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Strong negations and restricted equivalence functions revisited: An analytical and topological approach

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Abstract

Throughout this paper, our main idea is to analyze the concepts of a strong negation and a restricted equivalence function, that appear in a natural way when dealing with theory and applications of fuzzy sets and fuzzy logic. Here we will use an analytical and topological approach, showing how to construct them in an easy way. In particular, we will also analyze some classical functional equation related to those key concepts.

Keywords: Fuzzy sets; Strong negations; Restricted equivalence functions; Admissible distances; Implication operators; Real valued functions

1. Introduction and motivation

When dealing with fuzzy sets and systems and fuzzy logic and their applications in practice, one is naturally confronted to classical concepts as negations, restricted equivalence functions, implications, etc. Sometimes, those concepts have been characterized through some functional equations. However, it seems helpful to look at them in an alternative manner: since they are functions from the unit interval $[0, 1]$ into itself in the case of negations, or from the unit square $[0, 1]^2$ into $[0, 1]$ in the case of restricted equivalence functions, implications or dissimilarities, so giving raise either to curves in the unit plane or surfaces in the unit cube $[0, 1]^3$, we may wonder which are those curves and surfaces, analyzing their main properties and features. To do so, we will follow a topological and analytical approach.

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The use of negations and restricted equivalence functions is typical in applications. A relevant case is that of image processing in black and white. Thus, a picture can be understood there as a bivariate function $g : [0, 1] \times [0, 1] \to [0, 1]$ so that if $(x, y) \in [0, 1]^2$ represents a point (or a pixel) in the unit plane, $g(x, y)$ represents the amount of black, graded with values between 0 and 1 in the unit interval.[1] A negation N here can be used to get the "negative" of the photograph defined by the function g, by assigning now the value $N(g(x, y))$ to a point (x, y). As a matter of fact N acts on values of the unit interval $[0, 1]$, understood as amounts of black, and takes values in $[0, 1]$ too, so that given an amount $t \in [0, 1]$, $N(t)$ should be interpreted as its "complementary" or negative amount. It is quite usual here to use the standard negation $N(t) = 1 - t \quad (t \in [0, 1])$, but the abstract definition of a negation on $[0, 1]$ is more general (see Definition 2.3 below), so that the standard negation is just a particular case.

Aside from negations, it is important to compare elements that belong to the unit square $[0, 1]^2$. This is typical in Image Processing, when dealing with pixels. Thus, following [5]:

<< A very important problem in image processing is the global comparison of two images. The measures used in order to make the comparison are normally demanded to fulfill the following set of properties (see [29]):

(a) The comparison measure between two images must not depend on the order in which they are compared.
(b) The comparison measure between an image (in black and white) and its negative must be zero.[2]
(c) The comparison measure should be one if and only if the two images are the same.
(d) The comparison measure between two images must give the same result when it is applied to the two original images as when it is applied to their respective negatives.>>

Bearing this in mind, the concept of a restricted equivalence function was introduced and analyzed in this literature (see [5,6] and Definition 2.7 below).

Summarizing, an important *objective* throughout the present manuscript is to revise the definitions and main properties related to the key concepts of strong negation and restricted equivalence functions, but now *using an analytical and topological view with a really outstanding economy of resources and tools used for this purpose*. It is indeed true that some of the results (re)-obtained were already known in the literature. However, in our opinion, this new way to prove them could be more explanatory and helpful for any researcher interested in this topic. Furthermore, some of the key constructions can now be "visualized" by means of suitable figures, as shown along the paper.

2. Preliminaries

Henceforward U will denote a nonempty set, also known as the *universe*.

Definition 2.1. (See e.g. [30,22].) A *fuzzy subset* X of U is defined as a function $\mu_X : U \to [0, 1]$. μ_X is said to be the *membership function* or the *indicator* of X. Also, a *fuzzy binary relation* defined on U is a bivariate map $\mathcal{F} : U \times U \to [0, 1]$, that is, $\mathcal{F}$ is actually a fuzzy subset of the Cartesian product U^2.

When dealing with fuzzy sets, it is typical to consider functions $f : [0, 1] \to [0, 1]$ as well as $F : [0, 1] \times [0, 1] \to [0, 1]$. In the first situation, f can also be considered as a *modifier* of fuzzy subsets of a given universe U. Indeed, if μ_X is the membership function of a fuzzy subset X, the composition $f \circ \mu_X$ is the indicator of a new fuzzy subset of U, understood as the modification of X through f. The second situation, namely the one that involves bivariate maps $F : [0, 1] \times [0, 1] \to [0, 1]$, arises in a natural way when dealing with fuzzy logic (i.e.: implication functions).

Remark 2.2. For standard and well-known definitions in fuzzy set theory we refer to [22].

Definition 2.3. (The original definition was launched in [27]. See also [4,13–15,18,19,26,30].)

A *strong negation* is a function $N : [0, 1] \to [0, 1]$ that satisfies the following properties:

[1] That is, 0 here means "totally white" while 1 means "totally black" and intermediate values correspond to a scale of gray.

[2] However, in [5,6] as well as the key Definition 2.7 below, we will weaken this condition. If the negative of a pixel is understood as got through a strong negation in [0, 1], see Definition 2.3), the comparison measure between a pixel and its negative is a helpful function in many contexts (see Definition 2.11), but, needless to say, that function is not always zero.

i) (Involutiveness) $N(N(x)) = x$ holds for every $x \in [0, 1]$,
ii) N is decreasing.

Remarks 2.4.

i) The condition of being involutive is so strong that it forces N to be bijective. As a matter of fact, if $N(x) = N(y)$, then $x = N(N(x)) = N(N(y)) = y \quad (x, y \in [0, 1])$, so N is injective. Conversely, given $x \in [0, 1]$, we observe that $N(N(x)) = x$, so x is in the range of N. Therefore, N is surjective, too.
ii) Notice also that $N(0) = 1$. Indeed, if $N(0) = a < 1$, we would have that $N(t) \in [0, a]$ holds for every $t \in [0, 1]$ since N is decreasing. But then for every $b \in (a, 1]$ there is no $x \in [0, 1]$ such that $N(x) = b$. And this violates the fact of N being surjective, just proved in Remark 2.4 i) above.
iii) Some other authors (see e.g. [4,27]) do not impose a priori the condition of involutiveness when defining *negations* (but not necessarily "strong negations"). Instead, they usually ask the function N to be injective. Sometimes they do not ask $N(0)$ to be 1, either (see e.g. [27,28]).
iv) Notice here that a bijective and decreasing function from [0, 1] onto [0, 1] may fail to be involutive. An example is the function $G : [0, 1] \to [0, 1]$ given by $G(x) = 1 - x^2 \quad (x \in [0, 1])$.
v) The word "strong" is here associated to involutiveness. Along the present paper we will always deal with strong negations in the sense of Definition 2.3 above, so that they are involutive.
vi) Being N bijective, a strong negation is not only decreasing, but indeed it is *strictly* decreasing (i.e. $a < b \Leftrightarrow N(b) < N(a)$ holds true for every $a, b \in [0, 1]$). Moreover, its inverse N^{-1} is actually N because N is involutive. Furthermore, a strong negation is always *continuous* with respect to the usual Euclidean topology of the unit interval. This is a consequence of an elementary result of Real Analysis (see e.g. [24], Section 1.6). Another easy way to see this, now using general topology, is the following: just notice that given $a \in (0, 1)$ we have that $N^{-1}([0, a)) = N([0, a)) = (N(a), 1]$; and also $N^{-1}((a, 1]) = [0, N(a))$ are open sets in the usual topology of the unit interval. In fact, the axiom of continuity that some authors impose in the very definition of a strong negation (see e.g. [5]) is redundant, by these topological reasons. Finally, a strong negation N has a *fixed point* $c \in (0, 1)$ such that $N(c) = c$. This is a clear consequence of the well-known Bolzano's theorem for continuous functions: If we consider the function $g : [0, 1] \to [0, 1]$ given by $g(t) = N(t) - t \quad (t \in [0, 1])$ we may observe that g is continuous and $g(0).g(1) = -1 < 0$, so there exists some $d \in (0, 1)$ with $g(d) = 0$, or equivalently $N(d) = d$. In addition, this d is unique because N is bijective. Some authors (see e.g. [4,5]) call this fixed point an *equilibrium point* of the function N.

Examples 2.5.

i) Let $0 < a < 1$. Being $t \in [0, a]$ define $N(t) = \frac{(a-1)t}{a} + 1$, and being $t \in [a, 1]$ define[3] $N(t) = \frac{a(t-a)}{a-1} + a$. A straightforward checking shows that N is actually a strong negation.
ii) Define $N(t) = 1 - t^3$ if $t \in [0, \frac{1}{2}]$; $N(t) = \frac{11}{8} - t$ if $t \in [\frac{1}{2}, \frac{7}{8}]$; $N(t) = (1-t)^{\frac{1}{3}}$ if $t \in [\frac{7}{8}, 1]$. Again, one may check that N is actually a strong negation.

