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Admissible orders on fuzzy numbers

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***Abstract*—From the more than two hundred partial orders for fuzzy numbers proposed in the literature, only a few are total. In this paper, we introduce the notion of admissible order for fuzzy numbers equipped with a partial order, i.e. a total order which refines the partial order. In particular, it is given special attention to the partial order proposed by Klir and Yuan in 1995. Moreover, we propose a method to construct admissible orders on fuzzy numbers in terms of linear orders defined for intervals considering a strictly increasing upper dense sequence, proving that this order is admissible for a given partial order. Finally, we use admissible orders to ranking the path costs in fuzzy weighted graphs.**

***Index Terms*—Fuzzy numbers, orders on fuzzy numbers, admissible orders, fuzzy weighted graphs.**

I. INTRODUCTION

FUZZY numbers were introduced by Zadeh [1] to deal with imprecise numerical quantities in a practical way. The concept of a fuzzy number plays a fundamental role in formulating quantitative fuzzy variables, i.e. variables whose states are fuzzy numbers.

The study of admissible orders over the set of closed subintervals of $[0,1]$, i.e. orders which refine the natural order for intervals, starts with the work of Bustince *et al.* [2] and from then several pieces of research on this topic have been made, for example in [3], [4]. Lately, this notion was adapted for other domains in [5], [6], [7], [8], [9].

From the more than two hundred partial orders for fuzzy numbers proposed in the literature, only a few are total, for example [10], [11], [12], [13], [14]. Moreover, no study on admissible orders for fuzzy numbers or a subclass of them has been made so far. In order to overcome this lack and motivated mainly by the application potential of this subject, in this work we introduce and analyze the notion of admissible orders for fuzzy numbers with respect to a partial order and in particular, we explore the case where this partial order is the given in [15].

On the other hand, fuzzy weighted graphs are a generalization of the weighted graphs where fuzzy numbers are used to model the uncertainty in the weights of the edges (c.f. [16]). The fuzzy shortest path problem was first enunciated in [17] and since then several algorithms have been proposed to determine the fuzzy shortest path length in fuzzy weighted graphs (see for example [16], [18], [19]). It is worth to note that the order considered on fuzzy numbers is fundamental to such algorithms. In [16] it is formalized and proposed an algorithm to determine the fuzzy shortest path (routes) length on fuzzy weighted graphs. It pays special attention to the ranking methods of the routes, based in a defuzzification method. Nevertheless, the approach presents a problem with the center of gravity defuzzification method. In this paper, we present a solution to what was raised in [16]. It does not consider defuzzification thanks to the given definition of admissible order.

This paper is organized as follows: In Section 2, in addition to establishing the notation used, we recall some essential notions for the remaining sections. In Section 3 we see the most basic partial order on fuzzy numbers, and a total order proposal in [14]. The notion of admissible order for fuzzy numbers is studied in Section 4. Section 5, presents an application of admissible orders in the Shortest Path problem. Section 6, we present another application in graphs, this time for the travelling salesman problem considering the capitals of the Brazilian Northeast. Finally, Section 7 provides some final remarks.

II. PRELIMINARY CONCEPTS

In this section, we introduce notations, definitions and preliminary facts which are used throughout this work.

Given a poset $\langle P, \leqq \rangle$ where $\leqq$ is a partial order, or just an order and $a, b \in P$, we denote by $a \parallel b$ when a and b are incomparable, i.e. when neither $a \leqq b$ nor $b \leqq a$. When all the elements of P are comparable we will call the order of linear or total. We will denote the set of real numbers by $\mathbb{R}$ and $\mathbb{N}$ will denote the set of natural numbers.

Based on [20], [21] we consider the following definitions of left and right continuity of real functions.

Definition II.1. *Let $a \in \mathbb{R}$, $f :]-\infty, a[\to \mathbb{R}$ and $g :]a, +\infty[\to \mathbb{R}$ be functions. Then f is left-continuous if for each $x \in]-\infty, a[$ and increasing sequence $(x_i)_{i\in\mathbb{N}}$ of real numbers with $x_i \to x$, as $i \to \infty$ we have that $\lim\limits_{x_i \to x} f(x_i) = f(\lim\limits_{x_i \to x} x_i)$. Dually, g is right-continuous if for each $x \in]-\infty, a[$ and decreasing sequence $(x_i)_{i\in\mathbb{N}}$ of real numbers with $x_i \to x$, as $i \to \infty$ we have that $\lim\limits_{x_i \to x} g(x_i) = g(\lim\limits_{x_i \to x} x_i)$.*

A. Admissible Orders on the Real Closed Interval Set

Let $\mathbb{IR}$ be the set of all the non-empty, closed and bounded intervals of real numbers, i.e.

$$\mathbb{IR} = \{[a, b] \, : \, a, b \in \mathbb{R}, \; a \leq b\}.$$

Dedicated to the memory of Professor Roberto Callejas-Bedregal, a Covid 19 victim (1959 –2021).

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Closed intervals of real numbers will be called just of intervals. Degenerate intervals, that is, intervals $[a,a]$ will be written in the simplified form $[a]$. Given an interval $[a,b]$, its lower bound is denoted by $\underline{[a,b]}$, and its upper bound is denoted by $\overline{[a,b]}$, i.e. $\underline{[a,b]} = a$ and $\overline{[a,b]} = b$ for every $[a,b] \in \mathbb{IR}$.

Since intervals are sets, the inclusion determines an order. Observe that the inclusion order for intervals can be determined exclusively on their extremes as follows

$$[a,b] \subseteq [c,d] \Leftrightarrow c \leq a \text{ and } b \leq d.$$

Auxiliarly, we also define the following strict order on $\mathbb{IR}$:

$$[a,b] \Subset [c,d] \Leftrightarrow c < a \text{ and } b < d.$$

Notice that, $[3,4] \subseteq [3,5]$ but $[3,4] \not\Subset [3,5]$. Therefore, $\subseteq \neq \Subset$.

In [22] consider the following order for $\mathbb{IR}$:

$$[a,b] \leq_{KM} [c,d] \Leftrightarrow a \leq c \text{ and } b \leq d.$$

Nowadays, this order, which is inherited from the usual one between real numbers, is the most widely used for $\mathbb{IR}$. But this order is not linear, i.e. is not total, and, in some situations a linear order is fundamental (see for example [23]). Of course, there are infinitely many linear orders on $\mathbb{IR}$. This motived [2], in the context of interval-valued fuzzy sets, i.e. in $L([0,1]) = \{[a,b] \in \mathbb{IR} : 0 \leq a \leq b \leq 1\}$, to introduce the notion of admissible linear orders. For Bustince et al., in [2], an order only is admissible if it is linear and refines or encompasses the usual order on $L([0,1])$, i.e., $\langle L([0,1]), \leq_{KM}\rangle$.

But, it is clear that this notion can be adapted in a straightforward way for $\mathbb{IR}$:

Definition II.2. *A relation $\preceq$ on $\mathbb{IR}$ is called an admissible order, if*

(i) $\preceq$ is a linear order on $\mathbb{IR}$;
(ii) for all A, B on $\mathbb{IR}$, $A \preceq B$ whenever $A \leq_{KM} B$.

In addition, $\preceq$ is compatible with the addition if, for each $[a,b],[c,d],[e,f] \in \mathbb{IR}$, we have that $[a+e,b+f] \prec [c+e,d+f]$ whenever $[a,b] \prec [c,d]$.

Example II.1. *Admissible orders on $\mathbb{IR}$*

1) *Lexical 1: $[a,b] \preceq_{Lex1} [c,d] \Leftrightarrow a < c$ or $(a = c$ and $b \leq d)$;*
2) *Lexical 2: $[a,b] \preceq_{Lex2} [c,d] \Leftrightarrow b < d$ or $(b = d$ and $a \leq c)$;*
3) *Xu-Yager (adapted from [24]):*

$$[a,b] \preceq_{XY} [c,d] \Leftrightarrow a+b < c+d \text{ or } (a+b = c+d \text{ and } b-a \leq d-c);$$

4) *Twice Xu-Yager (adapted from [3, Ex. 4]):*

$$[a,b] \preceq_{2XY} [c,d] \Leftrightarrow a+3b < c+3d \text{ or } (a+3b = c+3d \text{ and } b-a \leq d-c).$$

Observe that all this order are compatible with the addition.

B. Fuzzy sets

The following definitions can be found in [15], [25] and in most of the introductory books on fuzzy sets theory. In all this section X will be a non-empty reference set with generic elements denoted by x.

Definition II.3. *A fuzzy set A on X is a function $A : X \longrightarrow [0,1]$. In addition,*

(i) The support of A, is the set $supp(A) = \{x \in X : A(x) > 0\}$;
(ii) The kernel of A, is the set $\ker(A) = \{x \in X : A(x) = 1\}$;
(iii) Given $\alpha \in]0,1]$, the α-cut of A is the set ${}^{A}/_{\alpha} = \{x \in X : A(x) \geq \alpha\}$;
(iv) The height of A is $h(A) = \sup_{x \in X} A(x)$.

If $h(A) = 1$, then the fuzzy set A is called of normal fuzzy set. Clearly, in a finite set X, we have that, A is normal if and only if $\ker(A) \neq \emptyset$.

C. Fuzzy numbers

There are several different definitions of fuzzy numbers in the literature, for instance [13], [15], [25], [26], [27], [28], [29], [30]. Most of them vary in the kind of continuity required for the membership function. For example, in [13], [29] is considered upper semi-continuity whereas in [26], [27] is required piecewise continuity and in [15], [25], [28] no continuity constraint is required. Another difference can be that some require that the kernel of the fuzzy number be a singleton, another one is that it should be non-empty. Besides, in some definitions the support is bounded (c.f. [28]) whereas in others it is unbounded (c.f. [30]). Here we adopted the approach given in [15] which only considers fuzzy numbers with bounded support.

Definition II.4. *A fuzzy set A on $\mathbb{R}$ is called a fuzzy number if it satisfies the following conditions*

(i) A is normal;
(ii) ${}^{A}/_{\alpha}$ is a closed interval for every $\alpha \in]0,1]$;
(iii) the support of A is bounded.

Henceforth, $\mathcal{F}(\mathbb{R})$ will denote the set of all fuzzy numbers.

We note that Definition II.4 is equivalent to that which appears in [25, p.44].

Remark II.1. *Since the support of a fuzzy number of A is bounded, there exist ω_1, ω_2 in $\mathbb{R}$, s.t. $cl(supp(A)) = [\omega_1, \omega_2]$, where cl is a topological closure operator with respect to the usual topology on $\mathbb{R}$. In addition, we will use the notation $supp(A)^-$ and $supp(A)^+$ for ω_1 and ω_2, respectively. Analogously, since the kernel of A is a closed interval $[a,b]$, we will use the notation $\ker(A)^-$ and $\ker(A)^+$ for a and b, respectively.*

The next theorem gives a full characterization of fuzzy numbers.

