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Analytic Approximations of Integral Transforms in Terms of Elementary Functions: Application to Special Functions

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"De pocas partidas he aprendido tanto como de la mayoría de mis derrotas"

"You may learn much more from a game you lose than from a game you win"

– José Raúl Capablanca, III World Chess Champion

To my parents,
my brother,
my friends,
and Sara.

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From the bottom of my heart... THANKS!!!

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Abstract

This thesis focuses on the study of new analytical methods for the approximation of integral transforms and, in particular, of special functions that admit an integral representation. The importance of these functions lies in the fact that they are solutions to a great variety of functional equations that model specific physical phenomena. Moreover, they play an important role in pure and applied mathematics, as well as in other branches of science such as chemistry, statistics or economics. Usually, the integrals defining these special functions depend on various parameters that have a specific physical meaning. For this reason, it is important to have analytical techniques that allow their computation in a quick and easy manner. The most commonly used analytical methods are based on series expansions of local validity: Taylor series and asymptotic (divergent) expansions that are, respectively, valid for small or large values of the physically relevant variable. However, neither of them is, in general, simultaneously valid for large and small values of the variable.

In this thesis we seek new methods for the computation of analytic expansions of integral transforms satisfying the following three properties:

(a) The expansions are uniformly valid in a large region of the complex plane. Ideally, these regions should be unbounded and contain the point 0 in their interior.

(b) The expansions are convergent. Therefore, it is not necessary to obtain error bounds or to study the optimal term to truncate the expansion: the more terms considered, the smaller the error committed.

(c) The expansions are given in terms of elementary functions.

We develop a theory of *uniform expansions* that shows the necessary and sufficient conditions to obtain expansions of integral transforms fulfilling the three conditions (a), (b) and (c) above. This theory is applied to obtain new series approximations satisfying (a), (b) and (c) of a large number of special functions. The new expansions are compared with other known representations that we may find in the literature to show their advantages and drawbacks. In contrast to the Taylor and asymptotic expansions, the main benefit of the uniform expansions is that they are valid in a large region of the complex plane. For this reason, they may be used to replace the function they approximate (which is often difficult to work with) when it appears in certain calculations, such as a factor of an integral or in a differential equation. Since these developments are also given in terms of elementary functions, such calculations may be carried out easily.

Next, we consider a particularly important case: when the kernel of the integral transform is given by an exponential. We develop a new Laplace's method for integrals that produces asymptotic and convergent expansions, in contrast to the classical Laplace method which produces divergent developments. The expansions obtained with this new method satisfy (a) and (b) but not (c), since the asymptotic sequence is given in terms of incomplete beta functions. Finally, we develop a new uniform asymptotic method "saddle point near an end point" which does not satisfy (b) and (c) but, unlike the classical "saddle point near an end point" method, allows us to calculate the coefficients of the expansion by means of a simple and systematic formula.

Chapter 1

Introduction

There is not a formal definition of what a special function is, although they can be defined as those functions that have a more or less established names and notation and that are solution to a large variety of functional equations of great significance in mathematics and physics. They constitute a vast researching field in the area of applied mathematics as they appear in a large variety of disciplines, not only in mathematics and physics, but also in others fields such as economy and statistics. The easiest examples of special functions are elementary functions such as the exponential, the logarithm or the trigonometric functions. However, when we talk about special functions we are typically thinking in non-elementary functions like, for example, Euler's gamma function $\Gamma(z)$, the error function or Bessel functions. They usually admit and integral or a series representation and, due to their appearance in a lot of different scientific areas, their analytic approximation is very important.

1.1 Historical background

The origin of non-elementary special functions dates back some hundreds of years ago, to the days of great mathematicians like Euler, Legendre, Gauss and Riemann. They introduced the first examples of special functions:

Euler faced the problem of generalizing the factorial of a number to non-natural numbers. Besides, motivated by the formula of the sum of the first n natural numbers, he was looking for a similar, closed, simple formula, given in terms of elementary functions, to compute the factorial of a number. He proved however that such a simple formula does not exist for $n!$ [33, 41] and gave as solution an integral representation:

$$n! = \int_0^1 (-\log x)^n \, dx.$$

The integral representation is indeed valid for non-negative real values of n. Nowadays, this function is called Euler's gamma function, it is usually written in the form $\Gamma(z) = \int_0^\infty e^{-t} t^{z-1} \, dt$ and plays a crucial role in mathematics. G. Michon [87] wrote "Arguably, the most common special function, or the least *special* of them. The other transcendental functions [...] are called *special* because you could conceivably avoid some of them by staying away from many specialized mathematical topics. On the other hand, tha gamma function $\Gamma(z)$ is most difficult to avoid". Just to name some applications, it is used to evaluate a lot of integrals and compute products, and it is extensively used in quantum physics, astrophysics and statistics.

On the other hand, Euler also studied the sum of the inverses of integer numbers to the power of a real number. In particular, he solved the famous Basel problem linking the summation of the reciprocals of the squares of all the natural numbers with the square of π. Riemann extended the study to the complex plane by defining the famous Riemann zeta function [130]. He proved that it is a meromorphic function and found the functional equation too. He stablished a relation between this function and the distribution of prime numbers and he also conjectured that all non-trivial zeros of the zeta function have real part equal to $1/2$. This problem, known as *Riemann's conjecture*, is still an open problem considered by many mathematicians to be the most important unsolved problem in pure mathematics, being the only problem listed in both, Hilbert's twenty-three Problems of 1900 and the Millenium Prize Problems created in 2000 by the Clay Mathematics Institute. Apart from its importance in pure mathematics, the Riemann zeta function has also many applications in physics. For example, it determines the critical gas temperature and density for the Bose-Einstein condensation phase transition in a dilute gas [64] and it is usually used in quantum field theory to perform regularizations to evaluate formally divergent series.

Legendre studied elliptic integrals [63], that is, integrals of the form $\int R(x,y)dx$ where $R(x,y)$ is a rational function of the two variables x and y, and y^2 is a polynomial of the third or fourth degree in x. He was able to show that, in general, those integrals can not be expressed in terms of elementary functions but he also revealed that, using suitable transformations, all elliptic integrals may be written in terms of three standard integrals that are nowadays named after him. The name *elliptic integrals* is due to the fact that these integrals are used to evaluate the lenght of an ellipse. However, they can also be used to obtain the lenght of other plane curves such as an hyperbola or a Bernoulli's lemniscate [28]. But, moreover, these special functions play an important role in other branches of mathematics and physics. For example, they appear in a natural way in geometrical and statistical problems [51, 128]; they appear in certain electromagnetic problems [151] and their zeros may be used to obtain an upper bound for the number of limit cycles of certain hamiltonian systems [146]. Many more applications of elliptic integrals may be found in [61].

For his part, Gauss studied an infinite series starting by $1+\frac{\alpha\beta}{1\cdot\gamma}x+\frac{\alpha(\alpha+1)\beta(\beta+1)}{1\cdot 2\cdot\gamma(\gamma+1)}x\,x+\ldots$, that was presented to the mathematical comunity in 1813 [50]. It was the first full systematic treatment of such a series and, although some transformation formulas were found by Gauss, it was clear that, the nowadays called Gauss hypergeometric function, could not be expressed by means of a closed, simple formula, in terms of elementary funcions. This function was later studied by Riemann who characterised the second-order differential equation it satisfies by its three regular singularities (on the Riemann sphere) [129]. A lot of mathematical functions are special or limiting cases of the Gauss hypergeometric function. For example, trigonometric functions and the logarithm function, the Kummer and Bessel functions, the complete elliptic integrals, and a lot of orthogonal polynomials such as Jacobi, Gegenbauer, Legendre, and Meixner polynomials, are all especial cases of the Gauss hypergeometric ${}_2F_1$ function [104].

Hence, the first investigations on special functions were made in the begining of the eighteenth century. The number of special functions, their importance and applications started to increase during the next centennial, but it was only on the twentieth century, with the development of quantum mechanics, that special functions became a vast topic of interest, as they appear as fundamental solutions of differential equations derived from spectral problems in atomic potentials. For example, Hermite polynomials are associated with the eigenfunctions of the harmonic oscillator [57, §18.39] and the spherical harmonics are eigenfunctions of the Laplacian operator in spherical coordinates. Furthermore, Bessel and hypergeometric functions arise when solving fundamental equations such as Schrödinger, Helmholtz or Dirac equations by the method of separation of variables [98]. Since then, special functions had appeared not only in pure mathematical problems or in quantum mechanics, but also in a lot of differents field of study related with engineering problems, numerical simulations, economic analysis, statistics, optics, chemistry... For this reason, the list of special function is endless: error functions, Fresnel integrals, Airy functions, Bessel functions, Kelvin's function, Struve functions, parabolic cylinder functions, confluent hypergeometric functions, generalized hypergeometric functions, Meijer G-function, orthogonal polynomials, polylogarithms, theta functions, Mathieu functions, Lamé functions, catastrophes integrals,... and many more [40, 106, 108]

1.2 State of the art

Many books and articles dedicated to the study of special functions have been written: definitions, representations, inequalities, differential equations, recurrence relations, symmetries, analytic approximations, numerical evaluations, applications, etc. Among the most modern books, we highlight [140], where the most important analytical properties of a large family of special functions relevant in mathematical physics is exhibited; [16] that includes a summary of formulas related with special functions; or [106] the most modern and complete, elaborated by the biggest experts on special functions around the world with the aim to update and extend the highly cited "A&S" [1]. In [106] almost everything which is known about special functions is referenced. Its on-line version [108] is under constant revision and periodically incorporates the new researches on special functions.

Due to the significance of special functions, their analytical and numerical approximation is very important. In this regard, many techniques have been developed: numerical methods for differential equations or integrals, continued fractions, lower and upper bounds, power series, asymptotics expansions...

On a modern computer, numerical evaluation of special functions or integral transforms can be easily carried out. However, there are a lot of scientific problems where it is important to have an analytic expression of integral transforms and where a numerical evaluation would be useless because there are several physical parameters involved; because we need to know its behavior for small or large values of a certain parameter or because we have to perform further analytic computations with them. For example and without the aim of being exhaustive, we give some such situations although we may find many more in the literature:

- In acoustics and optical diffraction, wave movement is described by high oscillatory integrals (catastrophe integrals [12]) whose numerical computation becomes unstable or highly time-consuming, but analytic representations give and insight into their structure and supplement numerical computations [11]. Moreover, it is important to have analytical representations of the catastrophe integrals due to their importance in the uniform asymptotic analysis of oscillatory integrals with several coalescing saddle points [102].

- When working with a signal in optical physics it is important to have an analytic representation of a Fourier transform to compute the frequency spectrum of the signal.

- The use of numerical methods to approximate integrals for accurate real-time rendering of surfaces lit by non-occluded area light sources leads to results with noise, hardly compatible with real-time rendering constraints [62].

- In quantum field theory in theoretical physics, it is important to have an analytic representation of the beta function [126, Ch. 16] in order to fully understand the dependence of the coupling parameter on the energy scale of a given physical process.

- In the problem of calculating the contributions to the specific heat of a crystal from the vibrations of the atoms, an analytical approximation of the energy of the quantum harmonic crystal for small temperatures shows its relation with the black-body radiation [138, p.301], whereas a numerical evaluation does not.

- In singular perturbation problems it is usually necessary to have an analytic solution of a simplified problem to compute its partial derivatives and derive bounds that are used afterwards to design numerical methods of the real, non-simplified problem.

- In astronomy, analytic representations of elliptic functions are required to make accurate descriptions and predictions of celestial mechanics of Keplerian orbital motions as well as to describe pendular movement in a constant gravitational field [140].

Due to the significance of analytical approximations of integral tranforms (or special functions having an integral representation), they are the object of study of this thesis. In practice, the most used techniques to derive analytic expansions are based on local approximations that are either convergent (Taylor series) or divergent (asymptotic expansions), usually given in terms of elementary functions. These expansions are usually derived from the theory of differential equations or integrals and are still an important researching field. Most of them can be found in [106, 108].

Both, Taylor series and asymptotic expansions are usually very accurate near the point of approximation, but they are useless when one moves away that point. Then, these approximations are not useful when one needs to know the behavior of the function in a large set of the complex plane. For example, in spectral problems of atomic models in quantum mechanics where the computation of the spectrum requires the knowledge of the analytic behavior of the eigenfunctions for both, large and small values of the radius. Moreover, if a certain special funcion appears as a factor in the integrand of an integral or in a differential equation, it is usually hard to work with it. It would be nice to replace that function by an elementary function and to work with that simplified expression. That simpler expression could be, for example, the first few terms of a series approximation. However, in general, this can not be done by using Taylor series or asymptotic expansions as they do not hold uniformly in a large enough set. Instead, it would be great to have at our disposal expansions valid for small and large values of the variable. Ideally, those expansions should be valid in unbounded sets of the complex plane that contain the origin in its interior.

Therefore, we study parametric integrals of the form

$$F(z) = \int_a^b g(t)h(t,z)dt, \tag{1.1}$$

for certain functions $g(\cdot)$ and $h(\cdot, z)$. This integral may correspond with an integral representation of a special function $F(z)$ or it may be a general integral transform. On the one hand, a lot of special funcions may be cast under this form and even several special functions are defined via such an integral representation. On the other hand, a lot of important mathematical integral transforms such as Laplace, Fourier or Stieltjes transforms are given by the integral (1.1), for a certain kernel $h(t,z)$. Moreover, many physical quantities in classical mechanics, astronomy or quantum mechanics are also given by an integral like (1.1).

The most used analytical approximations for integrals of the form (1.1) are Taylor series and asymptotic expansions. The former are found, in the particular case $h(t,z) = h(tz)$, by considering the Maclaurin expansion of $h(tz)$ at the point $t = 0$ or by an application of the Frobenius method in the differential equation satisfied by the special function. In general, the function $h(t,z)$ has some finite singularities and therefore, the classical Taylor expansion is only valid for small values of the variable $|z|$. Contrarily, asymptotic expansions are found by considering an expansion of a factor in the integrand at its asymptotically relevant point or by an application of Olver's method in the differential equation satisfied by the special function. Typically, the integration interval is unbounded and therefore, when we consider the Taylor expansion of a factor in the integrand and we interchange summation and integration, we obtain an asymptotic expansion at the cost of losing the convergence of the expansion.

