Type-$(2, k)$ overlap indices

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***Abstract*—Automatic image detection is one of the most important areas in computing due to its potential application in numerous real-world scenarios. One important tool to deal with that is called *overlap indices*. They were introduced as a procedure to provide the maximum lack of knowledge when comparing two fuzzy objects. They have been successfully applied in the following fields: image processing, fuzzy rule-based systems, decision making and computational brain interfaces.**

This notion of *overlap indices* is also necessary for applications in which type-2 fuzzy sets are required. In this paper we introduce the notion of *type-*$(2, k)$ *overlap index* ($k \in \{0, 1, 2\}$) in the setting of type-2 fuzzy sets. We describe both the reasons that have led to this notion and the relationships that naturally arise among the algebraic underlying structures. Finally, we illustrate how type-$(2, k)$ overlap indices can be employed in the setting of fuzzy rule-based systems when the involved objects are type-2 fuzzy sets.

***Index Terms*—Type-2 fuzzy set, Overlap index, Overlap function**

I. Introduction

Fusion functions [1], in particular aggregation functions [2], [3], [4], are useful tools for applications handling data in the fuzzy context. One special kind of such functions that have been attracted the attention in the literature are the *overlap functions*, introduced by Bustince et al. [5]. Roughly speaking, they can be seen as non necessarily associative bivariate functions (defined over the unit square), which are nondecreasing, continuous, and satisfy appropriate boundary conditions.

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In general, *overlap functions* are measurements of situations in which the lack of knowledge (when someone is comparing two fuzzy objects) is maximum. For instance, when two membership functions follow a distribution similar to the normal distribution, then an overlap function is useful to measure the degree of overlapping between them. This measurement can be read as the expert's doubt about the belonging of each point with respect to each membership function.

The nice properties that the class of overlap functions presents, such as the closeness with respect to the convex sum and the aggregation by internal generalized composition [6], [7], allow its higher applicability, in comparison with other classes of conjunctive aggregation functions (e.g., t-norms [8]). Besides that, overlap functions were shown to satisfy several interesting properties, some of them necessary for different applications, as properly studied by, for example, Bustince et al. [9], Bedregal et al. [10], Dimuro et al. [11], [12], [13], Qiao et al. [14], [15], [16], [17], [18], Wang and Hu [19], Zhou and Yan [20] and Zhu et al. [21].

Then, since their introduction, overlap functions have been successfully applied in many fields like: image processing [22], [23], decision making [24], [25], computational brain interfaces [26], forest fire detection [27], wavelet-fuzzy power quality diagnosis system [28], fuzzy community detection [29], social networks [30] and classification [31], [32], [33], [34], [35], [36].

One of the most interesting contexts in which overlap functions have been used is that of interpolative fuzzy systems [37], which can be interpreted as particular cases of fuzzy rule-based systems. In this class of systems there is a finite set of rules expressed in terms of fuzzy sets. Using them, it is possible to deduce a consequence (the conclusion) from an initial premise (the fact), which are also expressed as fuzzy sets. Usually, this process requires the application of a consistency index, firstly, between the rules and the fact and, later, between the resulting value and the consequent.

The *consistency index*, introduced by Zadeh in [38] as a reasonable extension of the *Boolean overlap index* when the universe of discourse is finite, was improved by several authors (e.g., Dubois et al. [39] and Pal and Pal [40]) to be successfully applied in the setting of interpolative fuzzy systems. In order to overcome some contradiction that naturally appeared by using previous definitions, these studies highlighted the necessity of introducing the notion of *overlap index* (see [37]) as a function that associates a real number on the interval $[0, 1]$ to each pair of fuzzy sets on the (finite) referential set and satisfying certain conditions (see also [7], [24], [41], [42]). Such kind of indices allowed the development of inference algorithms for interpolative systems according to some of the properties

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proposed by Fukami et al. in [43] and, almost simultaneously, by Baldwin and Pilsworth in [44].

The algebraic constructions we have commented until now involve real numbers or, at most, fuzzy sets (usually, fuzzy numbers of the real line). This last kind of mathematical objects associates a real number belonging to $[0,1]$ to each point of the universe of discourse. However, to determine a unique and precise real number that perfectly represents the ambiguity of the considered point could be interpreted as a very strong constraint [45]. In some contexts, taking into account the intrinsic imprecise nature of the elements of the referential set, it is better to associate them fuzzy sets (usually, fuzzy numbers whose supports are included on $[0,1]$) rather than real numbers (for instance, the opinion of an expert about some characteristic of a concrete wine). It naturally appears the notion of *type-2 fuzzy set*, which has been successfully applied in several contexts, like the works by Mendel and co-authors (e.g., on forecasting time-series [46] and pattern recognition [47]), by Garibaldi and co-authors (e.g., on decision making [48]), by Hagras and co-authors (e.g., on ambient intelligence [49]), and by Wagner and co-authors (e.g., on multiobjective optimization [50]).

The objectives of this paper are the following ones:

1. To introduce the notion of type-$(2,k)$-overlap index (with $k\in\{0,1,2\}$) as an extension of previous consistency indices to the setting of type-2 fuzzy sets.
2. To study the relation between these new type-$(2,k)$-overlap indices and the overlap indices that were already defined for type-1 fuzzy sets and for real numbers.

The structure of this paper is as follows. In Section II, we present some preliminary definitions and results that are necessary for the paper. Section III is devoted to the new concept of type-$(2,k)$ overlap index. In Section IV we discuss an illustrative example on the possible application of the type-$(2,k)$ overlap indices to interpolative fuzzy rule systems with type-2 fuzzy sets. We finish with some conclusions, future research lines and some relevant references.

II. Preliminaries

In this section we describe the basic algebraic background that we will use.

A. Aggregation functions and overlap functions

Let $\mathbb{R}$ denote the set of all real numbers and let $[0,1]$ be the real, closed and bounded interval whose bounds are 0 and 1. Henceforth, X and Y denote non-empty sets and $f:X\to Y$ represents a function whose domain is X and whose codomain is Y.

A function $T:[0,1]\times[0,1]\to[0,1]$:

- is *associative* if $T(t,T(s,r))=T(T(t,s),r)$ for all $t,s,r\in[0,1]$;
- is *commutative* if $T(t,s)=T(s,t)$ for all $t,s\in[0,1]$;
- is *increasing* whenever $T(t_1,s_1)\le T(t_2,s_2)$ for all $t_1,t_2,s_1,s_2\in[0,1]$ such that $t_1\le t_2$ and $s_1\le s_2$;
- is *continuous* if it is continuous with respect to the usual (Euclidean) topology;
- is *averaging* if $\min\{t,s\}\le T(t,s)\le\max\{t,s\}$ for all $t,s\in[0,1]$;
- is *0-dependent* if $T^{-1}(\{0\})=\{(0,0)\}$, that is, $T(t,s)=0\Leftrightarrow t=s=0$;
- is *normal* (or *1-dependent*) if $T^{-1}(\{1\})=\{(1,1)\}$, that is, $T(t,s)=1\Leftrightarrow t=s=1$.

Given $n\in\mathbb{N}$, a function $T:[0,1]^n\to[0,1]$ is:

- *increasing* if $T(t_1,t_2,\ldots,t_n)\le T(s_1,s_2,\ldots,s_n)$ for all $(t_1,t_2,\ldots,t_n),(s_1,s_2,\ldots,s_n)\in[0,1]^n$ such that $t_j\le s_j$ for all $j\in\{1,2,\ldots,n\}$;
- *symmetric* if $T(t_{\sigma(1)},t_{\sigma(2)},\ldots,t_{\sigma(n)})=T(t_1,t_2,\ldots,t_n)$ for all $(t_1,t_2,\ldots,t_n)\in[0,1]^n$ and all permutations $\sigma:\{1,2,\ldots,n\}\to\{1,2,\ldots,n\}$;
- *conjunctive* if $T\le\min$, that is, $T(t_1,t_2,\ldots,t_n)\le\min\{t_1,t_2,\ldots,t_n\}$ for all $(t_1,t_2,\ldots,t_n)\in[0,1]^n$;
- *disjunctive* if $T\ge\max$, that is, $T(t_1,t_2,\ldots,t_n)\ge\max\{t_1,t_2,\ldots,t_n\}$ for all $(t_1,t_2,\ldots,t_n)\in[0,1]^n$;
- *averaging* if $\min\{t_1,t_2,\ldots,t_n\}\le T(t_1,t_2,\ldots,t_n)\le\max\{t_1,t_2,\ldots,t_n\}$ for all $(t_1,t_2,\ldots,t_n)\in[0,1]^n$;

Notice that henceforth we will use the term *increasing* for *non-decreasing* functions. Also notice that *commutativity* and *symmetry* are the same for functions involving two arguments.

Definition 1: [2] An *aggregation function* is an increasing mapping $T:[0,1]^n\to[0,1]$ such that $T(0,0,\ldots,0)=0$ and $T(1,1,\ldots,1)=1$.

Definition 2: [8] A *triangular norm* (or *t-norm* is an associative, commutative and increasing function $T:[0,1]\times[0,1]\to[0,1]$ with neutral element 1 (that is, $T(1,s)=s$ for all $s\in[0,1]$).

Examples of t-norms are the following ones:

$$T_{\min}(t,s)=\min\{t,s\},\quad T_{\text{prod}}(t,s)=t\cdot s,$$
$$T_{\text{L}}(t,s)=\max(t+s-1,0).$$

Remark 3: Each t-norm is an aggregation function. In fact, it can be symmetrically extended to any number of arguments.

Any t-norm T satisfies $0\le T(0,s)\le T(0,1)=0$, so 0 is a *zero element for* T, that is, $T(0,s)=0$ for all $s\in[0,1]$. The t-norms $T_{\min}$ and T_{prod} are especially useful in practical examples. They satisfy the following properties:

- $T(t,s)=0\ \Leftrightarrow\ t\cdot s=0\ \Leftrightarrow\ \min\{t,s\}=0\ \Leftrightarrow\ [\ t=0\ \text{ or }\ s=0\ ];$
- $T(t,s)=1\quad\Leftrightarrow\quad t\cdot s=1\quad\Leftrightarrow\quad t=s=1.$

The first property is not satisfied by all t-norms (for instance, the t-norm T_L satisfies $T_L(t,s)=0$ for each $t,s\in[0,1]$ such that $t+s<1$). The second one is. Such functions have to be continuous.Their properties inspired the following notion.

Definition 4: [5] An *overlap function* is a mapping $G:[0,1]\times[0,1]\to[0,1]$ such that:

(G_1) G is commutative;
(G_2) G is increasing;
(G_3) $G(t,s)=0$ if and only if $t\cdot s=0$ (that is, $t=0$ or $s=0$);
(G_4) G is normal (or 1-dependent);
(G_5) G is continuous.

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B. Fuzzy sets of type 0, 1 or 2

Given a non-empty set X, a *fuzzy set* (or *type*-1 *fuzzy set*) is a function $A : X \to [0,1]$. Sometimes, it is written as the set $\{(x, \mu_A(x)) : x \in X\}$, where $\mu_A : X \to [0,1]$ is called its *membership function* (in fact, $\mu_A(x) = A(x)$ for all $x \in X$). The family of all fuzzy sets on X is denoted here by both $\mathrm{FS}(X)$ and $\mathrm{FS}_1(X)$.

In 1971, Zadeh [51] settled that the problem of estimating the membership degrees of the elements in a fuzzy set is related to abstraction, a problem that plays a central role in pattern recognition. Therefore, the determination of the membership degree of each element to the set is the biggest problem for applying fuzzy sets theory. Taking these considerations into account, the concept of type-2 fuzzy set was given as follows: a *type-2 fuzzy set* is a fuzzy set for which the membership degrees are expressed as fuzzy sets in $[0,1]$. Mathematically, we have the following definition.

Definition 5: A *type-2 fuzzy set* is a mapping $\mathcal{A} : X \to \mathrm{FS}([0,1])$. We denote by $\mathrm{FS}_2(X)$ the family of all type-2 fuzzy sets on X.

In practice, if $\mathcal{A}$ is a type-2 fuzzy set and $x \in X$, then $\mathcal{A}(x) : [0,1] \to [0,1]$ is a fuzzy set on $[0,1]$ (it is usually called an *slice* of $\mathcal{A}$, see [52]). Hence, it is a function whose domain and codomain is the interval $[0,1]$. This means that, algebraically, $\mathrm{FS}_2(X)$ coincides to $\mathrm{FS}_1(X \times [0,1])$ by the identification:

$$\mathcal{A}(x)(t) = \mathcal{A}(x,t) \quad \text{for all } x \in X \text{ and all } t \in [0,1].$$

This point of view was adopted in [53, Definition 6.1] and it shows that $\mathrm{FS}(X)$ and $\mathrm{FS}_2(X) \equiv \mathrm{FS}(X \times [0,1])$ are very different in nature: for instance, the universe X can be a finite set (in this case, we denote $X = U = \{u_1, u_2, \ldots, u_n\}$), but $X \times [0,1]$ is always an infinite set (in fact, it is not enumerable).

¿From our point of view and for the sake of clarity, it is very important the notation we will employ, so we explicitly declare it right now. Let X be a non-empty set.

- Elements in X will be denoted by $x, y \in X$.
- If X is finite, we will denote it by U, and elements in U are denoted by $u, u_i \in U$.
- Scalars in $[0,1]$ are denoted by $t, s, r \in [0,1]$.
- Type-1 fuzzy sets in $\mathrm{FS}(X)$ will be denoted by $A, B, A_i \in \mathrm{FS}(X)$.
- Type-2 fuzzy sets in $\mathrm{FS}_2(X)$ are denoted using calligraphic letters by $\mathcal{A}, \mathcal{B}, \mathcal{A}_i \in \mathrm{FS}_2(X)$.

