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Pareto rationalizability by two single-peaked preferences☆

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ABSTRACT

We study, in a finite setting, the problem of Pareto rationalizability of choice functions by means of a preference profile that is single-peaked with respect to an exogenously given linear order over the alternatives. This problem requires a new condition to be added to those that characterize Pareto rationalizability in the general domain of orders (Moulin (1985)). This new condition appeals to the existence of a central range of options such that the choice function excludes alternatives which are distant from that range.

1. Introduction

This paper seeks to study axiomatically the class of choice functions that are Pareto rationalizable by a profile of single-peaked orders. By this we mean that the choice function selects those alternatives which are undominated in such a profile, where "undominated" means that there is no alternative which is superior according to all the orders in the profile. We interpret this problem in a standard social choice setting where the orders in the profile are individual single-peaked preferences over a finite set of alternatives.[1]

The topic of rationalizability of a choice function is classical in economics and choice theory since the pioneering papers by Samuelson (1938, 1948). Rational preferences (according to a principle, such as transitivity or acyclicity) are a starting point of the consumer theory in economics, and therefore a cornerstone in understanding and explaining the functioning of markets. However, preferences have the serious drawback of not being observable. Thus, broadly speaking, rationalization seeks to induce underlying preferences from the information given by the choices made by an individual, and establishes minimum conditions for the choices to lead to preferences that are rational. *Pareto* rationalizability extends the concept of rationalizability to a multicriterion setting. It typically applies to choices that are the outcome of the interplay of *several* preferences. According to the Pareto principle, an option x is chosen in a set if there is no alternative option in the set that is preferred to x according to all the preference orderings at stake. Thus, Pareto rationalizability accounts for the conditions that a choice function must satisfy to be interpreted as if there were a profile of underlying preferences such that the choices are Pareto optimal according to the Pareto principle.

A result by Moulin (1985) enables the conclusion to be drawn that a choice function is Pareto rationalizable by a profile of strict preferences if and only if it satisfies three well-known conditions in choice theory. (i) condition α, stating that every option that is selected from a set K should be selected from any subset of K that contains it; (ii) condition γ, saying that if an option is selected from two different sets it should also be selected from their union; and (iii) "Aizermann", which establishes that the subset of options chosen from a set K cannot expand when non-chosen options are removed from K. Noteworthy examples of later developments of the concept of Pareto rationalizability are Sprumont (2001), Echenique and Ivanov (2011) and Qi (2015).

On the other hand, single-peakedness of preferences is a natural, widespread assumption in many contexts where the alternatives are ordered along a certain dimension, for example when they are candidates in a classic left–right spectrum, when they consist of the location of a public facility, or when they are the different options in a public-good-versus-private-good space (see Moulin, 1988).

The assumption of single-peakedness also has a long tradition in economics and political science, especially in the field of social choice. It is well-known that when individual preferences are

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[1] An alternative interpretation as an individual decision problem of choice under conflicting motivations is discussed in Section 6.

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single-peaked strategy-proof voting rules can be obtained (see Black, 1948; Moulin, 1980), thus overcoming the impossibility results of Gibbard (1973) and Satterthwaite (1975) for the general domain. The single-peakedness assumption has been the focus of continual attention ever since those seminal papers (see Achunthankutty and Roy, 2018; Puppe, 2018; Savaglio and Vanucci, 2019 for recent references), and has been extended to other contexts such as matching problems (Bade, 2019) and computational social choice (Brandt et al., 2015).

This paper connects those two broadly studied topics: Pareto rationalizability of a choice function and single-peakedness. The plausibility and significant implications of the latter make it worth investigating the conditions for a choice function to be Pareto rationalizable by a profile of preferences that have a single-peaked structure. We obtain characterization results for Pareto rationalizability by a profile of n single-peaked preferences, paying special attention to the case $n = 2$. In our model the dimension along which the preferences are single-peaked is exogenously given, but we set out the basis for the case where the dimension is unknown, as explained in Section 5.

The paper is structured as follows: Section 2 introduces the problem of Pareto rationalizability by means of orders that are not necessarily single-peaked. Section 3 studies Pareto rationalizability by a profile of single-peaked strict preferences. The case of two single-peaked preferences is analyzed first, then the general n-agents case. Section 4 applies the results obtained in Section 3 to the case of single-peaked *weak* preferences. Section 5 discusses three possible extensions of the model and Section 6 concludes. The proofs of all the theorems are relegated to Appendix. In the case of Theorem 3 we have considered it of interest to include a sketch of the proof after its statement.

2. Pareto rationalizability of choice functions

Throughout the paper we consider the following types of binary relations: A *partial order* is an asymmetric and transitive binary relation. A *strict order* (or *linear order*) is a total, antisymmetric, and transitive binary relation, and a *weak order* is a total, reflexive, and transitive binary relation.

Let $\mathcal{X}$ be a finite set of alternatives and $\pi(\mathcal{X})$ the set of all non-empty subsets of $\mathcal{X}$. We assume that $\mathcal{X}$ is equipped with an exogenously given linear order $<$ and use the following notation $[a, \dots, b] = \{x \in \mathcal{X} : a \leq x \leq b\}$ and $(a, \dots, b) = \{x \in \mathcal{X} : a < x < b\}$.

Let C be a *choice function* that maps each set $K \in \pi(\mathcal{X})$ to a non-empty subset of K. Let $\succcurlyeq$ be a binary relation defined on $\mathcal{X}$ and $\succ$ its asymmetric factor, that is, $\forall x, y \in \mathcal{X}, x \succ y$ if not $(y \succcurlyeq x)$. The binary relation $\succcurlyeq$ is said to *rationalize* C if, for all $K \in \pi(X)$, $C(K) = \{x \in K : \nexists y \in K \text{ such that } y \succ x\}$.

We use $\mathcal{R} = (\succcurlyeq_1, \dots, \succcurlyeq_n)$ to denote a vector of strict orders,[2] interpreted as a *preference profile* in the standard social choice sense (see Arrow, 1951; Aleskerov, 2002).

We study the case where C is Pareto rationalizable by a preference profile $\mathcal{R}$. In this sense, C is interpreted as a *social* choice function that selects a subset of optimal alternatives from every set on the basis of the information given by the profile of individual preferences.

Formally, from $\mathcal{R}$, the binary relation $\succ_P$ is defined as follows: for all $x, y \in \mathcal{X}$, $x \succ_P y$ if $x \succ_i y$ for all $i \leq n$. In this case x is said to *Pareto dominate* y in $\mathcal{R}$. Notice that $\succ_P$ is a partial order

Definition 1. Let C be a choice function defined on $\pi(\mathcal{X})$ and let $\mathcal{R} = (\succcurlyeq_1, \dots, \succcurlyeq_n)$ be a preference profile. $\mathcal{R}$ is said to *Pareto rationalize* C (or C is said to *be Pareto rationalized by* $\mathcal{R}$) when the following condition holds: $\forall K \in \pi(\mathcal{X})$, $\forall x \in K$, $x \in C(K)$ *if and only* $\nexists y \in K$ *such that* $y \succ_P x$.

[2] In Section 4 we study the case where $\succcurlyeq_1, \dots, \succcurlyeq_n$ are weak orders.

The interpretation of Pareto rationalizability is straightforward: No alternative dominated by any other alternative in the same set is chosen.

Clearly, every profile $\mathcal{R}$ of preferences Pareto rationalizes some choice function C. This is guaranteed by the existence of Pareto optimal alternatives for every set in our setting. In particular, it suffices to define, for all $K \in \pi(\mathcal{X})$, $C(K) = \{x \in K : \nexists y \in K \text{ such that } y \succ_P x\}$. However, the inverse implication is not necessarily true: Not every choice function C is Pareto rationalizable by a profile $\mathcal{R}$ of preferences, e.g., a choice function C such that $C(\{a, b\}) = a$; $C(\{b, c\}) = b$; $C(\{a, c\}) = a$, and $C(\{a, b, c\}) = \{a, b, c\}$.

In order to determine which are the necessary and sufficient conditions accordingly, Moulin (1985), based on previous result by Roberts (1979) and Schwartz (1976), presented a corollary that enables the Pareto rationalizability of a choice function to be characterized by means of the following three axioms[3]:

Chernoff (α): For all $K', K \in \pi(\mathcal{X})$, $K' \subset K$ implies $(C(K) \cap K') \subset C(K')$.

Expansion (γ): For all $K', K \in \pi(\mathcal{X})$, $C(K) \cap C(K') \subset C(K \cup K')$.

Aizermann (AZ): For all $K, K' \in \pi(\mathcal{X})$, $C(K) \subset K' \subset K$ implies $C(K') \subset C(K)$.

The three conditions and their interpretation are well-known in choice theory, and appear for example in Chernoff (1954), Sen (1971) and Aizerman and Malishevski (1981). α requires that if an alternative x is selected from a set of alternatives it should also be selected in any subset that contains it. γ states that every alternative that is selected from two different sets should also be selected from their union. AZ requires that the subset of alternatives chosen from a set cannot expand when non selected alternatives are removed from that set.[4]

Theorem 1 (*Moulin, 1985*)**.** *Let C be a choice function defined on $\pi(\mathcal{X})$. There exists a profile $\mathcal{R}$ of strict preferences that Pareto rationalizes C if and only if C satisfies α, γ and AZ.*

3. Single-peaked preferences

Next we focus on the case where the elements of $\mathcal{R}$ are not only strict orders but also single-peaked with respect to the specified linear order $<$. Formally, single-peakedness is defined as follows (see Black, 1958; Inada, 1969; Moulin, 1980):

Let $\succcurlyeq_i$ be a preference defined on $\mathcal{X}$, and let $\succ_i$ be its asymmetric factor. Let $<$ be the linear order defined on $\mathcal{X}$. We say that $\succcurlyeq_i$ is *single-peaked* (with respect to $<$) with peak $\hat{x}(\succcurlyeq_i)$ if, for all $x \in X$ such that $x \neq \hat{x}(\succcurlyeq_i)$, $\hat{x}(\succcurlyeq_i) \succ_i x$, and for all y such that $\hat{x}(\succcurlyeq_i) < y < x$ or $x < y < \hat{x}(\succcurlyeq_i)$, $y \succ_i x$.

