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LIPSCHITZ ALGEBRAS AND LIPSCHITZ-FREE SPACES OVER UNBOUNDED METRIC SPACES

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ABSTRACT. We investigate a way to turn an arbitrary (usually, unbounded) metric space $\mathcal{M}$ into a bounded metric space $\mathcal{B}$ in such a way that the corresponding Lipschitz-free spaces $\mathcal{F}(\mathcal{M})$ and $\mathcal{F}(\mathcal{B})$ are isomorphic. The construction we provide is functorial in a weak sense and has the advantage of being explicit. Apart from its intrinsic theoretical interest, it has many applications in that it allows to transfer many arguments valid for Lipschitz-free spaces over bounded spaces to Lipschitz-free spaces over unbounded spaces. Furthermore, we show that with a slightly modified pointwise multiplication, the space $\mathrm{Lip}_0(\mathcal{M})$ of scalar-valued Lipschitz functions vanishing at zero over any (unbounded) pointed metric space is a Banach algebra with its canonical Lipschitz norm.

1. INTRODUCTION

Lipschitz spaces over metric spaces and their canonical preduals form by now a fundamental class of Banach spaces. By a Lipschitz space over a pointed metric space $(\mathcal{M}, 0)$ we understand the Banach space of all scalar-valued Lipschitz functions vanishing at 0 with the minimal Lipschitz constant as a norm; further denoted by $\mathrm{Lip}_0(\mathcal{M})$. We recall that the canonical predual of $\mathrm{Lip}_0(\mathcal{M})$ is the Lipschitz-free space $\mathcal{F}(\mathcal{M})$, also known as the Arens-Eells space. The geometry of Lipschitz-free spaces is nowadays one of the most active fields of study within Banach space theory (see, e.g., [1, 8–10, 14, 16, 19, 20, 26] for a non-exhaustive list of some recent developments).

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Although Lipschitz-free spaces can be defined over any metric space, their theory can be developed more smoothly over bounded metric spaces. Indeed, the space $\mathrm{Lip}_0(\mathcal{M})$ is closed under pointwise multiplication if and only if $\mathcal{M}$ is bounded, in which case $\mathrm{Lip}_0(\mathcal{M})$ becomes a topological algebra (even a Banach algebra after renorming). This additional algebraic structure makes these spaces more interesting and enriches their study. We refer the reader to the monograph [25] and to the papers [11–13, 21, 23, 24] for some more information and recent advances on the subject. One does not face a priori such problems when dealing with Lipschitz-free spaces over unbounded metric spaces. However, a recurring pattern in the life of a Lipschitz-free space researcher is that the proofs are often much simpler in the bounded case, where one does not need to deal with the difficulties of dealing with the geometry of metric spaces at large scales.

In the literature we find classes of metric spaces whose Lipschitz-free space is isomorphic to the Lipschitz-free space over some bounded set. Perhaps the best-known example is the class of Banach spaces, whose Lipschitz free spaces are isomorphic to the Lipschitz free spaces over their unit balls (see [18]). A very useful way to deal with Lipschitz free spaces over unbounded metric spaces is Kalton's decomposition of Lipschitz-free spaces from [17, Proposition 4.3] (see also [3, §3]). This important tool inspired some authors to develop a way of passing from the bounded to the unbounded case (see, e.g., [7]). However, in all known cases so far the isomorphism is not explicit because it relies on Pełczyński's decomposition method. Our first main result in this article addresses this problem by showing that Lipschitz-free spaces, or more generally Lipschitz free p-spaces $\mathcal{F}_p(\mathcal{M})$ over bounded metric spaces for $0 < p \le 1$ (see Section 2 for the precise definition), are universal in the following sense.

Theorem A. *For an arbitrary metric space $\mathcal{M}$ there is a* bounded *metric space $\mathcal{B}(\mathcal{M})$ such that the Lipschitz free p-spaces $\mathcal{F}_p(\mathcal{M})$ and $\mathcal{F}_p(\mathcal{B}(\mathcal{M}))$ are isomorphic, for $0 < p \le 1$. Moreover, the association $\mathcal{M} \mapsto \mathcal{B}(\mathcal{M})$ is a functor from the category of metric spaces with bi-Lipschitz maps as morphisms.*

We point out that the isomorphism between $\mathcal{F}(\mathcal{M})$ and $\mathcal{F}(\mathcal{B}(\mathcal{M}))$ advertised in Theorem A is given by an explicit simple formula and preserves many properties of isometric nature. Namely, if $\delta_{\mathcal{M}} \colon \mathcal{M} \to \mathcal{F}_p(\mathcal{M})$ is the canonical isometric embedding, we may put

$$\mathcal{B}(\mathcal{M}) := \{0\} \cup \left\{ \frac{\delta_{\mathcal{M}}(x)}{d(x,0)} \colon x \in \mathcal{M} \setminus \{0\} \right\} \subset \mathcal{F}_p(\mathcal{M})$$

and the isomorphism between $\mathcal{F}_p(\mathcal{M})$ and $\mathcal{F}_p(\mathcal{B}(\mathcal{M}))$ is given by

$$T(\delta(x)) := d(x,0)\delta_{\mathcal{B}(\mathcal{M})}\left(\frac{\delta_{\mathcal{M}}(x)}{d(x,0)}\right), \qquad x \in \mathcal{M} \setminus \{0\}.$$

The above formulas for $\mathcal{B}(\mathcal{M})$ and T can be adjusted by some additional parameters, which is useful in some particular situations. We refer the reader to Section 3 for more details.

Although we obtained our result independently, with hindsight the isomorphism from the statement of Theorem A in the $p = 1$ case could also be deduced from [25, Theorem 2.20], which establishes an isomorphism between $\mathrm{Lip}_0(\mathcal{M})$ and $\mathrm{Lip}_0(\mathcal{B}(\mathcal{M} \setminus 0))$, and it can be proved that the given isomorphism is w^*-w^* continuous. This gives an isomorphism between $\mathcal{F}(\mathcal{M})$ and $\mathcal{F}(\mathcal{B}(\mathcal{M}\setminus\{0\}))$ which, in turn, is isomorphic to $\mathcal{F}(\mathcal{B}(\mathcal{M}))$. Here we focus on the Lipschitz-free space $\mathcal{F}(\mathcal{B}(\mathcal{M}))$ instead of on its dual $\mathrm{Lip}_0(\mathcal{B}(\mathcal{M}))$, and push in another direction the techniques derived from the reductions that the metric space $\mathcal{B}(\mathcal{M})$ provides. See Remark 3.6 for more details.

Our second main result is that $\mathrm{Lip}_0(\mathcal{M})$ becomes a Banach algebra when equipped with a simply defined product for any metric space $\mathcal{M}$, without needing to change the regular norm of the space $\mathrm{Lip}_0(\mathcal{M})$.

Theorem B. *For an arbitrary metric space $\mathcal{M}$ there is an explicit modification of the pointwise multiplication on $\mathrm{Lip}_0(\mathcal{M})$ given by a simple formula which turns $\mathrm{Lip}_0(\mathcal{M})$ with its regular norm into a Banach algebra.*

As in Theorem A, there are different ways to define this multiplication. The most basic one is to define the product $\odot$ of $f, g \in \mathrm{Lip}_0(\mathcal{M})$ by the formula

$$(f \odot g)(x) := \frac{f(x)g(x)}{3d(x,0)}, \qquad x \in \mathcal{M} \setminus \{0\}.$$

Both Theorems A and B hold in the context of real as well as complex Banach spaces; Theorem A holds even in the general setting of p-Banach spaces. Usually in the literature on the subject, it is considered the case of real spaces only, but in the context of Banach algebras the complex field is more natural.

Let us now describe the contents of the paper in some detail. In Section 2 we gather the most heavily used terminology and notation. Section 3 is devoted to the proof of Theorem A. We show the isomorphism between $\mathcal{F}_p(\mathcal{M})$ and $\mathcal{F}_p(\mathcal{B}(\mathcal{M}))$ with a quantitative estimate of $3^{1/p}$, and obtain $d_{\mathcal{B}_p(\mathcal{M})}(x,0) \leq 1$ for $x \in \mathcal{B}_p(\mathcal{M})$. The functoriality of $\mathcal{B}$ is shown at the end of the section. In Section 4, we compare the topology and geometry of the metric spaces $\mathcal{M}$ and $\mathcal{B}(\mathcal{M})$, which will be helpful for applications. In Section 5 we prove Theorem B, that is, we define a product on $\operatorname{Lip}_0(\mathcal{M})$ so that $\operatorname{Lip}_0(\mathcal{M})$ becomes a Banach algebra when $\mathcal{M}$ is an unbounded metric space, and we study the duality with respect to $\mathcal{F}(\mathcal{M})$. In Section 6 we then use our construction to transfer well-known results about the Lipschitz algebra $\operatorname{Lip}_0(\mathcal{B})$, for a bounded metric space $\mathcal{B}$, to $\operatorname{Lip}_0(\mathcal{M})$ when $\mathcal{M}$ is unbounded. For instance, we deal with w^*-closed ideals in $\operatorname{Lip}_0(\mathcal{M})$ (see Theorem 6.3 and Theorem 6.7), with the fact that the algebraic structure of $\operatorname{Lip}_0(\mathcal{M})$ determines the linear topological structure of $\mathcal{F}(\mathcal{M})$ (see Proposition 6.9 and Theorem 6.14), and we identify the spectrum of the Banach algebra $\operatorname{Lip}_0(\mathcal{M})$ (see Theorem 6.12 and Theorem 6.13). Finally, in Section 7 we present a small selection of known results where the proofs for bounded metric spaces are much easier than for unbounded ones. The usefulness of our techniques has to be seen in that they allow an easy transfer of the simpler proofs to the unbounded case.

2. Prerequisites on terminology and notation

In this section we collect together some of the definitions and background results which we will employ later in the paper and set the notation and the terminology.

Let $0 < p \le 1$. We say that $(\mathcal{M}, d)$ is a p-metric space if $(\mathcal{M}, d^p)$ is a metric space. Let us also recall that a p-Banach space is a vector space X over the real or complex field $\mathbb{F}$ equipped with a map $\|\cdot\|_X \colon X \to [0,\infty)$ that satisfies all the usual properties of the norm with the exception that the triangle law is replaced with

$$\|\gamma_1 + \gamma_2\|_X^p \le \|\gamma_1\|_X^p + \|\gamma_2\|_X^p, \quad \gamma_1, \gamma_2 \in X,$$

and, moreover, $(X, \|\cdot\|_X)$ is complete.

Let $(\mathcal{M}, d)$ and $(\mathcal{N}, d')$ be p-metric spaces. A map $f \colon \mathcal{M} \to \mathcal{N}$ is said to be Lipschitz if

$$\operatorname{Lip}(f) := \sup_{x \neq y} \frac{d'(f(x), f(y))}{d(x,y)} < \infty.$$

If $\operatorname{Lip}(f) \le C$, we say that f is a C-Lipschitz map. A bi-Lipschitz map is a one-to-one map f such that both f and $f^{-1} \colon f(\mathcal{M}) \to \mathcal{M}$ are Lipschitz. We say that $\mathcal{M}$ and $\mathcal{N}$ are *Lipschitz isomorphic*, and we put $\mathcal{M} \simeq_{\mathrm{Lip}} \mathcal{N}$, if there is a bi-Lipschitz bijection from $\mathcal{M}$ onto $\mathcal{N}$.

If $(\mathcal{M}, d, 0)$ is a pointed metric space, the linear space $\operatorname{Lip}_0(\mathcal{M})$ of all Lipschitz maps from $\mathcal{M}$ into $\mathbb{F}$ which vanish at the distinguished point $0 \in \mathcal{M}$ is a Banach space with the norm $\operatorname{Lip}(\cdot)$. The particular choice of $0 \in \mathcal{M}$ does not affect the linear isometric structure of $\mathrm{Lip}_0(\mathcal{M})$ (see, e.g., [25, p. 36]).

Given a pointed p-metric space $(\mathcal{M}, d, 0)$ one can construct the so-called *Lipschitz free p-space over $\mathcal{M}$* as the completion of the linear span, $\mathcal{P}(\mathcal{M}) := \mathrm{span}\{\delta(x) \colon x \in \mathcal{M} \setminus \{0\}\}$, in the linear dual $(\mathbb{F}^{\mathcal{M}})^{\#}$ of the evaluations $\delta(x) \colon \mathbb{F}^{\mathcal{M}} \to \mathbb{F}$, $f \to \delta(x)(f) = f(x)$, with the p-norm

$$\left\| \sum_{i=1}^{n} a_i\, \delta(x_i) \right\| = \sup \left\| \sum_{i=1}^{n} a_i\, f(x_i) \right\|_X ,$$

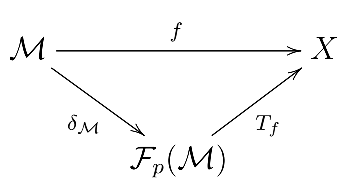
the supremum being taken over all p-Banach spaces $(X, \|\cdot\|_X)$ and all choices of 1-Lipschitz maps $f \colon \mathcal{M} \to X$ with $f(0) = 0$. In the case when $p = 1$ and $\mathcal{M}$ is a metric space, we set $\mathcal{F}(\mathcal{M}) = \mathcal{F}_1(\mathcal{M})$ and $\mathcal{F}(\mathcal{M})$ is just the Lipschitz-free space over $\mathcal{M}$. By the Hahn-Banach theorem, to define the norm of $\sum_{i=1}^{n} a_i\, \delta(x_i)$ in $\mathcal{F}(\mathcal{M})$ it is enough to take the supremum over all 1-Lipschitz maps in $\mathrm{Lip}_0(\mathcal{M})$. Alternatively, the norm on $\mathcal{P}(\mathcal{M})$ can be defined using a Kantorovich-Rubinstein type-formula (see [2] for details).