Definition 6: A real number $r \in [0,1]$ will be called here *type*-0 *fuzzy set*. For $k \in \{0,1,2\}$, a *type-k fuzzy set on* X is any element in $\mathrm{FS}_k(X)$, where:

$$\begin{aligned}&\mathrm{FS}_0(X) = [0,1] \text{ is the set of all type-0 fuzzy sets,}\\ &\mathrm{FS}_1(X) = \mathrm{FS}(X) \text{ is the set of all type-1 fuzzy sets, and}\\ &\mathrm{FS}_2(X) \text{ is the set of all type-2 fuzzy sets.}\end{aligned}$$

Given $\mathcal{A} \in \mathrm{FS}_2(X)$, the type-2 fuzzy set $\mathcal{A}_c \in \mathrm{FS}_2(X)$, defined by $\mathcal{A}_c(x)(t) = 1 - \mathcal{A}(x)(t)$ for all $x \in X$ and all $t \in [0,1]$, is said to be *the complementary set of* $\mathcal{A}$.

Given a bounded function $f : Y \to \mathbb{R}$, we denote

$$\begin{aligned}\sup f &= \sup\left(\{\, f(y) : y \in Y \,\}\right) \in \mathbb{R} \quad \text{and}\\ \inf f &= \inf\left(\{\, f(y) : y \in Y \,\}\right) \in \mathbb{R}.\end{aligned}$$

In particular, for all $A \in \mathrm{FS}(X)$,

$$\begin{aligned}\sup A &= \sup\left(\{\, A(x) : x \in X \,\}\right) \quad \text{and}\\ \inf A &= \inf\left(\{\, A(x) : x \in X \,\}\right),\end{aligned}$$

and for all $\mathcal{A} \in \mathrm{FS}_2(X)$,

$$\begin{aligned}\sup \mathcal{A} &= \sup\left(\{\, \mathcal{A}(x)(t) : x \in X,\, t \in [0,1] \,\}\right) \quad \text{and}\\ \inf \mathcal{A} &= \inf\left(\{\, \mathcal{A}(x)(t) : x \in X,\, t \in [0,1] \,\}\right).\end{aligned}$$

We have to distinguish between $\sup \mathcal{A} \in [0,1]$ and $\widetilde{\sup}\mathcal{A} \in \mathrm{FS}(X)$, where

$$\left(\widetilde{\sup}\mathcal{A}\right)(x) = \sup\left(\mathcal{A}(x)\right) \text{ for all } x \in X.$$

Given a mapping $T : [0,1]^n \to [0,1]$ and $\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_n \in \mathrm{FS}_2(X)$, we denote by $T(\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_n)$ the type-2 fuzzy set on X defined, for all $x \in X$ and all $t \in [0,1]$, by:

$$\begin{aligned}&T\left(\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_n\right)(x)(t)\\ &\quad = T\left(\mathcal{A}_1(x)(t), \mathcal{A}_2(x)(t), \ldots, \mathcal{A}_n(x)(t)\right).\end{aligned}$$

Given $t \in [0,1]$, we respectively denote by F^t and $\mathcal{F}^t$ the constant type-1 and -2 fuzzy sets:

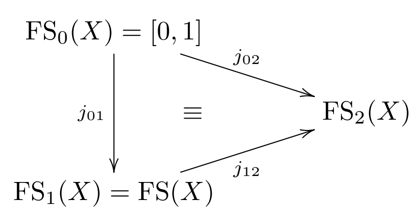
$$\begin{aligned}&F^t \in \mathrm{FS}_1(X), \quad F^t(x) = t \text{ for all } x \in X;\\ &\mathcal{F}^t \in \mathrm{FS}_2(X), \quad \mathcal{F}^t(x)(s) = t \text{ for all } x \in X \text{ and } s \in [0,1].\end{aligned}$$

Given $A \in \mathrm{FS}_1(X)$, we denote:

$$\mathcal{F}^A \in \mathrm{FS}_2(X),\ \mathcal{F}^A(x)(t) = A(x) \text{ for all } x \in X \text{ and } t \in [0,1].$$

There are embeddings and a commutative diagram:

$$\begin{aligned}&j_{01} : \mathrm{FS}_0(X) \longrightarrow \mathrm{FS}_1(X), &&j_{01}(t) = F^t\\ &j_{02} : \mathrm{FS}_0(X) \longrightarrow \mathrm{FS}_2(X), &&j_{02}(t) = \mathcal{F}^t\\ &j_{12} : \mathrm{FS}_1(X) \longrightarrow \mathrm{FS}_2(X), &&j_{12}(A) = \mathcal{F}^A\end{aligned}$$



$j_{12} \circ j_{01} = j_{02}$

The usual partial order $\leq$ in $\mathrm{FS}_0(X) = [0,1]$ can be extended to $\mathrm{FS}_1(X)$ and to $\mathrm{FS}_2(X)$ by:

- $A \leq B$ if $A(x) \leq B(x)$ for all $x \in X$;
- $\mathcal{A} \leq \mathcal{B}$ if $\mathcal{A}(x) \leq \mathcal{B}(x)$ for all $x \in X$, that is, $\mathcal{A}(x)(t) \leq \mathcal{B}(x)(t)$ for all $x \in X$ and all $t \in [0,1]$.

It is only a *total order* ($t \leq s$ or $s \leq t$, for all $t, s \in \mathrm{FS}_0(X)$) on $\mathrm{FS}_0(X)$.

The injections $j_{01} : (\mathrm{FS}_0(X), \leq) \to (\mathrm{FS}_1(X), \leq)$, $j_{02} : (\mathrm{FS}_0(X), \leq) \to (\mathrm{FS}_2(X), \leq)$ and $j_{12} : (\mathrm{FS}_1(X), \leq) \to (\mathrm{FS}_2(X), \leq)$ are *order preserving*.

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Each fuzzy set on $[0,1]$ can be seen as a type-2 fuzzy set on any set X with constant slices in the following way.

$$j_X : \mathrm{FS}([0,1]) \to \mathrm{FS}_2(X),$$
$$j_X(A)(x) = A \quad \text{for all } A \in \mathrm{FS}([0,1]) \text{ and all } x \in X. \quad (1)$$

Notice that j_X is injective and order-preserving, and by composing $X \xrightarrow{\mathcal{A}} \mathrm{FS}([0,1]) \xrightarrow{j_X} \mathrm{FS}_2(X)$, it satisfies, for all $\mathcal{A} \in \mathrm{FS}_2(X)$, all $x \in X$ and all $t \in [0,1]$,

$$[(j_X \circ \mathcal{A})(x)](t) = \mathcal{A}(x)(t)$$

because $[(j_X \circ \mathcal{A})(x)](t) = [j_X(\mathcal{A}(x))](t) = \mathcal{A}(x)(t)$.

The set $\mathrm{FS}_k(X)$ has absolute minimum and maximum elements w.r.t. $\leq$, which will be denoted by:

$$\mathfrak{m}_0 = 0 \quad \text{and} \quad \mathfrak{M}_0 = 1, \quad \text{if } k = 0,$$
$$\mathfrak{m}_1 = F^0 \quad \text{and} \quad \mathfrak{M}_1 = F^1, \quad \text{if } k = 1,$$
$$\mathfrak{m}_2 = \mathcal{F}^0 \quad \text{and} \quad \mathfrak{M}_2 = \mathcal{F}^1, \quad \text{if } k = 2.$$

Notice that $\mathfrak{m}_k \leq \boldsymbol{x} \leq \mathfrak{M}_k$ for all $\boldsymbol{x} \in \mathrm{FS}_k(X)$. Therefore, $(\mathrm{FS}_k(X), \leq)$ is a bounded partially ordered set.

Since $(\mathrm{FS}_2(X), \leq)$ is a bounded lattice with absolute minimum $\mathfrak{m}_2 = \mathcal{F}^0$ and absolute maximum $\mathfrak{M}_2 = \mathcal{F}^1$, then an aggregation function on $(\mathrm{FS}_2(X), \leq)$ is an increasing mapping $\mathcal{T} : \mathrm{FS}_2(X)^p \to \mathrm{FS}_2(X)$ such that $\mathcal{T}(\mathfrak{m}_2, \mathfrak{m}_2, \ldots, \mathfrak{m}_2) = \mathfrak{m}_2$ and $\mathcal{T}(\mathfrak{M}_2, \mathfrak{M}_2, \ldots, \mathfrak{M}_2) = \mathfrak{M}_2$.

Proposition 7: Each aggregation function $T : [0,1]^p \to [0,1]$ induces an aggregation function $\mathcal{T}_T : \mathrm{FS}_2(X)^p \to \mathrm{FS}_2(X)$ defined by

$$\mathcal{T}_T(\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_p)(x)(t)$$
$$= T(\mathcal{A}_1(x)(t), \mathcal{A}_2(x)(t), \ldots, \mathcal{A}_p(x)(t))$$

for all $\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_p \in \mathrm{FS}_2(U)$, all $x \in X$ and all $t \in [0,1]$.

Proof. Clearly, $\mathcal{T}_T$ is increasing, $\mathcal{T}_T(\mathcal{F}^0, \mathcal{F}^0, \ldots, \mathcal{F}^0) = \mathcal{F}^0$ and $\mathcal{T}_T(\mathcal{F}^1, \mathcal{F}^1, \ldots, \mathcal{F}^1) = \mathcal{F}^1$. ■

Definition 8: We say that:

- two type-0 fuzzy sets $t, s \in [0,1]$ are *completely disjoint* if $t \cdot s = 0$;
- two type-1 fuzzy sets $A, B \in \mathrm{FS}(X)$ are *completely disjoint* if $A(x) \cdot B(x) = 0$ for all $x \in X$;
- two type-2 fuzzy sets $\mathcal{A}, \mathcal{B} \in \mathrm{FS}_2(X)$ are *completely disjoint* if $\mathcal{A}(x)(t) \cdot \mathcal{B}(x)(t) = 0$ for all $x \in X$ and all $t \in [0,1]$.

Notice that:

- $A, B \in \mathrm{FS}_1(X)$ are completely disjoint if and only if $A(x)$ and $B(x)$ are completely disjoint for all $x \in X$;
- $\mathcal{A}, \mathcal{B} \in \mathrm{FS}_2(X)$ are completely disjoint if and only if $\mathcal{A}(x)$ and $\mathcal{B}(x)$ are completely disjoint for all $x \in X$, which is equivalent to say that $\mathcal{A}(x)(t)$ and $\mathcal{B}(x)(t)$ are completely disjoint for all $x \in X$ and all $t \in [0,1]$.

III. Type-$(2,k)$ overlap indices

A. Discussion about the notion of type-$(2,k)$ overlap index

Triangular norms were introduced in order to extend the triangle inequality to Menger *statistical metric spaces* (see [54]) and, later, to *probabilistic metric spaces* (see [55]). Their properties were intensively studied in order to adapt them to any practical settings. Each context needs distinct properties of the involved bivariate functions $T : [0,1] \times [0,1] \to [0,1]$.

Taking into account more flexibility in applications to several contexts, *overlap functions* were introduced by Bustince *et al* in [5]. The main advantage of this kind of functions is the large list of interesting properties that can be deduced from the five axioms (G_1)-(G_5) and the family of distinct examples that can be considered (see the discussion we have done in the Introduction). Many of such properties are also verified by *quasi overlap functions*, which are functions satisfying (G_1)-(G_4) (see [56]). Nevertheless, two comments must be done. On the one hand, the continuity property was introduced taking into account the possible application of overlap functions in the image processing setting, although some applications in other contexts do not require it (see the discussion by Asmus et al. [35], [36], [57]). On the other hand, the *normality* condition given by (G_4) may be very restrictive in some cases, as properly discussed by De Miguel et al. [58]. Having in mind that condition (G_4) can be decomposed into the following two properties:

$$(G_{4,1}) \quad G(1,1) = 1,$$
$$(G_{4,2}) \quad \text{if } G(t,s) = 1, \text{ then } t = s = 1,$$

García-Jiménez et al. [37] decided to introduce the notion of *overlap index* in the setting of a finite set $U = \{u_1, u_2, \ldots, u_n\}$ as a function $O : \mathrm{FS}(U) \times \mathrm{FS}(U) \to [0,1]$ satisfying the following properties (for practical reasons, we reorder their axioms).

- (O_1) O is symmetric, i.e. $O(A,B) = O(B,A)$;
- (O_2) O is increasing;
- (O_3) $O(A,B) = 0$ if and only if A and B are completely disjoint.

Under (O_1), the increasing monotonicity is equivalent to $O(A_1, B) \leq O(A_2, B)$ for all $A_1, A_2, B \in \mathrm{FS}(U)$ whenever $A_1 \leq A_2$. Clearly, properties (O_1)-(O_3) generalize properties (G_1)-(G_3) associated to an overlap function (recall Definition 4). Condition (G_4) was weakened by condition $(G_{4,1})$ by means of the following concept:

An overlap index $O : \mathrm{FS}(U) \times \mathrm{FS}(U) \to [0,1]$ is *normal* if it satisfies:

- (O_4') if $A, B \in \mathrm{FS}(U)$ are such that there exists $u_{j_0} \in U$ verifying $A(u_{j_0}) = B(u_{j_0}) = 1$, then $O(A,B) = 1$.

We highlight two facts. On the one hand, as the converse is not necessarily true, we have no additional information about A and B when $O(A,B) = 1$. We only know that, if O is normal, then $O(A, F^1) = 1$ for all $A \in \mathrm{FS}(U)$ such that $A(u_{j_0}) = 1$ for some $u_{j_0} \in U$ (in particular, $O(F^1, F^1) = 1$).

¿From our point of view, in order to establish a minimal collection of consistent axioms, we only consider the minimal quantity of properties that we will need to develop practical examples, so we are going to introduce more general functions than overlap functions by avoiding some axioms.

Definition 9: We say that a mapping $G : [0,1] \times [0,1] \to [0,1]$ is:

- a *lower overlap function* if it satisfies (G_1), (G_2) and (G_3);

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- a *normal lower overlap function* if it satisfies (G_1), (G_2), (G_3) and $(G_{4,1})$.