The structure of strong negations has already been characterized as follows (the original proof appears in [27], see also [4,7]).

Theorem 2.6. *A function* $N : [0, 1] \to [0, 1]$ *is a strong negation if and only if there exists a bijective, and increasing function* $\phi : [0, 1] \to [0, 1]$ *such that* $N(x) = \phi^{-1}(1 - \phi(x))$ *holds for every* $x \in [0, 1]$.

Definition 2.7. (See [5,6]) A function $F : [0, 1] \times [0, 1] \to [0, 1]$ is said to be a *restricted equivalence function*[4] (REF, for short) if it satisfies the following properties for every $x, y, z \in [0, 1]$:

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[3] This corresponds to fold the interval [0, 1] by the point a and stretch (or shrink) $[0, a]$ and, accordingly, shrink (or stretch) $[a, 1]$ to coincide.

[4] Similar concepts have been introduced in this literature with different purposes, and it seems interesting to compare their definitions and scopes. To put some example, we may analyze the pioneer concept of an *equivalence function* due to Fodor and Roubens, see [17] as well as the notions of an *indistinguishability operator* (see [21,28]) and that of a *monometric* (see [23]), and compare them to the concept of a REF.

i) $F(x, y) = F(y, x)$,
ii) $F(x, y) = 1 \Leftrightarrow x = y$,
iii) $F(x, y) = 0 \Leftrightarrow (x, y) = (0, 1) \vee (x, y) = (1, 0)$,
iv) $x \leq y \leq z \Rightarrow F(x, y) \geq F(x, z)$.

Definition 2.8.

i) If N is a strong negation and F is a REF, then F is said to be *compatible* with respect to N if $F(x, y) = F(N(x), N(y))$ holds true for every $x, y \in [0, 1]$.
ii) If F is a REF such that for every $x, y, z \in [0, 1]$ it holds true that $(x < y \leq z) \Rightarrow F(y, z) > F(x, z)$ and also $(x \leq y < z) \Rightarrow F(x, y) > F(x, z)$, then F is said to be *strict*.
iii) If F is a restricted equivalence function (REF), then it is said to be *continuous* if, as a map $F : [0, 1] \times [0, 1] \to [0, 1]$, it is continuous with respect to the usual topologies of the unit square and the unit plane.

Dually to the concept of a restricted equivalence function, and associated to a strong negation, the concept of a dissimilarity is introduced now (see [8]).

Definition 2.9. Let N, F respectively be a strong negation and a REF. The function $D_{N,F} : [0, 1] \times [0, 1] \to [0, 1]$ given by $D_{N,F}(x, y) = N(F(x, y))$ $(x, y \in [0, 1])$ is said to be the *dissimilarity* function associated to the pair (N, F).

Remark 2.10. Let N, F respectively be a strong negation and a REF. Let $D_{N,F}$ be the dissimilarity function associated to (N, F). Observe that it satisfies the following properties, for every $x, y \in [0, 1]$:

i) $D_{N,F}(x, y) = 0 \Leftrightarrow N(F(x, y)) = 0 \Leftrightarrow F(x, y) = 1$, since N is bijective. But $F(x, y) = 1 \Leftrightarrow x = y$.
ii) $D_{N,F}(x, y) = D_{N,F}(y, x)$.
iii) $D_{N,F}(x, y) = 1 \Leftrightarrow (x, y) = (0, 1) \vee (x, y) = (1, 0)$.
iv) $x \leq y \leq z \Rightarrow D_{N,F}(x, y) \leq D_{N,F}(x, z)$.

In the particular case in which $N(x) = 1 - x$ $(x \in [0, 1])$, consider now a restricted equivalence function F such that

$$F(x, y) + F(y, z) - F(x, z) \leq 1$$

holds true for every $x, y, z \in [0, 1]$. Denote $D = D_{N,F}$ and notice that in this case we have that $F(x, y) + F(y, z) - F(x, z) \leq 1 \Leftrightarrow -F(x, y) - F(y, z) + F(x, z) \geq -1 \Leftrightarrow 1 - F(x, y) + 1 - F(y, z) + F(x, z) \geq 1 \Leftrightarrow D(x, y) + D(y, z) \geq 1 - F(x, z) = D(x, z)$. Therefore D is actually a distance on $[0, 1]$. In addition the distance D is admissible with respect to the usual Euclidean order "$\leq$". (See Definition 2.13 below. Also, see e.g. [8] for more relationships between restricted equivalent functions, dissimilarities and distances.)

Definition 2.11. Let N, F respectively be a strong negation and a REF. The function $E_{N,F} : [0, 1] \to [0, 1]$ given by $E_{N,F}(x) = F(N(x), x)$ for every $x \in [0, 1]$ is said to be the *normal* function (see [8]) associated to the pair (N, F).

Example 2.12. Consider on $[0, 1]$ the Euclidean distance d given by $d(x, y) = |x - y|$. Now observe that the function $F_d : [0, 1]^2 \to [0, 1]$ given by

$$F_d(x, y) = 1 - |x - y| \quad (x, y \in [0, 1])$$

is a restricted equivalence function. Notice also that d is a dissimilarity in the sense of Definition 2.9 above. In this case, if we consider the standard negation $N(x) = 1 - x$ $(x \in [0, 1])$, the corresponding normal function E_{N,F_d} is given by $E_{N,F_d}(t) = 1 - |2t - 1|$ $(t \in [0, 1])$. Moreover, with respect to the standard negation, F_d is a compatible REF: Being $x, y \in [0, 1]$, observe that $F_d(N(x), N(y)) = F_d(1 - x, 1 - y) = 1 - |(1 - x) - (1 - y)| = 1 - |x - y| = F_d(x, y)$.

Definition 2.13. Let U be a universe. Let $\preceq$ be a total (linear) order defined on U. Let d be a metric on U. Then the metric d is said to be *admissible* as regards the linear order $\preceq$ if, for every $x, y, z \in U$ it holds that $x \preceq y \preceq z \Rightarrow d(x, y) \leq d(x, z)$.

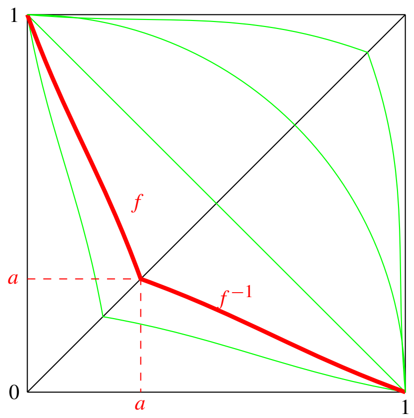

Fig. 1. Strong negations.

Remarks 2.14.

i) The Euclidean distance d on the real line $\mathbb{R}$ (and also its restriction to the unit interval $[0, 1]$) is admissible as regards the usual order $\leq$.
ii) If D is an admissible and translation-invariant[5] distance on $[0, 1]$ such that the function $f : [0, 1] \to [0, 1]$ given by $f(x) = D(0, x)$ $(x \in [0, 1])$ is strictly increasing and $D(0, 1) = 1$, then, for any strong negation N, the function $F_{D,N} : [0, 1]^2 \to [0, 1]$ given by $F_{D,N}(x, y) = N(D(x, y))$ $(x, y \in [0, 1]^2)$ is a restricted equivalence function that is compatible with respect to N. (See also Proposition 5.2 in Section 5).

3. On the structure of strong negations in [0, 1]

In this section we explain how to construct all the strong negations that may be defined on the unit interval $[0, 1]$.

In order to obtain a strong negation N on the unit interval $[0, 1]$ first we fix a point $a \in (0, 1)$. This point a will be the fixed (equilibrium) point of N, so that $N(a) = a$.

Since N is bijective and strictly decreasing, we should have that $N([0, a]) = [a, 1]$ and $N([a, 1]) = [0, a]$. Of course here we have $N(0) = 1$ and $N(1) = 0$.

Take now any (fixed) bijective and strictly decreasing function $f : [0, a] \to [a, 1]$. (In particular $f(0) = 1$; $f(a) = a$.) (See Fig. 1.)

Let $f^{-1} : [a, 1] \to [0, a]$ stand for the inverse map of f. (In particular, $f^{-1}(a) = a$; $f^{-1}(1) = 0$.)

Now define $N : [0, 1] \to [0, 1]$ as follows:

$$N(x) = f(x) \text{ if } x \in [0, a] \ ; \quad N(t) = f^{-1}(t) \text{ if } t \in [a, 1].$$

Theorem 3.1. *The function N so constructed is actually a strong negation on the unit interval $[0, 1]$. All strong negations can be constructed this way.*

Proof. By its own definition, $N(0) = f(0) = 1$ and $N(1) = f^{-1}(1) = 0$. Moreover N is bijective because f is bijective from $[0, a]$ onto $[a, 1]$ and, obviously, f^{-1} is also bijective from $[a, 1]$ onto $[0, a]$. In the same way, N is strictly decreasing because, by definition, f and f^{-1} are strictly decreasing in their respective domains. To conclude, let us prove that N is involutive: To see this, notice that if $x \in [0, a]$ we have that $N(x) = f(x) \in [a, 1]$. Hence $N(N(x)) = f^{-1}(N(x)) = f^{-1}(f(x)) = x$. Also, if $t \in [a, 1]$ we have that $N(t) = f^{-1}(t) \in [0, a]$. Thus $N(N(t)) = f(f^{-1}(t)) = t$.