By $\delta(0)$ we often denote the origin in $\mathcal{P}(\mathcal{M})$. Note that formally the *molecule* $m := \sum_{i=1}^{n} a_i \delta(x_i)$ belongs to $\mathcal{F}_p(\mathcal{M})$ for any pointed p-metric space space $(\mathcal{M}, d, 0)$ with $\{x_1, \dots, x_n\} \subset \mathcal{M}$, but the norm of m depends on the particular space $\mathcal{M}$. Thus, if we want to emphasize that $\delta(x)$ is considered as a member of $\mathcal{F}(\mathcal{M})$, we write $\delta_{\mathcal{M}}(x)$ instead of $\delta(x)$. Let us mention the crucial "universal property" of Lipschitz free p-spaces, $p \leq 1$. We point out that, although it was first stated for real scalars, the same proof works for Lipschitz free p-spaces over $\mathbb{C}$.

Theorem 2.1 ([2, Theorem 4.5])**.** *Let $(\mathcal{M}, d, 0)$ be a pointed p-metric space, $0 \leq p \leq 1$. Then:*

(a) The canonical map $\delta_{\mathcal{M}} \colon \mathcal{M} \to \mathcal{F}_p(\mathcal{M})$, $x \mapsto \delta_{\mathcal{M}}(x)$, is an isometric embedding.

(b) $\mathcal{F}_p(\mathcal{M})$ is the unique (up to isometric isomorphism) p-Banach space such that for every p-Banach space X and every Lipschitz map $f \colon \mathcal{M} \to X$ with $f(0) = 0$ there exists a unique linear map $T_f \colon \mathcal{F}_p(\mathcal{M}) \to X$ with $T_f \circ \delta_{\mathcal{M}} = f$. Moreover $\|T_f\| = \mathrm{Lip}(f)$. Pictorially,



Remark 2.2. Let $(\mathcal{N}, d, 0)$ and $(\mathcal{M}, d', 0')$ be pointed p-metric spaces and $f\colon \mathcal{N} \to \mathcal{M}$ be a Lipschitz map with $f(0) = 0'$. Applying Theorem 2.1 yields a canonical linear map $L_f\colon \mathcal{F}_p(\mathcal{N}) \to \mathcal{F}_p(\mathcal{M})$ with $\|L_f\| \le \operatorname{Lip}(f)$ (see [2, Lemma 4.8]). One of the main reasons why the theory of Lipschitz-free spaces has been developed for real scalars is that, in this case, whenever $\mathcal{N}$ is a subset of a metric space $\mathcal{M}$ then the canonical linearization $L_j\colon \mathcal{F}(\mathcal{N}) \to \mathcal{F}(\mathcal{M})$ of the inclusion map $j\colon \mathcal{N} \to \mathcal{M}$ is an isometric embedding, which is no longer true in the complex case (see, e.g., [25, comment below Theorem 3.7 on p. 86]). This fact is not used in our arguments, which makes it possible to consider complex Lipschitz free spaces as well as real ones. Of course, $L_j\colon \mathcal{F}(\mathcal{N}) \to \mathcal{F}(\mathcal{M})$ is still an isomorphic embedding in the complex case. However, for Lipschitz free p-spaces with $p < 1$, the canonical map from $\mathcal{F}_p(\mathcal{N})$ into $\mathcal{F}_p(\mathcal{M})$ needs not be an isometric embedding even in the real case (see [2, Theorem 6.1]), and its unknown whether it is an isomorphic embedding at all! (see [2, Question 6.2]).

Remark 2.3. Another consequence of Theorem 2.1 is that, regardless of the scalar field, $\operatorname{Lip}_0(\mathcal{M})$ is isometric to the dual space of $\mathcal{F}_p(\mathcal{M})$. The duality

$$\langle \cdot, \cdot \rangle \colon \mathcal{F}_p(\mathcal{M}) \times \operatorname{Lip}_0(\mathcal{M}) \to \mathbb{F}$$

is given by the *evaluation* $\langle \delta(x), f \rangle = f(x)$ for $x \in \mathcal{M}$ and $f \in \operatorname{Lip}_0(\mathcal{M})$ (see [2, Corollary 4.23]).

The dual pairing between the Banach spaces $\mathcal{F}(\mathcal{M})$ and $\operatorname{Lip}_0(\mathcal{M})$ allows us to define a w^*-topology on $\operatorname{Lip}_0(\mathcal{M})$, and to infer the following well-known observation which we record for further reference.

Lemma 2.4. *Let $(\mathcal{M}, d, 0)$ be a pointed metric space, $(f_i)_{i\in I}$ be a bounded net in $\operatorname{Lip}_0(\mathcal{M})$, and $f \in \operatorname{Lip}_0(\mathcal{M})$. Then $w^*\text{–}\lim_i f_i = f$ if and only if $\lim_i f_i = f$ pointwise.*

Given $0 < p < q \le 1$, there is a canonical norm-one linear map

$$E_{p,q,\mathcal{M}}\colon \mathcal{F}_p(\mathcal{M}) \to \mathcal{F}_q(\mathcal{M}).$$

It is known [2, Proposition 4.20] that the q-Banach envelope of $\mathcal{F}_p(\mathcal{M})$ is $\mathcal{F}_q(\mathcal{M})$ with q-envelope map $E_{p,q,\mathcal{M}}$. In particular the Banach envelope of $\mathcal{F}_p(\mathcal{M})$ is $\mathcal{F}(\mathcal{M})$.

3. Universality of Lipschitz free p-spaces over bounded metric spaces for $0 < p \le 1$

The aim of this section is to build a bounded metric space $\mathcal{B}$ from an unbounded metric space $\mathcal{M}$ and to prove that $\mathcal{F}(\mathcal{M}) \simeq \mathcal{F}(\mathcal{B})$. The main result here is Theorem 3.9. We emphasize that all of our results hold true both for real and complex p-Banach spaces when $p \in (0,1]$.

3.1. **Construction of the bounded metric space in Theorem A.** We start our construction of a bounded metric space from an arbitrary metric space (in general unbounded) by introducing the main ingredients.

Definition 3.1. Let $\alpha\colon (0,\infty)\to(0,\infty)$ be a map and $0<p\le 1$. Given a metric space $(\mathcal{M},d)$ we introduce the set

$$\mathcal{B}_p(\mathcal{M},\alpha)=\{0\}\cup\left\{\frac{\delta_{\mathcal{M}}(x)}{\alpha(d(x,0))}: x\in\mathcal{M}\setminus\{0\}\right\}\subset\mathcal{F}_p(\mathcal{M})$$

endowed with the p-metric it inherits from $\mathcal{F}_p(\mathcal{M})$, and we denote $\mathcal{B}_1(\mathcal{M},\alpha)$ by $\mathcal{B}(\mathcal{M},\alpha)$. Once α is clear, we write $\mathcal{B}_p(\mathcal{M})$ and $\mathcal{B}(\mathcal{M})$ instead of $\mathcal{B}_p(\mathcal{M},\alpha)$ and $\mathcal{B}(\mathcal{M},\alpha)$, respectively.

Convention 3.2. The map $\alpha\colon(0,\infty)\to(0,\infty)$ from Definition 3.1 will often be Lipschitz. In this case, we will denote by $\alpha(0)$ the limit $\lim_{t\to 0^+}\alpha(t)$.

Let $\alpha\colon(0,\infty)\to(0,\infty)$ be a function and $p\in(0,1]$. Given a pointed metric space $(\mathcal{M},d,0)$ we consider the maps $\zeta_\alpha\colon\mathcal{M}\to[0,\infty)$ given by

$$\zeta_\alpha(x):=\begin{cases}0 & \text{if } x=0,\\ \alpha(d(0,x)) & \text{if } x\neq 0,\end{cases} \tag{3.1}$$

and $\mu_\alpha\colon\mathcal{M}\to\mathcal{F}_p(\mathcal{M})$ given by

$$\mu_\alpha(x):=\begin{cases}0 & \text{if } x=0,\\ \dfrac{\delta_{\mathcal{M}}(x)}{\zeta_\alpha(x)} & \text{if } x\neq 0.\end{cases} \tag{3.2}$$

We will write ζ instead of ζ_α and μ instead of μ_α once α is clear. We have $\mathcal{B}_p(\mathcal{M},\alpha)=\mu(\mathcal{M})$.

We also need to introduce the constants

$$\mathrm{D}(\alpha):=\sup_{t>0}\frac{t}{\alpha(t)}$$

and

$$\mathrm{K}(\alpha)=\mathrm{Lip}(\alpha)\ \mathrm{D}(\alpha).$$

Note that if $\mathrm{D}(\alpha)<\infty$ then $\lim_{t\to\infty}\alpha(t)=\infty$ and so

$$\mathrm{K}(\alpha)\ge\sup_{t>1}\frac{t}{\alpha(t)}\frac{|\alpha(t)-\alpha(1)|}{t-1}\ge 1.$$

Lemma 3.5 below provides estimates for the distance in the p-metric space $\mathcal{B}_p(\mathcal{M},\alpha)$ in terms of $\mathrm{D}(\alpha)$, $\mathrm{Lip}(\alpha)$, and the distance d in $\mathcal{M}$.

Prior to proving it we give a couple of auxiliary results. With the convention $0/0 = 0$ we set

$$d_\alpha(x, y) = \frac{d(x, y)}{\max\{\zeta_\alpha(x), \zeta_\alpha(y)\}}, \quad x, y \in \mathcal{M}. \tag{3.3}$$

Lemma 3.3. *Let $\alpha\colon (0,\infty) \to (0,\infty)$ be a map, $(\mathcal{M}, d, 0)$ be a pointed metric space, and $0 < p \le 1$. Let ζ and μ be as in* (3.1) *and* (3.2)*, respectively.*

(i) If $\mathrm{D}(\alpha) < \infty$, then $\mathcal{B}_p(\mathcal{M}, \alpha)$ is a bounded subset of $\mathcal{F}_p(\mathcal{M})$. In fact, if $f \in \mathrm{Lip}_0(\mathcal{M})$ is given by $f(z) = d(0, z)$ for all $z \in \mathcal{M}$, then

$$\|\mu(x)\|_{\mathcal{F}_p(\mathcal{M})} = \langle \mu(x), f\rangle = \frac{d(0, x)}{\zeta(x)} \le \mathrm{D}(\alpha), \quad x \in \mathcal{M} \setminus \{0\}.$$

(ii) If α is Lipschitz, then ζ is Lipschitz on $\mathcal{M} \setminus \{0\}$. Quantitatively,

$$|\zeta(x) - \zeta(y)| \le \mathrm{Lip}(\alpha)\, d(x, y), \quad x, y \in \mathcal{M} \setminus \{0\}.$$

Proof. (i) is clear from the definition, and (ii) follows by combining the Lipschitz condition with the triangle inequality. □

Lemma 3.4. *Let $\alpha\colon (0,\infty) \to (0,\infty)$ be Lipschitz and $(\mathcal{M}, d, 0)$ be a pointed metric space. Let ζ, μ and d_α be as in* (3.1)*,* (3.2)*, and* (3.3) *respectively. Then, for every $y \in \mathcal{M} \setminus \{0\}$ there are f, $g \in \mathrm{Lip}_0(\mathcal{M})$ with $\mathrm{Lip}(f) \le 1$ such that:*

(i) $|\langle \mu(x) - \mu(y), f\rangle| \ge d_\alpha(x, y)$ whenever $x \in \mathcal{M} \setminus \{0\}$ satisfies $\zeta(x) \ge \zeta(y)$; and

(ii) for every $x \in \mathcal{M} \setminus \{0\}$ either $|\langle \mu(x) - \mu(y), f\rangle| \ge d_\alpha(x, y)$ or $|\langle \mu(x) - \mu(y), g\rangle| \ge d_\alpha(x, y)$.