Consider for example the Bessel function of the first kind $J_\nu(z)$ [107]. Its power series expansion is well-known [107, eq. 10.2.2]

$$\left(\frac{2}{z}\right)^\nu J_\nu(z) = \sum_{k=0}^{n-1} \frac{(-z^2/4)^k}{k!\Gamma(\nu+k+1)} + R_n^{(0)}(\nu,z). \tag{1.2}$$

On the other hand, its asymptotic expansion in terms of inverse powers of z is given by [107, eq. 10.17.3]

$$\sqrt{\frac{\pi z}{2}} J_\nu(z) = \cos\left(z - \frac{\nu\pi}{2} - \frac{\pi}{4}\right) \sum_{k=0}^{n-1} \frac{a_{2k}(\nu)}{(-z^2)^k} - \sin\left(z - \frac{\nu\pi}{2} - \frac{\pi}{4}\right) \sum_{k=0}^{n-1} \frac{a_{2k+1}(\nu)}{z(-z^2)^k} + R_n^{(\infty)}(\nu,z), \tag{1.3}$$

valid for $|\arg z| \le \pi - \delta < \pi$, and $a_0(\nu) = 1$ and, for $k = 1, 2, \ldots$,

$$a_k(\nu) = \frac{(4\nu^2 - 1^2)(4\nu^2 - 3^2)\cdots(4\nu^2 - (2k-1)^2)}{k! 8^k}.$$

These expansions are given in terms of elementary functions but they have a drawback: they are not uniformly valid for all values of z. More precisely, the remainder $R_n^{(0)}(\nu,z)$ is unbounded for large values of $|z|$, whereas the remainder $R_n^{(\infty)}(\nu,z)$ is unbounded for

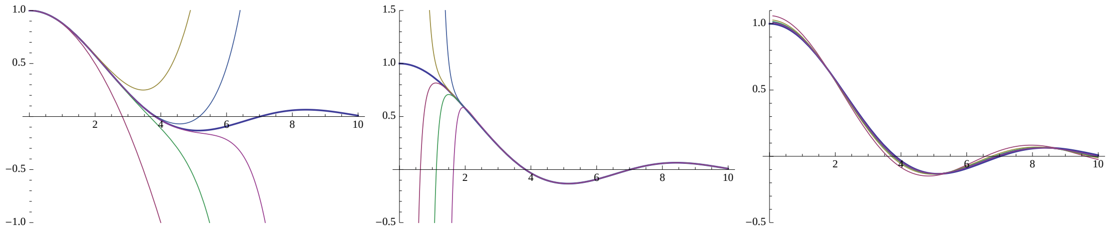


Figure 1.1: Approximation of $\frac{2}{x}J_1(x)$ (thicker graphics) given by the Taylor expansion (1.2) (left), the asymptotic expansion (1.3) (middle) and the uniform expansion (1.4) (right) for $x \in [0, 10]$ and five degress of approximation $n = 1, 2, 3, 4, 5$ (thinner graphics).

small values of $|z|$. On the other hand, in [66, §2, Theorem 1] we can find the following expansion given in terms of elementary functions and that holds uniformly in z in any fixed horizontal strip of the complex plane:

$$\frac{\sqrt{\pi}\Gamma(\nu+1/2)}{2(z/2)^\nu} J_\nu(z) = P_{n-1}(z,\nu)\frac{\sin z}{z} - Q_{n-1}(z,\nu)\cos z + R_n(z,\nu), \tag{1.4}$$

where $P_n(z, \nu)$ and $Q_n(z, \nu)$ are rational functions of z and ν [66, eq. 10]. Expansion (1.4) is valid for large and small values of $|z|$ as the remainder $R_n(z, \nu)$ can be bounded independently of z. The three expansions (1.2), (1.3) and (1.4) are compared in figure 1.1. As we may check from the figure expansion (1.4) is uniformly valid for small and large values of the variable x and produces expansions that are globally more satisfactory. As an illustration of the type of expansion (1.4) we may derive, for example, the following approximation valid for any real $x > 0$ [66, eq. 7]

$$\frac{15\pi}{2x^3} J_3(x) = \left[\frac{3x^4 - 140x^2 + 360}{8x^6} + \theta_1(x)\right] x\sin x + \left[\frac{5(x^2-18)}{2x^4} + \theta_2(x)\right] \cos x,$$

where the quantities θ_1 and θ_2 are uniformly bounded by means of $|\theta_1(x)| < 0.0062$ and $|\theta_2(x)| < 0.051$.

On the other hand, consider integral transforms of Laplace type, where the kernel $h(t, z) = e^{-zf(t)}$ is given by an exponential. In the particular case when the phase function $f(t) = t$ is simply a monomial, we can perform a logarithmic change of variables to find

$$F(z) = \int_0^\infty e^{-zt} g(t)dt = \int_0^1 u^{z-1} g(-\log u)du = \sum_{k=0}^{\infty} \frac{c_k k!}{(z)_{k+1}}, \tag{1.5}$$

where $(z)_n$ denotes the Pochhamer's symbol [4, §5.2(iii)] defined by $(z)_n = z(z + 1)(z + 2)\cdots(z + n - 1) = \Gamma(z + n)/\Gamma(z)$, and c_k are certain coefficients related with the Taylor coefficients of $g(-\log u)$ at $u = 1$ (see Appendix A). The series on the right hand side of (1.5) is not only convergent in a certain unbounded region of $\mathbb{C}$ (at the right of a vertical line), but it also is an asymptotic expansion of $F(z)$ as $z \to \infty$. Expansions of the form (1.5) are named *factorial series* and they were first studied by Newton and Stirling; and systematically analyzed by Nielsen [96, 97] and Nørlund [99].

Furthermore, many asymptotic methods involve a change of variables to obtain an asymptotic expansion of an integral. Then, the coefficients of the expansion are defined by reverting a series that is implicitly given by that change of variables. Therefore, simple, explicit formulas for those coefficients are not given in traditional text books on asymptotics [105, 154]. Instead, complicated formulas involving recursions and combinatorial objects whose complexity increase with the number of terms considered have been found [60, 92, 152, 153].

In this thesis, we face the challenge of designing, if possible, new analytic approximations of parametric integrals (special functions) satisfying the following three properties:

(a) The expansion is convergent and then, the more terms considered, the better, avoiding the study of the optimal truncation term, inherent to asymptotic (divergent) series.

(b) They hold uniformly for large and small values of a certain selected variable z, in contrast to the classical Taylor or asymptotic expansions that are only valid for small or large values of $|z|$ respectively. That is, the expansion is uniform for z in an unbounded set of the complex plane that contains the point $z = 0$. Then, for any orden of the approximation, its absolute error is bounded independently of z.

(c) The terms of the expansion are elementary functions of z whose coefficients can be computed by means of an explicit, simple and systematic formula.

In chapter 3 we investigate up to what point the derivation of an expansion satisfying requirements (a)-(b)-(c) is possible, elaborating a general theory. This theory is illustrated in chapter 4 with several examples of special functions, by deriving new and known uniform expansions of these functions. In chapter 5 we illustrate the interest of having a uniform expansion of a special function when it must be used in a later computation that requires the knowledge of the function in a large domain. In chapter 6 we investigate a dual version of the theory designed in chapter 3. In chapters 7 and 8 we introduce a new ingredient in the analysis: asymptotic expansions of integrals, in particular the method of Laplace and the uniform asymptotic method "saddle point near an end point". We investigate possible modifications of those asymptotic methods that produce expansions having some of the properties (a), (b) or (c) (maintaining of course the asymptotic property).

The methods investigated in this thesis may share with hyperasymptotic and Hadamard expansions [9, 10, 54, 93, 94, 95, 101, 113, 114, 115, 118, 120] part of the above approximation philosophy, but they are very different methods. Hyperasymptotic expansions are improvements of the classical asymptotic expansions, and may provide accurate approximations not only for large, but also for moderate values of the variables. But in general, their uniform character for large and small values of the variable is not studied, nor their convergence, nor their expression in terms of elementary functions. On the other hand, the goal of the theory of Hadamard expansions is to derive a convergent expansion of the integral in the form of a double series. It is a series of series (levels) where usually the first one encodes the standard asymptotic power series expansion. Upper levels (asymptotically less relevant) encode new corrections that become important at certain optimal truncation points of the previous series. These expansions are typically given in terms of incomplete gamma functions (not elementary functions) and their proper use require the study of optimal truncation points. Uniform aspects are not considered in general.

1.3 Structure of the thesis

In **Chapter 2** we give some preliminary results on Taylor series and asymptotic approximations, adapted to the applications that we need in the next chapters. In section 2.1 we summarize the theory of multi-point Taylor expansions [81, 82] of an analytic function. These expansions are preferable over the standard one-point Taylor series because the lemniscate of convergence of the multi-point Taylor expansion avoids the singular points of the function more efficiently than a disk. On the other hand, in section 2.2 we introduce the notion of asymptotic expansions of functions and we give two basic methods to obtain them: Watson's lemma and the Laplace's method for integrals, respectively described in subsections 2.2.1 and 2.2.2. As pointed out above, the coefficients of the method of Laplace are not given by a simple and explicit formula, but instead they are defined by reverting a certain series that is given implicitly by a change of variables. To solve this problem, a modification of the Laplace's method has been introduced in [74]. This method is summarized in subsection 2.2.3.

None of the expansions of integral transforms (1.1) obtained by means of the use of standard Taylor series or asymptotic expansions satisfy simultaneously the three properties (a), (b) and (c) described above. In particular, none of them is valid in a large region of the complex plane that contains large and small values of a certain selected variable $|z|$.

Chapter 3 of this thesis focuses on the development of a new general theory of *uniform*[1] convergent expansions of integral transforms (1.1) satisfying the three properties (a), (b) and (c) above.

The main idea is to consider the multi-point Taylor expansion of $g(t)$ at certain selected points in a way that the domain of convergence of the series contains the integration interval of (1.1), except possibly for one or both end points. When we replace $g(t)$ by its Taylor series and interchange the order of summation and integration we obtain a convergent expansion of $F(z)$. Moreover, the independence of z of the function $g(t)$ is somehow transferred to the remainder of the expansion of $F(z)$ and consequently the expansion is valid regardless of the value of z. In other words, the expansion is valid for small and large values of $|z|$.

We will show that this procedure is not only formal, but rigorous, finding accurate error bounds that do not depend on the uniform variable z. That is, in section 3.1 we state the hypotheses that the functions $g(\cdot)$ and $h(\cdot, z)$ must satisfy in order to find a convergent expansion of $F(z)$, given in terms of elementary functions, that holds uniformly in unbounded subsets of the complex plane that contain the point $z = 0$. Section 3.2 is the cornerstone of the new theory of uniform expansions. In it, we perform an accurate study of the remainder that shows not only the convergence of the new series, but also its rate of convergence.

Subsequently, a new ingredient for the approximation of integral transforms (1.1) or, in particular, for integral representations of special functions, is available: their *uniform* expansion. These approximations can be of interest to many scientific researchers around the world that work with special functions and it would be interesting to collect all of them in a new edition of the famous NIST Handbook of Mathematical Functions [106, 108].

[1]The classical meaning of *uniform* in mathematical analysis is used here. We shall not confuse it with the notion of *uniform* used in asymptotics, where the validity of an expansion as a particular parameter runs along a certain region is studied.

The seed of the idea to derive *uniform expansions* is [66] where one of the advisors of this thesis found new expansions of the Bessel functions of the first and second kind, $J_\nu(z)$ and $Y_\nu(z)$. Those expansions are convergent and hold uniformly in z in any fixed horizontal strip of the complex plane. The idea was later applied with sucess by my advisors and collaborators to many more special functions such as the incomplete gamma and beta functions or the hypergeometric Gauss function [19, 43, 44], obtaining convergent expansions, given in terms of elementary functions, that hold uniformly in z in unbounded subsets of the complex plane that also contain the point $z = 0$.

In **Chapter 4** we apply the theory of uniform approximations developed in the previous chapter to obtain new uniformly convergent representations of many special functions. In particular, we recover some uniform expansions derived by collaborators of my researching group [19, 20, 43, 44, 66]: the Bessel function of the first kind (section 4.2); the incomplete gamma function $\gamma(a, z)$ (section 4.3) uniformly valid in z; the hypergeometric confluent M function (section 4.5); and the Gauss hypergeometric function and the incomplete beta function (section 4.9). We also derive new uniform expansions for other special functions: the Struve $H_\nu(z)$ function (section 4.1); the incomplete gamma funcion $\Gamma(a, z)$ (section 4.4) uniformly valid in a; the hypergeometric confluent U function (section 4.6) and the symmetric elliptic integrals $R_F(x, y, z)$ and $R_D(x, y, z)$ (section 4.8). In all the special functions analyzed in this chapter, the uniform approximation is compared with the well-known power series and asymptotic expansions, respectively valid for small or large values of the uniform variable by means of figures and numerical tables.

As we may check from the graphics and tables presented in chapter 4, and the asymptotic behavior of the remainder of the uniform expansion may suggest, the speed of convergence of the uniform expansions is not impressive, specially in the case when one or both end points of the integration interval are singular points of the function $g(\cdot)$ in the integrand. Although uniform expansions can be useful in the numerical evaluation of special functions $F(z)$, their main advantage is that they are an analytical representation of $F(z)$ that hold uniformly in a large (possibly unbounded) set of the complex plane. Therefore, they can be used to replace the function $F(z)$ in a certain computation. More precisely, consider for example an integral or a differential equation that involves a particular function $F(z)$ whose analytical expression is complicated, and where the variable z runs in a large domain. In general, the difficult expression for $F(z)$ makes that calculation imposible. But we may just replace $F(z)$ by its uniform approximation, given in terms of elementary functions, and valid for the whole range of values of the variable z required in that calculation (range of integration, domain of the differential equation,...). In this way, that computation can be easily carried out.

In **Chapter 5** such an approach is used to derive a new convergent expansion of the Volterra function $\mu(t, \beta, \alpha)$ [40, Ch. 18, §18.3]. The Volterra function is defined by means of an integral representation having the inverse of the gamma function in its integrand. Then, we perform some changes in that integrand to get an integral representation of the Volterra function in terms of an incomplete gamma function. After that, a uniform convergent expansion of the incomplete gamma function valid in the whole domain of integration is used to interchange summation and integration, finding a new expansion of the Volterra function that is shown to be convergent.

In **Chapter 6** we consider a dual version of the uniform theory of integrals transform developed in chapter 3. That is, we study integral transforms of the form (1.1) and we consider the multi-point Taylor expansion not of $g(t)$ as in chapter 3, but of $h(t,z)$, in selected base points. If we considered the standard Taylor expansion of $h(t,z)=h(tz)$ at $t=0$, we would obtain the well-known power series expansion of the integral transform (or special function) $F(z)$. The region of convergence of this expansion is typically small. However, when we consider another, cleverly selected base point for the Taylor expansion, or even better, a multi-point Taylor expansion of the function $h(t,z)=h(tz)$, we may find larger regions of convergence that, in many cases, are unbounded and contain small values of the variable $|z|$. To illustrate this idea, in this chapter we consider the generalized hypergeometric ${}_{p+1}F_p(a,\vec{b},\vec{c};z)$ function and, using the one-, two- and three-point Taylor expansion of a factor in an integral representation, that we derive in Appendix C, we obtain different analytical representations of these functions in large (sometimes unbounded) regions of the complex plane that contain the indented closed unit disk $D^*=\{z\in\mathbb{C}:|z|\le 1,\ z\ne 1\}$. The speed of convergence of these expansions is exponential and they are shown to perform numerically better than other expansions of the ${}_{p+1}F_p(a,\vec{b},\vec{c};z)$ function that we may find in the literature. Furthermore, the three derived expansions are given in terms of rational functions and then, they satisfy the three properties (a), (b) and (c) described above.

