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N-Dimensional Admissibly Ordered Interval-valued Overlap Functions and its Influence in Interval-valued Fuzzy Rule-based Classification Systems

Tiago da Cruz Asmus, José Antonio Sanz, Graçaliz Pereira Dimuro *IEEE Member*, Benjamín Bedregal *IEEE Member*, Javier Fernández *IEEE Member*, Humberto Bustince *IEEE Senior Member*

***Abstract*—Overlap functions are a type of aggregation functions that are not required to be associative, generally used to indicate the overlapping degree between two values. They have been successfully used as a conjunction operator in several practical problems, such as fuzzy rule-based classification systems (FRBCSs) and image processing. Some extensions of overlap functions were recently proposed, such as general overlap functions and, in the interval-valued context, n-dimensional interval-valued overlap functions. The latter allow them to be applied in n-dimensional problems with interval-valued inputs, like interval-valued classification problems, where one can apply interval-valued FRBCSs (IV-FRBCSs). In this case, the choice of an appropriate total order for intervals, like an admissible order, can play an important role. However, neither the relationship between the interval order and the n-dimensional interval-valued overlap function (which may or may not be increasing for that order) nor the impact of this relationship in the classification process have been studied in the literature. Moreover, there is not a clear preferred n-dimensional interval-valued overlap function to be applied in an IV-FRBCS. Hence, in this paper we: (i) present some new results on admissible orders, which allow us to introduce the concept of n-dimensional admissibly ordered interval-valued overlap functions, that is, n-dimensional interval-valued overlap functions that are increasing with respect to an admissible order; (ii) develop a width-preserving construction method for this kind of function, derived from an admissible order and an n-dimensional overlap function, discussing some of its features; (iii) analyze the behaviour of several combinations of admissible orders and n-dimensional (admissibly ordered) interval-valued overlap functions when applied in IV-FRBCSs. All in all, the contribution of this paper resides in pointing out the effect of admissible orders and n-dimensional admissibly ordered interval-valued overlap functions, both from a theoretical and applied points of view, the latter when considering classification problems.**

***Index Terms*—n-dimensional overlap functions, interval-valued overlap functions, admissible orders, fuzzy rule-based classification systems**

T.C. Asmus is with the Departamento de Estadística, Informática y Matemáticas, Universidad Publica de Navarra, Pamplona, Spain and Instituto de Matemática, Estatística e Física, Universidade Federal do Rio Grande, Rio Grande, Brazil e-mail: tiago.dacruz@unavarra.es, tiagoasmus@furg.br

G.P. Dimuro is with the Centro de Ciências Computacionais, Universidade Federal do Rio Grande, Rio Grande, Brazil, and Departamento de Estadística, Informática y Matemáticas, Universidad Publica de Navarra, Pamplona, Spain e-mail: gracalizdimuro@furg.br, gracaliz.pereira@unavarra.es

J.A. Sanz, J. Fernández and H. Bustince are with the Departamento de Estadística, Informática y Matemáticas and the Institute of Smart Cities, Universidad Publica de Navarra, Pamplona, Spain e-mails: {joseantonio.sanz,fcojavier.fernandez,bustince}@unavarra.es.

B. Bedregal is with Departamento de Informática e Matemática Aplicada, Universidade Federal do Rio Grande do Norte, Natal, Brazil, e-mail: bedregal@dimap.ufrn.br.

I. Introduction

In 2010, Bustince et al. [1] introduced the concept of overlap functions in order to deal with the overlap problem that usually appears in image processing. For that, overlap functions were conceived as continuous aggregation functions [2] that are not required to be associative. In fact, the associativity is not a relevant property for many applications besides image processing, such as decision making based on fuzzy preference relations, as properly discussed by Dimuro et al. in [3], [4], [5]. Observe that the continuity property of overlap functions was essential for the application in image processing, in the context where the concept was born.

Overlap functions are more general than the well known t-norms [6], although the required continuity may be more restrictive. In fact, there is an intersection between those two families: any continuous positive t-norm is an overlap function and any associative overlap function with 1 as neutral element is a t-norm. Nevertheless, the class of overlap functions is reacher than that of t-norms in many aspects, considering, e.g., the idempotency and homogeneity properties [7]. Moreover, overlap functions are closed to the convex sum and the aggregation by generalized composition of overlap functions, whereas neither the convex sum of t-norms nor the aggregation of t-norms by a t-norm results in t-norms, in general [8], [9].

Since the appearance of the concept of overlap functions, many authors have dedicated time to the theoretical research on their properties and related concepts, such as Qiao [10], Qiao and Hu [11], Dimuro et al. [5], [8], [12], [13], Zhou and Yan [14], Zhu et al. [15], Zhang et al. [16] and Cao et al. [17]. Moreover, the application of overlap function is getting attention mainly because the associativity is not required during the information aggregation process, like in image processing [18], decision making [19], [20], wavelet-fuzzy power quality diagnosis system [21], forest fire detection [22] and classification by generalizations of the Choquet integral [23], [24], [25], [26], [27]. Observe that, in some of the mentioned applications (e.g., decision making and classification), the continuity of overlap functions is not required.

However, overlap functions are bivariate functions, which implies that they can only be applied in problems involving

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just two classes or objects. This becomes a serious drawback when one faces n-dimensional problems (e.g., classification [28]), since overlap functions may be not associative. In order to overcome this limitation, Gómez et al. [29] introduced the concept of n-dimensional overlap functions. More recently, De Miguel et al. [30] defined general overlap functions by relaxing the boundary conditions of n-dimensional overlap functions, providing a more flexible definition.

Now, observe that in some applications there may be uncertainty in providing either the membership grades or the definition of membership functions [31]. To deal with this problem, one may adopt interval-valued fuzzy sets (IVFSs) [32], [33], [34], since it is capable to model both vagueness (soft class boundaries) and uncertainty (with respect to the membership function), as discussed in [35], [36], [37]. That is the reason why IVFSs have been successfully applied in several problems, such as game theory [38], decision making [39], pest control [40] and, specially, classification [37].

To address the problem of working in the interval-valued fuzzy context, Qiao and Hu [41] and Bedregal et al. [35] introduced independently the concept of interval-valued (iv) overlap functions. Latter, in [42], Asmus et al. introduced the concepts of n-dimensional iv-overlap functions and general iv-overlap functions, which were applied to compute the interval matching degree in Interval-Valued Fuzzy Rule Based Classification Systems (IV-FRBCSs) [43], [44].

IV-FRBCSs are Fuzzy Rule Based Classification Systems (FRBCSs) [45] whose linguistic labels are modeled by means of IVFSs, as in the work of Sanz et al. [37]. In IV-FRBCSs, the ignorance/uncertainty inherent to the definition of the membership functions, represented by IVFSs, is taken into account in the whole reasoning process, which implies that in the end of the classification process one needs to compare intervals instead of numbers. To carry out this comparison, a total order relation between intervals is needed, instead of the usual partial orders (e.g., the product order [46]). For that, one may use admissible orders introduced by Bustince et al. [47], which are total orders that may be constructed by means of aggregation functions, that is, different total orders can be obtained by varying the aggregation functions used in their construction. Since their definition, several works took into account admissible orders, such as [48], [49].

When defining IV-FRBCSs, both the aggregation function used to compute the interval matching degree and the adopted total order play a key role, as they can change the behaviour of the system. However, in the literature, there is not a consensus regarding which are the recommended n-dimensional iv-overlap functions to be applied to compute the interval matching degree in IV-FRBCSs. Moreover, there is no previous study concerning the relation between the chosen interval total order and n-dimensional iv-overlap function (which may or may not be increasing for that order), and the impact of such relation in the whole classification process.

Considering the discussion above, in this paper we have the following objectives:

1. To define n-dimensional admissibly ordered iv-overlap functions, that is, n-dimensional iv-overlap functions that are increasing with respect to an admissible order, studying their properties and showing examples;
2. To introduce a construction method of n-dimensional admissibly ordered iv-overlap functions based on n-dimensional overlap functions and a chosen admissible order, aiming at obtaining width-preserving iv-functions, that is, the resulting interval is never wider than any of the aggregated inputs, which is a desirable property in many applications;
3. To analyze the influence of both the admissible orders and the n-dimensional admissibly ordered iv-overlap functions in IV-FRBCSs.

The paper is organized as follows. Section II presents some preliminary concepts that are necessary for the development of the paper. In Section III, we present new results on admissible orders and introduce the concept of n-dimensional admissibly ordered iv-overlap functions, studying properties and showing examples. In section IV we develop a width-preserving construction method for n-dimensional admissibly ordered iv-overlap functions. In Section V, we analyze the influence of the combination of admissible orders and n-dimensional (admissibly ordered) interval-valued overlap functions, in classification problems. Section VI is the Conclusion.

II. Preliminaries

A. Interval Representation

Let us denote as $L([0,1])$ the set of all closed subintervals of the unit interval $[0,1]$. Denote $\vec{x} = (x_1, \ldots, x_n) \in [0,1]^n$ and $\vec{X} = (X_1, \ldots, X_n) \in L([0,1])^n$. Given any $X = [x_1, x_2] \in L([0,1])$, $\underline{X} = x_1$ and $\overline{X} = x_2$ denote, respectively, the left and right projections of X. The product and inclusion partial orders are defined for all $X, Y \in L([0,1])$, respectively, by:

$$X \leq_{Pr} Y \quad \Leftrightarrow \quad \underline{X} \leq \underline{Y} \wedge \overline{X} \leq \overline{Y};$$
$$X \subseteq Y \quad \Leftrightarrow \quad \underline{X} \geq \underline{Y} \wedge \overline{X} \leq \overline{Y}.$$

We call as $\leq_{Pr}$-increasing a function that is increasing with respect to the product order $\leq_{Pr}$. The projections $F^-, F^+ : [0,1]^n \to [0,1]$ of $F : L([0,1])^n \to L([0,1])$ are defined, respectively, by:

$$F^-(x_1, \ldots, x_n) = \underline{F([x_1, x_1], \ldots, [x_n, x_n])}; \quad (1)$$
$$F^+(x_1, \ldots, x_n) = \overline{F([x_1, x_1], \ldots, [x_n, x_n])}. \quad (2)$$

Given two functions $f, g : [0,1]^n \to [0,1]$ such that $f \leq g$, we define the function $\widehat{f,g} : L([0,1])^n \to L([0,1])$ as

$$\widehat{f,g}(X_1, \ldots, X_n) = [f(\underline{X_1}, \ldots, \underline{X_n}), g(\overline{X_1}, \ldots, \overline{X_n})]. \quad (3)$$

Definition 1. *[36] Let $F : L([0,1])^n \to L([0,1])$ be an $\leq_{Pr}$-increasing interval function. F is said to be representable if there exist increasing functions $f, g : [0,1]^n \to [0,1]$ such that $f \leq g$ and $F = \widehat{f,g}$.*

The functions f and g are the *representatives* of the interval function F. When $F = \widehat{f,f}$, we denote simply as $\widehat{f}$.

