d-XC integrals: On the generalization of the expanded form of the Choquet integral by restricted dissimilarity functions and their applications

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***Abstract*—Restricted dissimilarity functions (RDFs) were introduced to overcome problems resulting from the adoption of the standard difference. Based on those RDFs, Bustince et al. introduced a generalization of the Choquet integral (CI), called d-Choquet integral, where the authors replaced standard differences with RDFs, providing interesting theoretical results. Motivated by such worthy properties, joint with the excellent performance in applications of other generalizations of the CI (using its expanded form, mainly), this paper introduces a generalization of the expanded form of the standard Choquet integral (X-CI) based on RDFs, which we named d-XC integrals. We present not only relevant theoretical results but also two examples of applications. We apply d-XC integrals in two problems in decision making, namely a supplier selection problem (which is a multi-criteria decision making problem) and a classification problem in signal processing, based on motor-imagery brain-computer interface (MI-BCI). We found that two d-XC integrals provided better results when compared to the original CI in the supplier selection problem. Besides that, one of the d-XC integrals performed better than any previous MI-BCI results obtained with this framework in the considered signal processing problem.**

***Index Terms*—Choquet integral, restricted dissimilarity functions, d-Choquet integral, d-XC integral, multi-criteria decision making, motor-imagery brain-computer interface**

I. INTRODUCTION

THE discrete Choquet integral [1] has been widely disseminated in the literature for its interesting property of taking into account the relationship among data when performing aggregation tasks [2], which is carried out by the fuzzy measure used in its definition. This popularity can be noticed in a large variety of applications of the Choquet integral and its generalizations and extensions [3], e.g., in classification [4], [5], [6], [7], multi-criteria (group) decision making [8], [9], [10], [11], [12], deep learning [13], [14], multi-source data fusion model [15], preference modeling [16], [17], multi-decision sorting [18], purchasing decision process [19], image processing [14], [20], risk analysis [21], ensemble-based solar irradiance forecasting [22], and multimodal brain-computer interface systems [23].

On the other hand, the introduction of generalizations and extensions of the Choquet integral has either improved the performance or adapted the concept to specific applications. For example, generalizations such as the C_T-integrals [4], CC-integrals [5], C_F-integrals [6], and $C_{F_1F_2}$-integrals [7], in which the product operator was replaced by aggregation or pre-aggregation functions [4], [24] have been applied to enhance the performance of fuzzy-rule based classification systems [7], multimodal brain-computer interface systems [23], [25], decision making problems [9], and image processing models [20]. Among those generalizations, the CC-integrals (based on copulas [26]) and the $C_{F_1F_2}$-integrals (based on pseudo pre-aggregation pairs) are built on the expanded form of the Choquet integral (X-CI).

Although such generalizations have allowed for considerable improvements in the performance of diverse applications, the difference operator (distance metric on the real line) in their formulation might be a drawback in their definitions, as properly discussed in [27]. In many knowledge-based applications (both algorithms and models), numerical data analysis and comparisons are usually performed by using the difference operator (as, for example, when measuring errors as stopping criteria of iterative methods). This may raise some issues when the difference is not properly defined in the application domain [27], or even by adding undesirable effects, like the width degradation/overestimation when dealing with interval-valued data [28], [29], [30]. The latter is a serious problem when the variables are correlated [31], since it provides interval results with too large widths and meaningless information [30], [32].

To solve this particular problem, Bustince et al. [33] introduced the restricted dissimilarity functions (RDFs), which generalizes the difference operation and open possibilities for performing data comparison in the unit interval in different ways. With this concept in hand, Bustince et al. [27] introduced the d-Choquet integrals, where the Choquet integral was generalized by restricted dissimilarity functions, replacing the difference operator in the standard Choquet integral by restricted dissimilarity functions, of which the difference is just a particular case. However, the authors have not provided

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an application of this generalization.

A. The theoretical objectives

Inspired by the generalizations of the expanded form of the Choquet integral (X-CI) and by the adoption of restricted dissimilarity functions in the d-Choquet integral, the theoretical objective of this paper is to introduce a generalization of the X-CI using RDFs, called d-XC integrals. We present a theoretical study of the most important properties of d-XC integrals, as the conditions for them to present some kind of increasingness (e.g, monotonicity, directional monotonicity [24], [34], and ordered directional monotonicity [35]), boundary conditions, idempotency, and the averaging property. Those properties are especially useful for the intended applications, namely, decision making and motor-imagery brain-computer interface framework. We study the behavior of d-XC integrals with six different restricted dissimilarity functions, one of then retrieving the original definition of the Choquet integral in its exoanded form (XCI).

B. The examples of applications

The objective of the applied part of this paper is to use d-XC integrals, including the original X-CI definition, in two distinct decision-making problems with an analysis and comparison among the behaviors of six different approaches of d-XC integrals. The first is a supplier selection problem and the second is a classification problem in signal processing:

1) A separation measure based on the d-XC integral for the Group Modular Choquet Random Technique for Order of Preference by Similarity to Ideal Solution (GMC-RTOPSIS) for multi-criteria decision making (MCDM): In the first application of d-XC integrals, we present a new version of GMC-RTOPSIS multi-criteria decision-making method, introduced by Lourenzutti et al. [8]. More precisely, we use the d-XC integral in the separation measure step of the method, where values for the criteria's separation are calculated. To evaluate the method, we apply it in a supplier selection problem [8], which is commonly found in businesses. Then, since we use six different restricted dissimilarity functions in the process, we proceed to classify them in order to select the best possible alternative by using the $\Delta_{R1,R2}$ [10] approach, that is, by using the d-XC integral that provides the highest difference between the alternative ranked first and the second one.

2) Motor-imagery brain-computer interface (MI-BCI) framework: In the second application, we show how the d-XC integrals can be applied in the decision-making phase of the MI-BCI Enhanced Multimodal Fusion (EMF) framework [25]. This framework presents a decision-making phase in which two aggregation functions are used: (1) first, to fuse the outputs from the classifiers trained on different wave bands, and (2) secondly, to fuse the outputs obtained in this procedure for different classifiers. Previous works showed that the best aggregation functions for this task are obtained by combining a fuzzy integral in the first fusion phase with an n-dimensional overlap function [36] in the second fusion phase. Here we use different combinations of d-XC integrals in the decision-making phase of the EMF to look for better alternatives than the actual Choquet and Sugeno [37] integrals of the original proposal.

C. Organization of the paper

This paper is organized as follows. Section II presents the preliminary concepts required for understanding the work. Section III introduces the concept of d-XC integrals and the theoretical study of its features, with several illustrative examples. Section IV presents the application to GMC-RTOPSIS, with a comprehensive discussion of the obtained results, and Section V presents the results obtained in the MI-BCI EMF framework. Finally, in Section VI we give our final remarks and conclusions for this work.

II. Preliminaries

In this section, the main concepts and notations related with the study are presented. A fuzzy set (FS) F on a universe U is given by a membership function $\mu_F : U \to [0,1]$, as follows [38]: $F = \{\langle x, \mu_F(x)\rangle \mid x \in U\}$.

In this paper, we also use the concept of intuitionistic fuzzy set (IFS) [39] IF on a universe U. It is defined by a membership function $\mu_{IF} : U \to [0,1]$ and a non-membership function $\nu_{IF} : U \to [0,1]$, such that $0 \leq \mu_{IF}(x) + \nu_{IF}(x) \leq 1$, for all $x \in U$: $IF = \{\langle x, \mu_{IF}(x), \nu_{IF}(x)\rangle \mid x \in U\}$.

Consider $U = [0,1]$, and let $\tilde{\mu}_{ITF}, \tilde{\nu}_{ITF} \in [0,1]$, such that $0 \leq \tilde{\mu}_{ITF} + \tilde{\nu}_{ITF} \leq 1$, be the maximum membership degree and the minimum non-membership degree of an IFS, respectively. Then, an intuitionistic trapezoidal fuzzy number (ITFN) ITF is defined by $ITF = \langle (a_1, a_2, a_3, a_4), (b_1, b_2, b_3, b_4), \tilde{\mu}_{ITF}, \tilde{\nu}_{ITF}\rangle$, where $a_1, a_2, a_3, a_4, b_1, b_2, b_3, b_4 \in \mathbb{R}$, $b_1 \leq a_1 \leq b_2 \leq a_2 \leq a_3 \leq b_3 \leq a_4 \leq b_4$, with $a_1 \neq a_2$, $a_3 \neq a_4$, $b_1 \neq b_2$, $b_3 \neq b_4$, and both μ_{ITF} and ν_{ITF} are given, respectively, for all $x \in \mathbb{R}$, by:

$$\mu_{ITF}(x) = \begin{cases} \frac{x-a_1}{a_2-a_1}\tilde{\mu}_{ITF}, & \text{if } a_1 \leq x < a_2 \\ \tilde{\mu}_{ITF}, & \text{if } a_2 \leq x \leq a_3 \\ \frac{a_4-x}{a_4-a_3}\tilde{\mu}_{ITF}, & \text{if } a_3 < x \leq a_4 \\ 0, & \text{otherwise,} \end{cases}$$

$$\nu_{ITF}(x) = \begin{cases} \frac{1-\tilde{\nu}_{ITF}}{b_1-b_2}(x-b_1)+1, & \text{if } b_1 \leq x < b_2 \\ \tilde{\nu}_{ITF}, & \text{if } b_2 \leq x \leq b_3 \\ \frac{1-\tilde{\nu}_{ITF}}{b_4-b_3}(x-b_4)+1, & \text{if } b_3 < x \leq b_4 \\ 1, & \text{otherwise.} \end{cases}$$

For convenience, when $a_1 = b_1$, $a_2 = b_2$, $a_3 = b_3$ and $a_4 = b_4$ we write the ITFN as $ITF = \langle (a_1, a_2, a_3, a_4), \tilde{\mu}_{ITF}, \tilde{\nu}_{ITF}\rangle$.

In what follows, let $N = \{1, \ldots, n\}$.

Definition 1. *[2] A function $A : [0,1]^n \to [0,1]$ is an aggregation function (AF) if: (A1) A is increasing in each argument: for each $i \in \{1, \ldots, n\}$, if $x_i \leq y$, then $A(x_1, \ldots, x_n) \leq A(x_1, \ldots, x_{i-1}, y, x_{i+1}, \ldots, x_n)$; (A2) A satisfies the boundary conditions: (i) $A(0, \ldots, 0) = 0$ and (ii) $A(1, \ldots, 1) = 1$.*

Notice that, a function $F : [0,1]^n \to [0,1]$ is said to be averaging if and only if: (AV) $\forall (x_1, \ldots, x_n) \in [0,1]^n$: $\min\{x_1, \ldots, x_n\} \leq F(x_1, \ldots, x_n) \leq \max\{x_1, \ldots, x_n\}$.

A function $G_O : [0,1]^n \to [0,1]$ is an n-dimensional overlap function [36] if, for all $x_1,\ldots,x_n \in [0,1]$, it holds that: (G1) G_O is symmetric; (G2) $G_O(x_1,\ldots,x_n) = 0$ if and only if $\prod_{i=1}^n x_i = 0$; (G3) $G_O(x_1,\ldots,x_n) = 1$ if and only if $x_i = 1$, for all $i \in \{1,\ldots,n\}$; (G4) G_O is increasing; (G5) G_O is continuous. An example of n-dimensional overlap function is the geometric mean defined by $GM(\boldsymbol{x}) = (\prod_{i=1}^n x_i)^{\frac{1}{n}}$.