In **Chapter 7** we consider a specially important case of the integral in the right hand side of (1.1): when the integration interval is $(0,\infty)$ and the kernel $h(t,z)$ is a exponential function of the form $h(t,z)=e^{-zf(t)}$. That is, we study parametric integrals of the form

$$F(z)=\int_0^\infty e^{-zf(t)}g(t)dt. \tag{1.6}$$

In the particular case when $f(t)=t$ we can derive a factorial series of $F(z)$ by introducing a logarithmic change of variables (see (1.5)), which is a convergent series that is also asymptotic for large z [145, §17.3]. However, generalizations of this results are not given in the literature. Therefore, we consider integral transforms of the form (1.6) with a general phase functions $f(t)$ and not simply a monomial of the first degree, and we derive an expansion of (1.6) that is both, asymptotic for large z and convergent. In contrast, the classical Laplace's method introduced in section 2.2.2 and usually used to approximate integrals of the form (1.6) produces divergent approximations. Moreover, the coefficients of the new *convergent and asymptotic Laplace's method* can be computed by means of a simple, closed formula, in terms of generalized Bernoulli polynomials. The derivation of that formula is relegated to appendix A. The price to pay for deriving a convergent Laplace's method is that the approximation is given in terms of (sometimes incomplete) beta functions.

The key point to derive this expansion is to split the phase function $f(t)$ into its asymptotically dominant part (as $z\to\infty$) and a remainder term, in the form summarized in section 2.2.3. In this way, the study is reduced to the case $f(t)=t^m$, with $m\in\mathbb{N}$ (section 7.2) that, after a change of variables, is reduced to compact Mellin transforms, analyzed in section 7.1. However, as we will see, the conditions on the functions $f(t)$ and $g(t)$ to apply the results of section 7.2 may be too restrictive. Therefore, in section 7.3 some tricks are performed to enlarge the applicability of the convergent and asymptotic Laplace's method. Finally, the method is summarized in the form of a theorem in section 7.4, where it is used to derive a convergent and asymptotic expansion of the hypergeometric confluent U function. In summary, we obtain asymptotic expansions that satisfy properties (a), (b) and partly (c) above, as the coefficients can be computed by means of an explicit formula but the asymptotic sequence is no longer given in terms of elementary functions, but incomplete beta functions.

In **Chapter 8** we continue the study of integrals (1.1) in the case when $h(t,z) = e^{-zf(t)}$ but from another point of view initiated by the advisors of this thesis: the systematization of classical asymptotic methods of integrals [68, 74, 75].

Using the modified Laplace's method introduced in [74] and summarized in section 2.2.3 simple and systematic formulas for the coefficients of the standard method of Laplace are found. The same idea was later applied in [68] to obtain a systematic formula for the coefficients of the "saddle point near a pole" uniform asymptotic method, whereas in [75] that idea was useful to simplify the computation of the paths of steepest descent in the saddle point method. We continue this line of research and revisit the uniform asymptotic method "saddle point near an end point" [154, Ch. 7, §3], [145, Ch. 22] where, depending on a certain parameter, the asymptotically dominant point of the integrand changes of nature from being an interior point to being an end point of the interval of integration. As a result, an application of the classical Laplace's method produces three expansions that are formally different. The classical uniform asymptotic method "saddle point near an end point" solves this problem by introducing the parabolic cylinder function into the game. This function encodes the abrupt transition from one case (the absolute minimum of $f(t)$ is an interior point of (a,b)) to another (the absolute minimum of $f(t)$ occurs at one end point, say $t=a$). However, as in the classical Laplace's method, simple and explicit formulas for the coefficients of the expansions are not known. Therefore, following the ideas of the modified Laplace's method of section 2.2.3 we obtain an asymptotic expansion of the integral (1.6) uniformly valid as that parameter varies. As in the classical method, the expansion is given in terms of parabolic cylinder functions but, on the other hand, the coefficients are given by means of an explicit, closed formula that is systematically computable.

Finally, in **Chapter 9** we draw some conclusions of the results of this thesis and we suggest some future work to continue this investigation.

Thoughout the thesis, for any complex variable z, $\arg z \in (-\pi,\pi]$ denotes its main argument and logarithms and fractional powers are assumed to take their principal value. The symbol $z\to\infty$ or the statement "large z" mean $|z|\to\infty$ with fixed $\arg z$. The symbol $\lfloor a \rfloor$ denotes the greatest integer less than or equal to a whereas the symbol $\lceil a \rceil$ denotes the least integer greater than or equal to a. All the computations of the thesis have been performed with the symbolic program *Wolfram Mathematica 12*, version 12.1.1.0.

Chapter 2

Preliminaries

The theory of Taylor series of analytical functions is well-known as it is a basic topic in any course of mathematical analysis. It allows to represent an analytic function $f(z)$ around a selected point $z = a$ by an infinite series (or a finite sum plus a remainder) given by integer powers of $(z - a)$, whose $n-$th coefficient is nothing but the $n-$th derivative of $f(z)$ evaluated at $z = a$ and divided by the factorial of n. It is well-known that the series is convergent in a disk around $z = a$ whose radius is determined by the closest singularity of $f(z)$ to the point $z = a$. With the use of the classical Taylor series, we find an approximation of $f(z)$ near a selected point $z = a$. But, what to do if there are two or more points that are important in the approximation of a function? We are thinking for example in the problem of finding an asymptotic expansion of an Airy-type integral with two (or more) nearby (or even coalescing) saddle points [29]; or in the study of second-order linear differential equations in the interval $[-1, 1]$ with initial conditions or boundary conditions of Dirichlet, Neumann or mixed Dirichlet-Neumann type given at the end points ± 1 [42, 45, 78, 80, 156].

Furthermore, the standard Taylor series is convergent in a disk that is too small if the singularities of the function are close to the center point $z = a$ of the Taylor expansion. Then, the use of multi-point Taylor expansions is much more appropriate because, as we will see below, the lemniscate of convergence of the multi-point Taylor expansion avoids those singular points more efficiently than the disk of convergence of the classical Taylor series.

Taylor and multi-point Taylor expansions of an analytic funcion $f(z)$ are valid in a bounded domain unless the function $f(z)$ is entire. In general they are not useful when the variable z becomes extremely large. However, in a lot of physical problems there are quantities that take values far from the rest. For example, in kinematics the speed of light is extremely large compared with other speeds; in certain central potential problems the central mass is extremely heavier than the other masses; the viscosity coefficient of a slightly viscous fluid is small compared with other constants in fluid mechanics... Moreover, small coefficients in boundary value problems lead to singularly perturbed problems [37, 38, 110] which appear, for example, in fluid or gas dynamics [84, 147], heat transfer [5, 6], atmospheric pollution [89]... Then, it is important to have solutions to this kind of problems. And these solutions should be easy to compute, in terms of the asymptotic parameter, as that parameter approaches a certain selected value (possibly infinity).

Most of the physical or engineering problems where asymptotic parameters occur are described by differential or integral equations whose solution admits an integral representation. For this reason, the theory of asymptotic approximations is divided in two main areas: the asymptotic approximation of solutions of differential equations [105, 149] and the asymptotic approximation of integrals [14, 145, 154].

The former can be directly applied to a large number of problems but, as there is not an explicit representation for the solution to be approximated, it is usually harder to apply. This is so because, apart from the differential equation, we need some more extra information to consider the right solution to be estimated. On the other hand, the solution to a lot of real problems can be written in terms of an integral by using integral transforms (Laplace, Fourier, Mellin, etc.) or Green's function techniques. Even further, the problem itself may be given by means of such an integral representation.

In this thesis, we focus on the approximation of functions given by means of an integral transform. For this reason, the asymptotic theory of integrals will be used throughout the next chapters. The main aim of the asymptotic theory of integrals is to obtain an approximation of a function $F(z)$ given in the form of a parametric integral, valid as a certain selected (asymptotic) parameter approaches a limit value. Typically, the analytic expression of $F(z)$ is difficult to work with and then, we seek expansions given in terms of simpler (ideally elementary) functions.

In the later chapters of this thesis, multi-point Taylor expansions and asymptotic expansions will be used. Then, for clarity in the exposition, the main results of these topics, adapted to the applications that we need in the next chapters, are given in the next two sections.

2.1 Multi-point Taylor expansions

The results of this section are a summary of the theory of multi-point Taylor expansions given in [81, 82] and also in [77].

Take m different points $z_1, z_2, \ldots, z_m \in \mathbb{C}$ and assume that $f(z)$ is an analytic function in an open set $\Omega \subset \mathbb{C}$ such that $z_j \in \Omega$, $\forall j = 1, \ldots, m$. Then, the function $f(z)$ has the following multi-point Taylor expansion at the m base points $z_1, z_2, \ldots, z_m$:

$$f(z) = \sum_{k=0}^{n-1} p_k(z) \left[\prod_{s=1}^{m} (z - z_s) \right]^k + r_n(z), \tag{2.1}$$

where $p_k(z)$ are polynomials of degree $m-1$ that can be represented by the following Lagrange-type formula:

$$p_k(z) := \sum_{j=1}^{m} a_{k,j} \frac{\prod_{s=1, s\neq j}^{m} (z - z_s)}{\prod_{s=1, s\neq j}^{m} (z_j - z_s)}, \tag{2.2}$$

being $a_{k,j}$ coefficients related with the first k derivatives of $f(z)$ evaluated at the base points:

$$a_{k,j} := \frac{1}{k!} \frac{d^k}{dz^k} \left[\frac{f(z)}{\prod_{s=1,s\neq j}^{m} (z - z_s)^k} \right]_{z=z_j} + \sum_{l=1, l\neq j}^{m} \frac{1}{(k-1)!} \frac{d^{k-1}}{dz^{k-1}} \left[\frac{f(z)/(z - z_j)}{\prod_{s=1,s\neq l}^{m} (z - z_s)^k} \right]_{z=z_l}. \tag{2.3}$$

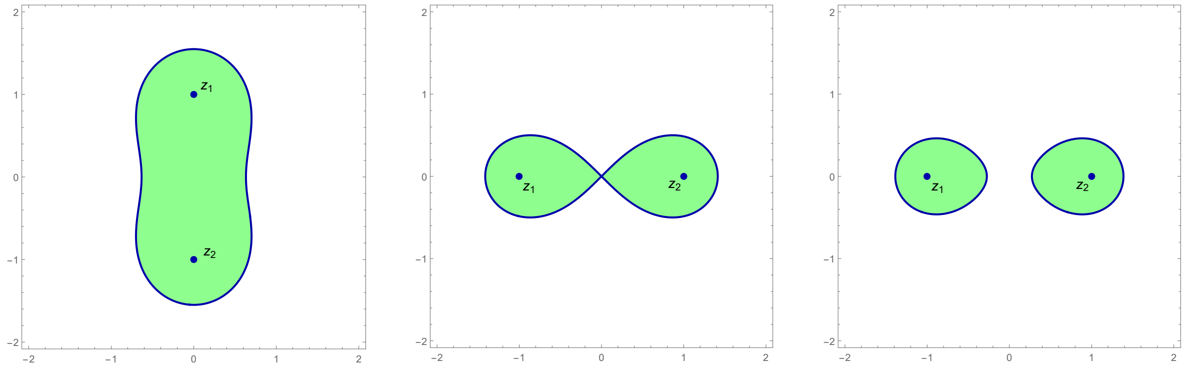


Figure 2.1: Different shapes of the Cassini oval of convergence O_m for $m = 2$ depending on the relative size of the "radius" r and the position of the two base points z_1 and z_2.

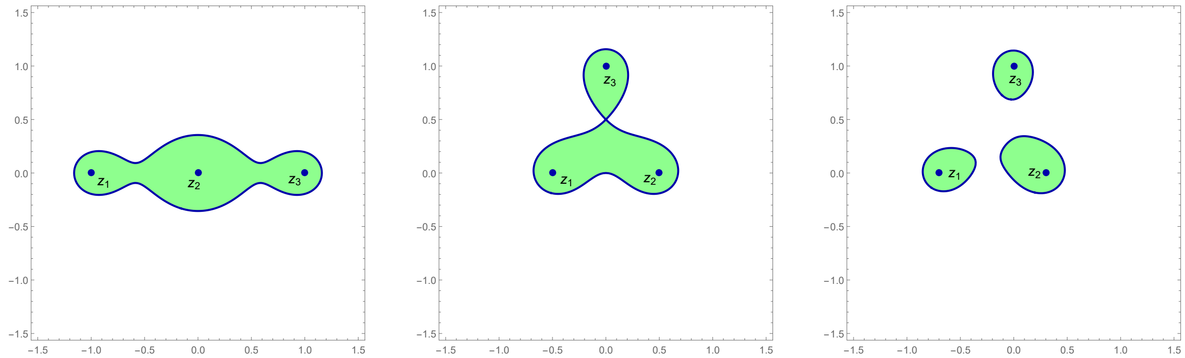


Figure 2.2: Different shapes of the lemniscate domain O_m for $m = 3$ depending on the relative size of the "radius" r and the position of the base points z_1, z_2 and z_3.

The multi-point Taylor remainder $r_n(z)$ in (2.1) is given by the following Cauchy's integral formula

$$r_n(z) = \frac{\prod_{s=1}^{m}(z - z_s)^n}{2\pi i} \oint_{\mathcal{C}} \frac{f(w)dw}{(w - z)\prod_{s=1}^{m}(w - z_s)^n}, \tag{2.4}$$

where $\mathcal{C}$ is a simple closed loop contained in Ω that encircles the points $z_1, z_2, \ldots, z_m$ and z in the counterclockwise direction.

Expansion (2.1) is uniformly and absolutely convergent inside a lemniscate O_m (see figures 2.1 and 2.2), where

$$O_m \equiv \left\{ z \in \Omega : \prod_{s=1}^{m} |z - z_s| < r \right\}, \qquad r \equiv \inf_{w \in \mathbb{C} \setminus \Omega} \left\{ \prod_{s=1}^{m} |w - z_s| \right\}.$$

If $m = 1$, the domain O_1 is a disk and we obtain the classical Taylor series expansion; if $m = 2$ the domain O_2 is a Cassini oval (see figure 2.1); for m greater than 2 the lemniscate takes different shapes depending on the relative size of the "radius" r and the position of the base points (see figure 2.2).