Proposition 1. *[42] For each $\leq_{Pr}$-increasing interval function $F : L([0,1])^n \to [0,1]$, F is representable if and only if F is inclusion monotonic.*

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Proposition 2. *[42] If an $\leq_{Pr}$-increasing interval function $F : L([0,1])^n \to L([0,1])$ is inclusion monotonic, then $\underline{F(X_1,\ldots,X_n)} = F^-(\underline{X_1},\ldots,\underline{X_n})$ and $\overline{F(X_1,\ldots,X_n)} = F^+(\overline{X_1},\ldots,\overline{X_n})$, for all $X_1,\ldots,X_n \in L([0,1])$.*

Definition 2. *[50] An interval-valued negation is a function $N : L([0,1]) \to L([0,1])$ that is $\leq_{Pr}$-decreasing and satisfies: **(N1)** $N([1,1]) = [0,0]$; **(N2)** $N([0,0]) = [1,1]$. If for all $X \in L([0,1]), N(N(X)) = X$, N is said to be involutive.*

Definition 3. *[51] An interval-valued restricted equivalence functions (IV-REF) is a function $IR : L([0,1])^2 \to L([0,1])$ satisfying: **(IR1)** IR is commutative; **(IR2)** $IR(X,Y) = [1,1] \Leftrightarrow X = Y$; **(IR3)** $IR(X,Y) = [0,0] \Leftrightarrow X = [0,0]$ and $Y = [1,1]$, or $X = [1,1]$ and $Y = [0,0]$; **(IR4)** $IR(X,Y) = IR(N(X),N(Y))$; **(IR5)** $\forall X,Y,Z \in L([0,1]), X \leq_{Pr} Y \leq_{Pr} Z \Rightarrow IR(X,Y) \geq_{Pr} IR(X,Z)$ and $IR(Y,Z) \geq_{Pr} IR(X,Z)$.*

Some interval operations that are used in this paper are defined, for all $X,Y \in L([0,1])$ as: [46], [52]

Sum: $X + Y = [\underline{X} + \underline{Y}, \overline{X} + \overline{Y}]$;
Product: $X \cdot Y = [\underline{X} \cdot \underline{Y}, \overline{X} \cdot \overline{Y}]$;
Generalized Hukuhara Division: with $\underline{Y} \neq 0$,
$X \div_H Y = [\min\{\underline{X}/\underline{Y}, \overline{X}/\overline{Y}\}, \max\{\underline{X}/\underline{Y}, \overline{X}/\overline{Y}\}]$.

B. Admissible orders

The notion of admissible orders for intervals came from the interest in extending the product order $\leq_{Pr}$ to a total order.

Definition 4. *[47] Let $(L([0,1]), \leq_{AD})$ be a partially ordered set. The order $\leq_{AD}$ is called an admissible order if*
(i) *$\leq_{AD}$ is a total order on $(L([0,1]), \leq_{AD})$;*
(ii) *For all $X,Y \in L([0,1])$, $X \leq_{AD} Y$ whenever $X \leq_{Pr} Y$.*

In other words, an order $\leq_{AD}$ on $L([0,1])$ is admissible, if it is total and refines the order $\leq_{Pr}$ [47].

Example 1. *The following relations on $L([0,1])$ are examples of admissible orders:*
(i) The lexicographical orders with respect to the first and second coordinate, defined, respectively, by:

$$X \leq_{Lex1} Y \Leftrightarrow \underline{X} < \underline{Y} \vee (\underline{X} = \underline{Y} \wedge \overline{X} \leq \overline{Y});$$
$$X \leq_{Lex2} Y \Leftrightarrow \overline{X} < \overline{Y} \vee (\overline{X} = \overline{Y} \wedge \underline{X} \leq \underline{Y}).$$

(ii) The order $\leq_{XY}$ introduced by Xu and Yager in [53], defined by:

$$X \leq_{XY} Y \Leftrightarrow \underline{X} + \overline{X} < \underline{Y} + \overline{Y} \text{ or } (\underline{X} + \overline{X} = \underline{Y} + \overline{Y} \text{ and } \overline{X} - \underline{X} \leq \overline{Y} - \underline{Y}).$$

(iii) Whenever one considers the comparison of the information quality [54] provided by the intervals X and Y in the order of Xu and Yager, it is possible to define, as in [43]:

$$X \leq_{IQ} Y \Leftrightarrow \underline{X} + \overline{X} < \underline{Y} + \overline{Y} \text{ or } (\underline{X} + \overline{X} = \underline{Y} + \overline{Y} \text{ and } \overline{Y} - \underline{Y} \leq \overline{X} - \underline{X}).$$

Proposition 3. *[47] Let $A, B : [0,1]^2 \to [0,1]$ be aggregation functions (see Def. 6), such that, for all $X,Y \in L([0,1])$, the equalities $A(\underline{X},\overline{X}) = A(\underline{Y},\overline{Y})$ and $B(\underline{X},\overline{X}) = B(\underline{Y},\overline{Y})$ can hold only if $X = Y$. Define the relation $\leq_{A,B}$ on $L([0,1])$ by*

$$X \leq_{A,B} Y \Leftrightarrow A(\underline{X},\overline{X}) < A(\underline{Y},\overline{Y}) \text{ or } (A(\underline{X},\overline{X}) = A(\underline{Y},\overline{Y}) \text{ and } B(\underline{X},\overline{X}) \leq B(\underline{Y},\overline{Y})).$$

Then $\leq_{A,B}$ is an admissible order on $L([0,1])$.

The pair (A,B) of aggregation functions that generates the order $\leq_{A,B}$ in Prop. 3 is called an admissible pair of aggregation functions [47]. Of particular interest is when the admissible order is generated by K_α mappings [47]. For $\alpha \in [0,1]$, the mapping $K_\alpha : [0,1]^2 \to [0,1]$ is defined by:

$$K_\alpha(x,y) = x + \alpha \cdot (y - x). \quad (4)$$

Definition 5. *[47] For $\alpha, \beta \in [0,1]$ such that $\alpha \neq \beta$, the relation $\leq_{\alpha,\beta}$ is defined by*

$$X \leq_{\alpha,\beta} Y \Leftrightarrow K_\alpha(\underline{X},\overline{X}) < K_\alpha(\underline{Y},\overline{Y}) \text{ or } (K_\alpha(\underline{X},\overline{X}) = K_\alpha(\underline{Y},\overline{Y}) \text{ and } K_\beta(\underline{X},\overline{X}) \leq K_\beta(\underline{Y},\overline{Y})).$$

Then, the relation $\leq_{\alpha,\beta}$ is an admissible order generated by an admissible pair of aggregation functions (K_α, K_β) [47].

Remark 1. *By varying the values of α and β one can recover some of the defined admissible orders, e.g., the lexicographical orders $\leq_{Lex1}$ and $\leq_{Lex2}$, and the orders $\leq_{XY}$ and $\leq_{IQ}$ are recovered, respectively, by $\leq_{0,1}$, $\leq_{1,0}$, $\leq_{0.5,1}$ and $\leq_{0.5,0}$.*

Lemma 1. *[47] For any $\alpha, \beta \in [0,1]$, $\alpha \neq \beta$, it holds that: (i)$\beta > \alpha \Rightarrow \leq_{\alpha,\beta} = \leq_{\alpha,1}$; (ii) $\beta < \alpha \Rightarrow \leq_{\alpha,\beta} = \leq_{\alpha,0}$.*

C. n-dimensional Overlap Functions

Definition 6. *[2] An aggregation function is a mapping $A : [0,1]^n \to [0,1]$ that is increasing in each argument and satisfying: **(A1)** $A(0,\ldots,0) = 0$; **(A2)** $A(1,\ldots,1) = 1$.*

Definition 7. *[28], [29] A function $On : [0,1]^n \to [0,1]$ is said to be an n-dimensional overlap function if the following conditions hold, for all $\vec{x} \in [0,1]^n$: **(On1)** On is commutative; **(On2)** $On(\vec{x}) = 0 \Leftrightarrow \prod_{i=1}^n x_i = 0$; **(On3)** $On(\vec{x}) = 1 \Leftrightarrow \prod_{i=1}^n x_i = 1$; **(On4)** On is increasing; **(On5)** On is continuous.*

If for all $x,y,z \in (0,1]$ one has that $x < y$ implies that $On(x,z,\ldots,z) < On(y,z,\ldots,z)$, then On is called a *strict n-dimensional overlap function*. In Table I we show some examples of n-dimensional overlap functions.

A 2-dimensional overlap function is just called overlap function. For properties on (n-dimensional) overlap functions and related concepts, see also: [3], [4], [9], [10], [11], [29].