Next we present an additional axiom. To that end two new definitions must be introduced:

For all $K, K' \subseteq \mathcal{X}$, K is said to be *left-contained in* K' and is denoted by $K \subseteq_L K'$, if $K \subseteq K'$ and $\forall x \in K' \setminus K, \forall y \in K, y < x$. Similarly, K is said to be *right-contained in* K' ($K \subseteq_R K'$), if $K \subseteq K'$ and $\forall x \in K' \setminus K, \forall y \in K, x < y$. In words, for $K \subseteq K'$, K is left-contained (right-contained) in K' if all the alternatives in $K' \setminus K$ are to the right (left) of the alternatives in K.

For all $x \in \mathcal{X}$, $D_L(x) = \{z \in \mathcal{X} : z < x \text{ and } z \notin C(\{x, z\})\}$, and $D_R(x) = \{z \in \mathcal{X} : x < z \text{ and } z \notin C(\{x, z\})\}$. That is, $D_L(x)$ (alternatively $D_R(x)$) contains all the alternatives to the left (right) of x that are not chosen when presented together with x. Consider the example of a set of political options that are ordered along

[3] In fact, Moulin (1985) is very modest about the originality of his result.

[4] Notice that this result is independent of the existence of the linear order $<$. Only finiteness of $\mathcal{X}$ is required for the theorem to hold.

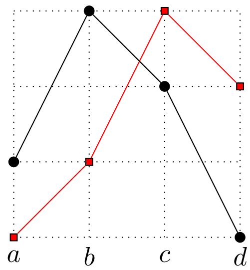

Fig. 1. A profile of two single-peaked preferences. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

the left–right line. In this case $D_L(x)$ ($D_R(x)$) would be the set of options that are further along the left-wing (right-wing) than x and would be rejected in the presence of x.

When C is Pareto rationalized by a preference profile, D_L (D_R) is the set of alternatives that are Pareto dominated and smaller (greater) than x. For example, in Fig. 1, $D_L(b) = D_L(c) = \{a\}$; $D_L(d) = \emptyset$; $D_R(a) = D_R(b) = \emptyset$ and $D_r(c) = \{d\}$.[5]

Now consider the following axiom imposed on C:

Centered Choice (CC): Let $<$ be the linear order defined on $\mathcal{X}$. There exist $\underline{x}$ and $\bar{x}$ in $\mathcal{X}$ with $\underline{x} \leq \bar{x}$, such that for all $x, y \in \mathcal{X}$ with $x < y$,

- (i) $y \leq \underline{x}$ implies $x \in D_L(y)$
- (ii) $\bar{x} \leq x$ implies $y \in D_R(x)$
- (iii) $\underline{x} \leq x$ implies $D_L(y) \subseteq_L D_L(x)$
- (iv) $y \leq \bar{x}$ implies $D_R(x) \subseteq_R D_R(y)$

First, a remarkable difference between axiom CC and axioms α, γ and AZ is that CC is not a purely choice theoretic condition, in the sense that it requires the assumption that X is equipped with a linear order $<$.[6]

CC appeals to the idea that the choice function establishes a compromise between the different preferences of the members of society. Given the unidimensional space where the alternatives are represented, CC can be interpreted as a normatively reasonable property. The compromise in social choice takes the form of a *centered* choice, where "centered" means that there is a certain range of centered options between $\underline{x}$ and $\bar{x}$ such that, (a) extreme choices are discarded if more centered alternatives are available (this is expressed by conditions (i) and (ii)); and (b) when the options "tilt" towards an extreme, then the number of choices that become socially acceptable on the other side increases (or at least that set does not shrink).

In other words, note that every alternative, x, induces a set of options to its left, $D_L(x)$, and to its right, $D_R(x)$, which are rejected in its presence. Conditions (iii) and (iv) imply that when moving right (left) within that central range this induced set shrinks on its left (right) side.

To better understand axiom CC, consider again the problem of selecting from a set of political parties ordered from left to right.[7] Assume that this set is {*Extreme Left* (*EL*), *Left* (*L*), *Moderate Left* (*ML*), *Moderate Right* (*MR*), *Right* (*R*), *Extreme Right* (*MR*)}. The social choice function could fulfill (CC) if, for example, *L* were chosen over *EL*; *ML* were chosen over *L* and *R* were chosen over *ER*. This would be consistent with $\underline{x} = ML$ and $\underline{x} = R$, showing some kind of social aversion to extreme options. In this case *ML* excludes *L* and *EL*, but according to (iii), if a unique option further right than *ML* were available, then more leftish options could be admitted in the chosen set. Indeed, if the only option were sufficiently extreme-right, the most leftish option *EL* could be admitted. That is, having to choose a too extreme right wing option enables more extreme left wing parties to be choosable, but when more moderate options are presented for choice those extreme options to the left tend to be rejected. The interpretation of (iv) is analogous.

3.1. Two single-peaked preferences

As in the case of two commodities or two production factors in basic microeconomic models, Pareto rationalizability by two preferences enables the main insights of the problem to be isolated from the mathematical complexity of the n-case (see Bossert and Sprumont, 2002; Echenique and Ivanov, 2011 and Qi, 2015). Furthermore, there are numerous environments where the binary case is plausible. Household decision problems, agreements in two-agent committees and elections with two well-identified social groups come to mind.

Moreover, reducing the problem to *two* single-peak preferences makes an important technical difference with respect to the n-preferences case because in this case, under the information given by the choice function, fixing one of the individual's preferences enables it to be automatically determined what the other preference must be for the choice function to be consistent with the Pareto optimality criterion (see Echenique and Ivanov, 2011; Sprumont, 2001).

Pareto rationalizability by means of two orders was first studied by Dushnik and Miller (1941). They provided necessary and sufficient conditions in an abstract setting for a partial order to be the intersection of at least two linear orders. However, those conditions are based on the existence of hard-to-check mathematical objects. Sprumont (2001) characterizes Pareto rationalizability in an infinite environment by means of two preferences that satisfy certain "regularity" restrictions. Qi (2015) uses conditions in the same spirit to obtain a close characterization in a discrete environment. Also in a discrete setting, Echenique and Ivanov (2011) propose two alternative characterizations of Pareto rationalizability by reducing the problem to a graph coloring problem. One characterization result is obtained by combining two standard revealed preference axioms with another two testable properties that take the form of graph coloring conditions. Alternatively, they reformulate one of these conditions as a problem of solvability of a particular system of quadratic equations.

Even though the study of Pareto rationalizability in the two-agents case is technically simpler, it remains far from trivial in the two *single-peaked* preferences case.

Notice also that, unlike the standard approach in voting problems, the choice function that selects the Pareto optimal alternatives of a profile of two single-peaked preferences depends crucially on their whole configuration, and not only on their peaks. Fig. 2 depicts two variants of Fig. 1 with the same peaks. It is easy to check that each of the profiles Pareto rationalizes a very different choice function.

We next introduce two binary relations. The *strict base relation* associated with C is defined by: xRy if $C(\{x, y\}) = \{x\}$, and the *indifference base relation* associated with C is defined by: $x \otimes y$ if $C(\{x, y\}) = \{x, y\}$. Notice that, by Definition 1, the Pareto relation associated with a profile $\mathcal{R}$ can be obtained as the base relation associated with C. The next lemma focuses on the relationship between CC and the base relation associated with C.

[5] We thank an anonymous referee for suggesting this example to motivate this section, even providing the tikz-code of the figure.

[6] In Section 5, devoted to extensions of the model, we discuss the possibility of $<$ being induced from the information given by the choice function.

[7] These choices usually take the form of voting for a single party, but certain procedures such as approval voting allow for multi-valued choice.

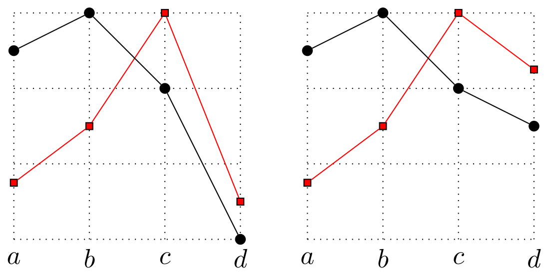

Fig. 2. Profiles with the same peaks may induce different choice functions. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

Lemma 1. *Let $<$ be the linear order defined on $\mathcal{X}$. If C satisfies CC, then there exist $\underline{x}$ and $\overline{x}$ such that*

(i) for all $x, y \le \underline{x}$, xRy if and only if $y < x$
(ii) for all $x, y \ge \overline{x}$, xRy if and only if $x < y$
(iii) for all $x, y \in [\underline{x}, \dots, \overline{x}]$, $x \otimes y$

Axiom CC enables it to be determined what the base relation associated with C looks like when both x and y are either within the range $[\underline{x}, \dots, \overline{x}]$ or both at the same side out of that range. (i) and (ii) follow directly from conditions (i) and (ii) of CC. The third statement of the lemma does not follow so directly from CC. The intuition is the following[8]: Notice that $D_L(z)$ is to the left of $\underline{x}$ for any alternative z. This is clear from condition (i) in CC when $z \le \underline{x}$. When $z \ge \underline{x}$ notice that what condition (iii) of CC is in fact saying is that when "moving to the right", set $D_L(\cdot)$ "shrinks to the left". Therefore, when "moving to the right" starting from $\underline{x}$, it is impossible to find an alternative z such that $D_L(z)$ contains an alternative to the right of $\underline{x}$. A similar reasoning leads to the conclusion that set $D_R(z)$ is to the right of $\overline{x}$. As a consequence, if two alternatives x, y such that $x < y$ are within $[\underline{x}, \dots, \overline{x}]$, then $x \notin D_L(y)$ and $y \notin D_R(x)$, and therefore $C(\{x, y\}) = \{x, y\}$.

CC does not enable it to be established univocally what the base relation should look like when one alternative is within the range $[\underline{x}, \dots, \overline{x}]$ and the other is not.

The next theorem shows the necessary condition that a profile $(\succcurlyeq_1, \succcurlyeq_2)$ must satisfy in order to Pareto rationalize a choice function C satisfying α, γ, AZ and CC. In particular, it must be a pair of what we call "mirrored preferences in the range (a, b)".

Definition 2. . Let $<$ be the linear order defined on $\mathcal{X}$ and let $\succcurlyeq_1, \succcurlyeq_2$ be a pair of strict preferences defined on $\mathcal{X}$. We say that $\succcurlyeq_1, \succcurlyeq_2$ are *a pair of mirrored strict preferences in the range* (a, b) *with respect to* $<$ if there exist $(a, b) \in \mathcal{X}$, $a \le b$, such that the following conditions hold:

$\forall x, y \le a$, x Pareto dominates y iff $y < x$.
$\forall x, y \ge b$, x Pareto dominates y iff $x < y$.
$\forall x, y \in (a, \dots, b)$, $x \succ_1 y$ implies $y \succ_2 x$.