In the next chapters we will need to approximate a function $f(z)$ by means of multi-point Taylor expansions in a way that the lemniscate of convergence D_r contains the

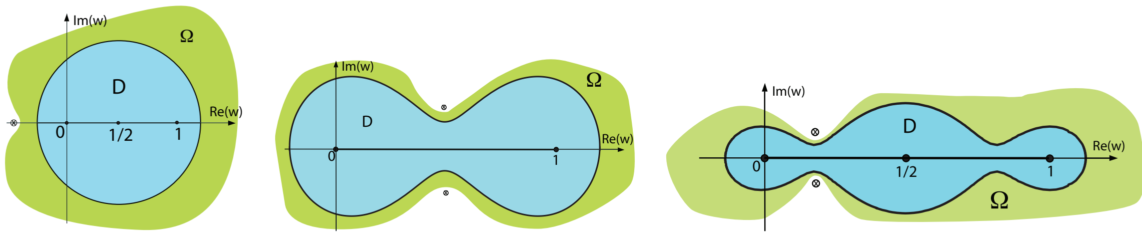


Figure 2.3: Lemniscates of convergence D_r with base points $t_1, \dots, t_m$ with $0 \le t_1 < t_2 < \dots t_m \le 1$ for different number of base points $m = 1$ (left), $m = 2$ (middle) and $m = 3$ (right). In the tree pictures, the crossed points represent the singularities of the given functions $f(w)$ and define the value of the maximal "radius" ρ. Left: $f(w) = (2w+1)^{-1}$. Disk of radius $r_0 = 1/2 < r < \rho = 1$ centered at $1/2$. Middle: $f(w) = (5 - 16w + 16w^2)^{-1}$. Cassini oval of radius $r_0 = 1/4 < r < \rho = 5/16$ and foci at $0, 1$. Right: $f(w) = (20w^2 - 8w + 1)^{-1}$ Lemniscate of radius $r_0 = 1/(12\sqrt{3}) < r < \rho = \sqrt{13}/(20\sqrt{10})$ and foci at $0, 1/2, 1$.
In every example, the lemniscate D_r contains the interval $(0,1)$ but avoids the singularities of $f(w)$. The more base points t_1, t_2,..., t_m the lemniscate D_r has, the better it avoids the singularities of $f(w)$, as D_r becomes more and more thinner (always containing the interval $(0,1)$) as we can see in the sequence of examples (left)-(middle)-(right).

interval $[0,1]$ while being contained in the analyticity region Ω of the function $f(z)$. In order to obtain it, we choose m different real points $0 \le t_1 < t_2 < \dots < t_m \le 1$ and we define the quantities

$$r_0 := \sup_{t\in(0,1)} \left\{ \prod_{s=1}^{m} |t - t_s| \right\} \tag{2.5}$$

and

$$\rho := \inf_{w\in\mathbb{C}\setminus\Omega} \left\{ \prod_{s=1}^{m} |w - t_s| \right\}. \tag{2.6}$$

We consider the lemniscate

$$D_r := \left\{ w \in \mathbb{C} : \prod_{s=1}^{m} |w - t_s| < r \right\} \tag{2.7}$$

with $r_0 \le r \le \rho$. The restriction $r_0 \le r$ assures that the interval $(0,1)$ is contained in D_r whereas the restriction $r \le \rho$ assures that D_r is contained in Ω. In other words, D_{r_0} and D_ρ are, respectively, the smallest and the largest possible lemniscates satisfying $(0,1) \subset D_r \subset \Omega$. Note that the more number of base points m considered, the thinner the convergence region D_r is (see figure 2.3). Thus, even if the function $f(z)$ has singularities that are located near the interval $[0,1]$, it is always possible to avoid those singularities by appropriately choosing a larger number of base points $t_1, \dots, t_m$, in order to assure that $\rho \ge r_0$, as explained in [80] in a different context.

On the other hand, we have seen above that the polynomials $p_k(z)$ in (2.1) can be represented by a Lagrange-type formula (2.2). Alternatively, if we write $p_k(z)$ as a sum of monomials

$$p_k(z) := \sum_{j=0}^{m-1} A_{k,j} z^j, \tag{2.8}$$

the coefficients $A_{k,j}$ can be computed using the following recurrent algorithm:

Define

$$U(z) = \prod_{s=1}^{m}(z-t_s)$$

and for $k = 1, 2, 3, \ldots,$ consider

$$\phi_k(z) := \frac{\phi_{k-1}(z) - p_{k-1}(z)}{U(z)}, \qquad \phi_0(z) = f(z).$$

Then, the coefficients $A_{k,j}$, $j = 0, 1, 2, \ldots, m-1$ are the solution to the following Vandermonde linear system

$$\begin{pmatrix} 1 & t_1 & t_1^2 & \ldots & t_1^{m-1} \\ 1 & t_2 & t_2^2 & \ldots & t_2^{m-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & t_m & t_m^2 & \ldots & t_m^{m-1} \end{pmatrix} \begin{pmatrix} A_{k,0} \\ A_{k,1} \\ \vdots \\ A_{k,m-1} \end{pmatrix} = \begin{pmatrix} \phi_k(t_1) \\ \phi_k(t_2) \\ \vdots \\ \phi_k(t_m) \end{pmatrix}, \tag{2.9}$$

where, for $k = 0, 1, 2, \ldots$ and $j = 1, 2, \ldots, m$, the numbers $\phi_k(t_j)$ are computed by means of

$$\phi_k(t_j) = \lim_{z\to t_j} \frac{\phi_{k-1}(z) - p_{k-1}(z)}{U(z)} = \frac{\phi'_{k-1}(t_j) - p'_{k-1}(t_j)}{\prod_{s=1,s\neq j}^{m}(t_j - t_s)}.$$

The matrix in the system (2.9) is a Vandermonde matrix which is, in general, ill-condicionated and some numerical care should be taken when using this method. However, in many practical examples, we take 1, 2 or at most 3 base points for the multi-point Taylor expansion that are, in general, well-distribuited along the interval $[0, 1]$, say $t_1 = 0$, $t_2 = 1/2$ and $t_3 = 1$. Therefore, numerical problems should not occur and this recurrent algorithm to compute the coefficients $A_{k,j}$ may be useful.

On the other hand, the function $f(z)$ usually satisfies a certain differential equation of the first or second order. Thence, it is usually possible to obtain a recurrence relation to compute the coefficients $A_{k,j}$ by replacing $f(z)$ and its derivatives by their corresponding multi-point Taylor expansion and equating the coefficients of equal powers.

2.2 Asymptotic expansions

In asymptotics, we usually use the notation due to Bachman and Landau [58, pp. 3-5] to describe the behavior of a complex function $f(z)$ by comparing it to another known and simpler function $g(z)$ as z approaches a selected value z_0 (possibly infinity). In particular, we write $f(z) = \mathcal{O}(g(z))$, as $z \to z_0$, to point out that, near the point $z = z_0$, the quotient $|f(z)/g(z)|$ is bounded by a positive constant. In other words,

$$f(z) = \mathcal{O}(g(z)), \quad \text{as } z \to z_0; \qquad \text{if and only if} \qquad \lim_{z\to z_0} |f(z)/g(z)| = L, \quad L < \infty.$$

Two particular values of L should be highlighted. On the one hand, if $L = 0$ we say that f is of order less than g, and we write $f(z) = o(g(z))$, as $z \to z_0$. That is,

$$f(z) = o(g(z)), \quad \text{as } z \to z_0; \qquad \text{if and only if} \qquad \lim_{z\to z_0} |f(z)/g(z)| = 0.$$

On the other hand, if $L = 1$ we say that f and g are asymptotically equal at z_0 and we denote it by $f(z) \sim g(z)$. In other words,

$$f(z) \sim g(z), \quad \text{as } z \to z_0; \qquad \text{if and only if} \qquad \lim_{z\to z_0} |f(z)/g(z)| = 1.$$

For properties on the O-symbols, we refer to [105, Ch. 1] or [119, Ch. 1].

These three symbols are also used in the definition of an asymptotic expansion of a function. First, if $\{\phi_n(z)\}_{n=0}^{\infty}$ is a sequence of continuous functions such that, for all $n = 0, 1, 2, \ldots$

$$\phi_{n+1}(z) = o\left(\phi_n(z)\right), \quad \text{as } z \to z_0, \tag{2.10}$$

we say that $\{\phi_n(z)\}_{n=0}^{\infty}$ is an *asymptotic sequence* or an *asymptotic scale*. Then, if $\{\phi_n(z)\}_{n=0}^{\infty}$ is such an asymptotic sequence as $z \to z_0$ and $f(z)$ is a function that, for every fixed $n = 0, 1, 2, \ldots$, verifies

$$f(z) - \sum_{k=0}^{n-1} a_k \phi_k(z) = O\left(\phi_n(z)\right), \quad \text{as } z \to z_0, \qquad \text{with } a_k \in \mathbb{C}, \ \forall k, \tag{2.11}$$

the formal series $\sum_{n=0}^{\infty} a_n \phi_n(z)$ is said to be an *asymptotic expansion* at $z = z_0$ of the function $f(z)$. In that case, we write

$$f(z) \sim \sum_{n=0}^{\infty} a_n \phi_n(z), \quad z \to z_0. \tag{2.12}$$

Many times in practice, the limit point z_0 will be clear by the context and we will simply write $f(z) \sim \sum_{n=0}^{\infty} a_n \phi_n(z)$. Besides, when the limit point z_0 tends to infinity we usually obtain the asymptotic sequence $\phi_n(z) = z^{-n}$. Asymptotic expansions of this form, where the asymptotic sequence is given by inverse powers of the asymptotic variable z that decrease monotonically with the order n of the approximation are called *Poincaré expansions* or *asymptotic expansion of Poincaré-type* [127].

Note that only those two properties (2.10) and (2.11) on the asymptotic scale and the remainder are important in asymptotics. It is not important if the asymptotic expansion on the right hand side of (2.12) is convergent for any value of z or not. Indeed, as a convergent Taylor series fits the definition of asymptotic expansion as well, the concept *asymptotic series* usually implies a divergent series at the point of infinity. Despite non-convergence, asymptotic expansions are still a powerful tool: if we truncate the series after a certain number of terms N we usually have as much precision as we need in practice and many times only the first few terms of the expansion are important to estimate the asymptotic behavior of $f(z)$. The approximation is usually given in terms of elementary functions and it allows to compute the approximated function in a fast way, at the vicinity of the point z_0. For this reason, it is important to find accurate bounds for the remainder and sometimes a study of the optimal truncation term of the expansion is needed, a problem inherent to divergent series.

As it has been said in the introduction of this chapter, the theory of asymptotic expansions is divided in two areas: the approximation of solutions to differential equations and the approximation of integrals. In this thesis, we are only interested in the latter case as we are interested in the approximation of integral transforms or special functions admitting an integral representation. In order to obtain an asymptotic expansion of a parametric integral we do not have a universal method that works for any given integral. Instead, there are a lot of methods that may be applied depending on the properties of the integrand and the integration path. Among others, we highlight Watson's lemma, the method of Laplace, the stationary phase method, the method of steepest descent, the Mellin transform methods and the distributional methods [105, 119, 145, 154], that may be found in [109] in a summarized manner.

In chapters 4, 5 and 6 of this thesis we will compare the new expansions derived in those chapters with well-known asymptotic expansions. On the other hand, in chapters 7 and 8 we focus on some modifications of the Laplace's method for integrals. For a better understanding of those chapters, in the next subsections we summarize the Watson's lemma, the method of Laplace and a modification of the latter, from a heuristic point of view.

2.2.1 Watson's lemma

Watson's lemma [150] is one of the simplest yet powerful results in the theory of asymptotic expansions of integrals. It can be directly applied to a lot of integrals, like for example any Laplace's transform. On the other hand, it is the basis of many other method as the Laplace's method or the steepest descent method. Basically, it consists on termwise integration of a series expansion of a factor of the integrand. More precisely:

Theorem 2.2.1. *Consider the integral*

$$F(z) = \int_0^\infty e^{-zt} g(t)dt, \qquad \Re z > 0, \tag{2.13}$$

and assume the following hypotheses.

(i) The function $g : \mathbb{R}^+ \to \mathbb{C}$ *has a finite number of discontinuities.*

(ii) The function $g(t)$ *has an asymptotic expansion at* $t = 0$ *of the form*

$$g(t) \sim \sum_{n=0}^\infty g_n t^{\frac{n+\lambda}{\mu}-1} \qquad \text{as } t \to 0^+,$$

being λ *a real or complex number with* $\Re\lambda > 0$ *and* $\mu > 0$ *a positive number.*

(iii) The integral (2.13) *converges for sufficiently large positive values of* $\Re z$.

Then the integral $F(z)$ *admits the following asymptotic expansion, for large* $|z|$

$$F(z) \sim \sum_{n=0}^\infty g_n \frac{\Gamma\left(\frac{n+\lambda}{\mu}\right)}{z^{\frac{n+\lambda}{\mu}}}, \qquad z \to \infty \tag{2.14}$$

in the sector $|\arg z| \le \frac{\pi}{2} - \delta < \frac{\pi}{2}$.

The sector of validity of expansion (2.14) can be extended when more information on the analytic properties of the function $f(t)$ is available [145, pp. 14]. Generalizations of the lemma of Watson may be found in [154, Ch. 1, §5]. Moreover, in [145, Ch. 2] we can find a Watson's lemma for loop integrals.

Example 1. Consider the function $F(z) = e^z E_1(z)$, where $E_1(z)$ denotes the exponential integral [142, eq. 6.2.2]

$$F(z) = \int_0^\infty \frac{e^{-zt}}{t+1} dt.$$

We have the expansion

$$g(t) = \frac{1}{t+1} = \sum_{n=0}^{\infty} (-1)^n t^n, \qquad |t| < 1,$$

that is convergent in any disk $D_r(0)$ centered at $t = 0$ of radius $r < 1$. Thus, in this particular example, expansion (2.14) reads [142, eq. 6.12.1]

$$F(z) \sim \sum_{n=0}^{\infty} \frac{(-1)^n n!}{z^{n+1}}, \qquad z \to \infty. \tag{2.15}$$

We highlight that the series (2.15) is asymptotic, for large z, but it is not convergent for any value of $z \in \mathbb{C}$. The divergence of expansion (2.15) may be regarded as a penalty for interchanging summation and integration when the disk of convergence of the Taylor series expansion of $g(t)$ does not contain the integration interval[1]. In any case, this is not important in asymptotics and only the two properties (2.10) and (2.11) for the asymptotic sequence and the remainder matters.

2.2.2 The classical Laplace's method

The method of Laplace is a generalization of the Watson's lemma when the phase function in (2.13) is not simply $f(t) = t$ but a general function $f(t)$. More precisely, we consider integrals of the form

$$F(x) = \int_a^b e^{-xf(t)} g(t)(t-a)^{s-1} dt, \qquad a < b, \qquad s > 0 \tag{2.16}$$

where either a or b or both may be infinite. For the sake of simplicity, we assume that the asymptotic variable $x > 0$ is positive and large, and also that the functions $f(t)$ and $g(t)$ are real for real t. On the other hand, we allow a branch point at the end integration point $t = a$ given by the factor $(t-a)^{s-1}$. We also assume that $f(t)$ and $g(t)$ are smooth enough for $t \in (a, b)$.