Example 10: Given $\lambda_1, \lambda_2, \lambda_3 \in (0,1]$ three real numbers such that $0 < \lambda_1 \leq \lambda_2 \leq \lambda_3 \leq 1$, let $G : [0,1] \times [0,1] \to [0,1]$ be the function defined by:

$$G(t,s) = \begin{cases} 0, & \text{if} \quad t = 0 \quad \text{or} \quad s = 0, \\ \lambda_1, & \text{if} \quad t, s \in (0,1), \\ \lambda_2, & \text{if} \quad 0 < s < t = 1 \quad \text{or} \quad 0 < t < s = 1, \\ \lambda_3, & \text{if} \quad t = s = 1. \end{cases}$$

Then G is a lower overlap function. Furthermore, it is normal if and only if $\lambda_3 = 1$. In fact, it satisfies the condition $(G_{4,2})$ if and only if $\lambda_2 < \lambda_3 = 1$. Notice that G is never continuous because $0 < \lambda_1$, so it is not an overlap function.

Example 11: Given $\lambda \in (0,1]$, let $G : [0,1] \times [0,1] \to [0,1]$ be the function defined, for each $t, s \in [0,1]$, by:

$$G(t,s) = \lambda\, t\, s.$$

Then G is a continuous lower overlap function. Notice that it is normal if and only if $\lambda = 1$. In such a case, it also satisfies the property $(G_{4,2})$, that is, it is an overlap function.

Example 12: Given $\lambda \in (0, 1/2]$ and $a, b \in (0, +\infty)$, let $G : [0,1] \times [0,1] \to [0,1]$ be the function defined, for each $t, s \in [0,1]$, by:

$$G(t,s) = \lambda\, t^a\, s^a (t^b + s^b).$$

Then G is a lower overlap function. In fact, it is continuous. Furthermore, it is normal if and only if $\lambda = 1/2$. In such a case, it is an overlap function.

Example 13: Given two real numbers $\lambda_1, \lambda_2 \in (0,1]$ such that $0 < \lambda_1 \leq \lambda_2 \leq 1$, let $G : [0,1] \times [0,1] \to [0,1]$ be the function defined, for each $t, s \in [0,1]$, by:

$$G(t,s) = \begin{cases} \lambda_1\, t^3 s^3, & \text{if } t + s \leq 1, \\ \lambda_2\, t^2 s^2, & \text{if } t + s > 1. \end{cases}$$

Then G is a lower overlap function. Furthermore, it is normal if and only if $\lambda_2 = 1$. Notice that G is not continuous along the points of the open segment $\{ (t, 1-t),\, t \in (0,1) \}$, so it is not an overlap function.

Example 14: Let $G : [0,1] \times [0,1] \to [0,1]$ be the function defined, for each $t, s \in [0,1]$, by:

$$G(t,s) = \begin{cases} \dfrac{2 - t + 2s - \sqrt{4(1-t) + (t-2s)^2}}{2}, & \text{if } t \geq s, \\[2ex] \dfrac{2 - s + 2t - \sqrt{4(1-s) + (s-2t)^2}}{2}, & \text{if } t < s. \end{cases}$$

Then G is a continuous, normal lower overlap function (see Figure 1). However, it does not satisfy the condition $(G_{4,2})$ because

$$G(t,s) = 1 \quad \Leftrightarrow \quad (\ t = 1 \text{ and } s \in [0.5, 1]\ ) \quad \text{or} \quad (\ s = 1 \text{ and } t \in [0.5, 1]\ ).$$

Notice that $G(t,t) = t$ for all $t \in [0,1]$.

Taking into account the previous comments and examples, we consider that the most natural definition for *indices* is the following one. Notice that we employ the minimum $\mathfrak{m}_k$ and the maximum $\mathfrak{M}_k$ of the bounded lattice $(\mathrm{FS}_k(X), \leq)$.

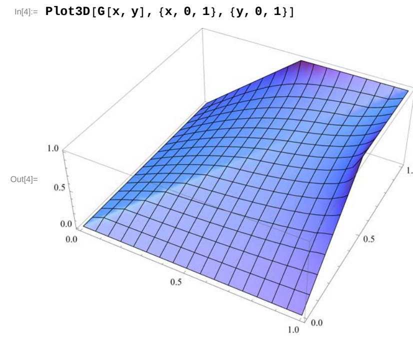

Figure 1: Graphic representation of the function of Example 14.

Definition 15: Given $k \in \{0,1,2\}$, a *type-$(2,k)$ overlap index* is a mapping:

$$\theta : \mathrm{FS}_2(X) \times \mathrm{FS}_2(X) \to \mathrm{FS}_k(X)$$

such that:

θ_1) θ is symmetric, that is, $\theta(\mathcal{A}, \mathcal{B}) = \theta(\mathcal{B}, \mathcal{A})$ for all $\mathcal{A}, \mathcal{B} \in \mathrm{FS}_2(X)$.

θ_2) θ is increasing, that is, if $\mathcal{A} \leq \mathcal{B}$ then $\theta(\mathcal{A}, \mathcal{C}) \leq \theta(\mathcal{B}, \mathcal{C})$ for all $\mathcal{C} \in \mathrm{FS}_2(X)$.

θ_3) $\theta(\mathcal{A}, \mathcal{B}) = \mathfrak{m}_k$ if and only if $\mathcal{A}$ and $\mathcal{B}$ are completely disjoint.

A type-$(2,k)$ overlap index θ is *normal* if it also satisfies:

θ_4) If $\mathcal{A}, \mathcal{B} \in \mathrm{FS}_2(X)$ are such that for all $x \in X$ there is $t_x \in [0,1]$ satisfying $\mathcal{A}(x)(t_x) = \mathcal{B}(x)(t_x) = 1$, then $\theta(\mathcal{A}, \mathcal{B}) = \mathfrak{M}_k$.

Remark 16: As we have commented, axiom (θ_4) is a reasonable generalization of condition (O_4') to the setting of type-2 fuzzy sets. However, other possibilities can also be taken into account in order to extend condition $(G_{4,1})$. We highlight the following two alternative conditions.

θ_4') If $\mathcal{A}, \mathcal{B} \in \mathrm{FS}_2(X)$ are such that there are $x_0 \in X$ and $t_0 \in [0,1]$ verifying $\mathcal{A}(x_0)(t_0) = \mathcal{B}(x_0)(t_0) = 1$, then $\theta(\mathcal{A}, \mathcal{B}) = \mathfrak{M}_k$.

θ_4'') $\theta(\mathcal{F}^1, \mathcal{F}^1) = \theta(\mathfrak{M}_2, \mathfrak{M}_2) = \mathfrak{M}_k$.

These definitions lead to distinct notions than we are going to handle, but the reader can also consider one of them depending on his/her own interests.

For the sake of simplicity,

- type-$(2,0)$ overlap indices will be denoted by $i : \mathrm{FS}_2(X) \times \mathrm{FS}_2(X) \to [0,1]$;
- type-$(2,1)$ overlap indices will be denoted by $I : \mathrm{FS}_2(X) \times \mathrm{FS}_2(X) \to \mathrm{FS}(X)$;
- type-$(2,2)$ overlap indices will be denoted by $\mathcal{I} : \mathrm{FS}_2(X) \times \mathrm{FS}_2(X) \to \mathrm{FS}_2(X)$;
- type-$(1,0)$ overlap indices will be denoted by $o : \mathrm{FS}(X) \times \mathrm{FS}(X) \to [0,1]$;
- type-$(1,1)$ overlap indices will be denoted by $O : \mathrm{FS}(X) \times \mathrm{FS}(X) \to \mathrm{FS}(X)$;
- type-$(0,0)$ overlap indices will be denoted by $f : [0,1] \times [0,1] \to [0,1]$.

The notions of type-$(1,0)$, type-$(1,1)$ and type-$(0,0)$ overlap indices, that are not encompassed by Definition 15, can

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be similarly posed on the corresponding families of fuzzy sets (for instance, a type-$(0,0)$ overlap index is a lower overlap function).

Example 17: Let $\theta : \mathrm{FS}_2(X) \times \mathrm{FS}_2(X) \to \mathrm{FS}(X)$ be the mapping defined, for all $\mathcal{A}, \mathcal{B} \in \mathrm{FS}_2(X)$, by:

$$\theta(\mathcal{A},\mathcal{B})(x) = \sup_{t\in[0,1]} \Big\{ \mathcal{A}(x)(t) \cdot \mathcal{B}(x)(t) \Big\}$$

for all $x \in X$. Then θ is a normal type-$(2,1)$ overlap index.

Example 18: Let $\theta : \mathrm{FS}_2(X) \times \mathrm{FS}_2(X) \to \mathrm{FS}(X)$ be the mapping defined, for all $\mathcal{A}, \mathcal{B} \in \mathrm{FS}_2(X)$, by:

$$\theta(\mathcal{A},\mathcal{B})(x) = \sup_{t\in[0,1]} \frac{\tan\left(\mathcal{A}(x)(t) \cdot \mathcal{B}(x)(t)\right)}{\tan 1}$$

for all $x \in X$. Then θ is a normal type-$(2,1)$ overlap index.

Example 19: Let $\theta : \mathrm{FS}_2(X) \times \mathrm{FS}_2(X) \to \mathrm{FS}(X)$ be the mapping defined, for all $\mathcal{A}, \mathcal{B} \in \mathrm{FS}_2(X)$, by:

$$\theta(\mathcal{A},\mathcal{B})(x) = \sup_{t\in[0,1]} \frac{1}{3\sin 1} \Big[\sin\left(\mathcal{A}(x)(t) \cdot \mathcal{B}(x)(t)\right) \cdot \left(1 + \mathcal{A}(x)(t)^2 + \mathcal{B}(x)(t)^2\right) \Big]$$

for all $x \in X$. Then θ is a normal type-$(2,1)$ overlap index.

Remark 20: A mapping $T : [0,1] \times [0,1] \to [0,1]$ is a type-$(0,0)$ overlap index if and only if it is a lower overlap function. Under this point of view, lower overlap functions are the seeds of overlap indices theory as it was introduced in [5]. In fact, lower overlap functions are very useful tools in order to construct many examples of high order indices. In this sense, high order indices have also been called *overlap indices* as the authors used in [37].

B. Type-$(2,0)$ indices

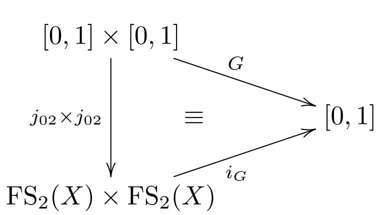
In the following result we introduce an extensive family of type-$(2,0)$ overlap indices.

Lemma 21: Given a lower overlap function $G : [0,1] \times [0,1] \to [0,1]$, let be the function $i_G : FS_2(X) \times FS_2(X) \to [0,1]$, s.t.

$$i_G(\mathcal{A},\mathcal{B}) = \sup\left(\{\, G(\mathcal{A}(x)(t), \mathcal{B}(x)(t)) : x \in X,\, t \in [0,1] \,\}\right). \quad (2)$$

Then i_G is a type-$(2,0)$ overlap index satisfying the following properties.

1) If G is normal, then i_G is also normal.
2) If $G \le \max$, then $i_G(\mathcal{A},\mathcal{B}) \le \max\{\sup\mathcal{A}, \sup\mathcal{B}\}$ for all $\mathcal{A},\mathcal{B} \in \mathrm{FS}_2(X)$. In this case, $i_G(\mathcal{A},\mathcal{A}) \le \sup\mathcal{A}$.
3) If $G(t_1,t_2) = s$, then $i_G(\mathcal{F}^{t_1}, \mathcal{F}^{t_2}) = s$. In particular, the following diagram commutes:



Proof. As G is symmetric, then i_G is also. Furthermore,

$$\begin{aligned} & i_G(\mathcal{A},\mathcal{B}) = 0 \\ \Leftrightarrow\ & [\, G(\mathcal{A}(x)(t), \mathcal{B}(x)(t)) = 0 \text{ for all } x \in X \text{ and } t \in [0,1] \,] \\ \Leftrightarrow\ & [\, \mathcal{A}(x)(t) \cdot \mathcal{B}(x)(t) = 0 \text{ for all } x \in X \text{ and } t \in [0,1] \,] \\ \Leftrightarrow\ & \ \mathcal{A} \text{ and } \mathcal{B} \text{ are completely disjoint.} \end{aligned}$$

If $\mathcal{A},\mathcal{B} \in \mathrm{FS}_2(X)$ are such that $\mathcal{A} \le \mathcal{B}$, then $\mathcal{A}(x)(t) \le \mathcal{B}(x)(t)$ for all $x \in X$ and all $t \in [0,1]$, so $G(\mathcal{A}(x)(t), \mathcal{C}(x)(t)) \le G(\mathcal{B}(x)(t), \mathcal{C}(x)(t))$ for all $x \in X$ and all $t \in [0,1]$, which leads to $i_G(\mathcal{A},\mathcal{C}) \le i_G(\mathcal{B},\mathcal{C})$.

(1) Suppose that G is normal and let $\mathcal{A},\mathcal{B} \in \mathrm{FS}_2(X)$ be such that for all $x \in X$ there is $t_x \in [0,1]$ satisfying $\mathcal{A}(x)(t_x) = \mathcal{B}(x)(t_x) = 1$. Then $G(\mathcal{A}(x)(t_x), \mathcal{B}(x)(t_x)) = 1$, so $i_G(\mathcal{A},\mathcal{B}) = 1$.