In words, a pair of mirrored strict preferences in a range (a, b) is a pair such that both are increasing to the left of a, decreasing to the right of b and, within $(a, \dots, b)$, one order increases if and only if the other decreases.

The profile of Fig. 3 is a pair of mirrored strict preferences in the range (b, f). However, it is *not* single-peaked because one of the preferences is single-peaked with $<$ but the other is not.[9]

[8] The formal proof is in Appendix.

[9] It could be that the profile is not single-peaked with the exogenously given linear order of the alternatives but is single-peaked with respect to some other linear order. In this example the profile is not single-peaked with respect to any of the possible linear orders that can be defined over the alternatives.

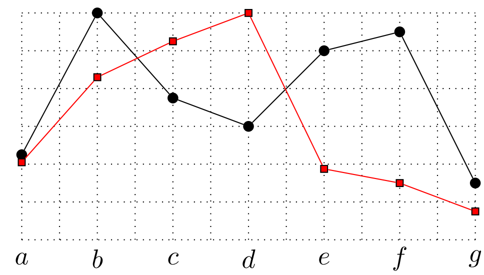

Fig. 3. A profile of "mirrored" strict preferences where CC is satisfied.

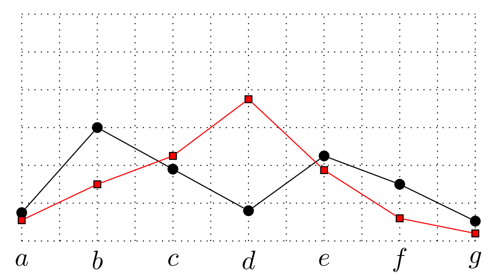

Fig. 4. A pair of mirrored preferences that fails to satisfy CC.

Theorem 2. *Let $<$ be the linear order defined on $\mathcal{X}$ and let C be a choice function defined on $\pi(\mathcal{X})$ such that C satisfies CC and is Pareto rationalized by a pair $\mathcal{R} = (\succcurlyeq_1, \succcurlyeq_2)$ of strict preferences. Then $(\succcurlyeq_1, \succcurlyeq_2)$ are mirrored in the range $(\underline{x}, \overline{x})$ with respect to $<$.*

Under the condition set in Theorem 2, for C to be Pareto rationalizable by $(\succcurlyeq_1, \succcurlyeq_2)$, there should not be pairwise dominance for the elements in $(\underline{x}, \dots, \overline{x})$.

Theorem 2 is an "if" result. This means that it is not true that any pair of mirrored strict preferences Pareto rationalizes a choice function C satisfying CC.

Clearly, the Paretian choice function associated with the profile in Fig. 3 satisfies conditions (i) and (ii) of CC for $\underline{x} = b$ and $\overline{x} = f$. Notice also that $D_L(b) = D_L(c) = D_L(d) = \{a\}$; $D_L(e) = D_L(f) = D_L(g) = \emptyset$ and $D_R(a) = D_R(b) = D_R(c) = D_R(d) = D_R(e) = D_R(f) = \{g\}$. Thus, conditions (iii) and (iv) of CC are also satisfied. The figure below shows another profile of mirrored strict preferences that induces a Paretian choice function that fails to satisfy CC: In that profile, for C to be Pareto rationalizable by the two preferences and satisfy conditions (i) and (ii) of CC, $\underline{x} = b$ and $\overline{x} = e$. However, in this case, C does not satisfy (iv) because $D_R(c) = \{f, g\}$ and $D_R(d) = \{g\}$.

Profiles where both preferences are single-peaked are a meaningful subclass of the family of mirrored preferences. Interestingly, it turns out that Axioms α, γ, AZ and CC axiomatically characterize (bidirectionally) the class of choice functions that are Pareto rationalizable by two single-peaked strict preferences.

Theorem 3. *Let $<$ be the linear order defined on $\mathcal{X}$ and let C be a choice function defined on $\pi(\mathcal{X})$. C is Pareto rationalizable by a profile $\mathcal{R} = (\succcurlyeq_1, \succcurlyeq_2)$ of single-peaked strict preferences with respect to $<$ if and only if C satisfies α, γ, AZ, and CC.*

The following simple example illustrates that the proving the sufficient part of the theorem is harder than it may seem at first sight. Let $\mathcal{X} = \{a, b, c, d, e\}$ (with $a < b < c < d < e$) and let C be such that b, c and d are always chosen in the sets that they belong to; a is chosen if and only if neither b, nor c, nor d is in the same set, and e is chosen if and only if d is not in the same set.

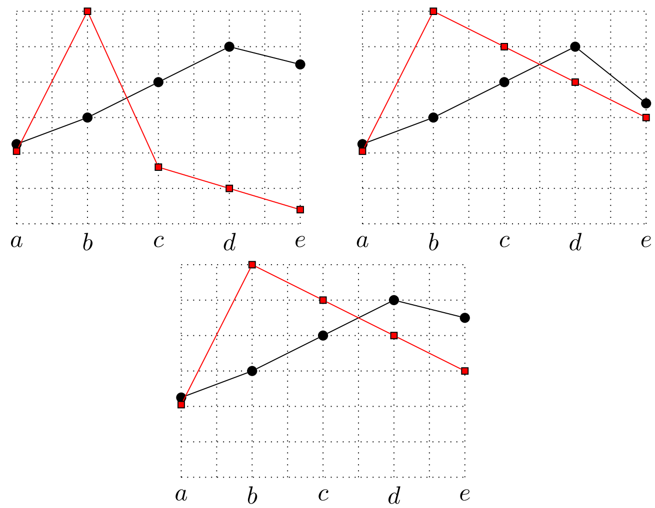

Fig. 5. Peaks are at b and d but the profiles do not Pareto rationalize C.

Such a choice function satisfies α, γ, AZ and also satisfies CC for $\underline{x} = b$ and $\bar{x} = d$. The first impression might be that by constructing two single-peaked preferences, one with a peak at b and the other with a peak at d, most of the proof of the sufficient part of the theorem would be solved. However, Fig. 5 shows three of many examples of such single-peaked profiles that do not Pareto rationalize C (for example, in the first profile $a \in C(\{a, c\})$ and in the other two $a \notin C(\{a, e\})$. The examples are also useful in highlighting again that, beyond the peaks, the whole configuration of the preferences is determinant in our problem

The difficulty of the proof lies in showing a systematic procedure to properly embed the utility values of all the alternatives. In the example above there is only one alternative on each side of the two peaks, but the presence of more alternatives at the "tails" complicates the job of properly "embedding" them. A large part of the proof is devoted to checking that, thanks to axiom CC, it is always possible to find a suitable interval where the alternatives can be placed, so that the construction is feasible and fits single-peakedness of the profile.

In particular, the proof of the sufficient part of the theorem can be sketched as follows: α, γ and AZ imply that there exists a partial order $\mathcal{P}$ that rationalizes the choice function and it corresponds to $\succ_P$. For the purposes of the proof the agent's preferences are represented by utility functions. Thus the proof consists of constructing two single-peaked utility functions compatible with the information contained in $\succ_P$. Their peaks turn out to be, precisely, $\underline{x}$ and $\bar{x}$ from axiom CC. The values of the function representing one of the preferences for the different alternatives are fixed in terms of the Pareto dominance of one alternative over the other, and the values of the other representing function are fixed in terms of the *absence* of Pareto dominance of one alternative over the other. The proof starts by setting arbitrary decreasing values of the two functions at the right tail to $\bar{x}$. Second, the values are set at the left tail to $\underline{x}$ in such a way that they are compatible with the Pareto dominance relationships with the alternatives to the right of $\bar{x}$. Finally, the values for both functions are embedded for the alternatives in the interval $(\underline{x}, \dots, \bar{x})$ in such a way that they are compatible with all the Pareto dominance relationships with the alternatives at the two tails. The conditions in CC enable such functions to be constructed in such a way that the two corresponding preferences are single-peaked with peaks at $\underline{x}$ and $\bar{x}$, respectively.

Theorem 3 enables us to ensure that any choice function that is Pareto rationalizable by a pair of mirrored preferences can also be Pareto rationalized by a profile of two single-peaked preferences. For example, it can be checked that the profile represented in Fig. 6 Pareto rationalizes the same choice function as the one represented in Fig. 3. However it is impossible to find a profile of single-peaked preferences that Pareto rationalizes the same choice function that is Pareto rationalized by the profile in Fig. 4.

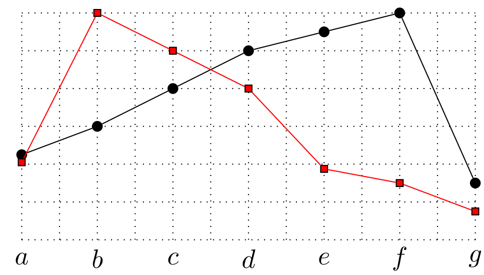

Fig. 6. This profile Pareto rationalizes the same choice function as the profile in Fig. 3.

It should be noted that a choice function that satisfies all the axioms in Theorem 3 might be Pareto rationalized by two single-peaked preferences in a trivial way. By "trivial" we mean that the two preferences are *identical* ($\forall x, y \in \mathcal{X}$, $x \succcurlyeq_1 y$ if and only if $x \succcurlyeq_2 y$). In that case society would be unanimous in its preference over any pair of alternatives and the choice function would also be rationalizable by a single preference (either $\succcurlyeq_1$ or $\succcurlyeq_2$).

The following theorem shows that if an additional, easily testable condition called *Local Indecisiveness* is fulfilled the choice function needs two *distinct*[10] preferences to be Pareto rationalized. This axiom just requires to be at least a two-fold set where both alternatives are selected by C. The result clearly shows the role of multivaluedness of C. If only single-valued choices were observable then Pareto rationalizability by distinct single-peaked preferences would be impossible.