Theorem II.1. *[15, Theorem 4.1] Let A be a fuzzy set on $\mathbb{R}$. Then, $A \in \mathcal{F}(\mathbb{R})$ if and only if there exist a closed interval $[a,b] \neq \emptyset$, a function l from $]-\infty, a[$ to $[0,1]$ which is right-continuous, increasing and $l(x) = 0$ for each $x \in]-\infty, supp(A)^-]$, and a function r from $]b,+\infty[$ to $[0,1]$ which is left-continuous, decreasing and $r(x) = 0$ for each $x \in [supp(A)^+, +\infty[$, such that*

$$A(x) = \begin{cases} 1, & \text{if } x \in [a,b], \\ l(x), & \text{if } x \in]-\infty, a[, \\ r(x), & \text{if } x \in]b,+\infty[. \end{cases} \quad (1)$$

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Corollary II.1. *For each interval $[a,b] \in \mathbb{IR}$ their characteristic function $\widetilde{[a,b]} : \mathbb{R} \to [0,1]$ defined by*

$$\widetilde{[a,b]}(x) = \begin{cases} 1, & \text{if } x \in [a,b], \\ 0, & \text{if } x \notin [a,b] \end{cases}$$

is a fuzzy number.

So, in some sense, we can think that fuzzy numbers generalize the set of closed intervals of real numbers, i.e. that $\mathbb{IR} \subseteq \mathcal{F}(\mathbb{R})$ and therefore $\mathbb{R} \subseteq \mathcal{F}(\mathbb{R})$ too, once degenerated intervals can be seen as real numbers and instead of writing $\widetilde{[a,a]}$ or $\widetilde{[a]}$ we just use $\widetilde{a}$. A fuzzy number $\widetilde{a}$ is called a crisp number or fuzzy singleton in [26]. A fuzzy number A is called a triangular fuzzy number whenever $ker(A) = [a]$, $l(x) = \frac{x - supp(A)^-}{a - supp(A)^-}$ for all $x \in]supp(A)^-, a[$ and $r(x) = \frac{supp(A)^+ - x}{supp(A)^+ - a}$ for all $x \in]a, supp(A)^+[$, and is denoted by the triple $(supp(A)^-, a, supp(A)^+)$.

Remark II.2. *From [25, Remark 3.3.2.] we have that each fuzzy number A is an upper semi-continuous function and therefore, the definition given in [13], [29] is equivalent to Definition II.4.*

In the proof of Theorem II.1, Klir and Yuan, provide a characterization of the α-cut of fuzzy numbers based on the functions $l^* : [0,1] \to]-\infty, a[$ and $r^* : [0,1] \to]b, \infty[$ defined by

$$l^*(\alpha) = \inf\{x \in]-\infty, a[\,: l(x) \geq \alpha\}$$

and

$$r^*(\alpha) = \sup\{x \in]b, \infty[\,: r(x) \geq \alpha\}$$

with the convention that if $\{x \in]-\infty, a[\,: l(x) \geq \alpha\} = \{x \in]b, \infty[\,: r(x) \geq \alpha\} = \varnothing$ then $l^*(\alpha) = a$ and $r^*(\alpha) = b$, and given by

$$^{A}/_{\alpha} = \begin{cases} [supp(A)^-, supp(A)^+], & \text{if } \alpha = 0, \\ [l^*(\alpha), r^*(\alpha)], & \text{if } \alpha \in]0,1[, \\ [a,b], & \text{if } \alpha = 1. \end{cases} \tag{2}$$

Example II.2. *Consider the fuzzy number A (see Figure 1) given by:*

$$A(x) = \begin{cases} 1, & \text{if } x \in [3,4], \\ l(x), & \text{if } x \in]-\infty, 3[, \\ r(x), & \text{if } x \in]4, +\infty[, \end{cases}$$

where l and r are:

$$l(x) = \begin{cases} \frac{x+1}{4}, & \text{if } 2 \leq x < 3, \\ \frac{1}{2}, & \text{if } 1 \leq x < 2, \\ 0, & \text{if } x < 1, \end{cases}$$

and

$$r(x) = \begin{cases} \frac{20-3x}{8}, & \text{if } 4 < x \leq 5 \\ \frac{6-x}{3}, & \text{if } 5 < x \leq 6, \\ 0, & \text{if } x > 6, \end{cases}$$

verify the conditions of the Theorem II.1. Let's calculate the α-cut of A, starting with l, i.e.:

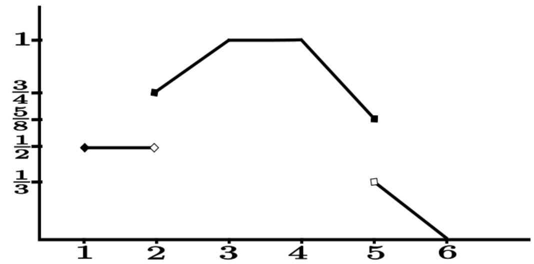

Figure 1. Fuzzy Number A.

$$l^*(\alpha) = \begin{cases} 4\alpha - 1, & \text{if } \alpha \in \left[\frac{3}{4}, 1\right], \\ 2, & \text{if } \alpha \in \left]\frac{1}{2}, \frac{3}{4}\right[, \\ 1, & \text{if } \alpha \in \left[0, \frac{1}{2}\right]. \end{cases} \tag{3}$$

To continue, with r it's analog, we have

$$r^*(\alpha) = \begin{cases} \frac{20-8\alpha}{3}, & \text{if } \alpha \in \left[\frac{5}{8}, 1\right], \\ 5, & \text{if } \alpha \in \left[\frac{1}{3}, \frac{5}{8}\right[, \\ 6 - 3\alpha, & \text{if } \alpha \in \left[0, \frac{1}{3}\right[. \end{cases} \tag{4}$$

Therefore, from Eq. (2)*, we express the α-cut of A given by:*

$$\begin{aligned} ^{A}/_{\alpha} &= \begin{cases} [l^*(\alpha), r^*(\alpha)], & \text{if } \alpha \in [0,1[, \\ [a,b], & \text{if } \alpha = 1 \end{cases} \\ &= \begin{cases} \left[4\alpha - 1, \frac{20-8\alpha}{3}\right], & \text{if } \alpha \in \left[\frac{3}{4}, 1\right], \\ \left[2, \frac{20-8\alpha}{3}\right], & \text{if } \alpha \in \left[\frac{5}{8}, \frac{3}{4}\right[, \\ [2,5], & \text{if } \alpha \in \left]\frac{1}{2}, \frac{5}{8}\right[, \\ [1,5], & \text{if } \alpha \in \left[\frac{1}{3}, \frac{1}{2}\right], \\ [1, 6-3\alpha], & \text{if } \alpha \in \left[0, \frac{1}{3}\right[. \end{cases} \end{aligned}$$

Proposition II.1. *[15, p. 109-110] Let $A, B \in \mathcal{F}(\mathbb{R})$ then the fuzzy sets $A \wedge B$ and $A \vee B$ defined for each $x \in \mathbb{R}$ by*

$$A \wedge B(x) = \sup_{x = \min\{y,z\}} \min\{A(y), B(z)\}$$

and

$$A \vee B(x) = \sup_{x = \max\{y,z\}} \min\{A(y), B(z)\},$$

are fuzzy numbers. In addition, $\langle \mathcal{F}(\mathbb{R}), \wedge, \vee \rangle$ is a distributive lattice.

The arithmetic operations on fuzzy numbers are defined based on the Zadeh extension principle. Let A and B be two fuzzy numbers and $\star \in \{+, -, \cdot, \div\}$ and the fuzzy set $A \star B$ defined for z in $\mathbb{R}$ [28] as $A \star B(z) = \sup_{z = x \star y} \min\{A(x), B(y)\}$.

Then $A \star B$ is also a fuzzy number [15, Theorem 4.2]. Nevertheless, to $A \div B$ be a fuzzy number it is necessary that $0 \notin supp(B)$ [28, pp. 64].

An alternative form to define the arithmetic operations on $\mathcal{F}(\mathbb{R})$ is based on the Klir and Yuan decompositional theorem [15, Theorem 2.5] which proves that each fuzzy set can be recovered from its α-cut. Thereby, when either $0 \notin supp(B)$ or $\star \neq \div$, $A \star B$ is the fuzzy set whose α-cut are

$$^{A \star B}/_{\alpha} = {}^{A}/_{\alpha} \diamond {}^{B}/_{\alpha}, \tag{5}$$

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where $\diamond \in \{+,-,\cdot,\div\}$ is the respective arithmetic operation on $\mathbb{IR}$ (see [31]) for all $\alpha \in]0,1]$. In addition, when A and B are triangular fuzzy numbers then $A+B$ and $A-B$ are also triangular fuzzy numbers, but $A\cdot B$ and $A\div B$ can not be triangular fuzzy numbers [25, Section 3.5] and [32]. Indeed, $A=(a_1,a_2,a_3)$ and $B=(b_1,b_2,b_3)$ then $A+B=(a_1{+}b_1,a_2{+}b_2,a_3{+}b_3)$ and $A-B=(a_1-b_3,a_2-b_2,a_3-b_1)$. Notice also that for each $r\in\mathbb{R}$ and a triangular fuzzy number $A=(a_1,a_2,a_3)$ we have that $\widetilde{r}\cdot A=(a_1\cdot r,a_2\cdot r,a_3\cdot r)$ if $r\geq 0$ and $\widetilde{r}\cdot A=(a_3\cdot r,a_2\cdot r,a_1\cdot r)$ if $r<0$. For the product of two fuzzy numbers see [15], [25], [32], [33].

D. Order on fuzzy numbers

The following partial order in $\mathcal{F}(\mathbb{R})$ was proposed by Zadeh in [1].

Definition II.5. *Let A and B be two fuzzy numbers. We write:*

$$A \leq_Z B \Longleftrightarrow A(x)\leq B(x) \text{ for all } x\in\mathbb{R}.$$

The Figure-2a shows a case where $A\leq_Z B$.

The Zadeh's order can be characterized in terms of the inclusion order on their α-cut.

Proposition II.2. *[15, Theorem 2.3-viii] Let A and $B\in\mathcal{F}(\mathbb{R})$. Then $A\leq_Z B$ if and only if $\forall\alpha\in]0,1]$ ${}^{A}/_{\alpha}\subseteq {}^{B}/_{\alpha}$.*

The problem with this order is that it does not generalize the usual order on the real numbers. In fact, given $x,y\in\mathbb{R}$ such that $x<y$, we have that $\widetilde{x}\nleq_Z \widetilde{y}$.

Klir and Yuan in [15] proposed the following partial order on $\mathcal{F}(\mathbb{R})$:

Let A and $B\in\mathcal{F}(\mathbb{R})$. Then

$$A\leq_{KY} B \Longleftrightarrow A\wedge B = A.$$

Proposition II.3. *[15, p. 114] Given fuzzy number A and B, the following assertions are equivalents*

1) $A\leq_{KY} B$;
2) $A\vee B = B$;
3) ${}^{A}/_{\alpha}\leq_{KM} {}^{B}/_{\alpha}$ *for each $\alpha\in]0,1]$.*

Observe that the Klir-Yuan partial order, when restricted to intervals, corresponds to the Kulisch-Miranker order and when restricted to real numbers it agrees with the usual order. The problem is that there are pairs of fuzzy numbers which are non-comparable under this order. The Figures 2a and 2b present the two generic cases of pairs of fuzzy numbers which are non-comparable by the partial order $\leq_{KY}$ and therefore, $\leq_{KY}$ is not a linear order.