[5] That is $D(x, y) = D(x + a, y + a)$ holds for every x, y, a such that $x, y, x + a, y + a \in [0, 1]$.

Finally, if N is a strong negation on $[0, 1]$, and a denotes its fixed point, it is clear that N maps $[0, a]$ onto $[a, 1]$ and $[a, 1]$ onto $[0, a]$. If we define $f : [0, a] \to [a, 1]$ as $f(x) = N(x)$ $(x \in [0, 1])$, it is clear that $f^{-1} : [a, 1] \to [0, a]$ is given by $f^{-1}(t) = N(t)$ ($t \in [a, 1]$, since N is involutive. □

Remark 3.2. As stated in Theorem 2.6, the structure of strong negations was already got in [27,28] using techniques of functional equations. However, in our opinion, our previous analytical and topological construction given in Theorem 3.1 is more direct and natural. Moreover, the formulation of this result, namely Theorem 3.1, looks similar, in a sense, to the one got in terms of symmetric functions with respect to the diagonal that appears in [14,15]. It seems appealing at this stage to compare the ideas and methodology used here and there. Other generalizations are also encountered in this literature (see [16]).

We may generalize classical results (see [4–7,16,27,28]) in several ways, as developed in the next paragraphs in this Section 3. To do so, first we introduce some helpful definitions (see also [4,7]):

Definition 3.3. A strictly increasing and bijective map $\Phi : [0, 1] \to [0, 1]$ is said to be an *automorphism* of the unit interval.

Proposition 3.4. *Any automorphism is continuous.*

Proof. It is a direct consequence of Remark 2.4 (vi). □

Definition 3.5. A strictly decreasing and bijective map $\Phi : [0, 1] \to [0, 1]$ is said to be an *anti-automorphism* of the unit interval.[6]

Remarks 3.6.

i) As suggested from a referee, perhaps the terms *homeomorphism* in Definition 3.3 and *anti-homeomorphism* in Definition 3.5 could have been employed to fit better the title of this manuscript in what concerns Analysis and/or Topology. Nevertheless we have opted to keep the terms automorphism and anti-automorphism because they had already been introduced in this literature in the context of strong negations (see [4]). Furthermore, topologically the notion of homeomorphism is different (namely, a bijective, continuous and open map). Last but not least, an anti-homeomorphism is indeed a topological homeomorphism, too.
ii) Observe that the composition of any even number of strong negations defined in the unit interval $[0, 1]$ gives rise to an automorphism. Also, the composition of any odd number of strong negations defined in the unit interval $[0, 1]$ gives rise to an anti-automorphism.

Proposition 3.7. *Any anti-automorphism is continuous.*

Proof. This is similar to Proposition 3.4. □

Lemma 3.8. *If M (respectively, Φ) is a strong negation (respectively, an automorphism) on* $[0, 1]$, *the composition* $\Phi^{-1} \circ M \circ \Phi$ *is also a strong negation on the unit interval* $[0, 1]$.

Proof. This is straightforward. For a formal proof, see e.g. Proposition 1.4.8 in [3]. □

As we will see below in Theorem 3.11, the compositions involved in the statement of Lemma 3.8 attain any strong negation.

[6] In the fuzzy literature, this concept is also known as a *strict negation*. Notice that it may fail to be involutive.

Definition 3.9. Given two strict negations N and M defined on the unit interval $[0, 1]$ we say that N and M are *conjugate* if there exist an automorphism Φ such that $N(x) = \Phi^{-1}(M(\Phi(x)))$ holds true for every $x \in [0, 1]$. (In other words, the composition $\Phi^{-1} \circ M \circ \Phi$ is N).

Lemma 3.10. *If N and M are conjugate, any automorphism Φ such that $N(x) = \Phi^{-1}(M(\Phi(x)))$ holds true for every $x \in [0, 1]$ must send the fixed (equilibrium) point of N to the fixed point of M.*

Proof. Let a (respectively b) denote the fixed point of N (respectively, of M). Then $a = N(a) = \Phi^{-1}(M(\Phi(a))$. Hence $\Phi(a) = \Phi(\Phi^{-1}(M(\Phi(a))) = M(\Phi(a))$. Therefore $\Phi(a) = b$, namely the unique fixed (equilibrium) point of M. □

Next result was already obtained in [27]. However, we furnish here an alternative proof for the sake of completeness, and also in order to use directly analytical and topological facts relative to the structure of strict negations, as explained in previous paragraphs as well as in Fig. 1 above, and, of course, in the spirit of the title of the present manuscript. In fact, the existence of a fixed point for any strong negation plays a decisive role in this construction, showing the forceful strength of Theorem 3.1 above. This has been done with economy of resources. In addition, the significance of the proof is clear: knowing the fixed points of any two strong negations N and M we can directly build a strictly increasing function from the unit interval into itself that transforms N into M conjugating these two strong negations in the sense of Definition 3.9. In a way, this also tells us that once we know any strong negation on the unit interval, we can construct any other one using a suitable automorphism.

Theorem 3.11. *Any two strong negations N and M on the unit interval $[0, 1]$ are conjugate one another.*

Proof. (This result was already obtained in [27]. We furnish here an alternative proof for the sake of completeness.)

Let a (respectively, b) be the fixed point of N (respectively, the fixed point of M). Bearing in mind Lemma 3.10 consider a bijective and strictly increasing function Θ from the interval $[0, a]$ onto the interval $[0, b]$. Given $t \in [a, 1]$ we may observe that $N(t) \in [0, a]$, because N is a strong negation. Let $x = N(t)$. Notice that $N(x) = N(N(t)) = t$ because N is involutive. Define now the map $\Gamma : [a, 1] \to [b, 1]$ given by $\Gamma(t) = M(\Theta(N(t))) = M(\Theta(x))$ $(t \in [a, 1])$.

Define the function $\Phi : [0, 1] \to [0, 1]$ as follows:

$$\Phi(x) = \Theta(x) \text{ if } x \in [0, a] \;\; ; \;\; \Phi(t) = \Gamma(t) \text{ if } t \in [a, 1] \,.$$

Let us analyze some properties of the so defined function Φ:

i) If $x < y \in [0, a]$, then $\Phi(x) = \Theta(x) < \Theta(y) = \Phi(y)$ holds true by definition.
ii) If $t < u \in [a, 1]$ then $N(u) < N(t) \in [0, a]$. Hence $\Theta(N(u)) < \Theta(N(t))$, so that $\Gamma(t) = M(\Theta(N(t))) < M(\Theta(N(u))) = \Gamma(u)$.
iii) If $x \in [0, a]$ (respectively, if $t \in [a, 1]$), then $\Phi(x) \in [0, b]$ (respectively $\Phi(t) \in [b, 1]$).

By the properties i) to iii) we may conclude that Φ is strictly increasing. It is also bijective, because Θ is a bijection from $[0, a]$ onto $[0, b]$ and Γ is a bijection from $[a, 1]$ onto $[b, 1]$.

Therefore, Φ is actually an automorphism of the unit interval $[0, 1]$.

Given now $t \in [a, 1]$ we have that $\Phi(t) = \Gamma(t) = M(\Theta(N(t))) = M(\Phi(N(t)))$. So $M(\Phi(t)) = M(M(\Phi(N(t)))) = \Phi(N(t))$ because M is involutive. Therefore $N(t) = \Phi^{-1}(M(\Phi(t)))$.

Also, given $x \in [0, a]$, we have that $N(x) \in [a, 1]$. So $\Phi(x) = \Phi(N(N(x))) = M(\Phi(N(x)))$. Thus $M(\Phi(x)) = M(M(\Phi(N(x)))) = \Phi(N(x))$ since M is involutive. Therefore $N(x) = \Phi^{-1}(M(\Phi(x)))$.

Hence $N = \Phi^{-1} \circ M \circ \Phi$, so that N and M are conjugate. □

Remark 3.12. As aforesaid, the result stated in Theorem 3.11 already appears in [27]. Furthermore, an alternative proof is available using existing results: from Theorem 2.6 (see also Theorem 1.4.3 in [3]), there exist two increasing bijections f, g on $[0, 1]$ such that $N(x) = f^{-1}(1 - f(x))$ and $M(x) = g^{-1}(1 - g(x))$, for all $x \in [0, 1]$. Take $\Phi = g^{-1} \circ f$. Then Φ is an automorphism on $[0, 1]$ and $\Phi^{-1} \circ M \circ \Phi = N$. One may think that this alternative proof of Theorem 3.11 looks simpler than ours. However, the proofs of the known results needed, mainly Theorem 2.6, are by no means so simple. They use different techniques as functional equations theory. As a matter of fact, one of our intentions through the present paper is showing that with an important economy of means, and just using simple but crucial and decisive analytical and topological concepts, we may (re)-obtain those classical results.