D. n-dimensional Interval-valued Overlap Functions

Recently, the concepts of n-dimensional interval-valued aggregation/overlap functions and general interval-valued overlap functions were introduced by Asmus et al. in [42]:

Definition 8. *[42] A function $IA : L([0,1])^n \to L([0,1])$ is an n-dimensional interval-valued aggregation function*

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TABLE I: Examples of n-dimensional overlap functions

Name	Definition
Product	$On_P(\vec{x}) = \prod_{i=1}^n x_i$
Minimum	$On_M(\vec{x}) = \min\{x_1, \ldots, x_n\}$
Hamacher	$On_{Hp}(\vec{x}) = \begin{cases} 0, & \text{if } x_1 = \ldots = x_n = 0; \\ \frac{\prod_{i=1}^n x_i}{\left(\sum_{i=1}^n \prod_{j \in N_i^n} x_j\right) - (n-1)\prod_{i=1}^n x_i}, & \text{otherwise} \\ \text{where } N_i^n = \{1, \ldots, n\} - \{i\} \end{cases}$
OB Overlap	$On_{OB}(\vec{x}) = \sqrt{\min\{x_1, \ldots, x_n\} \cdot \prod_{i=1}^n x_i}$
Geom. Mean	$On_{Gm}(\vec{x}) = \sqrt[n]{\prod_{i=1}^n x_i}$
Harm. Mean	$On_{Hm}(\vec{x}) = \begin{cases} \frac{n}{\frac{1}{x_i} + \ldots + \frac{1}{x_n}}, & \text{if } x_i \neq 0, \forall i \in \{1, \ldots, n\}; \\ 0, & \text{otherwise} \end{cases}$

whenever the following conditions hold: ***(IA1)*** IA *is* $\leq_{Pr}$*-increasing in each argument;* ***(IA2)*** IA *satisfies the boundary conditions:* ***(i)*** $IA([0,0], \ldots, [0,0]) = [0,0]$ *and* ***(ii)*** $IA([1,1], \ldots, [1,1]) = [1,1]$.

Definition 9. *[42] A function* $IOn : L([0,1])^n \rightarrow L([0,1])$ *is an n-dimensional interval-valued (iv) overlap function if, for all* $\vec{X} \in L([0,1])^n$ *and* $Y \in L([0,1])$*, it satisfies:* ***(IOn1)*** IOn *is commutative;* ***(IOn2)*** $IOn(\vec{X}) = [0,0] \Leftrightarrow \prod_{i=1}^n X_i = [0,0]$*;* ***(IOn3)*** $IOn(\vec{X}) = [1,1] \Leftrightarrow \prod_{i=1}^n X_i = [1,1]$*;* ***(IOn4)*** IOn *is* $\leq_{Pr}$*-increasing in the first component:* $X_1 \leq_{Pr} Y \Rightarrow IOn(X_1, X_2, \ldots, X_n) \leq_{Pr} IOn(Y, X_2, \ldots, X_n)$*;* ***(IOn5)*** IOn *is Moore continuous [46].*

Example 2. *Some examples of n-dimensional iv-overlap functions, for* $\vec{X} = \in L([0,1])^n$ *are:*

1. $IOn_M(\vec{X}) = \left[\min\{\underline{X_1}, \ldots, \underline{X_n}\}, \min\{\overline{X_1}, \ldots, \overline{X_n}\}\right]$;

2. $IOn_{Pp}(\vec{X}) = \left[\prod_{i=1}^n \underline{X_i}^p, \prod_{i=1}^n \overline{X_i}^p\right]$*, for* $p > 0$*;*

3. $IOn_{Mp}(\vec{X}) = IOn_M(\vec{X}) \cdot IOn_{Pp}(\vec{X})$.

Theorem 1. *[42] Let* $On_1, On_2 : [0,1]^n \rightarrow [0,1]$ *be n-dimensional overlap functions such that* $On_1 \leq On_2$*. Then, the function* $\widehat{On_1, On_2} : L([0,1])^n \rightarrow L([0,1])$*, as defined in Eq. (3), is an n-dimensional iv-overlap function.*

An n-dimensional iv-overlap function $IOn : L([0,1])^n \rightarrow L([0,1])$ is said to be *o*-representable if there exist n-dimensional overlap functions $On_1, On_2 : [0,1]^n \rightarrow [0,1]$ such that $On_1 \leq On_2$ and $IOn = \widehat{On_1, On_2}$.

Theorem 2. *[42] Let* $IOn : L([0,1])^n \rightarrow L([0,1])$ *be an n-dimensional iv-overlap function. Then,* IOn *is o-representable if and only if* IOn *is inclusion monotonic and satisfies, for all* $\vec{X} \in L([0,1])^n$*: (i)* $\underline{IOn(\vec{X})} = 0 \Leftrightarrow \prod_{i=1}^n \underline{X_i} = 0$*; (ii)* $\overline{IOn(\vec{X})} = 1 \Leftrightarrow \prod_{i=1}^n \overline{X_i} = 1$.

Corollary 1. *Let* $IOn : L([0,1])^n \rightarrow L([0,1])$ *be an n-dimensional iv-overlap function such that* $IOn^+ : L([0,1])^n \rightarrow L([0,1])$ *(Eq. (2)) is a strict n-dimensional overlap function. Then,* IOn *is o-representable if and only if it is inclusion monotonic and, for all* $\vec{X} \in L([0,1])^n$*:* $\underline{IOn(\vec{X})} = 0 \Leftrightarrow \prod_{i=1}^n \underline{X_i} = 0$.

Proof. It is immediate from Prop. 2 and Theorem 2. □

A 2-dimensional iv-overlap function is called iv-overlap function. For more properties on such functions, see [35], [41].

III. N-DIMENSIONAL ADMISSIBLY ORDERED INTERVAL-VALUED OVERLAP FUNCTIONS

In this section, we define the concept of n-dimensional admissibly ordered interval-valued overlap function, following by some properties and examples. But first, we introduce some new results regarding admissible orders.

Proposition 4. *For all* $\alpha_1, \alpha_2, \beta_1, \beta_2 \in [0,1]$ *such that* $\alpha_1 \neq \alpha_2, \alpha_1 \neq \beta_1$ *and* $\alpha_2 \neq \beta_2$*, one has that* $\leq_{\alpha_1,\beta_1} \neq \leq_{\alpha_2,\beta_2}$.

Proof. Consider $Y = [0,1]$ and $\alpha_1, \alpha_2 \in [0,1]$ such that $\alpha_1 < \alpha_2$. For all $X = [x,x]$ such that $\alpha_1 < x < \alpha_2$ one has that $Y \leq_{\alpha_1,\beta_1} X$ and $X \leq_{\alpha_2,\beta_2} Y$, for any $\beta_1 \neq \alpha_1$ and $\beta_2 \neq \alpha_2$. The proof for the case in which $\alpha_2 < \alpha_1$ is analogous. □

Proposition 5. *For all* $\alpha \in (0,1)$ *one has that* $\leq_{\alpha,0} \neq \leq_{\alpha,1}$.

Proof. For all $\alpha \in (0,1)$, it is possible to find $X, Y \in L([0,1])$, namely, $X = [\alpha, \alpha]$ and $Y = [0,1]$, such that $Y <_{\alpha,0} X$ and $X <_{\alpha,1} Y$. □

Corollary 2. *For all* $\alpha, \beta_1, \beta_2 \in [0,1]$ *such that* $\beta_1 < \alpha < \beta_2$*, one has that* $\leq_{\alpha,\beta_1} \neq \leq_{\alpha,\beta_2}$.

Proof. It is immediate from Lemma 1 and Prop. 5. □

From Prop. 4 and 5, and Lemma 1, it is clear that $\leq_{\alpha_1,\beta_1} \neq \leq_{\alpha_2,\beta_2}$, for all $\alpha_1, \alpha_2, \beta_1, \beta_2 \in [0,1]$ such that $\alpha_1 \neq \beta_1$ and $\alpha_2 \neq \beta_2$, except when $\alpha_1 = \alpha_2 = \alpha$, $\beta_1 < \alpha$ and $\beta_2 < \alpha$ or when $\alpha_1 = \alpha_2 = \alpha$, $\alpha < \beta_1$ and $\alpha < \beta_2$.

Here, we introduce the definition of n-dimensional admissibly ordered interval-valued overlap function.

Definition 10. *A function* $AOn : L([0,1])^n \rightarrow L([0,1])$ *is an n-dimensional admissibly ordered interval-valued overlap function for an admissible order* $\leq_{AD}$ *(n-dimensional* $\leq_{AD}$*-overlap function) if it satisfies the conditions* ***(IOn1)***, ***(IOn2)*** *and* ***(IOn3)*** *of Def. 9 and, for all* $Y, X_1, \ldots, X_n \in L([0,1])$*:*

(AOn4) AOn *is increasing for* $\leq_{AD}$ *in the first component:* $X_1 \leq_{AD} Y \Rightarrow AOn(X_1, \ldots, X_n) \leq_{AD} AOn(Y, X_2, \ldots, X_n)$.

Remark 2. *Condition* ***(IOn5)*** *from Def. 9 is not needed as the continuity was only a requirement in the original definition of overlap functions in order to enable them to be applied in image processing [1], which is not the case here. Besides that, the notion of continuity for admissible orders is still an open problem, and it is not the focus of this work.*

A 2-dimensional $\leq_{AD}$-overlap function is called $\leq_{AD}$-overlap function.