Local indecisiveness (LI): There exist $x, y \in \mathcal{X}$, $x \neq y$, such that $C(\{x, y\}) = \{x, y\}$

Theorem 4. *Let $<$ be the linear order defined on $\mathcal{X}$ and let C be a choice function defined on $\pi(\mathcal{X})$. C is Pareto rationalizable by two* distinct *single-peaked strict preferences $(\succcurlyeq_1, \succcurlyeq_2)$ if and only if C satisfies α, γ, AZ, CC and LI.*

3.2. The n single-peaked preferences case

Now we study now the problem of Pareto rationalizability by a profile of n single-peaked preferences. It turns out that the same axioms that characterize Pareto rationalizability by two single-peaked preferences also characterize Pareto rationalizability by an arbitrary number, n, of single-peaked preferences. A direct corollary of this result is that any choice function than can be Pareto rationalized by n single-peaked strict preferences is also Pareto rationalizable by a profile of two single-peaked strict preferences. We discuss this issue with further detail some paragraphs below.

Theorem 5. *Let $<$ be the linear order defined on $\mathcal{X}$ and let C be a choice function defined on $\pi(\mathcal{X})$. C is Pareto rationalizable by a profile $\mathcal{R}$ of n single-peaked preferences with respect to $<$ if and only if C satisfies α, γ, AZ and CC.*

10 By "distinct" we mean that there exist $x, y \in \mathcal{X}$ such that $x \succcurlyeq_1 y$ and not $(x \succcurlyeq_2 y)$.

The formal proof of Theorem 5 is in the Appendix, but the idea of the proof of the sufficient part is that, by Theorem 3, every choice function, C, that satisfies the four axioms can be Pareto rationalized by a profile of two single-peaked preferences, $(\succcurlyeq_1, \succcurlyeq_2)$. Thus, for any $n \geq 2$, the profile $\mathcal{R} = (\succcurlyeq'_1, \dots, \succcurlyeq'_n)$ such that

$$\succcurlyeq'_i = \begin{cases} \succcurlyeq_1 & \text{if } i = 1 \\ \succcurlyeq_2 & \text{otherwise} \end{cases}$$

also Pareto rationalizes C.[11]

This reasoning poses the question of whether the only way to Pareto rationalize a choice function that satisfies the four axioms is by means of one ordering and $n-1$ copies of another ordering. The answer is no. For example, let $\mathcal{X} = \{x, y, z\}$ be such that $x < y < z$, and let C be such that $C(K) = K$ for all $K \in \pi(X)$. C is Pareto rationalizable by the profile $\mathcal{R} = (\succcurlyeq_1, \succcurlyeq_2, \succcurlyeq_3)$ such that $x \succ_1 y \succ_1 z$; $y \succ_2 x \succ_2 z$ and $z \succ_3 y \succ_3 x$. On the other hand, let C' be such that $C'(\{x, y, z\}) = C'(\{x, y\}) = \{x, y\}$; $C'(\{x, z\}) = x$ and $C'(\{y, z\}) = y$. In this case, it is not possible to Pareto rationalize C' by means of more than two distinct linear orderings. In sum, a choice function *may* or *may not* be Pareto rationalizable by more than two distinct linear orderings. However, as already pointed out, the combination of Theorems 3 and 5 implies that every choice function that is Pareto rationalizable by more than two linear orders (distinct or not) is also Pareto rationalizable by exactly two linear orders. For example, the choice function C above can also be Pareto rationalized by $(\succcurlyeq_1, \succcurlyeq_2)$.

4. Weak preferences

In this section we show that, except for Theorem 4, the results in the preceding sections also hold with no need to change CC when the elements of $\mathcal{R}$ are single-peaked weak preferences[12] and Pareto domination is redefined as follows: For all $x, y \in \mathcal{X}$, alternative x *Pareto dominates* y in $\mathcal{R}$ if $x \succcurlyeq_i y$ for all $i \leq n$ and $x \succ_k y$ for some $k \leq n$.

In particular, the next theorem shows that Theorem 1 (on Pareto rationalizability in the general setting) also holds for weak preferences.

Theorem 6. *Let C be a choice function defined on $\pi(\mathcal{X})$. There exists a profile $\mathcal{R}$ of weak preferences that Pareto rationalizes C if and only if C satisfies α, γ and AZ.*

Theorem 2 also holds for weak preferences. The definition of a pair of mirrored weak preferences reads exactly like Definition 2 except that, as a consequence of the last condition, $x \sim_1 y$ implies $x \sim_2 y$ and that the meaning of Pareto domination is the one corresponding to weak preferences.

Theorem 7. *Let $<$ be the linear order defined on $\mathcal{X}$ and let C be a choice function defined on $\pi(\mathcal{X})$ such that C satisfies CC and is Pareto rationalizable by a pair $\mathcal{R} = (\succcurlyeq_1, \succcurlyeq_2)$ of weak preferences. Then $(\succcurlyeq_1, \succcurlyeq_2)$ are mirrored preferences in the range $(\underline{x}, \overline{x})$ with respect to $<$.*

The next theorem shows that in the case of Pareto rationalizability by two single-peaked strict preferences (Theorem 3) the axiomatic characterization is similar when those preferences are *weak*.

Theorem 8. *Let $<$ be the linear order defined on $\mathcal{X}$ and let C be a choice function defined on $\pi(\mathcal{X})$. C is Pareto rationalizable by a profile $\mathcal{R} = (\succcurlyeq_1, \succcurlyeq_2)$ of single-peaked weak preferences with respect to $<$ if and only if C satisfies α, γ, AZ, and CC.*

A direct corollary of Theorems 1 and 6 is that a choice function is Pareto rationalizable by a profile of strict preferences if and only if it is Pareto rationalizable by a profile of weak preferences. Similarly, a corollary of Theorems 3 and 8 is that a choice function is Pareto rationalizable by a pair of single-peaked weak preferences if and only if it is Pareto rationalizable by a pair of single-peaked strict preferences.

Those equivalences are closely related with another equivalence proven in Echenique and Ivanov (2011). There, it is shown that a choice function is Pareto rationalizable by two strict preferences if and only if it is Pareto rationalizable by two weak preferences, but without imposing any additional structure on them. Theorem 8 shows that this equivalence is preserved when those preferences are required to be single-peaked.[13]

Finally, Theorem 4 does not hold in the case of weak preferences. The following theorem shows that it is only possible to present a sufficient condition for a choice function to be Pareto rationalizable by two distinct single-peaked weak preferences.

Local composed indecisiveness (LCI): There exist $x, y, z \in \mathcal{X}$, $x \neq y \neq z$, such that $C(\{x, y\}) = \{x, y\}$ and $C(\{x, z\}) = \{x, z\}$

Theorem 9. *Let $<$ be the linear order defined on $\mathcal{X}$ and let C be a choice function defined on $\pi(\mathcal{X})$. If C satisfies α, γ, AZ, CC and LCI then it is Pareto rationalizable by two* distinct *single-peaked weak preferences with respect to $<$.*

Theorem 9 cannot be proved in the reverse direction. In fact, it cannot be ensured that every pair of distinct single-peaked weak preferences Pareto rationalizes a choice function that cannot be Pareto rationalizable by a unique single-peaked preference. To see this, take two distinct preferences, $\succcurlyeq_1, \succcurlyeq_2$, that differ only in the treatment of two options, x, y, in such a way that $x \succ_1 y$ and $y \succcurlyeq_2 x$ and take $\succcurlyeq'_1, \succcurlyeq'_2$ that only differ with $\succcurlyeq_1, \succcurlyeq_2$ in that $x \sim'_1 y$ and $x \sim'_2 y$. Then both pairs of preferences Pareto rationalize the same choice function. Preferences $\succcurlyeq_1$ and $\succcurlyeq_2$ are distinct, $\succcurlyeq'_1$ and $\succcurlyeq'_2$ are not.

Still, Theorem 9 allows to conclude that every choice function that satisfies α, γ, AZ and CC, but not LCI, can only be Pareto-rationalized by identical weak single-peaked preferences, which is equivalent to be rationalized by a unique single-peaked preference.

Finally, Theorem 5 (on Pareto rationalizability by an arbitrary number, n, of single-peaked strict preferences) also holds when preferences are weak. This means that it is still true that a choice function is Pareto rationalizable by n single-peaked *weak* preferences if and only if it is Pareto rationalizable by exactly two single-peaked weak preferences.

Theorem 10. *Let $<$ be the linear order defined on $\mathcal{X}$ and let C be a choice function defined on $\pi(\mathcal{X})$. C is Pareto rationalizable by a profile $\mathcal{R}$ of n single-peaked weak preferences with respect to $<$ if and only if C satisfies α, γ, AZ and CC.*

5. Extensions

In this section we give some pointers to three meaningful extensions of the model. First we discuss its applicability in an infinite environment where $\mathcal{X}$ is continuous. Secondly, we discuss

[11] We thank two anonymous referees for pointing out that the sufficient part of Theorem 5 could be proved in this way.

[12] Notice that, by definition, single-plateau preferences are *not* single-peaked weak preferences. In the latter the peak is unique and indifferences may hold only between elements that are located one on each side of the peak.

[13] Echenique and Ivanov (2011) apply a stronger notion of Pareto domination than ours, requiring $x \succ_i y$ for all i also when $\succcurlyeq_i$ are weak preferences.

the possibility that the linear order along which the preferences are single-peaked may be induced by the information given by the profile, instead of being exogenously given. Finally we propose an alternative interpretation of the model as a problem of individual choice by an agent that faces internal conflict between different motivations.

5.1. The continuous case

The four axioms of the model hold in general when $\mathcal{X}$ is continuous. However, starting from Moulin's (1985) theorem (Theorem 1), the proofs of the "only if" parts of the theorems rely very much on the finiteness assumption. We discuss this issue for each of the theorems in the following paragraphs.

The necessary part of Theorem 3 (characterization of the conditions for Pareto rationalizability by two single-peaked strict preferences) holds in the continuous case. However, the proof of the sufficient part makes extensive use of the finiteness assumption and is not true in the continuous case: In that setting it is possible to construct choice functions that satisfy α, γ, AZ and CC and cannot be Pareto rationalized by two single-peaked strict preferences. Thus, the question of what the sufficient conditions of C that guarantee that a choice function is Pareto rationalizable by a pair of such preferences are in a continuous environment remains open. The same can be said about the version of Theorem 3 for weak preferences (Theorem 8).

Theorems 2 and 7 hold in the continuous case (if C satisfies the four axioms then the corresponding two preferences should be mirrored, either in the strict preferences case or in the weak preferences case).