Laplace [59] made the observation that, for large x, the peak value of $e^{-xf(t)}$ occurs at the point $t = t_0$ where $f(t)$ attains its minimum value. For large x, the peak is very sharp and the main contribution of the integrand to the integral comes from the neighborhood of that point. Then, Laplace derived the approximation

$$F(x) \sim g(t_0)(t_0 - a)^{s-1} e^{-xf(t_0)} \sqrt{\frac{2\pi}{xf''(t_0)}}, \qquad \text{as } x \to \infty,$$

which is the first term of a full asymptotic expansion that was rigorously found by Erdélyi [39, Ch. 2, §4].

[1] Even if $g(t)$ was an entire function, the convergence of the asymptotic expansion found after an application of Watson's lemma is not assured.

Erdélyi's approach is the following: If the function $f(t)$ has a finite number of minima and maxima, break up the integral in a finite number of integrals and reverse the sign of x if necessary so that in each integral $f(t)$ reaches its minimum value at one (and only one) point $t = t_0 \in [a, b)$. Then, we can perform a change of variables and apply Watson's lemma to the resulting integral to find an asymptotic expansion of the integral $F(x)$ in (2.16).

From now on, the analysis strongly depends on the position of the absolute minimum t_0 of $f(t)$ with respect to the integration interval $[a, b)$. We distinguish three different cases: (i) the absolute minimum of $f(t)$ is an interior point $t_0 \in (a, b)$ and it is a simple zero of $f'(t)$; (ii) $t_0 = a$ and it is a simple zero of $f'(t)$; (iii) the function $f(t)$ is strictly increasing in $[a, b)$ and therefore the absolute minimum of $f(t)$ in $[a, b)$ occurs at $t_0 = a$, with $f'(a) > 0$.

(i) The absolute minimum of $f(t)$ occurs at the interior point $t_0 \in (a, b)$, with $f'(t_0) = 0$ and $f''(t_0) > 0$. Then, the graph of $f(t)$ near the point $t = t_0$ is that of a parabola. For this reason, we introduce in (2.16) the change of variable $t \mapsto u$ given by $f(t) - f(t_0) = u^2$ and we extend the integration limits to $\pm\infty$, since the contribution of the tails up to $\pm\infty$ to the integral are exponentially small. We find

$$F(x) = e^{-xf(t_0)} \int_{-\infty}^{+\infty} e^{-xu^2} h(u) du, \tag{2.17}$$

where $h(u) := 2u\,(t(u) - a)^{s-1} \frac{g(t(u))}{f'(t(u))}$. The main contribution of the integrand to the integral occurs near the point $u = 0$ and, under some mild hypotheses on the functions $f(t)$ and $g(t)$, it can be shown that the function $h(u)$ has an asymptotic expansion at $u = 0$ of the form

$$h(u) \sim \sum_{n=0}^{\infty} h_n u^n, \qquad \text{as } u \to 0^+,$$

where the coefficients h_n may be computed by reverting the series at $t = t_0$ of the function $u(t)$ implicitly defined by the change of variables. They are given in terms of the coefficients of the asymptotic expansion of the functions $f(t)$ and $g(t)$ at $t = t_0$. We apply Watson's lemma to the integral (2.17): we introduce the expansion of $h(u)$ at $u = 0$ into the integral (2.17) and interchange summation and integration. We find

$$F(x) \sim e^{-xf(t_0)} \sum_{n=0}^{\infty} h_{2n} \frac{\Gamma(n + 1/2)}{x^{n+1/2}}, \tag{2.18}$$

which is an asymptotic expansion of $F(x)$, as $x \to \infty$ [154, Ch. 7, §3].

(ii) The absolute minimum of the phase function $f(t)$ occurs at one of the end integration points, say $t = a$, and it is a simple, critical point of $f(t)$. We can repeat step by step the computations of the previous case, but with two differences:

1. After the change of variable $f(t) - f(t_0) = u^2$ and adding the exponentially negligible tails, the lower integration point in the integral (2.17) is not $u = -\infty$ but $u = 0$.

2. The algebraic singularity $t = a$ (which is transformed into $u = 0$) of the integrand has a certain influence on the asymptotic behavior of the function $F(x)$. Now, the function $h(u) := 2u\,(t(u) - a)^{s-1}\,\frac{g(t(u))}{f'(t(u))}$ has an asymptotic expansion at $u = 0$ of the form
$$h(u) \sim \sum_{n=0}^{\infty} h_n u^{n+s-1}, \qquad \text{as } u \to 0^+.$$

Therefore, instead of (2.18) we obtain a different asymptotic expansion of $F(x)$ for large x [154, Ch. 7, §3]:
$$F(x) \sim \frac{1}{2} e^{-xf(t_0)} \sum_{n=0}^{\infty} h_n \frac{\Gamma\left(\frac{n+s}{2}\right)}{x^{\frac{n+s}{2}}}, \qquad \text{as } x \to \infty. \tag{2.19}$$

(iii) The function $f(t)$ is strictly incresing in $[a, b)$ and therefore it attains its absolute minimum at the end point $t = a$, but with $f'(a) > 0$. Now, the shape of the graph of $f(t)$ near the absolute minimum $t = a$ is no longer a parabola, but a straight line. Thus, we no longer perform a quadratic transformation as in the previous cases, but a change of variable of the form $f(t) - f(t_0) = u$. Then, except for exponentially small terms, we have that
$$F(x) = e^{-xf(t_0)} \int_0^{\infty} e^{-xu} \bar{h}(u) du, \tag{2.20}$$
with $\bar{h}(u) = (t(u) - a)^{s-1}\,\frac{g(t(u))}{f'(t(u))}$. As in case (ii), the branch point $t = a$ (transformed into $u = 0$) has a certain influence on the asymptotic expansion of $\bar{h}(u)$ at $u = 0$ and we find
$$\bar{h}(u) = \sum_{n=0}^{\infty} h_n u^{n+s-1}, \qquad \text{as } u \to 0^+.$$
Then, Watson's lemma yields the asymptotic expansion [154, Ch. 7, §3]
$$F(x) \sim e^{-xf(t_0)} \sum_{n=0}^{\infty} h_n \frac{\Gamma(n+s)}{x^{n+s}}, \qquad \text{as } x \to \infty. \tag{2.21}$$

This exposition of the Laplace's method is more intuitive than rigorous. For details on the assumptions for the functions $f(t)$ and $g(t)$, the sector of validity of the expansions for complex values of z and also for a proof of these results, we refer to [105, Ch. 3, §7], [145, Ch. 3], [154, Ch. 2, §1]. If the integral (2.16) was defined along a complex path, the saddle point method should instead by applied [105, Ch. 4, §§6 and 7], [145, Ch. 4], [154, Ch. 2, §4].

Probably, the most important expansion derived by an application of Laplace's method is the one of the Euler's gamma function, defiend by [4, eq. 5.2.1.]
$$\Gamma(z) = \int_0^{\infty} e^{-t} t^{z-1} dt, \qquad (z > 0).$$

The method of Laplace can not be direcly applied to this integral, as the absolute extremum of the integrand is attained at a point that depends on the asymptotic variable

z. However, after a straightforward change of variables, the method can be applied and the following expansion may be derived [4, eq. 5.11.3]

$$\Gamma(z) \sim e^{-z} z^z \left(\frac{2\pi}{z}\right)^{1/2} \left[1 + \frac{1}{12z} + \frac{1}{288z^2} + \dots\right], \qquad \text{as } z \to \infty,$$

whose first term is the famous Stirling's formula [134] for the factorial of a large integer number. A simple and explicit formula for the coefficients between brackets of the right hand side of this asymptotic expansion may be found in [91].

Note that the expansions obtained by the classical Laplace's method are, in general, divergent. Even if the function $h(u)$ admits not only an asymptotic series at $u = 0$, but a Taylor series, the radius of convergence is usually small and does not contain the integration interval (see example 1). In other words, the price to pay for deriving expansions (2.18), (2.19) or (2.21) by interchanging summation and integration is that the expansions usually diverge. Consequently, accurate error bounds and a study of the optimal truncation term are usually needed in practice, although many times the first term of the expansion gives enough information on the asymptotic behavior of $F(x)$. Also, if the asymptotic variable x is large enough, we can obtain as many precision as required.

In any of the three cases (i), (ii) or (iii), the asymptotic sequence is easy to compute as it is nothing but inverse powers of the asymptotic variable x, that is, it is a Poincaré-type asymptotic sequence. In contrast, the coefficients h_n of the expansion involve the reversion of the series of $h(u)$ or $\bar{h}(u)$, a function defined implicitly by the change of variables $f(t) - f(t_0) = u^2$ or $f(t) - f(t_0) = u$. Traditional textbooks on asymptotics [105, 154] do not give explicit formulas for those coefficients, but rather an indication on how to compute the first few. Nevertheless, some more or less explicit representations of these coefficients can be found in the literature: Perron's formula [125] gives them in terms of the derivatives of an explicit function whereas Lauwerier [60] derives a formula for the coefficients in the form of an integral of an exponential, a power function and a polynomial p_n that must be computed recursively using a $n-$terms recurrence that involves integrals. On the other hand, Wojdylo [152, 153] found a recurrence formula for the coefficients of the Laplace expansion in terms of partial ordinary Bell polynomials. More recently, based on those expressions, Nemes [92] obtained a formula for the coefficients given in terms of ordinary potential polynomials. In any case, those formulas to calculate the coefficients h_n after an aplication of Laplace's method involve the computation of combinatorial objects whose complexity increases with the number of terms considered. In order to circumvent this problem, a modification of Laplace's method for integrals was introduced in [74].

2.2.3 A modification of the method of Laplace

The main drawback of the classical Laplace's method for integrals explained in the previous subsection is that there is not an explicit, closed formula for the coefficients h_n of the asymptotic expansion as those coefficients are defined by reverting a certain series that is given implicitly by a change of variables. In [74] this problem has been solved in the most simple way: by not creating it. That change of variables is an artifical tool used to obtain a monomial u or u^2 in the exponent of the exponential inside the integral in the right hand side of (2.16). Then, the key idea introduced in [74] is to avoid that change of variables and instead to obtain a monomial in the exponent of the exponential in (2.16) by considering the first non-constant and non-vanishing term of the Taylor expansion of $f(t)$ at the asymptotically relevant point $t = t_0$ (the absolute minimum of $f(t)$ in $[a, b]$).

More precisely, the modification of the Laplace's method introduced in [74] works as follows: we consider again the integral (2.16) being (a, b) a real (may be unbounded) interval and $f(t)$, $g(t)$ smooth enough functions. Without loss of generality we assume that $s \in (0, 1]$. We also assume that $f(t)$ is real-valued and that it has a unique absolute minimum at $t = t_0 \in [a, b)$. We also require that the functions $f(t)$ and $g(t)$ have a Taylor series expansion[2] at $t = t_0$ with common radius of convergence $r > 0$. Let $m \in \mathbb{N}$ denote the first non-vanishing derivative of $f(t)$ at $t = t_0$, that is, $f'(t_0) = f''(t_0) = \dots f^{(m-1)}(t_0) = 0$ and $f^{(m)}(t_0) > 0$. If $t_0 \in (a, b)$ is an interior point, then m is even. We consider also the *remainder* function

$$f_m(t) := f(t) - f(t_0) - \frac{f^{(m)}(t_0)}{m!}(t - t_0)^m$$

and we split the exponential in the form

$$e^{-xf(t)} = e^{-xf(t_0)} e^{-x\frac{f^{(m)}(t_0)}{m!}(t-t_0)^m} e^{-xf_m(t)}.$$

For large $x > 0$, the main contribution of the integrand to the integral comes from the neighborhood of the point $t = t_0$. But moreover, the contribution of the factor $e^{-xf_m(t)}$ to the integral is subdominant compared with the contribution of $e^{-x\frac{f^{(m)}(t_0)}{m!}(t-t_0)^m}$. Therefore, we re-write the integral (2.16) in the form

$$F(x) = e^{-xf(t_0)} \int_a^b e^{-x\frac{f^{(m)}(t_0)}{m!}(t-t_0)^m} h(t, x) dt, \qquad h(t, x) = e^{-xf_m(t)} g(t)(t - a)^{s-1}. \tag{2.22}$$

In this manner, we have obtained the dominant monomial $(t - t_0)^m$ in the exponent of the exponential while avoiding the change of variables that characterises the classical Laplace's method. The price to pay is that now, the asymptotic variable x also appears in the function $h(t, x)$ but, as we will see below, this will not spoil the asymptotic character of the expansion. We consider the Taylor series expansion of $h(t, x)$ at the asymptotically dominant point $t = t_0$, as $x \to \infty$:

$$h(t, x) = \sum_{n=0}^{\infty} h_n(x)(t - t_0)^{n+s-1}, \qquad |t - t_0| < r, \tag{2.23}$$

with $s \in (0, 1]$ if $t_0 = a$ and $s = 1$ otherwise.

The coefficients $h_n(x)$ can be computed by means of an explicit, closed formula in terms of the Taylor coefficients $A_n(x)$ and B_n at $t = t_0$ of the functions $e^{-xf(t)}$ and $g(t)(t - a)^{s-1}$ respectively [74, eq. 12]:

$$h_n(x) = e^{xf(t_0)} \sum_{k=0}^{\lfloor n/m \rfloor} \frac{x^k}{k!} \left[\frac{f^{(m)}(t_0)}{m!}\right]^k \sum_{j=0}^{n-mk} A_j(x) B_{n-mk-j}. \tag{2.24}$$

[2]This condition may be relaxed and require only that both, $f(t)$ and $g(t)$ have an asymptotic expansion at $t = t_0$.

When we introduce the expansion (2.23) into (2.22), interchange summation and integration and extend (if necessary) the integration interval up to $\pm\infty$, we obtain [74, eq. 19]

$$F(x) \sim e^{-xf(t_0)} \sum_{n=0}^{\infty} \left[h_n(x) \frac{1+(-1)^n \beta}{m} \Gamma\left(\frac{n+s}{m}\right) \left| \frac{m!}{f^{(m)}(t_0)x} \right|^{\frac{n+s}{m}} \right], \qquad \text{as } x \to \infty, \tag{2.25}$$

with $\beta = 0$ if $t_0 = a$ and $\beta = 1$ otherwise. Expansion (2.25) is in fact an asymptotic expansion of $F(x)$ for large x, although this is not obvious as the coefficients $h_n(x)$ depend on the asymptotic variable x. However, in [74] it has been shown that the coefficients $h_n(x)$ are polynomials in x of degree $\lfloor \frac{n}{p} \rfloor$, where $p > m$ denotes the first non-vanishing derivative of $f(t)$ at $t = t_0$ after the $m-$th derivative. Thus, the terms of the expansion (2.25) between brackets are $\mathcal{O}\left(x^{\lfloor n/p \rfloor - \frac{n+s}{m}}\right)$, as $x \to \infty$. They do not constitute a genuine Poincaré sequence that decrease monotonically with the order n of the approximation in the form x^{-n}, but in the form of a sawtooth.