Proof. For $z \in \mathcal{M}$ put $f(z) = d(z, y) - d(0, y)$. Then

$$\langle \mu(x) - \mu(y), f\rangle = \frac{d(x, y) - d(0, y)}{\zeta(x)} + \frac{d(0, y)}{\zeta(y)}$$

for all $x \in \mathcal{M} \setminus \{0\}$. We infer that $\langle \mu(x) - \mu(y), f\rangle \ge d_\alpha(x, y)$ in the case when $\zeta(y) \le \zeta(x)$ or $d(x, y) \ge d(0, y)$. This proves (i). To conclude the proof of (ii), we set

$$g_1 = \frac{\zeta}{\zeta(y)} \wedge 1, \quad g_2 = |f| \wedge d(0, y), \quad \text{and} \quad g = g_1 g_2.$$

Since g_1 is bounded and g_2 is continuous with $g_2(0) = 0$, we deduce that g is continuous at 0 with $g(0) = 0$. By Lemma 3.3 (ii), $g_1|_{\mathcal{M}\setminus\{0\}}$ is Lipschitz and so is $g_2|_{\mathcal{M}\setminus\{0\}}$. Taking into account that g_1 and g_2 are bounded we infer that $g|_{\mathcal{M}\setminus\{0\}}$ is Lipschitz. Hence, $g \in \mathrm{Lip}_0(\mathcal{M})$. Let $x \in \mathcal{M} \setminus \{0\}$ be so that $\zeta(x) < \zeta(y)$ and $d(x, y) < d(0, y)$. Then

$g_1(x) = \zeta(x)/\zeta(y)$, $g_1(y) = 1$, $g_2(x) = d(0, y) - d(x, y)$, and $g_2(y) = d(0, y)$. Therefore,

$$\langle \mu(x) - \mu(y), g\rangle = \frac{\zeta(x)(d(0,y) - d(x,y))}{\zeta(y)\zeta(x)} - \frac{d(0,y)}{\zeta(y)} = -d_\alpha(x,y). \qquad \square$$

Lemma 3.5. *Let $\alpha\colon (0,\infty) \to (0,\infty)$ be a Lipschitz map with $\mathrm{D}(\alpha) < \infty$, $(\mathcal{M}, d, 0)$ be a pointed metric space, and $0 < p \le 1$. Let ζ and μ be as in (3.1), (3.2), and (3.3) respectively. Then*

$$d_\alpha(x,y) \le \|\mu(x) - \mu(y)\|_{\mathcal{F}_p(\mathcal{M})} \le (1 + \mathrm{K}^p(\alpha))^{1/p} d_\alpha(x,y), \quad x, y \in \mathcal{M}.$$

Proof. It suffices to consider the case when x, $y \in \mathcal{M}\setminus\{0\}$ and $\zeta(x) \ge \zeta(y)$. The left hand-side inequality follows from Lemma 3.4 (i). Using Lemma 3.3, the expression

$$\mu(x) - \mu(y) = \frac{1}{\zeta(x)}(\delta_{\mathcal{M}}(x) - \delta_{\mathcal{M}}(y)) + \frac{\zeta(y) - \zeta(x)}{\zeta(x)}\mu(y),$$

yields

$$\begin{aligned}\|\mu(x) - \mu(y)\|^p_{\mathcal{F}_p(\mathcal{M})} &\le \frac{d^p(x,y)}{\zeta^p(x)} + \mathrm{D}^p(\alpha)\,\mathrm{Lip}^p(\alpha)\frac{d^p(x,y)}{\zeta^p(x)} \\ &= (1 + \mathrm{K}^p(\alpha))d^p_\alpha(x,y). \qquad \square\end{aligned}$$

Proposition 3.8 below exhibits that, up to Lipschitz isomorphism, it suffices to consider $\mathcal{B}_p(\mathcal{M},\alpha)$ in the case when $p = 1$ and α is either the identity map on $(0,\infty)$, which we will denote by $\alpha^{(0)}$, or the map $\alpha^{(1)} = 1 + \alpha^{(0)}$.

Remark 3.6. The metric space $\mathcal{B} := \mathcal{B}(\mathcal{M}, \alpha^{(0)})$ is in a sense natural and has previously appeared in the work of other authors. Weaver (see [25, Definition 2.17 and Lemma 2.18]) considered $\mathcal{B}$ when investigating the lattice structure of $\mathrm{Lip}_0(\mathcal{M})$. To be more specific, he identified the *complete lattice spectrum* of $\mathrm{Lip}_0(\mathcal{M})$ as the set $\mathcal{B}\setminus\{0\}$ ([25, Theorem 6.22]) and proved that there is a natural order-preserving surjective isomorphism between $\mathrm{Lip}_0(\mathcal{M})$ and $\mathrm{Lip}(\mathcal{B}\setminus\{0\})$ ([25, Theorem 6.23]). From a completely different approach, Aliaga et al. proved that the set of extreme points of the positive cone of the unit ball of $\mathcal{F}(\mathcal{M})$ coincides with our space $\mathcal{B}$ (see [9, Theorem 3.8]).

Given two real-valued functions f and g we write $f \approx g$ to denote that there are constants $C, D > 0$ such that $Cf \le g \le Df$.

Lemma 3.7. *Let $\alpha\colon (0,\infty) \to (0,\infty)$ be a Lipschitz map with $\mathrm{D}(\alpha) < \infty$. If $\alpha(0) = 0$, then $\alpha \approx \alpha^{(0)}$; and, if $\alpha(0) > 0$, then $\alpha \approx \alpha^{(1)}$.*

Proof. If $\alpha(0) = 0$ we have $\alpha(t) \approx t$ for $0 < t \le 1$. If $\alpha(0) > 0$ we have $\alpha(t) \approx 1$ for $0 < t \le 1$. Since $\alpha(t) \approx t$ for $t \ge 1$ we are done. $\square$

Proposition 3.8. *Let $(\mathcal{M}, d, 0)$ be a pointed metric space, let $0 < p \le 1$, and let $\alpha\colon (0,\infty) \to (0,\infty)$ be a Lipschitz map with $\mathrm{D}(\alpha) < \infty$. Then:*

(i) $\mathcal{B}_p(\mathcal{M},\alpha) \simeq_{\mathrm{Lip}} \mathcal{B}(\mathcal{M},\alpha)$. To be precise, the identity map is a bi-Lipchitz map with distortion at most $(1+\mathrm{K}^p(\alpha))^{1/p}$.
(ii) Either $\alpha(0) = 0$, in which case $\mathcal{B}(\mathcal{M},\alpha) \simeq_{\mathrm{Lip}} \mathcal{B}(\mathcal{M},\alpha^{(0)})$, or $\alpha(0) > 0$, in which case $\mathcal{B}(\mathcal{M},\alpha) \simeq_{\mathrm{Lip}} \mathcal{B}(\mathcal{M},\alpha^{(1)})$.
(iii) If 0 is an isolated point, $\mathcal{B}(\mathcal{M},\alpha^{(0)}) \simeq_{\mathrm{Lip}} \mathcal{B}(\mathcal{M},\alpha^{(1)})$.

Proof. Let ζ and d_α be as in (3.1) and (3.3), respectively. Let μ_q denote the function μ defined as in (3.2) corresponding to an index $q \in \{p, 1\}$. Let x, $y \in \mathcal{M}$. By lemma 3.5,

$$(1+\mathrm{K}^p(\alpha))^{-1/p}\|\mu_p(x)-\mu_p(y)\|_{\mathcal{F}_p(\mathcal{M})} \le d_\alpha(x,y) \le \|\mu_1(x)-\mu_1(y)\|_{\mathcal{F}(\mathcal{M})}.$$

Since $\|m\|_{\mathcal{F}(\mathcal{M})} \le \|m\|_{\mathcal{F}_p(\mathcal{M})}$ for every $m \in \mathcal{P}(\mathcal{M})$, (i) holds. Moreover,

$$\|\mu_1(x) - \mu_1(y)\|_{\mathcal{F}(\mathcal{M})} \approx d_\alpha(x,y), \quad x, y \in \mathcal{M}. \tag{3.4}$$

Combining Lemma 3.7 with (3.4) gives (ii). If $d := d(0, \mathcal{M}\setminus\{0\}) > 0$, then d_α only depends on the values of α in the interval $[d,\infty]$. Therefore, since $\alpha^{(0)}(t) \approx \alpha^{(1)}(t)$ for $t \ge d$, $d_{\alpha^{(0)}}(x,y) \approx d_{\alpha^{(1)}}(x,y)$ for all x, $y \in \mathcal{M}$. Applying (3.4) yields (iii). □

Theorem 3.9. *Let $(\mathcal{M}, d)$ be a metric space. There exists a bounded metric space $\mathcal{B}$ with $\mathcal{F}_p(\mathcal{M}) \simeq \mathcal{F}_p(\mathcal{B})$ for all $p \in (0,1]$. To be more specific, if $\alpha\colon (0,\infty) \to (0,\infty)$ is a Lipschitz function with $\mathrm{D}(\alpha) < \infty$, and ζ and μ are as in (3.1) and (3.2) respectively, then the linear operator $P_\alpha^F\colon \mathcal{F}_p(\mathcal{M}) \to \mathcal{F}_p(\mathcal{B}(\mathcal{M},\alpha))$ defined by*

$$P_\alpha^F(\delta_{\mathcal{M}}(x)) = \zeta(x)\delta_{\mathcal{B}}(\mu(x)), \quad x \in \mathcal{M}\setminus\{0\},$$

induces an isomorphism with inverse $Q_\alpha^F\colon \mathcal{F}_p(\mathcal{B}(\mathcal{M},\alpha)) \to \mathcal{F}_p(\mathcal{M})$ given by

$$Q_\alpha^F(\delta_{\mathcal{B}}(\mu(x))) = \mu(x), \quad x \in \mathcal{M}\setminus\{0\}.$$

Moreover,

$$\|P_\alpha^F\| \le (1+2\,\mathrm{K}^p(\alpha))^{1/p} \quad \text{and} \quad \|Q_\alpha^F\| \le \begin{cases} 1 & \text{if } p = 1, \\ (1+\mathrm{K}^p(\alpha))^{1/p} & \text{if } p < 1. \end{cases}$$

Proof. By Proposition 3.8 (i), we can replace $\mathcal{B}(\mathcal{M},\alpha)$ with $\mathcal{B} := \mathcal{B}_p(\mathcal{M},\alpha)$. By Lemma 3.3 (i), $\mathcal{B}$ is bounded. For ζ and μ as in (3.1) and (3.2) respectively, consider the map $f\colon \mathcal{M} \to \mathcal{F}_p(\mathcal{B})$ defined by the formula

$$f(x) = \zeta(x)\,\delta_{\mathcal{B}}(\mu(x)), \quad x \in \mathcal{M}.$$

Let us verify that f is $(1+2\,\mathrm{K}^p(\alpha))^{1/p}$-Lipschitz. Let x, $y \in \mathcal{M}$, and set $D = \|f(x)-f(y)\|_{\mathcal{F}_p(\mathcal{B})}$. Without loss of generality we may assume that $\zeta(x) \ge \zeta(y)$. The expansion

$$f(x)-f(y) = \zeta(x)\,(\delta_{\mathcal{B}}(\mu(x)) - \delta_{\mathcal{B}}(\mu(y))) + (\zeta(x)-\zeta(y))\,\delta_{\mathcal{B}}(\mu(y)),$$

combined with Lemmas 3.3 and 3.5 gives

$$\begin{aligned} D^p &\le \zeta^p(x)\|\mu(x)-\mu(y)\|^p_{\mathcal{F}_p(\mathcal{M})} + |\zeta(x)-\zeta(y)|^p\|\mu(y)\|^p_{\mathcal{F}_p(\mathcal{M})} \\ &\le (1+\mathrm{K}^p(\alpha))d^p(x,y) + \mathrm{Lip}^p(\alpha)d^p(x,y)\,\mathrm{D}^p(\alpha) \\ &= (1+2\,\mathrm{K}^p(\alpha))d^p(x,y). \end{aligned}$$

Then, Theorem 2.1 yields a bounded linear operator $P^F_\alpha \colon \mathcal{F}_p(\mathcal{M}) \to \mathcal{F}_p(\mathcal{B})$ such that

$$P^F_\alpha(\delta_{\mathcal{M}}(x)) = \zeta(x)\,\delta_{\mathcal{B}}(\mu(x)), \quad x \in \mathcal{M}.$$

Further, since the inclusion of $\mathcal{B}$ into $\mathcal{F}_p(\mathcal{M})$ is 1-Lipschitz, appealing again to Theorem 2.1, we infer that there is a norm-one linear operator $Q^F_\alpha \colon \mathcal{F}_p(\mathcal{B}) \to \mathcal{F}_p(\mathcal{M})$ with

$$Q^F_\alpha(\delta_{\mathcal{B}}(\mu(x))) = \mu(x), \quad x \in \mathcal{M}.$$

It is clear from the definition that P^F_α restricted to $\operatorname{span}\{\delta_{\mathcal{M}}(x) \colon x \in \mathcal{M}\setminus\{0\}\}$ is a linear bijection onto $V = \operatorname{span}\{\delta_{\mathcal{B}}(x) \colon x \in \mathcal{B}\setminus\{0\}\}$ with inverse $Q^F_\alpha|_V$. Therefore, P^F_α and Q^F_α are inverse isomorphisms of each other. □

We close this section with a couple of examples that for simplicity we work out in the case of real Banach spaces, that is, $\mathbb{F} = \mathbb{R}$.

Example 3.10. Consider the metric space $\mathbb{N}\cup\{0\}$ endowed with the Euclidean distance. The Lipschitz-free space $\mathcal{F}(\mathbb{N}\cup\{0\})$ is isometric to ℓ_1 via the map $\delta(n) \mapsto \sum_{j=1}^n \mathbf{e}_j$, where $\mathbf{e}_j$ denotes the jth unit vector, $n \in \mathbb{N}$. Hence, if we set

$$\mathcal{B} = \{0\} \cup \left\{ \frac{1}{n}\sum_{j=1}^n \mathbf{e}_j \colon n \in \mathbb{N} \right\} \subset \ell_1,$$

we have $\mathcal{F}_p(\mathbb{N}\cup\{0\}) \simeq \mathcal{F}_p(\mathcal{B})$ for every $0 < p \le 1$.

Example 3.11. Consider the metric space $\mathbb{R}^+ = [0,\infty)$ endowed with the Euclidean distance. The Lipschitz-free space $\mathcal{F}(\mathbb{R}^+)$ is isometric to $L_1(\mathbb{R}^+)$. Namely, the map $\delta(x) \mapsto \chi_{[0,x)}$ extends to an isometry. Hence, if we consider

$$\mathcal{B}_0 = \{0\} \cup \left\{ \frac{1}{x}\chi_{[0,x)} \colon x > 0 \right\},$$

$$\mathcal{B}_1 = \left\{ \frac{1}{x+1} \chi_{[0,x)} \colon x \geq 0 \right\}$$

as subsets of $L_1(\mathbb{R}^+)$, we have $\mathcal{F}_p(\mathbb{R}^+) \simeq \mathcal{F}_p(\mathcal{B}_0) \simeq \mathcal{F}_p(\mathcal{B}_1)$ for every $0 < p \leq 1$.