(2) Suppose that $G \le \max$. Then, for all $x \in X$ and all $t \in [0,1]$,

$$G(\mathcal{A}(x)(t), \mathcal{B}(x)(t)) \le \max\{\mathcal{A}(x)(t), \mathcal{B}(x)(t)\}.$$

Hence

$$\begin{aligned} & i_G(\mathcal{A},\mathcal{B}) \\ & \quad \le \sup\left(\{\, \max\{\mathcal{A}(x)(t), \mathcal{B}(x)(t)\} : x \in X,\, t \in [0,1] \,\}\right) \\ & \quad \le \max\{\, \sup\mathcal{A}, \sup\mathcal{B} \,\}. \end{aligned}$$

(3) If $G(t_1,t_2) = s$, then

$$\begin{aligned} & i_G\left(\mathcal{F}^{t_1}, \mathcal{F}^{t_2}\right) \\ & \quad = \sup\left(\left\{\, G\left(\mathcal{F}^{t_1}(x)(t), \mathcal{F}^{t_2}(x)(t)\right) : x \in X,\, t \in [0,1] \,\right\}\right) \\ & \quad = \sup\left(\{\, G(t_1,t_2) : x \in X,\, t \in [0,1] \,\}\right) = G(t_1,t_2) = s. \end{aligned}$$

■

Corollary 22: If G is an overlap function then i_G, defined as in (2), is a normal type-$(2,0)$ overlap index.

Corollary 23: If T is a t-norm satisfying (G_3) then i_T, defined as in (2), is a normal type-$(2,0)$ overlap index.

Furthermore, if T is averaging, then $i_T(\mathcal{A},\mathcal{A}) = \sup\mathcal{A}$ for each $\mathcal{A} \in \mathrm{FS}_2(X)$. And if $T = T_{\min}$, then $i_{T_{\min}}(\mathcal{A},\mathcal{B}) = \sup\mathcal{A}$ for each $\mathcal{A},\mathcal{B} \in \mathrm{FS}_2(X)$ such that $\mathcal{A} \le \mathcal{B}$.

The following particularization is especially interesting in applications.

Definition 24: The *Zadeh* $(2,0)$ *overlap index* is $i_{\min}$, that is, for all $\mathcal{A},\mathcal{B} \in \mathrm{FS}_2(X)$,

$$i_{\min}(\mathcal{A},\mathcal{B}) = \sup\left(\{\, \min(\mathcal{A}(x)(t), \mathcal{B}(x)(t)) : x \in X,\, t \in [0,1] \,\}\right).$$

Corollary 25: The Zadeh $(2,0)$ overlap index is a normal $(2,0)$ overlap index that satisfies the following properties for all $\mathcal{A},\mathcal{B} \in \mathrm{FS}_2(X)$:

1) if $\mathcal{A} \le \mathcal{B}$, then $i_{\min}(\mathcal{A},\mathcal{B}) = \sup\mathcal{A}$.
2) If $i_{\min}(\mathcal{A},\mathcal{B}) = 1$ then there are two sequences $\{x_n\} \subseteq X$ and $\{t_n\} \subset [0,1]$ such that $\{\mathcal{A}(x_n)(t_n)\} \to 1$ and $\{\mathcal{B}(x_n)(t_n)\} \to 1$. In such a case, $\sup\mathcal{A} = \sup\mathcal{B} = 1$.

Proof. The second item follows from the fact that $i_{\min}(\mathcal{A},\mathcal{B})$ is a supremum, so if $i_{\min}(\mathcal{A},\mathcal{B}) = 1$, then there are two sequences $\{x_n\} \subseteq X$ and $\{t_n\} \subset [0,1]$ such that $\{\, \min(\mathcal{A}(x_n)(t_n), \mathcal{B}(x_n)(t_n)) \,\} \to 1$, which implies that $\{\mathcal{A}(x_n)(t_n)\} \to 1$ and $\{\mathcal{B}(x_n)(t_n)\} \to 1$. ■

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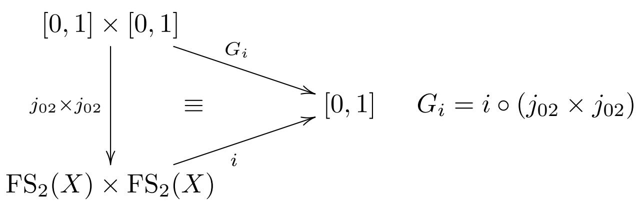
Lemma 21 illustrates a natural way to define type-$(2,0)$ overlap indices associated to lower overlap functions. Also, there exists a reciprocal process which is described in the following result.

Proposition 26: Given a type-$(2,0)$ overlap index $i : \mathrm{FS}_2(X)\times\mathrm{FS}_2(X)\to[0,1]$, let $G_i:[0,1]\times[0,1]\to[0,1]$ be the function defined, for all $t,s\in[0,1]$, by

$$G_i(t,s)=i\left(\mathcal{F}^t,\mathcal{F}^s\right).$$

Then G_i is a lower overlap function satisfying the following properties.

1) The function G_i can be expressed as $i\circ(j_{02}\times j_{02})$, so the following diagram commutes.



2) If i is normal, then G_i is normal.
3) If G is a lower overlap function, then $G_{i_G}=G$.

Proof. Clearly G_i is symmetric. If $t_1,t_2\in[0,1]$ are such that $t_1\le t_2$, then $\mathcal{F}^{t_1}(x)(s)=t_1\le t_2=\mathcal{F}^{t_2}(x)(s)$ for all $x\in X$ and all $s\in[0,1]$. Then $\mathcal{F}^{t_1}\le\mathcal{F}^{t_2}$. Therefore $G_i(t_1,s)=i\left(\mathcal{F}^{t_1},\mathcal{F}^s\right)\le i\left(\mathcal{F}^{t_2},\mathcal{F}^s\right)=G_i(t_2,s)$. Then G_i is increasing. Finally, let $t,s\in[0,1]$. Then:

$$\begin{aligned}&G_i(t,s)=0\\&\quad\Leftrightarrow\ i\left(\mathcal{F}^t,\mathcal{F}^s\right)=0\\&\quad\Leftrightarrow\ \mathcal{F}^t\text{ and }\mathcal{F}^s\text{ are completely disjoint}\\&\quad\Leftrightarrow\ \mathcal{F}^t(x)(r)\cdot\mathcal{F}^s(x)(r)=0\text{ for all }x\in X\text{ and }r\in[0,1]\\&\quad\Leftrightarrow\ t\cdot s=0.\end{aligned}$$

Hence G_i is a lower overlap function.

(2) Suppose that i is normal. Since $\mathcal{F}^1(x)(t)=1$ for all $x\in X$ and all $t\in[0,1]$, then

$$G_i(1,1)=i\left(\mathcal{F}^1,\mathcal{F}^1\right)=i\left(\mathfrak{M}_2,\mathfrak{M}_2\right)=\mathfrak{M}_0=1.$$

(3) If G is a lower overlap function, item 3 of Lemma 21 guarantees that, for all $t,s\in[0,1]$, $G_{i_G}(t,s)=i_G\left(\mathcal{F}^t,\mathcal{F}^s\right)=G(t,s)$. Therefore $G_{i_G}=G$. ■

Notice that, in general, if i is a type-$(2,0)$ overlap index, then $i_{G_i}\neq i$ because i_{G_i} loses information with respect to i, that is, i is richer than i_{G_i} (in the sense that the image of i is usually greater than the image of i_{G_i}).

Remark 27: In general, there is no relationship between a type-$(2,0)$ overlap index i and i_{G_i} because

$$\begin{aligned}&i_{G_i}(\mathcal{A},\mathcal{B})\\&\quad=\sup(\{\,i(\,\mathcal{F}^{\mathcal{A}(x)(t)},\mathcal{F}^{\mathcal{B}(x)(t)}\,):x\in X,\,t\in[0,1]\,\})\end{aligned}$$

for all $\mathcal{A},\mathcal{B}\in\mathrm{FS}_2(X)$. The only remarkable fact in this sense, that can be deduced from item 3 of Proposition 26, is that if $i'=i_G$ is a type-$(2,0)$ overlap index associated to a lower overlap function G, then $i_{G_{i'}}=i'$.

The following result completely characterizes type-$(2,0)$ overlap indices.

Theorem 28: A mapping $i:\mathrm{FS}_2(X)\times\mathrm{FS}_2(X)\to[0,1]$ is a type-$(2,0)$ overlap index if and only if there are two mappings $i^+,i^-:\mathrm{FS}_2(X)\times\mathrm{FS}_2(X)\to[0,1]$, related by:

$$i(\mathcal{A},\mathcal{B})=\frac{i^+(\mathcal{A},\mathcal{B})}{i^+(\mathcal{A},\mathcal{B})+i^-(\mathcal{A},\mathcal{B})}\quad\text{for all }\mathcal{A},\mathcal{B}\in\mathrm{FS}_2(X),\tag{3}$$

satisfying the following properties:

(a) i^+ and i^- are symmetric;
(b) i^+ is increasing and i^- is decreasing;
(c_1) $i^+(\mathcal{A},\mathcal{B})=0\Leftrightarrow i^-(\mathcal{A},\mathcal{B})=1\Leftrightarrow\mathcal{A}$ and $\mathcal{B}$ are completely disjoint;

Furthermore, i is normal if and only if i^- satisfies the following condition (that could be called *anti-normality*):

(c_2) If $\mathcal{A},\mathcal{B}\in\mathrm{FS}_2(X)$ are such that for all $x\in X$ there is $t_x\in[0,1]$ satisfying $\mathcal{A}(x)(t_x)=\mathcal{B}(x)(t_x)=1$, then $i^-(\mathcal{A},\mathcal{B})=0$.

Proof. $[\Rightarrow]$ Suppose that i is a type-$(2,0)$ overlap index and let define $i^+(\mathcal{A},\mathcal{B})=i(\mathcal{A},\mathcal{B})$ and $i^-(\mathcal{A},\mathcal{B})=1-i(\mathcal{A},\mathcal{B})$ for all $\mathcal{A},\mathcal{B}\in\mathrm{FS}_2(X)$. Then (3), (a), (b) and (c_1) clearly hold.

$[\Leftarrow]$ Suppose that (3) holds for some i^+ and i^- satisfying (a), (b) and (c_1). Notice that necessarily

$$i^+(\mathcal{A},\mathcal{B})+i^-(\mathcal{A},\mathcal{B})>0\qquad\text{for all }\mathcal{A},\mathcal{B}\in\mathrm{FS}_2(X).$$

Clearly i is symmetric. Also

$$\begin{aligned}i(\mathcal{A},\mathcal{B})=0\ &\Leftrightarrow\ i^+(\mathcal{A},\mathcal{B})=0\\&\Leftrightarrow\ \mathcal{A}\text{ and }\mathcal{B}\text{ are completely disjoint.}\end{aligned}$$

Finally, let $\mathcal{A},\mathcal{B}\in\mathrm{FS}_2(X)$ be such that $\mathcal{A}\le\mathcal{B}$. As i^+ is increasing and i^- is decreasing, then

$$0\le i^+(\mathcal{A},\mathcal{C})\le i^+(\mathcal{B},\mathcal{C})\quad\text{and}\quad0\le i^-(\mathcal{B},\mathcal{C})\le i^-(\mathcal{A},\mathcal{C}).$$

By multiplying,

$$i^+(\mathcal{A},\mathcal{C})\cdot i^-(\mathcal{B},\mathcal{C})\le i^+(\mathcal{B},\mathcal{C})\cdot i^-(\mathcal{A},\mathcal{C}).$$

Therefore,

$$\begin{aligned}&i^+(\mathcal{A},\mathcal{C})\cdot\left[\,i^+(\mathcal{B},\mathcal{C})+i^-(\mathcal{B},\mathcal{C})\,\right]\\&\quad=i^+(\mathcal{A},\mathcal{C})\cdot i^+(\mathcal{B},\mathcal{C})+i^+(\mathcal{A},\mathcal{C})\cdot i^-(\mathcal{B},\mathcal{C})\\&\quad\le i^+(\mathcal{B},\mathcal{C})\cdot i^-(\mathcal{A},\mathcal{C})+i^+(\mathcal{A},\mathcal{C})\cdot i^+(\mathcal{B},\mathcal{C})\\&\quad=i^+(\mathcal{B},\mathcal{C})\cdot\left[\,i^-(\mathcal{A},\mathcal{C})+i^+(\mathcal{A},\mathcal{C})\,\right],\end{aligned}$$

so we deduce that

$$\begin{aligned}i(\mathcal{A},\mathcal{C})&=\frac{i^+(\mathcal{A},\mathcal{C})}{i^-(\mathcal{A},\mathcal{C})+i^+(\mathcal{A},\mathcal{C})}\\&\le\frac{i^+(\mathcal{B},\mathcal{C})}{i^+(\mathcal{B},\mathcal{C})+i^-(\mathcal{B},\mathcal{C})}=i(\mathcal{B},\mathcal{C}).\end{aligned}$$

Hence i is a type-$(2,0)$ overlap index.

Now we study the normality. Suppose that i is normal and let $\mathcal{A},\mathcal{B}\in\mathrm{FS}_2(X)$ be such that for all $x\in X$ there is $t_x\in[0,1]$ satisfying $\mathcal{A}(x)(t_x)=\mathcal{B}(x)(t_x)=1$. Since i is normal,

$$1=i(\mathcal{A},\mathcal{B})=\frac{i^+(\mathcal{A},\mathcal{B})}{i^+(\mathcal{A},\mathcal{B})+i^-(\mathcal{A},\mathcal{B})},$$

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then $i^-(\mathcal{A},\mathcal{B}) = 0$.

Conversely, suppose that i^- satisfies (c_2). Let $\mathcal{A},\mathcal{B} \in \mathrm{FS}_2(X)$ be such that for all $x \in X$ there is $t_x \in [0,1]$ satisfying $\mathcal{A}(x)(t_x) = \mathcal{B}(x)(t_x) = 1$. Hence $i^-(\mathcal{A},\mathcal{B}) = 0$, which implies that $i(\mathcal{A},\mathcal{B}) = 1$, so i is normal. ■

Proposition 29: Under the hypotheses of Theorem 28, if i is a type-$(2,0)$ overlap index, the following properties hold.