Similarly to Lemma 3.8, Definition 3.9 and Theorem 3.11 we could also modify strong negations by means of anti-automorphisms.

Lemma 3.13. *If N (respectively, Ψ) is a strong negation (respectively, an anti-automorphism) on* $[0, 1]$*, the composition $\Psi^{-1} \circ N \circ \Psi$ is also a strong negation on the unit interval* $[0, 1]$.

Proof. It is similar to the proof of Lemma 3.8. □

Definition 3.14. Given two strict negations N and M defined on the unit interval $[0, 1]$ we say that N and M are *anti-conjugate* if there exist an anti-automorphism Ψ such that $N(x) = \Psi^{-1}(M(\Psi(x)))$ holds for every $x \in [0, 1]$.

Theorem 3.15. *Any two strong negations N and M on the unit interval* $[0, 1]$ *are anti-conjugate one another.*

Proof. It is similar to that of Theorem 3.11. □

Theorem 2.6 follows now as an easy corollary of the previous results.

Remark 3.16. Using strong negations different from the classical one ($M(y) = 1 - y \quad (y \in [0, 1])$) we may obtain other alternative expressions to describe any strong negation N that may be defined in the unit interval. For instance, if fixed $p > 0$ we consider the strong negation M defined by $M(x) = (1 - x^p)^{\frac{1}{p}} \quad (x \in [0, 1])$ then, as an alternative to Theorem 2.6 and a particular case of Theorem 3.11, we get that a function $N : [0, 1] \to [0, 1]$ is a strong negation if and only if there exists a bijective, and increasing function $\phi : [0, 1] \to [0, 1]$ such that

$$N(x) = \phi^{-1}((1 - (\phi(x))^p)^{\frac{1}{p}})$$

holds for every $x \in [0, 1]$. In particular, when $p = 2$ we obtain that there exists a bijective, and increasing function $\gamma : [0, 1] \to [0, 1]$ such that

$$N(x) = \gamma^{-1}(\sqrt{(1 - (\gamma(x))^2)}) \quad (x \in [0, 1]).$$

As an example of a different nature if we consider the strong negation[7] M defined by $M(x) = \frac{1-x}{1+3x} \quad (x \in [0, 1])$, then a function $N : [0, 1] \to [0, 1]$ is a strong negation if and only if there exists a bijective, and increasing function $\phi : [0, 1] \to [0, 1]$ such that

$$N(x) = \phi^{-1}(\frac{1 - \phi(x)}{1 + 3\phi(x)})$$

holds for every $x \in [0, 1]$.

In the same way, but now using Theorem 3.15 instead of Theorem 3.11, we also obtain the following corollary.

Corollary 3.17. *A function $N : [0, 1] \to [0, 1]$ is a strong negation if and only if there exists a bijective, and decreasing function $\psi : [0, 1] \to [0, 1]$ such that $N(x) = \psi^{-1}(1 - \psi(x))$ holds for every $x \in [0, 1]$.*

[7] This is the particular case with $b = 3$ of the so-called *Sugeno's strong negations* $M_b : [0, 1] \to [0, 1]$, that are defined when $b + 1 > 0$ (see e.g. [19]), given by $M_b(x) = \frac{1-x}{1+bx} \quad (x \in [0, 1])$.

Remarks 3.18.

i) Notice that in Theorem 3.11 (respectively, in Theorem 3.15 the function Φ (respectively, the function Ψ) is not unique.
ii) Notice also that once a strong negation has been fixed, by composing it with automorphisms or anti-automorphisms in a suitable way, as stated in Theorems 3.11 and 3.15, any other strong negation can be attained. This generalizes classical results (see e.g. [16,27,28]).

Given a strong negation $N : [0, 1] \to [0, 1]$ we immediately observe that for any $x, y \in [0, 1]$ we have that $x = N(y) \Leftrightarrow N(x) = y$. This idea inspires another way to find strong negations.

Definition 3.19. Let $f, g : [0, 1] \to [0, 1]$ be continuous and bijective functions such that f is strictly increasing and g is strictly decreasing. We say that f, g are *coupled* if for every $x, y \in [0, 1]$ it holds that $f(x) = g(y) \Leftrightarrow f(y) = g(x)$.

Example 3.20. The functions f, g where $f(x) = x^3$; $g(x) = \sqrt{1 - x^6}$ $(x \in [0, 1])$ are coupled. Of course there are much more sophisticated examples, e.g., $f(x) = \frac{x^6+x^4}{2}$; $g(x) = \frac{[32-(x^6+x^4)^5]^{\frac{1}{5}}}{2}$ $(x \in [0, 1])$.

Proposition 3.21. *If f and g are coupled, the composition $f^{-1} \circ g = g^{-1} \circ f$ is a strong negation on the unit interval* $[0, 1]$.

Proof. Due to the properties of f and g, that composition is bijective and strictly decreasing from $[0, 1]$ onto itself. Moreover, since $f^{-1} \circ g = g^{-1} \circ f$, the double composition $(f^{-1} \circ g) \circ (f^{-1} \circ g)$ is the identity map. In other words, $f^{-1} \circ g$ is involutive. $\square$

Remark 3.22. Proposition 3.21 also inspires the idea exposed in Fig. 4 in next Section 4.

Example 3.23. As in Example 3.20, if we take f, g where $f(x) = x^3$; $g(x) = \sqrt{1 - x^6}$ $(x \in [0, 1])$, the strong negation we get is $N : [0, 1] \to [0, 1]$ defined by $N(x) = (1 - x^6)^{\frac{1}{6}}$ $(x \in [0, 1])$.

4. On the structure of restricted equivalence functions in [0, 1]

Having in mind Definition 2.7 now we analyze the structure of restricted equivalence functions (REF) defined on the unit square $[0, 1]^2$.

To start with, taking into account the conditions involved in the definition, the geometrical aspect of a REF is as follows (see Fig. 2).

Remarks 4.1.

i) Unlike Definition 2.3 where the conditions imposed force a strong negation to be continuous (so we had automatic continuity), the conditions imposed in Definition 2.7 to introduce the concept of a restricted equivalence function (REF) do not carry the continuity, in general. (See Example 4.2.)
ii) Notice that, directly from the definitions of an automorphism $\Phi : [0, 1] \to [0, 1]$ and a restricted equivalence function $F : [0, 1] \times [0, 1] \to [0, 1]$, it immediately follows that the composition $\Phi \circ F$ also constitutes a REF. (See Proposition 3 in [5].)

Example 4.2. Consider the REF function $F : [0, 1] \times [0, 1] \to [0, 1]$ given by

$$F(x, y) = 1 - |x - y| \text{ if } |x - y| < \frac{1}{2} \;\; ; \;\; F(x, y) = \frac{1}{3}(1 - |x - y|) \text{ otherwise.}$$

Here all the points (x, y) where $|x - y| = \frac{1}{2}$ are points of discontinuity of F.

A restricted equivalence function (REF) may also fail to be strict. (See Example 4.3.)

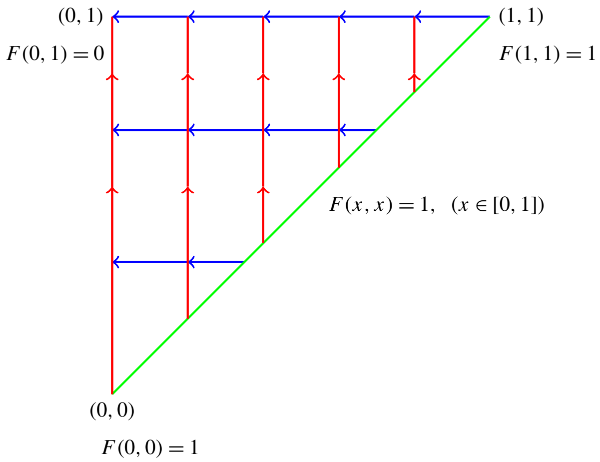

Fig. 2. Behavior of a restricted equivalence function F on the set $\{(x, y) : 0 \leq x \leq y \leq 1\}$.

Example 4.3. Consider the REF function $F : [0, 1] \times [0, 1] \to [0, 1]$ given by

$$F(0, 1) = F(1, 0) = 0; \quad F(x, x) = 1 \quad (x \in [0, 1]); \quad F(x, y) = \frac{1}{2} \text{ otherwise.}$$

By, the way, despite being too naive, this restricted equivalence function F is compatible with any strong negation N defined in the unit interval $[0, 1]$.

Let us analyze now the structure of restricted equivalence functions with good additional properties.

Thus, unless otherwise stated we will assume that we are given a strong negation N on $[0, 1]$ as well as a strict and continuous restricted equivalence function F that is compatible with respect to N.

Since F is strict and continuous we may observe that the functions $f : [0, 1] \to [0, 1]$ and $g : [0, 1] \to [0, 1]$ respectively given by $f(x) = F(x, 1)$ and $g(x) = F(0, x)$ $(x \in [0, 1])$ are, respectively an automorphism and an anti-automorphism from the unit interval $[0, 1]$ onto itself.