Let $\boldsymbol{r} = (r_1,\ldots,r_n)$ be a real n-dimensional vector such that $\boldsymbol{r} \neq \mathbf{0} = (0,\ldots,0)$. A function $F : [0,1]^n \to [0,1]$ is said to be $\boldsymbol{r}$-increasing if, for all $\boldsymbol{x} = (x_1,\ldots,x_n) \in [0,1]^n$ and $c > 0$ such that $\boldsymbol{x} + c\mathbf{r} = (x_1 + cr_1,\ldots,x_n + cr_n) \in [0,1]^n$, it holds that $F(\boldsymbol{x} + c\boldsymbol{r}) \geq F(\boldsymbol{x})$ [34]. Similarly, one defines an $\boldsymbol{r}$-decreasing function.

Then, based on the idea of directional monotonicity, Lucca et al. [4] introduced the concept of pre-aggregation functions:

Definition 2. *[4] A function $PA : [0,1]^n \to [0,1]$ is said to be a pre-aggregation function (PAF) if the following conditions hold: (PA1) PA is directional increasing, for some $\boldsymbol{r} = (r_1,\ldots,r_n) \in [0,1]^n$, $\boldsymbol{r} \neq \mathbf{0}$; (PA2) PA satisfies the boundary conditions: (i) $PA(0,\ldots,0) = 0$ and (ii) $PA(1,\ldots,1) = 1$. If F is a PAF with respect to a vector $\boldsymbol{r}$ we just say that F is an $\boldsymbol{r}$-PAF.*

A function $m : 2^N \to [0,1]$ is said to be a fuzzy measure (FM) [40] if, for all $X, Y \subseteq N$: (m1) m is increasing: if $X \subseteq Y$, then $m(X) \leq m(Y)$; (m2) m satisfies the boundary conditions: $m(\emptyset) = 0$, $m(N) = 1$.

Definition 3. *[1] Let $m : 2^N \to [0,1]$ be a FM. The discrete Choquet integral (CI) is the function $\mathfrak{C}_m : [0,1]^n \to [0,1]$, defined, for all of $\boldsymbol{x} \in [0,1]^n$, by:*

$$\mathfrak{C}_m(\boldsymbol{x}) = \sum_{i=1}^n \left(x_{(i)} - x_{(i-1)}\right) \cdot m\left(A_{(i)}\right), \tag{1}$$

where $\left(x_{(1)},\ldots,x_{(n)}\right)$ is an increasing permutation on the input $\boldsymbol{x}$, that is, $0 \leq x_{(1)} \leq \ldots \leq x_{(n)} \leq 1$, with $x_{(0)} = 0$ by convention, and $A_{(i)} = \{(i),\ldots,(n)\}$.

When the product operation is distributed in Eq. (1), we obtain the CI in its expanded form (X-CI), given by:

$$\mathfrak{C}_m(\boldsymbol{x}) = \sum_{i=1}^n \left(x_{(i)} \cdot m\left(A_{(i)}\right) - x_{(i-1)} \cdot m\left(A_{(i)}\right)\right). \tag{2}$$

In the MI-BCI application we also consider the Sugeno integral [37] and its generalization called FG-Functional [41].

The Sugeno integral $S_m : [0,1]^n \to [0,1]$ with respect to a fuzzy measure m is defined, for all $\boldsymbol{x} \in [0,1]^n$, by:

$$S_m(\boldsymbol{x}) = \bigvee_{i=1}^n \left(x_{(i)} \wedge m(A_{(i)})\right)$$

where $x_{(i)}$ and A_i, with $0 \leq i \leq n$, are stated as in Def. 3.

Let m be a symmetric FM, that is for any $A, B \subseteq N$, $|A| = |B|$ implies $m(A) = m(B)$. Let $F : [0,\infty[\times [0,1] \to [0,\infty[$ be a binary function and $G : [0,\infty[^n \to [0,\infty[$ be an n-ary function. A Sugeno-like FG-functional is a function $A : [0,\infty[^n \to [0,\infty[$ given, for all $\boldsymbol{x} \in [0,1]^n$, by:

$$A(\boldsymbol{x}) = G\left(F\left(x_{(1)}, m(A_{(1)})\right),\ldots,F\left(x_{(n)}, m(A_{(n)})\right)\right),$$

where $x_{(i)}$ and A_i, $0 \leq i \leq n$, are as in Def. 3. The function $A(\boldsymbol{x}) = \sum_{i=1}^n x_{(i)} \cdot m(A_{(i)})$ is an example of FG-functional.

Definition 4. *[33] A restricted dissimilarity function (RDF) $\delta : [0,1]^2 \to [0,1]$ is a function such that, for all $x, y, z \in [0,1]$: (d1) $\delta(x,y) = \delta(y,x)$; (d2) $\delta(x,y) = 1$ if and only if $\{x,y\} = \{0,1\}$; (d3) $\delta(x,y) = 0$ if and only if $x = y$; (d4) if $x \leq y \leq z$, then $\delta(x,y) \leq \delta(x,z)$ and $\delta(y,z) \leq \delta(x,z)$.*

Bustince et al. [27] introduced the discrete d-Choquet integral (d-CI) with respect to a FM $m : 2^N \to [0,1]$ and a RDF $\delta : [0,1]^2 \to [0,1]$, as a mapping $C_{m,\delta} : [0,1]^n \to [0,n]$, defined, for all $\boldsymbol{x} \in [0,1]^n$, by $C_{m,\delta}(\boldsymbol{x}) = \sum_{i=1}^n \delta\left(x_{(i)}, x_{(i-1)}\right) \cdot m\left(A_{(i)}\right)$, where $x_{(i)}$, $A_{(i)}$, with $0 \leq i \leq n$, are as in Def. 3.

For the sake of simplicity, we denote $m_{(i)} = m\left(A_{(i)}\right)$.

III. Generalizing the X-CI by RDFs

In this section, we consider the expanded form of the CI given in Eq. (2), replacing the difference operator by RDFs:

Definition 5. *The generalization of the expanded form of the CI by RDFs $\delta : [0,1]^2 \to [0,1]$ with respect to a FM $m : 2^N \to [0,1]$, named d-XChoquet integral (d-XC), is a mapping $X\mathfrak{C}_{\delta,m}$:$[0,1]^2 \to [0,n]$, defined, for all $\boldsymbol{x} \in [0,1]^n$, by:*

$$X\mathfrak{C}_{\delta,m}(\boldsymbol{x}) = x_{(1)} + \sum_{i=2}^n \delta\left(x_{(i)} \cdot m_{(i)},\ x_{(i-1)} \cdot m_{(i)}\right), \tag{3}$$

where $x_{(i)}$, $m_{(i)}$, with $0 \leq i \leq n$, were stated in Def. 3.

Proposition 1. *Under the conditions given in Def. 5, $X\mathfrak{C}_{\delta,m}$ is well defined, for all RDF δ and FM m.*

Proof. Observe that, for all $i = 1,\ldots,n$, one has that $0 \leq x_{(i)} \leq 1$, and for all $i = 2,\ldots,n$, we have that $0 \leq m_{(i)} \leq 1$. Also, one has that, for any $z_1, z_2 \in [0,1]$, $0 \leq \delta(z_1,z_2) \leq 1$. Then it is immediate that, for any $\boldsymbol{x} \in [0,1]^n$, $0 \leq X\mathfrak{C}_{\delta,m}(\boldsymbol{x}) \leq n$, for any RDF $\delta : [0,1]^2 \to [0,1]$ and FM $m : 2^N \to [0,1]$. Now consider an input vector $\boldsymbol{x} \in [0,1]^n$, for which there may be different increasing permutations, meaning that $\boldsymbol{x}$ has repeated elements. For the sake of simplicity, but without loss of generality, consider that there exists $r, s \in \{1,\ldots,n\}$ such that $x_r = x_s = z \in [0,1]$ and, for all $i \in \{1,\ldots,n\}$, with $i \neq r, s$, it holds that $x_i \neq x_r, x_s$. The only two possible increasing permutations are:

$$(x_{(1)},\ldots,x_{(k-1)} = x_r, x_{(k)} = x_s,\ldots,x_{(n)}) \tag{4}$$

$$(x_{(1)},\ldots,x_{(k-1)} = x_s, x_{(k)} = x_r,\ldots,x_{(n)}) \tag{5}$$

Denote by $m^{(1)}_{(i)} = m^{(1)}(A_{(i)})$ and $m^{(2)}_{(i)} = m^{(2)}(A_{(i)})$, with $i \in \{1,\ldots,n\}$, the fuzzy measures of the subsets of $A_{(i)}$ of indices corresponding to the $n-i+1$ largest components of $\boldsymbol{x}$ with respect to the permutations (4) and (5), respectively. Observe that it holds that

$$m^{(1)}_{(i)} = m^{(2)}_{(i)}, \tag{6}$$

for all $i \neq k$, and

$$m^{(1)}_{(k)} = m(\{s,(k+1),\ldots,(n)\}) \tag{7}$$

$$m^{(2)}_{(k)} = m(\{r,(k+1),\ldots,(n)\}), \tag{8}$$

which means that it may be the case that $m^{(1)}_{(k)} \neq m^{(2)}_{(k)}$. Now denote by $X\mathfrak{C}^{(1)}_{\delta,m}$ and $X\mathfrak{C}^{(2)}_{\delta,m}$ the d-XC integrals with respect to the permutations (4) and (5), respectively, and suppose that

$$X\mathfrak{C}^{(1)}_{\delta,m}(\boldsymbol{x}) \neq X\mathfrak{C}^{(2)}_{\delta,m}(\boldsymbol{x}). \tag{9}$$

From Eqs. (6), (7) and (8), whenever $k \neq 1$, it follows that:

$$\begin{aligned}
&X\mathfrak{C}^{(1)}_{\delta,m}(\boldsymbol{x}) - X\mathfrak{C}^{(2)}_{\delta,m}(\boldsymbol{x})\\
=&\delta\left(x_{(k)}\cdot m^{(1)}_{(k)},\ x_{(k-1)}\cdot m^{(1)}_{(k)}\right)\\
&\quad-\delta\left(x_{(k)}\cdot m^{(2)}_{(k)},\ x_{(k-1)}\cdot m^{(2)}_{(k)}\right)\\
=&\,\delta\Big(x_s\cdot m(\{s,(k+1),\ldots,(n)\}),\\
&\qquad x_r\cdot m(\{s,(k+1),\ldots,(n)\})\Big)\\
&\quad-\delta\Big(x_r\cdot m(\{r,(k+1),\ldots,(n)\}),\\
&\qquad x_s\cdot m(\{r,(k+1),\ldots,(n)\})\Big)\\
=&\delta\Big(z\cdot m(\{s,(k+1),\ldots,(n)\}), z\cdot m(\{s,(k+1),\ldots,(n)\})\Big)\\
&-\delta\Big(z\cdot m(\{r,(k+1),\ldots,(n)\}), z\cdot m(\{r,(k+1),\ldots,(n)\})\Big)\\
=&\,0 \qquad \text{by (d3)}
\end{aligned}$$

which is in contradiction to (9). Similarly, one can show that there is a contradiction for $k = 1$. The result can be easily generalized for any subsets of repeated elements in the input $\boldsymbol{x}$. The conclusion is that for any different increasing permutations of the same input $\boldsymbol{x}$ one always get the same output value of $X\mathfrak{C}_{\delta,m}(\boldsymbol{x})$. This completes the proof that $X\mathfrak{C}_{\delta,m}$ is well defined. □

Remark 1. *Observe that the first element of the summation in the definition of $X\mathfrak{C}_{\delta,m}$ is just $x_{(1)}$ instead of $\delta\left(x_{(1)}\cdot m_{(1)},\ x_{(0)}\cdot m_{(1)}\right)$. This is to avoid the initial discrepant behavior of non-averaging functions in the initial phase of the aggregation process, as pointed out in [7]. For example, consider an unitary vector $\boldsymbol{x} = (0.1)$ and $\delta_4(x,y) = |x^2 - y^2|$. If we included the first element inside the summation the result would be:*

$$\begin{aligned}
X\mathfrak{C}_{\delta,m}(\boldsymbol{x}) &= \delta_4\left(x_{(1)}\cdot m_{(1)},\ x_{(0)}\cdot m_{(1)}\right)\\
&= \left|(0.1\cdot 1)^2 - (0\cdot 1)^2\right| = 0.01.
\end{aligned}$$

Observe here the large discrepancy of the result (a relative error of 90%), since one expects that the aggregated value would be 0.1*. Using our definition of d-XC integral (Eq. (3)) this unexpected behavior is avoided and the result is 0.1.*

The following proposition gives an alternative way to express the d-XC integrals:

Proposition 2. *Let $\delta : [0,1]^2 \to [0,1]$ be an RDF and $X\mathfrak{C}_{\delta,m} : [0,1]^n \to [0,1]$ be the derived d-XC integral for any fuzzy measure $m : 2^N \to [0,1]$. Let $\delta_0 : [0,1]^2 \to [0,1]$, defined for all $x,y \in [0,1]$, by $\delta_0(x,y) = |x-y|$. If, for all $x,y \in [0,1]$ and $p,q \in \mathbb{R}^+$, it holds that:*

$$\delta(x,y) = (\delta_0(x^q, y^q))^p \tag{10}$$

TABLE I: RDFs and their respective d-XChoquet integrals, using Eq. (11), used in the applications of this study.

