theorem 7. *Let On be a strict n-dimensional overlap function, $\alpha, \beta \in [0,1]$ such that $\alpha \neq \beta$. Then $AGO^\alpha : L([0,1])^n \to L([0,1])$ defined, for all $\vec{X} \in L([0,1])^n$, by*

$$AGO^\alpha(\vec{X}) = [On(K_\alpha(X_1),\ldots,K_\alpha(X_n)) - \alpha m,\\ On(K_\alpha(X_1),\ldots,K_\alpha(X_n)) + (1-\alpha)m],$$

where

$$m = \min\{\overline{X_1} - \underline{X_1},\ldots,\overline{X_n} - \underline{X_n}, On(K_\alpha(X_1),\ldots,K_\alpha(X_n)),\\ 1 - On(K_\alpha(X_1),\ldots,K_\alpha(X_n))\},$$

is a general $\leq_{\alpha,\beta}$-overlap function.

Proof. Analogous to the proof of Theo. 4 in [35].

Remark 5. *Observe that* **(IGO2)** *and* **(IGO3)** *are only sufficient conditions, allowing for $\alpha \in [0,1]$ in the construction method presented in Theo. 7, differently than in Theo. 4, in which $\alpha \in (0,1)$. This means that, through Theo. 7, one can obtain general $\leq_{AD}$-overlap functions that are increasing with respect to either one of the lexicographical orders.*

Remark 6. *Regarding Theo. 7, one could think that it could be based on a general overlap function GO instead of a n-dimensional overlap function On, for it to be even more broad of a method. However, as the base function needs to be strictly increasing in order to the constructed interval-valued function AGO^{α} to be $\leq_{\alpha,\beta}$-increasing, by Lem. 1, one has that every strict general overlap function is also an n-dimensional overlap function, and that is why we chose to maintain On in Theo. 7 to reinforce this fact.*

Example 4. *Consider the general overlap function GO_P as defined in Ex. 1. Then, for $\alpha = 1$ and $\beta = 0$, the interval-valued function $AGO_P^1 : L([0,1])^n \to L([0,1])$ defined for all $\vec{X} \in L([0,1])^n$, by*

$$AGO_P^1(\vec{X}) = [GO_P(\overline{X_1}, \ldots, \overline{X_n}) - m, , GO_P(\overline{X_1}, \ldots, \overline{X_n})],$$

where

$$m = \min\{\overline{X_1} - \underline{X_1}, \ldots, \overline{X_n} - \underline{X_n}, GO_P(\overline{X_1}, \ldots, \overline{X_n}), \\ 1 - GO_P(\overline{X_1}, \ldots, \overline{X_n})\},$$

is a general $\leq_{1,0}$-overlap function, or in other words, a general $\leq_{Lex2}$-overlap function. It is noteworthy that AGO_P^1 is not an n-dimensional $\leq_{1,0}$-overlap function.

The next construction methods are inspired on Theo. 5. First, we will present a more restrictive construction method for n-dimensional $\leq_{\alpha,\beta}$-overlap functions:

Theorem 8. *Let $\alpha, \beta \in (0,1)$, such that, $\alpha \neq \beta$. Let $On : [0,1]^n \to [0,1]$ be a strict n-dimensional overlap function and $A : [0,1]^n \to [0,1]$ be a commutative aggregation function. Then $AOn_A^{\alpha} : L([0,1])^n \to L([0,1])$ defined by:*

$$AOn_A^{\alpha}(\vec{X}) = R, \text{ where,} \begin{cases} K_\alpha(R) = On(K_\alpha(X_1), \ldots, K_\alpha(X_n)), \\ \lambda_\alpha(R) = A(\lambda_\alpha(X_1), \ldots, \lambda_\alpha(X_n)), \end{cases}$$

for all $\vec{X} \in L([0,1])^n$, is an n-dimensional $\leq_{\alpha,\beta}$-overlap function.

Proof. *From Theo. 5, it is immediate that AOn_A^{α} is well defined and $\leq_{\alpha,\beta}$-increasing, thus, respecting condition **(AOn4)**. Now, let us verify if AOn_A^{α} respects the remainder conditions from Def. 10:*
(IOn1) *Immediate, since On and A are commutative.*
(IOn2) *($\Rightarrow$) Take $\vec{X} \in L([0,1])^n$ and suppose that $AOn_A^{\alpha}(\vec{X}) = R = [0,0]$. Then, we have that*

$$K_\alpha(R) = K_\alpha([0,0]) = 0 = On(K_\alpha(X_1), \ldots, K_\alpha(X_n)),$$