1) Given $t \in [0,1]$, $i(\mathcal{A},\mathcal{B}) = t$ if and only if $(1-t)i^+(\mathcal{A},\mathcal{B}) = t\, i^-(\mathcal{A},\mathcal{B})$. In particular, if $t < 1$,
$$i^+(\mathcal{A},\mathcal{B}) = \frac{t}{1-t}\, i^-(\mathcal{A},\mathcal{B}).$$
2) $i^+(\mathcal{A},\mathcal{B}) \cdot i^-(\mathcal{A},\mathcal{B}) = (1 - i(\mathcal{A},\mathcal{B}))\, i^+(\mathcal{A},\mathcal{B})$.

Proof. (1) Recall that $i^+(\mathcal{A},\mathcal{B}) + i^-(\mathcal{A},\mathcal{B}) > 0$ for all $\mathcal{A},\mathcal{B} \in \mathrm{FS}_2(X)$. Therefore:
$$\begin{aligned} &\frac{i^+(\mathcal{A},\mathcal{B})}{i^+(\mathcal{A},\mathcal{B}) + i^-(\mathcal{A},\mathcal{B})} = i(\mathcal{A},\mathcal{B}) = t \\ &\quad\Leftrightarrow\quad i^+(\mathcal{A},\mathcal{B}) = t\left(i^+(\mathcal{A},\mathcal{B}) + i^-(\mathcal{A},\mathcal{B})\right) \\ &\quad\Leftrightarrow\quad (1-t)i^+(\mathcal{A},\mathcal{B}) = t\, i^-(\mathcal{A},\mathcal{B}). \end{aligned}$$
■

Example 30: Suppose that $i^+, i^- : \mathrm{FS}_2(X) \times \mathrm{FS}_2(X) \to [0,1]$ are given, for all $\mathcal{A},\mathcal{B} \in \mathrm{FS}_2(X)$, by:
$$i^+(\mathcal{A},\mathcal{B}) = \sup(\{\sqrt[3]{\mathcal{A}(x)(t) \cdot \mathcal{B}(x)(t)} : x \in X,\ t \in [0,1]\}),$$
$$i^-(\mathcal{A},\mathcal{B}) = \inf(\{\max(1-\mathcal{A}(x)(t), 1-\mathcal{B}(x)(t)) : x \in X,\ t \in [0,1]\}).$$
Then i^+ and i^- satisfy all conditions of Theorem 28, so the mapping $i : \mathrm{FS}_2(X) \times \mathrm{FS}_2(X) \to [0,1]$, defined as in (3), is a normal type-$(2,0)$ overlap index.

C. *Type-$(2,1)$ overlap indices*

In [38], Zadeh introduced the *consistency index* between two fuzzy sets A and B over the same referential finite set $U = \{u_1, u_2, \ldots, u_n\}$ as the natural extension of the Boolean overlap index:
$$O_Z(A,B) = \max(\{\min\{A(u_j), B(u_j)\} : j \in \{1,2,\ldots,n\}\}).$$
This definition can be extended to an arbitrary set X as follows:
$$O_Z(A,B) = \sup(\{\min\{A(x), B(x)\} : x \in X\})$$
for all $A,B \in \mathrm{FS}(X)$. The reader can check that $O_Z : \mathrm{FS}(X) \times \mathrm{FS}(X) \to [0,1]$ is a normal type-$(1,0)$ overlap index associated to the overlap function $G = \min$. Zadeh's index, especially in the finite case, inspired many of the properties of overlap indices illustrated in [37]. In this section, inspired by the previous indices, we introduce some properties of type-$(2,1)$ overlap indices. First of all, we show some families of this kind of indices.

Lemma 31: Given a type-$(1,0)$ overlap index $o : \mathrm{FS}([0,1]) \times \mathrm{FS}([0,1]) \to [0,1]$ on $[0,1]$, the function $I_o : \mathrm{FS}_2(X) \times \mathrm{FS}_2(X) \to \mathrm{FS}(X)$ such that:
$$I_o(\mathcal{A},\mathcal{B})(x) = o(\mathcal{A}(x),\mathcal{B}(x))$$
for all $\mathcal{A},\mathcal{B} \in \mathrm{FS}_2(X)$ and all $x \in X$, is a type-$(2,1)$ overlap index satisfying the following properties.

1) o is normal if and only if I_o is normal.
2) If $j_X : \mathrm{FS}([0,1]) \to \mathrm{FS}_2(X)$ represents the injection defined in (1), then the following diagram commutes, that is, $j_{01} \circ o = I_o \circ (j_X \times j_X)$:

$$\begin{array}{ccc} \mathrm{FS}([0,1]) \times \mathrm{FS}([0,1]) & \xrightarrow{\ o\ } & [0,1] \\ \Big\downarrow{\scriptstyle j_X \times j_X} & \equiv & \Big\downarrow{\scriptstyle j_{01}} \\ \mathrm{FS}_2(X) \times \mathrm{FS}_2(X) & \xrightarrow[\ I_o\ ]{} & \mathrm{FS}(X) \end{array}$$

Proof. Clearly I_o is well defined because $\mathcal{A}(x),\mathcal{B}(x) \in \mathrm{FS}([0,1])$ for all $x \in X$. Furthermore, I_o is symmetric. Let $\mathcal{A}_1,\mathcal{A}_2,\mathcal{B} \in \mathrm{FS}_2(X)$ be such that $\mathcal{A}_1 \le \mathcal{A}_2$. Therefore $\mathcal{A}_1(x) \le \mathcal{A}_2(x)$ for all $x \in X$. Since o is a type-$(1,0)$ overlap index, then $I_o(\mathcal{A}_1,\mathcal{B})(x) = o(\mathcal{A}_1(x),\mathcal{B}(x)) \le o(\mathcal{A}_2(x),\mathcal{B}(x)) = I_o(\mathcal{A}_2,\mathcal{B})(x)$ for all $x \in X$. Hence $I_o(\mathcal{A}_1,\mathcal{B}) \le I_o(\mathcal{A}_2,\mathcal{B})$, so I_o is increasing. Finally,
$$\begin{aligned} &I_o(\mathcal{A},\mathcal{B}) = F^0 \\ &\quad\Leftrightarrow\ o(\mathcal{A}(x),\mathcal{B}(x)) = 0 \ \text{ for all } x \in X \\ &\quad\Leftrightarrow\ \mathcal{A}(x) \text{ and } \mathcal{B}(x) \text{ are completely disjoint for all } x \in X \\ &\quad\Leftrightarrow\ \mathcal{A} \text{ and } \mathcal{B} \text{ are completely disjoint.} \end{aligned}$$

(1) Suppose that o is normal and let $\mathcal{A},\mathcal{B} \in \mathrm{FS}_2(X)$ be such that for all $x \in X$ there is $t_x \in [0,1]$ satisfying $\mathcal{A}(x)(t_x) = \mathcal{B}(x)(t_x) = 1$. Given any $x_0 \in X$, let $A = \mathcal{A}(x_0), B = \mathcal{B}(x_0) \in \mathrm{FS}([0,1])$. As there is $t_{x_0} \in [0,1]$ such that $A(t_{x_0}) = B(t_{x_0}) = 1$ and o is normal, then $I_o(\mathcal{A},\mathcal{B})(x_0) = o(\mathcal{A}(x_0),\mathcal{B}(x_0)) = o(A,B) = 1$. Therefore $I_o(\mathcal{A},\mathcal{B}) = F^1$ on X, so I_o is normal.

Conversely, assume that I_o is a normal type-$(2,1)$ overlap index and let $A,B \in \mathrm{FS}([0,1])$ be such that $A(t_0) = B(t_0) = 1$ for some scalar $t_0 \in [0,1]$. Let define $\mathcal{A},\mathcal{B} \in \mathrm{FS}_2(X)$ as $\mathcal{A}(x) = A$ and $\mathcal{B}(x) = B$ for each $x \in X$. Then $\mathcal{A}$ and $\mathcal{B}$ are two type-2 fuzzy sets on X such that, for each $x \in X$, there is $t_0 \in [0,1]$ verifying $\mathcal{A}(x)(t_0) = A(t_0) = 1 = B(t_0) = \mathcal{B}(x)(t_0)$. Since I_o is normal, then $I_o(\mathcal{A},\mathcal{B}) = F^1 \in \mathrm{FS}(X)$. Hence, given an arbitrary $x_0 \in X$, $o(A,B) = o(\mathcal{A}(x_0),\mathcal{B}(x_0)) = I_o(\mathcal{A},\mathcal{B})(x_0) = F^1(x) = 1$. This proves that o is also normal.

(2) Let $A,B \in \mathrm{FS}([0,1])$ and let $x \in X$. Then:
$$[j_{01}(o(A,B))](x) = F^{o(A,B)}(x) = o(A,B)$$
and
$$\begin{aligned} &[I_o((j_X \times j_X)(A,B))](x) = [I_o(j_X(A), j_X(B))](x) \\ &\quad = o(j_X(A)(x), j_X(B)(x)) = o(A,B). \end{aligned}$$
Therefore, $j_{01} \circ o = I_o \circ (j_X \times j_X)$. ■

Definition 32: The *Zadeh* $(2,1)$ (*consistency*) *index* is the mapping $I_Z : \mathrm{FS}_2(X) \times \mathrm{FS}_2(X) \to \mathrm{FS}(X)$ defined, for all $\mathcal{A},\mathcal{B} \in \mathrm{FS}_2(X)$ and all $x \in X$, by:
$$I_Z(\mathcal{A},\mathcal{B})(x) = \sup(\{\min\{\mathcal{A}(x)(t), \mathcal{B}(x)(t)\} : t \in [0,1]\}).$$

Corollary 33: The Zadeh $(2,1)$ index on X is a normal type-$(2,1)$ overlap index satisfying the following properties.

1) If $\mathcal{A} \leq \mathcal{B}$ then $I_Z(\mathcal{A},\mathcal{B}) = \sup \mathcal{A} \in \mathrm{FS}(X)$ (that is, $I_Z(\mathcal{A},\mathcal{B})(x) = \sup(\mathcal{A}(x))$ for all $x \in X$).
2) $I_Z(\mathcal{A},\mathcal{A}) = \widetilde{\sup}\mathcal{A} \in \mathrm{FS}(X)$.
3) $I_Z\left(\mathcal{F}^0,\mathcal{A}\right) = F^0 \in \mathrm{FS}(X)$.
4) $I_Z\left(\mathcal{A},\mathcal{F}^1\right) = \widetilde{\sup}\mathcal{A} \in \mathrm{FS}(X)$.
5) $I_Z\left(\mathcal{F}^t,\mathcal{F}^s\right) = \widetilde{\sup}\mathcal{F}^{\min\{t,s\}} \in \mathrm{FS}(X)$ for all $t,s \in [0,1]$.

Proof. It follows from Lemma 31 taking into account that I_Z can be expressed as $I_{o_Z^{[0,1]}}$, where $o_Z^{[0,1]}$ is the Zadeh $(1,0)$ index on $[0,1]$, which is normal and is given, for all $A,B \in \mathrm{FS}([0,1])$, by:

$$o_Z^{[0,1]}(A,B) = \sup\left(\{\min\{A(t),B(t)\} : t \in [0,1]\}\right).$$

(1) Let $\mathcal{A},\mathcal{B} \in \mathrm{FS}_2(X)$ be such that $\mathcal{A} \leq \mathcal{B}$. Therefore, for all $x \in X$,

$$\begin{aligned} I_Z(\mathcal{A},\mathcal{B})(x) &= \sup\left(\{\min\{\mathcal{A}(x)(t),\mathcal{B}(x)(t)\} : t \in [0,1]\}\right) \\ &= \sup\left(\{\mathcal{A}(x)(t) : t \in [0,1]\}\right) \\ &= \sup(\mathcal{A}(x)) = \left(\widetilde{\sup}\mathcal{A}\right)(x). \end{aligned}$$

■

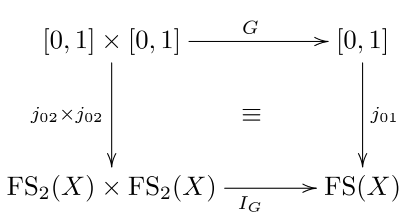
Zadeh $(2,1)$ index is an example of type-$(2,1)$ overlap indices generated by lower overlap functions as in the following result.

Lemma 34: Given a lower overlap function $G : [0,1] \times [0,1] \to [0,1]$, let define $I_G : \mathrm{FS}_2(X) \times \mathrm{FS}_2(X) \to \mathrm{FS}(X)$ as:

$$I_G(\mathcal{A},\mathcal{B})(x) = \sup\left(\{G(\mathcal{A}(x)(t),\mathcal{B}(x)(t)) : t \in [0,1]\}\right)$$

for all $\mathcal{A},\mathcal{B} \in \mathrm{FS}_2(X)$ and all $x \in X$. Then I_G is a type-$(2,1)$ overlap index satisfying the following properties.

1) If G is normal, then I_G is also normal.
2) The following diagram commutes, that is, $j_{01} \circ G = I_G \circ (j_{02} \times j_{02})$:



Proof. I_G is clearly symmetric and increasing. Furthermore,

$$\begin{aligned} & I_G(\mathcal{A},\mathcal{B}) = F^0 \\ \Leftrightarrow\ & \sup\left(\{G(\mathcal{A}(x)(t),\mathcal{B}(x)(t)) : t \in [0,1]\}\right) = 0 \\ & \qquad \text{for all } x \in X \\ \Leftrightarrow\ & G(\mathcal{A}(x)(t),\mathcal{B}(x)(t)) = 0 \ \text{ for all } x \in X \text{ and } t \in [0,1] \\ \Leftrightarrow\ & \mathcal{A}(x)(t) \cdot \mathcal{B}(x)(t) = 0 \ \text{ for all } x \in X \text{ and } t \in [0,1] \\ \Leftrightarrow\ & \mathcal{A} \text{ and } \mathcal{B} \text{ are completely disjoint.} \end{aligned}$$

Therefore, I_G is a type-$(2,1)$ overlap index.

