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Why using topological and analytical methods in aggregation of fuzzy preferences?

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Summary

The Arrow's Impossibility Theorem states that there is no function fusing individual preferences into a social one satisfying certain properties of *"common sense"*. On the contrary, in some of the **fuzzy extensions of the Arrovian model, possibility arises**.
We have developed a technique which has been able to prove new impossibility results in the fuzzy approach. In this poster, we will explain the fundaments of this technique and in which models we can apply it.
This technique, is based on **controlling the aggregation of fuzzy preferences through some aggregation functions of dichotomic preferences**. For each fuzzy aggregation function, we get a family of dichotomic aggregation functions. Studying this family, we obtain information about the initial aggregation function.
We will discuss why the fuzzy Arrovian models in which we can apply this technique are, in some sense, less fuzzy. Moreover, we will expose why **we should use topological and analytical methods** in the fuzzy models out of the scope of our technique.

Classic Arrovian model and Theorem of Impossibility

Let X be the set including all alternatives involved in a decision. They can be ordered by using binary relations satisfying certain properties. Particularly, in the Arrovian model, these binary relations are total preorders (reflexive, transitive and complete binary relations).

To give a total preorder on X is equivalent to give a ranking with ties on X.

Every binary relation $\succsim$ factors into the relations $\succ$ and $\sim$ defined as:

$$x \succ y \Leftrightarrow x \succsim y \wedge \neg(y \succsim x)$$
$$x \sim y \Leftrightarrow x \succsim y \wedge y \succsim x$$

and these binary relations are read as:

$x \succsim y$: x is at least as good as y
$x \succ y$: x is better than y
$x \sim y$: x and y are equally preferred

Arrow in [1] proved that given a finite set of agents $N = \{1, \dots, n\}$, each one expressing their preferences over a set of alternatives X with total preorders, there is no *"fair"* rule which aggregates all individual preferences obtaining a social one. Formally, if the set of all total preorders on X is denoted by $\mathcal{O}_X$:

Arrow's Impossibility Theorem: There is no function $f : {\mathcal{O}_X}^n \to \mathcal{O}_X$ on a set of alternatives with $|X| \geq 3$ satisfying for every $x, y \in X$ and profiles $\succsim, \succsim' \in {\mathcal{O}_X}^n$, the following conditions:

- Paretian: $\forall i \in N \; x \succ_i y \Rightarrow x \succ_{f(\succsim)} y$
- Independence of irrelevant alternatives (IIA):
$$\left[\forall i \in N \;\; \succsim_{i\rceil\{x,y\}} = \succsim'_{i\rceil\{x,y\}}\right] \Rightarrow f(\succsim)_{\rceil\{x,y\}} = f(\succsim')_{\rceil\{x,y\}}$$
- Non dictatoriship: $\nexists k \in N \left[x \succ_k y \Rightarrow x \succ_{f(\succsim)} y\right]$

Could Arrovian impossibility be walked around by using fuzzy preferences instead of dichotomic ones?

In the fuzzy setting, a preference is a fuzzy binary relation $R : X \times X \to [0,1]$. There are many generalizations of the crisp strict preference $\succ$ (of $\succsim$) to the fuzzy strict preference P_R (of R). For every fuzzy Arrovian model, we have to set a method of factorization.

Properties of preferences $\succsim$ can be generalized to the fuzzy setting in different ways. For example:

$$\text{Transitivity} \to \begin{cases} T\text{-transitivity (with } T \text{ a t-norm)} \\ \quad [\forall x, y, z \in X \; R(x,z) \geq T(R(x,y), R(y,z))] \\ \text{Weak transitivity} \\ \quad [\forall x, y, z \in X \; R(x,y) \geq R(y,x) \wedge R(y,z) \geq R(z,y) \Rightarrow R(x,z) \geq R(z,x)] \end{cases}$$

$$\text{Completeness} \to \begin{cases} S\text{-connected (with } S \text{ a t-conorm)} \\ \quad [\forall x, y \in X \; S(R(x,y), R(y,x)) = 1] \end{cases}$$