*for all $\alpha \in (0,1)$. Thus, by condition **(On2)**, $K_\alpha(X_i) = 0$ for some $i \in \{1, \ldots, n\}$, for all $\alpha \in (0,1)$, and, therefore, $\prod_{i=1}^n X_i = [0,0]$;*
*($\Leftarrow$) Consider $\vec{X} \in L([0,1])^n$ such that $\prod_{i=1}^n = [0,0]$. So, $K_\alpha(X_1) \cdot \ldots \cdot K_\alpha(X_n) = 0$, for all $\alpha \in (0,1)$. Then, by **(On2)**, one has that*

$$K_\alpha(R) = On(K_\alpha(X_1), \ldots, K_\alpha(X_n)) = 0,$$

for all $\alpha \in (0,1)$, meaning that $AOn_A^{\alpha}(\vec{X}) = R = [0,0]$;

(IOw3) *($\Rightarrow$) Take $\vec{X} \in L([0,1])^n$ such that $AOn_A^{\alpha}(\vec{X}) = R = [1,1]$. Then, one has that*

$$K_\alpha(R) = K_\alpha([1,1]) = 1 = On(K_\alpha(X_1), \ldots, K_\alpha(X_n)).$$

*By **(On3)**, $K_\alpha(X_1)\cdot\ldots\cdot K_\alpha(X_n)=1$, for all $\alpha\in(0,1)$, meaning that $\prod_{i=1}^n X_i=[1,1]$;*

*($\Leftarrow$) Consider $\vec{X}\in L([0,1])^n$ such that $\prod_{i=1}^n X_i=[1,1]$. So, $K_\alpha(X_1)\cdot\ldots\cdot K_\alpha(X_n)=1$, for all $\alpha\in(0,1)$. Then, by **(i)** and **(O3)**, one has that*

$$K_\alpha(R)=On(K_\alpha(X_1),\ldots,K_\alpha(X_n))=1,$$

for all $\alpha\in(0,1)$, meaning that $AOn_A^\alpha(\vec{X})=R=[1,1]$.

The following result is immediate, as it derives from a similar situation as discussed in Remarks 5 and 6.

***Theorem 9.** Let $\alpha,\beta\in[0,1]$, such that, $\alpha\neq\beta$. Let $On:[0,1]^n\to[0,1]$ be a strict n-dimensional overlap function and $A:[0,1]^n\to[0,1]$ be a commutative aggregation function. Then $AGO_A^\alpha:L([0,1])^n\to L([0,1])$ defined by:*

$$AGO_A^\alpha(\vec{X})=R,\ \text{ where, }\begin{cases}K_\alpha(R)=On(K_\alpha(X_1),\ldots,K_\alpha(X_n)),\\ \lambda_\alpha(R)=A(\lambda_\alpha(X_1),\ldots,\lambda_\alpha(X_n)),\end{cases}$$

for all $\vec{X}\in L([0,1])^n$, is an general $\leq_{\alpha,\beta}$-overlap function.

***Example 5.** Consider the general overlap functions GO_L and GO_{Gm} as defined in Ex. 1. Then, for $\alpha=1$ and $\beta=0$, the interval-valued function $AGm_{GO_L}^1:L([0,1])^n\to L([0,1])$ defined for all $\vec{X}\in L([0,1])^n$, by*

$$AGm_{GO_L}^1(\vec{X})=R,\ \text{ where, }\begin{cases}K_1(R)=GO_{Gm}(\overline{X_1},\ldots,\overline{X_n}),\\ \lambda_1(R)=GO_L(\lambda_1(X_1),\ldots,\lambda_1(X_n)),\end{cases}$$

is a general $\leq_{1,0}$-overlap function, but not an n-dimensional $\leq_{1,0}$-overlap function.

The following method allow the construction of general $\leq_{AD}$-overlap functions by the generalized composition of general $\leq_{AD}$-overlap functions by an $\leq_{AD}$-increasing iv-aggregation function.

***Theorem 10.** Consider $IM:L([0,1])^m\to L([0,1])$. For a tuple $\overrightarrow{AGO}=(AGO_1,\ldots,AGO_m)$ of general $\leq_{AD}$-overlap functions, define the mapping $IM_{\overrightarrow{AGO}}:L([0,1])^n\to L([0,1])$, for all $\vec{X}\in L([0,1])^n$, by:*

$$IM_{\overrightarrow{AGO}}(\vec{X})=IM(AGO_1(\vec{X}),\ldots,AGO_m(\vec{X})).$$

Then, $IM_{\overrightarrow{AGO}}$ is a general $\leq_{AD}$-overlap function if and only if IM is an $\leq_{AD}$-increasing iv-aggregation function.