From the above observation, we get the following characterization of the non-comparable fuzzy numbers for this order.

Corollary II.2. *Let A and B be fuzzy numbers. A and B are non-comparable in the order $\leq_{KY}$ if and only if one of the following assertions holds:*

1) *There exists $\alpha\in]0,1]$ such that ${}^{A}/_{\alpha}\Subset {}^{B}/_{\alpha}$ or ${}^{B}/_{\alpha}\Subset {}^{A}/_{\alpha}$;*
2) *There exist $\alpha,\beta\in]0,1]$ such that ${}^{A}/_{\alpha}<_{KM} {}^{B}/_{\alpha}$ and ${}^{B}/_{\beta}<_{KM} {}^{A}/_{\beta}$.*

Remark II.3. *Since for each positive real number r and $A\in\mathcal{F}(\mathbb{R})$ we have that ${}^{A_{-r}}/_{\alpha}<_{KM} {}^{A}/_{\alpha}<_{KM} {}^{A_{+r}}/_{\alpha}$, for all*

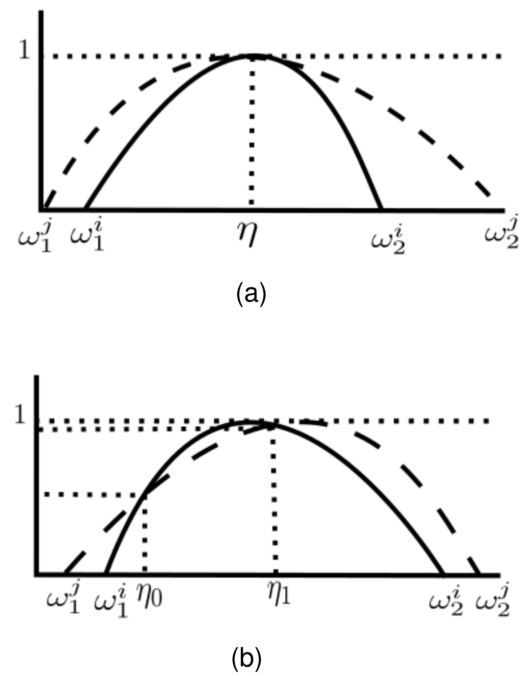

Figure 2. General cases of pairs of non-comparable fuzzy number with respect the Klir-Yuan order.

$\alpha\in]0,1]$ and where $A_{-r}(x)=A(x+r)$ and $A_{+r}(x)=A(x-r)$, the distributive lattice $\langle\mathcal{F}(\mathbb{R}),\wedge,\vee,\leq_{KY}\rangle$ is not bounded and therefore not a complete lattice. Also, $\langle\mathcal{F}(\mathbb{R}),+,\cdot,\leq_{KY}\rangle$ is a subdistributive lattice (see [15, p. 104, point 4]) which is not bounded and therefore it is not a complete lattice.

1) Wang-Wang order: Wei Wang and Zhenyuan Wang in [14] propose a total order for the set of fuzzy numbers based on α-cut from a special type of sequence in $[0,1]$.

Definition II.6. *([14]) Let $S=(\alpha_i)_{i\in\mathbb{N}}$ be a sequence in $]0,1]$. Then S is upper dense if, for every point $x\in]0,1]$ and any $\varepsilon>0$, there exists $i\in\mathbb{N}$ such that $\alpha_i\in[x,x+\varepsilon[$.*

Remark II.4. *If $S=(\alpha_i)_{i\in\mathbb{N}}$ is an upper dense sequence in $]0,1]$ then*

1) $\inf S=0$ *and* $\sup S=1$;
2) *for all $n\in\mathbb{N}$ there are $i<j\in\mathbb{N}$ such that $n\leq i$, $\alpha_i<\alpha_{i+1}$ and $\alpha_{j+1}<\alpha_j$;*
3) *for all $n\in\mathbb{N}$ the sequence $S_n=(\alpha'_i)_{i\in\mathbb{N}}$ with $\alpha'_i=\alpha_{n+i}$ is also an upper dense sequence in $]0,1]$;*
4) *for all $\alpha\in]0,1]$, the sequence $S_\alpha=(\alpha'_i)_{i\in\mathbb{N}}$ with $\alpha'_i=\alpha_{i-1}$ for each $i\geq 2$ and $\alpha'_1=\alpha$ is also an upper dense sequence in $]0,1]$;*
5) *in Definition II.6, when say $\alpha_i\in[x,x+\varepsilon[$, we have that $x{+}\varepsilon$ not need belong to $[0,1]$ but $\alpha_i\in]0,1]$ and therefore $\alpha_i\in[x,x+\varepsilon[\,\cap\,[x,1]$.*

Remark II.5. *When you take $x=1$ in the Definition II.6 then there is a $i\in\mathbb{N}$ such that $\alpha_i\in[1,1+\epsilon]$. Then in all upper dense sequence $S=(\alpha_i)_{i\in\mathbb{N}}$ exists k such that $\alpha_k=1$. But, by the Remark II.4-(3), the sequence S_{k+1} is upper dense and therefore $\alpha'_j=1$ for some $j\in\mathbb{N}$. So, each upper dense sequences has infinite copies of 1. Therefore, $max(S)=1$.*

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Examples of these sequences can be found in [14, Example 1 and 2].

Definition II.7 ([14])**.** *Let A be fuzzy numbers. For a given upper dense sequence $S = (\alpha_i)_{i\in\mathbb{N}}$ in $]0,1]$, we define $c_i : \mathcal{F}(\mathbb{R}) \longrightarrow \mathbb{R}$ given by*

$$c_i(A) = \begin{cases} r^*(\alpha_{\frac{i}{2}}) - l^*(\alpha_{\frac{i}{2}}), & \text{if } i \text{ is even,} \\ l^*(\alpha_{\frac{i+1}{2}}) + r^*(\alpha_{\frac{i+1}{2}}), & \text{if } i \text{ is odd.} \end{cases}$$

Definition II.8 ([14])**.** *Let A and B be two fuzzy numbers and an upper dense sequence $S = (\alpha_i)_{i\in\mathbb{N}}$ in $]0,1]$. We say that $A <_{WW}^S B$ when there exists a positive integer n_0 such that $c_{n_0}(A) < c_{n_0}(B)$ and $c_i(A) = c_i(B)$ for all positive integers $i < n_0$. We say that $A \leq_{WW}^S B$ if and only if $A <_{WW}^S B$ or $A = B$.*

As it is well know, any fuzzy set A can be fully identified with its α-cut in the following sense:

$$A(x) = \sup_{\alpha\in(0,1]} \alpha \cdot \chi_{A/\alpha}(x),$$

where $\chi_{A/\alpha}$ is the characteristic function of the interval A/α. This is called decomposition theorem [15, Theorems 2.5]. There are some variants of this theorem such as [14, Theorem 3] and [15, Theorems 2.6 and 2.7]. In particular, Wang and Wang variant proves that any fuzzy number is recovered from just a countably subset of their α-cut.

Theorem II.2. *[14, Theorem 3] Let A be a fuzzy number and $S = (\alpha_i)_{i\in\mathbb{N}}$ be an upper dense sequence in $]0,1]$. Then*

$$A(x) = \sup_{i\in\mathbb{N}} \alpha \cdot \chi_{A/\alpha_i}(x).$$

Corollary II.3. *Let A and B be two fuzzy numbers and $S = (\alpha_i)_{i\in\mathbb{N}}$ be an upper dense sequence in $]0,1]$. Then $A = B$ if and only if $A/\alpha_i = B/\alpha_i$, for all $i \in \mathbb{N}$.*

Proof. Straightforward from Theorem II.2. □

Theorem II.3. *[14] Let S be an upper dense sequence in $]0,1]$. Then $\leq_{WW}^S$ is a linear order on $\mathcal{F}(\mathbb{R})$.*

Remark II.6. *There exist several pairs of upper dense sequences S_1 and S_2 in $]0,1]$ which determine distinct linear orders, i.e. $\leq_{WW}^{S_1} \neq \leq_{WW}^{S_2}$. Therefore, we are dealing with a family of linear orders (see [14, Example 3]).*

III. Admissible orders on fuzzy numbers

Definition III.1. *Let $\leqq$ and $\preceq$ be two orders on $\mathcal{F}(\mathbb{R})$. The order $\preceq$ is called an admissible order w.r.t. $\langle \mathcal{F}(\mathbb{R}), \leqq \rangle$, if*
(i) $\preceq$ is a linear order on $\mathcal{F}(\mathbb{R})$;
(ii) for all A, B in $\mathcal{F}(\mathbb{R})$, $A \preceq B$ whenever $A \leqq B$.

Thus, an order $\preceq$ on $\mathcal{F}(\mathbb{R})$ is admissible for $\langle \mathcal{F}(\mathbb{R}), \leqq \rangle$ if it is linear and refines the order $\leqq$. In particular, when the order $\leqq$ is $\leq_{KY}$ we will call $\preceq$ just of admissible order on $\mathcal{F}(\mathbb{R})$. Furthermore, if $\leqq$ is a linear order then $\preceq$ and $\leqq$ are the same.

Proposition III.1. *Let $\preceq$ be an admissible order on $\mathcal{F}(\mathbb{R})$. Then, there are not greatest or smallest elements in $\mathcal{F}(\mathbb{R})$.*

Proof. Straightforward from Definition III.1 and Remark II.3. □

Definition III.2. *Let $A, B \in \mathcal{F}(\mathbb{R})$ and $S = (\alpha_i)_{i\in\mathbb{N}}$ be an upper dense sequence in $]0,1]$. Then define $m(A,B)$ by*

$$m(A,B) = \begin{cases} \min\{i \in \mathbb{N} : A/\alpha_i \neq B/\alpha_i\}, & \text{if } A \neq B, \\ 0, & \text{if } A = B. \end{cases}$$

Observe that $m(A,B) = m(B,A)$ and that, by Corollary II.3, $m(A,B)$ is well defined.