(1) Suppose that G is normal and let $\mathcal{A},\mathcal{B} \in \mathrm{FS}_2(X)$ be such that for all $x \in X$ there is $t_x \in [0,1]$ satisfying $\mathcal{A}(x)(t_x) = \mathcal{B}(x)(t_x) = 1$. Therefore, $G(\mathcal{A}(x)(t_x),\mathcal{B}(x)(t_x)) = G(1,1) = 1$, so, for all $x \in X$,

$$I_G(\mathcal{A},\mathcal{B})(x) = \sup\left(\{G(\mathcal{A}(x)(t),\mathcal{B}(x)(t)) : t \in [0,1]\}\right) = 1.$$

This means that $I_G(\mathcal{A},\mathcal{B}) = F^1$ on X and I_G is normal.

(2) Let $t,s \in [0,1]$ and let $x \in X$. Then $[j_{01}(G(t,s))](x) = F^{G(t,s)}(x) = G(t,s)$ and

$$\begin{aligned} [I_G((j_{02} \times j_{02})(t,s))](x) &= \left[I_G\left(\mathcal{F}^t,\mathcal{F}^s\right)\right](x) \\ &= \sup\left(\left\{G\left(\mathcal{F}^t(x)(r),\mathcal{F}^s(x)(r)\right) : r \in [0,1]\right\}\right) \\ &= \sup\left(\{G(t,s) : r \in [0,1]\}\right) = G(t,s). \end{aligned}$$

Therefore, $j_{01} \circ G = I_G \circ (j_{02} \times j_{02})$. ■

Example 35: The overlap function G, defined by $G(t,s) = \min(t,s)$ for all $t,s \in [0,1]$, provides the type-$(1,0)$ overlap index $o_G : \mathrm{FS}(X) \times \mathrm{FS}(X) \to [0,1]$ given by

$$o_G(A,B) = \sup\left(\{\min(A(x),B(x)) : x \in X\}\right)$$

for all $A,B \in \mathrm{FS}(X)$, which, at the same time, induces the type-$(2,1)$ overlap index $I_G : \mathrm{FS}_2(X) \times \mathrm{FS}_2(X) \to \mathrm{FS}(X)$ given by

$$I_G(\mathcal{A},\mathcal{B})(x) = \sup\left(\{\min(\mathcal{A}(x)(t),\mathcal{B}(x)(t)) : t \in [0,1]\}\right)$$

for all $\mathcal{A},\mathcal{B} \in \mathrm{FS}_2(X)$ and all $x \in X$.

Example 36: Similarly, the overlap function G, given by $G(t,s) = t \cdot s$, for all $t,s \in [0,1]$, generates the following superior structures:

$$\begin{aligned} & o_G : \mathrm{FS}(X) \times \mathrm{FS}(X) \to [0,1], \\ & \qquad o_G(A,B) = \sup\left(\{A(x) \cdot B(x) : x \in X\}\right); \\ & I_G : \mathrm{FS}_2(X) \times \mathrm{FS}_2(X) \to \mathrm{FS}(X), \\ & \qquad I_G(\mathcal{A},\mathcal{B})(x) = \sup\left(\{\mathcal{A}(x)(t) \cdot \mathcal{B}(x)(t) : t \in [0,1]\}\right), \end{aligned}$$

for all $A,B \in \mathrm{FS}(X)$, all $\mathcal{A},\mathcal{B} \in \mathrm{FS}_2(X)$ and all $x \in X$.

D. *Type-$(2,2)$ overlap indices*

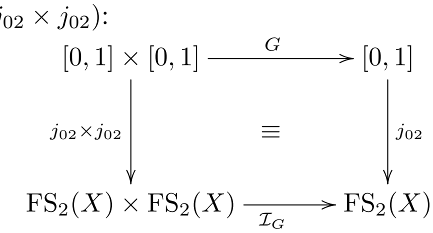
This subsection is devoted to study type-$(2,2)$ overlap indices. We present a family of such mappings.

Lemma 37: Given a lower overlap function $G : [0,1] \times [0,1] \to [0,1]$, let define $\mathcal{I}_G : \mathrm{FS}_2(X) \times \mathrm{FS}_2(X) \to \mathrm{FS}_2(X)$ as:

$$\mathcal{I}_G(\mathcal{A},\mathcal{B})(x)(t) = G(\mathcal{A}(x)(t),\mathcal{B}(x)(t))$$

for all $\mathcal{A},\mathcal{B} \in \mathrm{FS}_2(X)$, all $x \in X$ and all $t \in [0,1]$. Then $\mathcal{I}_G$ is a type-$(2,2)$ overlap index satisfying the following properties.

1) The following diagram commutes, that is, $j_{02} \circ G = \mathcal{I}_G \circ (j_{02} \times j_{02})$:



2) $\mathcal{I}_G\left(\mathcal{F}^t,\mathcal{F}^s\right) = \mathcal{F}^{G(t,s)}$ for all $t,s \in [0,1]$.
3) If G is normal then $\mathcal{I}_G\left(\mathcal{F}^1,\mathcal{F}^1\right) = \mathcal{F}^1$.

Proof. The proof is straightforward by using the same arguments of the proofs of Lemmas 31 and 34. ■

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Notice that $\mathcal{I}_G$ is not necessarily normal even if G is normal. To prove it, let $\mathcal{A}, \mathcal{B} \in \mathrm{FS}_2(X)$ be such that for all $x \in X$ there is $t_x \in [0,1]$ satisfying $\mathcal{A}(x)(t_x) = \mathcal{B}(x)(t_x) = 1$. Therefore, $G\left(\mathcal{A}(x)(t_x), \mathcal{B}(x)(t_x)\right) = G(1,1) = 1$, but

$$\mathcal{I}_G\left(\mathcal{A}, \mathcal{B}\right)(x)(t) = G\left(\mathcal{A}(x)(t), \mathcal{B}(x)(t)\right)$$

is not necessarily 1 for an arbitrary $t \in [0,1]$ even if G is normal. We can only assert that, if G is normal, then $\mathcal{I}_G\left(\mathcal{F}^1, \mathcal{F}^1\right) = \mathcal{F}^1$.

Example 38: Let define $\mathcal{I} : \mathrm{FS}_2(X) \times \mathrm{FS}_2(X) \to \mathrm{FS}_2(X)$, for each $\mathcal{A}, \mathcal{B} \in \mathrm{FS}_2(X)$, each $x \in X$ and each $t \in [0,1]$, by

$$\begin{aligned}\mathcal{I}(\mathcal{A},\mathcal{B})(x)(t) &= \frac{1}{2}\Bigg[2 - \max(\mathcal{A}(x)(t),\mathcal{B}(x)(t))\\ &+ 2\min(\mathcal{A}(x)(t),\mathcal{B}(x)(t)) - \Big(4(1-\max(\mathcal{A}(x)(t),\mathcal{B}(x)(t)))\\ &+(\max(\mathcal{A}(x)(t),\mathcal{B}(x)(t)) - 2\min(\mathcal{A}(x)(t),\mathcal{B}(x)(t)))^2\Big)^{\frac{1}{2}}\Bigg].\end{aligned}$$

Then $\mathcal{I}$ is a normal type-$(2,2)$ overlap index because $\mathcal{I} = \mathcal{I}_G$ is associated to the normal lower overlap function G defined, for all $t, s \in [0,1]$, as

$$G(t,s) = \frac{1}{2}\Big[2 - \max(t,s) + 2\min(t,s) - \sqrt{4(1-\max(t,s)) + (\max(t,s) - 2\min(t,s))^2}\ \Big]$$

(recall that G is not an overlap function as it was shown in Example 14).

IV. Inference Algorithms for Interpolative Type-2 Fuzzy Systems Using Overlap Indices

Overlap indices can be of great help in applications. In this Section we illustrate the applicability of overlap indices to a concrete framework of the fuzzy logic. We set this example in the context of the application of the *modus ponens* when the input data (rules and facts) are type-2 fuzzy sets rather than real scalars or type-1 fuzzy sets. The *modus ponens* is a celebrated technique in fuzzy logic in order to get a consequence from a set of rules and a concrete antecedent. In the following lines we introduce two distinct algorithms in order to face the problem of determining a consequent type-2 fuzzy set when the finite set of rules and the fact are performanced by type-2 fuzzy sets. To carry out this task, it will be of importance the usage of a type-$(1,0)$ overlap index (Algorithm 1) or a type-$(2,0)$ overlap index (Algorithm 2).

In recent years, several fuzzy interpolative reasoning methods have been proposed based on type-1 and (interval) type-2 fuzzy sets. For examples of the former see the works by Chang et al. [59], Chen and Adam [60], Chen and Chen [61], and the references therein. Now, for examples for the latter, see the papers by Chen and Barman [62], Chen and Shen [63], Chen et al. [64], and the references discussed by them. For other techniques, such as rough-fuzzy sets [65], see, e.g., [66]. Nevertheless, it was Garcia-Jimenez et al. [37] who proposed to use overlap indices in interpolative fuzzy systems.

The following development can be done for two arbitrary sets X and Y. However, due to the fact that the finite case is especially interesting in applications, we will only consider the finite sets $U = \{u_1, u_2, \ldots, u_n\}$ and $V = \{v_1, v_2, \ldots, v_m\}$, adapting the notation to this case.

Given $\mathcal{A} \in \mathrm{FS}_2(U)$ and $t \in [0,1]$, we denote by $\mathcal{A}(\cdot)(t) \in \mathrm{FS}(U)$ the fuzzy set on U defined by $\left[\mathcal{A}(\cdot)(t)\right](x) = \mathcal{A}(x)(t)$ for all $x \in U$.

Given $\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_p, \mathcal{A}' \in \mathrm{FS}_2(U)$ and $\mathcal{B}_1, \mathcal{B}_2, \ldots, \mathcal{B}_p, \mathcal{B}' \in \mathrm{FS}_2(V)$, let consider the following type-2 fuzzy rule-based system:

$$\begin{array}{llll}\text{Rule } R_1: & \text{If } (u,t) \text{ is } \mathcal{A}_1, & \text{then} & (v,s) \text{ is } \mathcal{B}_1\\ \text{Rule } R_2: & \text{If } (u,t) \text{ is } \mathcal{A}_2, & \text{then} & (v,s) \text{ is } \mathcal{B}_2\\ \ldots\ldots & \ldots\ldots & & \ldots\ldots\\ \text{Rule } R_p: & \text{If } (u,t) \text{ is } \mathcal{A}_p, & \text{then} & (v,s) \text{ is } \mathcal{B}_p\\ \text{Fact:} & (u,t) \text{ is } \mathcal{A}' & & \\ \hline \text{Conclusion:} & & & (v,s) \text{ is } \mathcal{B}'\end{array} \tag{4}$$

For short, we denote by "$R_j : \mathcal{A}_j \twoheadrightarrow \mathcal{B}_j$" the type-2 fuzzy rule: "if (u,t) is $\mathcal{A}_j$, then (v,s) is $\mathcal{B}_j$" (where $j \in \{1, 2, \ldots, p\}$). The main objective of this system is to compute the type-2 fuzzy output $\mathcal{B}'$ that can be deduced under the assumption of the fact and the rules. We describe two distinct procedures in order to carry out this task.

A. A first algorithm

We propose the following algorithm in order to compute the type-2 fuzzy set $\mathcal{B}'$ depending on the rules ($R_j : \mathcal{A}_j \twoheadrightarrow \mathcal{B}_j$) and the fact ($\mathcal{A}'$). This algorithm involves three main algebraic tools:

1) a lower overlap function $G : [0,1] \times [0,1] \to [0,1]$;
2) an aggregation function $M : [0,1]^p \to [0,1]$; and
3) a type-$(1,0)$ overlap index $O : \mathrm{FS}(U) \times \mathrm{FS}(U) \to [0,1]$ on the set U.

Algorithm 1

Input: A set of p rules $\{R_j : \mathcal{A}_j \twoheadrightarrow \mathcal{B}_j\}_{j=1}^p$, a fact $\mathcal{A}'$ and $t \in [0,1]$

Output: $\mathcal{B}'(\cdot)(t)$.

1: Select an aggregation function M, a lower overlap function G and a type-$(1,0)$ overlap index O on U
2: **for** $j \in \{1, 2, \ldots, p\}$ **and** $t \in [0,1]$ **do**
3: Compute $O\left(\mathcal{A}'(\cdot)(t), \mathcal{A}_j(\cdot)(t)\right)$
4: **end for**
5: Construct $\mathcal{B}'(\cdot)(t)$ given, for all $v \in V$, by
$$\mathcal{B}'\left(v\right)\left(t\right) = \underset{j=1}{\overset{p}{M}}\ G\left(\mathcal{B}_j\left(v\right)\left(t\right), O\left(\mathcal{A}'(\cdot)(t), \mathcal{A}_j(\cdot)(t)\right)\right).$$

Let us show that $\mathcal{B}'$ is well-defined, that is, it is a type-2 fuzzy set on V. Let $v \in V$ and $t \in [0,1]$ be given. Taking into account that $\mathcal{A}'(\cdot)(t) \in \mathrm{FS}(U)$ and $\mathcal{A}_j(\cdot)(t) \in \mathrm{FS}(U)$ for all $j \in \{1, 2, \ldots, p\}$, we can consider the real number $O\left(\mathcal{A}'(\cdot)(t), \mathcal{A}_j(\cdot)(t)\right) \in [0,1]$. Hence, as $\mathcal{B}_j\left(v\right)\left(t\right) \in [0,1]$, we can compute the lower overlap $G\left(\mathcal{B}_j\left(v\right)\left(t\right), O\left(\mathcal{A}'(\cdot)(t), \mathcal{A}_j(\cdot)(t)\right)\right) \in [0,1]$ for all $j \in \{1, 2, \ldots, p\}$, and, aggregating such numbers, we obtain $\mathcal{B}'\left(v\right)\left(t\right) \in [0,1]$.

Algorithm 1 naturally appears in the setting of type-2 fuzzy sets in order to be coherent to Algorithm 1 described in [37].

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As a consequence, it also satisfies many of the main properties of such procedure, that we describe in the following results.

