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$\mathcal{F}$-homogeneous functions and a generalization of directional monotonicity

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Abstract

A function that takes n numbers as input and outputs one number is said to be homogeneous whenever the result of multiplying each input by a certain factor λ yields the original output multiplied by that same factor. This concept has been extended by the notion of abstract homogeneity, which generalizes the product in the expression of homogeneity by a general function g and the effect of the factor λ by an automorphism. However, the effect of parameter λ remains unchanged for all the input values. In this study, we generalize further the condition of abstract homogeneity by introducing $\mathcal{F}$-homogeneity, which is defined with respect to a family of functions, enabling a different behavior for each of the inputs. Next, we study the properties that are satisfied by this family of functions and, moreover, we link this concept with the condition of directional monotonicity, which is a trendy property in the framework of aggregation functions. To achieve that, we generalize directional monotonicity by $\mathcal{F}$ directional monotonicity, which is defined with respect to a family of functions $\mathcal{F}$ and a family of vectors $\mathcal{V}$. Finally, we show how the introduced concepts could be applied

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in two different problems of computer vision: a snow detection problem and image thresholding improvement.

KEYWORDS
abstract homogeneity, F-homogeneity, homogeneity

1 | INTRODUCTION

A homogeneous function of degree 1 is the one that, when all the inputs are multiplied by a factor related to the domain, the output is multiplied by the same factor. Homogeneous functions and, more generally, the property of homogeneity[1] is a property that has caught the attention of many researchers. Indeed, in the literature, one can find works regarding this property both from the theoretical point of view,[2,3] and from the applied perspective.[4–6]

The notion of homogeneity has been generalized. Specifically, in the context of aggregation functions, in a recent work,[7] the concept of abstract homogeneity has been introduced, which is defined in terms of a function $g : [0, 1]^2 \to [0, 1]$ so that, in the specific case that g is given by $g(x, y) = xy$, abstract homogeneity coincides with standard homogeneity. Moreover, as exposed in Santiago et al.,[7] this concept generalizes other stability properties, such as shift-invariance[8] and power stability.[9] Finally, all the theoretical developments are applied in a multicriteria decision making problem.

Another research topic that has attracted the interest of many researchers in the framework of aggregation functions is that of relaxing the monotonicity condition that is required in the definition of an aggregation function. To that end, various relaxed forms of monotonicity have been presented in the literature: weak monotonicity,[10] directional monotonicity,[11] as well as some other extensions.[12,13]

In this paper, we explore a further generalization of homogeneity. The recent concept of abstract homogeneity generalizes standard homogeneity but it treats all the inputs in the same manner. We aim at defining a generalization of homogeneity in which the condition for each one of the inputs can vary. Additionally, this feature of varying the condition for each input is related to the concept of directional monotonicity, as it consists in studying the increasingness of a function when the inputs are modified according to a specific real vector. Thus, in this paper we aim at the following to objectives:

- generalizing the concept of abstract homogeneity, making the required condition specific for each input;
- linking the new generalization of homogeneity with the property of directional monotonicity.

To generalize homogeneity, we introduce $\mathcal{F}$-homogeneity, a condition that is defined with respect to a family of functions $\mathcal{F}$. This new property is a general case of abstract homogeneity and, thus, it recovers both the concepts of standard homogeneity and abstract homogeneity. To link this property with directional monotonicity, we propose a generalization of directional monotonicity: $\mathcal{F}$ directional monotonicity, which is defined with respect to a family of functions $\mathcal{F}$ and a family of vectors $\mathcal{V}$.

Finally, by means of an illustrative example, we show how the introduced concepts could be applied in a computer vision problem. Specifically, we show that the developed theoretical results can be applied to a snow detection problem and as an improvement to a general image thresholding algorithm.

The structure of the paper is the following: First, we fix the notation that is used throughout the rest of the manuscript and we expose some preliminary concepts. In Section 3, we introduce the concept of $\mathcal{F}$-homogeneity and, in Section 4, we study the main properties of the functions satisfying the proposed property. In Section 5, we present a generalization of the concept of directional monotonicity and we establish its link with $\mathcal{F}$-homogeneity. We end the manuscript by presenting two examples that illustrate the applicability of the proposed concepts to image thresholding in Section 6 and some concluding remarks in Section 7.

2 | NOTATION AND PRELIMINARIES

In this section, we present some basic concepts and notations that are used throughout the paper.

Definition 1. A function $\varphi : [0,1] \to [0,1]$ is said to be an automorphism on $[0,1]$ if it is an increasing bijection.

Definition 2. Let $f : [0,1]^n \to [0,1]$ and let $\varphi = (\varphi_0, \varphi_1, \ldots, \varphi_n)$ be a tuple of $n+1$ bijective functions $\varphi_i : [0,1] \to [0,1]$. Let us denote by f^{φ} the function given by $f^{\varphi}(x_1, \ldots, x_n) = \varphi_0^{-1}(f(\varphi_1(x_1), \ldots, \varphi_n(x_n)))$. If the functions φ_i are automorphisms, then f^{φ} is said to be the φ-conjugate function of f.

Definition 3. Let $\vec{x} = (x_1, \ldots, x_n) \in [0,1]^n$, the number of occurences (frequency) of x_i in $\vec{x}$ is denoted $k(i, \vec{x})$, that is, $k(i, \vec{x}) = \#\{j : x_i = x_j, 1 \leq j \leq n\}$—and $m = \max\{k(i, \vec{x}) : 1 \leq i \leq n\}$, the multimode of $\vec{x}$ is the set of all $x_i \in \vec{x}$ with the highest frequency; that is, $\mathbf{mmode}(\vec{x}) = \{x_i : k(i, \vec{x}) = m\}$; where "#" denotes the cardinality of a set. The MinMode of $\vec{x}$ is given by $MinMode(\vec{x}) = \min(mmode(\vec{x}))$ and the MaxMode of $\vec{x}$ is $MaxMode(\vec{x}) = \max(mmode(\vec{x}))$.

Example 1. For $\vec{x} = (0.2, 0.3, 0.5, 0.7, 0.3, 0.9, 0.7)$, it holds that $k(2, \vec{x}) = \#\{2, 5\} = 2$ and $m = \max\{1, 2\} = 2$. Also, it holds that $mmode(\vec{x}) = \{0.3, 0.7\}$, $MinMode(\vec{x}) = 0.3$ and $MaxMode(\vec{x}) = 0.7$.

Definition 4 (Kolesárová et al.[9]). A function $F : [0,1]^n \to [0,1]$ is power stable if for each $x_1, \ldots, x_n \in [0,1]$ and $r \in]0, \infty[$, $F(x_1^r, \ldots, x_n^r) = F(x_1, \ldots, x_n)^r$.

Definition 5. A function $A : [0,1]^n \to [0,1]$ is said to be an aggregation function whenever it satisfies the following conditions:

- $A(0, \ldots, 0) = 0$,
- $A(1, \ldots, 1) = 1$, and
- A is nondecreasing.

Definition 6. An associative and commutative bivariate aggregation function, $A : [0, 1]^2 \to [0, 1]$, is called a t-conorm whenever $A(x, 0) = x$.

2.1 | Homogeneity

The key notion in the present work is that of homogeneity. We start recalling the definition.

Definition 7. Let $\gamma \in [0, \infty[$. A function $F : [0, 1]^n \to [0, 1]$ is said to be homogeneous of order γ (or γ-homogeneous), whenever the identity

$$F(\lambda x_1, \ldots, \lambda x_n) = \lambda^\gamma F(x_1, \ldots, x_n)$$

holds for all $x_1, \ldots, x_n, \lambda \in [0, 1]$. We assume $0^0 = 1$.

Example 2. The following are examples of homogeneous functions:

- A constant function is homogeneous of order 0.
- The maximum and the minimum are examples of 1-homogeneous functions.
- The n-dimensional product:

$$\Pi_{\mathrm{n}}(x_1, \ldots, x_n) = \prod_{i=1}^{n} x_i = x_1 \cdot x_2 \cdot \ldots \cdot x_{n-1} \cdot x_n$$

  is homogeneous of order n.
- Given $\gamma > 0$, the function $G_\gamma : [0, 1]^n \to [0, 1]$ given by

$$G_\gamma(x_1, \ldots, x_n) = \left(\prod_{i=1}^{n} x_i \right)^{\frac{\gamma}{n}}$$

  is homogeneous of order γ.