The proof of sufficient part of Theorem 4 (necessary and sufficient conditions for Pareto rationalizability by two *distinct* single-peaked strict preferences) requires the use of the proof of the sufficient part of Theorem 3 and thus, like Theorem 3, only holds in its necessary part in the continuous case. Theorem 9 is the version of Theorem 4 for weak preferences, but it only applies in the form of sufficient conditions. Like Theorem 4, the proof of Theorem 9 requires the use of the proof of the sufficient part of Theorem 8. Thus, Theorem 9 does not apply in the continuous case.

The necessary parts of Theorems 5 and 10 (necessary conditions for a choice function that is Pareto rationalized by a given profile of n single-peak preferences, either strict or weak) apply to the continuous case. However, their sufficient parts do not hold in the continuous case: As explained in Section 3, there are choice functions that satisfy the axioms and that can *only* be Pareto rationalizable by a preference and $n-1$ copies of another preference. In those cases the proofs rely on the proof of the necessary part of Theorem 3 (alternatively Theorem 8) about Pareto rationalizability by two preferences which, as already pointed out, cannot be used in the continuous case.

5.2. Single-peakedness inducibility

In our model the dimension that orders the alternatives linearly is given. A challenging research issue is to identify the conditions under which a choice function is Pareto rationalizable by a preference profile that is single-peaked with respect to an unspecified dimension.[14] This connects with the question of when a preference profile is "single-peaked consistent", i.e. how to identify profiles whose preferences are single-peaked with respect to some unknown dimension. This issue has been approached by several authors. Ballester and Haeringer (2011) characterize the set of profiles of strict preferences that are single-peaked consistent by means of certain "local" conditions that the preferences over every triple and every quadruple of alternatives must satisfy. Puppe (2018) provides two necessary global conditions for single-peaked consistency. Escoffier et al. (2008) and Bartholdi III and Trick (1986) propose algorithmic solutions for recognizing whether a profile is single-peaked with respect to some unspecified dimension and identifying it.

As a consequence, the initial issue, namely, the conditions for a choice function to be Pareto rationalizable by a single-peaked consistent preference profile, can be answered in three steps: First by checking whether the choice function is Pareto rationalizable by a preference profile (it satisfies α, γ and AZ); second by checking whether the rationalizing profile satisfies the corresponding conditions for single-peaked consistency[15] (or the algorithms have a solution) and identifying the corresponding dimension on which preferences are single-peaked; and third by checking whether CC is satisfied with respect to the identified dimension.

A complication of this approach is that a choice function may be Pareto rationalized by more than one preference profile, and a preference profile can be single-peaked consistent with more than one dimension. For example, the profile $(\succcurlyeq_1, \succcurlyeq_2)$ with $a \succ_1 b \succ_1 c \succ_1 d$ and $b \succ_2 a \succ_2 c \succ_2 d$ is single-peaked consistent with the linear order $a < b < c < d$ and also with the linear order $b < a < c < d$. This means that axiom CC may hold for some of the admissible dimensions but not for others. Thus, in order to answer the main question, it must also be checked whether CC holds for at least one of the admissible dimensions. Notice that there can be an exponential number of compatible dimensions in relation to the number of alternatives. This makes the computational complexity aspect of the problem relevant. Escoffier et al. (2008) find that, given a preference profile, checking whether it is single-peaked or not, and obtaining a dimension if it is, can be done in polynomial time. Thus, the question to answer would be whether testing that the axioms hold for that dimension can be performed in polynomial time. At first sight, the problem seems to be at least in NP but, given that there can be an exponential number of dimensions, whether the problem is polynomial time solvable or NP-complete is not trivial.

5.3. Internal motivation conflict

The model can also be interpreted as an individual choice problem where: (i) choices are made in a unidimensional space, such as the level of effort devoted to a certain activity, the percentage of a budget devoted to the consumption of a certain good, the time allocated to either leisure or work or how many children to have; and (ii) behavior is the outcome of multiple underlying "principles" or "motivations" such as pleasure, self-image or the observance of social norms. In this case, and from the psychological point of view, single-peakedness is also a plausible way of representing motivations (see Ryan and Deci, 2000; Arlegi and Teschl, 2015). Thus, at the individual level our model can be interpreted as one of individual choice rationalizability by means of multiple single-peaked motivations. In this setting, Pareto rationalizability differs from the models of rationalizability by multiple preferences in what the former can select alternatives that are not maximal for any of the set of multiple rankings. In this context, the binary case where there are only two single-peaked orders is also plausible. Typical internal conflicts between what the individual *wants* do and what he/she thinks he/she *should* do can be reasonably represented by confronting motivations such as pleasure and social norms, or comfort and

[14] We thank an anonymous referee for suggesting this issue.

[15] All the references mentioned except Puppe (2018) apply to strict preferences only.

self-image (see e.g. Bazerman et al., 1998). The plausibility of axioms α, γ and AZ is well-understood as conditions of internal consistency of individual choice. CC can also be seen as a reasonable property in this setting, as a way of establishing a centered compromise between the different confronting motivations underlying individual behavior.

6. Conclusions

We present some results about the axiomatic structure of choice functions that can be Pareto rationalizable by a profile of single-peaked preferences. For this, a new condition called *Centered Choice* (CC) needs to be added to the classical conditions that enable to Pareto rationalizability to be proved in more general domains (Moulin, 1985). The new condition expresses the idea that the choice function is averse to options that are distant from a range of "centered" alternatives and the alternatives within the range maintain a certain consistency with the underlying criterion that orders the alternatives when looking at the other options that their presence excludes. CC is a necessary and sufficient condition for Pareto rationalizability of a choice function by a profile of single-peaked preferences. Moreover, the results show that if a choice function is Pareto rationalizable by a profile of n single-peaked preferences then it is also Pareto rationalizable by *two* single-peaked preferences.

Unlike the standard approach to single-peaked preferences in the context of voting, for Pareto rationalizability the whole configuration of the single preferences, and not only their peaks, is determinant, as well as the assumption that the choice function is multivalued. Taking observed choice as a reference, the results make it possible to test whether such choice can be interpreted as being the outcome of the interaction between several underlying single-peaked preferences. On the other hand, if the starting point of the problem is a profile of single-peaked preferences and Pareto dominance is to be implemented as a choice rule, the results inform about the necessary properties that such a criterion must satisfy.

Appendix

Throughout the proofs in the Appendix we will make a slight abuse of notation: for any $A \in \mathcal{X}$, and any $x \in \mathcal{X} \setminus A$, we will write "$A < x$" when all the elements of set A are to the left of x. Alternatively, "$A > x$" denotes that all the elements of A are to the right of x.

Proof of Theorem 1. By Theorem 4 in Moulin (1985) a choice function C is rationalizable by a partial order if and only if it satisfies α, γ and AZ. Assume that C satisfies α, γ and AZ. Let $\succ_P$ be the partial order that rationalizes C. By Szpilrajn's theorem (1930) every partial order has at least one linear extension, and by Dushnik and Miller's theorem (1941), every partial order is the intersection of all its possible linear extensions. The sufficient part of Theorem 1 is thus proved by taking $\mathcal{R}$ as the profile of the linear extensions of the partial order that rationalizes C.

Inversely, given a preference profile $\mathcal{R}$ that Pareto rationalizes C, by Theorem 4 in Moulin (1985) C satisfies α, γ and AZ. □

Proof of Lemma 1. Conditions (i) and (ii) of Lemma 1 follow directly from conditions (i) and (ii) in CC respectively; from the definitions of $D_L(\cdot)$ and $D_R(\cdot)$, and from the definitions of the base relations.

(iii) Take any $x, y \in [\underline{x}, \ldots, \overline{x}]$. Assume, without loss of generality, that $x < y$. Notice that, under CC, it is impossible that $x \in D_L(y)$. Otherwise, by condition (iii) of CC, $x \in D_L(x)$, which contradicts the definition of $D_L(x)$. Similarly, by condition (iv) of CC it is impossible that $y \in D_R(x)$. Therefore, $x \in C(\{x, y\})$ and $y \in C(\{x, y\})$. That is, $C(\{x, y\}) = \{x, y\}$. Therefore $x \otimes y$.

Proof of Theorem 2. We prove first the following Lemma, which is also used in the proofs of other theorems:

Lemma 2. *Let $<$ be the linear order defined on $\mathcal{X}$ and let C be a choice function defined on $\pi(\mathcal{X})$. If C satisfies CC, then $\forall x \in \mathcal{X}$, $D_L(x) < \underline{x}$ and $D_R(x) > \overline{x}$.*

Proof. If $x \le \underline{x}$, by condition (i) in CC, we have that $D_L(x) = \{y \in \mathcal{X} : y < x\} < \underline{x}$. If $\underline{x} < x$, by condition (iii) in CC, $D_L(x) < \underline{x}$. $D_R(x) > \overline{x}$ is proven analogously by using conditions (ii) and (iv) in CC.

Proof of Theorem 2. If C is Pareto rationalized by $\mathcal{R} = (\succcurlyeq_1, \succcurlyeq_2)$, then C satisfies α, γ and AZ by Theorem 1. This implies that, $\forall K \in \mathcal{X}$, $C(K) = \{x \in K : \nexists y \in K \text{ such that } y \succ_P x\}$.

By condition (i) in CC and the transitivity of $\succ_P$, for C to be Pareto rationalizable by $\mathcal{R} = (\succcurlyeq_1, \succcurlyeq_2)$, both $\succcurlyeq_1$ and $\succcurlyeq_2$ have to be strictly increasing to the left of $\underline{x}$. Similarly, by condition (ii), both $\succcurlyeq_1$ and $\succcurlyeq_2$ have to be strictly decreasing to the right of $\overline{x}$.

By Lemma 2, for all $x, y \in (\underline{x}, \ldots, \overline{x})$, $\neg(x \succ_P y)$ and $\neg(y \succ_P x)$. This implies that for C to be Pareto rationalizable by $\mathcal{R} = (\succcurlyeq_1, \succcurlyeq_2)$, there is no $x, y \in (\underline{x}, \ldots, \overline{x})$ such that x Pareto dominates y in $\mathcal{R}$. Thus, for all $x, y \in (\underline{x}, \ldots, \overline{x})$, if $x \succ_1 y$, then $x \succ_2 y$. Thus, $\succcurlyeq_1$ and $\succcurlyeq_2$ are a pair of mirrored strict preferences in the range $(\underline{x}, \overline{x})$. □

Proof of Theorem 3. We first prove three lemmas which will be used for the proof of the theorem.