Proposition III.2. *Let $S = (\alpha_i)_{i\in\mathbb{N}}$ be an upper dense sequence in $]0,1]$ and $A, B \in \mathcal{F}(\mathbb{R})$. Then $\{\alpha_i \in S : A/\alpha_i \neq B/\alpha_i\} \neq \{1\}$.*

Proof. If $A = B$ then $\{\alpha_i \in S : A/\alpha_i \neq B/\alpha_i\} = \varnothing \neq \{1\}$. If $A \neq B$ then, by Corollary II.3, we have that $A/\alpha_i \neq B/\alpha_i$ for some $i \in \mathbb{N}$. Suppose that $\alpha_i = 1$ then we have four cases: $\ker(A)^- < \ker(B)^-$, $\ker(A)^- > \ker(B)^-$, $\ker(A)^+ < \ker(B)^+$ or $\ker(A)^+ > \ker(B)^+$. In the first case, take $x = \ker(A)^-$. Then $A(x) = 1$ and $B(x) < 1$. So, for $\alpha = \frac{B(x)+1}{2}$, we have that $x \in A/\alpha$ and $x \notin B/\alpha$, i.e. $A/\alpha \neq B/\alpha$. So, since S is an upper sequence in $]0,1]$, there exists $j \in \mathbb{N}$ such that $\alpha \leq \alpha_j < 1$. Since, $x \in A/\alpha_j$ and $x \notin B/\alpha_j$, then $A/\alpha_j \neq B/\alpha_j$ and therefore, $\{\alpha_i \in S : A/\alpha_i \neq B/\alpha_i\} \neq \{1\}$.

The other three cases can be similarly proved. □

Definition III.3. *Let $S = (\alpha_i)_{i\in\mathbb{N}}$ be an upper dense sequence in $]0,1]$, $\preceq$ be an order on $\mathbb{IR}$, and $A, B \in \mathcal{F}(\mathbb{R})$. Then,*

$$A \trianglelefteq^S B \Longleftrightarrow A = B \text{ or } A/\alpha_{m(A,B)} \prec B/\alpha_{m(A,B)}.$$

Observe that, taking as convention that $A/0 = [supp(A)^-, supp(A)^+]$ then $A \trianglelefteq^S B \iff A/\alpha_{m(A,B)} \preceq B/\alpha_{m(A,B)}$.

Theorem III.1. *Let $\preceq$ be an admissible order on $\mathbb{IR}$ and $S = (\alpha_i)_{i\in\mathbb{N}}$ be an upper dense sequence in $]0,1]$. The relation $\trianglelefteq^S$ is an admissible order on $\mathcal{F}(\mathbb{R})$.*

Proof. Let $S = (\alpha_i)_{i\in\mathbb{N}}$ be an upper dense sequence in $]0,1]$.
Reflexivity: Straightforward from Definition III.3.
Antisymmetry: Let A, B be fuzzy numbers such that $A \trianglelefteq^S B$ and $B \trianglelefteq^S A$. Suppose that $A \neq B$ then, from Corollary II.3, $\{i \in \mathbb{N} : A/\alpha_i \neq B/\alpha_i\} \neq \varnothing$. Let $m = m(A,B) = \min\{i \in \mathbb{N} : A/\alpha_i \neq B/\alpha_i\} = m(B,A)$, then, from the former $A/\alpha_m \preceq B/\alpha_m$ and $B/\alpha_m \preceq A/\alpha_m$. So, because $\preceq$ is an order, $A/\alpha_m = B/\alpha_m$, which is a contradiction. Therefore, $A = B$.
Transitivity: Let A, B, and C be three fuzzy numbers such that $A \trianglelefteq^S B$ and $B \trianglelefteq^S C$. If $A = B$ or $B = C$ then trivially $A \trianglelefteq^S C$ and if $A = C$ then, by antisymmetry, $A = B = C$. If $A \neq B$ and $B \neq C$ then $A \triangleleft^S B$ and $B \triangleleft^S C$. So, $A/\alpha_k \prec B/\alpha_k$ and $B/\alpha_m \prec C/\alpha_m$, where $k = m(A,B)$ and $m = m(B,C)$. If $k \leq m$ then $B/\alpha_k \preceq C/\alpha_k$ and since $A/\alpha_k \prec B/\alpha_k$ then $A/\alpha_k \prec C/\alpha_k$. In addition, if $j < k$ then $A/\alpha_j = B/\alpha_j$ and $B/\alpha_j = C/\alpha_j$, and therefore $A/\alpha_j = C/\alpha_j$. Therefore, $A \trianglelefteq^S C$. Analogously, if $m < k$ we prove that $A \trianglelefteq^S C$. This means that the relation is transitive.
Totallity: Let A and B be two fuzzy numbers such that $A \neq B$. Then, $A/\alpha_m \neq B/\alpha_m$ for $m = m(A,B)$. Thus, because $\preceq$ is linear $A/\alpha_m \prec B/\alpha_m$ or $B/\alpha_m \prec A/\alpha_m$. Therefore, $A \trianglelefteq^S B$ or $B \trianglelefteq^S A$ for all $A, B \in \mathcal{F}(\mathbb{R})$.
Refinement: Let A and B be two fuzzy numbers such that $A \leq_{KY} B$. If $A = B$ then, since $\trianglelefteq^S$ is reflexive, we have that

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$A \trianglelefteq^S B$. If $A <_{KY} B$ then, by Proposition II.3, ${}^A/_\alpha \leq_{KM} {}^B/_\alpha$ for each $\alpha \in (0,1]$ and therefore, for $m = m(A,B)$, we have that ${}^A/_{\alpha_m} <_{KM} {}^B/_{\alpha_m}$ and ${}^A/_{\alpha_k} = {}^B/_{\alpha_k}$ for each $k \leq m$. So, as $\leq$ is admissible order on $\mathbb{IR}$, ${}^A/_{\alpha_m} \prec {}^B/_{\alpha_m}$ and ${}^A/_{\alpha_i} = {}^B/_{\alpha_i}$ for each $i < m$. Thereby, $A \trianglelefteq^S B$.
Therefore, $\trianglelefteq_S$ is an admissible order. □

We will denote as $\trianglelefteq^S_{Lex1}$, $\trianglelefteq^S_{Lex2}$, $\trianglelefteq^S_{XY}$ and $\trianglelefteq^S_{2XY}$ the admissible orders on $\mathcal{F}(\mathbb{R})$ generated by the admissible orders $\preceq_{Lex1}$, $\preceq_{Lex2}$, $\preceq_{XY}$ and $\preceq_{2XY}$ and an upper dense sequence $S = (\alpha_i)_{i\in\mathbb{N}}$ in $]0,1]$, respectively, according to Theorem III.1.

Definition III.4. *Let $\leq$ be an admissible order on $\mathcal{F}(\mathbb{R})$. $\leq$ is compatible with the addition, if for each $A, B, C \in \mathcal{F}(\mathbb{R})$, $A + C < B + C$ whenever $A < B$.*

Proposition III.3. *Let $S = (\alpha_i)_{i\in\mathbb{N}}$ be an upper dense sequence in $]0,1]$ and consider an admissible order $\leq$ on $\mathbb{IR}$. If $\leq$ is compatible with the addition, then $\trianglelefteq^S$ also is compatible with the addition.*

Proof. If $A \vartriangleleft^S B$ then ${}^A/_{\alpha_{m(A,B)}} \prec {}^B/_{\alpha_{m(A,B)}}$ and ${}^A/_{\alpha_i} = {}^B/_{\alpha_i}$ for each $i < m(A,B)$. So, because $\leq$ is compatible with the addition, ${}^A/_{\alpha_{m(A,B)}} + {}^C/_{\alpha_{m(A,B)}} \prec {}^B/_{\alpha_{m(A,B)}} + {}^C/_{\alpha_{m(A,B)}}$ and ${}^A/_{\alpha_i} + {}^C/_{\alpha_i} = {}^B/_{\alpha_i} + {}^C/_{\alpha_i}$ for each $i < m(A,B)$. Therefore, by Equation (5), ${}^{A+C}/_{\alpha_{m(A,B)}} \prec {}^{B+C}/_{\alpha_{m(A,B)}}$ and ${}^{A+C}/_{\alpha_i} = {}^{B+C}/_{\alpha_i}$ for each $i < m(A,B)$. Hence, $A + C \vartriangleleft^S B + C$. □

Corollary III.1. *The admissible orders $\trianglelefteq^S_{Lex1}$, $\trianglelefteq^S_{Lex2}$, $\trianglelefteq^S_{XY}$ and $\trianglelefteq^S_{2XY}$ are compatible with the addition.*

Proposition III.4. *Let $S = (\alpha_i)_{i\in\mathbb{N}}$ be an upper dense sequence in $]0,1]$. Then $\trianglelefteq^S_{XY} = \leq^S_{WW}$.*

Proof. Let $A, B \in \mathcal{F}(\mathbb{R})$. If $A <^S_{WW} B$ then there exists $n_0 \in \mathbb{N}$ such that $c_{n_0}(A) < c_{n_0}(B)$ and $c_i(A) = c_i(B)$ for each $i < n_0$. We consider two cases, namely:
Case 1. If n_0 is even then taking $m = \frac{n_0}{2}$ we have that $r^*_A(\alpha_m) - l^*_A(\alpha_m) < r^*_B(\alpha_m) - l^*_B(\alpha_m)$, $r^*_A(\alpha_m) + l^*_A(\alpha_m) = r^*_B(\alpha_m) + l^*_B(\alpha_m)$, $r^*_A(\alpha_{m-i}) - l^*_A(\alpha_{m-i}) = r^*_B(\alpha_{m-i}) - l^*_B(\alpha_{m-i})$ and $r^*_A(\alpha_{m-i}) + l^*_A(\alpha_{m-i}) = r^*_B(\alpha_{m-i}) + l^*_B(\alpha_{m-i})$ for each $i < m$. Hence, $r^*_A(\alpha_m) - l^*_A(\alpha_m) < r^*_B(\alpha_m) - l^*_B(\alpha_m)$, $r^*_A(\alpha_m) + l^*_A(\alpha_m) = r^*_B(\alpha_m) + l^*_B(\alpha_m)$, $r^*_A(\alpha_i) + l^*_A(\alpha_i) = r^*_B(\alpha_i) + l^*_B(\alpha_i)$ and $r^*_A(\alpha_i) - l^*_A(\alpha_i) = r^*_B(\alpha_i) - l^*_B(\alpha_i)$ for all $i < m$.
Case 2. If n_0 is odd then taking $m = \frac{n_0+1}{2}$ we have that $r^*_A(\alpha_m) + l^*_A(\alpha_m) < r^*_B(\alpha_m) + l^*_B(\alpha_m)$, $r^*_A(\alpha_{m-i}) - l^*_A(\alpha_{m-i}) = r^*_B(\alpha_{m-i}) - l^*_B(\alpha_{m-i})$ and $r^*_A(\alpha_{m-i}) + l^*_A(\alpha_{m-i}) = r^*_B(\alpha_{m-i}) + l^*_B(\alpha_{m-i})$ for each $i < m$. Hence, $r^*_A(\alpha_m) - l^*_A(\alpha_m) < r^*_B(\alpha_m) - l^*_B(\alpha_m)$, $r^*_A(\alpha_i) + l^*_A(\alpha_i) = r^*_B(\alpha_i) + l^*_B(\alpha_i)$ and $r^*_A(\alpha_i) - l^*_A(\alpha_i) = r^*_B(\alpha_i) - l^*_B(\alpha_i)$ for all $i < m$. Therefore, in both cases, ${}^A/_{\alpha_m} \prec_{XY} {}^B/_{\alpha_m}$ and ${}^A/_{\alpha_i} = {}^B/_{\alpha_i}$ for each $i < m$. Since, clearly $m = m(A,B)$ it follows that $A \vartriangleleft^S_{XY} B$.