The study of the notion of homogeneity from a theoretical point of view has led to different related concepts. In what follows we present two related concepts.

Definition 8 (Ebanks[14]). A function $F : [0, 1]^n \to [0, 1]$ is called quasi-homogeneous if there exist an automorphism $\varphi : [0, 1] \to [0, 1]$ and a strictly monotone and continuous function $P : [0, 1] \to [0, \infty[$ such that

$$F(\lambda x_1, \ldots, \lambda x_n) = P^{-1}(\varphi(\lambda) P(F(x_1, \ldots, x_n)))$$

for every $x_1, \ldots, x_n, \lambda \in [0, 1]$.

Note that if we take $\varphi(x) = x$ and $P(x) = x^{\frac{1}{\gamma}}$, then F comes out to be homogeneous of order γ.

Definition 9 (Xie et al.[15]). A function $F : [0, 1]^n \to [0, 1]$ is said to be pseudo-homogeneous if there exists a continuous and increasing function $P : [0, 1]^2 \to [0, 1]$ such that

$$F(\lambda x_1, ..., \lambda x_n) = P(\lambda, F(x_1..., x_n))$$

for every $x_1, ..., x_n, \lambda \in [0, 1]$.

Note that, if $\gamma \in [0, \infty[$ and we take $P(x, y) = x^{\gamma} \cdot y$, then a pseudo-homogeneous function F is homogeneous of order γ.

2.2 | Directional monotonicity and related concepts

In this section, we recall the notions of weak and directional monotonicity. Weak monotonicity emerged in the context of aggregation functions[8,16] seeking to relax their condition of monotonicity (to be increasing with respect to all the inputs), while maintaining that the output must increase whenever all the inputs increase by the same amount.

Definition 10 (Wilkin and Beliakov[10]). We say that a function $F : [0, 1]^n \to [0, 1]$ is weakly increasing (resp. weakly decreasing), if for all $c > 0$ and $(x_1, ..., x_n) \in [0, 1]^n$ such that $0 \leq x_i + c \leq 1$ for all $i \in \{1, ..., n\}$, it holds that $F(x_1, ..., x_n) \leq F(x_1 + c, ..., x_n + c)$ (resp. $F(x_1, ..., x_n) \geq F(x_1 + c, ..., x_n + c)$). If F is both weakly increasing and weakly decreasing, we say that F is weakly constant.

Interpreting this property as monotonicity along the vector $\vec{1} = (1, ..., 1)$, we can generalize it by taking any vector $\vec{0} \neq \vec{r} \in \mathbb{R}^n$, which led to the introduction of directional monotonicity.

Definition 11 (Bustince et al.[11]). Let $\vec{0} \neq \vec{r} \in \mathbb{R}^n$. We say that a function $F : [0, 1]^n \to [0, 1]$ is $\vec{r}$-increasing (resp. $\vec{r}$-decreasing) if for all $c > 0$ and $\vec{x} = (x_1, ..., x_n) \in [0, 1]^n$ such that $\vec{x} + c\vec{r} \in [0, 1]^n$, it holds that $F(\vec{x}) \leq F(\vec{x} + c\vec{r})$ (resp. $F(\vec{x}) \geq F(\vec{x} + c\vec{r})$). If F is both $\vec{r}$-increasing and $\vec{r}$-decreasing, we say that F is $\vec{r}$-constant.

There exist more relaxed forms of monotonicity in the literature, such as ordered directional monotonicity[17] and strengthened ordered directional monotonicity.[12] Additionally, pointwise directional monotonicity has also been proposed[18] and the concepts of weak and directional monotonicity have been extended to more general frameworks such as interval-valued functions, among other.[13]

3 | $\mathcal{F}$-HOMOGENEITY

In this section, we recall the definition of abstract homogeneity[7] and we provide a generalization in the sense that a function is considered homogeneous with respect to a family, $\mathcal{F}$, of functions and an automorphism φ.

Definition 12 (Santiago et al.[7]). Let $g : [0, 1]^2 \to [0, 1]$ and $F : [0, 1]^n \to [0, 1]$ be functions and $\varphi : [0, 1] \to [0, 1]$ be an automorphism. F is said to be abstractly homogeneous with respect to g and φ, or just (g, φ)-homogeneous if for every $\lambda, x_1, \ldots, x_n \in [0, 1]$:

$$F(g(\lambda, x_1), \ldots, g(\lambda, x_n)) = g(\varphi(\lambda), F(x_1, \ldots, x_n)).$$

In the case where φ is the identity, F is said to be g-homogeneous instead of (g, φ)-homogeneous.

(g, φ)-homogeneity generalizes the notion of homogeneity of order $\gamma \in [0, +\infty[$ at Definition 7.

We extend this concept to homogeneity with respect to a family of functions $g_1, \ldots, g_n$.

Definition 13. Let $\mathcal{F} = \{g_j : D \to [0, 1] | D \subseteq [0, 1]^2 \text{and} j \in \{1, \ldots, n\}\}$ be a family of functions, $\varphi : [0, 1] \to [0, 1]$ be an automorphism and $\psi : [0, 1]^n \to [0, 1]$ be an increasing function. A function $F : [0, 1]^n \to [0, 1]$ is said to be abstractly homogeneous with respect to $(\mathcal{F}, \varphi, \psi)$, or just $\mathcal{F}_\psi^\varphi$-homogeneous, if for every $x_1, \ldots, x_n \in [0, 1]$ and $\overrightarrow{\lambda} = (\lambda_1, \ldots, \lambda_n) \in [0, 1]^n$ it holds that

$$F(g_1(\lambda_1, x_1), \ldots, g_n(\lambda_n, x_n)) = \psi(g_1(\varphi(\lambda_1), F(x_1, \ldots, x_n)), \ldots, g_n(\varphi(\lambda_n), F(x_1, \ldots, x_n))). \tag{1}$$

If φ is the identity automorphism, then F is called $\mathcal{F}_\psi$-homogeneous.

Remark 1. Note that in the case that $\lambda_1 = \cdots = \lambda_n$, $g_1 = \cdots = g_n$ and ψ is such that it satisfies

$$\psi(g(\varphi(\lambda), F(x_1, \ldots, x_n)), \ldots, g(\varphi(\lambda), F(x_1, \ldots, x_n))) = g(\varphi(\lambda), F(x_1, \ldots, x_n)).$$

An example of such a ψ function would be a projection, or any increasing idempotent function.

Remark 2. Unless we explicitly state otherwise, from this point we assume $\mathcal{F}$, $\overrightarrow{\lambda}$, φ, and ψ as declared in Definition 13. Moreover, φ is not explicitly mentioned whenever $\varphi = id$.

Proposition 1. *Let $F : [0, 1]^n \to [0, 1]$ be a function which is homogeneous of order $\gamma \in [0, +\infty[$. Then it is (g, φ)-homogeneous for $g(x, y) = xy$ and $\varphi(x) = x^\gamma$. Moreover, for $k > 0$, $g(x, y) = x^k y$ and $\varphi(x) = x^{k \cdot \gamma}$, F is (g, φ)-homogeneous.*

Proof. Straightforward. □

Moreover, $\mathcal{F}$-homogeneity generalizes the notion of (g, φ)-homogeneity in Definition 12.

Proposition 2. *Let $g : [0, 1]^2 \to [0, 1]$. If $\mathcal{F}$ is such that $g_j = g$ for all $j \in \{1, \ldots, n\}$, ψ is idempotent* and $F : [0, 1]^n \to [0.1]$ is $\mathcal{F}_\psi^\varphi$-homogeneous, then F is also (g, φ)-homogeneous.*

Proof. In fact, $F(g(\lambda, x_1), ..., g(\lambda, x_n)) = \psi(g(\varphi(\lambda), F(x_1, ..., x_n)), g(\varphi(\lambda), F(x_1, ..., x_n)))$ $= g(\varphi(\lambda), F(x_1, ..., x_n))$. □

Example 3.