Chapter 3

Uniform Expansions of Integral Transforms

As it has been pointed out in the introduction 1, in this chapter we face the challenge of designing a new theory of uniform approximation of parametric integrals satisfying the following three properties:

(a) The expansion holds uniformly for large and small values of a certain selected variable z.

(b) The expansion is convergent.

(c) The terms of the expansion are elementary functions of z.

To this end, we consider an integral transform of a function $g(t)$ with kernel $h(t,z)$ of the form

$$F(z) = \int_a^b h(t,z)g(t)dt, \tag{3.1}$$

where we assume that (a,b) is a bounded or unbounded interval and $h(\cdot,z)g(\cdot)$ is integrable on (a,b). We also assume that $g(t)$ is analytic in a region $\Omega \subset \mathbb{C}$ that includes the open set $(a,b) \subset \Omega$. It could be that the function $F(z)$ depends on other extra variables that we assume to be included in h and/or g. So, we omit their explicit indication as we are interested in the function $F(z)$ as a function of a single (selected) variable z. Then, most of the special functions of the mathematical physics can be written in the form of an integral transform like (3.1) (see [106, 108]).

Moreover, after splitting the integration interval if necessary and performing an affine change of variables on (3.1), we may assume that the integration interval (a,b) is whether $[0,1]$ if $[a,b]$ was bounded or $[0,\infty)$ if (a,b) was unbounded. That is, we consider that the function $F(z)$ in (3.1) is one of the following two:

$$F(z) = \int_0^1 h(t,z)g(t)dt, \qquad \text{or} \qquad F(z) = \int_0^\infty h(u,z)g(u)du, \tag{3.2}$$

with $(0,1) \subset \Omega$ in the first case and $(0,\infty) \subset \Omega$ in the second case. But moreover, after a change of variable that transforms the unbounded interval $(0,\infty)$ into the unit interval $(0,1)$ we can reduce the study of the second case of (3.2) to the first one. Such a change of variable can be, for example, logarithmic: $u = -\log t$. In this case, we identify $h(-\log t, z)/t$ with $h(t, z)$ and $g(-\log t)$ with $g(t)$. Therefore, without loss of generality, we study integral transforms of a function $g(t)$ with kernel $h(t, z)$ of the form

$$F(z) = \int_0^1 h(t, z) g(t) dt, \qquad z \in \mathcal{D}, \tag{3.3}$$

where $\mathcal{D}$ is a bounded or unbounded region of the complex plane. In general, the logarithmic change of variables $u = -\log t$ or any other change that transforms the unbounded interval $(0, \infty)$ into the interval $(0, 1)$ makes the point $t = 0$ and/or $t = 1$ in (3.3) a singular point of $g(t)$. Then, in the later analysis of the integral (3.3) we consider four different cases, depending on whether the end points 0 and 1 of the integration interval are regular or singular points of the function $g(t)$, which, as we will see, will be an essential aspect of the analysis. In other words, we consider four different situations concerning the position of the end points 0 and 1 with respect to Ω:

1. $[0, 1] \subset \Omega$. That is, none of the end points are singular points of $g(t)$.
2. $(0, 1] \subset \Omega$ but $[0, 1] \subsetneq \Omega$. In other words, 0 is a singular point of $g(t)$.
3. $[0, 1) \subset \Omega$ but $[0, 1] \subsetneq \Omega$. Then, 1 is a singular point of $g(t)$.
4. $(0, 1) \subset \Omega$ but $[0, 1] \subsetneq \Omega$. Both end points 0 and 1 are singular points of $g(t)$.

The main idea to derive a uniform expansion of the integral $F(z)$ in (3.3) is to consider the multi-point Taylor expansion of the function $g(t)$ at selected points such that $(0, 1) \subset D_r \subset \Omega$, where D_r is the lemniscate of convergence of the chosen multi-point Taylor expansion of $g(t)$. Then, replace $g(t)$ in (3.3) by that expansion and interchange summation and integration. Under some hypotheses that we state in the next section and with accurate bounds for the remainder, we will show that this method produces an expansion of $F(z)$ satisfying the following three properties:

(a) The expansion is *uniform*[1] for z in an unbounded subset $\mathcal{D} \subset \mathbb{C}$ such that $\bar{\mathcal{D}}$ contains the point $z = 0$. Note that we are using the classical meaning of *uniform* of the mathematical analysis: for any order n of the approximation $F_n(z)$, the absolute error $|F(z) - F_n(z)| \leq M_n$ for any $z \in \mathcal{D}$, with M_n independent of z. That is, the expansion is valid for both large and small values of $|z|$.

(b) The expansion is convergent. That is, the remainder of the expansion, $R_n(z) := F(z) - F_n(z)$, vanishes as $n \to \infty$, for any $z \in \mathcal{D}$.

(c) The terms of the expansion are elementary functions of z.

To ensure that properties (a), (b) and (c) are satisfied, the following assumptions on the functions $h(t, z)$ and $g(t)$ are considered:

(i) The function $h(t, z)$ can be dominated by an integrable function $H(t)$ on $[0, 1]$ in the sense that $|h(t, z)| \leq H(t)$ for all $z \in \mathcal{D}$ and for all $t \in [0, 1]$.

[1] This use of *uniform expansion* shall not be confused with the notion of "uniform expansions" used in asymptotics. In asymptotics, the term "uniform expansion" is used to refer to an expansion of a function of several variables that remains valid as a certain selected parameter crosses a critical value.

(ii) The function $g(t)$ is analytic in a region $\Omega \subset \mathbb{C}$ that contains the open set $(0,1) \subset \Omega$.

(iii) The moments of the function $h(t,z)$, $M[h(\cdot,z);k] := \int_0^1 h(t,z)t^k dt$ are elementary functions of z.

Then, when $0 \in \mathcal{D}$ the requirement (i) implies the property (a). The requirement (ii) assures the property (b) and the requierement (iii) guarantees the property (c).

Compare the above mentioned idea with the convergent (Taylor) expansions or the asymptotic expansions of $F(z)$. Roughly speaking, the former are found by replacing $h(t,z)$ in the integrand of the integral (3.3) that defines $F(z)$ by its Taylor expansion, as a function of z, at $z=0$; whereas the latter are found in a similar way, but considering the asymptotic expansion of $g(t)$ at the asymptotically relevant point of $h(t,z)$. Then, interchanging summation and integration we find expansions of $F(z)$ that are, respectively, convergent or asymptotic. However, in general, those expansions do not hold uniformly in z. They are useful and accurate for small (convergent) or large (asymptotic) values of the variable z. But their validity is local, that is, they are useless when we want to evaluate the function away from small (convergent) or large (asymptotic) values of z. On the other, an expansion satisfying the three properties (a), (b) and (c) listed above is globally more satisfactory as it is valid in a large region of the complex plane, that contains small and large values of the selected variable z (see figure 1.1 in chapter 1 for an illustration of this discussion).

Moreover, imagine that a certain special function $F(z)$ admitting an integral representation of the form (3.3) appears in a certain computation like for example a factor of the integrand of an integral or in a differential equation. In general, neither the Taylor series expansions nor the asymptotic approximations can replace the function $F(z)$ in that computation, as they do not hold in a large enough region (range of integration, domain of the differential equation...). In contrast, an expansion satisfying the three properties (a), (b) and (c) is uniformly valid in a large set of the complex plane and therefore it can be used to replace the function $F(z)$ in that computation. As the approximation is given in terms of elementary functions, it should be easier to work with it instead of dealing with the function $F(z)$, whose analytical expresion can be more involved.

Using the above described procedure, uniform expansions of some particular special functions $F(z)$ (for particular functions $h(t,z)$ and $g(t)$ in the right hand side of (3.3)) and satisfying the properties (a), (b) and (c) above) have been obtained in [66] (Bessel functions), [19] (incomplete gamma function), [20] (confluent hypergeometric function), [44] (incomplete beta function), [43] (Gauss hypergeometric function), [69] (generalized hypergeometric functions ${}_{p-1}F_p$ and ${}_pF_p$) and [21, 22] (symmetric elliptic functions). A brief summary of these expansions is relegated to the next chapter but, as we will see, in all these examples, the derived expansions satisfy properties (a), (b) and (c) listed above. In this chapter of the thesis, we show that this is not a coincidence but those are just particular examples of a general theory: *the theory of uniform expansions of integral transforms*.

This general theory has been developed in the paper [77].

3.1 Hypotheses

In this section, we clearly set the hypotheses required for the two factors $h(t, z)$ and $g(t)$ in the integrand of (3.3) pointed out above. As we have already mentioned, we assume that $g(w)$ is an analytic function in an open region Ω that contains the integration interval $[0, 1]$ except, possibly, for an integrable singularity at $w = 0$ and/or at $w = 1$. More precisely:

Hypothesis 1. We assume that $g(w)$ is analytic in an open region Ω that contains the interval $(0, 1)$ and the function $f(w) := w^{1-\sigma}(1 - w)^{1-\gamma}g(w)$, with $0 < \sigma, \gamma \leq 1$ is bounded in Ω. In particular, according to the four different situations that we may find concerning the position of the end points 0 and 1 with respect to Ω, we have:

$$\begin{cases} \sigma = 1 \text{ and } \gamma = 1, & \text{in case 1, } [0,1] \subset \Omega. \\ \sigma < 1 \text{ and } \gamma = 1, & \text{in case 2, } (0,1] \subset \Omega \text{ but } [0,1] \subsetneq \Omega. \\ \sigma = 1 \text{ and } \gamma < 1, & \text{in case 3, } [0,1) \subset \Omega \text{ but } [0,1] \subsetneq \Omega. \\ \sigma < 1 \text{ and } \gamma < 1, & \text{in case 4, } (0,1) \subset \Omega \text{ but } [0,1] \subsetneq \Omega. \end{cases}$$

We have also mentioned above that the function $h(t, z)$ is majorized by an integrable function $H(t)$ on $[0, 1]$, for all $z \in \mathcal{D}$. In particular:

Hypothesis 2. We assume that $|h(t, z)| \leq Ht^{\alpha}(1 - t)^{\beta}$ for $(t, z) \in [0, 1] \times \mathcal{D}$, with $H > 0$ a positive number independent of z and t and the parameters α and β satisfying $\alpha + \sigma > 0$ and $\beta + \gamma > 0$.

Observe that it is natural to assume this form for the bound of the function $h(t, z)$ as the product $h(\cdot, z)g(\cdot)$ must be integrable in $[0, 1]$. Also, we could consider H not to be a constant, but an integrable function on $[0, 1]$ independent of z. The price to pay by doing so is too high, as working with that $H(t)$ function would be much more cumbersome and the advantage would not be that great, as the requirement "H constant" is usually enough in practice.

Hypothesis 3. We can choose m different base points $0 \leq t_1 < t_2 < \ldots < t_m \leq 1$ such that the lemniscate (2.7) of convergence of the multi-point Taylor expansion of $g(t)$ satisfies $(0, 1) \subset D_r \subset \Omega$, that is, $r_0 \leq r \leq \rho$ (see equations (2.5) and (2.6) in chapter 2 for the definition of the minimal and maximal "radius" r_0 and ρ respectively).

Note that, as it is explained in [80] (in a different context), if $(0, 1) \subset \Omega$ we can always find an appropriate selection of the base points $t_1, \ldots, t_m$ to assure that $r_0 \leq \rho$ and we can take a "radius" r such that $r_0 \leq r \leq \rho$ or, equivalently, $(0, 1) \subset D_r \subset \Omega$.

Recall the expresion (2.1) for the multi-point Taylor expansion of an analytic function $g(z)$:

$$g(z) = \sum_{k=0}^{n-1} p_k(z) \left[\prod_{s=1}^{m}(z - t_s)\right]^k + g_n(z), \tag{3.4}$$

where $p_k(z)$ is a polynomial in z of degree $m - 1$ (see section 2.1 in chapter 2) and $g_n(z)$ is the multi-point Taylor remainder that can be represented by the following Cauchy's integral (see (2.4))

$$g_n(z) := \frac{\prod_{s=1}^{m}(z - t_s)^n}{2\pi i} \oint_\Gamma \frac{g(w)dw}{(w - z)\prod_{s=1}^{m}(w - t_s)^n}, \qquad z \in D_r. \tag{3.5}$$

In this formula, the integration contour $\Gamma \subset \Omega$ is the boundary of a lemniscate $D_{r-\varepsilon}$, with $r-\varepsilon > 0$ and small enough ε such that $\Gamma \subset \Omega$:

$$\Gamma := \left\{ z \in \Omega : \prod_{s=1}^{m} |z - t_s| = r - \varepsilon \right\} \subset \Omega.$$

When $w = t \in (0,1)$ and $[0,1] \subset \Omega$ (case 1) we may choose the base points t_s, $s = 1, 2, \ldots, m$ such that $\Gamma \subset \Omega$ with $\varepsilon = 0$. In the other three cases (cases 2, 3 and 4), when $[0,1] \subsetneq \Omega$ it is not possible to take $\varepsilon = 0$ as the contour Γ would contain the singular points $t = 0$ and/or $t = 1$. Notwithstanding, for any $t \in (0,1)$ and small enough ε, the contour Γ encircles not only the base points $t_1, t_2, \ldots, t_m$ but also the point t. But, moreover, in any of the four cases, and in particular in cases 2, 3 and 4, the integral in the right hand side of (3.5) is a (constant) function of ε as it is the integral of an integrable function. Therefore, we can consider the limit $\varepsilon \to 0$ and consider that the "radius" of the lemniscate in the above integral equals r. Then, this limit lemniscate $D_r \subsetneq \Omega$ although $D_r \setminus \{0,1\} \subset \Omega$. Then, we may consider that the "radius" r of the lemniscate boundary Γ that defines the integration contour in (3.5) is such that the interval $[0,1] \subset D_r$.