Knowing N, and being a its fixed point ($N(a) = a$), by compatibility of F with respect to N we only need the values of $F(x, y)$ in the trapezoid limited by the lines $x = 0$, $y = x$, $x = a$ and $y = 1$ (see Fig. 3) to retrieve the whole function F. As a matter of fact, if $x < y$ and $x > a$ we have that $N(y) < N(x)$ and, in addition, $N(y), N(x) \in [0, a]$ because N maps $[a, 1]$ onto $[0, a]$.

Moreover, if we know that F is a strict and continuous REF, and it is compatible as regards the strong negation N, the knowledge of F helps us to retrieve N, as follows (see Proposition 3.21 and Fig. 4):

$$N(x) = t \in [0, 1] \text{ such that } g(t) = f(x) \quad (x \in [0, 1]).$$

In other words, we systematically use the equality $F(x, 1) = F(0, N(x))$ that comes from compatibility and the properties of a REF. Observe that $N(x)$ is unique because F is, by hypothesis, strict and continuous.

Remark 4.4. The conditions of being strict and continuous can not be dropped if we want the strong negation N to be well defined and univocally determined by the above procedure. For instance, if we consider the REF function F introduced in Example 4.3 we may observe that, by its own definition, F is compatible with any strong negation N that may be defined in $[0, 1]$.

Taking into account Definition 2.11, if a strict and continuous REF function F is compatible with respect to a strong negation N, on the surface defined by F in the unit cube (see Fig. 5 below) we may consider a special curve, namely that of the points $F(x, N(x))$, $(x \in [0, 1])$. This line is said to be the *normal function curve* relative to F and N.

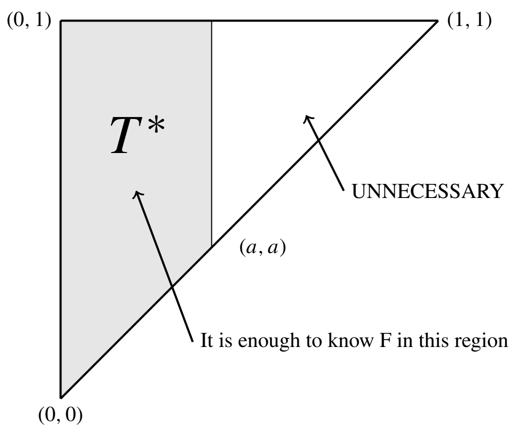

Fig. 3. It is enough to know F in the region T^*.

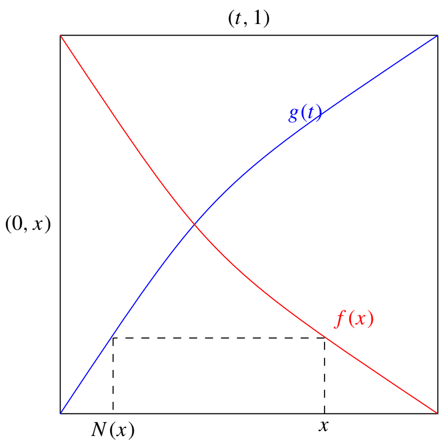

Fig. 4. Strong negation associated to a restricted equivalence function.

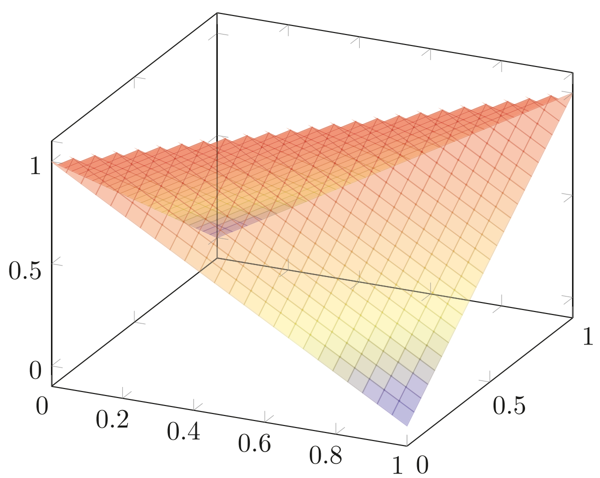

Fig. 5. $F(x, y) = 1 - |x - y|$ $(x, y \in [0, 1])$.

Remark 4.5. Unlike what happens to the naive restricted equivalence function introduced in Example 4.3, in general the compatibility of a REF function with respect to a strong negation N actually depends on the strong negation. In other words, if N and M are two different strong negations on the unit interval, and F is a restricted equivalence function, it may happen that F is compatible with N, but not with M.[8] See Example 4.6 below.

Example 4.6. Consider the restricted equivalence function F given by $F(x, y) = 1 - |x - y|$ $(x, y \in [0, 1])$ and the strong negations N and M respectively defined by $N(t) = 1 - t$ and $M(t) = \sqrt{1 - t^2}$ $(t \in [0, 1])$.

It is clear that $F(x, y) = 1 - |x - y| = 1 - |y - x| = 1 - |(1 - x) - (1 - y)| = F(N(x), N(y))$ $(x, y \in [0, 1])$. So F is compatible with respect to N.

However, taking $x = \frac{1}{4}$; $y = \frac{1}{2}$ we have that $F(\frac{1}{4}, \frac{1}{2}) = 1 - |\frac{1}{4} - \frac{1}{2}| = 1 - |(-\frac{1}{4})| = 1 - \frac{1}{4} = \frac{3}{4} = 0.75$, whereas $M(\frac{1}{4}) = \frac{\sqrt{15}}{4}$; $M(y) = \frac{\sqrt{3}}{2}$, so that $F(M(\frac{1}{4}), M(\frac{1}{2})) = F(\frac{\sqrt{15}}{4}, \frac{\sqrt{3}}{2}) = 1 - |\frac{\sqrt{15}}{4} - \frac{\sqrt{3}}{2}| = 1 - \frac{\sqrt{15}-\sqrt{12}}{4} = 1 - \frac{3.87-3.46}{4} = 0.8975$. Therefore, F is incompatible with the strong negation M.

We have already seen that if a strict and continuous REF, say F, is compatible with respect to a given strong negation N we may retrieve N from F.

At this stage, *we may wonder if given a strict and continuous restricted equivalence function F, there exists some strong negation N such that F is compatible with respect to N.*

The answer is negative.

Example 4.7. Define the restricted equivalence function $F : [0, 1]^2 \to [0, 1]$ as follows:

i) If $u \le v$, let $F(u, v) = 1 - a^2$ such that (u, v) belongs to the straight line whose equation is $Y = a + \frac{X}{1+a}$.
ii) If $u > v$, then define by symmetry $F(u, v) = F(v, u)$ where $F(v, u)$ is given as in case (i) above.

Assume the existence of a strong negation N such that F is compatible with N. Since $F(0, x) = F(N(x), 1)$ must hold for every $x \in [0, 1]$, we arrive at

$$x = a + \frac{0}{1 + a},$$

so that here $a = x$.

Also,

$$1 = x + \frac{N(x)}{1 + x},$$

so that $N(x) = 1 - x^2$.

However, the function $N : [0, 1] \to [0, 1]$ given by $N(x) = 1 - x^2$ $(x \in [0, 1])$, fails to be a strong negation, since it is not involutive. For instance, $N(\frac{1}{2}) = \frac{3}{4}$ and $N(N(\frac{1}{2})) = N(\frac{3}{4}) = \frac{7}{16} \neq \frac{1}{2}$.

Therefore, we may conclude that for this restricted equivalence function F, that is indeed continuous and strict, there is no strong negation N such that F is compatible with N.

Now we could analyze some additional conditions on a given restricted equivalence function F that may imply the existence of a strong negation N on the unit interval $[0, 1]$ such that F is compatible with respect to N.

However, and perhaps unfortunately, here we no longer have room for maneuver. If there were a strong negation N such that F is compatible with N, that N would be necessarily given in the terms of previous constructions already analyzed in Proposition 3.21 and Fig. 4. And, of course, F should meet the compatibility condition $F(x, y) = F(N(x), N(y))$ $(x, y \in [0, 1])$. In other words, once N has been described as in Proposition 3.21 and Fig. 4, if $F(x, y) = F(N(x), N(y))$ $(x, y \in [0, 1])$ holds true we can claim the existence of a strong negation on $[0, 1]$ such that F is compatible with it. Otherwise, provided that $F(x, y) \neq F(N(x), N(y))$ $(x, y \in [0, 1])$, we would have already exhausted all possibility for the restricted equivalence function F to be compatible with a strong negation.

[8] Perhaps this important feature should have been explicitly said in the definitions of a REF introduced in [5]. The negation c that appears in Definition 7 in [5] is not just anyone, it is given a priori, so fixed beforehand.

The following result summarizes the previous discussions and characterizes the restricted equivalence functions that are compatible with a strong negation.