- Consider the arithmetic mean:

$$M(x_1, ..., x_n) = \frac{x_1 + \cdots + x_n}{n}.$$

Then, if we take $g(x, y) = \frac{x+y}{2}$, from an easy calculation it follows that:

$$g(\lambda, M(x_1, ..., x_n) = M(g(\lambda, x_1), ..., g(\lambda, x_n))$$

for every $\lambda \in [0, 1]$ and M is g-homogeneous.

- Take $g(x, y) = \sqrt{xy}$. Then, if $\lambda \in [0, 1]$,

$$\max(\sqrt{\lambda x_1}, ..., \sqrt{\lambda x_n}) = \sqrt{\lambda \max(x_1, ..., x_n)}$$

and

$$\min(\sqrt{\lambda x_1}, ..., \sqrt{\lambda x_n}) = \sqrt{\lambda \min(x_1, ..., x_n)}.$$

So both max and min are g-homogeneous.

Proposition 3 (MaxMode)**.** *Consider the MaxMode function defined at Definition 3 and the weighted average $g_a(x, y) = a{\cdot}x + (1 - a){\cdot}y$, for $0 \le a \le 1$. Then MaxMode is g_a-homogeneous for any $a \in [0, 1]$.*

Proof. In fact, for any $\overrightarrow{x} = (x_1, ..., x_n) \in [0, 1]^n$ and $k \in [0, 1]$, let be: $k^n = \underbrace{(k, ..., k)}_{n}$, $k{\cdot}\{x_1, ..., x_n\} = \{k{\cdot}x_1, ..., k{\cdot}x_n\}$, and $k + \{x_1, ..., x_n\} = \{k + x_1, ..., k + x_n\}$, then $mmode(\lambda{\cdot}\overrightarrow{x}) = \lambda{\cdot}mmode(\overrightarrow{x})$ and $mmode(\lambda^n + \overrightarrow{x}) = \lambda + mmode(\overrightarrow{x})$. Hence, $\lambda{\cdot}a + (1 - a){\cdot}mmode(\overrightarrow{x}) = mmode((\lambda{\cdot}a)^n + (1 - a){\cdot}\overrightarrow{x}) = mmode(\lambda{\cdot}a + (1 - a){\cdot}x_1, ..., \lambda{\cdot}a + (1 - a){\cdot}x_n) = mmode(g_a(\lambda, x_1), ..., g_a(\lambda, x_n))$. Therefore, $\mathbf{MaxMode}(\mathbf{g_a}(\boldsymbol{\lambda}, \mathbf{x_1}), ..., \mathbf{g_a}(\boldsymbol{\lambda}, \mathbf{x_n})) = \max(mmode(g_a(\lambda, x_1), ..., g_a(\lambda, x_n))) = \max(\lambda \cdot a + (1 - a) \cdot mmode(\overrightarrow{x})) = \lambda \cdot a + \max((1 - a) \cdot mmode(\overrightarrow{x})) = \lambda \cdot a + (1 - a) \cdot \max(mmode(\overrightarrow{x})) = g_a(\lambda, \max(mmode(\overrightarrow{x}))) = g_a(\lambda, \mathbf{MaxMode}(x_1, ..., x_n))$. □

Example 4. Let be $\mathcal{F} = \{g_j : g_j(x, y) = x^j y\}$, $\psi(x_1, ..., x_n) = \sqrt[n]{\prod_{i=1}^n x_i}$ (the geometric mean), $\varphi(x) = x^n$, and $F(x_1, ..., x_n) = \prod_{i=1}^n x_i$. Then, F is $\mathcal{F}_\psi^\varphi$-homogeneous, since:

$$\begin{aligned} F(g_1(\lambda_1, x_1), ..., g_n(\lambda_n, x_n)) &= F\left(\lambda_1{\cdot}x_1, ..., \lambda_n^n{\cdot}x_n\right) \\ &= \prod_{i=1}^n \left(\lambda_i^i{\cdot}x_i\right), \end{aligned}$$

and

$$\begin{aligned}\psi(g_1(\varphi(\lambda_1), F(x_1, \ldots, x_n)), \ldots, g_n(\varphi(\lambda_n), F(x_1, \ldots, x_n))) &= \psi\left(\varphi(\lambda_1)\prod_{i=1}^{n} x_i, \ldots, \varphi(\lambda_n)^n \prod_{i=1}^{n} x_i\right)\\ &= \sqrt[n]{\lambda_1^{\,n}\prod_{i=1}^{n} x_i \;\cdot \ldots \cdot\; \lambda_n^{\,n^2}\prod_{i=1}^{n} x_i}\\ &= \sqrt[n]{\left(\prod_{i=1}^{n}\left(\lambda_i^{\,i}\cdot x_i\right)\right)^n}\\ &= \prod_{i=1}^{n}\left(\lambda_i^{\,i}\cdot x_i\right).\end{aligned}$$

4 | SOME PROPERTIES OF $\mathcal{F}$-HOMOGENEITY

In this section, we establish some relations between $\mathcal{F}$-homogeneity and other properties.

We start studying the relation of g-homogeneity and power stability.

Proposition 4. *A function $F : [0,1]^n \to [0,1]$ is power stable with respect to a power $r \in (0,1]$ if and only if it is abstractly homogeneous with respect to $g(x,y) = y^x$.*

Proof. It follows from the definitions of power stability and g-homogeneity. □

We also study the relations between $\mathcal{F}$-homogeneity and conjugates: given a function $F : [0,1]^n \to [0,1]$ and a bijection $\rho : [0,1] \to [0,1]$, we define the conjugate function of F, F^ρ, as $F^\rho(x_1, \ldots, x_n) = \rho^{-1}(F(\rho(x_1), \ldots, \rho(x_n)))$.

Proposition 5. *Let $F : [0,1]^n \to [0,1]$ be a function and $\rho : [0,1] \to [0,1]$ a bijection. If F is $(\mathcal{F}, \varphi, \psi)$-homogeneous, then F^ρ is such that for $1 \le i \le n$:*

$$\begin{aligned}&F^\rho\left(g_1^\rho(\lambda_1, x_1), \ldots, g_n^\rho(\lambda_n, x_n)\right)\\ &\quad= \rho^{-1}(\psi(g_1(\varphi\circ\rho(\lambda_1), \rho(F^\rho(x_1, \ldots, x_n)), \ldots, g_n(\varphi\circ\rho(\lambda_n), \rho(F^\rho(x_1, \ldots, x_n)))).\end{aligned} \tag{2}$$

Proof.

$$\begin{aligned}F^\rho\left(g_1^\rho(\lambda_1, x_1), \ldots, g_n^\rho(\lambda_n, x_n)\right) &\overset{def}{=} \rho^{-1}\left(F\left(\rho\left(g_1^\rho(\lambda_1, x_1)\right)\right), \ldots, \rho\left(g_n^\rho(\lambda_n, x_n)\right)\right)\\ &= \rho^{-1}(F(\rho(\rho^{-1}(g_1(\rho(\lambda_1), \rho(x_1)))), \ldots, \rho(\rho^{-1}(g_n(\rho(\lambda_n), \rho(x_n))))))\\ &= \rho^{-1}(F(g_1(\rho(\lambda_1), \rho(x_1)), \ldots, g_n(\rho(\lambda_n), \rho(x_n))))\\ &= \rho^{-1}(\psi(g_1(\varphi\circ\rho(\lambda_1), F(\rho(x_1), \ldots, \rho(x_n))), \ldots, g_n(\varphi\circ\rho(\lambda_n), F(\rho(x_1), \ldots, \rho(x_n)))))\\ &= \rho^{-1}(\psi(g_1(\varphi\circ\rho(\lambda_1), \rho\circ\rho^{-1}(F(\rho(x_1), \ldots, \rho(x_n)))), \ldots, \rho\circ\rho^{-1}(g_n(\varphi\circ\rho(\lambda_n),\\ &\qquad F(\rho(x_1), \ldots, \rho(x_n))))))\\ &= \rho^{-1}(\psi(g_1(\varphi\circ\rho(\lambda_1), \rho(F^\rho(x_1, \ldots, x_n))), \ldots, g_n(\varphi\circ\rho(\lambda_n), \rho(F^\rho(x_1, \ldots, x_n))))).\end{aligned}$$