Reciprocally, if $A \vartriangleleft^S_{XY} B$ then ${}^A/_{\alpha_m} \prec_{XY} {}^B/_{\alpha_m}$ where $m = m(A,B)$. So, either $l^*_A(\alpha_m) + r^*_A(\alpha_m) < l^*_B(\alpha_m) + r^*_B(\alpha_m)$ or $l^*_A(\alpha_m) + r^*_A(\alpha_m) = l^*_B(\alpha_m) + r^*_B(\alpha_m)$ and $r^*_A(\alpha_m) - l^*_A(\alpha_m) < r^*_B(\alpha_m) + l^*_B(\alpha_m)$. In the first case, take $n_0 = 2m-1$ and in the second case $n_0 = 2m$. In any of the cases we have that $c_{n_0}(A) < c_{n_0}(B)$ and since for each $i < m$ we have that ${}^A/_{\alpha_i} = {}^B/_{\alpha_i}$ then $c_i(A) = c_i(B)$. Thereby, $A \leq^S_{WW} B$. □

Corollary III.2. *For all upper dense sequence $S = (\alpha_i)_{i\in\mathbb{N}}$ in $]0,1]$, the relation $\leq^S_{WW}$ is an admissible order on $\mathcal{F}(\mathbb{R})$.*

Proof. Straightforward from Theorem III.1 and Proposition III.4. □

Definition III.5. *Let $\leq$ be an admissible order on $\mathcal{F}(\mathbb{R})$. A fuzzy number A is $\leq$-positive if $\widetilde{0} < A$, is $\leq$-negative if $A < \widetilde{0}$, is non $\leq$-negative if $\widetilde{0} \leq A$ and is non $\leq$-positive if $A \leq \widetilde{0}$.*

Remark III.1. *Clearly, each fuzzy number A is either $\leq$-positive, $\leq$-negative or $A = \widetilde{0}$. Nevertheless, some fuzzy number are $\leq_1$-positives for an admissible order $\leq_1$ but are $\leq_2$-negative for an admissible order $\leq_2$. For example, the fuzzy triangle number $(-1,0,2)$ is $\trianglelefteq^S_{Lex1}$-negative and $\trianglelefteq^S_{Lex2}$-positive for any upper dense sequence $S = (\alpha_i)_{i\in\mathbb{N}}$ in $]0,1]$.*

Proposition III.5. *Let $A \in \mathcal{F}(\mathbb{R})$. Then A is $\leq$-positive for each admissible order $\leq$ in $\mathcal{F}(\mathbb{R})$ if and only if $A \neq \widetilde{0}$ and $0 \leq supp(A)^-$.*

Proof. Firstly, we assume $A = \widetilde{0}$ then, trivially, A is non $\leq$-positive for each admissible order $\leq$. If $0 > supp(A)^-$ then $[supp(A)^-, supp(A)^+] \prec_{Lex1} [0]$ and therefore, for any upper dense sequence $S = (\alpha_i)_{i\in\mathbb{N}}$ in $]0,1]$, ${}^A/_{\alpha_m} \prec_{Lex1} [0] = {}^{\widetilde{0}}/_{\alpha_m}$ for $m = m(\widetilde{0}, A) = \min\{i \in \mathbb{N} : {}^{\widetilde{0}}/_{\alpha_i} \neq {}^A/_{\alpha_i}\}$. So, $A \vartriangleleft^S_{Lex1} \widetilde{0}$ and therefore A is not $\trianglelefteq^S_{Lex1}$-positive. Thus, this side of the proposition holds by contraposition.

On the other hand, if $A \neq \widetilde{0}$ and $0 \leq supp(A)^-$, then ${}^{\widetilde{0}}/_\alpha = [0] \leq_{KM} {}^A/_\alpha$ and $[0] <_{KM} {}^A/_{\alpha_m}$ for $m = m(\widetilde{0}, A)$. So, $\widetilde{0} <_{KY} A$ and therefore for any admissible order $\leq$ we have that $\widetilde{0} < A$, that is A is $\leq$-positive. □

Proposition III.6. *Let $A \in \mathcal{F}(\mathbb{R})$. Then A is $\leq$-negative for each admissible order $\leq$ in $\mathcal{F}(\mathbb{R})$ if and only if $A \neq \widetilde{0}$ and $supp(A)^+ < 0$.*

Proof. Using similar steps to Proposition III.5, we obtain the result. □

Corollary III.3. *Let $A \in \mathcal{F}(\mathbb{R})$ be such that $supp(A)^- < 0 < supp(A)^+$. Then there exist admissible orders $\leq_1$ and $\leq_2$ such that A is $\leq_1$-positive and A is $\leq_2$-negative*

Theorem III.2. *Let $A \in \mathcal{F}(\mathbb{R})$, $S = (\alpha_i)_{i\in\mathbb{N}}$ be an upper dense sequence in $]0,1]$ and $m = m(\widetilde{0}, A)$. Then*

1) *A is $\trianglelefteq^S_{Lex1}$-positive if and only if $0 \leq l^*_A(\alpha_m)$ and $A \neq \widetilde{0}$,*
2) *A is $\trianglelefteq^S_{Lex2}$-positive if and only if $0 < r^*_A(\alpha_m)$,*
3) *A is $\trianglelefteq^S_{XY}$-positive if and only if $-l^*_A(\alpha_m) \leq r^*_A(\alpha_m)$ and $A \neq \widetilde{0}$,*
4) *A is $\trianglelefteq^S_{2XY}$-positive if and only if $A \neq \widetilde{0}$ and $-l^*_A(\alpha_m) \leq 3r^*_A(\alpha_m)$.*

Proof. Let $A \in \mathcal{F}(\mathbb{R})$ and $m = m(\widetilde{0}, A) = \min\{i \in \mathbb{N} : {}^{\widetilde{0}}/_{\alpha_i} \neq {}^A/_{\alpha_i}\} = \min\{i \in \mathbb{N} : [0] \neq {}^A/_{\alpha_i}\}$, we have:

1) A is $\trianglelefteq^S_{Lex1}$-positive if and only if $\widetilde{0} \vartriangleleft^S_{Lex1} A$ if and only if ${}^{\widetilde{0}}/_{\alpha_m} = [0] \prec_{Lex1} {}^A/_{\alpha_m}$ if and only if $0 < l^*_A(\alpha_m)$ or, $l^*_A(\alpha_m) = 0$ and $0 < r^*_A(\alpha_m)$ if and only if $0 \leq l^*_A(\alpha_m)$ and $A \neq \widetilde{0}$.
3) A is $\trianglelefteq^S_{XY}$-positive if and only if ${}^{\widetilde{0}}/_{\alpha_m} = [0] \prec_{XY} {}^A/_{\alpha_m}$ if and only if $0 < l^*_A(\alpha_m) + r^*_A(\alpha_m)$ or, $l^*_A(\alpha_m) + r^*_A(\alpha_m) = 0$ and $0 < r^*_A(\alpha_m) - l^*_A(\alpha_m)$ if and only if $-l^*_A(\alpha_m) <$

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$r^*_A(\alpha_m)$ or, $-l^*_A(\alpha_m) = r^*_A(\alpha_m)$ and $l^*_A(\alpha_m) < r^*_A(\alpha_m)$ if and only if $-l^*_A(\alpha_m) \leq r^*_A(\alpha_m)$ and $A \neq \widetilde{0}$.

The proof for 2 and 4 is analagous. □

Corollary III.4. *Let $A \in \mathcal{F}(\mathbb{R})$ and $S = (\alpha_i)_{i\in\mathbb{N}}$ be an upper dense sequence in $]0,1]$. If $A \neq \widetilde{0}$ and $m = m(\widetilde{0}, A)$ then*

1) *A is $\trianglelefteq^S_{Lex1}$-positive if and only if $0 \leq l^*_A(\alpha_m)$,*
2) *A is $\trianglelefteq^S_{Lex2}$-positive if and only if $0 < r^*_A(\alpha_m)$,*
3) *A is $\trianglelefteq^S_{XY}$-positive if and only if $-l^*_A(\alpha_m) \leq r^*_A(\alpha_m)$,*
4) *A is $\trianglelefteq^S_{2XY}$-positive if and only if $-l^*_A(\alpha_m) \leq 3r^*_A(\alpha_m)$.*

Example III.1. *Consider the triangular fuzzy numbers $A = (-2,5,6)$, $B = \widetilde{0}$, $C = (0,2,4)$ and $D = (-1,2,5)$ which are illustrated in Figure 3.*

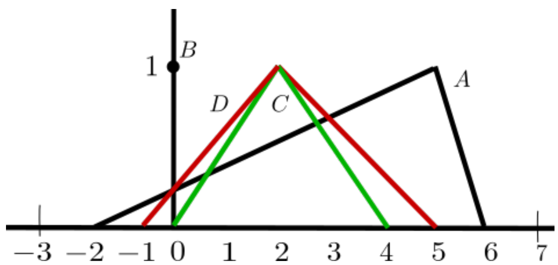

Figure 3. Fuzzy Numbers A, B, C and D.

Then, for each $\alpha \in]0,1]$, ${}^{A}/\alpha = [7\alpha-2, 6-\alpha]$, ${}^{B}/\alpha = [0,0]$, ${}^{C}/\alpha = [2\alpha, 4-2\alpha]$ and ${}^{D}/\alpha = [3\alpha-1, 5-3\alpha]$. Taking an upper dense sequence $S = (\alpha_i)_{i\in\mathbb{N}}$ in $]0,1]$ such that $\alpha_1 = 1$ and $\alpha_2 = 0.5$ then $m(C,D) = 2$ and $m(X,Y) = 1$ whenever $X \neq Y$ and $\{X,Y\} \neq \{C,D\}$. So, $B \lhd^S_{Lex1} D \lhd^S_{Lex1} C \lhd^S_{Lex1} A$ and $B \lhd^S_T C \lhd^S_T D \lhd^S_T A$ for each $T \in \{Lex2, XY, 2XY\}$. Therefore, A, C and D are $\trianglelefteq^S_T$-positives for each $T \in \{Lex1, Lex2, XY, 2XY\}$. Now, if $\alpha_1 = 0.1$ then $m(X,Y) = 1$ for each $X \neq Y \in \{A,B,C,D\}$. So,

- $A \lhd^S_{Lex1} D \lhd^S_{Lex1} B \lhd^S_{Lex1} C$;
- $B \lhd^S_{Lex2} C \lhd^S_{Lex2} D \lhd^S_{Lex2} A$;
- $B \lhd^S_{XY} A \lhd^S_{XY} C \lhd^S_{XY} D$;
- $B \lhd^S_{2XY} C \lhd^S_{2XY} D \lhd^S_{2XY} A$.