Let $\mathcal{FP}$ be a set of fuzzy preferences on X. An **aggregation fuzzy rule** is a function $f : \mathcal{FP}^n \to \mathcal{FP}$. Arrow axioms can also be generalized in various ways. For example:

$$\text{Paretian} \to \begin{cases} \text{Weak Pareto: } \forall x, y \in X \; P_i(x,y) > 0 \Rightarrow P_{f(\mathbf{R})}(x,y) > 0 \\ \text{Strong Pareto: } \forall x, y \in X \; P_{f(\mathbf{R})}(x,y) \geq \min_{i \in N} P_i(x,y) \end{cases}$$

$$\text{Dictatorship} \to \begin{cases} \text{Dictatorship: } \exists k \in N \; P_k(x,y) > 0 \Rightarrow P_{f(\mathbf{R})}(x,y) > 0 \\ \alpha - \text{dictatorship: } \exists k \in N \; \forall t \in [0,1] P_k(x,y) > t \Rightarrow P_{f(\mathbf{R})}(x,y) > t \end{cases}$$

$$\text{IIA} \to \forall x, y \in X [\forall i \in N R_i \approx_{\{x,y\}} R'_i \Rightarrow f(\mathbf{R}) \approx_{\{x,y\}} f(\mathbf{R}')]$$

were $\approx_{\{x,y\}}$ can be defined as:

- $R \approx^1_{\{x,y\}} R' \Leftrightarrow R_{\rceil\{x,y\}} = R'_{\rceil\{x,y\}}$
- $R \approx^2_{\{x,y\}} R' \Leftrightarrow supp(R_{\rceil\{x,y\}}) = supp(R'_{\rceil\{x,y\}})$
- $R \approx^3_{\{x,y\}} R' \Leftrightarrow R \approx^2_{\{x,y\}} R' \wedge [\forall \overline{z}, \overline{z}' \in \{x,y\}^2 R(\overline{z}) > R(\overline{z}') \Leftrightarrow R'(\overline{z}) > R(\overline{z}')]$

Studying fuzzy aggregation using crisp preferences

Consider a set of fuzzy preferences $\mathcal{FP}$ were all the preferences are reflexive and satisfy one type of fuzzy transitivity and one type of fuzzy connectedness. Then, we define a projection p from $\mathcal{FP}$ to a set of crisp preferences $\mathcal{B}$ on X. **These projections are interpreted as collapsing the fuzzy preferences** into its qualitative factor (a crisp binary relation). Some examples of projections are:

p_1) If R is a weak transitive and S-connected preferences, $\succsim^1_R$ defined as $x \succsim^1_R y \Leftrightarrow R(x,y) \geq R(y,x)$ is a total preorder.

p_2) If R is a T-transitive and max-connected preference, $\succsim^2_R$ defined as $x \succsim^2_R y \Leftrightarrow R(x,y) = 1$ is a total preorder.

p_3) If R is a min-transitive and S-complete preference, $\succsim^3_R$ defined as $x \succsim^3_R y \Leftrightarrow R(x,y) \geq R(y,x)$ is a quasi-transitive binary relation.

$$\begin{array}{ccc} \mathcal{FP}^n & \xrightarrow{\quad f \quad} & \mathcal{FP} \\ \uparrow \iota_1 \times \cdots \times \iota_n & & \downarrow p \\ \mathcal{B}^n & \xrightarrow{\quad f_\iota \quad} & \mathcal{B} \end{array}$$

The second steep is finding the same but applied to aggregation functions. Here, given a fuzzy aggregation function f and n embeddings $\iota_i : \mathcal{B} \to \mathcal{FP}$, we define $f_\iota := p \circ f \circ (\iota_i \times \cdots \times \iota_n)$. We have to choose the right embeddings in order to f_ι be an Arrovian aggregation function. **Then each f_ι is dictatorial**. However they may have different dictators. When all of them have the **same dictator**, and the image of all **embeddings cover $\mathcal{FP}$**, we can assure that f is dictatorial.

Let $\mathcal{P}$ be the set of weak transitive and S-connected fuzzy preferences on X. Using the strategy above, we proved in [7] the following theorem:

Theorem: Let $f : \mathcal{P}^n \to \mathcal{P}$ be a fuzzy aggregation function satisfying IIA defined by $\{\approx^3_{\{x,y\}}\}_{x,y \in X}$ and weakly Paretian, then f is dictatorial.