□

Remark 3. Proposition 5 can be rewritten in the following way:

$$F^{\rho}\left(g_1^{\rho}(\lambda_1, x_1), \ldots, g_n^{\rho}(\lambda_n, x_n)\right) = \rho^{-1}\left(\psi\left(g_1^{\beta}(\lambda_1, F^{\rho}(x_1, \ldots, x_n)), \ldots, g_n^{\beta}(\lambda_n, F^{\rho}(x_1, \ldots, x_n))\right)\right), \text{ for } \beta = (id, \varphi \circ \rho, id \circ \rho).$$

Indeed,

$$\begin{aligned}
& F^{\rho}\left(g_1^{\rho}(\lambda_1, x_1), \ldots, g_n^{\rho}(\lambda_n, x_n)\right) = \rho^{-1}(\psi(g_1(\varphi \circ \rho(\lambda_1), \rho(F^{\rho}(x_1, \ldots, x_n)), \ldots,\\
& g_n(\varphi \circ \rho(\lambda_n), \rho(F^{\rho}(x_1, \ldots, x_n))))\\
= \quad & \rho^{-1}(\psi(id^{-1}(g_1(\varphi \circ \rho(\lambda_1), id \circ \rho(F^{\rho}(x_1, \ldots, x_n))), \ldots,\\
& id^{-1}(g_n(\varphi \circ \rho(\lambda_n), id \circ \rho(F^{\rho}(x_1, \ldots, x_n)))))\\
\overset{\text{def}}{=} \quad & \rho^{-1}\left(\psi\left(g_1^{\beta}(\lambda_1, F^{\rho}(x_1, \ldots, x_n)), \ldots, g_n^{\beta}(\lambda_n, F^{\rho}(x_1, \ldots, x_n))\right)\right), \text{ for } \beta\\
& = (id, \varphi \circ \rho, id \circ \rho).
\end{aligned}$$

Corollary 1. *If* $\rho^{-1}(\psi(x_1, \ldots, x_n)) = \psi(\rho^{-1}(x_1), \ldots, \rho^{-1}(x_n))$, *then*

$$F^{\rho}\left(g_1^{\rho}(\lambda_1, x_1), \ldots, g_n^{\rho}(\lambda_n, x_n)\right) = \psi(\rho^{-1}(g_1(\varphi \circ \rho(\lambda_1), \rho(F^{\rho}(x_1, \ldots, x_n)))), \ldots,$$
$$\rho^{-1}(g_n(\varphi \circ \rho(\lambda_n), \rho(F^{\rho}(x_1, \ldots, x_n))))).$$

Moreover, if $\varphi \circ \rho = \rho \circ \varphi$, *then*

$$F^{\rho}\left(g_1^{\rho}(\lambda_1, x_1), \ldots, g_n^{\rho}(\lambda_n, x_n)\right) = \psi\left(g_1^{\rho}(\varphi(\lambda_1), F^{\rho}(x_1, \ldots, x_n)), \ldots, g_n^{\rho}(\varphi(\lambda_n), F^{\rho}(x_1, \ldots, x_n))\right),$$

that is, F^{ρ} *is* $(\mathcal{F}^{\rho}, \varphi, \psi)$*-homogeneous, where* $\mathcal{F}^{\rho} = \{g_i^{\rho} : g_i \in \mathcal{F}\}$.

Proof. Note that, by hypothesis,

$$F^{\rho}\left(g_1^{\rho}(\lambda_1, x_1), \ldots, g_n^{\rho}(\lambda_n, x_n)\right) = \psi(\rho^{-1}(g_1(\varphi \circ \rho(\lambda_1), \rho(F^{\rho}(x_1, \ldots, x_n)))), \ldots,$$
$$\rho^{-1}(g_n(\varphi \circ \rho(\lambda_n), \rho(F^{\rho}(x_1, \ldots, x_n))))).$$

Thus, if $\varphi \circ \rho = \rho \circ \varphi$, then it is straightforward that

$$F^{\rho}\left(g_1^{\rho}(\lambda_1, x_1), \ldots, g_n^{\rho}(\lambda_n, x_n)\right) = \psi\left(g_1^{\rho}(\varphi(\lambda_1), F^{\rho}(x_1, \ldots, x_n)), \ldots, g_n^{\rho}(\varphi(\lambda_n), F^{\rho}(x_1, \ldots, x_n))\right).$$

□

Proposition 6. *Let $\rho : [0,1] \to [0,1]$ be a bijective function and $g : [0,1]^2 \to [0,1]$ be a function. If ρ satisfies:*

$$\rho(g(x,y)) = g(\rho(x), \rho(y)), \tag{3}$$

then $\rho^{-1}(g(x,y)) = g(\rho^{-1}(x), \rho^{-1}(y))$. Moreover, if F is g-homogeneous and $\rho(g(x,y)) = g(\rho(x), \rho(y))$, then F^{ρ} is also g-homogeneous.

Proof. Just note that

$$g(\rho^{-1}(x), \rho^{-1}(y)) = \rho^{-1}(\rho(g(\rho^{-1}(x), \rho^{-1}(y))))$$

which, by hypothesis, is equal to

$$\rho^{-1}(g(\rho(\rho^{-1}(x)), \rho(\rho^{-1}(y)))) = \rho^{-1}(g(x,y)).$$

Moreover, if F is g-homogeneous and g satisfies (3), then,

$$F^{\rho}(g(\lambda, x_1), ..., g(\lambda, x_n)) = \rho^{-1}(F(\rho(g(\lambda, x_1)), ..., \rho(g(\lambda, x_n)))).$$

By hypothesis this expression is equal to

$$\rho^{-1}(F(g(\rho(\lambda), \rho(x_1)), ...g(\rho(\lambda), \rho(x_n))))$$

Since F is g-homogeneous, then we obtain $\rho^{-1}(g(\rho(\lambda), F(\rho(x_1), ...\rho(x_n))))$. Therefore,

$$g(\rho^{-1}(\rho(\lambda)), \rho^{-1}(F(\rho(x_1), ..., \rho(x_n)))) = g(\lambda, F^{\rho}(x_1, ..., x_n)).$$

□

5 | $\mathcal{F}$-HOMOGENEITY AND DIRECTIONAL MONOTONICITY

In this section, we link the concepts of $\mathcal{F}$-homogeneity and directional monotonicity. To achieve that we generalize the notions of weak and directional monotonicity to $\mathcal{F}$-monotonicity, a generalization of directional monotonicity with respect to a family of functions $\mathcal{F}$.

Definition 14. Let $\mathcal{F} = \{g_j : D \to [0,1] | D \subseteq [0,1]^2 \text{ and } j \in \{1, ..., n\}\}$ be a family of functions and $\mathcal{V} \subseteq \mathbb{R}^n \backslash \{\vec{0}\}$ a family of vectors. A function $F : [0,1]^n \to [0,1]$ is $\mathcal{F}$-increasing with respect to $\mathcal{V}$ or just $(\mathcal{F}, \mathcal{V})$-increasing if for every $\vec{\lambda} = (\lambda_1, ..., \lambda_n) \in \mathcal{V}$ it holds that

$$F(g_1(\lambda_1, x_1), ..., g_n(\lambda_n, x_n)) \geq F(x_1, ..., x_n), \tag{4}$$

whenever $(g_1(\lambda_1, x_1), ..., g_n(\lambda_n, x_n)) \in [0, 1]^n$. If $\mathcal{V} = [0, 1]^n \backslash \{\overrightarrow{0}\}$, we say that F is $(\mathcal{F}, \mathcal{V})$-weakly increasing or just $\mathcal{F}$-weakly increasing.