Lemma 3. *Let $<$ be the linear order defined on $\mathcal{X}$ and let C be a choice function defined on $\pi(\mathcal{X})$. If C satisfies CC then $\forall x \in \mathcal{X}$, $\forall y \in D_L(x)$, $y' \le \underline{x}$ and $y' \notin D_L(x)$ implies $y < y'$.*

Proof. If $y \in D_L(x)$, by Lemma 2, $y < \underline{x}$. Thus, by condition (i) in CC, $\forall z < y$, $z \in D_L(x)$. Therefore, if $y' < \underline{x}$ and $y' \notin D_L(x)$, necessarily $y \le y'$. Note that $y' = y$ leads to the contradiction that $y \in D_L(x)$ and $y \notin D_L(x)$. Therefore, $y < y'$.

Lemma 4. *Let $<$ be the linear order defined on $\mathcal{X}$ and let C be a choice function defined on $\pi(\mathcal{X})$. If C satisfies CC, then $\forall x \in \mathcal{X}$, $\forall y \in D_R(x)$, $y' \ge \overline{x}$ and $y' \notin D_R(x)$ implies $y' < y$.*

Proof. The proof is analogous to the proof of Lemma 3.

Lemma 5. *Let $<$ be the linear order defined on $\mathcal{X}$ and let C be a choice function defined on $\mathcal{X}$. If C satisfies CC, then $\forall x, x' \in \mathcal{X}$ such that $x < x'$, $\{z \in \mathcal{X} : z \ge \overline{x}\} \setminus \{y \in \mathcal{X} : y \ge \overline{x} \text{ and } x \in D_L(y)\} \subseteq_R \{z \in \mathcal{X} : z \ge \overline{x}\} \setminus \{y \in \mathcal{X} : y \ge \overline{x} \text{ and } x' \in D_L(y)\}$.*

Proof. For all $x \in \mathcal{X}$ denote by $R(x) = \{y \in \mathcal{X} : y \ge \overline{x}$ and $x \in D_L(y)\}$. We first prove that if $x < x'$, $\{z \in \mathcal{X} : z \ge \overline{x}\} \setminus R(x) \subseteq \{z \in \mathcal{X} : z \ge \overline{x}\} \setminus R(x')$. Assume that this is false. Then there must exist $y, x, x' \in \mathcal{X}$ such that $x < x'$, $y \ge \overline{x}$, $y \notin R(x)$ and $y \in R(x')$. That is, $x \notin D_L(y)$ and $x' \in D_L(y)$. This implies by Lemma 2 that $x' < \underline{x}$ and therefore $x < \underline{x}$. However, by Lemma 3, $x' < x$, which leads to a contradiction.

Now we prove that not only $\{z \in \mathcal{X} : z \ge \overline{x}\} \setminus R(x) \subseteq \{z \in \mathcal{X} : z \ge \overline{x}\} \setminus R(x')$, but also $\{z \in \mathcal{X} : z \ge \overline{x}\} \setminus R(x) \subseteq_R \{z \in \mathcal{X} : z \ge \overline{x}\} \setminus R(x')$. That is, if $y > \overline{x}$ and $y \notin R(x)$, it follows that for all $y < y'$, $y' \notin R(x)$. $y \notin R(x)$ implies that $x \notin D_L(y)$. Assume that $y' \in R(x)$. Then $x \in D_L(y')$. By condition (iii) of CC, $y < y'$ implies $D_L(y') \subseteq_L D_L(y)$. Therefore, $x \in D_L(y)$, and a contradiction is reached.

Proof of Theorem 3 (*Necessary Part*)**.** If C is Pareto rationalized by two single-peaked strict preferences, $\mathcal{R} = (\succcurlyeq_1, \succcurlyeq_2)$, then C

satisfies α, γ and AZ by Theorem 1. Now we prove that C satisfies CC: let $\underline{x} = \min_{\succcurlyeq_i \in \mathcal{R}}\{\hat{x}(\succcurlyeq_i)\}$ and $\overline{x} = \max_{\succcurlyeq_i \in \mathcal{R}}\{\hat{x}(\succcurlyeq_i)\}$.

(i) Take $x < y \leq \underline{x}$. By the single-peakedness of all $\succcurlyeq_i \in \mathcal{R}$, $y \succ_i x$ for all $\succcurlyeq_i \in \mathcal{R}$. Given that C is Pareto rationalized by $\mathcal{R}$, $x \in D_L(y)$. (ii) is proven analogously.

(iii) Assume that the statement is false. There must then exist $x, y, a \in \mathcal{X}$ such that $\underline{x} \leq x < y$; $a \in D_L(y)$ and $a \notin D_L(x)$. Notice that $a \in D_L(y)$ implies $a < y$. Given that $\underline{x} = \min_{\succcurlyeq_i \in \mathcal{R}}\{\hat{x}(\succcurlyeq_i)\}$, $a < \underline{x}$ also holds, otherwise, by the single-peakedness of all the preferences, there would be some preference $\succcurlyeq_k$ such that $a \succ_k y$ and, given that C is Pareto rationalized by $\mathcal{R}$, this would imply $\neg(a \in D_L(y))$. Now, given that C is Pareto rationalized by $\mathcal{R}$, $a \in D_L(y)$ implies $y \succ_i a$ for all $\succcurlyeq_i \in \mathcal{R}$. We have $a < \underline{x} \leq x < y$. Therefore, by the single-peakedness of all $\succcurlyeq_i \in \mathcal{R}$, it follows that $x \succ_i a$ for all $\succcurlyeq_i \in \mathcal{R}$ also holds, so a contradiction is reached with the assumption that $a \notin D_L(x)$.

At this point it has been proven that $D_L(y) \subseteq D_L(x)$. To prove that $D_L(y) \subseteq_L D_L(x)$ the next step is to prove that for all $z \in \mathcal{X}$, if $a, b \in D_L(z)$ and $a < b$, then $a \in D_L(z)$. That is, for all $z \in \mathcal{X}$, the set $D_L(z)$ always consists of a subset of smallest elements of $\mathcal{X}$: Notice that $b \in D_L(z)$ implies $b < \underline{x}$. By condition (i) it is known that $a \in D_L(b)$. Thus, given that C is Pareto rationalized by $\mathcal{R}$, $b \succ_i a$ for all $\succcurlyeq_i \in \mathcal{R}$. Similarly, $b \in D_L(z)$ implies $z \succ_i b$ for all $\succcurlyeq_i \in \mathcal{R}$. By transitivity, $z \succ_i a$ for all $\succcurlyeq_i \in \mathcal{R}$. Given that $\mathcal{R}$ Pareto rationalizes C this implies $a \in D_L(z)$.

Condition (iv) is symmetric to (iii) and is proven analogously.

Proof of Theorem 3 (*Sufficient Part*)**.** We have to prove that, if C satisfies α, γ, AZ and CC, then C is Pareto rationalizable by a profile of two single-peaked strict preferences, $\mathcal{R} = (\succcurlyeq_1, \succcurlyeq_2)$. For each $\succcurlyeq_i \in \mathcal{R}$, let $u_i : \mathcal{X} \to \mathbb{R}_{++}$ be a utility function that represents $\succcurlyeq_i$. We prove here that the two preferences that Pareto rationalize C are such that $\hat{x}(\succcurlyeq_1) = \underline{x}$ and $\hat{x}(\succcurlyeq_2) = \overline{x}$, and we construct the utility functions that are compatible with the preferences that Pareto rationalize C.

We distinguish two cases: $\underline{x} < \overline{x}$ and $\underline{x} = \overline{x}$. The latter case is easy and will be proved later. The former is longer and has the following structure:

– *Step 1*: We set $u_1(\underline{x}) > u_1(\overline{x})$ and $u_2(\underline{x}) < u_2(\overline{x})$ and a sequence of arbitrary strictly decreasing values of $u_2(y)$ for all $y \geq \overline{x}$.
– *Step 2*: We embed the values of $u_2(x)$ for all $x < \underline{x}$ into appropriate intervals, prove that they are not empty and that the utilities obtained are strictly increasing.
– *Step 3*: We set a sequence of strictly decreasing values of $u_1(x)$ for all $x \geq \overline{x}$.
– *Step 4*: We embed the values of $u_1(x)$ for all $x < \underline{x}$ into appropriate intervals, prove that they are not empty and that the utilities are strictly increasing.
– *Step 5*: We embed the values of $u_1(x)$ and $u_2(x)$ for all $x \in (\underline{x}, \ldots, \overline{x})$.

CASE 1. $\underline{x} < \overline{x}$.

Step 1: Let $u_1(\underline{x}) > u_1(\overline{x})$ and $u_2(\underline{x}) < u_2(\overline{x})$ and let a sequence of arbitrary strictly decreasing and strictly positive values of $u_2(y)$ for all $y \geq \overline{x}$.

Step 2: $\forall x < \underline{x}$, we look for a value of $u_2(x)$ in the following intervals:

(a): If $D_R(x) \neq \emptyset$, $u_2(x) \in (\max_{y \in D_R(x)}\{u_2(y)\}, \min_{y \geq \overline{x}: y \notin D_R(x)}\{u_2(y)\})$
(b): If $D_R(x) = \emptyset$, $u_2(x) \in (0, \min_{y \geq \overline{x}: y \notin D_R(x)}\{u_2(y)\})$.

Notice that $\forall x < \overline{x}$, $\overline{x} \notin D_R(x)$. The reason is that $\overline{x} \notin D_R(\overline{x})$ by definition and, by applying condition (iv) of CC, $\overline{x} \notin D_R(x)$. This also implies that $\forall x \leq \underline{x}$, $\{z \in \mathcal{X} : z \geq \overline{x}\} \setminus D_R(x) \neq \emptyset$ for all $x < \overline{x}$ and therefore $\min_{y \geq \overline{x}: y \notin D_R(x)}\{u_2(y)\}$ always exists.

We first prove that in both cases (a) and (b), the corresponding interval is non empty and therefore it is possible to find such a value of $u_2(x)$. In case (a) this is a consequence of the fact that u_2 is strictly decreasing to the right of $\overline{x}$ and Lemma 4. In case (b) it is because $u_2(y) > 0$ $\forall y \in \mathcal{X}$.