δ	RDF	d-XC
δ_0	$\lvert x-y\rvert$	$x_{(1)} + \sum_{i=2}^n m_{(i)}(x_{(i)} - x_{(i-1)})$
δ_1	$(x-y)^2$	$x_{(1)} + \sum_{i=2}^n m^2_{(i)}\left(x_{(i)} - x_{(i-1)}\right)^2$
δ_2	$\sqrt{\lvert x-y\rvert}$	$x_{(1)} + \sum_{i=2}^n \sqrt{m_{(i)}}\sqrt{x_{(i)} - x_{(i-1)}}$
δ_3	$\lvert\sqrt{x}-\sqrt{y}\rvert$	$x_{(1)} + \sum_{i=2}^n \sqrt{m_{(i)}}\left(\sqrt{x_{(i)}} - \sqrt{x_{(i-1)}}\right)$
δ_4	$\lvert x^2-y^2\rvert$	$x_{(1)} + \sum_{i=2}^n m^2_{(i)}\left(x^2_{(i)} - x^2_{(i-1)}\right)$
δ_5	$(\sqrt{x}-\sqrt{y})^2$	$x_{(1)} + \sum_{i=2}^n m_{(i)}\left(\sqrt{x_{(i)}} - \sqrt{x_{(i-1)}}\right)^2$

then the d-XC integral can be given, for all $\boldsymbol{x} \in [0,1]^n$, by:

$$X\mathfrak{C}_{\delta,m}(\boldsymbol{x}) = x_{(1)} + \sum_{i=2}^n \delta(0, m_{(i)})\delta(x_{(i)}, x_{(i-1)}). \tag{11}$$

Proof. Let $\delta : [0,1]^2 \to [0,1]$ be an RDF such that Eq. (10) is true. Then, for any $\boldsymbol{x} \in [0,1]$ and FM m, we have that:

$$\begin{aligned}
&X\mathfrak{C}_{\delta,m}(\boldsymbol{x})\\
&= x_{(1)} + \sum_{i=2}^n \delta(x_{(i)}\cdot m_{(i)}, x_{(i-1)}\cdot m_{(i)})\\
&= x_{(1)} + \sum_{i=2}^n \left[\delta_0((x_{(i)}\cdot m_{(i)})^q, (x_{(i-1)}\cdot m_{(i)})^q)\right]^p\\
&= x_{(1)} + \sum_{i=2}^n \left|x^q_{(i)}\cdot m^q_{(i)} - x^q_{(i)}\cdot m^q_{(i)}\right|^p\\
&= x_{(1)} + \sum_{i=2}^n \left|m^q_{(i)}\right|^p \left|x^q_{(i)} - x^q_{(i-1)}\right|^p\\
&= x_{(1)} + \sum_{i=2}^n \delta(0, m_{(i)})\delta(x_{(i)}, x_{(i-1)}) \qquad \text{by (10).}
\end{aligned}$$

And this proves the proposition. □

Remark 2. *Notice that all RDFs presented in the left column of Table I satisfy Eq. (10) and, therefore, the derived d-XC integral can be given using Eq. (11), as shown in the right column of the table. In fact, all such RDFs were constructed accordingly to [42, Prop. 2]. Nevertheless, it is possible to define an RDF that is not derived from δ_0. For example, let $\delta : [0,1]^2 \to [0,1]$ be given, for all $x,y \in [0,1]$, $c \in (0,1)$, by*

$$\delta(x,y) = \begin{cases} 1, & \text{if } \{x,y\} = \{0,1\},\\ 0, & \text{if } x = y,\\ c, & \text{otherwise.}\end{cases}$$

The d-XC integral derived from this RDF is not a transformation from the one derived from δ_0, and therefore can not be used in the form of Eq. (11).

Now observe that since the range of d-XC integrals are in the interval $[0,n]$, it makes no sense to talk about their boundary conditions in general, unless one considers increasing d-XC integrals. Then, in the context of this paper, the boundary conditions of AF and PAF (conditions (A2) and (PA2)), are refereed just by $0,1$-conditions. Furthermore, notice that for the applications presented in this work the range of the d-XC integral need not be in $[0,1]$.

Theorem 1 $(0,1$-conditions$)$**.** *The d-XC integral satisfies the* $0,1$*-conditions for any FM* m *and any RDF* δ.

Proof. (i) Consider $\boldsymbol{x} = \boldsymbol{0} = (0,\dots,0)$. Then, we have: $X\mathfrak{C}_{\delta,m}(\boldsymbol{0}) = 0 + \sum_{i=2}^{n} \delta\left(0\cdot m_{(i)}, 0\cdot m_{(i)}\right) = 0$.
(ii) Consider $\boldsymbol{x} = \boldsymbol{1} = (1,\dots,1)$. Then, we have: $X\mathfrak{C}_{\delta,m}(\boldsymbol{1}) = 1 + \sum_{i=2}^{n} \delta\left(1\cdot m_{(i)}, 1\cdot m_{(i)}\right) = 1+0 = 1$. Therefore, the $0,1$-conditions are satisfied. □

In what follows, denote the range of a d-XC integral $X\mathfrak{C}_{\delta,m}$ by Ran$(X\mathfrak{C}_{\delta,m})$.

Remark 3. *Whenever the range of a d-XC integral is* $[0,1]$*, then the* $0,1$*-conditions are equivalent to boundary conditions. Also, if a d-XC integral is increasing then it is immediate that its range is* $[0,1]$*. Now, notice that whenever a d-XC integral* $X\mathfrak{C}_{\delta,m}$ *is not increasing then, even if it satisfies the* $0,1$*-conditions, there may exist* $\boldsymbol{y}\in[0,1]^n$, $\boldsymbol{0}<\boldsymbol{y}<\boldsymbol{1}$ *such that* $X\mathfrak{C}_{\delta,m}(\boldsymbol{y})>1$*. In fact, consider the RDF* δ_2 *of Table I, and a FM* m *given, for all* $X\subseteq 2^N$ *and* $X\neq\emptyset$*, by* $m(X)=1$*. Take* $\boldsymbol{y}=(0.1,0.5,0.8)$*. It follows that:*

$$X\mathfrak{C}_{\delta_2,m}(\boldsymbol{y}) = 0.1 + \sqrt{0.5-0.1} + \sqrt{0.8-0.5} \simeq 1.28.$$

The following proposition states the condition to be verified whenever it is necessary to guarantee Ran$(X\mathfrak{C}_{\delta,m})\subseteq[0,1]$.

Proposition 3. *Let* m *and* δ *be a FM and an RDF, respectively. Then, Ran*$(X\mathfrak{C}_{\delta,m})\subseteq[0,1]$ *if the following condition holds, for any* $0\le z_1\le\ldots\le z_n\le 1$*:*

$$\sum_{i=2}^{n} \delta\left(z_i\cdot m_i, z_{i-1}\cdot m_i\right) \le 1 - z_1, \tag{12}$$

where $m_i = m(A_i)$*, for* $A_i=\{i,\dots,n\}$.

Remark 4. *Observe that Prop. 3 states a sufficient (but not necessary) condition for having Ran*$(X\mathfrak{C}_{\delta,m})\subseteq[0,1]$*. That is, it may be the case that Ran*$(X\mathfrak{C}_{\delta,m})\subseteq[0,1]$ *and it holds that* $\sum_{i=2}^{n}\delta\left(x_{(i)}\cdot m_{(i)},\ x_{(i-1)}\cdot m_{(i)}\right) > 1-x_{(1)}$*, for some* $\boldsymbol{x}\in[0,1]^n$ *and* $0\le x_{(1)}\le\ldots\le x_{(n)}\le 1$.

A. Directional Monotonicity of d-XC integrals

In this subsection, we analyze the conditions under which d-XC integrals are directional monotonic, in particular, $\boldsymbol{1}$-increasing. In addition, we discuss the cases in which d-XC integrals satisfy the $0,1$-conditions and have the range in the unit interval, so satisfying all the requirements to be a PAF.

Theorem 2 $(\boldsymbol{1}$-increasingness$)$**.** *Let* m *and* δ *be a FM and an RDF, respectively.* $X\mathfrak{C}_{\delta,m}$ *is* $\boldsymbol{1}$*-increasing if and only if one of the following conditions hold:*
(i) the RDF δ *is* $\boldsymbol{1}$*-increasing;*
(ii) For all $0\le z_1\le\ldots\le z_n\le 1$ *and* $c>0$*, such that* $z_i+c\in[0,1]$*, for all* $i=1,\dots,n$*, it holds that:*

$$\sum_{i=2}^{n} \delta((z_i+c)\cdot m_i, (z_{i-1}+c)\cdot m_i) \tag{13}$$

$$\ge \sum_{i=2}^{n} \delta(z_i\cdot m_i, z_{i-1}\cdot m_i) - c,$$

where $m_i = m(A_i)$*, for* $A_i=\{i,\dots,n\}$.