Dually we obtain the notion of $(\mathcal{F}, \mathcal{V})$-decreasing functions.

Example 5.

- Every constant function is $\mathcal{F}$-weakly increasing for any family $\mathcal{F}$.
- If F is a nondecreasing function, then it is $\mathcal{F}$-weakly increasing with respect to the family $\mathcal{F} = \{g_j : g(x, y) = x + y\}$.

Proposition 7. *Given a family, $\mathcal{F}$, of functions g_j such that $g_j(x, y) \geq y$, if a function $F : [0, 1]^n \to [0, 1]$ is isotonic, then it is $\mathcal{F}$-weakly increasing.*

Proof. Straightforward. □

Example 6. Any binary isotonic function is $\mathcal{F}$-weakly increasing for $\mathcal{F} = \{S_Ł, \max\}$, where $S_Ł$ is the Łukasiewicz t-conorm, defined by $S_Ł(x, y) = \min\{x + y, 1\}$.

Proposition 8. *Every function $F : [0, 1]^n \to [0, 1]$ is $\mathcal{F}$-weakly increasing/decreasing for $\mathcal{F} = \{g_j : g_j(x, y) = y\}$.*

Proof. $F(g_1(\lambda_1, x_1), ..., g_n(\lambda_n, x_n)) \stackrel{def}{=} F(x_1, ..., x_n)$. □

The following result exposes how $\mathcal{F}$-monotonicity generalizes directional monotonicity.

Proposition 9. *Let $\overrightarrow{0} \neq \overrightarrow{r} = (r_1, ..., r_n) \in \mathbb{R}^n$. A function F is $\overrightarrow{r}$-increasing if and only if it is $(\mathcal{F}, \mathcal{V}_{\overrightarrow{r}})$-weakly increasing, for $\mathcal{V}_{\overrightarrow{r}} = \{(c \cdot r_1, ..., c \cdot r_n) : c > 0\}$ and $\mathcal{F} = \{g_j : g_j(x, y) = x + y\}$.*

Proof. Let $c > 0$ and $\overrightarrow{r} = (r_1, ..., r_n) \neq \overrightarrow{0}$, making $\overrightarrow{\lambda} = c \cdot \overrightarrow{r} = (c \cdot r_1, ..., c \cdot r_n)$, if F is $\overrightarrow{r}$-increasing, then for $(x_1 + c \cdot r_1, ..., x_n + c \cdot r_n) \in [0, 1]^n$, $F(x_1 + c \cdot r_1, ..., x_n + c \cdot r_n) \stackrel{def}{=} F(g(\lambda_1, x_1), ..., g(\lambda_n, x_n)) \geq F(x_1, ..., x_n)$. The reciprocal is analogous. □

Proposition 10. *Let $\lambda \in [0, 1]$ and $F_\lambda : [0, 1]^n \to [0, 1]$ be such that $F_\lambda(x_1, ..., x_n) = MinMode(\min(\lambda, x_1), ..., \min(\lambda, x_n))$. Let $\mathcal{V} = \{(\lambda, ..., \lambda)\}$ and $\mathcal{F}$ be a family of functions $\mathcal{F} = \{g_j : [0, 1]^n \to [0, 1] | g_j(x, y) = g(x, y), \text{for} j \in \{1, ..., n\}\}$, where the function g is injective with respect to the second variable. If $g(x, y) \geq y$, then F_λ is $(\mathcal{F}, \mathcal{V})$-weakly increasing. If $g(x, y) \leq y$, then F_λ is $(\mathcal{F}, \mathcal{V})$-weakly decreasing.*

Proof. Suppose that $g(x, y) \geq y$, then $g_i(\lambda, x_i) \geq x_i$ and $\min(\lambda, g(\lambda, x_i)) \geq \min(\lambda, x_i)$. Therefore, since g is injective with respect to the second variable, it holds that

$$\begin{aligned} F_\lambda(g_1(\lambda, x_1), ..., g_n(\lambda, x_n)) &\overset{def}{=} MinMode(\min(\lambda, g_1(\lambda, x_1)), ..., \min(\lambda, g_n(\lambda, x_n))) \\ &\geq MinMode(\min(\lambda, x_1), ..., \min(\lambda, x_n)) \\ &\overset{def}{=} F_\lambda(x_1, ..., x_n). \end{aligned}$$

Similarly, one can prove the case of $(\mathcal{F}, \mathcal{V})$-weakly decreasingness. □

The next proposition connects $\mathcal{F}$-homogeneity with $\mathcal{F}$-increasing functions.

Proposition 11. *Let $\mathcal{F} = \{g_j : D \to [0, 1] | D \subseteq [0, 1]^2 \text{and} j \in \{1, ..., n\}\}$ be a family of functions such that $g_i(x, y) \geq y$, $\varphi : [0, 1] \to [0, 1]$ be an automorphism and $\psi : [0, 1]^n \to [0, 1]$ be an increasing function, such that $\psi(x, ..., x) \geq x$. If $F : [0, 1]^n \to [0, 1]$ is $\mathcal{F}_\psi^\varphi$-homogeneous, then it is $\mathcal{F}$-increasing.*

Proof.

$$\begin{aligned} F(g_1(\lambda_1, x_1), ..., g_n(\lambda_n, x_n)) &\overset{\text{hom}}{=} \psi(g_1(\varphi(\lambda_1), F(x_1, ..., x_n)), ... \\ &\quad , g_n(\varphi(\lambda_n), F(x_1, ..., x_n))) \\ &\geq \psi(F(x_1, ..., x_n), ..., F(x_1, ..., x_n)) \\ &\geq F(x_1, ..., x_n). \end{aligned}$$

□

Corollary 2. *For any T-conorm S, if F is S-homogeneous, then it is S-weakly increasing.*

Proof. All T-conorms satisfy the conditions of Proposition 11. □

Corollary 3. *If F is $S_Ł$-homogeneous, then it is weakly increasing.*

Proof. Let $(x_1, ..., x_n) \in [0, 1]^n$ and $\lambda > 0$ be such that $(x_1 + \lambda, ..., x_n + \lambda) \in [0, 1]^n$, since $S_Ł(x, y) = \min(x + y, 1)$, then:

$$\begin{aligned} F(x_1 + \lambda, ..., x_n + \lambda) &= F(S_Ł(\lambda, x_1), ..., S_Ł(\lambda, x_n)) \\ &= S_Ł(\lambda, F(x_1, ..., x_n)) \\ &\geq F(x_1, ..., x_n). \end{aligned}$$

□

Corollary 4. *Given $c, r > 0$ and $\vec{r} = (r, ..., r) \in \mathbb{R}^n$, if a function $F : [0, 1]^n \to [0, 1]$ is $S_Ł$-homogeneous, then F is $\vec{r}$-increasing.*

Proof. Given $c, r > 0$, $\overrightarrow{r} = (r, ..., r)$ and $\overrightarrow{\lambda} = c \cdot \overrightarrow{r}$, then $F(x_1 + c \cdot r, ..., x_n + c \cdot r) = F(\lambda_1 + x_1, ..., \lambda_n + x_n) = F(S_{Ł}(\lambda, x_1), ..., S_{Ł}(\lambda, x_n))$. And the result holds by Corollary 3. □

Corollary 5. *For every function F, there exists ψ and a family $\mathcal{F}$ such that F is $\mathcal{F}_\psi$-homogeneous and*

$$F(g_1(\lambda_1, x_1), ..., g_n(\lambda_n, x_n)) \geq F(x_1, ..., x_n).$$

Proof. Let $\psi(x_1, ..., x_n) = x_1$ and $\mathcal{F} = \{g_j(x, y) = y : j \in \{1, ... n\}\}$. Then,

$$\begin{aligned} F(x_1, ..., x_n) &= F(g_1(\lambda_1, x_1), ..., g_n(\lambda_n, x_n)) \\ &= \psi(g_1(\lambda_1, F(x_1, ..., x_n)), ..., g_n(\lambda_n, F(x_n, ..., x_n))). \end{aligned}$$