3.2. **The functorial character of our construction.** Recall that the construction of Lipschitz-free spaces is functorial in nature. Indeed, there is a functor $\mathcal{F}$ from the category of pointed metric spaces equipped with Lipschitz maps between them preserving the base points to the category of Banach spaces. To each pointed metric space $\mathcal{M}$, the functor $\mathcal{F}$ associates a Lipschitz-free space $\mathcal{F}(\mathcal{M})$ and to each Lipschitz map $f \colon \mathcal{M} \to \mathcal{N}$, with $f(0) = 0$, it associates its linear extension between $\mathcal{F}(\mathcal{M})$ and $\mathcal{F}(\mathcal{N})$. Moreover, $\mathcal{F}$ is a left-adjoint functor to the forgetful functor from the category of Banach spaces to the category of pointed metric spaces.

It is very natural to investigate to which extent the construction of a bounded metric space $\mathcal{B}(\mathcal{M})$ from a metric space $\mathcal{M}$ is functorial too. We shall next show that it is functorial in a weak sense - namely, when we restrict the class of morphisms to bi-Lipschitz zero-preserving maps only (not necessarily bijections).

For any zero-preserving Lipschitz map $f \colon \mathcal{M} \to \mathcal{N}$ between two pointed metric spaces and any $0 < p \leq 1$ there is a canonical linear extension $L_f \colon \mathcal{F}_p(\mathcal{M}) \to \mathcal{F}_p(\mathcal{N})$. If $P_{p,\mathcal{M}}$ and $P_{p,\mathcal{N}}$ denote the linear isomorphisms provided by Theorem 3.9, we obtain a bounded linear map $\mathcal{B}(f) \colon \mathcal{F}_p(\mathcal{B}(\mathcal{M})) \to \mathcal{F}_p(\mathcal{B}(\mathcal{N}))$ making the diagram

$$\begin{array}{ccc} \mathcal{F}_p(\mathcal{M}) & \xrightarrow{L_f} & \mathcal{F}_p(\mathcal{N}) \\ \downarrow{\scriptstyle P_{p,\mathcal{M}}} & & \downarrow{\scriptstyle P_{p,\mathcal{N}}} \\ \mathcal{F}_p(\mathcal{B}(\mathcal{M})) & \overset{\mathcal{B}(f)}{\dashrightarrow} & \mathcal{F}_p(\mathcal{B}(\mathcal{N})) \end{array}$$

commute. A straightforward computation yields

$$\mathcal{B}(f) \left(\delta_{\mathcal{B}(\mathcal{M})}(\mu(x)) \right) = \frac{\zeta(f(x))}{\zeta(x)} \delta_{\mathcal{B}(\mathcal{N})} \left(\mu(f(x)) \right), \qquad x \in \mathcal{M} \setminus \{0\}.$$

Therefore $\mathcal{B}(f)$ is not a linear extension of a Lipschitz map between $\mathcal{B}(\mathcal{M})$ and $\mathcal{B}(\mathcal{N})$ unless $\zeta(f(x)) = \zeta(x)$ for all $x \in \mathcal{M} \setminus \{0\}$, as it happens for example when f is an isometric embedding, and the construction is carried out with the same function α in both metric spaces.

However, when the category is restricted to the class of bi-Lipschitz morphisms, then the construction is functorial in the regular sense. In the proposition below, id denotes the identity map on a given metric space.

Proposition 3.12. *Let $\alpha\colon (0,\infty)\to(0,\infty)$ be a Lipschitz map. For any bi-Lipchitz zero-preserving map $f\colon \mathcal{M}\to\mathcal{N}$ between pointed metric spaces let us consider $\mathcal{B}(f)\colon \mathcal{B}(\mathcal{M})\to\mathcal{B}(\mathcal{N})$ defined as*

$$\mathcal{B}(f)(\mu(x)) := \mu(f(x)), \quad x\in\mathcal{M},$$

where μ is as in (3.2). Then for every bi-Lipchitz zero-preserving maps f, $g\colon \mathcal{M}\to\mathcal{N}$ between pointed metric spaces we have:

(i) $\mathcal{B}(\mathrm{id})=\mathrm{id}$ *and* $\mathcal{B}(f\circ g)=\mathcal{B}(f)\circ\mathcal{B}(g)$, *and*

(ii) $\mathcal{B}(f)$ *is a bi-Lipchitz zero-preserving map.*

Proof. We only verify that $\mathcal{B}(f)$ is a Lipschitz map between $\mathcal{B}(\mathcal{M})$ and $\mathcal{B}(\mathcal{N})$. By symmetry, it will follow that it is bi-Lipschitz, and it is clear that $\mathcal{B}(\mathrm{id})=\mathrm{id}$ and $\mathcal{B}(f\circ g)=\mathcal{B}(f)\circ\mathcal{B}(g)$.

Fix $x,y\in\mathcal{M}$. Let us estimate the ratio

$$\frac{\|\mathcal{B}(f)(\mu(x))-\mathcal{B}(f)(\mu(y))\|}{\|\mu(x)-\mu(y)\|}=\frac{\|\mu(f(x))-\mu(f(y))\|}{\|\mu(x)-\mu(y)\|}.$$

By Lemma 3.5,

$$d_\alpha(x,y)\le\|\mu(x)-\mu(y)\|_{\mathcal{F}_p(\mathcal{M})}\le(1+\mathrm{K}^p(\alpha))^{1/p}d_\alpha(x,y),$$

and analogously for $f(x)$ and $f(y)$. It follows that

$$\frac{\|\mu(f(x))-\mu(f(y))\|}{\|\mu(x)-\mu(y)\|}\le\frac{(1+\mathrm{K}^p(\alpha))^{1/p}d_\alpha(f(x),f(y))}{d_\alpha(x,y)},$$

so it is enough to provide a uniform bound for

$$\frac{d_\alpha(f(x),f(y))}{d_\alpha(x,y)}=A(x,y)B(x,y),$$

where

$$A(x,y)=\frac{d(f(x),f(y))}{d(x,y)} \text{ and } B(x,y)=\frac{\max\{\zeta(x),\zeta(y)\}}{\max\{\zeta(f(x)),\zeta(f(y))\}}.$$

Since $A(x,y)\le\mathrm{Lip}(f)$, it suffices to obtain a uniform bound for $B(x,y)$. To that end, without loss of generality we assume that $\zeta(y)\le\zeta(x)$. By Lemma 3.7 we may assume that $\alpha=\alpha^{(0)}$ or $\alpha=\alpha^{(1)}$; in particular α is increasing. Then,

$$B(x,y)\le\frac{\zeta(x)}{\zeta(f(x))}\le\frac{\alpha(d(x,0))}{\alpha(\mathrm{Lip}(f^{-1})d(x,0))},$$

and considering separately the cases $\alpha=\alpha^{(0)}$ and $\alpha=\alpha^{(1)}$ it is routine to check that the right-hand side term is uniformly bounded. Note that the last inequality is the only place where we used that the inverse of f is Lipschitz. □

The following example shows that the requirement that f be bi-Lipschitz in Proposition 3.12 cannot be weakened. That is, an analogous result to Proposition 3.12 just for Lipschitz maps does not hold.

Example 3.13. Consider the set $\mathcal{M} := (\mathbb{N} \cup \{0\}) \times \{0,1\}$ equipped with the metric d defined by

$$d((n,i),(m,j)) = \begin{cases} 2^n + 2^m - 1 & \text{if } m \neq n, \\ |i-j| & \text{if } m = n. \end{cases}$$

Let $\alpha = \alpha^{(0)}$ be the identity map. Choosing $(0,0)$ as distinguished point of $\mathcal{M}$ we have

$$\mu(n,i) = \frac{1}{2^n} \delta_{\mathcal{M}}(n,i), \quad (n,i) \in \mathcal{M}.$$

The map $f \colon \mathcal{M} \to \mathcal{M}$ given by $f(n,i) = (0,i)$ for all $(n,i) \in \mathcal{M}$ is 1-Lipschitz. Since $\|\mu(n,1) - \mu(n,0)\| = 2^{-n}$ for all $n \in \mathbb{N} \cup \{0\}$,

$$\|\mathcal{B}(f)(\mu(n,1)) - \mathcal{B}(f)(\mu(n,0))\| = 2^n \|\mu(n,1) - \mu(n,0)\|, \quad n \in \mathbb{N} \cup \{0\}.$$

Letting n tend to ∞ shows that $\mathcal{B}(f)$ is not Lipschitz.

Remark 3.14. Let $\mathcal{M}$ be a metric space, $\mathcal{N}$ be a subset of $\mathcal{M}$, and let $0 < p < 1$. An interesting consequence of Proposition 3.12 is that the aforementioned question of whether the canonical linearization $L_j \colon \mathcal{F}_p(\mathcal{N}) \to \mathcal{F}_p(\mathcal{M})$ of the inclusion map $j \colon \mathcal{N} \to \mathcal{M}$ is an isomorphism (see Remark 2.2) reduces now to check with bounded metric spaces. Indeed, since $\mathcal{B}(j) = L_j|_{\mathcal{B}_p(\mathcal{N})}$, if the answer were positive for bounded metric spaces $\mathcal{M}$ then the answer would be positive also for unbounded ones.

Remark 3.15. It follows directly from the construction that if $(\mathcal{M}, d, 0)$ is a metric space and $0 \in \mathcal{N} \subset \mathcal{M}$, then $\mathcal{B}(\mathcal{N}) \subset \mathcal{B}(\mathcal{M})$ via the canonical isometry. It also follows from the proof of Proposition 3.12 that if $f \colon \mathcal{M} \to \mathcal{N}$ is a zero-preserving isometry and $\alpha = \alpha^{(0)}$, then $\mathcal{B}(f) \colon \mathcal{B}(\mathcal{M}) \to \mathcal{B}(\mathcal{N})$ is an isometry as well.

Remark 3.16. Our construction also has a functorial behavior with respect to the family of spaces that arise if p is allowed to take values in interval $(0,1]$: for any pointed metric space $\mathcal{M}$ and any $0 < p < q \leq 1$

the diagram

$$\begin{array}{ccc} \mathcal{F}_p(\mathcal{M}) & \xrightarrow{P_{p,\mathcal{M}}} & \mathcal{F}_p(\mathcal{B}(\mathcal{M})) \\ \Big\downarrow{\scriptstyle E_{p,q,\mathcal{M}}} & & \Big\downarrow{\scriptstyle E_{p,q,\mathcal{B}(\mathcal{M})}} \\ \mathcal{F}_q(\mathcal{M}) & \xrightarrow{P_{q,\mathcal{M}}} & \mathcal{F}_q(\mathcal{B}(\mathcal{M})). \end{array} \tag{3.5}$$

commutes. Another open problem from [2] is whether the envelope map $E_{p,1,\mathcal{M}} \colon \mathcal{F}_p(\mathcal{M}) \to \mathcal{F}(\mathcal{M})$ is one-to-one for any metric space $\mathcal{M}$ and any $0 < p < 1$. By (3.5), to address this question it suffices to consider the case when $\mathcal{M}$ is bounded. We recall that $E_{p,1,\mathcal{M}}$ is always one-to-one on $\mathcal{P}(\mathcal{M})$, but this does not guarantee its injectivity on its completion $\mathcal{F}_p(\mathcal{M})$ (see [4] for a discussion about the injectivity of envelope maps).

4. The metric space $\mathcal{M}$ versus the bounded metric space $\mathcal{B}(\mathcal{M}, \alpha)$

In this section we compare the Lipschitz structures and the topological structures of the unbounded metric space $\mathcal{M}$ on one hand, and the bounded metric space $\mathcal{B}(\mathcal{M}, \alpha)$ built from $\mathcal{M}$ and a Lipschitz function α with $\mathrm{D}(\alpha) < \infty$ on the other. We prove that $\mathcal{M}$ and $\mathcal{B}(\mathcal{M}, \alpha)$ are topologically homeomorphic whenever $0 \in \mathcal{M}$ is an isolated point.

The assumption that $0 \in \mathcal{M}$ is isolated is not so strong. Indeed, if $\mathcal{M}$ does not have isolated points we add an isolated point $*$ to our space $\mathcal{M}$ so that $\mathcal{F}_p(\mathcal{M} \cup \{*\}) \simeq \mathcal{F}_p(\mathcal{M})$ for every $0 < p \le 1$ (see e.g. [3, Lemma 2.8]). This way we obtain a homeomorphism between $\mathcal{M}$ and $\mathcal{B} := \mathcal{B}(\mathcal{M} \cup \{*\}, \alpha) \setminus \{0\}$ such that $\mathcal{F}_p(\mathcal{M}) \simeq \mathcal{F}_p(\mathcal{B})$.

Everything that follows in this section holds for both real and complex spaces, because we are using exclusively the properties of Lipchitz-free spaces summarized in Section 2.

Lemma 4.1. *Let $(\mathcal{M}, d, 0)$ be a metric space, and let α be a map from $(0,\infty)$ into $(0,\infty)$.*

(i) Let $d > 0$ and $x \in \mathcal{M} \setminus \{0\}$ be such that $d(0,x) \ge d$. If α is Lipschitz then

$$\|\mu(x)\|_{\mathcal{F}(\mathcal{M})} \ge \min\left\{\frac{1}{\mathrm{Lip}(\alpha)}, \frac{d}{\alpha(d)}\right\}.$$

(ii) Suppose that α is Lipschitz, that $\mathrm{D}(\alpha) < \infty$, and that either $\alpha(0) > 0$ or 0 is an isolated point of $\mathcal{M}$. Then there is a constant $C < \infty$ such that

$$\|\mu(x)\|_{\mathcal{F}(\mathcal{M})} \le C d(0,x), \quad x \in \mathcal{M} \setminus \{0\}.$$

In fact, we can choose $\inf_{x\neq 0} \alpha(d(0,x)) = 1/C$.