Proof. $(\Leftarrow)$(i) Suppose that δ is $\boldsymbol{1}$-increasing and let $\boldsymbol{c} = (c,\dots,c)$, $c>0$, such that $\boldsymbol{x},\boldsymbol{x}+\boldsymbol{c}\in[0,1]^n$. Then:

$$\begin{aligned}
&X\mathfrak{C}_{\delta,m}(\boldsymbol{x}+\boldsymbol{c})\\
=&(x_{(1)}+c) + \sum_{i=2}^{n}\delta\left((x_{(i)}+c)\cdot m_{(i)}), (x_{(i-1)}+c)\cdot m_{(i)}\right)\\
=&(x_{(1)}+c) + \sum_{i=2}^{n}\delta\left(x_{(i)}\cdot m_{(i)} + c\cdot m_{(i)},\right.\\
&\qquad\qquad\left. x_{(i-1)}\cdot m_{(i)} + c\cdot m_{(i)}\right)\\
>&x_{(1)} + \sum_{i=2}^{n}\delta\left(x_{(i)}\cdot m_{(i)}, x_{(i-1)}\cdot m_{(i)}\right)\\
=&X\mathfrak{C}_{\delta,m}(\boldsymbol{x}).
\end{aligned}$$

Now suppose that (ii) holds. Then, for all $\boldsymbol{x}\in[0,1]^n$ and $\boldsymbol{c} = (c,\dots,c)$, with $c>0$, such that $x_i+c\in[0,1]$, $\forall i=1,\dots,n$, it follows that:

$$\begin{aligned}
&\sum_{i=2}^{n}\delta((x_{(i)}+c)\cdot m_{(i)}, (x_{(i-1)}+c)\cdot m_{(i)})\\
&\qquad\ge \sum_{i=2}^{n}\delta(x_{(i)}\cdot m_{(i)}, x_{(i-1)}\cdot m_{(i)}) - c\\
\Rightarrow&(x_{(1)}+c) + \sum_{i=2}^{n}\delta((x_{(i)}+c)\cdot m_{(i)}, (x_{(i-1)}+c)\cdot m_{(i)})\\
&\qquad\ge x_{(1)} + \sum_{i=2}^{n}\delta(x_{(i)}\cdot m_{(i)}, x_{(i-1)}\cdot m_{(i)})\\
\Rightarrow&X\mathfrak{C}_{\delta,m}(\boldsymbol{x}+\boldsymbol{c}) \ge X\mathfrak{C}_{\delta,m}(\boldsymbol{x}).
\end{aligned}$$

Therefore, if (i) or (ii) holds, then $X\mathfrak{C}_{\delta,\mathrm{m}}$ is $\boldsymbol{1}$-increasing.
$(\Rightarrow)$ Suppose that $X\mathfrak{C}_{\delta,\mathrm{m}}$ is $\boldsymbol{1}$-increasing, that is $X\mathfrak{C}_{\delta,m}(\boldsymbol{x}+\boldsymbol{c})\ge X\mathfrak{C}_{\delta,m}(\boldsymbol{x})$. Then it follows that

$$\begin{aligned}
&(x_{(1)}+c) + \sum_{i=2}^{n}\delta((x_{(i)}+c)\cdot m_{(i)}, (x_{(i-1)}+c)\cdot m_{(i)})\\
\ge\ \ & x_{(1)} + \sum_{i=2}^{n}\delta(x_{(i)}\cdot m_{(i)}, x_{(i-1)}\cdot m_{(i)}),
\end{aligned}$$

which implies that condition (ii) holds. □

Example 1. *Consider the RDF* δ_4 *of Table I, which is* $\boldsymbol{1}$*-increasing, since, whenever* $x\ge y$*, it holds that:*

$$\begin{aligned}
&\delta_4(x+c,y+c)\\
&\quad= (x+c)^2-(y+c)^2 = (x^2-y^2)+2c(x-y)\\
&\quad\ge x^2-y^2 = \delta_4(x,y),
\end{aligned}$$

for any $c>0$ *with* $x,y,x+c,y+c\in[0,1]$*. Therefore, by Theorem 2, we have that* $X\mathfrak{C}_{\delta_4,m}$ *is* $\boldsymbol{1}$*-increasing, for any FM* m*. The same holds for the RDFs* δ_0*,* δ_1 *and* δ_2 *of Table I.*

From Theorems 1 and 2, and Prop. 3 it is immediate that:

Theorem 3 (PAF). *Let $m : 2^N \to [0,1]$ and $\delta : [0,1]^2 \to [0,1]$ be a fuzzy meaure and an RDF, respectively. Then, the d-XChoquet integral $X\mathfrak{C}_{\delta,m}$ is an $\mathbf{1}$-PAF of signature $[0,1]^2 \to [0,1]$ if and only if one of the conditions (i) or (ii) of Theorem 2 holds and, additionally, Prop. 3 holds.*

Example 2. *Take the RDF δ_3 from Table I and the FM m given by $m(X) = 1$ (which is the greatest FM on N), for any $X \subseteq 2^N$ and $X \neq \emptyset$. Then, for any $\boldsymbol{x} \in [0,1]^n$, we have that:*

$$\sum_{i=2}^{n} \sqrt{m_{(i)}} \left(\sqrt{x_{(i)}} - \sqrt{x_{(i-1)}}\right) = \sqrt{x_{(n)}} - \sqrt{x_{(1)}} \leq 1 - \sqrt{x_{(1)}} \leq 1 - x_{(1)}, \tag{14}$$

and, then, Prop. 3 holds. However, δ_3 is not $\mathbf{1}$-increasing, since, for example, for $x = 0.4$, $y = 0.2$ and $c = 0.5$ one has

$$\delta_3(0.4 + 0.5, 0.2 + 0.5) \simeq 0.112 \leq 0.185 \simeq \delta_3(0.4, 0.2), \tag{15}$$

and, thus, the condition (i) of Theorem 2 does not hold. Also, it is easy to verify that neither the condition (ii) of Theorem 2 holds. In fact, consider, for example, the input $\boldsymbol{x} = (0.1, 0.2, 0.8)$ and $c = 0.00001$. Then, by Equations (13) and (14), it follows that:

$$\sum_{i=2}^{3} \delta_3(x_{(i)} + 0.00001,\ x_{(i-1)} + 0.00001) = \sqrt{0.8 + 0.00001} - \sqrt{0.2 + 0.00001} = 0.44715706.$$

However,

$$\sum_{i=2}^{3} \delta_3(x_{(i)},\ x_{(i-1)}) - 0.00001 = \sqrt{0.8} - \sqrt{0.2} - 0.00001 = 0.4472035955.$$

and $0.44715706 < 0.44720360$. Therefore, although the corresponding d-XChoquet integral satisfies the $0,1$-conditions, it is not a PAF, according to Theorem 3.

Example 3. *An example of $\mathbf{1}$-increasing d-XChoquet integral that is not a PAF is when one considers the RDF δ_4, which, by Example 1, is $\mathbf{1}$-increasing. In fact, take the input $\boldsymbol{x} = (0.15, 0.23, 0.99)$ with the FM given $m(X) = 1$, for any $X \subseteq 2^N$ such that $X \neq \emptyset$. Then, by Eq. (12), we have that:*

$$\sum_{i=2}^{3} \delta(x_{(i)} \cdot 1, x_{(i-1)} \cdot 1) = 0.23^2 - 0.15^2 + 0.99^2 - 0.23^2 = 0.9576 > 1 - 0.15 = 0.85,$$

and, thus, Prop. 3 does not hold, and, then, neither Theorem 3 is satisfied. Now, by Remark 3 and Example 1, whenever one takes the RDF δ_2 and the same FM as above, then the corresponding d-XChoquet integral is also $\mathbf{1}$-increasing but its range is not included in the interval $[0,1]$, and, thus, although it satisfies the $0,1$-conditions, it is not a PAF.

Example 4. *Consider the RDF δ_0 of Table I and the FM m given by $m(X) = 1$ for any $X \subseteq 2^N$ and $X \neq \emptyset$. By Example 1, δ_0 is $\mathbf{1}$-increasing then the condition (i) of Theorem 2 holds. Also, by an analogous argument used in Example 2, Prop. 3 holds. Therefore, by Theorem 3, the corresponding d-XChoquet integral is a PAF.*

B. Monotonicity of d-XC integrals

In this subsection, we analyse the conditions under which d-XC integrals are full monotonic and satisfy the $0,1$-conditions, so satisfying all the requirements to be an AF.

Theorem 4 (Monotonicity). *Consider a FM $m : 2^N \to [0,1]$. $X\mathfrak{C}_{\delta,m}$ is non-decreasing if and only if the following conditions hold for the RDF $\delta : [0,1]^2 \to [0,1]$:*
(i) For all $z_1, z_2, z_3, z_4 \in [0,1]$, with $z_1 \leq z_2 \leq z_3 \leq z_4$, $w_1, w_2 \in [0,1]$, with $w_1 \geq w_2$:

$$\delta(z_1 \cdot w_1, z_3 \cdot w_1) + \delta(z_3 \cdot w_2, z_4 \cdot w_2) \geq \delta(z_1 \cdot w_1, z_2 \cdot w_1) + \delta(z_2 \cdot w_2, z_4 \cdot w_2); \tag{16}$$

(ii) For all $z_1, z_2, z_3 \in [0,1]$, with $z_1 \leq z_2 \leq z_3$, $w \in [0,1]$:

$$z_2 + \delta(z_3 \cdot w, z_2 \cdot w) \geq z_1 + \delta(z_3 \cdot w, z_1 \cdot w). \tag{17}$$

Proof. ($\Leftarrow$) Take $\boldsymbol{x}, \boldsymbol{y} \in [0,1]^n$, where for some $k \in \{1, \ldots, n\}$ and $h \geq 0$, we have that $x_{(k)} = y_{(k)} + h$ and, for every $i \in \{1, \ldots, n\}$, $i \neq k$, $x_{(i)} = y_{(i)}$, such that

$$x_{(k-1)} = y_{(k-1)} \leq y_{(k)} \leq x_{(k)} = y_{(k)} + h \leq x_{(k+1)} = y_{(k+1)}.$$

Supposing the conditions in the theorem hold, we have three cases to prove:
(i) $k = 1$: In this case, denote $z_1 = y_{(1)}$, $z_2 = y_{(1)} + h$, $z_3 = y_{(2)}$ and $w = m_{(2)}$. Then, we have that:

$$\begin{aligned}
&X\mathfrak{C}_{\delta,m}(\boldsymbol{x}) \\
&= x_{(1)} + \delta\left(x_{(2)} \cdot m_{(2)}, x_{(1)} \cdot m_{(2)}\right) + \sum_{i=3}^{n} \delta\left(x_{(i)} \cdot m_{(i)}, x_{(i-1)} \cdot m_{(i)}\right) \\
&= (y_{(1)} + h) + \delta\left(y_{(2)} \cdot m_{(2)}, (y_{(1)} + h) \cdot m_{(2)}\right) + \sum_{i=3}^{n} \delta\left(y_{(i)} \cdot m_{(i)}, y_{(i-1)} \cdot m_{(i)}\right) \\
&= z_2 + \delta\left(z_3 \cdot w, z_2 \cdot w\right) + \sum_{i=3}^{n} \delta\left(y_{(i)} \cdot m_{(i)}, y_{(i-1)} \cdot m_{(i)}\right) \\
&\geq z_1 + \delta\left(z_3 \cdot w,\ z_1 \cdot w\right) + \sum_{i=3}^{n} \delta\left(y_{(i)} \cdot m_{(i)}, y_{(i-1)} \cdot m_{(i)}\right) \quad \text{by (17)} \\
&= y_{(1)} + \delta\left(y_{(2)} \cdot m_{(2)},\ y_{(1)} \cdot m_{(2)}\right) + \sum_{i=3}^{n} \delta\left(y_{(i)} \cdot m_{(i)}, y_{(i-1)} \cdot m_{(i)}\right) \\
&= X\mathfrak{C}_{\delta,m}(\boldsymbol{y})
\end{aligned}$$