□

Theorem 1. *Let $\mathcal{F}$ be such that $g_j \in \mathcal{F}$ are increasing in the first variable for all $1 \leq j \leq n$ and let F be an $\mathcal{F}_\psi^\varphi$-homogeneous function. Then, for all $\lambda_1, \lambda_2 \in [0, 1]$ such that $\lambda_1 \geq \lambda_2$, it holds that*

$$F(g_1(\lambda_1, x_1), ..., g_n(\lambda_1, x_n)) \geq F(g_1(\lambda_2, x_1), ..., g_n(\lambda_2, x_n)).$$

Proof. Since F is $\mathcal{F}_\psi^\varphi$-homogeneous, then

$$\begin{aligned} F(g_1(\lambda_1, x_1), ..., g_n(\lambda_1, x_n)) &\overset{def}{=} \psi(g_1(\varphi(\lambda_1), F(x_1, ..., x_n)), ... \\ &\qquad , g_n(\varphi(\lambda_1), F(x_1, ..., x_n))) \\ &\geq \psi(g_1(\varphi(\lambda_2), F(x_1, ..., x_n)), ... \\ &\qquad , g_n(\varphi(\lambda_2), F(x_1, ..., x_n))) \\ &\overset{def}{=} F(g_1(\lambda_2, x_1), ..., g_n(\lambda_2, x_n)). \end{aligned}$$

□

6 | TWO ILLUSTRATIVE EXAMPLES OF APPLICATION

In this section, we show two examples that illustrate the applicability of our proposal to two different problems, both related to image processing. The first one is using function F_{x_0} of Proposition 10 as a tool for snow detection in an image. The second application consists in an improvement of a given image thresholding algorithm. Specifically, once the threshold is computed, we propose to use F_{x_0} to alter the way in which the segmented image is obtained. This second illustrative example is supported by the comparison with four image thresholding algorithms in the literature. In all cases, it is shown that our proposal is valid to improve the results obtained by each method.

6.1 | Snow detection

The amount of snow in a given mountain may be very important to determine the existing reserve of water, a critical part in the generation of accurate hydrological models. There exist several methods to gather such knowledge, many of them using images from which the available snow is determined. The first problem in this case is to locate the areas of the studied image where snow appears. To do so, it is usually taken into account the fact that snow can be distinguished by means of two properties: the high reflectivity in the visible part of the spectrum and the low reflectivity in the infrared one.[19] The combination of these two spectral features is typically made by means of the Normalized Difference Snow Index (NDSI). In general, NDSI is much greater for snow than for other types of surfaces, so the first criterion to determine the presence of snow is to study whether a NDSI threshold is surpassed.

This fact leads us to settle that, if we are working with images in the RGB color space (with each channel taking values in a scale from 0 to 255), the presence of snow is determined by the existence of several pixels in the considered area whose intensities are, in each channel, greater than a threshold x_0 which has been fixed beforehand. This threshold must be high for each of the channels since we want to detect a bright object, and the maximum of intensity in the RGB channels, corresponding to white, is obtained for (255, 255, 255).

In this setting, we select the threshold x_0 over which we assume the existence of snow. We take $n \in \mathbb{N}$ and choose windows of size $(2n + 1) \times (2n + 1)$. We calculate, for each channel, $\min(x_{11}, x_0)..., \min(x_{nn}, x_0)$. If the mode in each of the three channels is equal to x_0, then we conclude that we have snow and we mark the central pixel of the window in white, and, in other case, we mark the central pixel in black. Observe that this amounts to consider the function $F_{x_0}(x_{11}, ..., x_{nn}) = MinMode(\min(x_{11}, x_0), ..., \min(x_{nn}, x_0))$. According to Proposition 10, F_{x_0} is $(\mathcal{F}, \mathcal{V})$-weakly increasing for $\mathcal{F} = \{g\}$ and $\mathcal{V} = \{(x_0, ..., x_0)\}$, where $g : [0, 1]^2 \to [0, 1]$ is a function that is injective with respect to the second variable and satisfies $g(x, y) \geq y$. An algorithmic representation of this procedure is included in Algorithm 1.

Algorithm 1 Algorithm using the function F_{x_0} of Proposition 10.

Require: An image f in RGB color space, threshold x_0, and F.
Ensure: A segmented image.

```
for each pixel(x, y) do
  for each of the color components of the pixel, R,G,B do
    Take the corresponding submatrix ((2n + 1) × (2n + 1)) of the image f centered on the pixel (x, y).
    Compute the value obtained by applying the function F_{x0} to all the pixels of the matrix.
  endfor
  Assign to the pixel (x, y) the value of 255 if the F value is equal to
  x0 for R,G,B and 0 otherwise.
endfor
```

Figure 1 shows the original images (first column) and the resulting output considering $x_0 = 170$ (second column) in our proposed algorithm.

6.2 | Image thresholding improvement

Another possible application of our proposal is image thresholding. Image thresholding is a simple method of image segmentation, which is used to differenciate between an object that is present in an image and the background. The general idea of an image thresholding algorithm is to replace each pixel of the image with a black pixel if the intensity of said pixel is less than a given threshold or, with a white pixel, if its intensity is greater than the threshold.

Different thresholding algorithms yield different thresholds for an image and, thus, some perform better than others depending on the image. In this study, we propose a methodology to to improve the results of a given thresholding algorithm by means of Algorithm 1.

Thus, we use Algorithm 1 as a final step of a given thresholding algorithm to improve the segmented image of in grayscale images. Specifically, our proposal consists in using Algorithm 1 as a final step of a given thresholding algorithm. Once the threshold value is obtain, it could be used as x_0 to perform the segmentation. Note that in this case, as images are grayscale, there is only one color component or channel. This procedure is exposed in Algorithm 2.

Algorithm 2 Algorithm to improve a given methodology of image thresholding.

Require: An image f in RGB color space, an image thresholding algorithm H and F.
Ensure: A segmented image.
1: Apply algorithm H to obtain a threshold x_0
2: Apply Algorithm 1 with f, x_0 and F to obtain the segmented image.

We put this methodology to the test in two differente frameworks:

- Using two recent thresholding algorithms: one based on equivalence measures (EM)[20] and one based on a fuzzy entropy approach (FE)[21];
- Using two well-known image thresholding algorithms in the literature: Otsu[22] and Tizhoosh.[23]

To check whether our proposal improves the thresholded images, we consider 10 images that are commonly used to test image thresholding algorithms (see the first column of Figures 2 and 3).

For our experiments, we set x_0 as the threshold value provided by each of the thresholding algorithm and, as before, we use $F_{x_0}(x_{11}, ..., x_{nn}) = MinMode(\min(x_{11}, x_0), ..., \min(x_{nn}, x_0))$.

In Figure 2, we show the thresholded images by each of the two recent methods (EM and FE) and the result of applying Algorithm 1 to obtain the segmented image (EM + F_{x_0} and FE + F_{x_0}).

In Figure 3, we show the thresholded images by each of the two well-known methods (Otsu and Tizhoosh) and the result of applying Algorithm 1 to obtain the segmented image (Otsu + F_{x_0} and Tizhoosh + F_{x_0}).

By visual inspection, we get the impression that our proposal yields images that are more similar to the ground-truth image for each of the base methodologies that we have considered, thus improving the results. Moreover, we have computed a series of evaluation metrics that confirms our first intuition. In Table 1, we can compare the accuracy, precision, recall and

FIGURE 1 Original images in RGB and their corresponding segmented images [Color figure can be viewed at wileyonlinelibrary.com]

F-score of the EM original approach with our proposal. In Tables 2–4 we show the analogous results for the FE, Otsu and Tizhoosh approaches, respectively.

From the results of both Tables 1–4 we see that the global accuracy rates, the recall and the F-score have improved using our proposal. It is important to point out that these improvements have been produced for all the methods that we have considered. On the other hand, although the average precision has decreased, we can say that the overall results are positive since the F-score is a measure that takes into account both the precision and recall and it has increased. Therefore, the increase in recall compensates the loss in precision.