Therefore, A and D are $\trianglelefteq^S_{Lex1}$-negatives whereas C is $\trianglelefteq^S_{Lex1}$-positive and A, C and D are all $\trianglelefteq^S_T$-positives for each $T \in \{Lex2, XY, 2XY\}$. So, the ranking of these fuzzy numbers is fully dependent of the generator order and of S. Nevertheless, there are fuzzy numbers where S is unrelevant. Take for example, trapezoidal fuzzy numbers whose corners are $E = (0,4,6,10)$ and $F = (1,3,7,9)$. Then for any upper dense sequence S we have that $F \lhd^S_{Lex1} E$, $F \lhd^S_{Lex2} E$, $E \lhd^S_{XY} F$ and $E \lhd^S_{2XY} F$.

A natural property of the arithmetic of real numbers is that the sum of two positive numbers is always positive and the sum of two negative numbers is also a negative number. However, this natural property does not work for each admissible order considering the addition of fuzzy numbers given in the Preliminary section. Indeed, take an upper dense sequence $S = (\alpha_i)_{i\in\mathbb{N}}$ in $]0,1]$ such that $\alpha_1 = 0.8$ (By item 3 of Remark II.4 such S there exists), and consider the triangular fuzzy numbers $A = (-3,1,2)$ and $B = (-4,0,2)$. Then, ${}^{A}/\alpha_1 = [0.2, 1.2]$, ${}^{B}/\alpha_1 = [-0.8, 0.4]$ and therefore both are $\trianglelefteq^S_{2XY}$-positive. However, ${}^{A+B}/\alpha_1 = [-0.6, 1.6]$ and hence $A+B$ is $\trianglelefteq^S_{2XY}$-negative.

So, the next result analyses which of the other three admissible orders verifies this natural property.

Proposition III.7. *Let $S = (\alpha_i)_{i\in\mathbb{N}}$ be an upper dense sequence in $]0,1]$ and $A, B \in \mathcal{F}(\mathbb{R})$. Then*

1) *If A and B are $\trianglelefteq^S_{Lex1}$-positive then $A+B$ is also $\trianglelefteq^S_{Lex1}$-positive;*
2) *If A and B are $\trianglelefteq^S_{Lex1}$-negative then $A+B$ is also $\trianglelefteq^S_{Lex1}$-negative;*
3) *If A and B are $\trianglelefteq^S_{Lex2}$-positive then $A+B$ is also $\trianglelefteq^S_{Lex2}$-positive;*
4) *If A and B are $\trianglelefteq^S_{Lex2}$-negative then $A+B$ is also $\trianglelefteq^S_{Lex2}$-negative;*
5) *If A and B are $\trianglelefteq^S_{XY}$-positive then $A+B$ is also $\trianglelefteq^S_{XY}$-positive;*
6) *If A and B are $\trianglelefteq^S_{XY}$-negative then $A+B$ is also $\trianglelefteq^S_{XY}$-negative.*

Proof. Let $A, B \in \mathcal{F}(\mathbb{R})$ and $m_A = m(\widetilde{0}, A)$, $m_B = m(\widetilde{0}, B)$. From Theorem III.2 we have:

1. If A and B are $\trianglelefteq^S_{Lex1}$-positive then by Theorem III.2-(1) we have that $A \neq \widetilde{0}$, $B \neq \widetilde{0}$, $0 \leq l^*_A(\alpha_{m_A})$ and $0 \leq l^*_B(\alpha_{m_B})$. Without loss of generality we can suppose that $m_A \leq m_B$. Then, for each $i < m_A$, by Eq. (5) we have that ${}^{A+B}/\alpha_i = {}^{A}/\alpha_i + {}^{B}/\alpha_i = [0] + [0] = [0]$ and, since $0 \leq l^*_B(\alpha_{m_B})$, then $0 \leq l^*_A(\alpha_{m_A}) \leq l^*_A(\alpha_{m_A}) + l^*_B(\alpha_{m_A}) = l^*_{A+B}(\alpha_{m_A})$. Therefore, $m_A = m(A+B, \widetilde{0})$ and thereby $\widetilde{0} \lhd^S_{Lex1} A+B$.
3. If A and B are $\trianglelefteq^S_{Lex2}$-positive then $0 < r^*_A(\alpha_{m_A})$ and $0 < r^*_B(\alpha_{m_B})$. Without loss of generality we can suppose that $m_A \leq m_B$. Then, for each $i < m_A$, by Eq. (5) we have that ${}^{A+B}/\alpha_i = {}^{A}/\alpha_i + {}^{B}/\alpha_i = [0] + [0] = [0]$ and since $r^*_{A+B}(\alpha_{m_A}) = r^*_A(\alpha_{m_A}) + r^*_B(\alpha_{m_A}) > 0$ then $m_A = m(A+B, \widetilde{0})$ and therefore, $\widetilde{0} \lhd^S_{Lex2} A+B$.
5. If A and B are $\trianglelefteq^S_{XY}$-positive then $-l^*_A(\alpha_{m_A}) < r^*_A(\alpha_{m_A})$ and $-l^*_B(\alpha_{m_B}) < r^*_B(\alpha_{m_B})$. Without loss of generality we can suppose that $m_A \leq m_B$. Then, for each $i < m_A$, by Eq. (5) we have that ${}^{A+B}/\alpha_i = {}^{A}/\alpha_i + {}^{B}/\alpha_i = [0] + [0] = [0]$ and since $l^*_{A+B}(\alpha_{m_A}) = l^*_A(\alpha_{m_A}) + l^*_B(\alpha_{m_A})$ and $r^*_{A+B}(\alpha_{m_A}) = r^*_A(\alpha_{m_A}) + r^*_B(\alpha_{m_A})$ then $l^*_{A+B}(\alpha_{m_A}) < r^*_{A+B}(\alpha_{m_A})$. So, $m_A = m(A+B, \widetilde{0})$ and therefore, $\widetilde{0} \lhd^S_{XY} A+B$.

The proof for 2, 4 and 6 is analogous. □

IV. RANKING PATH COSTS IN FUZZY WEIGHTED GRAPHS

Weighted graphs arise from the necessity to model practical problems where the edges in a graph have an associated cost as, for example, the well-known problem of the traveling salesman. This problem consists of going through a list of towns, visiting each one exactly once and featuring the origin and in such a way that the travelling total time (the cost) is minimized [34]. When we consider that such cost is imprecise, as the time dispensed in the travel of car from a city X to a city Y, the use of $\mathcal{F}(\mathbb{R})$ to model the costs is more appropriate.

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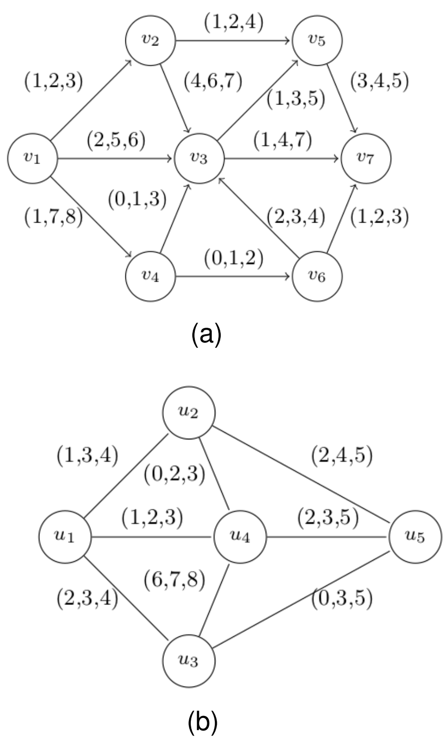

Figure 4. Examples of directed and undirected fuzzy weighted graphs.

Definition IV.1. *[16] A fuzzy weighted graph is a triple $G = \langle V, E, c\rangle$ where V is a set whose elements are called vertices, $E \subseteq V \times V$ is a set of edges and $c : E \to \mathcal{F}(\mathbb{R})$ is the cost (or weight) function. Given $v, u \in V$, a (v, u)-path in G is a finite and non empty sequence of edges $p = (e_1, \dots, e_n) = ((v_1, u_1), \dots, (v_n, u_n))$ such that $v_i = u_{i-1}$ for each $i = 2, \dots, n$, $v_1 = v$ and $u_n = u$. A (v, u)-path is a cycle if $v = u$.*

In order to simplify the notation, we will denote the (v_1, u_n)-path $p = ((v_1, u_1), (v_2, u_2), \dots, (v_n, u_n))$ by $p = (v_1, \dots, v_n, u_n)$.

A fuzzy weighted graph $G = \langle V, E, c\rangle$ such that E is symmetric, i.e. $(v, u) \in E$ if and only if $(u, v) \in E$, and $c(v, u) = c(u, v)$ for each $(v, u) \in E$ will be called of undirected fuzzy weighted graph.

Example IV.1. *The Figure 4 is an example of a directed and of an undirected fuzzy weighted graph with triangular fuzzy numbers as cost.*

The cost of a (v, u)-path $p = (e_1, \dots, e_n)$, denoted by $c(p)$ is given by the addition of the cost of each edge in the path, i.e. $c(p) = \sum_{i=1}^{n} c(e_i)$. Given a pair of vertices (v, u) in a fuzzy weighted graph G there can be several, or even none, (v, u)-paths. Given an admissible order $\leq$ on $\mathcal{F}(\mathbb{R})$, a fuzzy weighted graph $G = \langle V, E, c\rangle$, and $v, u \in V$, we say that a (v, u)-path p is $\leq$-minimal if $c(p) \leq c(q)$ for each (v, u)-path q in G.

Example IV.2. *In the case of the (directed) fuzzy weighted graph in Figure 4, the Table I presents all the possible paths from v_1 to v_7 and their costs. Given an upper dense sequence $S = (\alpha_i)_{i\in\mathbb{N}}$ in $]0, 1]$ such that $\alpha_1 = 0.8$ we have the following ranking with respect to the orders:*
For $\trianglelefteq^S_{Lex1}$:

$$p_1 \lhd^S_{Lex1} p_4 \lhd^S_{Lex1} p_6 \trianglelefteq^S_{Lex1} p_7 \lhd^S_{Lex1} p_5 \lhd^S_{Lex1} p_3 \lhd^S_{Lex1} p_2 \lhd^S_{Lex1} p_9 \lhd^S_{Lex1} p_8 \lhd^S_{Lex1} p_{10}.$$

For $\trianglelefteq^S_{Lex2}$:

$$p_1 \lhd^S_{Lex2} p_4 \lhd^S_{Lex2} p_6 \lhd^S_{Lex2} p_5 \lhd^S_{Lex2} p_7 \lhd^S_{Lex2} p_3 \lhd^S_{Lex2} p_2 \lhd^S_{Lex2} p_9 \lhd^S_{Lex2} p_8 \lhd^S_{Lex2} p_{10}.$$

For $\trianglelefteq^S_{XY}$:

$$p_1 \lhd^S_{XY} p_4 \lhd^S_{XY} p_6 \lhd^S_{XY} p_7 \lhd^S_{XY} p_5 \lhd^S_{XY} p_3 \lhd^S_{XY} p_2 \lhd^S_{XY} p_9 \lhd^S_{XY} p_8 \lhd^S_{XY} p_{10}.$$

Observe that ranking based on $\trianglelefteq^S_{Lex1}$ and $\trianglelefteq^S_{XY}$ is the same and differs from the given by $\trianglelefteq^S_{Lex2}$ in the positions of p_5 and p_7. So, the minimal path from v_1 to v_7, with respect to all the three admissible orders, is p_1.