(ii) $1 < k < n$: In this case, denote $z_1 = y_{(k-1)}$, $z_2 = y_{(k)}$, $z_3 = y_{(k)} + h$, $z_4 = y_{(k+1)}$, $w_1 = m_{(k)}$ and $w_2 = m_{(k+1)}$. Then we have:

$$X\mathfrak{C}_{\delta,m}(\boldsymbol{x}) = x_{(1)} + \sum_{i=2}^{k-1} \delta\left(x_{(i)} \cdot m_{(i)}, x_{(i-1)} \cdot m_{(i)}\right)$$

$$
\begin{aligned}
&+ \delta\left(x_{(k)} \cdot m_{(k)}, x_{(k-1)} \cdot m_{(k)}\right) \\
&+ \delta\left(x_{(k+1)} \cdot m_{(k+1)}, x_{(k)} \cdot m_{(k+1)}\right) \\
&+ \sum_{i=k+2}^{n} \delta\left(x_{(i)} \cdot m_{(i)}, x_{(i-1)} \cdot m_{(i)}\right) \\
= {}& y_{(1)} + \sum_{i=2}^{k-1} \delta\left(y_{(i)} \cdot m_{(i)}, y_{(i-1)} \cdot m_{(i)}\right) \\
&+ \delta\left((y_{(k)} + h) \cdot m_{(k)}, y_{(k-1)} \cdot m_{(k)}\right) \\
&+ \delta\left(y_{(k+1)} \cdot m_{(k+1)}, (y_{(k)} + h) \cdot m_{(k+1)}\right) \\
&+ \sum_{i=k+2}^{n} \delta\left(y_{(i)} \cdot m_{(i)}, y_{(i-1)} \cdot m_{(i)}\right) \\
= {}& y_{(1)} + \sum_{i=2}^{k-1} \delta\left(y_{(i)} \cdot m_{(i)}, y_{(i-1)} \cdot m_{(i)}\right) \\
&+ \delta\left(z_3 \cdot w_1, z_1 \cdot w_1\right) + \delta\left(z_4 \cdot w_2, z_3 \cdot w_2\right) \\
&+ \sum_{i=k+2}^{n} \delta\left(y_{(i)} \cdot m_{(i)}, y_{(i-1)} \cdot m_{(i)}\right) \\
\geq {}& y_{(1)} + \sum_{i=2}^{k-1} \delta\left(y_{(i)} \cdot m_{(i)}, y_{(i-1)} \cdot m_{(i)}\right) \\
&+ \delta\left(z_2 \cdot w_1, z_1 \cdot w_1\right) + \delta\left(z_4 \cdot w_2,\ z_2 \cdot w_2\right) \\
&+ \sum_{i=k+2}^{n} \delta\left(y_{(i)} \cdot m_{(i)}, y_{(i-1)} \cdot m_{(i)}\right) \quad \text{by (16)} \\
= {}& y_{(1)} + \sum_{i=2}^{k-1} \delta\left(y_{(i)} \cdot m_{(i)}, y_{(i-1)} \cdot m_{(i)}\right) \\
&+ \delta\left(y_{(k)} \cdot m_{(k)}, y_{(k-1)} \cdot m_{(k)}\right) \\
&+ \delta\left(y_{(k+1)} \cdot m_{(k+1)}, y_{(k)} \cdot m_{(k+1)}\right) \\
&+ \sum_{i=k+2}^{n} \delta\left(y_{(i)} \cdot m_{(i)}, y_{(i-1)} \cdot m_{(i)}\right) \\
= {}& X\mathfrak{C}_{\delta,m}(\boldsymbol{y})
\end{aligned}
$$

(iii) $k = n$: In this case, we have:

$$
\begin{aligned}
X\mathfrak{C}_{\delta,m}(\boldsymbol{x}) &= x_{(1)} + \sum_{i=2}^{n-1} \delta\left(x_{(i)} \cdot m_{(i)}, x_{(i-1)} \cdot m_{(i)}\right) \\
&\quad + \delta\left(x_{(n)} \cdot m_{(n)}, x_{(n-1)} \cdot m_{(n)}\right) \\
&= y_{(1)} + \sum_{i=2}^{n-1} \delta\left(y_{(i)} \cdot m_{(i)}, y_{(i-1)} \cdot m_{(i)}\right) \\
&\quad + \delta\left((y_{(n)} + h) \cdot m_{(n)}, y_{(n-1)} \cdot m_{(n)}\right) \\
&= y_{(1)} + \sum_{i=2}^{n-1} \delta\left(y_{(i)} \cdot m_{(i)}, y_{(i-1)} \cdot m_{(i)}\right) \\
&\quad + \delta\left(y_{(n)} \cdot m_{(n)} + h \cdot m_{(n)}, y_{(n-1)} \cdot m_{(n)}\right) \\
&\geq y_{(1)} + \sum_{i=2}^{n-1} \delta\left(y_{(i)} \cdot m_{(i)}, y_{(i-1)} \cdot m_{(i)}\right) \\
&\quad + \delta\left(y_{(n)} \cdot m_{(n)}, y_{(n-1)} \cdot m_{(n)}\right) \quad \text{by (d4)} \\
&= X\mathfrak{C}_{\delta,m}(\boldsymbol{y})
\end{aligned}
$$

(⇒) It is analogous. □

Theorem 5 (AF)**.** *Let $m : 2^N \to [0,1]$ be a FM. Let $\delta : [0,1]^2 \to [0,1]$ be an RDF satisfying the conditions (i) and (ii) of Theorem 4. Then, the d-XChoquet integral $X\mathfrak{C}_{\delta,m}$ is an AF of signature $[0,1]^2 \to [0,1]$.*

Proof. If the conditions (i) and (ii) of Theorem 4 hold, then $X\mathfrak{C}_{\delta,m}$ is increasing. Furthermore, by Theorem 1, it satisfies the boundary conditions and its range is $[0,1]$. It follows that $X\mathfrak{C}_{\delta,m}$ is an aggregation function. □

Example 5. *The d-XChoquet $X\mathfrak{C}_{\delta_0,m}$ is an AF for any FM m, since δ_0 satisfies the conditions (i) and (ii) of Theorem 4. In fact, by Eq. (16), for all $z_1, z_2, z_3, z_4 \in [0,1]$, with $z_1 \leq z_2 \leq z_3 \leq z_4$, $w_1, w_2 \in [0,1]$, with $w_1 \geq w_2$ one has that*

$$
\begin{aligned}
&\delta(z_1 \cdot w_1, z_3 \cdot w_1) + \delta(z_3 \cdot w_2, z_4 \cdot w_2) \\
&\quad = \ w_1(z_3 - z_1) + w_2(z_4 - z_3) \\
&\quad = \ z_3(w_1 - w_2) - w_1 z_1 + w_2 z_4 \\
&\quad \geq \ z_2(w_1 - w_2) - w_1 z_1 + w_2 z_4 \\
&\quad = \ \delta(z_1 \cdot w_1, z_2 \cdot w_1) + \delta(z_2 \cdot w_2, z_4 \cdot w_2).
\end{aligned}
$$

Also, by Eq. (17), for all $z_1, z_2, z_3 \in [0,1]$, with $z_1 \leq z_2 \leq z_3$, $w \in [0,1]$, it holds that:

$$
\begin{aligned}
z_2 + \delta(z_3 \cdot w, z_2 \cdot w) &= z_2 + w(z_3 - z_2) = z_2(1 - w) + w z_3 \\
&\geq z_1(1 - w) + w z_3 = z_1 + \delta(z_3 \cdot w, z_1 \cdot w).
\end{aligned}
$$

C. Ordered Directional Monotonicity of d-XC integrals

One important feature of aggregation-like operators is to present some kind of "increasingness property" to guarantee that the more information is provided the higher is the aggregated value in the considered direction (conditions **(A1)** of Def. 1 and **(PA1)** of Def. 2). See [3] for more details.

One can notice (see Table II) that there may exist d-XC integrals that are neither increasing nor directional increasing, which is the case of $X\mathfrak{C}_{\delta_3,m}$, for instance. Nevertheless, any d-XC integral do present some kind of "increasingness property". In fact, those are denoted as Ordered Directionally (OD) monotone functions [35]. Such functions are monotonic along different directions according to the ordinal size of the coordinates of each input.

Definition 6. *[35] Consider a function $F : [0,1]^n \to [0,1]$ and let $\boldsymbol{r} = (r_1, \ldots, r_n)$ be a real n-dimensional vector, $\boldsymbol{r} \neq \boldsymbol{0}$. F is said to be ordered directionally (OD) $\boldsymbol{r}$-increasing if, for each $\boldsymbol{x} \in [0,1]^n$, any permutation $\sigma : \{1, \ldots, n\} \to \{1, \ldots, n\}$ with $x_{\sigma(1)} \geq \ldots \geq x_{\sigma(n)}$, and $c > 0$ such that $1 \geq x_{\sigma(1)} + c r_1 \geq \ldots \geq x_{\sigma(n)} + c r_n$, it holds that $F(\boldsymbol{x} + c\boldsymbol{r}_{\sigma^{-1}}) \geq F(\boldsymbol{x})$, where $\boldsymbol{r}_{\sigma^{-1}} = (r_{\sigma^{-1}(1)}, \ldots, r_{\sigma^{-1}(n)})$. Similarly, one defines an ordered directionally (OD) $\boldsymbol{r}$-decreasing function.*

Theorem 6. *For any FM $m : 2^N \to [0,1]$, RDF $\delta : [0,1]^2 \to [0,1]$ and $k > 0$, the d-XChoquet integral $X\mathfrak{C}_{\delta,m}$ is an (OD) $(k, 0, \ldots, 0)$-increasing function.*

Proof. For all $\boldsymbol{x} \in [0,1]^n$ and permutation $\sigma : \{1, \ldots, n\} \to \{1, \ldots, n\}$, with $x_{\sigma(1)} \geq \ldots \geq x_{\sigma(n)}$, and $c > 0$ such that $x_{\sigma(i)} + c r_i \in [0,1]$, for $i \in \{1, \ldots, n\}$, and $1 \geq x_{\sigma(1)} + c r_1 \geq \ldots \geq x_{\sigma(n)} + c r_n$, for $\boldsymbol{r}_{\sigma^{-1}} = (r_{\sigma^{-1}(1)}, \ldots, r_{\sigma^{-1}(n)})$, one has:

$$X\mathfrak{C}_{\delta,m}(\boldsymbol{x} + c\boldsymbol{r}_{\sigma^{-1}})$$

TABLE II: Properties of d-XChoquet integrals for any fuzzy measure m.

d-XC	$0,1$-cond.	$\mathbf{1}$-inc.	Ran $\subseteq [0,1]$	PAF	inc.	AF	(OD)-inc	$\geq \min$	$\leq \max$	Aver.
$X\mathfrak{C}_{\delta_0,m}$	yes	yes	yes	yes	yes	yes	yes	yes	yes	yes
$X\mathfrak{C}_{\delta_1,m}$	yes	yes	yes	yes	no	no	yes	yes	yes	yes
$X\mathfrak{C}_{\delta_2,m}$	yes	yes	no	no	no	no	yes	yes	no	no
$X\mathfrak{C}_{\delta_3,m}$	yes	no	yes	no	no	no	yes	yes	no	no
$X\mathfrak{C}_{\delta_4,m}$	yes	yes	no	no	no	no	yes	yes	no	no
$X\mathfrak{C}_{\delta_5,m}$	yes	no	yes	no	no	no	yes	yes	no	no

$$
\begin{aligned}
&= x_{(1)} + c \cdot r_{\sigma^{-1}(1)} \\
&\quad + \sum_{i=2}^{n-1} \delta((x_{(i)} + c \cdot r_{\sigma^{-1}(i)}) \cdot m_{(i)}, \\
&\qquad\qquad (x_{(i-1)} + c \cdot r_{\sigma^{-1}(i-1)}) \cdot m_{(i)}) \\
&\quad + \delta((x_{(n)} + c \cdot r_{\sigma^{-1}(n)}) \cdot m_{(n)}, \\
&\qquad\qquad (x_{(n-1)} + c \cdot r_{\sigma^{-1}(n-1)}) \cdot m_{(n)}) \\
&= x_{(1)} + c \cdot 0 \\
&\quad + \sum_{i=2}^{n-1} \delta((x_{(i)} + c \cdot 0) \cdot m_{(i)}, (x_{(i-1)} + c \cdot 0) \cdot m_{(i)}) \\
&\quad + \delta(x_{(n)} \cdot m_{(n)} + c \cdot k \cdot m_{(n)}, (x_{(n-1)} + c \cdot 0) \cdot m_{(n)}) \\
&\geq x_{(1)} + \sum_{i=2}^{n-1} \delta(x_{(i)} \cdot m_{(i)}, x_{(i-1)} \cdot m_{(i)}) \\
&\quad + \delta(x_{(n)} \cdot m_{(n)}, x_{(n-1)} \cdot m_{(n)}) \quad \text{by (d4)} \\
&= X\mathfrak{C}_{\delta,m}(\boldsymbol{x}).
\end{aligned}
$$