Theorem 4.8. *Given a restricted equivalence function F and assuming that F is continuous and the functions f, g such that $f(x) = F(x, 1)$ and $g(x) = F(0, x)$ $(x \in [0, 1])$ are, respectively, an automorphism and an anti-automorphism from the unit interval $[0, 1]$ onto itself, and they are coupled, then there exists a strong negation N such that F is compatible with respect to N if and only if $F(x, y) = F(f^{-1}(F(x, 0)), f^{-1}(F(y, 0))$ holds true for every $x, y \in [0, 1]$). In this case $N(x) = f^{-1}(g(x)) = f^{-1}(F(x, 0))$ $(x \in [0, 1])$.*

Proof. First, observe that if F is a REF that is compatible with a strong negation N, we have that $F(x, 0) = F(N(x), N(0)) = F(N(x), 1) = f(N(x))$, for every $x \in [0, 1]$ so that $N(x) = f^{-1}(F(x, 0)) = f^{-1}(F(0, x)) = f^{-1}(g(x))$. Conversely, by Proposition 3.21, since f and g are coupled, the function $N : [0, 1] \to [0, 1]$ given by $N(x) = f^{-1}(g(x))$ $(x \in [0, 1])$ is a strong negation. Moreover $F(x, y) = F(f^{-1}(F(x, 0)), f^{-1}(F(y, 0)) = F(f^{-1}(F(0, x)), f^{-1}(F(0, y)) = F(f^{-1}(g(x)), f^{-1}(g(y)) = F(N(x), N(y))$ holds true for any pair $(x, y) \in [0, 1]^2$, so that F is compatible with respect to N. □

5. Restricted equivalence functions vs. admissible distances

Inspired by applications into Image Processing, the concept of a restricted equivalence function was introduced in the specialized literature (see [5]) having in mind classical concepts of fuzzy logic, and, in particular, logical connectives as implications or negations. This is a typical approach in Theoretical Computer Science. Nevertheless, if we ask an analyst or topologist about which could be the device most frequently used to compare points that belong to an abstract set, a quite natural answer would involve topological concepts as neighborhoods, distances, metric spaces or even uniformities (see e.g. [12]).

Both approaches are not disparate. They have points in common (see e.g. [8]) that, in the approach we follow throughout the present paper, should be analyzed in detail. This is the aim of this Section 5.

As a matter of fact, unlike a restricted equivalence function F, that is telling us how two points are alike, so that $F(x, x) = 1$ for every $x \in [0, 1]$, a distance d is measuring just the opposite, that is, how different they are, so that in particular $d(x, x) = 0$.

Remark 5.1. Other topological aspect of a restricted equivalence function F is that it defines a topology τ on the unit interval $[0, 1]$. Given a point $x \in [0, 1]$ a basis of neighborhoods of x, say $\tau(x)$, with respect to that topology is $\tau(x) = \{\tau_\epsilon(x), \epsilon > 0\}$, with $\tau_\epsilon(x) = \{y \in [0, 1] : F(x, y) > 1 - \epsilon\}$. In the present paper we leave as an open problem to explore the main features and properties of this topology τ.

A suitable distance will give rise to restricted equivalence functions, as follows:

Proposition 5.2. *Let d stand for a distance defined on $[0, 1]$ such that it satisfies the following properties:*

i) $d(x, y) = 1 \Leftrightarrow (x, y) = (0, 1) \vee (x, y) = (1, 0)$,
ii) $d(x, y) \le 1$ *for every* $(x, y) \in [0, 1]^2$,
iii) *d is admissible as regards the usual Euclidean order in $[0, 1]$, that is, given $x \le y \le z \in [0, 1]$ it holds true that $d(x, y) \le d(x, z)$.*

Then, for any strong negation N defined on $[0, 1]$, the function $F : [0, 1]^2 \to [0, 1]$ given by $F(x, y) = N(d(x, y))$ $(x, y \in [0, 1])$ is a restricted equivalence function.

Proof. Given $x, y \in [0, 1]$, it is clear that $F(x, y) = F(y, x)$, because d is symmetric. Moreover $F(x, y) = 0 \Leftrightarrow N(d(x, y)) = 0 \Leftrightarrow d(x, y) = 1$, and similarly $F(x, y) = 1 \Leftrightarrow d(x, y) = 0 \Leftrightarrow x = y$. Finally, if $x \le y \le z$ we have that $d(x, y) \le d(x, z) \Leftrightarrow F(x, y) = N(d(x, y)) \ge N(d(x, z)) = F(x, z)$. □

Remark 5.3. It is true that this Proposition 5.2 can be considered as direct or straightforward because the most sufficient conditions on d have been assumed by just negating the axioms of restricted equivalence function introduced in Definition 2.7. However we have finally decided to include it, since we think that an account of properties and relationships between restricted equivalence functions and distances is relevant to be, at least, exposed. The same comment also applies to Proposition 6.3 and Proposition 6.4 in next section, dealing with relationships between restricted equivalence functions and implications.

Suppose now that we start with a restricted equivalence function F that is compatible with a strong negation F. We wonder when the dissimilarity $D : [0, 1]^2 \to [0, 1]$ given by $D(x, y) = N(F(x, y))$ $(x, y \in [0, 1])$ is actually a distance. At this stage, it is clear that all the properties that define a distance are met, except maybe the triangular inequality, namely

$$D(a, b) + D(b, c) \geq D(a, c) \quad (a, b, c \in [0, 1]).$$

Here, we only know a priori that given $x \leq y \leq z \in [0, 1]$ it holds true that $F(x, y) \geq F(x, z)$. Since N is strictly decreasing, $F(x, y) \geq F(x, z) \Leftrightarrow D(x, y) \leq D(x, z)$. Thus, if D indeed were a distance, it should be admissible with respect to the usual order $\leq$ on the unit interval.

Moreover, for any given $a, b, c \in [0, 1]$, the following six situations may appear:

i) $a \leq b \leq c$,
ii) $a \leq c \leq b$,
iii) $b \leq a \leq c$,
iv) $b \leq c \leq a$,
v) $c \leq a \leq b$,
vi) $c \leq b \leq a$.

Due to the properties of F, in all the situations except maybe i) and vi) we already have that $D(a, b) + D(b, c) \geq D(a, c)$. However in the first or sixth situations we cannot guarantee that the triangular inequality is satisfied.

Remark 5.4. Indeed, there are practical situations in real life in which we can have a D that does not meet this triangular inequality in the first and sixth situations. This can occur in applications that have to do with computer security. For instance, in actions in which the State Administration asks us to enter our data in a document sometimes we are forced to turn off the computer after some pre-established period of time, and restart then. If, let us say, the data uploading will last forty minutes, it is very possible that the system forces us to save the data already entered after the first twenty minutes, turn off the computer, and then start over by entering the data that still have to be uploaded and saved, in the last twenty minutes. This is done for security reasons, because the longer the application is open, the more vulnerable it is to cyber attacks. And in a situation like this, in this way, two periods of twenty minutes can be much safer than one of forty minutes. If $x \leq y$ represent time instants and $D(x, y)$ is a measure of computer vulnerability in the interval of time $[x, y]$, it is quite reasonable to think that perhaps $D(a, b) + D(b, c)$ could actually be smaller than $D(a, c)$ if $[a, b]$ and $[b, c]$, are intervals of time shorter than $[a, c]$.

As already seen in Remark 2.10, the satisfaction of the triangular inequality by D may depend on some functional equation or inequality satisfied by F. But also, it could depend on the strong negation N involved. As commented then, if $N(x) = 1 - x$ $(x \in [0, 1])$ and F satisfies $F(x, y) + F(y, z) - F(x, z) \geq 1$ $(x, y, z \in [0, 1])$ then the dissimilarity D given by $D(x, y) = 1 - F(x, y)$ $(x, y \in [0, 1])$ is a distance, and it is admissible with respect to the usual order $\leq$ on $[0, 1]$. With other different strong negations, the corresponding functional inequality to be satisfied by F could be much more complicated.

For instance, with $N(x) = \sqrt{1 - x^2}$ $(x \in [0, 1])$, if F satisfies

$$2\sqrt{1 - (F(x, y))^2 - (F(y, z))^2 + (F(x, y))^2(F(y, z))^2} \geq$$
$$(F(x, y))^2 + (F(y, z))^2 - (F(x, z))^2 - 1 \quad (x, y, z \in [0, 1]),$$

then the corresponding dissimilarity D is a distance, too.