Theorem 39: Let consider the type-2 fuzzy rule-based system detailed in (4), where $\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_p, \mathcal{A}' \in \mathrm{FS}_2(U)$ and $\mathcal{B}_1, \mathcal{B}_2, \ldots, \mathcal{B}_p \in \mathrm{FS}_2(V)$ are given. Then the following properties are fulfilled under Algorithm 1.

1) If $\mathcal{A}'' \in \mathrm{FS}_2(U)$ are such that $\mathcal{A}' \leq \mathcal{A}''$, then $\mathcal{B}' \leq \mathcal{B}''$.
2) If $\mathcal{A}'$ is completely disjoint to each $\mathcal{A}_j$, $j \in \{1,2,\ldots,p\}$, then $\mathcal{B}' = \mathcal{F}^0$ on V.
3) If $M \geq \max$, $G(t,1) = t$ for all $t \in [0,1]$ and $O(\mathcal{A}'(\cdot)(t), \mathcal{A}_j(\cdot)(t)) = 1$ for all $t \in [0,1]$ and all $j \in \{1,2,\ldots,p\}$, then $\mathcal{B}' \geq \max(\mathcal{B}_1, \mathcal{B}_2, \ldots, \mathcal{B}_p)$.
4) If $M = \max$, $G(t,1) = t$ for all $t \in [0,1]$ and $O(\mathcal{A}'(\cdot)(t), \mathcal{A}_j(\cdot)(t)) = 1$ for all $t \in [0,1]$ and all $j \in \{1,2,\ldots,p\}$, then $\mathcal{B}' = \max(\mathcal{B}_1, \mathcal{B}_2, \ldots, \mathcal{B}_p)$.
5) If $M \geq \max$, $G \geq \min$ and $\mathcal{B}_j(v)(t) \leq O(\mathcal{A}'(\cdot)(t), \mathcal{A}_j(\cdot)(t))$ for all $t \in [0,1]$ and all $j \in \{1,2,\ldots,p\}$, then $\mathcal{B}' \geq \max(\mathcal{B}_1, \mathcal{B}_2, \ldots, \mathcal{B}_p)$.

Proof. **(1)** Suppose that $\mathcal{A}' \leq \mathcal{A}''$, that is, $\mathcal{A}'(u)(t) \leq \mathcal{A}''(u)(t)$ for all $u \in U$ and all $t \in [0,1]$. Then $\mathcal{A}'(\cdot)(t) \leq \mathcal{A}''(\cdot)(t)$ for all $t \in [0,1]$. As O is a type-$(1,0)$ overlap index, it is increasing, so $O(\mathcal{A}'(\cdot)(t), \mathcal{A}_j(\cdot)(t)) \leq O(\mathcal{A}''(\cdot)(t), \mathcal{A}_j(\cdot)(t))$. Also, as the lower overlap function G and the aggregation function are increasing, then, for all $v \in V$ and all $t \in [0,1]$,

$$\begin{aligned}
\mathcal{B}'(v)(t) &= \mathop{M}_{j=1}^{p} G\left(\mathcal{B}_j(v)(t), O\left(\mathcal{A}'(\cdot)(t), \mathcal{A}_j(\cdot)(t)\right)\right) \\
&\leq \mathop{M}_{j=1}^{p} G\left(\mathcal{B}_j(v)(t), O\left(\mathcal{A}''(\cdot)(t), \mathcal{A}_j(\cdot)(t)\right)\right) \\
&= \mathcal{B}''(v)(t).
\end{aligned}$$

(2) Suppose that $\mathcal{A}'$ and $\mathcal{A}_j$ are completely disjoint for all $j \in \{1,2,\ldots,p\}$. Then $\mathcal{A}'(u)(t) \cdot \mathcal{A}_j(u)(t) = 0$ for all $u \in U$ and all $t \in [0,1]$. In particular, the fuzzy sets $\mathcal{A}'(\cdot)(t)$ and $\mathcal{A}_j(\cdot)(t)$ are completely disjoint for all $j \in \{1,2,\ldots,p\}$ and all $t \in [0,1]$. As O is a type-$(1,0)$ overlap index, then $O(\mathcal{A}'(\cdot)(t), \mathcal{A}_j(\cdot)(t)) = 0$ for all $j \in \{1,2,\ldots,p\}$ and all $t \in [0,1]$. Therefore, as G is a lower overlap function,

$$G\left(\mathcal{B}_j(v)(t), O\left(\mathcal{A}'(\cdot)(t), \mathcal{A}_j(\cdot)(t)\right)\right) = G\left(\mathcal{B}_j(v)(t), 0\right) = 0$$

for all $j \in \{1,2,\ldots,p\}$, all $t \in [0,1]$ and all $v \in V$. Therefore,

$$\begin{aligned}
\mathcal{B}'(v)(t) &= \mathop{M}_{j=1}^{p} G\left(\mathcal{B}_j(v)(t), O\left(\mathcal{A}'(\cdot)(t), \mathcal{A}_j(\cdot)(t)\right)\right) \\
&= M(0,0,\ldots,0) = 0
\end{aligned}$$

for all $v \in V$ and all $t \in [0,1]$, which means that $\mathcal{B}' = \mathcal{F}^0$.

(3) In this case, for all $v \in V$ and all $t \in [0,1]$,

$$\begin{aligned}
\mathcal{B}'(v)(t) &= \mathop{M}_{j=1}^{p} G\left(\mathcal{B}_j(v)(t), O\left(\mathcal{A}'(\cdot)(t), \mathcal{A}_j(\cdot)(t)\right)\right) \\
&= \mathop{M}_{j=1}^{p} G\left(\mathcal{B}_j(v)(t), 1\right) = \mathop{M}_{j=1}^{p} \mathcal{B}_j(v)(t) \\
&= M\left(\mathcal{B}_1(v)(t), \mathcal{B}_2(v)(t), \ldots, \mathcal{B}_p(v)(t)\right) \\
&\geq \max\left\{\mathcal{B}_1(v)(t), \mathcal{B}_2(v)(t), \ldots, \mathcal{B}_p(v)(t)\right\} \\
&= \left(\max\left\{\mathcal{B}_1, \mathcal{B}_2, \ldots, \mathcal{B}_p\right\}\right)(v)(t). \qquad (5)
\end{aligned}$$

(4) If $M = \max$, then inequality in (5) is, in fact, an equality.

(5) It follows from the fact that, for all $v \in V$ and all $t \in [0,1]$,

$$\begin{aligned}
\mathcal{B}'(v)(t) &= \mathop{M}_{j=1}^{p} G\left(\mathcal{B}_j(v)(t), O\left(\mathcal{A}'(\cdot)(t), \mathcal{A}_j(\cdot)(t)\right)\right) \\
&\geq \mathop{M}_{j=1}^{p} \min\left(\mathcal{B}_j(v)(t), O\left(\mathcal{A}'(\cdot)(t), \mathcal{A}_j(\cdot)(t)\right)\right) \\
&= \mathop{M}_{j=1}^{p} \mathcal{B}_j(v)(t) \\
&\geq \max\left\{\mathcal{B}_1(v)(t), \mathcal{B}_2(v)(t), \ldots, \mathcal{B}_p(v)(t)\right\} \\
&= \left(\max\left\{\mathcal{B}_1, \mathcal{B}_2, \ldots, \mathcal{B}_p\right\}\right)(v)(t).
\end{aligned}$$

■

A first consequence of the previous theorem occurs when we only consider a unique rule and the fact is equal to the antecedent in the rule.

Corollary 40: If $G = \min$ and $O(\mathcal{A}(\cdot)(t), \mathcal{A}(\cdot)(t)) = 1$ for all $t \in [0,1]$, then the conclusion of the particular type-2 modus ponens:

Rule:	If (u,t) is $\mathcal{A}$, then (v,s) is $\mathcal{B}$
Fact:	(u,t) is $\mathcal{A}$

under Algorithm 1 is $\mathcal{B}' = \mathcal{B}$.

Proof. It follows from item 4 of Theorem 39 by using $G = \min$, $p = 1$ and $\mathcal{A}_1 = \mathcal{A}'$. ■

Applying items 3 and 4 of Theorem 39 to the case in which all type-2 fuzzy sets $\mathcal{B}_1, \mathcal{B}_2, \ldots, \mathcal{B}_p$ are equal (that is, $\mathcal{B}_1 = \mathcal{B}_2 = \ldots = \mathcal{B}_p = \mathcal{B} \in \mathrm{FS}_2(U)$), we deduce the following result.

Corollary 41: Under the type-2 fuzzy rule-based system (4), if $M \geq \max$, $G(t,1) = t$ for all $t \in [0,1]$, $O(\mathcal{A}'(\cdot)(t), \mathcal{A}_j(\cdot)(t)) = 1$ for all $t \in [0,1]$ and all $j \in \{1,2,\ldots,p\}$ and $\mathcal{B}_1 = \mathcal{B}_2 = \ldots = \mathcal{B}_p = \mathcal{B}$, then the conclusion $\mathcal{B}'$ under Algorithm 1 satisfies $\mathcal{B}' \geq \mathcal{B}$.

Furthermore, if we additionally assume that $M = \max$, then $\mathcal{B}' = \mathcal{B}$.

In the following result, we involve the absolute minimum $\mathcal{F}_U^0 \in \mathrm{FS}_2(U)$ and $\mathcal{F}_V^0 \in \mathrm{FS}_2(V)$, and the absolute maximum $\mathcal{F}_U^1 \in \mathrm{FS}_2(U)$, $\mathcal{F}_V^1 \in \mathrm{FS}_2(V)$ and $F_U^1 \in \mathrm{FS}(U)$.

Corollary 42: Given $\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_p \in \mathrm{FS}_2(U)$ and $\mathcal{B}_1, \mathcal{B}_2, \ldots, \mathcal{B}_p \in \mathrm{FS}_2(V)$, let

$$\Phi : \mathrm{FS}_2(U) \to \mathrm{FS}_2(V)$$

be the mapping that associates to each $\mathcal{A}' \in \mathrm{FS}_2(U)$ the conclusion $\mathcal{B}'$ of the type-2 fuzzy rule-based system (4) under Algorithm 1. Then Φ satisfies the following properties.

1) Φ is increasing.
2) $\Phi\left(\mathcal{F}_U^0\right) = \mathcal{F}_V^0$.
3) If $M \geq \max$, $G(t,1) = t$ for all $t \in [0,1]$, $\max(\mathcal{B}_1, \mathcal{B}_2, \ldots, \mathcal{B}_p) = \mathcal{F}_V^1$ and $O\left(F_U^1, \mathcal{A}_j(\cdot)(t)\right) = 1$ for all $t \in [0,1]$ and all $j \in \{1,2,\ldots,p\}$, then $\Phi\left(\mathcal{F}_U^1\right) = \mathcal{F}_V^1$.

Proof. These properties directly follow from items 1, 2 and 3 of Theorem 39. ■

Corollary 42 recall us that given $A_1, A_2, \ldots, A_p \in \mathrm{FS}(U)$ and $B_1, B_2, \ldots, B_p \in \mathrm{FS}(V)$, there exists a mapping

$$\phi : \mathrm{FS}(U) \to \mathrm{FS}(V)$$

that associates to each $A' \in \mathrm{FS}(U)$ the conclusion $B' \in \mathrm{FS}(V)$ of the fuzzy rule-based system:

$$\begin{array}{llll} \text{Rule } R_1: & \text{If } u \text{ is } A_1, & \text{then} & v \text{ is } B_1 \\ \text{Rule } R_2: & \text{If } u \text{ is } A_2, & \text{then} & v \text{ is } B_2 \\ \cdots\cdots & \cdots\cdots & & \cdots\cdots \\ \text{Rule } R_p: & \text{If } u \text{ is } A_p, & \text{then} & v \text{ is } B_p \\ \text{Fact:} & \quad u \text{ is } A' & & \\ \hline \text{Conclusion:} & & & v \text{ is } B' \end{array} \tag{6}$$

under any algorithm we can imagine.

B. A second algorithm

One of the main characteristic of the previous algorithm is that the number $O\left(\mathcal{A}'(\cdot)(t), \mathcal{A}_j(\cdot)(t)\right)$ directly depends on $t \in [0,1]$. This fact can be interpreted as an advantage (because it directly takes into account the effective values of $\mathcal{A}'$ and each $\mathcal{A}_j$) or as a drawback (because, as we will see by an example, it maybe requires a considerable computing effort). In the second case, it is possible to reduce the needs of computation by using a number that does not depend on t. A reasonable way to carry out this change on the point of view is by replacing the type-$(1,0)$ overlap index $O : \mathrm{FS}(U) \times \mathrm{FS}(U) \to [0,1]$ on the set U by a type-$(2,0)$ overlap index $i : \mathrm{FS}_2(U) \times \mathrm{FS}_2(U) \to [0,1]$ on the same set. In this case, we consider the following second approach, that depends on:

1) a lower overlap function $G : [0,1] \times [0,1] \to [0,1]$;
2) an aggregation function $M : [0,1]^p \to [0,1]$; and
3) a type-$(2,0)$ overlap index $i : \mathrm{FS}_2(U) \times \mathrm{FS}_2(U) \to [0,1]$ on the set U.

Algorithm 2

Input: A set of p rules $\{R_j : \mathcal{A}_j \twoheadrightarrow \mathcal{B}_j\}_{j=1}^{p}$, a fact $\mathcal{A}'$ and $t \in [0,1]$.
Output: $\mathcal{B}'(\cdot)(t)$.
1: Select an aggregation function M, a lower overlap function G and a type-$(2,0)$ overlap index i on U
2: **for** $j \in \{1,2,\ldots,p\}$ **do**
3: Compute $i\left(\mathcal{A}', \mathcal{A}_j\right)$
4: **end for**
5: Construct $\mathcal{B}'(\cdot)(t)$ given, for all $v \in V$, by

$$\mathcal{B}'(v)(t) = \mathop{M}_{j=1}^{p} G\left(\mathcal{B}_j(v)(t), i\left(\mathcal{A}', \mathcal{A}_j\right)\right).$$

This second algorithm also satisfies similar properties than described in Theorem 39.