3.2 Analysis of the remainder $g_n(t)$

In this section we derive a bound for the remainder $g_n(t)$ (3.5), $t \in (0,1)$ of the multi-point Taylor expansion of $g(t)$ (3.4). The analysis strongly depends on the case 1-4 under consideration as it is more involved when one or both end points of the integration interval, $t = 0$ and/or $t = 1$ are singular points of $g(t)$ (cases 2, 3, 4). For each case, we choose different base points $t_1, t_2, \ldots, t_m$ and we consider a different lemniscate D_{r_j}, $j = 1, 2, 3, 4$, whose "radius" must satisfy $r_0 \le r_j \le \rho$, $j = 1, 2, 3, 4$ (recall the definition of the minimal and the maximal radius r_0 and ρ respectively (see (2.5) and (2.6))). As we will see below, the first and/or last base points chosen, t_1 and/or t_m will play a crucial role in the later analysis of the remainder. More precisely, the base points and "radius" are chosen satisfying:

- Case 1: We choose m different base points $0 \le t_1 < t_2 < \ldots < t_m \le 1$. The "radius" r_1 of the lemniscate D_{r_1} must satisfy the inequality $r_1 \ge r_0 \ge \max\{\prod_{s=1}^{m} t_s, \prod_{s=1}^{m}(1-t_s)\}$.
- Case 2: We have to choose $t_1 > 0$. The "radius" r_2 of the lemniscate D_{r_2} must satisfy $r_2 := \prod_{s=1}^{m} t_s \ge r_0 > \prod_{s=1}^{m}(1-t_s)$. This condition assures that the interval $(0,1] \subset D_{r_2}$. The point $0 \notin D_{r_2}$ but $0 \in \bar{D}_{r_2}$.
- Case 3: We have to choose $t_m < 1$. The "radius" r_3 of the lemniscate D_{r_3} must satisfy $r_3 := \prod_{s=1}^{m}(1-t_s) \ge r_0 > \prod_{s=1}^{m} t_s$. This condition assures that the interval $[0,1) \subset D_{r_3}$. The point $1 \notin D_{r_3}$ but $1 \in \bar{D}_{r_3}$.
- Case 4: We have to choose $0 < t_1 < t_m < 1$. The "radius" r_4 of the lemniscate D_{r_4} must satisfy $r_4 := \prod_{s=1}^{m} t_s = \prod_{s=1}^{m}(1-t_s)$. This condition assures that the interval $(0,1) \subset D_{r_4}$. The points $0, 1 \notin D_{r_4}$ but $0, 1 \in \bar{D}_{r_4}$.

With this election of the base points $t_1, \ldots, t_m$ and of the "radius" r_j of the lemniscate D_{r_j} in each of the four cases $j = 1, 2, 3, 4$, and taking into account the above observation

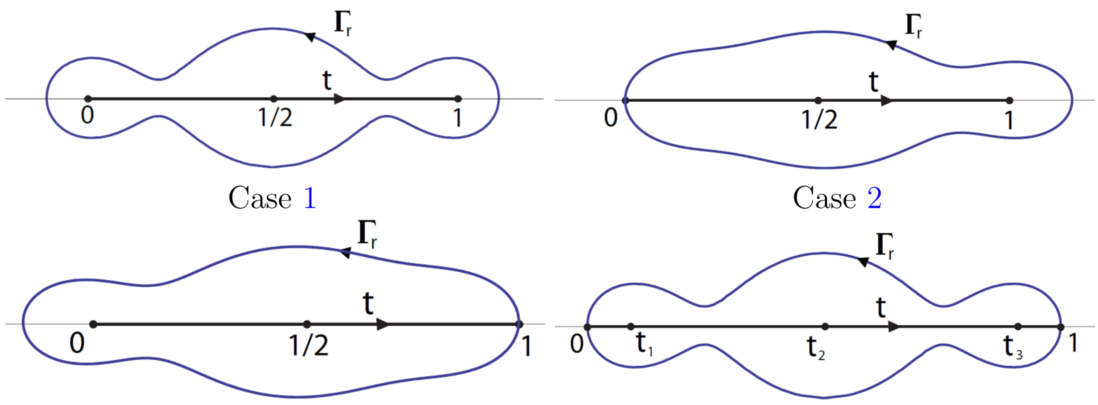


Figure 3.1: A possible election of appropriate base points in the case $m = 3$. We have taken: Case 1 (top left) $t_1 = 0$, $t_2 = 1/2$, $t_3 = 1$ and $r_1 = 1/20 > r_0$. Case 2 (top right) $t_1 = 1/10$, $t_2 = 1/2$, $t_3 = 1$ and $r_2 = t_1t_2t_3 = 1/20 = r_0$. Case 3 (bottom left) $t_1 = 0$, $t_2 = 1/2$, $t_3 = 9/10$ and $r_3 = (1 - t_1)(1 - t_2)(1 - t_3) = 1/20 = r_0$. Case 4 (bottom right) $t_1 = 1/14$, $t_2 = 1/2$, $t_3 = 13/14$ and $r_4 = t_1t_2t_3 = 13/392 = r_0$.

on the integration contour Γ, we assume that the integration path is, in each of the four cases, the boundary of the corresponding lemniscate D_{r_j}, that is, $\Gamma_{r_j} := \partial D_{r_j}$. In figure 3.1 we show a possible selection of the base points and "radius" with $m = 3$ in the four different cases 1-4.

In the remaining of this section we find sharp bounds for the remainder $g_n(t)$ in all four cases 1-4.

- Case 1. From the definition of D_{r_1} it is clear that $\prod_{s=1}^m |t-t_s| < \prod_{s=1}^m |w-t_s| = r_1$ for any $t \in [0,1]$ and $w \in \Gamma_{r_1}$. Therefore, $r_1/\sup_{t\in(0,1)} \prod_{s=1}^m |t - t_s| := a > 1$ and from (3.5)
$$|g_n(t)| \leq \frac{1}{2\pi a^n} \oint_{\Gamma_{r_1}} \frac{|g(w)|dw}{|w-t|} = \frac{M}{a^n}, \qquad t \in [0,1], \tag{3.6}$$
with $M > 0$ a constant independent of t and n.

- Case 2. From the definition of D_{r_2} we have that $\prod_{s=1}^m |t-t_s| < \prod_{s=1}^m |w-t_s| = r_2$ for any $t \in [t_1, 1]$, but $\prod_{s=1}^m |t - t_s| \leq \prod_{s=1}^m |w - t_s| = r_2 := \prod_{s=1}^m |t_s|$ for any $t \in [0, t_1]$ and $w \in \Gamma_{r_2}$. Then, for any $t \in [t_1, 1]$ we may derive a similar bound to the one derived in case 1:
$$|g_n(t)| \leq \frac{1}{2\pi a^n} \oint_{\Gamma_{r_2}} \frac{|g(w)|dw}{|w-t|} = \frac{M}{a^n}, \qquad t \in [t_1, 1],$$
with $M > 0$ and $a > 1$ independent of t and n.

  However, for $t \in [0, t_1]$ we must be more careful because if we proceeded in the same way, we would obtain a similar bound but with $a = 1$, that is, $|g_n(t)| \leq M$. Then, we would not have enough information to deduce that the remainder vanishes as $n \to \infty$. That is, we would not be able to show that the expansion is convergent.

  Instead, we seek a sharper bound for the remainder $g_n(t)$. We note that $\prod_{s=1}^m |w - t_s| = \prod_{s=1}^m |t_s|$ for $w \in D_{r_2}$ and also that $|t - t_s| \leq t_s$ for $s = 2, 3, \ldots, m$ and

$t \in [0, t_1]$. We also use hypothesis 1, namely, that $f(w) := w^{1-\sigma} g(w)$ is bounded in D_{r_2}, with $0 < \sigma < 1$. Then, from (3.5) we find, for $t \in [0, t_1]$:

$$|g_n(t)| \leq \frac{1}{2\pi} \frac{\prod_{s=1}^m |t - t_s|^n}{\prod_{s=1}^m |t_s|^n} \oint_{\Gamma_{r_2}} \frac{|w^{\sigma-1} f(w)|}{|w - t|} dw \leq M \frac{(t_1 - t)^n}{t_1^n} \oint_{\Gamma_{r_2}} \frac{|w^{\sigma-1} f(w)|}{|w - t|} dw,$$

with $M > 0$ independent of t and n. We perform now the change of variables $w \mapsto tw$ to find

$$|g_n(t)| \leq M \frac{(t_1 - t)^n}{t_1^n} t^{\sigma-1} \oint_{\Gamma_{r_2}/t} \frac{|w^{\sigma-1} f(tw)|}{|w - 1|} dw, \qquad t \in (0, t_1],$$

where the integration contour is the scaled lemniscate boundary Γ_{r_2}/t:

$$\Gamma_{r_2}/t = \left\{ w \in \mathbb{C} : \left| \left(w - \frac{t_1}{t} \right) \left(w - \frac{t_2}{t} \right) \cdots \left(w - \frac{t_m}{t} \right) \right| = \frac{r_2}{t^m} \right\}.$$

For any $t > 0$, the most left point of this scaled lemniscate is the point $w = 0$ and the most right point is the point $w = t_0/t$, where t_0 is the most right point of the lemniscate boundary Γ_{r_2}. In the limit $t \to 0$ the scaled lemniscate boundary Γ_{r_2}/t becomes the imaginary axis traversed downwards. In this path, we have that $|f(wt)| \leq M_0$, with $M_0 > 0$ independent of $w \in \Gamma_{r_2}/t$ and $t > 0$ and the integral

$$\oint_{\Gamma_{r_2}/t} \frac{|w^{\sigma-1}|}{|w - 1|} dw, \qquad 0 < \sigma < 1 \tag{3.7}$$

can be bounded by a constant that is independent of t. Therefore, for any $t \in (0, t_1]$ we find the bound

$$|g_n(t)| \leq \frac{M (t_1 - t)^n t^{\sigma-1}}{t_1^n}, \qquad t \in (0, t_1], \tag{3.8}$$

for some constant $M > 0$ independent of t and n.

- Case 3. It is similar to the previous studied case 2, but interchanging the roles of the points $w = 0$ and $w = 1$ and the lemniscates D_{r_2} and D_{r_3}. That is, case 3 becomes case 2 after the change of variables $w \mapsto 1 - w$, considering the factorization $g(w) = (1 - w)^{\gamma-1} f(w)$ instead of $g(w) = w^{\sigma-1} f(w)$, the "radius" r_3 instead of r_2 and reversing the order of the points $t_1, t_2, \ldots, t_m$. Then, for $t \in [0, t_m]$ we obtain a bound similar to the bound obtained in the case 1:

$$|g_n(t)| \leq \frac{M}{a^n}, \qquad t \in [0, t_m],$$

with $M > 0$ and $a > 1$ independent of t and n. On the other hand, for $t \in [t_m, 1)$ we find

$$|g_n(t)| \leq M \frac{(t - t_m)^n (1 - t)^{\gamma-1}}{(1 - t_m)^n}, \qquad t \in [t_m, 1), \tag{3.9}$$

with $0 < \gamma < 1$ given in hypothesis 1 and $M > 0$ independent of t and n.

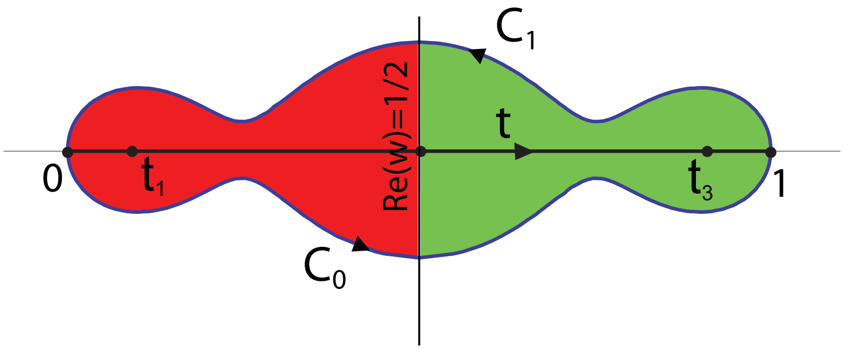


Figure 3.2: Case 4: The lemniscate Γ_{r_4} is divided into two mirror half-lemniscates C_0 and C_1 after cutting Γ_{r_4} with the vertical line $\Re w = 1/2$. In this example, $m = 3$, $t_1 > 0$, $t_2 = 1/2$ and $t_3 = 1 - t_1$.

- Case 4. From the definition of D_{r_4} we have that $\prod_{s=1}^{m} |t - t_s| < \prod_{s=1}^{m} |w - t_s| = r_4$ for any $t \in [t_1, t_m]$ and $w \in \Gamma_{r_4}$. Then, for $t \in [t_1, t_m]$ we may derive a similar bound to the one derived in case 1:

$$|g_n(t)| \le \frac{M}{a^n}, \qquad t \in [t_1, t_m],$$

with $M > 0$ and $a > 1$ independent of n and t.

For $t \in (0, t_1]$ and $t \in [t_m, 1)$ we must find a sharper bound for $g_n(t)$, similar to the ones found in cases 2 and 3. We only consider now the case $t \in (0, t_1]$, as the case $[t_m, 1)$ is similar, since the same simmetry between cases 2 and 3 applies here as well. We also consider that the base points t_s are symmetrically distribuited with respect to the middle point of the integration interval $t = 1/2$. This condition is superfluous and could be eliminated but it would not provide more generality. Then, for the sake of simplicity in the analysis, we assume that symmetric distribution of the base points t_s. We divide the lemniscate Γ_{r_4} into two mirror half-lemniscate C_0 and C_1, obtained after cutting Γ_{r_4} with the vertical line $\Re w = 1/2$. Then, $\Gamma_{r_4} = C_0 \cup C_1$ (see figure 3.2).

We use, in both half-lemniscates C_0 and C_1, that $|t - t_s| \le |t_s|$ for $s = 2, 3, \ldots, m$ and that $|w - t_1| \ge t_1$. Moreover, in C_0 we use the factorization $g(w) = w^{\sigma-1} f_0(w)$, with $f_0(w) = (1 - w)^{\gamma-1} f(w)$ bounded in C_0. Then, we have

$$|g_n(t)| \le \frac{(t_1 - t)^n}{2\pi t_1^n} \oint_{C_0} \frac{|w^{\sigma-1} f_0(w)|}{|w - t|} dw + \frac{(t_1 - t)^n}{2\pi t_1^n} \oint_{C_1} \frac{|g(w)|}{|w - t|} dw, \qquad t \in (0, t_1]. \tag{3.10}$$

In the integral along C_1 we simply use that $|w - t| \ge c > 0$, for any $t \in [0, t_1]$ and $w \in C_1$, with c independent of t. Besides, as $g(w)$ is integrable, the second integral in (3.10) can be bounded by a certain constant $M > 0$ independent of t and n. But, on the first integral, we have to be more careful. We perform a change of variable $w \mapsto tw$, similar to the one performed in the previous case 2. We find

$$\oint_{C_0} \frac{|w^{\sigma-1} f_0(w)|}{|w - t|} dw = t^{\sigma-1} \oint_{C_0/t} \frac{|w^{\sigma-1} f_0(wt)|}{|w - 1|} dw \le M_0 t^{\sigma-1} \oint_{C_0/t} \frac{|w^{\sigma-1}|}{|w - 1|} dw \tag{3.11}$$

where we have used that $f_0(t)$ is bounded in C_0. In this formula, $M_0 > 0$ is a constant independent of t and n and the integration contour C_0/t is the scaled half-lemniscate contour. For any $t > 0$ the most left point of the scaled half-lemniscate is the point $w = 0$ whereas the most right points are the two points w that satisfy $\Re w = 1/(2t)$. In the limit $t \to 0$ the scaled half-lemniscate becomes the imaginary axis traversed downwards. Then, for any $t \in [0, t_1]$ the integral on the right hand side of (3.11) is finite. Therefore, the remainder $g_n(t)$ in (3.10) is bounded by means of

$$|g_n(t)| \leq M' \frac{(t_1 - t)^n t^{\sigma-1}}{t_1^n} + \bar{M} \frac{(t_1 - t)^n}{t_1^n} \leq M \frac{(t_1 - t)^n t^{\sigma-1}}{t_1^n}, \qquad t \in (0, t_1],$$

with $0 < \sigma < 1$ given in hypothesis 1 and a certain constant $M > 0$ independent of t and n.