Summarizing, from a distance d that is admissible as regards the usual total order $\leq$ on the unit interval it is direct the obtention of restricted equivalence functions, as stated in Proposition 5.2 above. Conversely, given a restricted

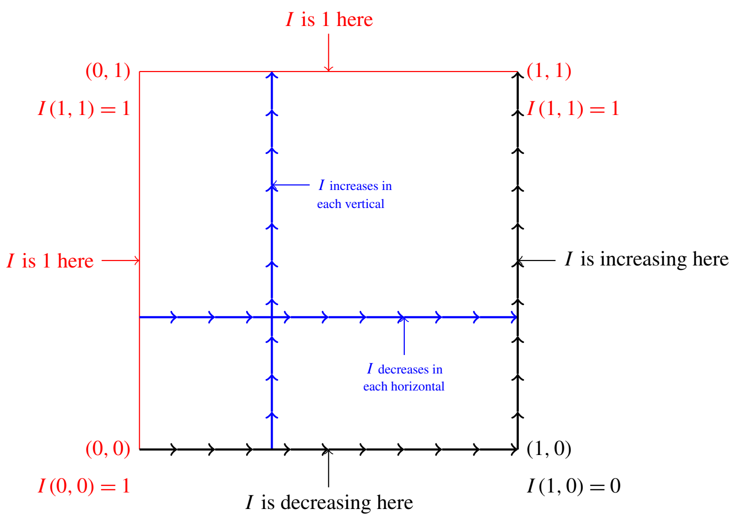

Fig. 6. Properties of an implication.

equivalence function F that is compatible with a strong negation N, in order for the corresponding dissimilarity $D = N \circ F$ to be a distance a – perhaps complicated – functional inequality should be accomplished. Thus, a first glance, one could say that distances that are admissible with respect to $\leq$ are *particular cases* of dissimilarities that come from a restricted equivalence function. The converse is not true in general. In other words, the concept of a restricted equivalence function introduced in [5,6] is less restrictive, in order to measure similarity between points, than the concept of a distance, even when the distance is admissible as regards the usual order $\leq$. Therefore, the restricted equivalence functions can be considered a reasonable and useful device in the applications of this theory.

6. Restricted equivalence functions vs. implication operators

As analyzed in the previous sections, restricted equivalence functions are closely related to some classical logical connectives in the fuzzy setting, namely strong negations. However, as already considered in several papers in this literature (see e.g. [5,6,8]), restricted equivalence functions can be also obtained from some particular class of *implication* operators (see Definition 6.1 below).

Definition 6.1. A *fuzzy implication operator* is a function $I : [0,1]^2 \to [0,1]$ that satisfies the following properties:

i) $I(0,0) = I(0,1) = I(1,1) = 1$,
ii) $I(1,0) = 0$,
iii) If $x \leq y$ then $I(x,z) \geq I(y,z)$ $(x, y, z \in [0,1])$,
iv) If $x \leq y$ then $I(z,x) \leq I(z,y)$ $(x, y, z \in [0,1])$.

(See Fig. 6.)

Given a strong negation N defined on $[0,1]$, a fuzzy implication operator I is said to be *compatible with respect to N* if for any $x, y \in [0,1]$ it holds true that $I(x,y) = I(N(y), N(x))$. This is also called the *contraposition property* with respect to N.

Remark 6.2. As commented and analyzed in [5] many other properties (or axioms) have been used in the literature to define fuzzy implications (see e.g. [3]). In fact, some of these additional axioms give rise to several non-equivalent definitions. Depending on the contexts and applications different types of fuzzy implications have been considered in each case.

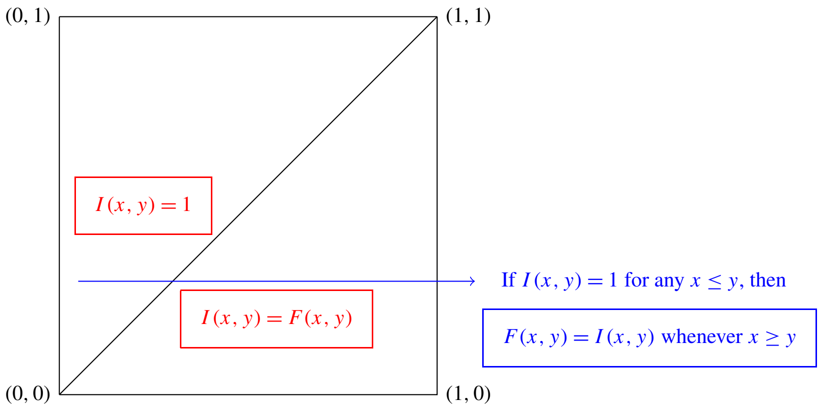

Fig. 7. Implication operators that induce restricted equivalence functions.

Taking into account Definition 2.7, Definition 6.1 and Fig. 6, we may relate restricted equivalence functions and fuzzy implication operators as follows (see also Proposition 7 in [5]):

Proposition 6.3. *Let $I : [0, 1]^2 \to [0, 1]$ be a fuzzy implication such that $I(x, y) = 0 \Leftrightarrow x = 1$ and $y = 0$, and in addition $I(x, y) = 1 \Leftrightarrow x \leq y$ $(x, y \in [0, 1])$. Then the map $F : [0, 1]^2 \to [0, 1]$ given by*

$$F(x, y) = \min\{I(x, y), I(y, x)\} \quad (x, y \in [0, 1])$$

is actually a restricted equivalence function.

Moreover, if I is compatible with respect to a strong negation N, so is F.

Proof. (This result was essentially proved in [5], Proposition 7. We include a proof here for the sake of completeness.)

Notice that F is symmetric by definition. Given $x, y \in [0, 1]$, if $F(x, y) = 1$, by definition we have both $I(x, y) = 1$ and $I(y, x) = 1$. Thus $x \leq y$ and $y \leq x$. Hence $x = y$. Also, if $F(x, y) = 0$ we have that either $I(x, y) = 0$ or $I(y, x) = 0$, so that $(x, y) = (1, 0)$ or else $(y, x) = (1, 0)$. Finally, if $x \leq y \leq z$, we have that and $I(y, x) \geq I(z, x)$. Moreover $I(x, y) = I(y, z) = I(x, z) = 1$. This implies, in particular, that $F(x, y) = I(y, x)$ and $F(x, z) = I(z, x)$. Therefore $F(x, y) \geq F(x, z)$ and we are done.

Moreover, if I is compatible with N, given $x, y \in [0, 1]$ it follows that:

$$F(x, y) = \min\{I(x, y), I(y, x)\} = \min\{I(N(y), N(x)), I(N(x), N(y))\} =$$
$$= \min\{I(N(x), N(y)), I(N(y), N(x)) = F(N(x), N(y)).$$

So F is compatible as regards the strong negation N. □

The converse of Proposition 6.3 is also true, in the following sense (see Fig. 7):

Proposition 6.4. *Let $F : [0, 1]^2 \to [0, 1]$ be a restricted equivalence function on the unit interval $[0, 1]$. Define the function $I : [0, 1]^2 \to [0, 1]$ by declaring that $I(x, y) = 1$ if $x \leq y$ and $I(x, y) = F(x, y)$ if $x > y$ $(x, y \in [0, 1])$. Then I is a fuzzy implication operator. If, in addition, F is compatible with respect to a strong negation N, so is I.*

Proof. It follows from Definition 2.7, Definition 6.1 and Fig. 6, and it is entirely analogous to the proof of Proposition 6.3, so it will be omitted here. (See also Proposition 7 in [5].) □

7. Discussion

It seems necessary to study in a total abstract manner, the analogous of strong negations and restricted equivalence functions when defined on a nonempty set X. Here X has no structure given a priori. The corresponding additional structures (e.g., a partial order, a topology, etc.) will be implemented then in order to define the corresponding concepts and equations in parallel. Needless to say, particular cases different from $X = [0, 1]$ of this more general setting have already been analyzed in the literature (see e.g. [2,4,7]).

To put just an example, if we have in mind, say, the motivation of the introduction of the concept of a restricted equivalence function (REF) from the study of images in different scales of gray, if we think of color (RGB) pictures, we may see that now a picture can be understood as a map $G : [0, 1]^2 \to [0, 1]^3$ so that given $(x, y) \in [0, 1]^2$, now $G(x, y) = (a, b, c)$ where, respectively, a, b and c correspond to the amount of red (respectively: of green, of blue), graded between 0 and 1, at the point (x, y). At this stage, we should interpret in a suitable way what a "negation" of (a, b, c) could be. Thus, if we call $L = [0, 1]^3$ we should extend the concept of a negation by dealing with functions from L^2 into L accomplishing some properties that could remind us those that appear in Definition 2.3. However, to do so, we may immediately notice that $L = [0, 1]^3$ has not the same order structure as, just, the unit interval $[0, 1]$. Indeed, on $[0, 1]^3$ we might perhaps consider the *partial order* $\leq_3$ so that $(a, b, c) \leq_3 (a', b', c') \Leftrightarrow (a \leq a') \wedge (b \leq b') \wedge (c \leq c') \quad (a, b, c, a', b', c' \in [0, 1])$. But, being $\leq_3$ only partial, this would force us to interpret concepts as "decreasing" or "increasing" in a new suitable way.

Here is a suitable option to do so when working with $L = [0, 1]^3$:

Definition 7.1. A linear order $\preceq$ defined on $L = [0, 1]^3$ is said to be *admissible* (see e.g. [2,11]) if it extends the partial order $\leq_3$. That is, $\leq_3 \subset \preceq$ so that $(a, b, c) \preceq (a', b', c') \Rightarrow (a, b, c) \leq_3 (a', b', c')$ holds true for every $a, b, c, a', b', c' \in [0, 1]$.