(iii) Suppose that α *is Lipschitz and* $\alpha(0) = 0$. *Then*

$$\|\mu(x)\|_{\mathcal{F}(\mathcal{M})} \geq \frac{1}{\operatorname{Lip}(\alpha)}, \quad x \in \mathcal{M} \setminus \{0\}.$$

Proof. Note that $\|\mu(x)\|_{\mathcal{F}(\mathcal{M})} = d(0,x)/\alpha(d(0,x))$ so (ii) easily holds. To prove (i) we set $u = d(0,x) - d$, so that

$$\|\mu(x)\|_{\mathcal{F}(\mathcal{M})} = \frac{u+d}{\alpha(u+d)} \geq \frac{u+d}{\operatorname{Lip}(\alpha)u + \alpha(d)} \geq \min\left\{\frac{1}{\operatorname{Lip}(\alpha)}, \frac{d}{\alpha(d)}\right\}.$$

Finally, under the assumptions in (iii), we have $\alpha(t) \leq \operatorname{Lip}(\alpha)t$ for all $t > 0$, and the result is a consequence of this inequality. □

Proposition 4.2. *Let* $\alpha\colon (0,\infty) \to (0,\infty)$ *be a Lipschitz map with* $\mathrm{D}(\alpha) < \infty$ *and let* $(\mathcal{M}, d, 0)$ *be a pointed metric space. Set* $\mathcal{B} = \mathcal{B}(\mathcal{M}, \alpha)$. *Let* μ *be as in* (3.2). *Then:*

(i) μ *is one-to-one.*

(ii) $\mu|_{\mathcal{M}_{[r,R]}}$ *is bi-Lipschitz for all choices of* $0 < r < R < \infty$, *where*

$$\mathcal{M}_{[r,R]} = \{x \in \mathcal{M} \colon r \leq d(0,x) \leq R\}.$$

(iii) If 0 *is an isolated point of* $\mathcal{M}$ *or* $\alpha(0) > 0$, μ *is Lipschitz, and* $\mu|_{\mathcal{M}_{[0,R]}}$ *is bi-Lipschitz for every* $R < \infty$.

(iv) $\mu|_{\mathcal{M}\setminus\{0\}}$ *is a topological homeomorphism onto* $\mathcal{B} \setminus \{0\}$.

(v) If $0 \in K \subset \mathcal{M}$, K *is closed in* $\mathcal{M}$ *if and only if* $\mu(K)$ *is closed in* $\mathcal{B}$.

(vi) If 0 *is an isolated point of* $\mathcal{M}$ *or* $\alpha(0) > 0$, μ *is a topological homeomorphism.*

(vii) Given a net $(x_i)_{i\in I}$ *in* $\mathcal{M} \setminus \{0\}$ *the following are equivalent:*

(a) $\lim_i \mu(x_i) = 0$ *in norm.*

(b) $\lim_i \mu(x_i) = 0$ *weakly.*

(c) $\alpha(0) > 0$ *and* $\lim_i x_i = 0$.

(viii) 0 *is an isolated point of* $\mathcal{B}$ *if and only if either* $\alpha(0) = 0$ *or* 0 *is an isolated point of* $\mathcal{M}$.

(ix) The inverse of μ *is continuous.*

(x) A sequence $(x_n)_{n=1}^\infty$ *in* $\mathcal{M} \setminus \{0\}$ *is Cauchy if and only if either it converges to* 0 *or* $(\mu(x_n))_{n=1}^\infty$ *is Cauchy in* $\mathcal{B}$.

(xi) If 0 *is an isolated point of* $\mathcal{M}$ *or* $\alpha(0) > 0$, $\mathcal{M}$ *is complete if and only if* $\mathcal{B}$ *is complete.*

(xii) The norm and weak topologies coincide on $\mathcal{B} \subset \mathcal{F}(\mathcal{M})$.

Proof. Let d_α be as in (3.3). (i) is clear, because $\{\delta_{\mathcal{M}}(x) \colon x \neq 0\}$ is a linearly independent set. (ii) and (iii) follow from Lemma 3.5. In turn, (iv) follows easily from (ii), (v) is a consequence of (iv), and (vi) follows from (iii).

The equivalence between (a) and (b) in (vii) is a consequence of Lemma 3.3 (i). Let $(x_i)_{i\in I}$ be a net in $\mathcal{M} \setminus \{0\}$. We infer from Lemma 4.1 (ii) that $\lim_i \mu(x_i) = 0$ whenever $\lim_i x_i = 0$ and $\alpha(0) > 0$, from Lemma 4.1 (i) that $(\mu(x_i))_{n=1}^\infty$ does not converge to 0 whenever $(x_i)_{i\in I}$ does not converge to 0, and from Lemma 4.1 (iii) that $(\mu(x_i))_{i\in I}$ does not converge to 0 whenever $\alpha(0) = 0$. Thus, (a) and (c) in (vii) are equivalent.

We obtain (viii) as a consequence of (vii), and (ix) follows from (iv) and (vii). We now prove (x). Let $(x_n)_{n=1}^\infty$ be a sequence in $\mathcal{M} \setminus \{0\}$. Suppose that it is Cauchy and does not converge to 0. Then, $\inf_n d(0, x_n) > 0$. Applying Lemma 3.5 gives that $(\mu(x_n))_{n=1}^\infty$ is a Cauchy sequence. Suppose that $(\mu(x_n))_{n=1}^\infty$ is a Cauchy sequence and assume by contradiction that $(x_n)_{n=1}^\infty$ is not Cauchy. Then, using Lemma 3.5,

$$\lim_{k\to\infty} \sup_{m,n\geq k} d_\alpha(x_n, x_m) = 0 < \lim_{k\to\infty} \sup_{m,n\geq k} d(x_n, x_m),$$

which easily implies $\sup_n \alpha(d(0, x_n)) = \infty$. Passing to a subsequence we can assume that $(\alpha(d(0, x_n)))_{n=1}^\infty$ increases to ∞. Then, if $m \leq n$,

$$d_\alpha(x_n, x_m) = \frac{d(x_n, x_m)}{\alpha(d(0, x_n))} \geq \frac{d(0, x_n) - d(0, x_m)}{\alpha(d(0, x_n))}.$$

Thus, by Lemma 3.7, $\varepsilon_m := \liminf_n d_\alpha(x_n, x_m) > 0$ for every $m \in \mathbb{N}$ and so applying Lemma 3.5 we obtain

$$\liminf_n \|\mu(x_n) - \mu(x_m)\| \geq \varepsilon_m, \quad m \in \mathbb{N}.$$

This absurdity concludes the proof of (x).

Clearly, (xi) follows from (x) and (vi). By (vii), in order to prove (xii) it suffices to show that the norm and weak topologies coincide on $\mathcal{B} \setminus \{0\}$. To that end, taking into account Lemma 3.5, it suffices to show that for every $y \in \mathcal{M} \setminus \{0\}$ and every $\varepsilon > 0$ there is a finite family $(f_i)_{i=1}^n$ in $\operatorname{Lip}_0(\mathcal{M})$ and $\delta > 0$ such that $d_\alpha(x, y) \leq \varepsilon$ whenever $x \in \mathcal{M} \setminus \{0\}$ satisfies $|\langle \mu(x) - \mu(y), f_i\rangle| \leq \delta$ for all $1 \leq i \leq n$. But this fact is immediate from Lemma 3.4 (ii). Indeed, for $y \in \mathcal{M} \setminus \{0\}$ and $\varepsilon > 0$ we pick functions $f, g \in \operatorname{Lip}_0(\mathcal{M})$ from Lemma 3.4 (ii) and put $\delta := \varepsilon$, so if $|\langle \mu(x) - \mu(y), \psi\rangle| \leq \delta$ for $\psi \in \{f, g\}$ we obtain that $d_\alpha(x, y) \leq \delta = \varepsilon$. □

Roughly speaking, the following corollary of Proposition 4.2 (iii) shows that applying our construction to an already bounded metric space barely alters the original space.

Corollary 4.3. *Let α be a Lipschitz map with $\mathrm{D}(\alpha) < \infty$ and let $(\mathcal{M}, d, 0)$ be a pointed metric space. Suppose that $\mathcal{M}$ is bounded and that either 0 is an isolated point of $\mathcal{M}$ or $\alpha(0) > 0$. Then $\mathcal{M} \simeq_{\mathrm{Lip}} \mathcal{B}(\mathcal{M}, \alpha)$.*

It is obvious that if two metric spaces $\mathcal{M}$ and $\mathcal{N}$ are bi-Lipschitz equivalent, then the corresponding Lipschitz free p-spaces $\mathcal{F}_p(\mathcal{M})$ and $\mathcal{F}_p(\mathcal{N})$ are linearly isomorphic for all $0 < p \le 1$. It is well-known that the converse does not hold, i.e., a linear isomorphism between $\mathcal{F}_p(\mathcal{M})$ and $\mathcal{F}_p(\mathcal{N})$ does not imply the bi-Lipschitz equivalence between $\mathcal{M}$ and $\mathcal{N}$. Theorem 3.9 provides examples of this situation. With the tools obtained so far, we next show how the bi-Lipschitz equivalence between $\mathcal{B}(\mathcal{M}, \alpha)$ and $\mathcal{B}(\mathcal{N}, \alpha)$ stands in this comparison.

Theorem 4.4. *Let $\mathcal{M}$ and $\mathcal{N}$ be pointed metric spaces, let $\alpha\colon (0,\infty) \to (0,\infty)$ be a Lipschitz function with $D(\alpha) < \infty$, and let $0 < p \le 1$. Then the following implications hold:*

$$\mathcal{M} \simeq_{\mathrm{Lip}} \mathcal{N} \Rightarrow \mathcal{B}(\mathcal{M}, \alpha) \simeq_{\mathrm{Lip}} \mathcal{B}(\mathcal{N}, \alpha) \Rightarrow \mathcal{F}_p(\mathcal{M}) \simeq \mathcal{F}_p(\mathcal{N}).$$

Moreover, none of the converse implications is true even if the metric spaces $\mathcal{M}$ and $\mathcal{N}$ are complete.

Proof. The implications are a consequence of Proposition 3.12 and Theorem 3.9. Let us show that there exist metric spaces $\mathcal{M}$ and $\mathcal{N}$ with $\mathcal{F}(\mathcal{M}) \simeq \mathcal{F}(\mathcal{N})$ but such that $\mathcal{B}(\mathcal{M}, \alpha)$ and $\mathcal{B}(\mathcal{N}, \alpha)$ are not bi-Lipschitz equivalent. Pick any compact metric space K and a non-compact metric space X such that $\mathcal{F}_p(X) \simeq \mathcal{F}_p(K)$ (e.g., by [2], it suffices to put $K = [0, 1]$ and $X = [0,\infty)$). In particular, by attaching isolated points to both these spaces, there are a non-compact metric space $\mathcal{M}$ with isolated 0 and a compact metric space $\mathcal{N}$ with isolated 0 such that $\mathcal{F}_p(\mathcal{M}) \simeq \mathcal{F}_p(\mathcal{N})$. By Proposition 4.2 (vi), $\mathcal{B}(\mathcal{M}, \alpha)$ and $\mathcal{M}$ are homeomorphic. The same is true for $\mathcal{B}(\mathcal{N}, \alpha)$ and $\mathcal{N}$. Therefore $\mathcal{B}(\mathcal{M}, \alpha)$ and $\mathcal{B}(\mathcal{N}, \alpha)$ cannot be bi-Lipschitz equivalent.

Finally, we find $\mathcal{M}$ and $\mathcal{N}$ such that $\mathcal{M}$ and $\mathcal{N}$ are not bi-Lipschitz equivalent, while $\mathcal{B}(\mathcal{M}, \alpha)$ and $\mathcal{B}(\mathcal{N}, \alpha)$ are. Let $\mathcal{M}$ be a countable bounded metric space. Suppose that $\mathcal{M}$ is uniformly separated, i.e.,

$$\inf\{d(x, y)\colon x, y \in \mathcal{M}, x \ne y\} > 0.$$

By Proposition 4.2 (ii), $\mathcal{B}(\mathcal{M}, \alpha) \simeq_{\mathrm{Lip}} \mathcal{M}$. Therefore $\mathcal{B}(\mathcal{M}, \alpha)$ also is countable, bounded and uniformly separated. Consider $\mathcal{N} = \mathbb{N} \cup \{0\}$ endowed with the metric

$$d(m, n) = |2^m - 2^n|, \quad m, n \in \mathbb{N} \cup \{0\}.$$

For m, $n \in \mathbb{N} \cup \{0\}$ with $m \neq n$ we have

$$d(m,n) \approx 2^{\max\{m,n\}} \approx \max\{\alpha(d(n,0)), \alpha(d(m,0))\}.$$

Thus, an application of Lemma 3.5 shows that $\mathcal{B}(\mathcal{N},\alpha)$ is countable, bounded and uniformly separated. Therefore, $\mathcal{B}(\mathcal{M},\alpha)$ and $\mathcal{B}(\mathcal{N},\alpha)$ are bi-Lipschitz equivalent. On the other hand, since $\mathcal{M}$ is bounded and $\mathcal{N}$ is not, these spaces are not bi-Lipschitz equivalent. □

In Section 6, we shall see how the Lipschitz equivalence between $\mathcal{B}(\mathcal{M},\alpha)$ and $\mathcal{B}(\mathcal{N},\alpha)$ is tightly related to the algebraic isomorphism between the algebras $\mathrm{Lip}_0(\mathcal{M})$ and $\mathrm{Lip}_0(\mathcal{N})$ (see Theorem 6.14).