7 | CONCLUSION

We have introduced the concept of $\mathcal{F}$-homogeneity which generalizes the previous generalization of the property of homogeneity for functions: abstract homogeneity. This new proposal permits to add variability to the condition that is required to each of the individual inputs of the function. Moreover, we have carried out a theoretical study of the properties of the functions that satisfy the proposed notion of $\mathcal{F}$-homogeneity. Furthermore, we have related this concept with the property of directional monotonicity by generalizing it by $\mathcal{F}$ directional monotonicity, that relates to a monotonicity condition with respect to a family of funcions $\mathcal{F}$ and a family of vectors $\mathcal{V}$. Finally, we have shown an illustrative example of a possible application of the introduced concepts in snow detection and as a final step in image thresholding. The proposed image thresholding algorithm yields better results when compared to four image thresholding algorithms, two recent and two well-known. Our proposal specially improves the recall, which suggests that our proposal can lead to a reduction of false negatives.

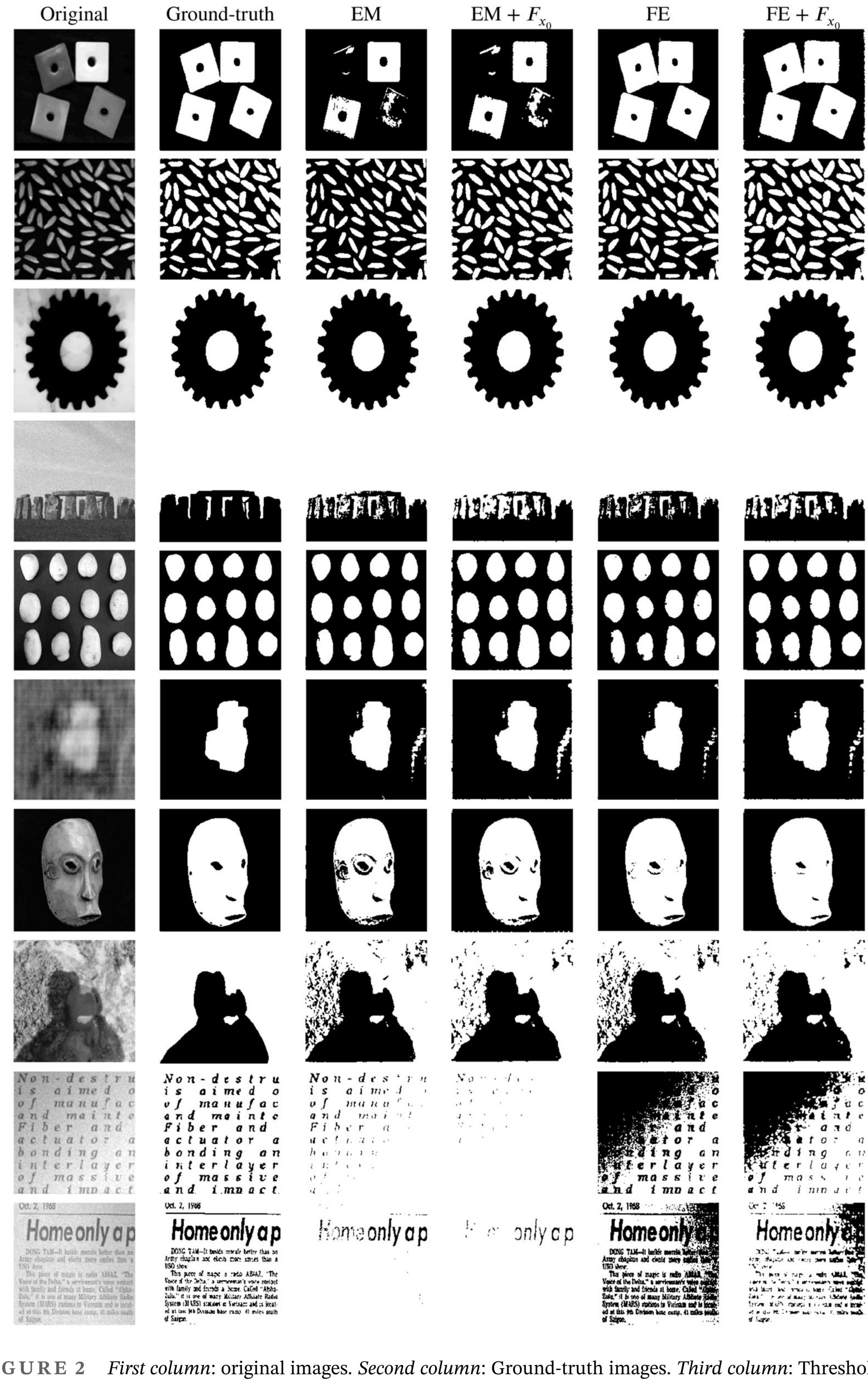

FIGURE 2 *First column*: original images. *Second column*: Ground-truth images. *Third column*: Thresholded images using EM. *Fourth column*: Thresholded images using EM and our proposal. *Fifth column*: Thresholded images using FE. *Sixth column*: Thresholded images using FE and our proposal

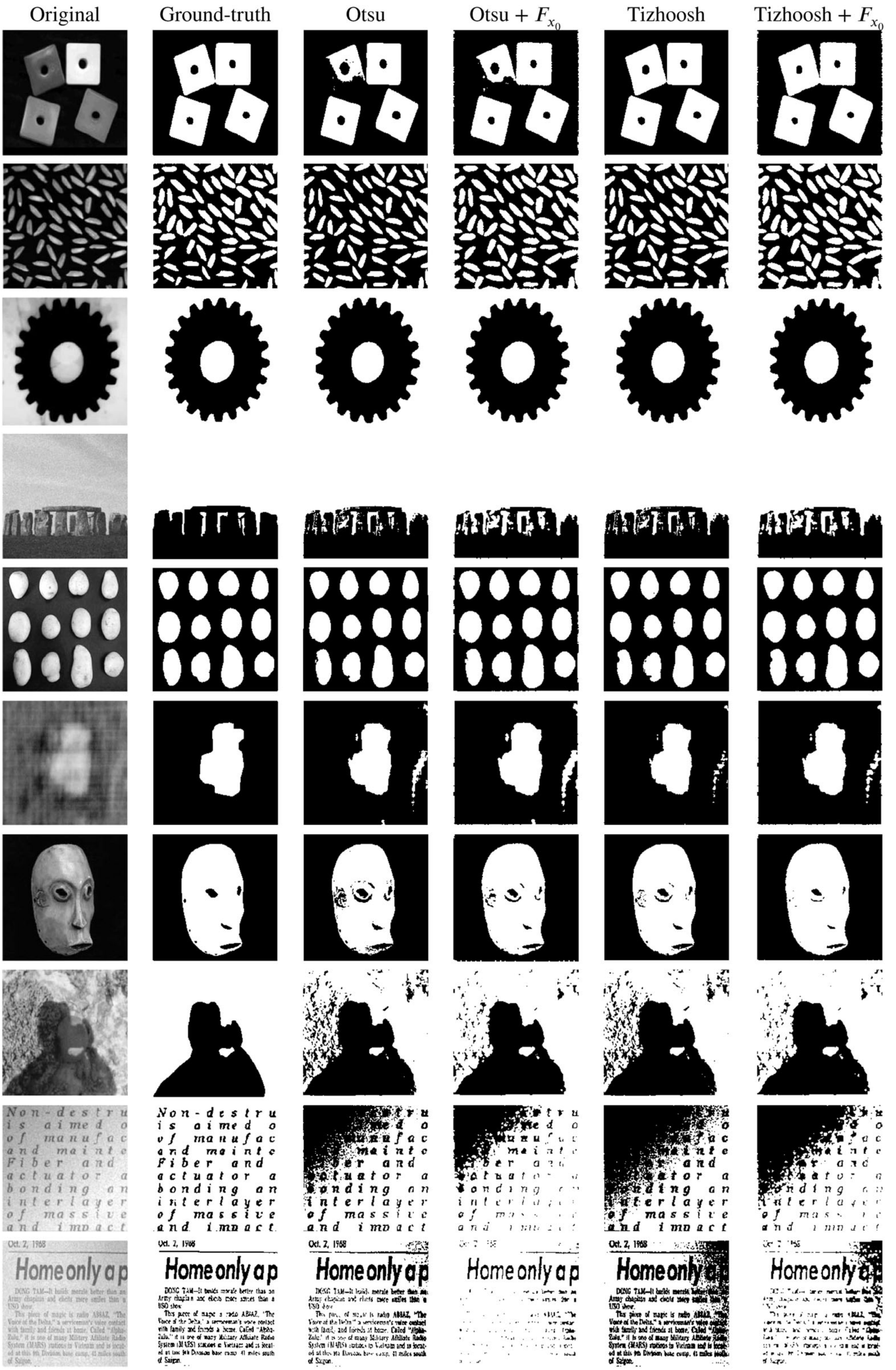

FIGURE 3 *First column*: original images. *Second column*: Ground-truth images. *Third column*: Thresholded images using Otsu. *Fourth column*: Thresholded images using Otsu and our proposal. *Fifth column*: Thresholded images using Tizhoosh. *Sixth column*: Thresholded images using Tizhoosh and our proposal