Example 7.2. Let $f_1, f_2, f_3 : [0, 1]^3 \to [0, 1]$ be three continuous functions such that the map $F : [0, 1]^3 \to [0, 1]^3$ given by $F(x, y, z) = (f_1(x, y, z), f_2(x, y, z), f_3(x, y, z)) \quad (x, y, z \in [0, 1])$ is injective. Assume also that f_i $(i = 1, 2, 3)$ is strictly increasing in each variable with respect to the usual Euclidean order $\leq$. Then the order $\preceq_F$ defined on $[0, 1]^3$ as the lexicographic linear order induced by F, that is, $(x, y, z) \preceq_F (a, b, c) \Leftrightarrow (f_1(x, y, z) < f_1(a, b, c)) \vee (f_1(x, y, z) = f_1(a, b, c) \; ; \; f_2(x, y, z) < f_2(a, b, c)) \vee (f_1(x, y, z) = f_1(a, b, c) \; ; \; f_2(x, y, z) = f_2(a, b, c) \; ; \; f_3(x, y, z) \leq f_3(a, b, c)) \quad (x, y, z, a, b, c \in [0, 1])$ is actually an admissible ordering on $L = [0, 1]^3$.

A particular case of this situation appears when given $(x, y, z) \in L = [0, 1]^3$ we take $f_i(x, y, z) = a_i x + b_i y + c_i z$ with $0 \leq a_i, b_i, c_i \leq 1$; $a_i + b_i + c_i = 1$ $(i = 1, 2, 3)$ and

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \neq 0 \, .$$

However, in a much more abstract setting, with a nonempty set X and perhaps to ordering defined a priori, some conditions that were considered to introduce concepts as strong negations or restricted equivalence functions on $[0, 1]$ (e.g. $F(x, y) \geq F(x, z)$ if $x \leq y \leq z$ $(x, y, z \in [0, 1])$ are now void, or should be redefined someway.

The new definitions should be as close as possible to the typical concepts arising in relation to $[0, 1]$ or to a lattice L when working in the fuzzy approach. If we analyze the definitions of a strong negation and a restricted equivalence function on the unit interval $[0, 1]$ we observe that some of the conditions involved are actually functional equations (e.g.: $F(x, y) = F(N(x), N(y))$ $(x, y \in [0, 1])$).

In a more general and abstract setting we may also expect that the analogous concepts be also closely related to some functional equation, but now defined on an abstract set X. The properties of those functional equations, when established in a general abstract setting, will be valid for any particular case of a lattice L as well as for the typical fuzzy sets and systems that lean on the unit interval $[0, 1]$. Also, as studied in [10], solutions of suitable functional equations on a set can also be used to define and understand better some kinds of binary relations defined on the set. We may notice, for instance, that a strong negation N defined on $[0, 1]$ can indeed be interpreted as a *fuzzy* binary relation on $[0, 1]$, so taking values in the whole interval $[0, 1]$ instead of, just, in $\{0, 1\}$. In the particular case of $[0, 1]$ those binary relations coming from functional equations could give rise to some special kind of modifiers (see e.g. [22], pp. 111 and ff.).

When dealing with applications as image processing in color (RGB system), instead of working with the unit interval, the space considered is $L = [0, 1]^3$, that endowed with the partial order $\leq_3$ is a lattice with minimum $(0, 0, 0)$ and maximum $(1, 1, 1)$. In order to compare triplets $(a, b, c), (a', b', c') \in L$ we should extend the concept of a REF

(a) img A (b) img B (c) img out (d) img out 2

Fig. 8. In Fig. 8a we see the image (named A) of a flying plane. In Fig. 8b we see a modified version of Fig. 8a (named B). Fig. 8c comes from applying Algorithm 1 with input images A and B, and choosing as restricted equivalence function the one presented in Example 4.2, namely $F(x,y) = 1 - |x-y|$ if $|x-y| < \frac{1}{2}$; $F(x,y) = \frac{1}{3}(1-|x-y|)$ otherwise. Finally, Fig. 8d is the result of applying Algorithm 1 with input images A and B, but now choosing as REF the classical $F(x,y) = 1 - |x-y|$; $x, y \in [0,1]$.

to functions that may go[9] from L^2 to [0, 1]. However, in this theory after studying how different two given triplets are, in practice it is necessary to aggregate them in a suitable way, getting accordingly a new triplet (α, β, γ) satisfying certain restrictions ad hoc. These restrictions, in some particular studies, look like those in Definition 2.7. Anyways, all this forces us to consider functions that go from L^2 into L, or perhaps suitable restrictions of positive operators (see [1]) between the Riesz spaces[10] $\mathcal{L}^2$ and $\mathcal{L}$, where $\mathcal{L} = \mathbb{R}^3$, and also to extend the concept of a REF to functions that go from L^2 (or $\mathcal{L}^2$) to L (or to $\mathcal{L}$).

We leave open the generalization of concepts as strong negations and restricted equivalence functions to a totally abstract setting, namely starting from a nonempty set X. This would be the starting point for further research on these topics.

8. Illustrative example

The aim of this section is to illustrate the utility of the restricted equivalence functions, introduced in the previous sections, by means of an explanatory example. In particular, and in the same way as in [25], we show how those functions can detect differences between two grayscale images. To that end, we are going to show how they can be used to detect new objects in a given image.

Let A be a grayscale image of size $n \times m$ and $P_{A_{ij}}$ be the intensity value on [0, 1] of the ij-th pixel (situated on the i-th row and j-th column) of A. Let B be another grayscale image of size $n \times m$. Then, A and B can be compared pixel to pixel with respect to a restricted equivalence function, using the next algorithm:

Algorithm 1: Compare all pixels of the images A and B, pixel to pixel.

Input: Two grayscale images of equal size A and B.
Result: A grayscale image with size equal to the input images.
Choose a REF to compare intensities between pixel values;
for *each pixel position* (i, j) **do**
  Compare pixel intensities $P_{A_{ij}}$ and $P_{B_{ij}}$ using the chosen restricted equivalence function;
  Assign the resulting value to the ij-th pixel of the output image;
end

For this illustrative example (see Fig. 8) we have taken an image from the Berkeley Segmentation Dataset [20]. This dataset consists of color images, but we have reconverted the given image to grayscale and used it as input image A (see Fig. 8a). By contrast, image in Fig. 8b is a modified version of image A, in which a second plane has been manually added to the background.

9 This is the spirit of the so-called indistinguishability operators (see e.g. [21]).

10 A real vector space E endowed with a partial order $\preceq$ is said to be an *ordered vector space* provided that $x \preceq y \Rightarrow x + z \preceq y + z$ and $ax \preceq ay$ hold true for every $x, y, z \in E$ and $a > 0$. A *Riesz space* is an ordered vector space E such that the partial order $\preceq$ is latticial, that is, for every $x, y \in E$ the supremum and the infimum of the set $\{x, y\}$ both exist in E. An element x of a real vector space is said to be *positive* if $\bar{0} \preceq x$, where $\bar{0}$ is the null element in E. An operator $T : E \to E'$ between two ordered vector spaces E and E' is said to be *positive* if it maps positive elements of E into positive elements of E'.

Fig. 8c shows the output image obtained after applying the Algorithm 1 choosing the restricted equivalence function considered in Example 4.2. Otherwise, Fig. 8d shows the output image obtained after applying the Algorithm 1 when choosing the REF given by $F(x, y) = 1 - |x - y|; \quad x, y \in [0, 1]$.

As seen in Fig. 8, differences between the results we have got are pretty close. However, while in Fig. 8d the luminosity is homogeneous among all parts of the object, in Fig. 8c (which is in general clearer, although it presents small dark areas), the contrast is higher [9]. This may be of interest when trying to detect the most distinct areas of two images, thus, giving less importance to the most subtle differences. The capability of choosing different restricted equivalence functions allows us to get greater tunability and adaptability of the algorithm to different problems.

9. Conclusion

This article being mainly theoretical, we have revised the concepts of a strong negation and a restricted equivalence function in $[0, 1]$ from an analytical and topological point of view. This allows us to get new characterizations, as well as old ones but proved in a much more direct manner. Moreover, this topological approach also conciliates the use of techniques coming from fuzzy logic, mainly using devices related to logical connectives as negations or implications, with the consideration of other analytical and topological devices as, for instance, distances.

Several open problems or lines for future research come to our mind now. To put just a few examples, we could explore the main features of topologies induced by a restricted equivalence function in $[0, 1]$. Are they metrizable? Also, we may consider functions defined on $[0, 1] \times [0, 1]$ into $[0, 1]$ that also come from implication operators, but, unlike Proposition 6.3, they involve triangular norms different from the minimum. For instance, if we are given an implication operator I and a t-norm T, we may wonder whether or not the function $F : [0, 1]^2 \to [0, 1]$ given by $F(x, y) = T(I(x, y), I(y, x)) \ \ (x, y \in [0, 1])$ is a restricted equivalence function on the unit interval $[0, 1]$.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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