Moreover, such values of u_2 can always be found in such a way that u_2 is strictly increasing to the left of $\underline{x}$. That is, let $\{x \in \mathcal{X} : x \leq \underline{x}\} = \{x_1, \ldots, x_k\}$ with $x_i < x_j$ iff $i < j$. We prove that, $\forall x_r < \underline{x}$, there exists a value of $u_2(x_r)$ in the corresponding interval such that $u_2(x_r) < u_2(x_{r+1})$.

If $k = 1$ the solution is trivial because $x_k = \underline{x}$, which is already fixed. If $k > 1$, the first step is to arbitrarily fix $u_2(x_1)$ in its corresponding interval. By condition (iv) $D_R(x_1) \subseteq_R D_R(x_2)$ and, given that u_2 is strictly decreasing to the right of $\overline{x}$, this implies that $\min_{y \geq \overline{x}: y \notin D_R(x_2)}\{u_2(y)\} \geq \min_{y \geq \overline{x}: y \notin D_R(x_1)}\{u_2(y)\}$. Therefore, it is possible to define $u_2(x_2) = \min_{y \geq \overline{x}: y \notin D_R(x_2)}\{u_2(y)\} - \epsilon_2$ for a sufficiently small and positive ϵ_2 such that $u_2(x_2) > u_2(x_1)$ and such that $u_2(x_2) > \max_{y \in D_R(x_2)}\{u_2(y)\} > 0$. If $k = 2$ the proof is complete. If $k > 2$ the reasoning can be completed to continue successively fixing the value of each $u_2(x_r) = \min_{y \geq \overline{x}: y \notin D_R(x_r)}\{u_2(y)\} - \epsilon_r$ for a sufficiently small and positive ϵ_r such that $u_2(x_r) > u_2(x_{r-1})$ and such that $u_2(x_r) > \max_{y \in D_R(x_r)}\{u_2(y)\} > 0$.

Step 3: We fix a set of arbitrary strictly decreasing values of $u_1(y)$ for all $y \in \mathcal{X} : y \geq \overline{x}$.

Step 4: We try to find appropriate values of u_1 for all $x \in \mathcal{X} : x < \underline{x}$:

Recall the definition, $\forall x \leq \underline{x}$, $R(x) = \{y \in \mathcal{X} : y \geq \overline{x}$ and $x \in D_L(y)\}$. Notice that $\forall y, y' \in \mathcal{X}$, $y \in R(x)$ and $y' \notin R(x)$ implies $y < y'$. The reason is that, if $x \in D_L(y)$, $y' < y$ and $x \notin D_L(y')$, then $\neg(D_L(y) \subseteq D_L(y'))$, which contradicts condition (iii) in CC. If $y = y'$ the contradiction $x \in D_L(y)$ and $x \notin D_L(y')$ is reached.

For all $x < \underline{x}$, we look for a value of $u_1(x)$ in the following intervals:

(a): If $R(x) = \emptyset$, $u_1(x) \in (\max_{y \geq \overline{x}: y \notin R(x)}\{u_1(y)\}, u_1(\underline{x}))$.
(b): If $\{y \geq \overline{x} : y \notin R(x)\} = \emptyset$, $u_1(x) \in (0, \min_{y \in R(x)}\{u_1(y)\})$.
(c): If $R(x) \neq \emptyset$ and $\{y \geq \overline{x} : y \notin R(x)\} \neq \emptyset$, $u_1(x) \in (\max_{y \geq \overline{x}: y \notin R(x)}\{u_1(y)\}, \min_{y \in R(x)}\{u_1(y)\})$.

Notice that the case $R(x) = \{y \geq \overline{x}\} \setminus R(x) = \emptyset$ is not possible because either $\overline{x} \in R(x)$ or $\overline{x} \notin R(x)$.

In all cases the corresponding interval is non empty: In case (a) this is due to the fact that $u_1(\overline{x}) < u_1(\underline{x})$ is fixed and by construction u_1 is strictly decreasing to the right of $\overline{x}$, thus $\max_{y \geq \overline{x}}\{u_1(y)\} < u_1(\underline{x})$, which implies that, $\forall x \leq \underline{x}$, $\max_{y \geq \overline{x}: y \notin R(x)}\{u_1\} < u_1(\underline{x})$. In case (b) the interval is non empty because $u_1(y) > 0$ $\forall y \in \mathcal{X}$. In case (c) it is a consequence of the fact that u_1 is strictly decreasing to the right of $\overline{x}$ and Lemma 5.

Next we prove that such values of u_1 can always be found in such a way that u_1 is strictly increasing to the left of $\underline{x}$. That is, let $\{x \in \mathcal{X} : x \leq \underline{x}\} = \{x_1, \ldots, x_k\}$ with $x_i < x_j$ iff $i < j$. We prove that, $\forall x_r < \underline{x}$, there exists a value of $u_1(x_r)$ in the corresponding interval such that $u_1(x_r) < u_1(x_{r+1})$.

If $k = 1$ the solution is trivial. If $k > 1$, the first step is to arbitrarily fix $u_1(x_1)$ in its corresponding interval and look for an appropriate value of $u_1(x_2)$. Notice that, by Lemma 5, $\{z \in \mathcal{X} : x \geq \overline{x}$ and $z \notin R(x_1)\} \subseteq_R \{z \in \mathcal{X} : x \geq \overline{x}$ and $z \notin R(x_2)\}$ and, given that u_1 is strictly decreasing to the right of $\overline{x}$, this implies that $\min_{y \in R(x_2)}\{u_1(y)\} \geq \min_{y \in R(x_1)}\{u_1(y)\}$ and $\max_{y \geq \overline{x}: y \notin R(x_2)}\{u_1(y)\} \geq \max_{y \geq \overline{x}: y \notin R(x_1)}\{u_1(y)\}$.

Assume that $R(x_2) = \emptyset$:

Given that $\max_{y \geq \overline{x}: y \notin R(x_2)}\{u_1(y)\} \geq \max_{y \geq \overline{x}: y \notin R(x_1)}\{u_1(y)\}$, we can fix $u_1(x_2) = u_1(x_1) + \epsilon'_2$ for a sufficiently small and positive ϵ'_2 such that $u_1(x_2) > \max_{y \geq \overline{x}: y \notin R(x_2)}\{u_1(y)\} > 0$ and $u_1(x_2) < u_1(\underline{x})$.

Assume now that $R(x_2) \neq \emptyset$ (cases (b) and (c)). In this case, $R(x_1) \neq \emptyset$ by Lemma 5 and it is known that $\min_{y \in R(x_2)}\{u_1(y)\} \geq$

$\min_{y\in R(x_1)}\{u_1(y)\}$. It is thus possible to fix $u_1(x_2) = \min_{y\in R(x_2)}\{u_1(y)\} - \epsilon_2''$ for a sufficiently small and positive ϵ_2'' such that $u_1(x_2) > u_1(x_1)$ and $u_1(x_2) > \max_{y\geq\bar{x}:y\notin R(x_2)}\{u_1(y)\} > 0$. If $k = 2$, the proof is complete. If $k > 2$ the reasoning can be repeated to continue successively fixing the value of each $u_1(x_r)$ in reference to $u_1(x_{r-1})$.

Step 5: Once the values of $u_1(x)$ and $u_2(x)$ are fixed for all $x \leq \underline{x}$ and for all $x \geq \bar{x}$, we fix the values of $u_1(x)$ and $u_2(x)$ for all $x \in (\underline{x}, \dots, \bar{x})$: Let $\underline{x} = x_0$ and $(\underline{x}, \dots, \bar{x}) = \{x_1, \dots, x_k\}$ with $x_i < x_j$ iff $i < j$.

The first step is to fix any value $u_1(x_1)$ such that $u_1(x_1) \in (\max\{\max_{y\in D_L(x_1)}\{u_1(y)\}, u_1(\bar{x})\}, \min\{\min_{y\leq\underline{x}:y\notin D_L(x_1)}\{u_1(y)\}, u_1(x_0)\})$.

By construction, $u_1(\bar{x}) < u_1(x_1) < u_1(x_0)$. What must be proven is that the interval is non empty. We distinguish four cases:

– $\max_{y\in D_L(x_1)}\{u_1(y)\} \geq u_1(\bar{x})$ and $\min_{y\leq\underline{x}:y\notin D_L(x_1)}\{u_1(y)\} \leq u_1(x_0)$. In this case, $\max_{y\in D_L(x_1)}\{u_1(y)\} < \min_{y\leq\underline{x}:y\notin D_L(x_1)}\{u_1(y)\}$ as a consequence of Lemma 3 and the fact that u_1 is strictly increasing to the left of $\underline{x}$.

– $\max_{y\in D_L(x_1)}\{u_1(y)\} \geq u_1(\bar{x})$ and $\min_{y\leq\underline{x}:y\notin D_L(x_1)}\{u_1(y)\} \geq u_1(x_0)$. In this case, $y \in D_L(x_1)$ implies $y \in D_L(x_0)$ by condition (iii) of CC, thus by construction $u_1(x_0) > \max_{y\in D_L(x_1)}\{u_1(y)\}$.

– $\max_{y\in D_L(x_1)}\{u_1(y)\} \leq u_1(\bar{x})$ and $\min_{y\leq\underline{x}:y\notin D_L(x_1)}\{u_1(y)\} \leq u_1(x_0)$. In this case, notice that $y \leq \underline{x}$ and $y \notin D_L(x_1)$ implies by condition (iii) that $y \leq \underline{x}$ and $y \notin D_L(\bar{x})$. That is, $\bar{x} \geq \bar{x}$ and $\bar{x} \notin R(y)$. Recall that, by construction, $\forall y \leq \underline{x}$, $u_1(y) > \max_{z\geq\bar{x}:z\notin R(y)}\{u_1(z)\}$. Thus, for all $y \leq \underline{x}$ such that $y \notin D_L(x_1)$, $u_1(y) > u_1(\bar{x})$ and therefore $\min_{y\leq\underline{x}:y\notin D_L(x_1)}\{u_1(y)\} > u_1(\bar{x})$.

– $\max_{y\in D_L(x_1)}\{u_1(y)\} \leq u_1(\bar{x})$ and $\min_{y\leq\underline{x}:y\notin D_L(x_1)}\{u_1(y)\} \geq u_1(x_0)$. In this case, by construction we know that $u_1(x_0) > u_1(\bar{x})$.