Example 3. *Some examples of* $\leq_{AD}$*-overlap functions are: (1) The interval minimum with respect to the order* $\leq_{IQ}$*, defined by*

$$\min_{\leq_{0.5,0}}(X,Y) = \begin{cases} X, & \textit{if } X \leq_{0.5,0} Y \\ Y, & \textit{otherwise} \end{cases}$$

is an $\leq_{0.5,0}$*-overlap function (see Prop. 6);*

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(2) The interval product is an n-dimensional $\leq_{1,0}$-overlap function, that is, it is increasing with respect to the order $\leq_{Lex2}$ (see Theorem 2);
(3) For a given $\alpha \in [0,1]$ and On_{OB} defined in Table I, the function $AOn_{OB}^{0.5}$ defined by

$$AOn_{OB}^{0.5}(X_1,\dots,X_n) = [On_{OB}(K_{0.5}(\underline{X_1},\overline{X_1}),\dots,K_{0.5}(\underline{X_n},\overline{X_n})) - 0.5\cdot m, On_{OB}(K_{0.5}(\underline{X_1},\overline{X_1}),\dots,K_{0.5}(\underline{X_n},\overline{X_n})) + 0.5\cdot m],$$

with

$$m = \min\{\overline{X_1}-\underline{X_1},\dots,\overline{X_n}-\underline{X_n}, On_{OB}(K_{0.5}(\underline{X_1},\overline{X_1}),\dots,K_{0.5}(\underline{X_n},\overline{X_n})), 1 - On_{OB}(K_{0.5}(\underline{X_1},\overline{X_1}),\dots,K_{0.5}(\underline{X_n},\overline{X_n}))\},$$

is an n-dimensional $\leq_{0.5,1}$-overlap function (see Theorem 4).

It is immediate that:

Proposition 6. *The interval minimum defined, for all $X,Y \in L([0,1])$, by*

$$\min_{\leq_{AD}}(X,Y) = \begin{cases} X, & \text{if } X \leq_{AD} Y \\ Y, & \text{otherwise} \end{cases}$$

is an $\leq_{AD}$-overlap function for any admissible order $\leq_{AD}$.

The result in Prop. 6 holds for a similarly defined n-dimensional interval minimum, that is, the function that returns the least interval from n interval-valued inputs accordingly to an admissible order $\leq_{AD}$.

Lemma 2. *Let $On : [0,1]^n \to [0,1]$ be an n-dimensional overlap function. Then, there exists $b \in (0,1)$ such that, for all $a \in (0,b)$, it holds that $On(a,1,\dots,1) < On(b,1,\dots,1)$.*

Proof. By condition $(\boldsymbol{On3})$ of Def. 7, one has that $On(x,1,\dots,1) < 1$, for each $x \in (0,1)$, and by $(\boldsymbol{On5})$, we have that there exists $x_0 \in (0,1)$ such that, for each $y \in (x_0,1)$, it holds that $O_n(x_0,1\dots,1) < O_n(y,1,\dots,1) < 1$. So, taking $b = \frac{x_0+1}{2}$ we have that, for each $a \in (0,b)$, it holds that $On(a,1,\dots,1) < On(b,1,\dots,1)$. ☐

Remark 3. *Consider $X,Y \in L([0,1])$. Observe that whenever $X <_{\alpha,\beta} Y$, with $\overline{X} > \overline{Y}$ and $\underline{X}+\alpha(\overline{X}-\underline{X}) < \underline{Y}+\alpha(\overline{Y}-\underline{Y})$, then it is immediate that*

$$\alpha < \frac{\underline{Y}-\underline{X}}{(\underline{Y}-\underline{X})-(\overline{Y}-\overline{X})}. \tag{5}$$

The following theorem presents an important result regarding o-representable n-dimensional iv-overlap functions and the conditions for them to be increasing with respect to an admissible order $\leq_{\alpha,\beta}$.

Theorem 3. *Let $IOn : L([0,1])^n \to L([0,1])$ be an o-representable n-dimensional iv-overlap function and $\alpha,\beta \in [0,1]$, $\alpha \neq \beta$. Then, IOn is $\leq_{\alpha,\beta}$-increasing if and only if $\alpha = 1$ and IOn^+ is a strict n-dimensional overlap function.*

Proof. ($\Rightarrow$) Based on Lemma 2, there exists $b \in (0,1)$ such that for all $a \in (0,b)$, $IOn^+(a,1,\dots,1) < IOn^+(b,1,\dots,1)$ holds. Consider $\alpha < 1$. It is possible to find $X,Y \in L([0,1])$ such that $X <_{\alpha,\beta} Y$, with $\underline{X} < \underline{Y} < \overline{Y} < \overline{X}$ and $\underline{X}+\alpha(\overline{X}-\underline{X}) < \underline{Y}+\alpha(\overline{Y}-\underline{Y})$. In fact, that is the case when $X = [\frac{b}{4}, b]$ and $Y = [\frac{b}{2}, \frac{b}{2-0.9\underbrace{9\dots9}_{n-times}}]$, for n sufficiently great.

Next, suppose that $Z = [0,1]$. Then, it follows that $IOn(X,Z,\dots,Z) = [0, IOn^+(\overline{X},1,\dots,1)] >_{Pr} [0, IOn^+(\overline{Y},1,\dots,1)] = IOn(Y,Z,\dots,Z)$. As $\leq_{\alpha,\beta}$ is an admissible order, then, one has that $IOn(X,Z,\dots,Z) >_{\alpha,\beta} IOn(Y,Z,\dots,Z)$, showing that IOn is not $\leq_{\alpha,\beta}$-increasing. By the contrapositive, if IOn is $\leq_{\alpha,\beta}$-increasing then $\alpha = 1$.

Now, let us suppose that IOn^+ is not strict. Then, there exist $x_1,\dots,x_n,y,z \in (0,1]$ such that $y < z$ and $IOn^+(x_1,\dots,x_{n-1},y) = IO^+(x_1,\dots,x_{n-1},z)$. As IOn is $\leq_{\alpha,\beta}$-increasing, one has that $\alpha = 1$, and thus, by Lemma 1, IOn is $\leq_{1,0}$-increasing. Since $[y,y] \leq_{1,0} [0,z]$, then:

$$IOn([x_1,x_1],\dots,[x_{n-1},x_{n-1}],[y,y]) \leq_{1,0} IOn([x_1,x_1],\dots,[x_{n-1},x_{n-1}],[0,z]).$$

As IOn is o-representable, one has that

$$IOn([x_1,x_1],\dots,[x_{n-1},x_{n-1}],[y,y]) = [IOn^-(x_1,\dots,x_{n-1},y), IOn^+(x_1,\dots,x_{n-1},y)],$$

$$IOn([x_1,x_1],\dots,[x_{n-1},x_{n-1}],[0,z]) = [IOn^-(x_1,\dots,x_{n-1},0), IOn^+(x_1,\dots,x_{n-1},z)].$$

Since $IOn^+(x_1,\dots,x_{n-1},y) = IOn^+(x_1,\dots,x_{n-1},z)$, it follows that:

$$IOn([x_1,x_1],\dots,[x_{n-1},x_{n-1}],[y,y]) \leq_{1,0} IOn([x_1,x_1],\dots,[x_{n-1},x_{n-1}],[0,z]) \Leftrightarrow IOn^-(x_1,\dots,x_{n-1},y) \leq IOn^-(x_1,\dots,x_{n-1},0),$$

which is a contradiction as $IOn^-(x_1,\dots,x_{n-1},0) = 0$ and $IOn^-(x_1,\dots,x_{n-1},y) > 0$, showing that IOn is not $\leq_{\alpha,\beta}$-increasing. Then, if IOn is $\leq_{\alpha,\beta}$-increasing then IOn^+ must be a strict n-dimensional overlap function.
($\Leftarrow$) Consider $X_1,\dots,X_{n-1},[a,b],[c,d] \in L([0,1])$ such that $[a,b] \leq_{1,0} [c,d]$. Then, one has the following cases:
(i) $a > c$ and $b < d$: In this case, one has that $[a,b] \subset [c,d]$, and, thus, by Theorem 2,

$$IOn(X_1,\dots,X_{n-1},[a,b]) \subseteq IOn(X_1,\dots,X_n,[c,d]).$$

First, consider $X_i \neq [0,0]$ for all $i \in \{1,\dots,n\}$. By Prop. 2, since IOn^+ is a strict overlap function, one has that:

$$\overline{IOn(X_1,\dots,X_{n-1},[a,b])} = IOn^+(\overline{X_1},\dots,\overline{X_{n-1}},b) < IOn^+(\overline{X_1},\dots,\overline{X_{n-1}},d) = \overline{IOn(X_1,\dots,X_{n-1},[c,d])}.$$

If X_i=$[0,0]$ for some $i \in \{1,\dots,n\}$, then

$$IOn(X_1,\dots,X_{n-1},[a,b]) = IOn(X_1,\dots,X_{n-1},[c,d]) = [0,0].$$

Thus, for any $X \in L([0,1])$, one concludes that

$$IOn(X_1,\dots,X_{n-1},[a,b]) \leq_{1,0} IOn(X_1,\dots,X_{n-1},[c,d]).$$

(ii) $b \leq d$ and $a \leq c$: In this case, $[a,b] \leq_{Pr} [c,d]$, and, thus,

$$IOn(X_1,\dots,X_{n-1},[a,b]) \leq_{Pr} IOn(X_1,\dots,X_{n-1},[c,d]).$$

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Since $\leq_{1,0}$ is an admissible order, one concludes that

$$IOn(X_1,\ldots,X_{n-1},[a,b]) \leq_{1,0} IOn(X_1,\ldots,X_{n-1},[c,d]).$$

Since IOn is o-representable and IOn^+ is a strict overlap function, then the result follows Lemma 1. $\square$

Now, let us show an example to illustrate Theorem 3 with a specific o-representable n-dimensional $\leq_{AD}$-overlap function.