Thus, any d-XC integral is (OD) $(k, 0, \ldots, 0)$-increasing. □

D. Other Important Properties

In this section, we provide the study of some additional relevant properties. The following three results are immediate:

Proposition 4. *For any RDF δ and FM m, $X\mathfrak{C}_{\delta,m}(\boldsymbol{x}) \geq \min(\boldsymbol{x})$, for all $\boldsymbol{x} \in [0,1]$.*

Proposition 5. *$X\mathfrak{C}_{\delta,m}(\boldsymbol{x}) \leq \max(\boldsymbol{x})$ if and only if the RDF δ satisfies, for all $0 \leq a_1 \leq \ldots \leq a_n$ and for a FM m: $\sum_{i=2}^{n} \delta(a_i \cdot m_i,\ a_{i-1} \cdot m_i) \leq a_n - a_1$, where $m_i = m(A_i)$, for $A_i = \{i, \ldots, n\}$.*

Corollary 1 (Averaging)**.** *$X\mathfrak{C}_{\delta,m}$ is averaging if and only if it satisfies Prop. 5.*

Proposition 6 (Idempotency)**.** *For any RDF δ and FM m, it holds that $X\mathfrak{C}_{\delta,m}$ is idempotent.*

Proof. If $\boldsymbol{x} = (x, \ldots, x)$, then, by **(d3)**, one has that: $X\mathfrak{C}_{\delta,m}(\boldsymbol{x}) = x + \sum_{i=2}^{n} \delta\left(x \cdot m_{(i)}, x \cdot m_{(i)}\right) = x + 0 = x$. Therefore, the d-XChoquet is idempotent. □

Table II summarizes the properties satisfied by the d-XChoquet integrals that are studied in this paper, considering the RDFs of Table I and any fuzzy measure m. Notice that d-XChoquet integrals are richer than other generalizations of the CI found in the literature, in the sense that they satisfy the properties studied in this paper (which are the same properties studied for other previous generalizations of the CI) under a lower number of restrictions when compared to other generalizations.

IV. Modified GMC-RTOPSIS

In this section, we present an application of the d-XC integrals in the Group Modular Choquet Random Technique for Order of Preference by Similarity to Ideal Solution (GMC-RTOPSIS) [8] method for decision-making.

A. The modified method to use the d-XC integral

We begin by defining the notation used in this subsection. We represent by q the q-th decision maker of a total of $Q \in \mathbb{N} = \{1, 2, 3, \ldots\}$. Additionally, let $\boldsymbol{A} = \{A_1, \ldots, A_m\}$ be the set of m alternatives for the problem, which are the same for all decision makers. Let $\boldsymbol{C}_q = \{C_1, \ldots, C_{n_q}\}$ represent the criteria set for the q-th decision maker, and $\boldsymbol{C} = \bigcup_{q=1}^{Q} \boldsymbol{C}_q = \{C_1, \ldots, C_n\}$, where $n = \sum_{q=1}^{Q} n_q$, represents the criteria set of all the decision makers.

Example 6. *Take for example the problem described in the Section IV-C. In that case, we have three decision makers, $Q = 3$, and also have four distinct alternatives, represented by $\boldsymbol{A} = \{A_1, A_2, A_3, A_4\}$. With respect to the criteria, we have the following criteria set for each decision maker: $\boldsymbol{C}_1 = \{price^{(1)}, warranty^{(1)}, payment\ options^{(1)}\}$, $\boldsymbol{C}_2 = \{ price^{(2)}, delivery\ time^{(2)}, production\ capacity^{(2)}, product\ quality^{(2)}, support\ waiting^{(2)}\}$ and $\boldsymbol{C}_3 = \{ product\ lifespan^{(3)}, responsibilities^{(3)}, certifications^{(3)}, price^{(3)}\}$. This means that $n = \sum_{q=1}^{3} n_q = 12$, where $n_1 = 3$, $n_2 = 5$ and $n_3 = 4$, and the complete criteria set is $\boldsymbol{C} = \boldsymbol{C}_1 \cup \boldsymbol{C}_2 \cup \boldsymbol{C}_3$.*

Each q-th decision maker provides his/hers rating for each criterion and alternative in the form of the following matrix, denoted as a decision matrix (DM), where each value $s_{ij}^{q}(\boldsymbol{Y}^q)$, with $1 \leq i \leq m$ and $1 \leq j \leq n_q$, is the rating of the criterion j for alternative i:

$$
DM^q = \begin{array}{c} \\ A_1 \\ A_2 \\ \vdots \\ A_m \end{array}
\begin{array}{c}
\begin{array}{cccc} C_1 & C_2 & \cdots & C_{n_q} \end{array} \\
\begin{pmatrix}
s_{11}^{q}(\boldsymbol{Y}^q) & s_{12}^{q}(\boldsymbol{Y}^q) & \cdots & s_{1n_q}^{q}(\boldsymbol{Y}^q) \\
s_{21}^{q}(\boldsymbol{Y}^q) & s_{22}^{q}(\boldsymbol{Y}^q) & \cdots & s_{2n_q}^{q}(\boldsymbol{Y}^q) \\
\vdots & \vdots & \cdots & \vdots \\
s_{m1}^{q}(\boldsymbol{Y}^q) & s_{m2}^{q}(\boldsymbol{Y}^q) & \cdots & s_{mn_q}^{q}(\boldsymbol{Y}^q)
\end{pmatrix}
\end{array}.
$$

Additionally, each rating can be a function which depends on factors that model random and deterministic events, given by $\boldsymbol{Y} = (\boldsymbol{Y}_{rand},\ \boldsymbol{Y}_{det})$. Random events are modeled by stochastic processes, while deterministic events are those which are not random, such as time, location or even a parameter of a random event. Each fixed value $x \in \mathcal{X}$ of $\boldsymbol{Y}_{det}$ is called a state, where $\mathcal{X}$ is the set of states for the problem.

After all the decision makers have provided their opinions in the form of DMs, the method (steps below) is applied. Notice that, since the GMC-RTOPSIS is an extension of the standard TOPSIS, the steps are similar. The methods work by finding the alternative that is closer to the Positive Ideal Solution (**PIS**) and farther from the Negative Ideal Solution (**NIS**), as defined in step 2 below. One of the advantages in using the GMC-RTOPSIS is that for each criterion one may use a different data type. Additionally, by using the d-XC integral, the interaction among the criteria can be used in the process.

In the following, we present our modified version from the original method [8], where the main modification is done in the step 4, when we use the d-XC integral instead of the standard CI (see Fig. 1 for the diagram of the below steps):

0) Select a state $x \in \mathcal{X}$ not yet processed;

1) Normalize the complete DM matrix;

2) Select the Positive Ideal Solution (PIS), denoted by $s_j^+(\boldsymbol{Y})$, and the Negative Ideal Solution (NIS), denoted by $s_j^-(\boldsymbol{Y})$, considering, for each $j \in \{1, \ldots, n\}$, respectively:

$$s_j^+(\boldsymbol{Y}) = \begin{cases} \max\limits_{1 \le i \le m} s_{ij}, & \text{if it is a benefit criterion,} \\ \min\limits_{1 \le i \le m}, s_{ij} & \text{if it is a cost/loss criterion,} \end{cases} \quad (18)$$

$$s_j^-(\boldsymbol{Y}) = \begin{cases} \min\limits_{1 \le i \le m} s_{ij}, & \text{if it is a benefit criterion,} \\ \max\limits_{1 \le i \le m} s_{ij}, & \text{if it is a cost/loss criterion;} \end{cases}$$

3) Calculate the distance measure of each criterion C_j, with $j \in \{1, \ldots, n\}$, to the PIS and NIS solutions:

$$d_{ij}^+ = d\left(s_j^+(\boldsymbol{Y}), s_{ij}(\boldsymbol{Y})\right), \; d_{ij}^- = d\left(s_j^-(\boldsymbol{Y}), s_{ij}(\boldsymbol{Y})\right),$$

where $i \in \{1, \ldots, m\}$ and $d : \mathbb{R}^n \to [0, 1]$ is a distance measure associated with the criterion data type.

4) For each alternative $i \in \{1, \ldots, m\}$, calculate the separation measure using the d-XC integral, with an RDF δ, as follows:

$$S_i^+(\boldsymbol{Y}) = \Bigg[\left(d_{i(1)}^+\right)^2 + \sum_{j=2}^{n} \delta\left(\left(d_{i(j)}^+\right)^2 \cdot m_{\boldsymbol{Y}}\left(\boldsymbol{C}_{(j)}^+\right), \left(d_{i(j-1)}^+\right)^2 \cdot m_{\boldsymbol{Y}}\left(\boldsymbol{C}_{(j)}^+\right)\right)\Bigg]^{\frac{1}{2}}$$

$$S_i^-(\boldsymbol{Y}) = \Bigg[\left(d_{i(1)}^-\right)^2 + \sum_{j=2}^{n} \delta\left(\left(d_{i(j)}^-\right)^2 \cdot m_{\boldsymbol{Y}}\left(\boldsymbol{C}_{(j)}^-\right), \left(d_{i(j-1)}^-\right)^2 \cdot m_{\boldsymbol{Y}}\left(\boldsymbol{C}_{(j)}^-\right)\right)\Bigg]^{\frac{1}{2}}$$

where $d_{i(1)}^+ \le \ldots \le d_{i(n)}^+$, $d_{i(1)}^- \le \ldots \le d_{i(n)}^-$, for each $j \in \{1, \ldots, n\}$, $C_{(j)}^+$ is the criterion correspondent to $d_{i(j)}^+$, $C_{(j)}^-$ is the criterion correspondent to $d_{i(j)}^-$, m_Y is the learned fuzzy measure by a particle swarm optimization (PSO) algorithm [43] (for more details on how the PSO algorithm works for finding the measure see [8]) and $\boldsymbol{C}_{(j)}^+ = \{C_{(j)}^+, C_{(j+1)}^+, \ldots, C_{(n)}^+\}$, $\boldsymbol{C}_{(j)}^- = \{C_{(j)}^-, C_{(j+1)}^-, \ldots, C_{(n)}^-\}$, $\boldsymbol{C}_{(n+1)}^+ = \boldsymbol{C}_{(n+1)}^- = \emptyset$, $d_{i(0)}^+ = d_{i(0)}^- = 0$. Notice that for each state we may have a different fuzzy measure, which means that the fuzzy measure is dependent on $\boldsymbol{Y}_{det}$;

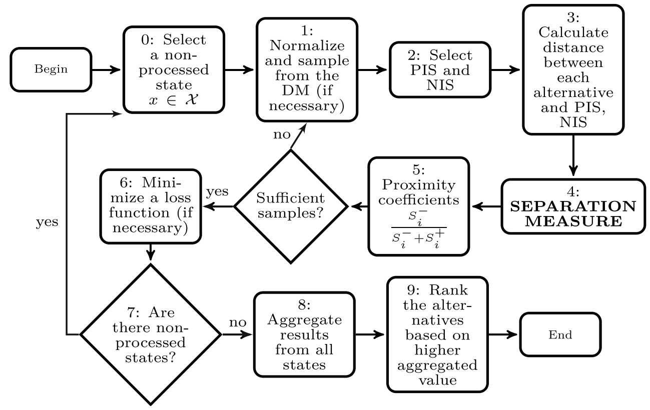

Fig. 1: Diagram of the steps of the GMC-RTOPSIS method described in Section IV-A. Source: [10]

5) For each alternative $i \in \{1, \ldots, m\}$, calculate the relative closeness coefficient to the ideal solution with:

$$CC_i(\boldsymbol{Y}) = \frac{S_i^-(\boldsymbol{Y})}{S_i^-(\boldsymbol{Y}) + S_i^+(\boldsymbol{Y})};$$

6) When using probability distributions in the DM, it is introduced a bootstrapped probability distribution in the CC_i values, so as a point representation for this distribution we minimize a pre-defined risk function:

$$\begin{aligned} cc_i &= \arg\min_c R(c) \\ &= \arg\min_c \int_{\mathbb{R}} L(c,\; CC_i(\boldsymbol{Y}))\; dF(CC_i(\boldsymbol{Y})); \end{aligned} \quad (19)$$

7) If there is at least one non-processed state x, return to 0;

8) Aggregate the cc_i values from all the states with $\widehat{cc_i} = f_{x \in \mathcal{X}}(cc_i(x))$, where f is an aggregation function;

9) Rank the alternatives from the highest to the lowest $\widehat{cc_i}$ values.