5. $\mathrm{Lip}_0(\mathcal{M})$ as a Banach algebra for $\mathcal{M}$ unbounded metric space

In this section we examine the (real or complex) Banach space $\mathrm{Lip}_0(\mathcal{M})$ as an algebra over an arbitrary metric space $\mathcal{M}$. Recall that $\mathrm{Lip}_0(\mathcal{M})$ is the dual space of $\mathcal{F}(\mathcal{M})$ (see Remark 2.3). In the case when the pointed metric space $(\mathcal{M}, d, 0)$ is bounded, $\mathrm{Lip}_0(\mathcal{M})$ is a Banach algebra in the weak sense, i.e., there is a constant $C > 1$ such that

$$\mathrm{Lip}(fg) \le C\mathrm{Lip}(f)\mathrm{Lip}(g), \quad f, g \in \mathrm{Lip}_0(\mathcal{M}).$$

In fact, we can choose $C = 2\max_{x\in\mathcal{M}} d(0,x)$. These are called *Gelfand algebras* in [25]. Note that the Banach algebra law requires that the above estimate holds with $C = 1$. As it is discussed in [25, Chapter 7], when $\mathcal{M}$ is bounded it is possible to define a submultiplicative equivalent norm on $\mathrm{Lip}_0(\mathcal{M})$ so that $\mathrm{Lip}_0(\mathcal{M})$ becomes a Banach algebra. As Weaver explains, the drawbacks of changing the natural norm are the loss of the lattice structure of the unital ball of $\mathrm{Lip}_0(\mathcal{M})$ as well as the breakdown of several isometric results (see [25, Chapter 7] for a discussion on this topic).

Here we keep the standard Lipschitz norm on $\mathrm{Lip}_0(\mathcal{M})$. However, we redefine the product by a simple natural formula so that the resulting algebra is a Banach algebra for any (even unbounded) metric space $\mathcal{M}$. In doing so, some of the results of isometric nature are preserved.

In the case when $\mathcal{M}$ is unbounbed we shall use Theorem 3.9 as a vehicle to define a multiplication on $\mathrm{Lip}_0(\mathcal{M})$. Given a Lipschitz map $\alpha\colon (0,\infty) \to (0,\infty)$ with $\mathrm{D}(\alpha) < \infty$, let $Q_\alpha^F\colon \mathcal{F}(\mathcal{B}(\mathcal{M})) \to \mathcal{F}(\mathcal{M})$ and $P_\alpha^F\colon \mathcal{F}(\mathcal{M}) \to \mathcal{F}(\mathcal{B}(\mathcal{M}))$ be as in Theorem 3.9. We know that Q_α^F and P_α^F are linear isomorphisms inverse of each other. Let $Q_\alpha^L\colon \mathrm{Lip}_0(\mathcal{M}) \to \mathrm{Lip}_0(\mathcal{B}(\mathcal{M}))$ and $P_\alpha^L\colon \mathrm{Lip}_0(\mathcal{B}(\mathcal{M})) \to \mathrm{Lip}_0(\mathcal{M})$ be their respective dual maps via the canonical isometries between Lipschitz spaces and the duals of Lipschitz-free spaces. Since pointwise multiplication is a well-defined operation on $\mathrm{Lip}_0(\mathcal{B}(\mathcal{M}))$, the operation

$$P^L_\alpha(Q^L_\alpha(f)Q^L_\alpha(g)), \quad f,g \in \mathrm{Lip}_0(\mathcal{M}).$$

is well-defined on $\mathrm{Lip}_0(\mathcal{M})$. Nevertheless, let us define a multiplication by a more transparent formula and show that it is equivalent to what is displayed above.

Definition 5.1. For $f,g \in \mathrm{Lip}_0(\mathcal{M})$ and $x \in \mathcal{M}$ we define a new operation $\odot_\alpha$ on $\mathrm{Lip}_0(\mathcal{M})$ as $f \odot_\alpha g(0) = 0$ and

$$f \odot_\alpha g(x) := \frac{f(x)g(x)}{\zeta(x)}$$

for $x \in \mathcal{M} \setminus \{0\}$, where ζ is as in (3.1)

A routine computation yields

$$P^L_\alpha(h)(x) = \zeta(x)\, h(\mu(x)), \quad h \in \mathrm{Lip}_0(\mathcal{B}(\mathcal{M})), \tag{5.6}$$

and

$$Q^L_\alpha(f)(\mu(x)) = \frac{f(x)}{\zeta(x)}, \quad f \in \mathrm{Lip}_0(\mathcal{M}). \tag{5.7}$$

As the alert reader might have noticed, expression (5.7) connects the operator Q^L_α with the work of Weaver (see [25, Theorem 2.20]).

Thus, for $x \in \mathcal{M}$,

$$f \odot_\alpha g(x) = P^L_\alpha(Q^L_\alpha(f)Q^L_\alpha(g))(x), \tag{5.8}$$

which immediately gives that $f \odot_\alpha g \in \mathrm{Lip}_0(\mathcal{M})$. Moreover, using the estimates for $\|P^L_\alpha\|$ and $\|Q^L_\alpha\|$ from Theorem 3.9, and the fact that for Lipschitz functions on a bounded metric space $\mathcal{N}$ we have the estimate

$$\mathrm{Lip}(f'g') \leq 2\,\mathrm{rad}(\mathcal{N})\mathrm{Lip}(f')\mathrm{Lip}(g'),$$

where $\mathrm{rad}(\mathcal{N}) := \sup\{d(0,x) \colon x \in \mathcal{N}\}$, we get

$$\mathrm{Lip}(f \odot_\alpha g) \leq 2\,\mathrm{D}(\alpha)(1 + 2\,\mathrm{K}(\alpha))\mathrm{Lip}(f)\mathrm{Lip}(g).$$

The following result is aimed at improving the constant $2\,\mathrm{D}(\alpha)(1 + 2\,\mathrm{K}(\alpha))$ in the last inequality.

Lemma 5.2. *Let $(\mathcal{M}, d, 0)$ be an unbounded pointed metric space and let $\alpha \colon (0,\infty) \to (0,\infty)$ be a Lipschitz map with $\mathrm{D}(\alpha) < \infty$. Then*

$$\mathrm{Lip}(f \odot_\alpha g) \leq \mathrm{D}(\alpha)(\mathrm{K}(\alpha) + 2)\mathrm{Lip}(f)\mathrm{Lip}(g)$$

for all $f, g \in \mathrm{Lip}_0(\mathcal{M})$.

Proof. Let x, $y \in \mathcal{M}$. If $0 \notin \{x,y\}$, using Lemma 3.3, we estimate $E := |f \odot_\alpha g(x) - f \odot_\alpha g(y)|$ by

$$\left|\frac{(f(x)-f(y))g(x)}{\zeta(x)}\right| + \left|\frac{(g(x)-g(y))f(y)}{\zeta(y)}\right| + \left|\frac{f(y)g(x)(\zeta(y)-\zeta(x))}{\zeta(x)\zeta(y)}\right|$$
$$\le \mathrm{Lip}(f)\mathrm{Lip}(g)d(x,y)\left(\frac{d(0,x)}{\zeta(x)} + \frac{d(0,y)}{\zeta(y)} + \mathrm{Lip}(\alpha)\frac{d(0,x)d(0,y)}{\zeta(x)\zeta(y)}\right)$$
$$\le (2\,\mathrm{D}(\alpha) + \mathrm{D}^2(\alpha)\mathrm{Lip}(\alpha))\mathrm{Lip}(f)\mathrm{Lip}(g)d(x,y).$$

If $y = 0$ and $x \neq 0$,

$$E \le \frac{\mathrm{Lip}(f)\mathrm{Lip}(g)d(0,x)d(0,x)}{\zeta(x)} \le \mathrm{D}(\alpha)\mathrm{Lip}(f)\mathrm{Lip}(g)d(y,x). \qquad \square$$

Notice that there are choices of maps α such that $D(\alpha)(K(\alpha)+2) \le 1$ for which the constant obtained in Lemma 5.2 is optimal. Indeed, the function $\alpha\colon (0,\infty) \to (0,\infty)$ defined by $\alpha(t) = 3t$ satisfies

$$D(\alpha)(K(\alpha)+2) = 1. \tag{5.9}$$

Morover, if we pick $\mathcal{M} = \mathbb{R}^+$, $f(t) = 1 - |t-1|$ and $g(t) = -f(t)$, we have $\mathrm{Lip}(f) = \mathrm{Lip}(g) = 1$ and

$$\lim_{\varepsilon\to 0^+} \frac{|(f \odot_\alpha g)(1) - (f \odot_\alpha g)(1+\varepsilon)|}{\varepsilon} = (f \odot_\alpha g)'_+(1) = 1,$$

so that $\mathrm{Lip}(f \odot_\alpha g) \ge D(\alpha)(K(\alpha)+2)\mathrm{Lip}(f)\mathrm{Lip}(g)$.

Our next result is a ready consequence of our construction. Recall that given a metric space $\mathcal{M}$ we have a natural ordering on $\mathrm{Lip}_0(\mathcal{M})$, namely, $f \ge 0$ if $f(x) \ge 0$ for every $x \in \mathcal{M}$. Then $\mathrm{Lip}_0(\mathcal{M})$ is an ordered Banach algebra, i.e., if we put

$$\mathrm{Lip}_0^+(\mathcal{M}) = \{f \in \mathrm{Lip}_0(\mathcal{M})\colon f \ge 0\},$$

the sets

$$\mathbb{R}^+ \cdot \mathrm{Lip}_0^+(\mathcal{M}), \quad \mathrm{Lip}_0^+(\mathcal{M}) + \mathrm{Lip}_0^+(\mathcal{M}), \quad \mathrm{Lip}_0^+(\mathcal{M}) \cdot \mathrm{Lip}_0^+(\mathcal{M})$$

are contained in the positive cone $\mathrm{Lip}_0^+(\mathcal{M})$.

Let $\mathcal{N}$ be another metric space. A map $T\colon \mathrm{Lip}_0(\mathcal{M}) \to \mathrm{Lip}_0(\mathcal{N})$ is said to be *normal* if whenever $(f_i)_{i\in I}$ is a norm-bounded net in $\mathrm{Lip}_0(\mathcal{M})$ which increases pointwise towards $f \in \mathrm{Lip}_0(\mathcal{M})$, then the net $(T(f_i))_{i\in I}$ increases pointwise towards $T(f)$. Any normal operator is positive.

Proposition 5.3. *(cf.* [25, Theorem 6.23]*). Let* $(\mathcal{M},d)$ *be an unbounded metric space, and let* $\alpha\colon (0,\infty) \to (0,\infty)$ *be a Lipschitz function with* $\mathrm{D}(\alpha) < \infty$. *Then, the map* $Q_\alpha^L\colon \mathrm{Lip}_0(\mathcal{M}) \to \mathrm{Lip}_0(\mathcal{B})$ *defined as in* (5.7) *is a* w^*-w^* *continuous normal isomorphism with* $Q_\alpha^L(f \odot_\alpha$

$g) = Q^L_\alpha(f)Q^L_\alpha(g)$ for all f and $g \in \mathrm{Lip}_0(\mathcal{M})$. Its inverse operator P^L_α defined in (5.6) satisfies the same properties.

Let us now investigate the existence of a unit for the Banach algebra $\mathrm{Lip}_0(\mathcal{M})$.

Lemma 5.4. *Let $(\mathcal{M}, d)$ be an unbounded metric space, and let α be a Lipschitz function with $\mathrm{D}(\alpha) < \infty$. Consider $\mathrm{Lip}_0(\mathcal{M})$ endowed with the multiplication $\odot_\alpha$. Let ζ be defined as in (3.1). The following are equivalent:*

(i) $\mathrm{Lip}_0(\mathcal{M})$ has a unit, in which case such a unit is ζ.
(ii) ζ is a Lipschitz function.
(iii) ζ is continuous at zero.
(iv) Either 0 is an isolated point or $\alpha(0) = 0$.

Proof. Suppose that $e \in \mathrm{Lip}_0(\mathcal{M})$ is a unit. Then $e(x) = \zeta(x)$ for all $x \in \mathcal{N} := \cup_{f\in \mathrm{Lip}_0(f)}(\mathcal{M} \setminus f^{-1}(0))$. Since $\mathcal{N} = \mathcal{M} \setminus \{0\}$, $e = \zeta$ and so ζ is Lipschitz. This proves (i) $\Rightarrow$ (ii), and the converse is clear. The equivalence between (ii) and (iii) follows from Lemma 3.3 (ii), and the equivalence between (iii) and (iv) follows from Lemma 3.7. □

If we choose $\alpha\colon (0,\infty) \to (0,\infty)$ satisfying $\mathrm{D}(\alpha)(\mathrm{K}(\alpha) + 2) \le 1$ (see (5.9)), then, in light of Lemma 5.2, the space $\mathrm{Lip}_0(\mathcal{M})$ becomes a Banach algebra with the multiplication $\odot_\alpha$. However, there is no choice of α for which $\mathrm{Lip}_0(\mathcal{M})$ becomes a unital Banach algebra (that is, a Banach algebra with a norm-one unit). Indeed, if $\mathrm{Lip}_0(\mathcal{M})$ had a unit e with $\mathrm{Lip}(e) = 1$, by Lemma 5.4 it would have to be $e = \zeta$. Then we would have $\alpha(t) \le t$ for all $t \in \{(d(0,x)\colon x \in \mathcal{M}\}$. Therefore $\mathrm{D}(\alpha) \ge 1$ and so $\mathrm{D}(\alpha)(\mathrm{K}(\alpha) + 2) \ge 2$.