Once the value of $u(x_1)$ has been fixed the previous reasoning can be repeated to sequentially fix the values of $u_1(x_2), \dots, u_1(x_k)$ for all $x_2, \dots, x_k \in (\underline{x}, \dots, \bar{x})$ such that $u_1(\underline{x}) > u_1(x_i) > u_1(x_{i+1}) > u_1(\bar{x})$ for all $x_i \in (\underline{x}, \dots, \bar{x})$.

In conclusion, by construction, u_1 is strictly increasing to the left of $\underline{x}$ and strictly decreasing to the right of $\underline{x}$. Thus, $\succcurlyeq_1$ is single-peaked and $\hat{x}(\succcurlyeq_1) = \underline{x}$.

To fix the values of $u_2(x)$ for all $x \in (\underline{x}, \dots, \bar{x})$, an analogous procedure can be followed, applying Lemma 4 instead of Lemma 3 and condition (iv) and the fact that u_2 is strictly decreasing to the right of $\bar{x}$ instead of condition (iii) and strict increasingness of u_1 to the left of $\underline{x}$. In this case, the first step is to fix any value of $u_2(x_k)$ such that $u_2(x_k) \in (\max\{\max_{y\in D_R(x_k)}\{u_2(y)\}, u_2(\underline{x})\}, \min\{\min_{y\geq\bar{x}:y\notin D_R(x_k)}\{u_2(y)\}, u_2(x_{k+1})\})$ (note that $x_{k+1} = \bar{x}$), and then by sequentially fixing the values of $u_2(x_{k-1}), \dots, u_2(x_1)$ for all $x_{k-1}, \dots, x_1 \in (\underline{x}, \dots, \bar{x})$ such that $u_2(\bar{x}) > u_2(x_i) > u_2(x_{i-1}) > u_1(\underline{x})$ for all $x_i \in (\underline{x}, \dots, \bar{x})$.

CASE 2. $\underline{x} = \bar{x} = x^*$: In this case we fix $u_1(x^*) = u_2(x^*)$; a sequence of arbitrary strictly decreasing values and strictly positive values of $u_1(y)$ for all $y \geq \bar{x}$, and another sequence of arbitrary strictly decreasing values of $u_2(y)$ for all $y \geq \bar{x}$. Then, $\forall y < \underline{x}$ the values of $u_1(y)$ and $u_2(y)$ are fixed as in CASE 1.

In sum, either if $\underline{x} = \bar{x}$ or $\underline{x} \neq \bar{x}$, two single-peaked preferences $\succcurlyeq_1$ and $\succcurlyeq_2$ have been constructed whose peaks are $\underline{x}$ and $\bar{x}$, respectively. By construction, the profile $(\succcurlyeq_1, \succcurlyeq_2)$ Pareto rationalizes C. □

Proof of Theorem 4. First, assume that C satisfies α, γ, AZ and CC. We know by Theorem 3 that there exists a pair of single-peaked strict preferences, $\mathcal{R} = (\succcurlyeq_1, \succcurlyeq_2)$ that Pareto rationalizes C. Assume that the statement of the theorem is false in such a way that LI also holds but $\succcurlyeq_1$ and $\succcurlyeq_2$ are identical. Then, for all $x, y \in \mathcal{X}$, either ($x \succ_1 y$ and $x \succ_2 y$) or ($y \succ_1 x$ and $y \succ_2 x$). That is, given that $(\succcurlyeq_1, \succcurlyeq_2)$ Pareto rationalizes C, for all $x, y \in \mathcal{X}$, either $C(\{x, y\}) = \{x\}$ or $C(\{x, y\}) = \{y\}$, so a contradiction is reached with LI.

Now, assume that C is Pareto rationalized by two distinct single-peaked strict preferences, $(\succcurlyeq_1, \succcurlyeq_2)$. By Theorem 3 it is known that C satisfies α, γ, AZ and CC. If the two preferences are distinct, then there exist $x, y \in \mathcal{X}$ such that $x \succ_1 y$ and $y \succ_2 x$. Given that C is Pareto rationalized by those preferences, $C(\{x, y\}) = \{x, y\}$. Thus LI also holds. □

Proof of Theorem 5. The proof of the necessary part of Theorem 3 is also valid for proving the necessary part of Theorem 5.

For the proof of the sufficient part, notice that if C satisfies α, γ, AZ and CC then, by Theorem 3, C is Pareto rationalized by a profile of two single-peaked preferences, $(\succcurlyeq_1, \succcurlyeq_2)$. For any arbitrary number $n \geq 2$, consider the profile $\mathcal{R} = (\succcurlyeq_1', \dots, \succcurlyeq_n')$ such that

$$\succcurlyeq_i' = \begin{cases} \succcurlyeq_1 & \text{if } i = 1 \\ \succcurlyeq_2 & \text{otherwise} \end{cases}$$

Given that $(\succcurlyeq_1, \succcurlyeq_2)$ Pareto rationalizes C, for all $K \in \pi(\mathcal{X})$, for all $x \in K$, $x \in C(K)$ if and only if $\nexists y \in K$ such that ($y \succ_1 x$ and $y \succ_2 x$). Thus, by the definition of $\mathcal{R}$, $x \in C(K)$ if and only if $\nexists y \in K$ such that $y \succcurlyeq_i' x$ for all $\succcurlyeq_i' \in \mathcal{R}$. Thus, $\mathcal{R}$ Pareto rationalizes C. □

Proof of Theorem 6. First we prove the necessary part of the theorem: If there exists a profile of weak preferences $\mathcal{R}$ that Pareto rationalizes C, then C satisfies α, γ and AZ[16]:

α: Consider any two sets $K', K \in \pi(\mathcal{X})$ such that $K' \subset K$. Assume that $x \in C(K) \cap K'$. Thus $\nexists y \in K$ that Pareto dominates x in $\mathcal{R}$. Since $K' \subset K$ this is also true for K'. Thus $x \in C(K')$.

γ: Consider any two sets $K', K \in \pi(\mathcal{X})$ and any element $x \in C(K) \cap C(K')$. Given that C is Pareto rationalizable by $\mathcal{R}$, $\nexists y \in K \cup K'$ that Pareto dominates x in $\mathcal{R}$. Thus $x \in C(K \cup K')$.

AZ: Consider any two sets $K', K \in \pi(\mathcal{X})$ such that $C(K) \subset K' \subset K$ and assume that $\neg(C(K') \subset C(K))$. This implies that $\exists x \in K$ such that $x \in C(K')$ and $x \notin C(K)$. Given that $x \notin C(K)$, $\exists y \in C(K)$ that Pareto dominates x in $\mathcal{R}$. By hypothesis $C(K) \subseteq K'$, therefore $\exists y \in K'$ that Pareto dominates x in $\mathcal{R}$. Thus, $x \notin C(K')$ and a contradiction is reached.

The proof of the sufficiency part of Theorem 1 also applies in this case, because the concept of weak order generalizes that of strict order. □

Proof of Theorem 7. As seen in the proof of Theorem 6, if C is Pareto rationalizable by a pair of weak preferences $\mathcal{R} = (\succcurlyeq_1, \succcurlyeq_2)$, then C satisfies α, γ and AZ, which implies that, $\forall K \in \mathcal{X}$, $C(K) = \{x \in K : \nexists y \in K \text{ such that } y \succ_P x\}$.

By condition (i) in CC, for C to be Pareto rationalizable by $\mathcal{R} = (\succcurlyeq_1, \succcurlyeq_2)$, at least one of the preferences is increasing while the other is not decreasing to the left of $\underline{x}$. Similarly, by condition (ii), at least one of the preferences is decreasing while the other is not increasing to the right of $\bar{x}$.

By conditions (i) to (iv) in CC, for all $x, y \in (\underline{x}, \dots, \bar{x})$, $\neg(x \succ_P y)$ and $\neg(y \succ_P x)$. This implies that for C to be Pareto rationalizable by $\mathcal{R} = (\succcurlyeq_1, \succcurlyeq_2)$, there is no $x, y \in (\underline{x}, \dots, \bar{x})$ such that x Pareto dominates y in $\mathcal{R}$. Thus, for all $x, y \in (\underline{x}, \dots, \bar{x})$, if $x \succ_1 y$, then $y \succ_2 x$ and if $x \sim_1 y$, then $y \sim_2 x$. Thus, $\succcurlyeq_1$ and $\succcurlyeq_2$ are a pair of mirrored weak preferences in the range $(\underline{x}, \bar{x})$. □

Proof of Theorem 8. The first step of the proof of the necessary part is similar to the corresponding step of the proof of Theorem 3 but with Theorem 6 applied instead of Theorem 1. The proof that C satisfies conditions (i) and (ii) of CC is also the same, just noticing that, even though all $\succcurlyeq_i \in \mathcal{X}$ are single-peaked weak preferences, for all $x, y \in \mathcal{X}$ with $x \neq y$, $x \succcurlyeq y$ and $y \succcurlyeq x$ only

[16] This part of the proof does not in fact depend on whether the elements $\succcurlyeq_i \in \mathcal{R}$ are strict or weak preferences. We provide the proof to avoid ambiguities.

holds if either $y < \hat{x}(\succcurlyeq_i) < x$ or $x < \hat{x}(\succcurlyeq_i) < y$, thus, it is still true that both, $x_j < x_i < \underline{x}$ and $\overline{x} < x_i < x_j$ imply $x_i \succ_i x_j$ for all $i \leq n$. The proof of the sufficient part of Theorem 3 is valid as proof of the sufficient part of Theorem 8 because strict orders are particular cases of weak orders. □

Proof of Theorem 9. If C satisfies α, γ, AZ and CC then, by Theorem 8, there exists a pair of single-peaked weak preferences, $\succcurlyeq_1, \succcurlyeq_2$, that Pareto rationalizes C. Assume that the statement of the theorem is false. Then there exist x, y, z such that $C(\{x, y\}) = \{x, y\}$, $C(\{x, z\}) = \{x, z\}$ and $\succcurlyeq_1$ is identical to $\succcurlyeq_2$. This is only possible if $x \sim_1 y \sim_1 z$ and $x \sim_2 y \sim_2 z$. This is in contradiction with the fact that both are single-peaked preferences, because in that case it is impossible for more than two alternatives to be indifferent according to either $\succcurlyeq_1$ or $\succcurlyeq_2$. □

Proof of Theorem 10. The proof of the necessary part is like the proof of the necessary part of Theorem 8. The proof of the sufficient part is similar to the corresponding part of the proof of Theorem 5. □

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