Since we run the algorithm above for multiple d-XC integrals defined by the functions δ in Table I, we use the $\Delta_{R1,R2}$ [10] to calculate the biggest separation between alternative ranked first and the second one: $\Delta_{R1,R2} = \max(\hat{c}_1) - \max(\hat{c}_2)$, where $\hat{c}_1 = \{\widehat{cc_i} \mid i \in \{1, \ldots, m\}\}$ and $\hat{c}_2 = \hat{c}_1 - \{\max(\hat{c}_1)\}$. We, then, consider the d-XC integral with δ that has the biggest $\Delta_{R1,R2}$ value. Observe that, although the $\Delta_{R1,R2}$ value does not give the "full picture" of the rank in the problem, it gives a straightforward way to compare the ranks from multiple d-XC integrals. This means that the higher the $\Delta_{R1,R2}$ value, the more likely the d-XC integral has separated the two better ranked alternatives.

B. Methodology

The parameters for the simulation are described in this section and are the same as those presented in [8], [10]. Since we are using probability distributions in the DM, we use 10,000 samples from the complete DM matrix in each of the states. We apply a particle swarm optimization [43] to learn the fuzzy measure using 30 particles and 100 interactions. The PSO is used since the original method presented good results.

For the risk function, given in Eq. (19), we use the squared loss: $L(cc,\ CC_i) = (cc - CC_i)^2$. This gives the mean function as the point estimator for the values. Also, we use the weighted arithmetic mean as the aggregation function in step 8: $WAM_i = w(S_1) \cdot cc_i(S_1) + w(S_2) \cdot cc_i(S_2)$, where S_1 and S_2 are the states of the problem.

Lastly, notice that we only changed the Choquet function to the d-XC integral. Therefore, for each used RDF δ, the method maintains its original computation complexity.

C. The considered problem

In this subsection we describe the problem used as an example in this article, which was first studied in [8]. A company is evaluating four suppliers, namely A_1, A_2, A_3 and A_4, for a provision and it asks three of its managers to give their opinions by specifying ratings in the DM.

The first is the *budget manager*. He/she considered the following criteria: $C_1^{(1)}$ - price per batch (in thousands); $C_2^{(1)}$ - warranty (in days); and $C_3^{(1)}$ - payment conditions (in days). Since it is known that the demand for the product is higher in December, he/she used a binary variable τ, defined as:

$$\tau = \begin{cases} 0, & \text{if the month is from January to November,} \\ 1, & \text{if the month is December.} \end{cases}$$

Lastly, he/she assigned a weight for each of his/her criteria with the following weighting vector: $\boldsymbol{w}^{(1)} = (0.5, 0.25, 0.25)$.

The next is the *product manager*; he/she considered the following criteria: $C_1^{(2)}$ - price; $C_2^{(2)}$ - delivery time (in hours); $C_3^{(2)}$ - production capacity; $C_4^{(2)}$ - product quality; and $C_5^{(2)}$ - time to respond to a support request (in hours). Also, the reliability in the supplier's production was modeled by a random variable P_i such that:

$$P_i = \begin{cases} 0, & \text{no failures in production of supplier } A_i, \\ 1, & \text{failures occurred in production of supplier } A_i. \end{cases}$$

Additionally, since in December the production accelerates, the chance of a failure increases as a result. Therefore he/she included this characteristic as a stochastic process using the following function: $f_i(x, y) = x\big(1 + y(P_i + \tau)^2\big)$.

Furthermore, the suppliers' production capacity was modeled by ITFNs as follows:

$$\begin{aligned} s_{13}^2 &= \big((0.8^{1+P_1}, 0.9^{1+P_1}, 1.0^{1+P_1}, 1.0^{1+P_1}),\ 1.0,\ 0.0\big) \\ s_{23}^2 &= \big((0.8^{1+4P_2}, 0.9^{1+4P_2}, 1.0^{1+4P_2}, 1.0^{1+4P_2}),\ 0.7,\ 0.1\big) \\ s_{33}^2 &= \big((0.6^{1+2P_3}, 0.7^{1+2P_3}, 0.8^{1+2P_3}, 1.0^{1+2P_3}),\ 0.8,\ 0.0\big) \\ s_{43}^2 &= \big((0.5^{1+3P_4}, 0.6^{1+3P_4}, 0.8^{1+3P_4}, 0.9^{1+3P_4}),\ 0.8,\ 0.1\big). \end{aligned}$$

Lastly, this manager assigned the same weight for all their criteria, i.e, $\boldsymbol{w}^{(2)} = (0.2, 0.2, 0.2, 0.2, 0.2)$.

The third is the *commercial manager*, which adopted the following criteria: $C_1^{(3)}$ - product lifespan (in years); $C_2^{(3)}$ - social and environmental responsibility; $C_3^{(3)}$ - quantity of quality certifications; and $C_4^{(3)}$ - price. The weighting vector for the criteria provided by this manager was $\boldsymbol{w}^{(3)} = (0.25, 0.12, 0.23, 0.4)$.

TABLE III: Decision matrices as given by three managers when analyzing four alternative suppliers based on their criteria

(a) Budget manager

Alternatives	$C_1^{(1)}$	$C_2^{(1)}$	$C_3^{(1)}$	
			$\tau = 0$	$\tau = 1$
A_1	$260.00(1 + 0.15\tau)$	90	G	G
A_2	$250.00(1 + 0.25\tau)$	90	P	W
A_3	$350.00(1 + 0.20\tau)$	180	G	I
A_4	$550.00(1 + 0.10\tau)$	365	I	W

(b) Production manager

Alternatives	$C_1^{(2)}$	$C_2^{(2)}$	$C_3^{(2)}$	$C_4^{(2)}$	$C_5^{(2)}$
A_1	260.00	U($f_1(48,\ 0.10),\ f_1(96,\ 0.10)$)	s_{13}^2	I	[24, 48]
A_2	250.00	U($f_2(72,\ 0.20),\ f_2(120,\ 0.20)$)	s_{23}^2	P	[24, 48]
A_3	350.00	U($f_3(36,\ 0.15),\ f_3(72,\ 0.15)$)	s_{33}^2	G	[12, 36]
A_4	550.00	U($f_4(48,\ 0.25),\ f_4(96,\ 0.25)$)	s_{34}^2	E	[0, 24]

(c) Commercial manager

Alternatives	$C_1^{(3)}$	$C_2^{(3)}$	$C_3^{(3)}$	$C_4^{(3)}$
A_1	Exp(3.5)	W	1	260.00
A_2	Exp(3.0)	W	0	250.00
A_3	Exp(4.5)	P	3	350.00
A_4	Exp(5.0)	I	5	550.00

TABLE IV: Linguistic variables and their respective trapezoidal fuzzy numbers used in criteria $C_3^{(1)}$, $C_4^{(2)}$ and $C_2^{(3)}$

Linguistic variables	Trapezoidal fuzzy numbers
Worst (W)	(0.0, 0.0, 0.2, 0.3)
Poor (P)	(0.2, 0.3, 0.4, 0.5)
Intermediate (I)	(0.4, 0.5, 0.6, 0.7)
Good (G)	(0.6, 0.7, 0.8, 1.0)
Excellent (E)	(0.8, 0.9, 1.0, 1.0)

To define the P_i distribution the company used historical data of each supplier as follows: for $\tau = 0$: $p(P_1 = 0|S_1) = 0.98$, $p(P_2 = 0|S_1) = 0.96$, $p(P_3 = 0|S_1) = 0.97$, $p(P_4 = 0|S_1) = 0.95$. And for $\tau = 1$: $p(P_1 = 0|S_2) = 0.96$, $p(P_2 = 0|S_2) = 0.92$, $p(P_3 = 0|S_2) = 0.96$, $p(P_4 = 0|S_2) = 0.90$.

Notice that, considering all the DMs, the underlying factors are given by a random component $\boldsymbol{Y}_{rand} = (P_1, P_2, P_3, P_4)$ and a deterministic component $Y_{det} = \tau$. Since τ can have two distinct values, there are two states for the problem: the first occurs when $\tau = 0$ and the second when $\tau = 1$, represented by S_1 and S_2 respectively. Therefore, the underlying factors can be represented by $\boldsymbol{Y} = (\boldsymbol{Y}_{rand},\ \boldsymbol{Y}_{det})$. Additionally, since the production is higher in December, the managers decided that the state S_2 is more important than S_1, hence they gave it a higher weight in the aggregation step (step 8 of the method) by setting $w(S_1) = 0.4$ and $w(S_2) = 0.6$.

Table III shows the DMs for each manager. Linguistic variables W, P, I, G, and E are defined in Table IV.

Thereafter, the company favored the opinion of the product manager over the others by assigning a higher weight for him/her in the weighting vector for the decision makers: $\boldsymbol{w} = (0.3,\ 0.4,\ 0.3)$. Lastly, the company allowed a variation of up to 30% for each fuzzy measure, when finding it via the

TABLE V: Results from the execution of the algorithm using the d-XC integral for each RDF δ presented in Table I.

d-XC	Rank 1	Rank 2	Rank 3	Rank 4	$\Delta_{R1,R2}$
δ_4	$A_3(0.6812)$	$A_4(0.5747)$	$A_1(0.4439)$	$A_2(0.3522)$	0.1064
δ_1	$A_3(0.6217)$	$A_4(0.5723)$	$A_2(0.4168)$	$A_1(0.4102)$	0.0494
δ_3	$A_3(0.5527)$	$A_4(0.5368)$	$A_1(0.4769)$	$A_2(0.4401)$	0.0159
δ_0	$A_3(0.5821)$	$A_4(0.5701)$	$A_1(0.4346)$	$A_2(0.3977)$	0.0120
δ_5	$A_3(0.5454)$	$A_4(0.5429)$	$A_2(0.4575)$	$A_1(0.4486)$	0.0024
δ_2	$A_4(0.5610)$	$A_3(0.5609)$	$A_1(0.4622)$	$A_2(0.3906)$	0.0001

PSO algorithm, in respect to the coefficient in the additive fuzzy measure [44]. This means that the fuzzy measure $m_{\boldsymbol{Y}}$ can be in the range $0.7 \cdot m_A \leq m_{\boldsymbol{Y}} \leq 1.3 \cdot m_A$ of the additive fuzzy measure m_A. This was done to allow interaction among the criteria.

D. Results

The aggregated results in Table V are ordered by the highest to the lowest $\Delta_{R1,R2}$ in the last column. For each RDF δ used in the d-XC integral, the alternatives ranked first to fourth are shown in columns Rank 1 to Rank 4. Inside the parenthesis are the final aggregated values for the alternatives.