Remark 5.5. Given a Banach algebra X there is an explicit formula for an equivalent norm $|\cdot|$ on X such that $(X, |\cdot|)$ becomes a unital Banach algebra (see [15, Proposition 2.1.9]). However, as we mentioned above, this renorming leads to the loss of the lattice structure of the unit ball of $\mathrm{Lip}_0(\mathcal{M})$.

Remark 5.6. Using Lemma 2.4, the multiplication $\odot_\alpha$ on $\mathrm{Lip}_0(\mathcal{M})$ is w^*-w^* separately continuous, or more precisely it is w^*-w^* separately continuous on bounded sets and then we use the Banach-Dieudonné theorem (see, e.g., [5, Appendix G.8]). So in the terminology of [22, Chapter 5], $(\mathrm{Lip}_0(\mathcal{M}), \mathcal{F}(\mathcal{M}))$ is a dual Banach algebra. Moreover, in the complex scalar case it is easy to see that the involution on $\mathrm{Lip}_0(\mathcal{M})$ defined by $f \mapsto \overline{f}$ shows that $\mathrm{Lip}_0(\mathcal{M})$ is also a Banach $*$-algebra (see [15, Definition 3.1.1] for the corresponding definition).

6. From Lipschitz algebras over unbounded metric spaces to Lipschitz algebras over bounded metric spaces

In this section we deal only with Banach spaces over the real field. In [25, Chapter 7] there are many results about the algebra $\mathrm{Lip}_0(\mathcal{B})$ for bounded metric spaces $\mathcal{B}$, which now easily transfer to the ordered Banach algebra $\mathrm{Lip}_0(\mathcal{M})$ over an unbounded metric space $\mathcal{M}$. Note however that in [25] the case of complex Banach spaces is not dealt with, which is the reason why we deal here with real Banach spaces only. The problems that arise in the complex case are mentioned below in Remark 6.16, which shows that further research is needed to generalize the contents of this section to complex scalars.

Troughout this section $(\mathcal{M}, d, 0)$ will be a pointed metric space, $\alpha\colon (0,\infty) \to (0,\infty)$ will be a map with $\mathrm{D}(\alpha)(\mathrm{K}(\alpha)+2) \le 1$, and μ and ζ will be the functions defined in (3.1) and (3.2) respectively. We will not a priori assume that $\mathrm{Lip}_0(\mathcal{M})$ has a natural unit, i.e., we do not impose ζ to be continuous at 0.

Definition 6.1. Let $(\mathcal{N}, d, 0)$ be a pointed metric space and let $Y \subset \mathrm{Lip}_0(\mathcal{N})$ be a subalgebra.

- (i) We say that Y is *order complete*, if it is stable under pointwise convergence of norm-bounded increasing nets.
- (ii) We say that Y is a *linear complete sublattice* if it is closed under taking the supremum of an arbitrary, possibly infinite, set of functions that are uniformly bounded in the Lipschitz norm.

Lemma 6.2. *Suppose that* $\mathrm{Lip}_0(\mathcal{M})$ *has a natural unit. Let* f, $g \in \mathrm{Lip}_0(\mathcal{M})$, *and* $c \in [0,\infty)$. *Then, if* $\mathcal{B} = \mathcal{B}(\mathcal{M},\alpha)$ *and* Q_α^L *is as in* (5.7), *we have* $Q_\alpha^L(f) \vee (Q_\alpha^L(g) - c\,\mathbf{1}_\mathcal{B}) \in \mathrm{Lip}_0(\mathcal{B})$ *and, if* P_α^L *is as in* (5.6),

$$P_\alpha^L(Q_\alpha^L(f) \vee (Q_\alpha^L(g) - c\,\mathbf{1}_\mathcal{B})) = f \vee (g - c\,\zeta).$$

Proof. It is a routine check. □

Theorem 6.3. *A subalgebra* $Y \subset \mathrm{Lip}_0(\mathcal{M})$ *is order complete if and only if it is* w^**-closed. Moreover, if* Y *is order complete then:*

- *(i)* Y *is linear complete sublattice.*
- *(ii) If* $\mathrm{Lip}_0(\mathcal{M})$ *has a natural unit, we have* $f \vee (g - c\,\zeta) \in Y$ *for all* f, $g \in Y$ *and all* $c \ge 0$.

Proof. (i) follows from combining [25, Lemma 7.6, Corollary 7.7 and Lemma 7.9] with Lemma 2.4 and Proposition 5.3, which asserts that Q_α^L is a w^*-w^*-homeomorphism that preserves multiplication and ordering. To see (ii), we need to take into account also Lemma 6.2. □

Definition 6.4. A subalgebra Y of a Banach algebra X is said to be an *ideal* if xy, $yx \in I$ for every $x \in X$ and $y \in Y$, and it is said to be a *complete band* if it is a linear complete sublattice, and for every y_1, $y_2 \in Y$ and $x \in X$ with $y_1 \le x \le y_2$ we have $x \in Y$.

Definition 6.5. Given a metric space $\mathcal{N}$ and a subspace Y of $\operatorname{Lip}_0(\mathcal{N})$, define the *hull* of Y as the closed set

$$\mathcal{H}_{\mathcal{N}}(Y) := \{x \in \mathcal{N} \colon f(x) = 0 \text{ for all } f \in Y\}.$$

Given a closed set $K \subset \mathcal{N}$, we put

$$\mathcal{I}_{\mathcal{N}}(K) := \{f \in \operatorname{Lip}_0(\mathcal{N}) \colon f(x) = 0 \text{ for all } x \in K\}.$$

The indices in $\mathcal{I}_{\mathcal{N}}$, resp. $\mathcal{H}_{\mathcal{N}}$, will be omitted if the metric space $\mathcal{N}$ is clear from the context.

Note that $\mathcal{H}(\mathcal{I}(K)) = K$ for every closed set $K \subset \mathcal{N}$ as shown by the Lipschitz map $x \mapsto d(x, K)$.

Lemma 6.6. *Let* $P^L_\alpha \colon \operatorname{Lip}_0(\mathcal{B}) \to \operatorname{Lip}_0(\mathcal{M})$ *be as in* (5.6)*, where* $\mathcal{B} = \mathcal{B}(\mathcal{M}, \alpha)$*. Then,*

$$P^L_\alpha(\mathcal{I}_{\mathcal{B}}(\mathcal{H}_{\mathcal{B}}(Y))) = \mathcal{I}_{\mathcal{M}}(\mathcal{H}_{\mathcal{M}}(P^L_\alpha(Y)))$$

for any subspace Y *of* $\operatorname{Lip}_0(\mathcal{B})$.

Proof. It is clear from Proposition 5.3 and the definition of P^L_α. □

Theorem 6.7. *Let* $Y \subset \operatorname{Lip}_0(\mathcal{M})$ *be an ideal. Then the following conditions are equivalent.*

(i) Y *is order complete.*
(ii) Y *is* w^**-closed.*
(iii) Y *is a complete band.*
(iv) $Y = \mathcal{I}(\mathcal{H}(Y))$.
(v) There exists a closed set $K \subset \mathcal{M}$ *such that* $Y = \mathcal{I}(K)$.

Proof. The equivalence between (i) and (ii) follows from Theorem 6.3. (ii)⇒(iii) follows from [25, Lemma 7.16], where it is proved for bounded metric spaces, and from Proposition 5.3. The implication (iii)⇒(iv) follows from [25, Proof of Theorem 6.19] and Lemma 6.6. Finally, (iv)⇒ (v) is obvious, and (v)⇒ (i) is immediate from the definition. □

Theorem 6.8. *If* $Y \subset \operatorname{Lip}_0(\mathcal{M})$ *is an ideal, then* $\overline{Y}^{w^*} = \mathcal{I}(\mathcal{H}(Y))$.

Proof. The result follows by combining the corresponding result for bounded metric spaces proved in [8, Proposition 3.2] with Proposition 5.3 and Lemma 6.6. □

Let $\mathcal{N}$ be another metric space. A map $T\colon \mathrm{Lip}_0(\mathcal{M}) \to \mathrm{Lip}_0(\mathcal{N})$ is an *algebra homomorphism with respect to* α if it is linear and preserves the multiplication $\odot_\alpha$. If α is known, we simply write *algebra homomorphism*

Proposition 6.9. *If $T\colon \mathrm{Lip}_0(\mathcal{M}) \to \mathrm{Lip}_0(\mathcal{N})$ is an algebra homomorphism then it is bounded (i.e., continuous) and preserves ordering.*

Proof. It follows from [25, Proposition 7.29], where it is proved for bounded metric spaces, and Proposition 5.3. □

By $\Delta(\mathcal{M})$ we denote the set of all nonzero algebra homomorphisms from $\mathrm{Lip}_0(\mathcal{M})$ into $\mathbb{R}$, where $\mathbb{R}$ is of course equipped with its standard addition and multiplication. By $\Delta_0(\mathcal{M})$ we denote the set $\Delta(\mathcal{M})\cup\{0\}$, where 0 is the zero map. By $\Delta^*(\mathcal{M})$ and $\Delta_0^*(\mathcal{M})$ we denote the set of all members of $\Delta(\mathcal{M})$ and $\Delta_0(\mathcal{M})$ respectively that are w^*-continuous.

A straightforward computation gives that, if we regard $\mathcal{F}(\mathcal{M})$ as a subspace of $(\mathrm{Lip}_0(\mathcal{M}))^*$, then $\mu(x) \in \Delta(\mathcal{M})$ for all $x \in \mathcal{M} \setminus \{0\}$. The following lemma identifies the set $\Delta_0^*(\mathcal{M})$.

Lemma 6.10. *Let μ be as in* (3.2)*. For $\omega \in \Delta(\mathcal{M})$ the following are equivalent:*

(i) ω is normal.
(ii) ω is w^*-continuous.
(iii) $\omega = \mu(x)$ for some $x \in \mathcal{M} \setminus \{0\}$.

In particular, we have $\Delta_0^*(\mathcal{M}) = \mathcal{B}(\mathcal{M}, \alpha)$.

Proof. Just combine [25, Lemma 7.22] with Proposition 5.3. □

The following is an analogue to [25, Theorem 7.23].

Theorem 6.11. *Let $\mathcal{N}$ be another metric space, and assume that the multiplication $\odot_\alpha$ admits a natural unit in both $\mathcal{M}$ and in $\mathcal{N}$. Let $T\colon \mathrm{Lip}_0(\mathcal{M}) \to \mathbb{R}^{\mathcal{N}}$ be a mapping. The following are equivalent:*

(i) $T\colon \mathrm{Lip}_0(\mathcal{M}) \to \mathrm{Lip}_0(\mathcal{N})$ is a unital normal algebra homomorphism.
(ii) There exists $g\colon \mathcal{N} \setminus \{0\} \to \mathcal{M} \setminus \{0\}$ such that $T = D_g$ and $D_g(\mathrm{Lip}_0(\mathcal{M})) \subset \mathrm{Lip}_0(\mathcal{N})$, where $D_g\colon \mathrm{Lip}_0(\mathcal{M}) \to \mathbb{R}^{\mathcal{N}}$ is the map given by

$$D_g(f)(x) = \frac{\alpha(d(0,x))}{\alpha(d(0,g(x)))} f(g(x)) = \frac{\zeta(x)}{\zeta(g(x))} f(g(x)), \quad x \in \mathcal{N} \setminus \{0\}.$$

Proof. The implication (ii)⇒(i) is straightforward. Let us assume that (i) holds. Let ζ' and μ' be the maps defined in (3.1) corresponding to the metric space $\mathcal{N}$. Fix $x \in \mathcal{N} \setminus \{0\}$. Since $\mu'(x) \in \Delta(\mathcal{N})$ and

$T(\zeta) = \zeta'$ (because T is unital and ζ, ζ' are the units), we obtain $\mu'(x) \circ T \neq 0$ and so $\mu'(x) \circ T \in \Delta(\mathcal{M})$. Since $\mu'(x)$ is normal, $\mu'(x) \circ T$ is normal. Applying Lemma 6.10 yields $g(x) \in \mathcal{M} \setminus \{0\}$ such that $\mu'(x) \circ T = \mu(g(x))$, i.e.,

$$\delta_{\mathcal{N}}(x) \circ T = \frac{\zeta'(x)}{\zeta(g(x))} \delta_{\mathcal{M}}(g(x)). \qquad \square$$

The following is an analogue of [25, Theorem 7.26]. It seems this is actually interesting even outside of the framework of Banach algebras.

Theorem 6.12. *The norm and weak* topologies of* $(\mathrm{Lip}_0(\mathcal{M}))^*$ *coincide on* $\Delta_0^*(\mathcal{M})$. *Moreover, if* $\mathrm{Lip}_0(\mathcal{M})$ *has a natural unit, then* 0 *is an isolated point of* $\Delta_0^*(\mathcal{M})$.

Proof. The first part follows from combining Proposition 4.2 (xii) with Lemma 6.10. Combining Lemma 5.4 with Proposition 4.2 (viii) yields the 'moreover' part. $\square$

Theorem 6.13. *Suppose* $\mathrm{Lip}_0(\mathcal{M})$ *has a natural unit. Then the spectrum (considered to be either* $\Delta_0(\mathcal{M})$ *or* $\Delta(\mathcal{M})$*) of the Banach algebra* $\mathrm{Lip}_0(\mathcal{M})$ *verifies*

$$\Delta_0(\mathcal{M}) = \overline{\mathcal{B}(\mathcal{M}, \alpha)}^{w^*} \subset \mathrm{Lip}_0(\mathcal{M})^{**},$$

and

$$\Delta(\mathcal{M}) = \overline{\mathcal{B}(\mathcal{M}, \alpha) \setminus \{0\}}^{w^*} \subset \mathrm{Lip}_0(\mathcal{M})^{**}.$$

Proof. By Lemma 6.10 and Theorem 6.12, $\mathcal{B} := \mathcal{B}(\mathcal{M}, \alpha) = \Delta_0^*(\mathcal{M})$, and 0 is an isolated point of $\Delta_0^*(\mathcal{M})$.