Table I
PATHS OF THE FUZZY WEIGHTED GRAPH IN FIGURE 4A.

Identification	**Paths**	**Cost**	**α_1-cut**
p_1	(v_1, v_2, v_5, v_7)	$(5, 8, 12)$	$[7.4, 8.8]$
p_2	$(v_1, v_2, v_3, v_5, v_7)$	$(9, 15, 20)$	$[12.2, 16]$
p_3	(v_1, v_2, v_3, v_7)	$(6, 12, 17)$	$[10.8, 16]$
p_4	(v_1, v_3, v_7)	$(3, 9, 13)$	$[7.8, 9.8]$
p_5	(v_1, v_3, v_5, v_7)	$(6, 12, 16)$	$[10.8, 12.8]$
p_6	(v_1, v_4, v_6, v_7)	$(2, 10, 13)$	$[8.4, 10.6]$
p_7	(v_1, v_4, v_3, v_7)	$(2, 12, 18)$	$[10, 13.2]$
p_8	$(v_1, v_4, v_3, v_5, v_7)$	$(5, 15, 21)$	$[13, 16.2]$
p_9	$(v_1, v_4, v_6, v_3, v_7)$	$(4, 15, 21)$	$[12.8, 16.2]$
p_{10}	$(v_1, v_4, v_6, v_3, v_5, v_7)$	$(7, 18, 24)$	$[15.8, 19.2]$

We note, according to a Figure 4a, that we obtain

$$W = \begin{pmatrix} 0 & (1,2,3) & (2,5,6) & (1,7,8) & - & - & - \\ - & 0 & (2,6,7) & - & (1,2,4) & - & - \\ - & - & 0 & - & (1,3,5) & - & (1,4,7) \\ - & - & (0,1,3) & 0 & - & (0,1,2) & - \\ - & - & - & - & 0 & - & (3,4,7) \\ - & - & (2,3,4) & - & - & 0 & (1,2,3) \\ - & - & - & - & - & - & 0 \end{pmatrix}$$

where $-$ denotes that there is no path and 0 is the null distance.

The Algorithm 1 is used to determine the $\leq$-minimal (v, u)-path in a fuzzy weighted graph $G = \langle V, E, c\rangle$ which is based on an adaptation of Floyd-Warshall algorithm (see [35], [36]).

V. Illustrative Example

Consider the road distances between the neighbor capitals of the 9 Brazilian Northeast States obtained from the sites

1) http://www.distanciasentrecidades.com/,
2) https://www.rotamapas.com.br/,
3) https://www.melhoresrotas.com/s/distancia-entre-cidades,
4) http://rotasbrasil.com.br,

From such site, we observed, for example, that the distance between the cities of Natal and João Pessoa vary (178 km, 179 km, 181 km, 182 km, 189 km, 214 km). From this date we generate the triangular fuzzy number $(178, 181.5, 214)$ by taking the minimum, median and maximum of such distances,

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i.e., $(\min, \mathrm{Me}, \max)$. In the Table II and Figure 5, abbreviations of cities will be used, i.e., São Luis by SL, Teresina by T, Fortaleza by F and so on. Besides, in Figure 5, each c_{ij} is the TFN of the position (i, j) in the Table II, i.e. the cost or weight to go from the i-th to j-th cities in such Table. For example, $c_{45} = (178.8, 180, 182)$ is the cost to go from Natal to João Pessoa. There are 362,880 possible routes for the travelling salesman problem.

Algorithm 1 Minimal path between two nodes in a fuzzy weighted graph

Require: A fuzzy wegthed graph $G = \langle V, E, c\rangle$ and $v_s, v_f \in V$ with $V = \{v_1, ..., v_n\}$
Ensure: Solution alternative: vector $p = min{-}path(v_s, v_f)$ and $d^{(n)}_{s,f}$

$M = \widetilde{1} + \sum_{(v_i,v_j)\in E} c(v_i, v_j)$
for $i = 1$ to n
  for $j = 1$ to n
    if $i = j$
    then $d^{(0)}_{ij} = \widetilde{0}$
      $\pi^{(0)}_{ij} = NIL$
    else if $(v_i, v_j) \in E$
      then $d^{(0)}_{ij} = c(v_i, v_j)$
        $\pi^{(0)}_{ij} = i$
      else $d^{(0)}_{ij} = M$
        $\pi^{(0)}_{ij} = NIL$
for $k = 1$ to n
  for $i = 1$ to n
    for $j = 1$ to n
      If $d^{(k-1)}_{ij} \le d^{(k-1)}_{ik} + d^{(k-1)}_{kj}$
        then $d^{(k)}_{ij} = d^{(k-1)}_{ij}$
  $\pi^{(k)}_{ij} = \pi^{(k-1)}_{ij}$
      else $d^{(k)}_{ij} = d^{(k-1)}_{ik} + d^{(k-1)}_{kj}$
        $\pi^{(k)}_{ij} = \pi^{(k-1)}_{kj}$
$k = 1$
$m = 2$
Repeat
  While $\pi^{(n)}_{p(k)p(k+1)} \ne p(k)$
    for $i = k + 1$ to m
      $p(m - i + k + 2) = p(m - i + k + 1)$
    $p(k + 1) = \pi^{(n)}_{p(k)p(k+1)}$
    $m = m + 1$
  $k = k + 1$
Until $k = m$
Return p, $d^{(n)}_{sf}$

From the Table II we observe that the distance from Fortaleza to João Pessoa is different from Fortaleza to Natal and from Natal to João Pessoa, we have:

$$(665, 668, 698.3) \ne (511, 515, 531.8) + (178.8, 180, 182) = (689.8, 695, 713.5),$$

from where $(665, 668, 698.3) <_{KY} (689.8, 695, 713.5)$. Then by definition of admissible order $(665, 668, 698.3) \prec (689.8, 695, 713.5)$.

In the case of the (directed) fuzzy weighted graph in Figure 5, the Table III presents some routes chosen possibilities among the 362,880 randomly and their cost from any capital city of the Brazilian northeast until returning to it, having passed through all the other capitals.

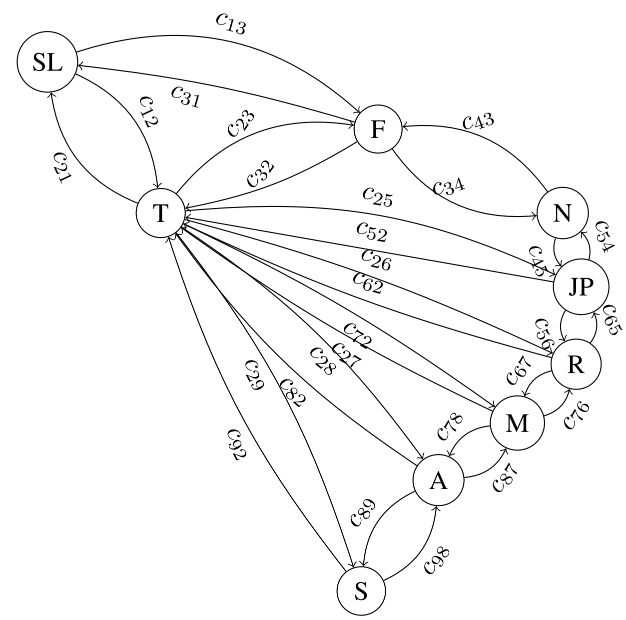

Figure 5. Directed weighted graph based on Table II.

We note that the possible routes of the Table III r_1 and r_{10} cannot be compared with the partial order $\le_{KY}$. Indeed, since ${}^{r_1}\!/\!_\alpha \le_{KM} {}^{r_{10}}\!/\!_\alpha$ for $\alpha \in \left]\alpha_0, 1\right]$ and ${}^{r_{10}}\!/\!_\alpha \subseteq {}^{r_1}\!/\!_\alpha$ for $\alpha \in \left]0, \alpha_0\right[$, with $\alpha_0 = \frac{938}{2027}$, then by Proposition II.3, $r_1 \parallel r_{10}$.

In Figure 6 we have ordered the costs in Table III from the smallest (r_2 and r_{17})to the greatest (r_7), according to the $\le_{KY}$ partial order. We note that the routes r_1, r_2, r_5, r_{10}, r_{12}, r_{13}, r_{14}, r_{15}, r_{17}, r_{18}, r_{19} and r_{20} are incomparable with at least another route. Given an upper dense sequence $S = (\alpha_i)_{i\in\mathbb{N}}$ in $]0, 1]$ such that $\alpha_1 = 0.8$ we have the following ranking with respect the orders:

For $\trianglelefteq^S_{Lex1}$:

$$r_{17} \triangleleft^S_{Lex1} r_2 \triangleleft^S_{Lex1} r_{14} \triangleleft^S_{Lex1} r_8 \triangleleft^S_{Lex1} r_{11} \trianglelefteq^S_{Lex1} r_6 \triangleleft^S_{Lex1} r_{16} \triangleleft^S_{Lex1} r_4 \triangleleft^S_{Lex1} r_{15} \triangleleft^S_{Lex1} r_{20} \trianglelefteq^S_{Lex1} r_{12} \triangleleft^S_{Lex1} r_{18} \triangleleft^S_{Lex1} r_5 \triangleleft^S_{Lex1} r_3 \triangleleft^S_{Lex1} r_{19} \triangleleft^S_{Lex1} r_{13} \triangleleft^S_{Lex1} r_1 \triangleleft^S_{Lex1} r_{10} \triangleleft^S_{Lex1} r_9 \triangleleft^S_{Lex1} r_7.$$

For $\trianglelefteq^S_{Lex2}$:

$$r_{17} \trianglelefteq^S_{Lex2} r_2 \triangleleft^S_{Lex2} r_{14} \triangleleft^S_{Lex2} r_8 \triangleleft^S_{Lex2} r_{11} \triangleleft^S_{Lex2} r_6 \triangleleft^S_{Lex2} r_{16} \triangleleft^S_{Lex2} r_4 \triangleleft^S_{Lex2} r_{15} \triangleleft^S_{Lex2} r_{20} \triangleleft^S_{Lex2} r_{12} \triangleleft^S_{Lex2} r_{18} \triangleleft^S_{Lex2} r_5 \triangleleft^S_{Lex2} r_3 \triangleleft^S_{Lex2} r_{19} \triangleleft^S_{Lex2} r_1 \triangleleft^S_{Lex2} r_{13} \triangleleft^S_{Lex2} r_{10} \triangleleft^S_{Lex2} r_9 \triangleleft^S_{Lex2} r_7.$$