One can notice that 5 of the 6 results agreed in the top of the rank order. The one that disagreed, δ_2, had a quite small difference in the values of the alternatives ranked first and second, which may be the reason to this change in the top of the rank, as also discussed in [10]. Notice that when using δ_4 we have the highest $\Delta_{R1,R2}$ with 0.1064. It is more than double the second highest value, δ_1, which resulted $\Delta_{R1,R2} = 0.0494$.

It is important to remark that the standard X-Choquet integral, which is the d-XC integral given by using the RDF δ_0, has achieved one of the lowest separations between positions 1 and 2 of the rank. Thus, non standard d-XC integrals (with RDFs different from δ_0) offered further trustful decision making, in the sense that there are no doubts when choosing alternative rank first instead of second.

V. Enhanced-Multimodal Fusion BCI Framework

In this section we present an application of d-XC integrals in the Enhanced Multimodal Fusion (EMF) BCI Framework. The EMF is a Motor Imagery based BCI framework that classifies electroencephalogram (EEG) signals [25] into different commands, and has been tested in tasks consisting of discriminating among left hand, right hand, foot and tongue movements. The EMF consists of 5 different phases:

1) Apply the Fast Fourier transform (FFt) to the EEG signals. Then, perform a differentiation of the FFt output.

2) Divide the data into five wave bands: δ $(1-4\text{Hz})$, θ $(4-8\text{Hz})$, α $(8-14\text{Hz})$, $\beta(14-30\text{Hz})$ and All $(1-30\text{Hz})$.

3) Compute the common spatial pattern (CSP) on each wave band to extract features with maximal spatial separation [45].

4) Train a set of classifiers for each wave band: linear discriminant analysis (LDA), quadratic discriminant analysis (QDA), support vector machines (SVM), k-nearest neighbours (KNN) and gaussian process (GP). Where one of each kind was trained for each wave band. For example, considering the case of the LDA we would obtain a δ-LDA, θ-LDA, ..., All-LDA; and so on with QDA, KNN, SVM and GP.

5) Perform the multimodal decision with two aggregation functions. First, we fuse all the classifiers from the same kind in the frequency fusion phase. Then, we fuse the outputs of the frequency fusion phase using another aggregation function. So, we first fuse all the LDA classifiers and the same for the remaining classifiers. Then, we aggregate the output for each of these fusions using another aggregation function. The best results in [25] were obtained using Choquet/Sugeno integrals in the frequency fusion phase and an n-dimensional overlap function in the classifier fusion phase.

A visual scheme of the components of EMF can be found in Fig. 2. We used different d-XC integrals in the multimodal decision step of the EMF. We have tested all possible combinations of the d-XC integrals in both phases of the multimodal decision making phase, and studied how the d-XC performs when used alongside n-dimensional overlap functions.

A. Methodology

For our experiment, we have considered the BCI Competition IV 2a dataset, consisting of 4 classes of motor imagery tasks: tongue, foot, left-hand and right-hand, performed by 9 subjects. For each task, 22 EEG channels were collected. A total of 288 trials for each participant was equally distributed among the 4 classes. In the experimental setup, we used 4 out of the 22 channels (C3, C4, CP3, CP4), according to [25]. From each subject, we generated 10 partitions of the 288 trials consisting of 50% for train (144 trials) and 50% for test (144 trials) chosen randomly. Since there are 9 subjects, this produces 90 different datasets. The average accuracy of all partitions was used as evaluation metric.

B. Results

In Table VI, we show the results using the different d-XC integrals in both phases of the decision making. We found that $X\mathfrak{C}_{\delta_1,m}$ used in both phases provided the best results, with an average accuracy of 77.03, followed by the results obtained using $X\mathfrak{C}_{\delta_1,m}$ in the first phase of the fusion and, finally, the results from $X\mathfrak{C}_{\delta_1,m}$.

In Table VII, we have reported the results in the EMF using other existing aggregation functions and the d-XC integrals with n-dimensional overlap functions. We tested some combinations of fuzzy integrals with n-dimensional overlap functions, as they reported very satisfactory results when considering the original EMF. Some classical aggregation functions, such as the arithmetic mean, were also adopted. It was revealed that the best result overall was obtained using the $X\mathfrak{C}_{\delta,m}$ integral with δ_1 in the first phase of the decision making phase, and then the geometric mean in the second phase.

VI. Conclusions

This paper brought a novel approach for generalizing the expanded form of the CI using RDFs, namely the d-XChoquet integral, of which the X-CI standard form based on the

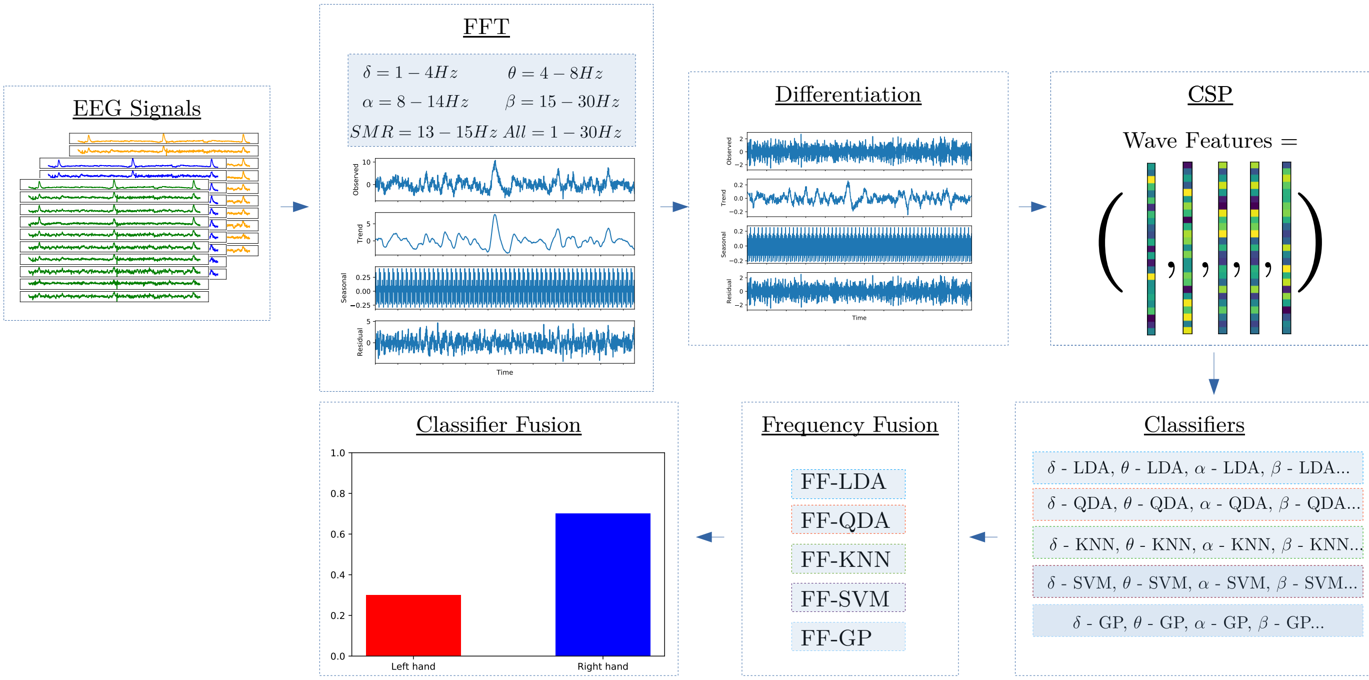

Fig. 2: Visual scheme for the Enhanced Multimodal Fusion BCI framework (Fig. taken from [25] with permission from the authors). **1.** EEG signals are measured and preprocessed. **2.** We compute the FFt over the signals in order to obtain five different wave bands. **3.** We differentiate the signals. **4.** We use CSP to extract a feature vector for each signal. **5.** We train different kinds of classifiers for each wave band. **6.** We fuse the classifiers output in two phases: first we fuse all the classifiers of the same kind, and then we fuse the resulting outputs.

TABLE VI: Average accuracy obtain for all the partitions in the BCI Competition IV 2a dataset using two d-XC integrals in the decision making phase of the EMF framework. The rows correspond to the RDF δ used in the d-XC integral in the first phase, and the columns to the one used in the second phase. The highest value is in **bold**.

$X\mathfrak{C}_{\delta,m}$	δ_0	δ_1	δ_2	δ_3	δ_4	δ_5
δ_0	68.93	68.04	65.54	70.42	68.03	68.26
δ_1	69.84	**77.03**	42.43	31.17	76.72	73.81
δ_2	64.53	58.30	63.32	63.35	61.60	57.81
δ_3	52.37	62.13	35.54	50.02	58.00	62.26
δ_4	70.40	75.97	48.65	46.98	75.74	74.52
δ_5	63.71	67.31	43.66	56.63	67.17	67.41

difference operator is a special case. The definition of the X-CI in terms of restricted dissimilarity functions may enlarge its application to several cases in which the difference in not properly defined and/or causes the degradation of the information quality of the result. Moreover, since one may define different types of restricted dissimilarity functions, this approach offers further flexibility in the choice of the function to use according to the considered application.

Concretely, the theoretical contributions of this paper were the definition of the d-XChoquet integral and the study of the main properties usually required for aggregation functions in applications, such as some kind of monotonicity, $0, 1$-conditions, idempontency and the averaging property. In particular, we analyzed the behaviors of d-XC integrals concerning six different restricted dissimilarity functions. We showed that the

TABLE VII: Average accuracy obtained for all the partitions in the BCI Competition IV 2a dataset using fuzzy and d-XC integrals in the decision-making phase of the EMF framework. We also use the geometric mean in the second fusion phase, as the n-ary overlaps performed best in this part of the decision making in the original results.

Agg. 1	Agg. 2	Avg. Accuracy
Arith. Mean	Arith. Mean	78.38
Choquet Int.	Choquet Int.	77.98
Sugeno Int.	Sugeno Int.	67.60
FG-Sugeno	FG-Sugeno	78.80
Choquet Int.	Geo. mean	82.57
Sugeno Int.	Geo. mean	67.60
FG-Sugeno	Geo. mean	82.84
d-XC with δ_0	Geo. mean	82.91
d-XC with δ_1	**Geo. mean**	**85.55**
d-XC with δ_2	Geo. mean	83.28
d-XC with δ_3	Geo. mean	79.15
d-XC with δ_4	Geo. mean	85.20
d-XC with δ_5	Geo. mean	82.56

range of d-XC integrals may be larger than the unit interval for some restricted dissimilarity functions. Nevertheless, this issue is unremarkable when d-XC integrals are used in a decision-making step.

To showcase the role of our developments in concrete applications, we presented two case studies. First, we showed an application to MCDM, introducing a novel version of the GMC-RTOPSIS decision making method, by using the d-XC integral in the separation measure step of the proposed algorithm. We showed that the standard X-CI using the difference presented the lowest separation value between ranks 1 and 2. The results showed that non standard d-XC integrals based on restricted dissimilarity functions different from the difference may provide a more trustful decision making, guaranteeing no doubts when choosing rank 1 over of rank 2.

Finally, we showed how the d-XC integral performed on the multimodal decision making scheme of the EMF MI-BCI framework. In this case, it was revealed that the combination of a d-XC integral and an n-dimensional overlap function performed better than any of the previous MI-BCI classification results obtained with this framework.

Ongoing work is concerned with the use of d-XC integrals in fuzzy-rule based classification systems (as in [7]) and to image processing, e.g., to edge feature fusion (as in [20]).

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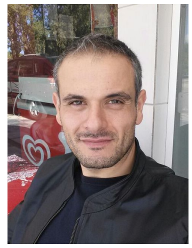

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