Example 4. *Let IOn_{Pp} be an n-dimensional iv-overlap function as defined in Example 2, and $\alpha,\beta \in [0,1]$, $\alpha \neq \beta$. As*

$$\begin{aligned} IOn_{Pp}(\vec{X}) &= \left[IOn^-_{Pp}(\vec{X}), IOn^+_{Pp}(\vec{X})\right] \\ &= \left[\underline{X_1}^p \cdot \ldots \cdot \underline{X_n}^p, \overline{X_1}^p \cdot \ldots \cdot \overline{X_n}^p\right], \end{aligned}$$

it is clear that IOn^+_{Pp} is a strict n-dimensional overlap function. Furthermore, suppose $\alpha < 1$ and consider $Z = [0,1]$, $X = [0.1, 0.5], Y = [0.4, 0.4\underbrace{9\ldots9}_{n-times}]$, for $n > 0$, where, clearly, $\underline{X} < \underline{Y}$ and $\overline{X} > \overline{Y}$. Then, there exists a sufficiently great n such that $X <_{\alpha,\beta} Y$, and, by Remark 3, Eq. (5) holds. However, one has that

$$\begin{aligned} IOn_{Pp}(X,Z) &= [0, 0.5^p] >_{Pr} [0,(0.4\underbrace{9\ldots9}_{n-times})^p] \\ &= IOn_{Pp}(Y,Z). \end{aligned}$$

As $\leq_{\alpha,\beta}$ is an admissible order, then, it follows that $IOn_{Pp}(X,Z) >_{\alpha,\beta} IOn_{Pp}(Y,Z)$, showing that IOn_{Pp} is not $\leq_{\alpha,\beta}$-increasing. Then, if IOn_{Pp} is $\leq_{\alpha,\beta}$-increasing then $\alpha = 1$. Since IOn_{Pp} is o-representable and IOn^+_{Pp} is a strict n-dimensional overlap function, from Theorem 3 and Lemma 1 one has that IOn_{Pp} is $\leq_{1,\beta}$-increasing. Thus, IOn_{Pp} is $\leq_{\alpha,\beta}$-increasing if and only if $\alpha = 1$.

IV. A Construction Method

In this section, we present a construction method to obtain n-dimensional $\leq_{\alpha,\beta}$-overlap functions, with $\alpha \neq \beta$, for a given α and a strict n-dimensional overlap function. For the sake of simplicity, let us denote $K_\alpha(\underline{X},\overline{X})$ simply as $K_\alpha(X)$.

Theorem 4. *Let On be a strict n-dimensional overlap function, $\alpha \in (0,1)$ and $\beta \in [0,1]$ such that $\alpha \neq \beta$. Then $AOn^\alpha : L([0,1])^n \to L([0,1])$ defined, for all $\vec{X} \in L([0,1])^n$, by*

$$\begin{aligned} AOn^\alpha(\vec{X}) = [&On(K_\alpha(X_1),\ldots,K_\alpha(X_n)) - \alpha m, \\ &On(K_\alpha(X_1),\ldots,K_\alpha(X_n)) + (1-\alpha)m], \end{aligned}$$

where

$$\begin{aligned} m = &\min\{\overline{X_1} - \underline{X_1},\ldots,\overline{X_n} - \underline{X_n}, On(K_\alpha(X_1),\ldots,K_\alpha(X_n)), \\ &1 - On(K_\alpha(X_1),\ldots,K_\alpha(X_n))\}, \end{aligned}$$

is an n-dimensional $\leq_{\alpha,\beta}$-overlap function.

Proof. Consider $\alpha \in (0,1)$ and $\beta \in [0,1]$ such that $\alpha \neq \beta$. By Lemma 1 it is sufficient to consider the case $\beta = 0$ and $\beta = 1$. Clearly, AOn^α is well defined and commutative. Also:

$$\begin{aligned} &K_\alpha(AOn^\alpha(\vec{X})) \\ &= On(K_\alpha(X_1),\ldots,K_\alpha(X_n)) - \alpha m \\ &\quad +\alpha(On(K_\alpha(X_1),\ldots,K_\alpha(X_n)) + (1-\alpha)m \\ &\quad -On(K_\alpha(X_1),\ldots,K_\alpha(X_n)) + \alpha m) \\ &= On(K_\alpha(X_1),\ldots,K_\alpha(X_n)). \end{aligned}$$

Furthermore,

$$\begin{aligned} K_0(AOn^\alpha(\vec{X})) &= On(K_\alpha(X_1),\ldots,K_\alpha(X_n)) - \alpha m, \\ K_1(AOn^\alpha(\vec{X})) &= On(K_\alpha(X_1),\ldots,K_\alpha(X_n)) + (1-\alpha)m. \end{aligned}$$

Now, consider $\vec{X} \in L([0,1])^n$. Then, since $\alpha \neq 0$,

$$\begin{aligned} &AOn^\alpha(\vec{X}) = [0,0] \\ &\Leftrightarrow K_\alpha(AOn^\alpha(\vec{X})) = 0 \\ &\Leftrightarrow On(K_\alpha(X_1),\ldots,K_\alpha(X_n)) = 0 \\ &\Leftrightarrow K_\alpha(X_i) = 0 \text{ for some } i \in \{1,\ldots,n\} \\ &\Leftrightarrow X_i = [0,0] \text{ for some } i \in \{1,\ldots,n\}. \end{aligned}$$

Therefore, AOn^α satisfies **(IOn2)**.

Consider $\vec{X} \in L([0,1])^n$. Then, since $\alpha \neq 1$,

$$\begin{aligned} &AOn^\alpha(\vec{X}) = [1,1] \\ &\Leftrightarrow K_\alpha(AOn^\alpha(\vec{X})) = 1 \\ &\Leftrightarrow On(K_\alpha(X_1),\ldots,K_\alpha(X_n)) = 1 \\ &\Leftrightarrow K_\alpha(X_i) = 1 \text{ for each } i \in \{1,\ldots,n\} \\ &\Leftrightarrow X_i = [1,1] \text{ for each } i \in \{1,\ldots,n\}. \end{aligned}$$

Thus, AOn^α satisfies **(IOn3)**. In order to prove that AOn^α satisfies **(AOn4)** for $\leq_{\alpha,0}$, consider $Y <_{\alpha,0} Z$ and $X \in L([0,1])$ such that $K_\alpha(X) = 0$. Then,

$$\begin{aligned} &On(K_\alpha(Y),K_\alpha(X),\ldots,K_\alpha(X)) = 0 \\ &\qquad = On(K_\alpha(Z),K_\alpha(X),\ldots,K_\alpha(X)) \end{aligned}$$

and, therefore,

$$\begin{aligned} &\min\{\overline{Y} - \underline{Y}, \overline{X} - \underline{X},\ldots,\overline{X} - \underline{X}, \\ &\quad On(K_\alpha(Y),K_\alpha(X),\ldots,K_\alpha(X)), \\ &\quad 1 - On(K_\alpha(Y),K_\alpha(X),\ldots,K_\alpha(X))\} = 0 \\ &= \min\{\overline{Z} - \underline{Z}, \overline{X} - \underline{X},\ldots,\overline{X} - \underline{X}, \\ &\quad On(K_\alpha(Z),K_\alpha(X),\ldots,K_\alpha(X)), \\ &\quad 1 - On(K_\alpha(Z),K_\alpha(X),\ldots,K_\alpha(X))\}. \end{aligned}$$

Hence, $AOn^\alpha(Y,X,\ldots,X) = [0,0] = AOn^\alpha(Z,X,\ldots,X)$.

Now take $X \in L([0,1])$ such that $K_\alpha(X) > 0$. By definition, we have the following two cases:

1) $K_\alpha(Y) < K_\alpha(Z)$. Since On is strict, one has that

$$\begin{aligned} &On(K_\alpha(Y),K_\alpha(X),\ldots,K_\alpha(X)) < \\ &\qquad On(K_\alpha(Z),K_\alpha(X),\ldots,K_\alpha(X)) \end{aligned}$$

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and, therefore,

$$\begin{aligned} & K_\alpha(AOn^\alpha(Y, X, \ldots, X)) \\ & \quad = \; On(K_\alpha(Y), K_\alpha(X), \ldots, K_\alpha(X)) < \\ & \qquad On(K_\alpha(Z), K_\alpha(X), \ldots, K_\alpha(X)) \\ & \quad = \; K_\alpha(AOn^\alpha(Z, X, \ldots, X)). \end{aligned}$$

So, $AOn^\alpha(Y, X, \ldots, X) <_{\alpha,0} AOn^\alpha(Z, X, \ldots, X)$.

2) $K_\alpha(Y) = K_\alpha(Z)$ and $K_0(Y) < K_0(Z)$. Then, $\underline{Y} < \underline{Z} \leq \overline{Z} < \overline{Y}$ and, therefore $\overline{Y} - \underline{Y} > \overline{Z} - \underline{Z}$. Thus, since

$$On(K_\alpha(Y), K_\alpha(X), \ldots, K_\alpha(X)) = On(K_\alpha(Z), K_\alpha(X), \ldots, K_\alpha(X)),$$

it holds that

$$\begin{aligned} m_1 = \min\{ & \overline{Y} - \underline{Y}, \overline{X} - \underline{X}, \ldots, \overline{X} - \underline{X}, \\ & On(K_\alpha(Y), K_\alpha(X), \ldots, K_\alpha(X)), \\ & 1 - On(K_\alpha(Y), K_\alpha(X), \ldots, K_\alpha(X))\} \geq \\ & \min\{\overline{Z} - \underline{Z}, \overline{X} - \underline{X}, \ldots \overline{X} - \underline{X}, \\ & On(K_\alpha(Z), K_\alpha(X), \ldots, K_\alpha(X)), \\ & 1 - On(K_\alpha(Z), K_\alpha(X), \ldots, K_\alpha(X))\} \; = \; m_2. \end{aligned}$$

Hence,

$$\begin{aligned} & K_\alpha(AOn^\alpha(Y, X, \ldots, X)) \\ & \quad = \; On(K_\alpha(Y), K_\alpha(X), \ldots, K_\alpha(X)) \\ & \quad = \; On(K_\alpha(Z), K_\alpha(X), \ldots, K_\alpha(X)) \\ & \quad = \; K_\alpha(AOn^\alpha(Z, X, \ldots, X)), \end{aligned}$$

and

$$\begin{aligned} & K_0(AOn^\alpha(Y, X, \ldots, X)) \\ & \quad = \; On(K_\alpha(Y), K_\alpha(X), \ldots, K_\alpha(X)) - \alpha m_1 \\ & \quad \leq \; On(K_\alpha(Y), K_\alpha(X), \ldots, K_\alpha(X)) - \alpha m_2 \\ & \quad = \; On(K_\alpha(Z), K_\alpha(X), \ldots, K_\alpha(X)) - \alpha m_2 \\ & \quad = \; K_0(AOn^\alpha(Z, X, \ldots, X)). \end{aligned}$$

So, $AOn^\alpha(Y, X, \ldots, X) \leq_{\alpha,0} AOn^\alpha(Z, X, \ldots, X)$.