The theorem above is an example about when we can reduce the study of a fuzzy model to the study of a family of crisp functions from the Arrovian model (and we obtain an impossibility result), then the fuzziness of the model is an illusion.

Aggregation funcions using ordinal expressions

These *illusory fuzziness* arises when we study the fuzzy Arrovian aggregation functions in the literature. We can consider some of these expressions:

$$\text{In [5]:} \quad f(\mathbf{R})(x,y) = \begin{cases} 1 & \text{if } \forall i \in N \; R_i(x,y) > R_i(y,x) \\ 0.5 & \text{otherwise} \end{cases} \tag{1}$$

$$\text{In [6]:} \quad f(\mathbf{R})(x,y) = \frac{1}{n} \sum_{i \in N} R_i(x,y) \tag{2}$$

$$\text{In [4]:} \quad f(\mathbf{R})(x,y) = med\left\{\min_i\{R_i(x,y)\}, h, \max_i\{R_i(x,y)\}\right\} \quad (\text{where } T(h,h) = 0) \tag{3}$$

Notice that in the functions 1 and 3 the same expression we used in $\succsim^1_R$ and $\succsim^3_R$ is employed, and 2 is the well-known arithmetic mean. These three examples represent the present situation in the existing literature. **All functions are built using the reasoning based on crisp binary relations** or **testing pre-existing well-known algebraic expressions** as means.
If we look for functions capturing the vagueness, **we should think out of the box of crisp binary relations**. Moreover, testing the functions with an algebraic expression we know does not seem a suitable method.

Conclusions

- In order to get more satisfactory results and classify the fuzzy Arrovian models, we can not rely on functions build as algebraic expressions or close to binary relations. **We need a richer framework able to express the vagueness**, and it can not be constrained by human dichotomic thinking.
- We propose **using topological or analytical tools** to build this general framework. Using the fact that the degrees of a preference are in $[0,1]$, we can interpret a preferences as a point in the cube $[0,1]^{X^2}$, the spaces of preferences as topological subspaces of $[0,1]^{X^2}$, and the aggregation functions as continuous functions (see [2] for an extended discussion).
- Using this framework, we expect to find suitable aggregation functions with no need to write them explicitly. For example, **using differential equations**.
- It is important to remark that our approach is different from the topological models proposed by Chichilnisky in [3]. We depart from a model with no topological structure, whereas Chichilnisky built her models using a topological background.

Future research

- Find a general framework to create suitable binary relation form fuzzy preferences and use them to study fuzzy aggregation functions.
- Continue the study initiated in [2] about how fuzzy Arrovian models can be translated to differential equations.

References

[1] K. J. Arrow. *Social Choice and Individual Values*. Cowles Commission Monograph No. 12. John Wiley & Sons, Inc., New York, N. Y.; Chapman & Hall, Ltd., London, 1951.

[2] M. J. Campión, E. Induráin, and A. Raventós-Pujol. Fusing Fuzzy Preferences in Contexts of Social Choice: Towards a Topological Structure. In C. Kahraman, S. Cevik Onar, B. Oztaysi, I. U. Sari, S. Cebi, and A. C. Tolga, editors, *Intelligent and Fuzzy Techniques: Smart and Innovative Solutions*, pages 1301–1308, Cham, 2021. Springer International Publishing.

[3] G. Chichilnisky. Social choice and the topology of spaces of preferences. *Advances in Mathematics*, 37(2):165–176, 1980.

[4] C. Duddy, J. Perote-Peña, and A. Piggins. Arrow's theorem and max-star transitivity. *Social Choice and Welfare*, 36(1):25–34, 2011.

[5] B. Dutta. Fuzzy preferences and social choice. *Mathematical Social Sciences*, 13(3):215–229, 1987.

[6] M. B. Gibilisco, A. M. Gowen, K. E. Albert, J. N. Mordeson, M. J. Wierman, and T. D. Clark. *Fuzzy Social Choice Theory*. Springer International Publishing, Cham, 2014.

[7] A. Raventos-Pujol, M. J. Campion, and E. Indurain. Arrow theorems in the fuzzy setting. *Iranian Journal of Fuzzy Systems*, 17(5):29–41, 2020.