Theorem 43: Let consider the type-2 fuzzy rule-based system detailed in (4), where $\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_p, \mathcal{A}' \in \mathrm{FS}_2(U)$ and $\mathcal{B}_1, \mathcal{B}_2, \ldots, \mathcal{B}_p \in \mathrm{FS}_2(V)$ are given. Then the following properties are fulfilled under Algorithm 2.

1) If $\mathcal{A}', \mathcal{A}'' \in \mathrm{FS}_2(U)$ are such that $\mathcal{A}' \leq \mathcal{A}''$, then $\mathcal{B}' \leq \mathcal{B}''$.
2) If $\mathcal{A}'$ is completely disjoint to each $\mathcal{A}_j$, $j \in \{1,2,\ldots,p\}$, then $\mathcal{B}' = \mathcal{F}^0$ on V.
3) If $M \geq \max$, $G(t,1) = t$ for all $t \in [0,1]$ and $i\left(\mathcal{A}', \mathcal{A}_j\right) = 1$ for all $j \in \{1,2,\ldots,p\}$, then $\mathcal{B}' \geq \max(\mathcal{B}_1, \mathcal{B}_2, \ldots, \mathcal{B}_p)$.
4) If $M = \max$, $G(t,1) = t$ for all $t \in [0,1]$ and $i\left(\mathcal{A}', \mathcal{A}_j\right) = 1$ for all $j \in \{1,2,\ldots,p\}$, then $\mathcal{B}' = \max(\mathcal{B}_1, \mathcal{B}_2, \ldots, \mathcal{B}_p)$.
5) If $M \geq \max$, $G \geq \min$ and $\mathcal{B}_j(v)(t) \leq i\left(\mathcal{A}', \mathcal{A}_j\right)$ for all $j \in \{1,2,\ldots,p\}$, $v \in V$ and all $t \in [0,1]$, then $\mathcal{B}' \geq \max(\mathcal{B}_1, \mathcal{B}_2, \ldots, \mathcal{B}_p)$.

Proof. For the sake of completeness, we include the proof of the first item (the rest are similar).

(1) Suppose that $\mathcal{A}' \leq \mathcal{A}''$. Since i is increasing, then $i\left(\mathcal{A}', \mathcal{A}_j\right) \leq i\left(\mathcal{A}'', \mathcal{A}_j\right)$ for all $j \in \{1,2,\ldots,p\}$. Furthermore, the increasing character of G and M leads to $\mathcal{B}' \leq \mathcal{B}''$. ■

Corollary 44: If $G \geq \min$ and $\mathcal{B}(v)(t) \leq i(\mathcal{A}, \mathcal{A})$ for all $v \in V$ and all $t \in [0,1]$, then the conclusion of the particular type-2 modus ponens:

$$\begin{array}{lll} \text{Rule:} & \text{If } (u,t) \text{ is } \mathcal{A}, \text{ then} & (v,s) \text{ is } \mathcal{B} \\ \text{Fact:} & (u,t) \text{ is } \mathcal{A} & \end{array}$$

under Algorithm 2 satisfies $\mathcal{B}' \geq \mathcal{B}$. And if $G = \min$, then $\mathcal{B}' = \mathcal{B}$.

Proof. In this particular case, as $\mathcal{A}' = \mathcal{A}_1 = \mathcal{A}$ and $\mathcal{B}_1 = \mathcal{B}$, then

$$\begin{aligned} \mathcal{B}'(v)(t) &= G\left(\mathcal{B}_1(v)(t), i\left(\mathcal{A}', \mathcal{A}_j\right)\right) \\ &= G\left(\mathcal{B}(v)(t), i(\mathcal{A}, \mathcal{A})\right) \geq \min\left(\mathcal{B}(v)(t), i(\mathcal{A}, \mathcal{A})\right) \\ &= \mathcal{B}(v)(t) \end{aligned}$$

for all $v \in V$ and all $t \in [0,1]$, so $\mathcal{B}' \geq \mathcal{B}$. And if $G = \min$, then the equality holds. ■

C. Example and discussion

In this section, given $a, b, c \in [0,1]$ such that $a \leq b \leq c$, we use the notation (a,b,c) to refer to the triangular fuzzy number on $\mathrm{FS}([0,1])$ whose membership function is, for each $t \in [0,1]$,

$$(a,b,c)[t] = \begin{cases} \dfrac{t-a}{b-a}, & \text{if } a < t < b, \\ 1, & \text{if } t = b, \\ \dfrac{c-t}{c-b} & \text{if } b < t < c, \\ 0, & \text{otherwise.} \end{cases}$$

Let $U = \{u_1, u_2, u_3\}$ and $V = \{v_1, v_2\}$ be two finite sets. Let consider the following two type-2 fuzzy rules $\{R_j : \mathcal{A}_j \to$

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$\mathcal{B}_j\}_{j=1}^{2}$ $(p=2)$, where:

$$\mathcal{A}_1 \equiv \begin{cases} \mathcal{A}_1(u_1) = (0.3, 0.5, 0.7), \\ \mathcal{A}_1(u_2) = (0.2, 0.6, 0.8), \\ \mathcal{A}_1(u_3) = (0.5, 0.75, 1); \end{cases}$$

$$\mathcal{B}_1 \equiv \begin{cases} \mathcal{B}_1(v_1) = (0.4, 0.5, 0.7), \\ \mathcal{B}_1(v_2) = (0.8, 0.9, 1); \end{cases}$$

$$\mathcal{A}_2 \equiv \begin{cases} \mathcal{A}_2(u_1) = (0.2, 0.3, 0.5), \\ \mathcal{A}_2(u_2) = (0.3, 0.4, 0.8), \\ \mathcal{A}_2(u_3) = (0.7, 0.8, 0.9); \end{cases}$$

$$\mathcal{B}_2 \equiv \begin{cases} \mathcal{B}_2(v_1) = (0.5, 0.7, 0.8), \\ \mathcal{B}_2(v_2) = (0.3, 0.5, 0.8), \end{cases}$$

and the fact

$$\mathcal{A}' \equiv \begin{cases} \mathcal{A}'(u_1) = (0.4, 0.6, 0.8), \\ \mathcal{A}'(u_2) = (0.2, 0.5, 0.7), \\ \mathcal{A}'(u_3) = (0.8, 0.9, 1). \end{cases}$$

For simplicity on computations, in this example we use the lower overlap function $G(t,s) = t \cdot s$ for all $t, s \in [0,1]$, the aggregation function $M(t,s) = (t+s)/2$ for all $t, s \in [0,1]$ (using $p = 2$ arguments), the type-$(1,0)$ overlap index O : $\mathrm{FS}(U) \times \mathrm{FS}(U) \to [0,1]$ given, for all $A, B \in \mathrm{FS}(U)$, by

$$O(A,B) = \sup_{u \in U} \min\{A(u), B(u)\}$$

(for Algorithm 1) and the type-$(2,0)$ overlap index i : $\mathrm{FS}_2(U) \times \mathrm{FS}_2(U) \to [0,1]$ given, for $\mathcal{A}, \mathcal{B} \in \mathrm{FS}_2(U)$, by:

$$i(\mathcal{A},\mathcal{B}) = \sup \Big(\min\{\mathcal{A}(u)(t), \mathcal{B}(u)(t)\} : u \in U,\ t \in [0,1] \Big)$$

(for Algorithm 2). After carrying out all computations, the type-2 fuzzy set $\mathcal{B}'_1$ obtained by Algorithm 1 is:

$$\mathcal{B}'_1(v_1)(t) = \begin{cases} 1 - 7.5t + 12.5t^2, & \text{if } 0.4 < t \le 0.5, \\ -0.25 + 1.25t, & \text{if } 0.5 < t \le 8/15, \\ 1.75 - 2.5t, & \text{if } 8/15 < t \le 0.7, \\ 0, & \text{otherwise}; \end{cases}$$

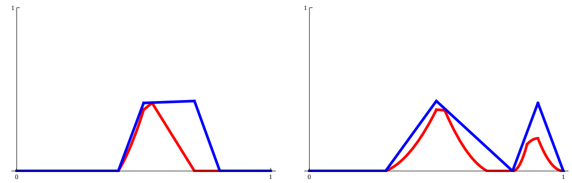
$$\mathcal{B}'_1(v_2)(t) = \begin{cases} (3 - 25t + 50t^2)/8, & \text{if } 0.3 < t \le 0.5, \\ (-4 + 25t - 25t^2)/6, & \text{if } 0.5 < t \le 8/15, \\ (28 - 75t + 50t^2)/6, & \text{if } 8/15 < t \le 0.7, \\ 32 - 80t + 50t^2, & \text{if } 0.8 < t \le 6/7, \\ -16 + 36t - 20t^2, & \text{if } 6/7 < t \le 0.9, \\ 20(1-t)^2, & \text{if } 0.9 < t \le 1, \\ 0, & \text{otherwise}; \end{cases}$$

and the type-2 fuzzy set $\mathcal{B}'_2$ obtained by using Algorithm 2 is:

$$\mathcal{B}'_2(v_1)(t) = \begin{cases} 5(5t-2)/6, & \text{if } 0.4 < t \le 0.5, \\ 5(2t+13)/168, & \text{if } 0.5 < t \le 0.7, \\ 6(4-5t)/7, & \text{if } 0.7 < t \le 0.8, \\ 0, & \text{otherwise}; \end{cases}$$

$$\mathcal{B}'_2(v_2)(t) = \begin{cases} 0, & \text{if } 0 \le t \le 0.3, \\ 3(10t-3)/14, & \text{if } 0.3 < t \le 0.5, \\ 2(4-5t)/7, & \text{if } 0.5 < t \le 0.8, \\ 5(5t-4)/6, & \text{if } 0.8 < t \le 0.9, \\ 25(1-t)/6, & \text{if } 0.9 < t \le 1. \end{cases}$$

Figure 2 represents the fuzzy sets $\mathcal{B}'_1(v_1)$ and $\mathcal{B}'_2(v_2)$ by using either Algorithm 1 (in red) or Algorithm 2 (in blue).



(a) $\mathcal{B}'(v_1)$ (b) $\mathcal{B}'(v_2)$

Figure 2: Comparison of the type-2 fuzzy outputs $\mathcal{B}'$ by employing Algorithm 1 (red) and 2 (blue).

We can observe that, by using the lower overlap function $G(t,s) = t \cdot s$, Algorithm 1 is more complicated because it leads to a moment in which the membership functions are multiplied. Therefore, linear functions (associated to triangular fuzzy numbers) are transformed into (convex or concave) parabolic functions, which are more difficult to handle.

As all functions we have employed are continuous, then the obtained type-2 fuzzy sets are also continuous.

In the example below it was shown that the fuzzy sets $\mathcal{B}'(v_1)$ and $\mathcal{B}'(v_2)$ with Algorithm 1 are less than or equal to their corresponding functions by employing Algorithm 2.

To finish this Section, we highlight that there are other settings in which type-$(2,k)$ overlap indices can be of great help, especially in applications. For instance, there are many contexts where type-1 or type-2 fuzzy sets are operated through aggregation functions. For this, it is usually necessary to descend to the real hyperplane and there, by considering real numbers, to apply the aggregation function. In other words, two type-2 fuzzy sets $\mathcal{A}, \mathcal{B} \in \mathrm{FS}_2(X)$ are operated by applying an aggregation function M to the real numbers $\mathcal{A}(x)(t)$ and $\mathcal{B}(x)(t)$, where $x \in X$ and $t \in [0,1]$. Nevertheless, overlap indices open new ways in which the type-2 fuzzy sets $\mathcal{A}$ and $\mathcal{B}$ can be directly operated by obtaining a type-2 fuzzy set $\mathcal{I}(\mathcal{A},\mathcal{B}) \in \mathrm{FS}_2(X)$, a type-1 fuzzy set $I(\mathcal{A},\mathcal{B}) \in \mathrm{FS}_1(X)$, or even a scalar $i(\mathcal{A},\mathcal{B}) \in \mathrm{FS}_0(X)$. Such families of mappings enrich the possibilities of the research. This is the case of *similarity measures* among type-1 or type-2 fuzzy sets, that can be seen as algebraic structures that associate a real number to each pair of type-1 or type-2 fuzzy sets trying to evaluate how similar or different the fuzzy sets are (see [67], [68], [69]). Such fuzzy measures usually satisfy an overlap property (see [67], [69]). It seems reasonable in prospect works to study the possible use of type-$(2,k)$ overlap indices on these applied scenarios.

V. Conclusions and Further Research

Classifying objects is a complex task, especially when overlapping is observed and fuzzy information is involved. In such case, it is necessary to consider appropriate (adapted) criteria to the context in which we are working. ¿From the computational point of view, overlap indices and overlap functions are two of the main tools that we can use to address this problem (when the involved objects are fuzzy sets). One of their main advantages is that such families of functions can

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be used in different contexts such as image classification and fuzzy rule-based systems. However, in the literature there is no such approach to support the overlapping problem whenever the uncertainty of the involved objects is handled as type-2 fuzzy sets.

In this manuscript we have introduced the notion of type-$(2, k)$ overlap index (where $k \in \{0, 1, 2\}$) as a coherent extension of previous indices that had been successfully applied in situations in which two overlapped fuzzy objects are compared. The proposed definition has a clear connection with previous interpretations, so many interrelationships naturally appear from distinct levels of fuzziness. We have also illustrated through two distinct algorithms how to apply type-$(2, k)$ overlap indices to obtain a conclusion as the result of a type-2 fuzzy rule-based system. In addition to this, we have discussed about the arguments that led us to the proposed definition.

However, taking into account the natural interest on the applicability of overlap indices and overlap functions in computational developments related to real life problems, further research must be carried out in order both to deep in the main properties of the given notion and to propose some other approaches that could also be coherent with human behavior and human decision making (see Remark 16 for alternative definitions that deserve investigation).

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