Thus, for each $\alpha \in (0, 1)$, AOn^α is an n-dimensional $\leq_{\alpha,0}$-overlap function. The proof that AOn^α satisfies **(AOn4)** for $\leq_{\alpha,1}$ and $\alpha \in (0, 1)$ is obtained analogously. □

Now, let us see an example of a $\leq_{\alpha,\beta}$ order and overlap function that do not allow for the construction of an $\leq_{\alpha,\beta}$-increasing o-representable iv-overlap function IO, but in which one can obtain an $\leq_{\alpha,\beta}$-overlap function AO^α via the method presented in Theorem 4.

Example 5. *Consider the admissible order $\leq_{0.4,0}$ and be the overlap function $Op : [0,1]^2 \to [0,1]$ defined, for all $x, y \in [0, 1]$, by $Op(x, y) = x \cdot y$. From Theorem 4, for $On = Op$:*

$$AOp^{0.4}(X, Y) = [K_{0.4}(X) \cdot K_{0.4}(Y) - 0.4 \cdot m, \; K_{0.4}(X) \cdot K_{0.4}(Y) + (0.6)m],$$

where

$$m = \min\{\overline{X} - \underline{X}, \overline{Y} - \underline{Y}, K_{0.4}(X) \cdot K_{0.4}(Y), \; 1 - K_{0.4}(X) \cdot K_{0.4}(Y)\}.$$

Now, for $X = [0, 1], Y = [0.2, 0.2]$ and $Z = [0, 0.4]$ one has that $Z \leq_{0.4,0} Y$, $AOp^{0.4}(X, Z) = [0.0384, 0.1024]$ and $AOp^{0.4}(X, Z) = [0.08, 0.08]$, meaning that

$$Z \leq_{0.4,0} y \Leftrightarrow AOp^{0.4}(X, Z) \leq_{0.4,0} AOp^{0.4}(X, Y),$$

which is expected for an $\leq_{0.4,0}$-overlap function.

However, if we try to construct an (admissibly ordered) o-representable interval-valued overlap function IOp in which $IOp^- = IOp^+ = Op$, one can observe that $IOp(X, Z) = [0, 0.4]$ and $IOp(X, Y) = [0, 0.2]$, meaning that $Y \leq_{0.4,0} Z$ and $IOp(X, Z) >_{0.4,0} IOp(X, Y)$, proving that IOp is not an $\leq_{0.4,0}$-overlap function. This happens because $\alpha \neq 1$, which fails to follow the conditions stated in Theorem 3.

Remark 4. *The construction method introduced in Theorem 4 allows us to obtain different n-dimensional $\leq_{\alpha,\beta}$-overlap functions with respect to any $\leq_{\alpha,\beta}$ order. Thus, its adaptability allows for it to be employed in various applications with different approaches to the ranking of intervals, determined by the choice of different α and β.*

Remark 5. *As stated in Theorem 4, the n-dimensional overlap function that is the core of the construction method must be strict. Yet, this requirement does not present itself as a hindrance, as most n-dimensional overlap functions are, in fact, strict. One notable exception is the minimum operator. However, the interval minimum as show in Prop. 6 is an $\leq_{AD}$-overlap function for any admissible order $\leq_{AD}$, and turns out to be a more suitable interval representation of the minimum.*

Remark 6. *It is noteworthy that the width of the resulting interval when applying AOn^α is given by*

$$\begin{aligned} m \;\; = \;\; \min\{ & \overline{X_1} - \underline{X_1}, \ldots, \overline{X_n} - \underline{X_n}, \\ & On(K_\alpha(X_1), \ldots, K_\alpha(X_n)), \\ & 1 - On(K_\alpha(X_1), \ldots, K_\alpha(X_n))\}. \end{aligned}$$

Thus, AOn^α is a width-preserving operation, as the resulting interval will never be wider than any of the aggregated inputs, which is a desirable property in many applications. On the other hand, by the way m is defined, if at least one of the aggregated intervals is degenerate, than the resulting interval when applying AOn^α will also be degenerate.

V. Analysis of the Influence of the Studied Concepts in IV-FRBCSs

The objective of this section is to analyze the behaviour of different admissible orders and n-dimensional (admissibly ordered) iv-overlap functions applied on the interval-valued fuzzy reasoning method (IV-FRM) of an IV-FRBCS. In order to do that, first we are going to review the main points of FRBCSs and IV-FRBCSs, highlighting the steps where we apply our new theoretical results.

A. Interval-Valued Fuzzy Rule-based Classification Systems

A classification problem is composed by P training examples $\vec{x_p} = (x_{p1}, \ldots, x_{pn}), p \in \{1, \ldots, P\}$ where x_{pi} is the value of the i-th variable of the p-th example. Each example

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belongs to one of M classes in $C=\{C_1,\dots,C_M\}$. The learned classifier aims to identify the class of new testing examples.

FRBCSs are one most frequently adopted technique to deal with classification problems. They provide a good balance between accuracy and interpretability, since the antecedents of their rules are composed of linguistic labels, while still providing accurate results [44]. We adopt the following structure for the fuzzy rules:

$$\text{Rule } R_j : \text{If } x_1 \text{ is } A_{j1} \text{ and } \dots \text{ and } x_n \text{ is } A_{jn} \quad \text{then } Class = C'_j \text{ with } RW_j, \tag{6}$$

where R_j is the label of the j-th rule, $x = (x_1, \dots, x_n)$ is an n-dimensional example vector, A_{ji} is the fuzzy set representing the linguistic term of the j-th rule in the i-th antecedent, $C'_j \in C$ is a class label, and $RW_j \in [0, 1]$ is the rule weight [55]. Specifically, we consider the computation of the rule weight using the fuzzy confidence value or certainty factor, given by:

$$RW_j = \frac{\sum_{x_p \in C'_j} A_j(x_p)}{\sum_{p=1}^{P} A_j(x_p)}, \tag{7}$$

where $A_j(x_p)$ is the matching degree of the pattern x_p with the antecedent part of the fuzzy rule R_j, computed as

$$A_j(x_p) = \mathfrak{c}(A_{j1}(x_{p1}), \cdots, A_{jn}(x_{pn})), \tag{8}$$

where $\mathfrak{c}$ is an n-dimensional conjunction operator and $j \in \{1,\dots,L\}$.

IV-FRBCSs are FRBCSs where some of the linguistic labels (or all of them) are modelled using IVFSs. Furthermore, the FRM must work with intervals instead of numbers to take into account the degree of uncertainty throughout the whole inference process (see Section V-B).

B. New Interval-valued Fuzzy Reasoning Method

In this paper, we apply our new theoretical results in the IVTURS algorithm[1], which is a state of the art IV-FRBCS. Its learning process is composed of three steps:

1) To build an IV-FRBCS. This step involves the following tasks:

• The generation of an initial FRBCS by applying FARC-HD [56], whose first learning stage is based on the Apriori algorithm [57] that builds fuzzy rules using the support and confidence (Eq. (7)). In this process, the product t-norm is usually used as the conjunction operator $\mathfrak{c}$ in Eq. (8). In this present paper, we propose to replace the product t-norm by different n-dimensional overlap functions On. Those functions are considered in the construction of the n-dimensional (admissibly ordered) iv-overlap functions used in the IV-FRM (described in the sequence). This change is important because in this manner we can learn different fuzzy rules (resulting in different IV-FRBCSs) depending on the function On.

• Modelling the linguistic labels of the learned FRBCS by means of IVFSs;

• The generation of an initial IV-REF for each variable of the problem.

2) To apply an optimization approach with a double purpose:

• To learn the best values of the IV-REFs' parameters;

• To apply a rule selection process in order to decrease the system's complexity.

Once the interval-valued fuzzy rules composing the system have been created, let us modify the mechanism for classifying new examples. Thus, let $\vec{x_p} = (x_{p1}, \dots, x_{pn})$ be a new example to be classified, L being the number of rules in the rule base and M being the number of classes of the problem. The steps of the new IV-FRM are the following:

(1) *Interval matching degree*: It represents the strength of the activation of the if-part of the rules for each x_p. We use an IV-REF IR to compute the similarity between the interval membership degrees (of each variable of the pattern to the corresponding IVFS) and the ideal membership degree $[1, 1]$, and then, we apply an interval-valued function $F_O : L([0, 1])^n \to L([0, 1])$, for $j \in \{1, ..., L\}$ as follows:

$$\left[\underline{\mathcal{A}_j(x_p)}, \overline{\mathcal{A}_j(x_p)}\right] = F_O\left(IR\left(\left[\underline{\mathcal{A}_{j1}(x_{p1})}, \overline{\mathcal{A}_{j1}(x_{p1})}\right], [1,1]\right), \dots, IR\left(\left[\underline{\mathcal{A}_{jn}(x_{pn})}, \overline{\mathcal{A}_{jn}(x_{pn})}\right], [1,1]\right)\right),$$

with F_O being an interval conjunction operator that can be defined in two different ways:

a) IOn, an o-representable n-dimensional iv-overlap function;

b) AOn^{α}, an n-dimensional $\leq_{\alpha,\beta}$-overlap function, with $\leq_{\alpha,\beta}$ being the same order applied in Step (4) of the IV-FRM.

As we have mentioned previously, F_O is defined based on the n-dimensional overlap function On applied as the conjunction operator when generating the initial FRBCS.