Proof.It follows that:

*($\Rightarrow$) Suppose that $IM_{\overrightarrow{AGO}}$ is a general $\leq_{AD}$-overlap function. Then it is immediate that IM $\leq_{AD}$-increasing, and, also, $\leq_{Pr}$-increasing **(IA2)**. Now consider $\vec{X}\in L([0,1])^n$ such that $\prod_{i=1}^n X_i=[0,0]$. Then, by **(IGO2)**, one has that: $IM_{\overrightarrow{AGO}}(\vec{X})=IM(AGO_1(\vec{X}),\ldots,AGO_m(\vec{X}))=[0,0]$ and $AGO_1(\vec{X})=\ldots=AGO_m(\vec{X})=[0,0]$. Thus, it holds that $IM([0,0],\ldots,[0,0])=[0,0]$. Now, consider $\vec{X}\in L([0,1])^n$, such that $X_i=[1,1]$ for all $i\in\{1,\ldots,n\}$. Then, by **(IGO3)**, one has that: $IM_{\overrightarrow{AGO}}(\vec{X})=IM(AGO_1(\vec{X}),\ldots,AGO_m(\vec{X}))=[1,1]$ and $AGO_1(\vec{X})=\ldots=AGO_m(\vec{X})=[1,1]$. Therefore, it holds that $IM([1,1],\ldots,[1,1])=[1,1]$. This proves that IM also satisfies condition **(IA1)**, and, thus, an $\leq_{AD}$-increasing iv-aggregation function.*

*($\Leftarrow$) Suppose that IM is an $\leq_{AD}$-increasing iv-aggregation function. Then it is immediate that $IM_{\overrightarrow{AGO}}$ is commutative (by **(IGO1)**), and respects **(AGO4)**. It remains to prove:*

(IGO2) *Consider* $\vec{X} \in L([0,1])^n$ *such that* $\prod_{i=1}^n X_i = [0,0]$. *Then, by* ***(IGO2)***, *one has that* $AGO_1(\vec{X}) = \ldots = AGO_m(\vec{X}) = [0,0]$. *It follows that:* $IM_{\overrightarrow{AGO}}(\vec{X}) = IM(AGO_1(\vec{X}), \ldots, AGO_m(\vec{X})) = IM([0,0], \ldots, [0,0]) = [0,0]$, *by condition* ***(IA1)***, *since* IM *is an iv-aggregation function.*

(IGO3) *Take* $\vec{X} \in L([0,1])^n$ *such that* $X_i = [1,1]$ *for all* $i \in \{1, \ldots, n\}$. *Then,* ***(IGO3)***, *it holds that* $AGO_1(\vec{X}) = \ldots = AGO_m(\vec{X}) = [1,1]$. *It follows that:* $IM_{\overrightarrow{AGO}}(\vec{X}) = IM(AGO_1(\vec{X}), \ldots, AGO_m(\vec{X})) = IM([1,1], \ldots, [1,1]) = [1,1]$, *by condition* ***(IA1)***. *This proves that* $IM_{\overrightarrow{AGO}}(\vec{X})$ *is a general* $\leq_{AD}$*-overlap function.*

Example 6. *Consider the general* $\leq_{1,0}$*-overlap functions* AGO_P, AGO_P^1 *and* $AGm_{GO_L}^1$, *from Ex.s 3, 4 and 5. Then, the interval-valued function* $AGO : L([0,1])^n \to L([0,1])$ *defined, for all* $\vec{X} \in L([0,1])^n$, *by*

$$AGO(\vec{X}) = AGO_P(AGO_P^1(\vec{X}), AGm_{GO_L}^1(\vec{X})),$$

is a general $\leq_{1,0}$*-overlap function.*

5. Conclusion

In this paper we presented the concept of general admissibly ordered interval-valued overlap functions, a more flexible definition of n*-dimensional interval-valued overlap functions that are increasing with respect to an admissible order. This new definition allowed us to construct several interval-valued overlap operations taking into account different admissible orders, in particular,* $\leq_{\alpha,\beta}$ *orders with any* $\alpha, \beta \in [0,1]$ *such that* $\alpha \neq \beta$. *Finally, those constructed functions can be combined by generalized composition to obtain new general admissibly ordered interval-valued overlap functions, showcasing their adaptability.*

Most construction methods for $\leq_{\alpha,\beta}$*-increasing functions are based on the aggregation of the* K_α *values of the inputs by strictly increasing aggregation functions, which is a restriction that could be interesting to overcome in our future work. We also intend to apply the developed functions (with different combination of construction methods) in classification problems with interval-valued data.*

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