TABLE 1 Evaluation metrics of the comparison between the original EM algorithm and EM + our proposal

	Accuracy		Precision		Recall		F-score	
	EM	EM+F_{x_0}	EM	EM+F_{x_0}	EM	EM+F_{x_0}	EM	EM+F_{x_0}
Im_0	0.803464	0.832415	1.000000	0.990915	0.454899	0.540149	0.625334	0.699176
Im_1	0.914218	0.920289	0.966039	0.877638	0.790091	0.905364	0.869251	0.891285
Im_2	0.983666	0.979620	0.989342	0.968774	0.978227	0.991649	0.983753	0.980078
Im_3	0.939911	0.905202	0.916733	0.869680	0.994580	0.998563	0.954071	0.929676
Im_4	0.958499	0.951536	0.959188	0.907189	0.929201	0.970433	0.943956	0.937746
Im_5	0.964085	0.947097	0.886510	0.789752	0.896936	0.925410	0.891692	0.852216
Im_6	0.953620	0.962799	0.979785	0.961351	0.888762	0.933601	0.932056	0.947273
Im_7	0.882129	0.934318	0.976113	0.968195	0.833285	0.926160	0.899062	0.946711
Im_8	0.882602	0.869501	0.889514	0.869368	0.987065	0.999526	0.935754	0.929915
Im_9	0.826813	0.817070	0.828562	0.816609	0.990278	0.997268	0.902230	0.897942
Average	0.910901	0.911985	0.939178	0.901947	0.874332	0.918812	0.893716	0.901202

TABLE 2 Evaluation metrics of the comparison between the original FE algorithm and FE + our proposal

	Accuracy		Precision		Recall		F-score	
	FE	FE+F_{x_0}	FE	FE+F_{x_0}	FE	FE+F_{x_0}	FE	FE+F_{x_0}
Im_0	0.967401	0.968932	0.921348	0.921931	0.994482	0.921931	0.956519	0.958631
Im_1	0.927482	0.903224	0.895868	0.812067	0.904163	0.921931	0.899996	0.876578
Im_2	0.983339	0.977635	0.982107	0.963056	0.984988	0.921931	0.983546	0.978228
Im_3	0.954753	0.924012	0.938419	0.893427	0.993060	0.921931	0.964967	0.942799
Im_4	0.945949	0.955923	0.995371	0.957414	0.860301	0.921931	0.922920	0.940364
Im_5	0.968881	0.954316	0.922469	0.826415	0.885639	0.921931	0.903679	0.868473
Im_6	0.969080	0.968003	0.942114	0.931388	0.973426	0.921931	0.957514	0.956510
Im_7	0.895726	0.941388	0.974562	0.964812	0.856843	0.921931	0.911919	0.952906
Im_8	0.565532	0.623090	0.944321	0.912427	0.529629	0.921931	0.678639	0.741721
Im_9	0.710055	0.790227	0.922021	0.878166	0.699879	0.921931	0.795737	0.868604
Average	0.888820	0.900675	0.943860	0.906110	0.868241	0.921931	0.897544	0.908481

As future research lines, we have two paths in mind. On the one hand, abstract homogeneity has been applied to a multi-expert decision making problem.[7] We intend to study whether the variability that our new generalization provides is beneficial in such a problem. On the other hand, the relation between $\mathcal{F}$-homogeneity and directional monotonicity suggests that $\mathcal{F}$-homogeneity could be extended to domains that are more abstract than real numbers, in the same way that directional monotonicity was extended to Riesz spaces to be able to handle multidimensional data in Sesma-Sara et al.[13]

TABLE 3 Evaluation metrics of the comparison between the original Otsu algorithm and Otsu + our proposal

	Accuracy		Precision		Recall		F-score	
	Otsu	Otsu+F_{x_0}	Otsu	Otsu+F_{x_0}	Otsu	Otsu+F_{x_0}	Otsu	Otsu+F_{x_0}
Im_0	0.936614	0.953347	0.967496	0.967909	0.852848	0.900460	0.906561	0.932967
Im_1	0.922227	0.917508	0.953061	0.861065	0.825147	0.919853	0.884503	0.889489
Im_2	0.980148	0.983389	0.995742	0.979023	0.964856	0.988318	0.980056	0.983649
Im_3	0.958283	0.929449	0.943919	0.900589	0.992486	0.997700	0.967593	0.946661
Im_4	0.958474	0.953984	0.968902	0.918077	0.919097	0.963651	0.943342	0.940312
Im_5	0.956408	0.938654	0.840717	0.752962	0.907459	0.934386	0.872814	0.833920
Im_6	0.959748	0.967157	0.966072	0.958092	0.919848	0.949789	0.942394	0.953923
Im_7	0.890935	0.939135	0.975326	0.966538	0.848333	0.935781	0.907408	0.950911
Im_8	0.700031	0.743206	0.939850	0.906041	0.698378	0.784928	0.801318	0.841147
Im_9	0.799405	0.834530	0.886085	0.846001	0.862267	0.971850	0.874014	0.904569
Average	0.906227	0.916036	0.943717	0.905630	0.879072	0.934672	0.908000	0.917755

TABLE 4 Evaluation metrics of the comparison between the original Tizhoosh algorithm and Tizhoosh + our proposal

	Accuracy		Precision		Recall		F-score	
	Tiz.	Tiz.+F_{x_0}	Tiz.	Tiz.+F_{x_0}	Tiz.	Tiz.+F_{x_0}	Tiz.	Tiz.+F_{x_0}
Im_0	0.974212	0.978803	0.949822	0.948188	0.980262	0.995614	0.964802	0.971322
Im_1	0.927660	0.909014	0.916562	0.829001	0.879638	0.942257	0.897721	0.882009
Im_2	0.983565	0.979947	0.990128	0.969569	0.977233	0.991450	0.983639	0.980387
Im_3	0.958283	0.929449	0.943919	0.900589	0.992486	0.997700	0.967593	0.946661
Im_4	0.957224	0.961968	0.992093	0.951803	0.893395	0.946833	0.940161	0.949312
Im_5	0.966789	0.951408	0.905405	0.809958	0.891674	0.921387	0.898487	0.862086
Im_6	0.966721	0.969542	0.946226	0.937342	0.961679	0.980446	0.953890	0.958409
Im_7	0.893603	0.939964	0.975043	0.965517	0.852938	0.938207	0.909912	0.951666
Im_8	0.589350	0.646882	0.943931	0.911664	0.559111	0.655873	0.702259	0.762898
Im_9	0.726618	0.802892	0.917975	0.873196	0.726091	0.884127	0.810836	0.878627
Average	0.894402	0.906987	0.948111	0.909683	0.871451	0.925389	0.902930	0.914338

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ENDNOTE

*A function $f : [0, 1]^n \to [0, 1]$ is said to be idempotent if $f(x, ..., x) = x$ for all $x \in [0, 1]$.

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