(2) *Interval association degree*: For the class of each rule, the interval matching degree is weighted with the corresponding iv-rule weight $IRW_j^k \in L([0, 1])$, through an interval-valued function $F_P : L([0, 1])^n \to L([0, 1])$, resulting in:

$$\left[\underline{b_j^k}, \overline{b_j^k}\right] = F_P\left(\left[\underline{\mathcal{A}_j(x_p)}, \overline{\mathcal{A}_j(x_p)}\right], \left[\underline{IRW_j^k}, \overline{IRW_j^k}\right]\right), \tag{9}$$

with $k = 1,\dots,M$, $j = 1,\dots,L$ and F_P being defined according to the function F_O applied to obtain the interval matching degree in Step (1), resulting in two possibilities:

a) If $F_O = IOn$, then $F_P = IOn_P = \widehat{On_P}$ (representable interval product overlap), with On_p shown in Table I;

b) If $F_O = AOn^{\alpha}$, then $F_P = AOn_P^{\alpha}$ (admissibly ordered interval product overlap), with AOn_P^{α} being an n-dimensional $\leq_{\alpha,\beta}$-overlap function defined through the construction method presented in Theorem. 4 considering On_p as shown in Table I and the same α as the one in the chosen $\leq_{\alpha,\beta}$ order to be applied in Step (4) of the IV-FRM.

For the rule weight, we utilize the interval-valued confidence value as in [58]. The resulting equation is shown as follows:

$$IRW_j = \sum_{x_p \in C'_j} \left[\underline{\mathcal{A}_j(x_p)}, \overline{\mathcal{A}_j(x_p)}\right] \div_H \sum_{p=1}^{P} \left[\underline{\mathcal{A}_j(x_p)}, \overline{\mathcal{A}_j(x_p)}\right].$$

(3) *Interval pattern classification soundness degree for all classes*: We aggregate the interval association degrees of each

[1]For an in-depth look at each step of the IVTURS algorithm, see [43].

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class (obtained in Step (2)) in which the upper bound is greater than 0 by applying an interval-valued aggregation function IA:

$$\left[\underline{Y_k}, \overline{Y_k}\right] = IA\left(\left[\underline{b_j^k}, \overline{b_j^k}\right], j = 1, \ldots, L \text{ and } \overline{b_j^k} > 0\right),$$

with $k = 1, \ldots, M$.

(4) *Classification*: A decision function F is applied over the interval soundness degree of the system for the pattern classification for all classes, given by:

$$F\left(\left[\underline{Y_1}, \overline{Y_1}\right], \ldots, \left[\underline{Y_M}, \overline{Y_M}\right]\right) = \arg\max_{k=1,\ldots,M}\left(\left[\underline{Y_k}, \overline{Y_k}\right]\right).$$

The last step of the IV-FRM consists of selecting the maximum interval soundness degree. To avoid a stalemate, the usage of a total order for intervals is preferred in this step. So, we use an admissible order $\leq_{\alpha,\beta}$ as defined in Def. 5.

One can observe that there are many possibilities of configuration of this new IV-FRM, based on the chosen functions F_O, F_P and the admissible order $\leq_{\alpha,\beta}$. However, as some of those choices are interconnected, we first decide on the admissible order $\leq_{\alpha,\beta}$ to be used in Step (4), as it determines the α applied in the construction of the n-dimensional $\leq_{\alpha,\beta}$-overlap functions when $F_O = AOn^{\alpha}$ and $F_P = AOn_P^{\alpha}$.

Notice that the interval-valued function F_O plays a key role because it determines the choice of: 1) the n-dimensional overlap function On used in the rule learning process; 2) the interval-valued function F_P used in Step (2) of the IV-FRM.

C. Experimental Framework

To analyze the behaviour of a classification system when applying different n-dimensional (admissibly ordered) iv-overlap functions and different admissible orders, we have selected 31 real-world data-sets from the KEEL repository [59], which are publicly available on the webpage (http://www.keel.es/dataset.php). Table II summarizes the properties of the selected data-sets, showing for each data-set the number of attributes (Atts.), the number of examples (Ex.), and the number of classes (Class.). We must point out that the *magic, page-blocks, penbased, ring, satimage, shuttle*, and *twonorm* data-sets have been stratified sampled at 10% in order to improve the learning process efficiency. Missing values from *bands*, *cleaveland* and *wisconsin* data-sets have been removed before the experimentation.

A *fivefold cross-validation model* has been applied in order to carry out the different experiments. This was done by splitting the data-set into five random partitions of data, employing a combination of four of them (80%) to train the system and the remaining one (20%) to test it. This process is carried out 5 times, changing the testing partition in each iteration. The performance measure was done through the accuracy rate.

The set-up of the IVTURS classifier is as recommended in [43], but we apply our new theoretical developments described in Sections V-A and V-B. We study the behaviour of the classifier using several combinations of the new theoretical concepts, as shown in Table III. Looking at Table III, we can clearly observe that the interval-valued conjunction operator (F_O) used in Step (1) the IV-FRM determines the overlap function (On) used when generating the initial fuzzy rules as well as the interval product (F_P) used in the Step (2) of the IV-FRM. We must point out in the case of ADM-Min, $\min_{\leq_{\alpha,\beta}}$ is simply the n-dimensional interval minimum with respect to the $\leq_{\alpha,\beta}$ order at hand, as in Def. 6. Finally, for each combination we check the influence of the admissible order used in Step (4) of the IV-FRM. Specifically, we test three linear orders for intervals: $\leq_{Lex1}$ ($\alpha = 0, \beta = 1$), $\leq_{IQ}$ ($\alpha = 0.5, 0$) and $\leq_{Lex2}$ ($\alpha = 1, \beta = 0$)[2].

To give statistical support to our analysis, we use the aligned Friedman ranks test [60] to detect statistical differences among a group of results and report the obtained ranks of each method (with lower ranks being preferable). Next, we apply the Holm's post-hoc test [61] to compare the best ranking method with the other considered methods. Finally, we apply a Wilcoxon

TABLE II: Summary of the employed datasets

id	Data-set	Atts.	Ex.	Class.
app	appendicitis	7	106	2
bal	balance	4	625	3
ban	banana	2	5300	2
bds	bands	19	365	2
bup	bupa	6	345	2
clv	cleveland	13	297	5
con	contraceptive	9	1473	3
eco	ecoli	7	336	8
gla	glass	9	214	7
hab	haberman	3	306	2
hay	hayes-hoth	4	160	3
ion	ionosphere	33	351	2
iri	iris	4	150	3
led	led7digit	7	500	10
mag	magic	10	19020	2
new	newthyroid	5	215	3
pag	pageblocks	10	5472	5
pen	penbased	16	10992	10
pho	phoneme	5	5404	2
pim	pima	8	768	2
rin	ring	20	7400	2
sah	saheart	9	462	2
sat	satimage	36	6435	7
shu	shuttle	9	58000	7
spe	spectfheart	44	267	2
tit	titanic	3	2201	2
two	twonorm	20	7400	2
veh	vehicle	18	846	4
win	wine	13	178	3
wis	wisconsin	9	683	2
yea	yeast	8	1484	10

TABLE III: Configuration schemes for the used classifiers

Classifier identifier	On	F_O	F_P
REP-Prod	On_P	$IOn_P = \widehat{On_P}$	$IOn_P = \widehat{On_P}$
REP-Min	On_M	$IOn_M = \widehat{On_M}$	$IOn_P = \widehat{On_P}$
REP-Hp	On_{Hp}	$IOn_{Hp} = \widehat{On_{Hp}}$	$IOn_P = \widehat{On_P}$
REP-OB	On_{OB}	$IOn_{OB} = \widehat{On_O B}$	$IOn_P = \widehat{On_P}$
REP-Gm	On_{Gm}	$IOn_{Gm} = \widehat{On_{Gm}}$	$IOn_P = \widehat{On_P}$
REP-Hm	On_{Hm}	$IOn_{Hm} = \widehat{On_{Hm}}$	$IOn_P = \widehat{On_P}$
ADM-Prod	On_P	AOn_P^{α}	$F_P = AOn_P^{\alpha}$
ADM-Min	On_M	$\min_{\leq_{\alpha,\beta}}$	AOn_P^{α}
ADM-Hp	On_{Hp}	AOn_{Hp}^{α}	AOn_P^{α}
ADM-OB	On_{OB}	AOn_{OB}^{α}	AOn_P^{α}
ADM-Gm	On_{Gm}	AOn_{Gm}^{α}	AOn_P^{α}
ADM-Hm	On_{Hm}	AOn_{Hm}^{α}	AOn_P^{α}

[2]To respect Theorem 4, in all experiments with ADM classifiers we consider $\alpha = 0 + 1^{-10}$ and $\alpha = 1 - 1^{-10}$, for $\leq_{Lex1}$ and $\leq_{Lex2}$, respectively.

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TABLE IV: Results in testing for the different methods

Method	$\leq_{Lex1}$	$\leq_{IQ}$	$\leq_{Lex2}$
REP-Prod	78.96	79.67	79.17
REP-Min	78.92	79.52	79.51
REP-Hp	79.08	79.34	79.35
REP-OB	79.00	79.83	79.33
REP-Gm	79.14	79.48	79.57
REP-Hm	79.19	79.39	79.41
ADM-Prod	**79.49**	79.14	79.19
ADM-Min	79.22	79.39	79.40
ADM-Hp	79.28	79.47	79.54
ADM-OB	79.25	79.57	**79.64**
ADM-Gm	79.17	**79.93**	79.49
ADM-Hm	78.93	79.23	79.16

Signed-Ranks test [62] in order to do pairwise comparisons. This selection of tests is suggested in [63], where it is shown that its use in machine learning is highly recommended.