For $\trianglelefteq^S_{XY}$:

$$r_{17} \triangleleft^S_{XY} r_2 \triangleleft^S_{XY} r_{14} \trianglelefteq^S_{XY} r_8 \triangleleft^S_{XY} r_{11} \triangleleft^S_{XY} r_6 \trianglelefteq^S_{XY} r_{16} \triangleleft^S_{XY} r_4 \trianglelefteq^S_{XY} r_{20} \trianglelefteq^S_{XY} r_{15} \triangleleft^S_{XY} r_{12} \trianglelefteq^S_{XY} r_{18} \triangleleft^S_{XY} r_5 \triangleleft^S_{XY} r_3 \trianglelefteq^S_{XY} r_{19} \triangleleft^S_{XY} r_{13} \trianglelefteq^S_{XY} r_1 \triangleleft^S_{XY} r_{10} \trianglelefteq^S_{XY} r_9 \triangleleft^S_{XY} r_7.$$

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Km	SL	T	F	N	JP	R	M	A	S	lines
SL		(433, 439, 610.8)	(878, 1169.1, 1177.9)	(1395, 1408, 1753.9)	(1549, 1556, 1825.6)	(1557, 1664, 1738.4)	(1552, 1563, 1733.3)	(1563, 1584, 1819)	(1627, 1658, 1834.7)	1
T	(432, 435, 609.1)		(592, 593.5, 594.4)	(1040, 1097.6, 1146.2)	(1156, 1160.8, 1251.5)	(1127, 1127.5, 1130.7)	(1125.7, 1126.5, 1139)	(1133, 1143.15, 1159)	(1196, 1226.55, 1233)	2
F	(879, 900, 1179)	(592, 593.9, 594)		(511, 515, 531.8)	(665, 668, 698.3)	(771.9, 773, 798.5)	(957.4, 1027, 1050.5)	(1089.9, 1119, 1130)	(1190, 1203.25, 1205)	3
N	(1396, 1418, 1756.9)	(1041, 1107, 1149)	(511, 520, 530)		(178.8, 180, 182)	(285, 285.1, 286)	(535, 539.85, 543)	(782, 788.5, 794.8)	(1093, 1095, 1101.5)	4
JP	(1550, 1571, 1831.4)	(1155, 1158.8, 1257)	(665, 671.05, 684)	(175.9, 180, 182)		(116, 116.6, 117)	(366, 370.85, 375)	(613, 620.5, 625.8)	(924, 927, 932.4)	5
R	(1557, 1673.5, 1735.6)	(1126, 1127.5, 1127.7)	(773, 781, 803.2)	(285.8, 286, 290)	(116, 118, 119.2)		(154.7, 253, 258)	(497, 504.9, 510)	(808, 811, 816.4)	6
M	(1552, 1561.5, 1733.2)	(1124, 1126.2, 1137)	(964, 1028, 1051.2)	(537, 539.05, 543)	(367, 371.25, 373)	(251, 253.8, 258)		(271, 279.25, 291)	(581, 588.1, 593)	7
A	(1561, 1583, 1821.7)	(1131, 1142.05, 1158)	(1095, 1119.5, 1130)	(782, 783.4, 794)	(613, 615.1, 624)	(497, 497.65, 508)	(270, 271, 289)		(325, 325.95, 338)	8
S	(1626, 1655, 1831.3)	(1196, 1222.75, 1232)	(1188, 1200.4, 1204)	(1091, 1093.55, 1096)	(922, 924.75, 926)	(805.6, 807.5, 810)	(580, 581, 591)	(321.8, 326, 349)		9
rows	1	2	3	4	5	6	7	8	9	

Table II

TRIANGULAR FUZZY NUMBERS DEFINED BY $(\min, \mathrm{Me}, \max)$ FOR DISTANCES BETWEEN CAPITALS IN THE BRAZILIAN NORTHEAST .

Route	Paths	Cost
r_1	SL,JP,R,S,M,F,A,T,N,SL	$(8673.9, 8869.25, 9592.3)$
r_2	N,T,SL,F,A,R,JP,S,M,N	$(6094.9, 6492.84, 6759.6)$
r_3	N,F,JP,T,A,R,S,SL,M,N	$(8484, 8653.85, 8983.3)$
r_4	N,JP,A,F,SL,S,M,T,R,N	$(7509.6, 7598.7, 8100)$
r_5	JP,SL,A,F,T,R,N,M,S,JP	$(8250.8, 8334.6, 8857.1)$
r_6	JP,T,F,R,A,M,S,SL,N,JP	$(7066.7, 7132, 7809.1)$
r_7	JP,N,S,T,M,F,A,R,SL,JP	$(9247.5, 9498.4, 9904.9)$
r_8	JP,S,N,A,R,F,SL,T,M,JP	$(6871.7, 6924.45, 7436.2)$
r_9	A,JP,T,N,M,SL,S,R,F,A	$(9190.5, 9333.35, 9881.3)$
r_{10}	S,T,M,A,F,JP,SL,R,N,S	$(8838.5, 9092, 9351.6)$
r_{11}	N,R,F,JP,M,SL,T,S,A,N	$(6988.7, 7051.05, 7638.7)$
r_{12}	A,N,JP,S,SL,R,M,T,F,A	$(8028.4, 8301.1, 8597.5)$
r_{13}	M,SL,F,S,R,JP,T,A,N,M	$(8681.6, 9073.55, 9341.3)$
r_{14}	F,R,T,SL,N,JP,S,A,M,F	$(6383.5, 6475.5, 7092.8)$
r_{15}	M,F,R,S,JP,T,SL,A,N,M	$(7932.9, 8037.8, 8612.2)$
r_{16}	M,A,N,S,JP,SL,F,T,M	$(7213.7, 7592.9, 7854.8)$
r_{17}	SL,S,R,F,N,JP,A,M,T,SL	$(6334.4, 6394.2, 6822.6)$
r_{18}	A,N,M,T,R,SL,F,JP,S,A	$(7918.3, 8340.05, 8497.9)$
r_{19}	F,R,N,SL,M,S,JP,T,A,F	$(8028.4, 8301.1, 8597.5)$
r_{20}	F,M,S,T,A,SL,JP,R,N,F	$(7890.2, 8040.6, 8618.8)$

Table III

PATHS WITH THEIR COST OF THE FUZZY WEIGHTED GRAPH IN FIGURE 5 AND $\alpha \in]0, 1]$.

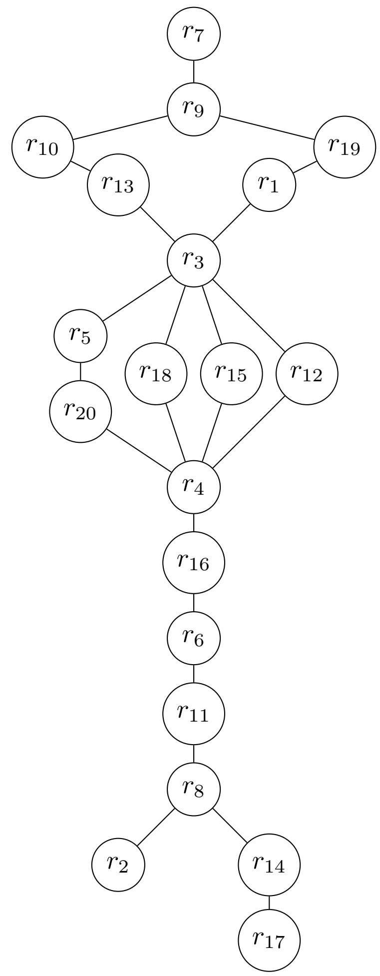

Figure 6. Ranking of the routes in Table III considering the order $\leq_{KY}$.

For $\trianglelefteq^S_{2XY}$:

$$
\begin{aligned}
r_{17} \triangleleft^S_{2XY} r_2 \triangleleft^S_{2XY} r_{14} \triangleleft^S_{2XY} r_8 \triangleleft^S_{2XY} r_{11} \triangleleft^S_{2XY} r_6 \\
\triangleleft^S_{2XY} r_{16} \triangleleft^S_{2XY} r_4 \triangleleft^S_{2XY} r_{15} \triangleleft^S_{2XY} r_{20} \triangleleft^S_{2XY} r_{12} \\
\triangleleft^S_{2XY} r_{18} \triangleleft^S_{2XY} r_8 \triangleleft^S_{2XY} r_3 \triangleleft^S_{2XY} r_{19} \triangleleft^S_{2XY} r_{13} \\
\triangleleft^S_{2XY} r_1 \triangleleft^S_{2XY} r_{10} \triangleleft^S_{2XY} r_9 \triangleleft^S_{2XY} r_7.
\end{aligned}
$$

Observe that rankings based on $\trianglelefteq^S_{Lex1}$, $\trianglelefteq^S_{XY}$ and $\trianglelefteq^S_{2XY}$ are the same and differs from the given by $\trianglelefteq^S_{Lex2}$ in the position of r_1 and r_{13}. So, in the case of Table III, the minimum route to travel the capitals of the Brazilian Northeast of the Travelling salesman problem is r_{17}.

VI. FINAL REMARKS

In this paper, we generalize the notion of admissible order on the set of closed subintervals of $[0, 1]$ to the set of fuzzy numbers equipped with an arbitrary order. Although the Klir and Yuan order is not consensually accepted as the natural order for the set of fuzzy numbers, most of the orders proposed for fuzzy numbers refine this order. So we deal with the Klir-Yuan order as the "natural" one for $\mathcal{F}(\mathbb{R})$ and explore the admissible order with respect to this order.

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a) Answer:: Applications of admissible orders on several domains have been successfully developed in several areas as can be seen in [6], [8], [37], [38], [39], [40]. Thus, since there are many application of fuzzy numbers where the order is important, such as [10], [32], [41], [42], [43], [44], the present study on admissible orders for fuzzy numbers can be useful in such applications. Thus, it may be expected that in a future efforts can be made to develop interesting applications of admissible orders on $\mathcal{F}(\mathbb{R})$.

In [2] a construction method of admissible orders over the set of closed subintervals of $[0,1]$ based on aggregation functions is provided and lately generalized in [3]. As a future work, we will intend to introduce a generation method of admissible orders on $\mathcal{F}(\mathbb{R})$.

Namely, in [16] is one of the many proposals in the literature regarding weighted fuzzy graphs. Our use of admissible orders is what guarantees that the final cost of all possible paths can be ordered linearly, thus always obtaining the shortest way.

Finally, in [45] are listed 9 properties that methods to rank classes of fuzzy numbers must satisfy. Trivially, each admissible order satisfies the 5 firsts, the 6th (A_5) is not applicable in this context, and the 7th and 8th follows from Proposition III.3 for each admissible order $\trianglelefteq^S$. Nevertheless, the last property deserves to be investigated in a future work, but it is clear that for some admissible orders of the type $\trianglelefteq^S$ it will be held whereas for others not.

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