By [25, Lemma 7.28] the set $\Delta_0^*(\mathcal{B})$ is dense in the w^*-closed set $\Delta_0(\mathcal{B}) \subset \mathrm{Lip}_0(\mathcal{B})^{**}$. Let Q_α^L be as in (5.7). By Proposition 5.3,

$$(Q_\alpha^L)^{**}(\Delta_0^*(\mathcal{B})) = \Delta_0^*(\mathcal{M}) \text{ and } (Q_\alpha^L)^{**}(\Delta_0(\mathcal{B})) = \Delta_0(\mathcal{M}).$$

Therefore, $\overline{\mathcal{B}}^{w^*} = \Delta_0(\mathcal{M})$. Moreover, since $\{0\}$ is a closed and open set, we have $0 \notin \overline{\mathcal{B} \setminus \{0\}}^{w^*}$ and $\overline{\mathcal{B} \setminus \{0\}}^{w^*} \cup \{0\} = \overline{\mathcal{B}}^{w^*}$. Thus,

$$\Delta(\mathcal{M}) = \overline{\mathcal{B}}^{w^*} \setminus \{0\} = \overline{\mathcal{B} \setminus \{0\}}^{w^*}. \qquad \square$$

The following result is an interesting analogue of [25, Corollary 7.27]. In particular it shows that the linear topological structure of Lipschitz-free spaces is completely determined by the purely algebraic structure of their duals.

Theorem 6.14. *Let* $\mathcal{N}$ *be another metric space. Then there exists an algebra isomorphism between* $\mathrm{Lip}_0(\mathcal{M})$ *and* $\mathrm{Lip}_0(\mathcal{N})$ *if and only if* $\mathcal{B}(\mathcal{M}, \alpha) \simeq_{\mathrm{Lip}} \mathcal{B}(\mathcal{N}, \alpha)$.

Proof. Suppose first that $\mathcal{B}(\mathcal{M},\alpha) \simeq_{\mathrm{Lip}} \mathcal{B}(\mathcal{N},\alpha)$. Then it is clear that $\mathrm{Lip}_0(\mathcal{B}(\mathcal{M},\alpha))$ and $\mathrm{Lip}_0(\mathcal{B}(\mathcal{N},\alpha))$ are algebraically isomorphic with their standard pointwise product, which in turn implies that $\mathrm{Lip}_0(\mathcal{M})$ and $\mathrm{Lip}_0(\mathcal{N})$ are algebraically isomorphic with the product $\odot_\alpha$.

Conversely, suppose that there exists an algebra isomorphism

$$T\colon \mathrm{Lip}_0(\mathcal{M}) \to \mathrm{Lip}_0(\mathcal{N}).$$

By Proposition 6.9, T is an isomorphism between the Banach spaces $\mathrm{Lip}_0(\mathcal{M})$ and $\mathrm{Lip}_0(\mathcal{N})$. Since T preserves multiplication and its dual map T^* is w^*-w^* continuous, we claim that $T^*(\Delta_0^*(\mathcal{N})) = \Delta_0^*(\mathcal{M})$. Indeed, it suffices to check that for every $m \in \Delta_0^*(\mathcal{N})$ the map

$$T^*(m)\colon \mathrm{Lip}_0(\mathcal{M}) \to \mathbb{R}$$

preserves multiplication and is w^*-continuous, which is routine using the properties of T. By Theorem 6.12 we obtain that $\mathcal{B}(\mathcal{M},\alpha)$ and $\mathcal{B}(\mathcal{N},\alpha)$ are Lipschitz isomorphic. □

An immediate consequence of Theorems 6.14 and 3.9 is the following.

Corollary 6.15. *If there is an algebra isomorphism between* $\mathrm{Lip}_0(\mathcal{M})$ *and* $\mathrm{Lip}_0(\mathcal{N})$*, then* $\mathcal{F}(\mathcal{M}) \simeq \mathcal{F}(\mathcal{N})$*.*

Note also that a similar result holds as well for (nonlinear) lattice isomorphisms. This follows from [11, Theorem 1], where it is proved that two bounded metric spaces $\mathcal{M}$ and $\mathcal{N}$ are Lipschitz isomorphic if and only if there is a (nonlinear) lattice isomorphism betweeen $\mathrm{Lip}(\mathcal{M})$ and $\mathrm{Lip}(\mathcal{N})$.

Remark 6.16. In this section we dealt with real Banach algebras only. Let us discuss the problems that arise when we consider complex Banach algebras. To begin with, some of the results that rely on the notion of order-completenes are meaningless in this case, while others make sense. In general, our proofs show that it suffices to consider the case of bounded metric spaces $\mathcal{M}$ with pointwise multiplication on the space $\mathrm{Lip}_0(\mathcal{M})$ (over the complex field). The obstructions that appear in this case are as follows.

- To obtain the equivalence between (ii), (iv), and (v) in Theorem 6.7 one needs to show that whenever $Y \subset \mathrm{Lip}_0(\mathcal{M})$ is a w^*-closed ideal, then $\mathcal{I}(\mathcal{H}(Y)) \subset Y$. In order to prove that, we use [25, Theorem 6.19], where lattice structure on $\mathrm{Lip}_0(\mathcal{M})$ is used. Note that if the equivalence between (ii) and (iv) holds in Theorem 6.7, then Theorem 6.8 holds as well (the same proof would work for complex scalars as well).

- Proposition 6.9 cannot be transferred to complex Banach algebras. Indeed, in the proof of [25, Proposition 7.29] the author uses [25, Lemma 7.28], which relies on order completeness.
- The same could be said about Lemma 6.10, since the proof of [25, Lemma 7.22] also uses order completeness. Note that if Lemma 6.10 holds, then Theorem 6.11 and Theorem 6.12 hold as well (the same proof would work for complex scalars as well).
- In order to prove Theorem 6.13 in the complex scalar case we would need Lemma 6.10 and an analogue of [25, Lemma 7.28], where order completeness is used in the proof.

7. Simplifications of existing proofs using our construction

There are several results on Lipschitz spaces and on Lipschitz free-spaces that work for unbounded metric spaces but whose proofs are much easier for bounded ones. This section is devoted to exhibiting that our methods permit to circumvent the technicalities that one encounters when proving some of this results for unbounded spaces. Our choice of the known results below is rather arbitrary and non exhaustive. We believe there are many more applications of our techniques in this direction and encourage the reader to further exploit them. In this section we deal with real Banach spaces only (the complex-scalar case is not considered here).

7.1. **Normal functionals.** The main result of [7] consists of extending to any $\omega \in (\operatorname{Lip}_0(\mathcal{M}))^*$ the equivalence between (i) and (ii) in Lemma 6.10, that is, in proving that every normal functional $\phi \in \mathcal{F}(\mathcal{M})^{**}$ is w^*-continuous. The proof carried out by the authors of [7] is easy for bounded metric spaces but for unbounded metric spaces the authors need [7, Lemma 4] and [7, Lemma 5] (in particular a very deep analogue of "Kalton's decomposition") which reduces the result to investigating Lipschitz-free spaces over annuli.

Let $\alpha = \alpha^{(0)}$, so that 0 is an isolated point of $\mathcal{B}(\mathcal{M}, \alpha)$. Since by Proposition 5.3 the isomorphism Q_α^L defined as in (5.7) preserves normality and w^*-continuity, it suffices to prove the result for bounded metric spaces $\mathcal{B}$ with $d(0, x) = 1$ for all $x \in \mathcal{B} \setminus \{0\}$. Thus, our construction would allow the authors to consider only the much easier case of bounded metric spaces with an isolated point. In order for the reader to appreciate this reduction, we briefly sketch the most important simplifications that it facilitates.

Theorem 7.1. *Let $\mathcal{M}$ be a pointed metric space such that $d(x,0)=1$ for all $x\in\mathcal{M}\setminus\{0\}$. A functional $\phi\in\mathcal{F}(\mathcal{M})^{**}$ is normal if and only if it is w^*-continuous.*

Sketch of the simplifications of the proof from [7]. We refer the reader to [7, proof of Theorem 2]. Simplifications of the proof in this special case are the following:

- We circumvent [7, Lemmas 4 and 5]. Moreover, in this particular case we have $r=1=R$.
- Our assumptions yield $\|f\|_\infty\leq \mathrm{Lip}(f)$ for all $f\in\mathrm{Lip}_0(\mathcal{M})$ and $\mathrm{Lip}(\mathbf{1}_{\mathcal{M}\setminus\{0\}})=1$, so instead of the functions e and e' we would use simply the function $\mathbf{1}_{\mathcal{M}\setminus\{0\}}$ which would further simplify several computations. □

7.2. **The Intersection Theorem and supports.** For Lipschitz-free spaces there is a well-defined notion of *support* developed very recently in [9]. Let us recall that the *support* of $\gamma\in\mathcal{F}(\mathcal{M})$, denoted by $\mathrm{supp}(\gamma)$, is the smallest closed set K such that $\gamma\in\mathcal{F}(K\cup\{0\})$. Note that if L is the smallest closed set with $\gamma\in\mathcal{F}(L)$, then $\mathrm{supp}(\gamma)=L\setminus\{0\}$ in the case when 0 is an isolated point of L, and $\mathrm{supp}(\gamma)=L$ otherwise. The existence of the support is ensured by the following theorem.

Theorem 7.2 (cf. [8,9])**.** *Let $\{K_i\colon i\in I\}$ be a family of closed subsets of $\mathcal{M}$ with nonempty intersection. Then*

$$\cap_{i\in I}\mathcal{F}(K_i)=\mathcal{F}\left(\cap_{i\in I}K_i\right).$$

To properly understand Theorem 7.2, called the Intersection Theorem, we must take into account that for every subset $\mathcal{N}$ of a metric space $\mathcal{M}$ there is a canonical isometric embedding of $\mathcal{F}(\mathcal{N})$ into $\mathcal{F}(\mathcal{M})$. This crucial property does not transfer to the case $p<1$ (see Remark 3.14).

The Intersection Theorem was proved for bounded metric spaces in [8] and extended to its full generality more recently in [9]. Our construction shows that the theorem for bounded spaces immediately implies the general case. Moreover, the notion of a support is preserved by our construction (see Proposition 7.4). Recall that $\mu\colon\mathcal{M}\to\mathcal{B}$ is the map from (3.2) and $P_\alpha^F\colon\mathcal{F}(\mathcal{M})\to\mathcal{F}(\mathcal{B})$ is the isomorphism in Theorem 3.9. For further reference we write down a lemma that we will use a couple of times.

Lemma 7.3. *Let $\mathcal{M}$ be a metric space, and let $0\in\mathcal{N}\subset\mathcal{M}$. Then $P_\alpha^F(\mathcal{F}(\mathcal{N}))=\mathcal{F}(\mu(\mathcal{N}))$.*

Proposition 7.4. *If the Intersection Theorem holds for bounded metric spaces, then it holds for all metric spaces. Moreover, for any metric space $\mathcal{M}$ and $\gamma \in \mathcal{F}(\mathcal{M})$,*

$$P_\alpha^F(\operatorname{supp}(\gamma) \cup \{0\}) = \operatorname{supp}(P_\alpha^F(\gamma)) \cup \{0\}.$$

Proof. Let $\mathcal{M}$ be an arbitrary metric space and let $\{K_i : i \in I\}$ be a family of closed subsets of $\mathcal{M}$ with non-empty intersection. Without loss of generality we may assume that $0 \in \bigcap_{i\in I} K_i$. By Lemma 4.2 (v), each $\mu(K_i)$ is closed in $\mathcal{B}$, and so is the set $\mu(\cap_{i\in I} K_i) = \cap_{i\in I}\mu(K_i)$. By the Intersection Theorem for bounded spaces,

$$\cap_{i\in I}\mathcal{F}(\mu(K_i)) = \mathcal{F}(\cap_{i\in I}\mu(K_i)) = \mathcal{F}(\mu(\cap_{i\in I} K_i)).$$

Applying Lemma 7.3, and using that P_α^L is an isomorphism, puts an end to the proof. The "moreover" part is shown similarly. □

Remark 7.5. Our construction actually shows that the proof of the Intersection Theorem, which works for bounded metric spaces, also works for unbounded metric spaces: we need only Theorem 6.8 instead of [8, Proposition 3.2]; the remainder of the proof is exactly the same as in the proof presented in [8, proof of Theorem 3.3].

7.3. **Compact reduction.** Let W be a subset of a Lipschitz-free space $\mathcal{F}(\mathcal{M})$. The set W is said to be *tight* if for every $\varepsilon > 0$ there is $K \subset \mathcal{M}$ compact such that

$$W \subset \mathcal{F}(K) + \varepsilon B_{\mathcal{F}(\mathcal{M})}.$$

Aliaga et al. proved in [6, Theorem 2.3] that every weakly precompact subset of $\mathcal{F}(\mathcal{M})$ is tight. Let us explain how our construction helps to simplify the proof of this result. If $W \subset \mathcal{F}(\mathcal{M})$ is weakly precompact then $P_\alpha^F(W)$ is a weakly precompact subset of $\mathcal{F}(\mathcal{B}(\mathcal{M}))$. Combining [6, Proposition 3.3 and Theorem 3.2] (which are stated for bounded metric spaces!) gives that $P_\alpha^F(W)$ is tight. By Lemma 7.3 and Proposition 4.2 (ix), we infer that W is tight.

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