D. Discussion of the Results

In Table IV we show the averaged results in testing for all the possible combinations among the three orders (by columns) and the configurations shown in Table III (by rows). The result we show is the averaged behaviour of the system in the 31 datasets considered in the study. For each admissible order, we highlight in bold face the best result, that is, the best n-dimensional (admissibly ordered) overlap function. The detailed results, that is, the results in all the datasets (in all the partitions) for all the combinations can be queried on the webpage (https://github.com/tiagoasmus/TestingResults-Adm-Overlaps/find/master?q=).

By looking at the highlighted results in Table IV, we see that, for each admissible order, the best performing configuration of the algorithm (regarding the global mean) was based on an $\leq_{AD}$-overlap function (ADM-Prod, ADM-Gm and ADM-OB). Furthermore, it appears that both the admissible order and the (interval-valued) conjunction operators have an impact on the accuracy obtained by each classifier.

In first place we studied if there are differences in the accuracy for a given method when we vary the chosen admissible order. In order to do so, we applied the aligned test to compare the three total orders for each configuration. The obtained ranks, as well as the Adjusted P-Values (APVs, presented in brackets) provided by the Holm's post hoc test are shown in Table V, where we have highlighted in **bold-face** the best rank (the least one) and we have stressed with an asterisk (*) those cases in which there are statistical differences (using $\alpha = 0.05$) between the control method (the one associated with the best rank) and the method in the corresponding total order.

From the results in Table V, one can observe:

1) The order $\leq_{Lex1}$ is the control method for only one configuration (ADM-Prod), being the worst ranking method in most cases, with statistical differences in several comparisons;
2) Although the order $\leq_{Lex2}$ is considered the control method in six configurations, in all those cases there are no significant differences with respect to the order $\leq_{IQ}$, with both orders presenting similar ranks;
3) The order $\leq_{IQ}$ is the control method in five configurations, and in two of those cases, it presents statistical differences versus $\leq_{Lex2}$ (and a low APV for ADM-Gm). Furthermore, it produces comparable results with the other orders even when it is not the control method.

TABLE V: Average Rankings of the algorithms (Aligned Friedman) - Comparing $\leq_{\alpha,\beta}$ orders

Method	$\leq_{Lex1}$	$\leq_{IQ}$	$\leq_{Lex2}$
REP-Prod	55.16 (0.011)*	**36.11** (-)	49.73 (0.047)*
REP-Min	57.94 (0.033)*	41.55 (0.996)	**41.52** (-)
REP-Hp	52.29 (0.466)	44.60 (0.944)	**44.11** (-)
REP-OB	57.81 (0.001)*	**33.73** (-)	49.47 (0.022)*
REP-Gm	56.24 (0.065)	**41.60** (-)	43.16 (0.819)
REP-Hm	51.32 (0.567)	45.71 (0.799)	**43.9677** (-)
ADM-Prod	**43.05** (-)	48.95 (0.771)	49.00 (0.771)
ADM-Min	47.11 (1.000)	48.61 (1.000)	**45.27** (-)
ADM-Hp	49.69 (0.9230)	46.66 (0.9230)	**44.65** (-)
ADM-OB	52.58 (0.4129)	44.50 (0.9325)	**43.92** (-)
ADM-Gm	57.95 (0.002)*	**35.13** (-)	47.92 (0.062)
ADM-Hm	45.00 (0.079)	**40.89** (-)	45.11 (0.538)

TABLE VI: Average Rankings of the algorithms (Aligned Friedman)

Method	Group REP Rank	APV	Method	Group ADM Rank	APV
REP-Prod	84.82	0.453	ADM-Prod	100.74	0.186
REP-Min	98.24	0.250	ADM-Min	99.02	0.187
REP-Hp	104.61	0.112	ADM-Hp	97.03	0.187
REP-OB	**74.57**	-	ADM-OB	87.65	0.302
REP-Gm	92.11	0.399	ADM-Gm	**73.52**	-
REP-Hm	106.65	0.095	ADM-Hm	103.05	0.154

In summary, we can conclude that $\leq_{Lex1}$ is not a suitable choice and $\leq_{IQ}$ is providing a robust behaviour regardless of the configuration. For these reasons, we decided to investigate the behaviour of our classifiers by varying the n-dimensional (admissibly ordered) iv-overlap functions used in the IV-FRM, taking in consideration the admissible order $\leq_{IQ}$.

To do it, we divided the methods into two groups, based on the interval conjunction operator (F_O) applied in Step (1) of the IV-FRM: representable (REP) and admissibly ordered (ADM) n-dimensional iv-overlap functions. We applied the Aligned Friedman and Holm's tests to compare the six n-dimensional (admissibly ordered) iv-overlap function belonging to each group. The results obtained for the functions of groups REP and ADM are shown in Tables VI, with the best ranking method in each group highlighted in **bold-face**.

From the results presented in Table VI, one can observe:

1) The behaviours of the representable n-dimensional iv-overlap functions are similar, but REP-OB seems to be the best option among its group;
2) Though there are not statistical differences among the n-dimensional admissibly ordered iv-overlap functions, ADM-

TABLE VII: Pairwise comparisons via Wilcoxon test

Comparison	R^+	R^-	p-value
REP-OB vs ADM-Gm	230	266	0.649
REP-Prod vs ADM-Gm	195	301	0.285

Gm stands out as it obtains low APVs versus the remained functions in its group, except for ADM-OB.

An interesting observation is that both control methods (REP-OB and ADM-Gm) and their respective interval conjunction operations (IOn_{OB} and $AOn^{0.5}_{Gm}$) are based on non-associative operations (OB overlap and geometric mean), pointing out that n-dimensional overlap functions and their interval extensions are suitable to be applied in IV-FRBCS.

Next, we carry out a pairwise comparison between the two representatives of each group (control methods), using the Wilcoxon test. We also compare the best performing method overall (ADM-Gm) with the original IVTURS (which is obtained using the REP-Prod configuration and the order $\leq_{IQ}$). The results for of these two last pairwise comparisons can be seen in Table VII.

As first indicated by the global means and afterwards confirmed by the statistical analysis, the combination of the admissible order $\leq_{IQ}$ and the n-dimensional $\leq_{IQ}$-overlap function $AOn^{0.5}_{Gm}$ in the ADM-Gm method produces the most accurate classification results. It does not statistically improve the performance over all other configurations, but in the light of the obtained results, we can recommend it as the best option for this type of IV-FRBCS.

VI. Conclusion

In this paper, we presented new results regarding admissible orders and defined the concept of n-dimensional admissibly ordered interval-valued overlap functions. A width-preserving construction method for this type of function for a given admissible order was also presented, which allowed us to define different n-dimensional $\leq_{AD}$-overlap functions to be applied in the IV-FRM of IVTURS.

On the application side, our experimentation made clear the impact of the chosen admissible order on IV-FRBCSs, with the order $\leq_{IQ}$ presenting itself as the most robust one. We also conclude that n-dimensional (admissibly ordered) interval-valued overlap functions, particularly the non-associative ones, are recommended to be applied on the IV-FRM of an IV-FRBCSs, with a special mention to the n-dimensional $\leq_{IQ}$-overlap function $AOn^{0.5}_{Gm}$.

All of the aforementioned contributions aimed to address: (i) the theoretical and applied gap in the literature regarding the configuration possibilities of IV-FRBCSs; (ii) the characteristics of the applied interval-valued functions and related interval orders.

As future work, we intend to further research on the effect of different n-dimensional interval-valued aggregation functions (such as the ones studied in this paper) in the interval pattern classification soundness degree for all classes (Step (3) of the IV-FRM). Particularly, the relation between such interval functions applied in this third stage and the admissible orders chosen for the decision making in the classification phase (last stage of the IV-FRM).

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Tiago da Cruz Asmus received the M.Sc. degree in computational modelling from the Universidade Federal do Rio Grande, Brazil. In 2014, he became an Assistant Teacher in Departamento de Matemática, Estatística e Física, Universidade Federal do Rio Grande, Brazil. He is currently working toward the Ph.D. degree with the Universidad Pública de Navarra, Spain, under the advising of Prof. José A. Sanz.

José Antonio Sanz received the M.Sc. degree in computer sciences and the Ph.D. degree, both form the Public University of Navarre. He is currently an Associate Lecturer with the Department of Automatics and Computation, Public University of Navarre. He is the author of 37 published original articles in international journals. He received the best paper award in the FLINS 2012 international conference and the Pepe Millá award in 2014.

Graçaliz Pereira Dimuro received the M.Sc. and Ph.D. degrees from the Instituto de Informática of Universidade Federal do Rio Grande do Sul, Brazil. In 2015, she had a Pos-Doc post-doctorate grant from the Science Without Borders Program from the Brazilian Research Funding Agency CNPq, to join GIARA research group at Universidad Publica de Navarra, and, in 2017, she had a talent grant at the Institute of Smart Cities of Universidad Publica de Navarra, Spain. Currently, she is a full professor with Universidade Federal do Rio Grande, Brazil, and a Researcher of CNPq, Brazil.

Benjamín Bedregal received the Engineering degree in Computing from the Tarapac University (UTA), in Arica, Chile, the M.Sc. degree in informatics and the Ph.D. degree in computer sciences from the Federal University of Pernambuco, Brazil, in 1987 and 1996, respectively. He is a titular Professor at the Department of Informatics and Applied Mathematics, Federal University of Rio Grande do Norte, Natal, Brazil.

Javier Fernández received the M.Sc. and Ph.D. degrees in mathematics from the University of Zaragoza, Spain. He is currently an Associate Lecturer with the Department of Automatics and Computation, Public University of Navarre, Spain. He is involved with teaching artificial intelligence and computational mathematics for students of the computer sciences.

Humberto Bustince is full professor in the Public University of Navarra and Honorary Professor in the University of Nottingham. He has authored more than 210 works. He is associated editor of the IEEE Transactions on Fuzzy Systems journal and member of the editorial board of the journals Fuzzy Sets and Systems, Information Fusion, International Journal of Computational Intelligence Systems and Journal of Intelligent & Fuzzy Systems.