A similar analysis shows that, for $t \in [t_m, 1)$,

$$|g_n(t)| \leq M \frac{(t - t_m)^n (1 - t)^{\gamma-1}}{(1 - t_m)^n}, \qquad t \in [t_m, 1),$$

with $0 < \gamma < 1$ and a certain constant $M > 0$ independent of t and n.

These sharp bounds for the remainder $g_n(t)$ in the four analyzed cases 1-4 are going to be used in the next section, where we derive the main result of this chapter: a uniform convergent expansion of the integral $F(z)$ given in (3.3).

3.3 A uniform convergent expansion of the integral $F(z)$

We state the result in the form of a theorem:

Theorem 3.3.1. *Assume that hypotheses 1-3 of section 3.2 for the functions $g(t)$ and $h(t, z)$ and for the base points $t_1, t_2, \ldots, t_m$ of the multi-point Taylor expansion of $g(t)$ hold. Then, for $n = 1, 2, 3, \ldots,$ and $z \in \mathcal{D}$,*

$$F(z) = \int_0^1 h(t, z) g(t) dt = \sum_{k=0}^{n-1} \sum_{j=0}^{m-1} A_{k,j} M[h(\cdot, z); k, j] + R_n(z), \tag{3.12}$$

where $A_{k,j}$ are the multi-point Taylor coefficients of the function $g(t)$ at the base points $t_1, t_2, \ldots, t_m$ (see formulas (2.1), (2.2), (2.8) and (2.9) in chapter 2), and $M[h(\cdot, z); k, j]$ are the multi-point moments:

$$M[h(\cdot, z); k, j] := \int_0^1 h(t, z) \left[\prod_{s=1}^{m} (t - t_s) \right]^k t^j dt \tag{3.13}$$

On the other hand, the remainder $R_n(z)$ may be bounded in the form

$$|R_n(z)| \leq MH \left[\frac{1}{a^n} + A \frac{n! \, \Gamma(\alpha + \sigma)}{\Gamma(n + \alpha + \sigma + 1)} + B \frac{n! \, \Gamma(\beta + \gamma)}{\Gamma(n + \beta + \gamma + 1)} \right], \tag{3.14}$$

with

$$(A,B) := \begin{cases} (0,0) & \textit{in case } 1, \\ (1,0) & \textit{in case } 2, \\ (0,1) & \textit{in case } 3, \\ (1,1) & \textit{in case } 4. \end{cases}$$

Furthermore, the constants $H>0$ and $M>0$ are independent of t and n and have been respectively introduced in hypothesis 2 of section 3.1 and in the analysis of the remainder $g_n(t)$ in section 3.2. The parameters α, β, σ and γ are defined in hypotheses 1 and 2 and $a>1$, derived in section 3.2, is independent of t and n.

Therefore, expansion (3.12) is uniformly convergent for $z \in \mathcal{D}$ in any of the four cases 1-4, but we note that the rate of convergence strongly depends on the case under consideration: the convergence is of exponential type in case 1, that is, $R_n(z) = \mathcal{O}(a^{-n})$; whereas it is of power type in the remaining cases, $R_n(z) = \mathcal{O}(n^{-\delta})$, for some $\delta > 0$. More precisely, as $n \to \infty$,

$$R_n(z) = \mathcal{O}\left(a^{-n} + An^{-\sigma-\alpha} + Bn^{-\gamma-\beta}\right). \tag{3.15}$$

Proof. Consider the multi-point Taylor expansion (2.1) of the function $g(t)$ at the base points $t_1 < t_2 < \ldots < t_m$, with representation (2.8) of $p_k(t)$, that converges in the lemniscate D_r, with $(0,1) \subset D_r \subset \Omega$ and $r = r_1, r_2, r_3$ or r_4 depending on the case 1-4 under consideration. Replace $g(t)$ on the left hand side of (3.12) by that multi-point Taylor expansion and interchange summation and integration to find the right hand side of (3.12), with

$$R_n(t) := \int_0^1 h(t,z) g_n(t) dt. \tag{3.16}$$

Note that, by hypotesis 2, the multi-point moments (3.13) of $h(t,z)$ exists. It remains to show that the remainder $R_n(z)$ can be bounded as in (3.14). The analysis is different in the four cases 1-4.

- Case 1. From the analysis of the remainder $g_n(t)$ in section 3.2, case 1, it is clear that $|g_n(t)| \leq Ma^{-n}$ for all $t \in [0,1]$, with $M>0$ and $a>1$ independent of t and n. Then, if we introduce this bound into (3.16) we get (3.14) with $A=B=0$.

- Case 2. From the analysis of the remainder $g_n(t)$ in section 3.2, case 2, we have that $|g_n(t)| \leq Ma^{-n}$, with $M>0$ and $a>1$ independent of t and n, for $t \in [t_1, 1]$ and we have that $|g_n(t)| \leq M\frac{(t_1-t)^n t^{\sigma-1}}{t_1^n}$, for $t \in (0, t_1]$. Then, we write

$$R_n(z) = \int_0^{t_1} h(t,z) g_n(t) dt + \int_{t_1}^1 h(t,z) g_n(t) dt.$$

We introduce the above bounds of $g_n(t)$ and the bound $h(t,z) \leq Ht^\alpha$ given in hypothesis 2 to find

$$|R_n(z)| \leq \frac{MH}{t_1^n} \int_0^{t_1} (t_1-t)^n t^{\alpha+\sigma-1} dt + \frac{MH}{a^n} = MHt_1^{\alpha+\sigma} \int_0^1 t^{\alpha+\sigma-1}(1-t)^n dt + \frac{MH}{a^n}.$$

Using the definition of Euler's beta function [4, eq. 5.12.1] and the fact that $0 < t_1^{\alpha+\sigma} \leq 1$ we find the bound (3.14) with $A=1$, $B=0$.

- Case 3. It is similar to the previous case. We write

$$R_n(z) = \int_0^{t_m} h(t,z)g_n(t)dt + \int_{t_m}^1 h(t,z)g_n(t)dt.$$

In the first integral, we have $|g_n(t)| \le Ma^{-n}$, with $M > 0$ and $a > 1$ independent of t and n, whereas in the second integral we have $|g_n(t)| \le M\frac{(t-t_m)^n(1-t)^{\gamma-1}}{(1-t_m)^n}$. Then, it follows that

$$\begin{aligned}|R_n(z)| &\le \frac{MH}{a^n} + \frac{MH}{(1-t_m)^n}\int_{t_m}^1 (1-t)^{\beta+\gamma-1}(t-t_m)^n dt \\ &\le \frac{MH}{a^n} + \frac{MH}{(1-t_m)^n}\int_0^{1-t_m} t^{\beta+\gamma-1}(1-t_m-t)^n dt \\ &\le \frac{MH}{a^n} + MH(1-t_m)^{\beta+\gamma}\int_0^1 t^{\beta+\gamma-1}(1-t)^n dt.\end{aligned}$$

Now, formula (3.14) follows immediately with $A=0$, $B=1$.

- Case 4. In this case, we write

$$R_n(z) = \int_0^{t_1} h(t,z)g_n(t)dt + \int_{t_1}^{t_m} h(t,z)g_n(t)dt + \int_{t_m}^1 h(t,z)g_n(t)dt.$$

The derivation of bound (3.14) follows the same steps that in the previous cases: In the first integral we use the bound $|g_n(t)| \le M\frac{(t_1-t)^n t^{\sigma-1}}{t_1^n}$; in the second one we have $|g_n(t)| \le Ma^{-n}$ whereas in the last integral $|g_n(t)| \le M\frac{(t-t_m)^n(1-t)^{\gamma-1}}{(1-t_m)^n}$ holds. Then, using hypothesis 2 and these bounds, formula (3.14) with $A=B=1$ follows.

Finally, using the asymptotic behavior of the quotient of two gamma functions [4, eq. 5.11.12] in the right hand side of (3.14) we find the convergence rate (3.15). □

Some remarks are to be done:

Remark 3.3.2. *The bound (3.14) of the remainder $R_n(z)$ of the expansion given in the right hand side of (3.12) is independent of z. Then, no matter how small or large $|z|$ is, the expansion (3.12) is valid as long as $z \in \mathcal{D}$. That is, the expansion given in the theorem is valid uniformly on z, for all $z \in \mathcal{D}$. And it is also convergent, as shown by the rate of convergence (3.15).*

Remark 3.3.3. *In cases 2-4 the proof is more involved than in case 1. We could have repeated step by step the simpler proof of case 1, but with $a=1$. Then, that proof would not have shown the convergence of expansion (3.12) as the parameter a in (3.14) would not have been large enough (>1). Therefore, the more involved proof in cases 2-4 is neccesary to show the convergence.*

Remark 3.3.4. *In case 1 when both end points of the integration interval $[0,1]$ are contained in the region Ω of analyticity of the function $g(w)$, the convergence of the expansion (3.12) is of exponential type, that is, the remainder $R_n(z) = \mathcal{O}(a^{-n})$, with $a > 1$. In contrast, when one or both end points are not contained in Ω, the scenario is worse: Expansion (3.12) is still convergent, but the convergence is only of power type, that is, the remainder $R_n(z) = \mathcal{O}(n^{-\delta})$, for some $\delta > 0$.*

As shown in (3.15) *the value of* δ *is related with the behavior of the integrand in the left hand side of* (3.12) *at the end points* $t = 0$ *and/or* $t = 1$*. That behavior is somehow transferred to the remainder* $R_n(z)$ *in the right hand side of expansion* (3.12)*. This analysis reminds [46, 47] where it is proved that the asymptotic behavior of a function* $g(t)$*, with integrable singularities in the unit disk, is transferred to the asymptotic behavior of its standard Taylor coefficients.*

Remark 3.3.5. *It follows from* (3.14) *that, the larger* α *and* β *are, the faster the convergence of the expansion of* $F(z)$ *in cases 2-4 is. In case 1, we have considered the possibility* $\alpha = \beta = 0$ *because the bound* $|h(t,z)| \leq Ht^\alpha(1-t)^\beta$ *with* α *and/or* $\beta > 0$ *does not mean any improvement in the convergence speed of expansion* (3.12)*: regardless* α *and/or* β *vanish or not, we would derive formula* (3.14) *for the remainder in case 1.*

The same applies to the second integral in the analysis of case 2: we have used the bound $|h(t,z)| \leq H$ *because the bound* $|h(t,z)| \leq Ht^\alpha(1-t)^\beta$ *with* α *and/or* $\beta > 0$ *does not mean any improvement in the convergence speed of expansion* (3.12)*. On the other hand, in the first integral of case 2 we have considered* $\beta = 0$*, but* $\alpha \geq 0$ *as the factor* t^α *accelerates the speed of convergence but the factor* $(1-t)^\beta$ *does not. The situation is analogous in cases 3 and 4.*

Remark 3.3.6. *Given an integral, we have to factorize the integrand in two functions* $g(t)$ *and* $h(t,z)$ *satisfying hypotheses 1-3 in section 3.1, being* z *the uniform variable. For example, consider the following integral representation of the incomplete gamma function [116, eq. 8.2.1]*

$$\gamma(a,z) = z^a \int_0^1 e^{-zt} t^{a-1} dt, \qquad \Re a > 0.$$

It is straightforward to check that the inmediate choice $g(t) = t^{a-1}$ *and* $h(t,z) = e^{-zt}$ *satisfies the hypotheses 1-3 in* $\mathcal{D} = \{z \in \mathbb{C} : \Re z \geq \Lambda\}$ *for any* $\Lambda \in \mathbb{R}$*. Indeed, we can take* $H = \max\{e^{-\Lambda}, 1\}$*,* $\alpha = \beta = 0$ *and* $\gamma = 1$*,* $\sigma = a - \lceil \Re a \rceil + 1$ *(recall that* $0 < \sigma \leq 1$*). Then, we are in case 1 if* $a \in \mathbb{N}$ *and in case 2 if* $a \notin \mathbb{N}$*. In the former case the incomplete gamma function reduces to an elementary function whereas in the latter the point* $t = 0$ *is a branch-point of* $g(t)$*. Therefore, we assume that* $a \notin \mathbb{N}$ *and we apply directly the theorem 3.3.1 with* $t_1 = 1/2$ *as the only base point of the Taylor expansion of* $g(t)$ *(*$m = 1$*) to find the uniform expansion [19, eq. 8]. However, with this election of the functions* $g(t)$ *and* $h(t,z)$*, using* (3.15) *in theorem 3.3.1 we would find that the remainder* $R_n(z) = \mathcal{O}(n^{-(a-\lceil \Re a \rceil+1)})$*. This exponent is too small (between* -1 *and* 0*) and does not agree with the order given in [19, eq. 13] for the remainder: We are losing information on the speed of convergence. The reason is that the parameter* σ *must satisfy* $0 < \sigma < 1$*.*

Taking into account the previous remark 3.3.5, we note that the greater α *and* β *are, the faster the convergence is. Then, a clever factorization of the integrand is the following: take* $g(t) = t^{a-\lceil \Re a \rceil}$ *and* $h(t,z) = e^{-zt} t^{\lceil \Re a \rceil - 1}$*. Considering* $\mathcal{D} = \{z \in \mathbb{C} : \Re z \geq \Lambda\}$ *for any* $\Lambda \in \mathbb{R}$*, it is easy to show again that hypotheses 1-3 hold by taking the only base point* $t_1 = 1/2$ *(and therefeore* $m = 1$*). We again find* $H = \max\{e^{-\Lambda}, 1\}$*,* $\beta = 0$ *and* $\gamma = 1$ *but* $\alpha = \lceil \Re a \rceil - 1$ *and* $\sigma = \Re a - \lceil \Re a \rceil + 1$*, case 2 assuming that* $a \notin \mathbb{N}$*. That is, the value of* σ *is the same that on the previous choice, but the value of* α *is greater. In this way, we find a uniform and convergent expansion of the incomplete gamma function, similar to the one given in [19, eq. 8]. The remainder satisfies* $R_n(z) = \mathcal{O}(n^{-\sigma-\alpha}) = n^{-\Re a}$*, which agrees with [19, eq. 15] and shows the speed of convergence of the expansion without losing information, in contrast to the first election of* $g(t)$ *and* $h(t,z)$ *taken above in this remark.*

The restriction $0 < \sigma < 1$ is neccesary from a theoretical point of view as it is needed to assure that the integral (3.7) in the proof of theorem 3.3.1 is finite. On the other hand, we are not losing generality as we can take σ to be the fractional part of the exponent a in t^{a-1} as in the example of remark 3.3.6. But moreover, in many practical situation, as in the case of the incomplete gamma function $\gamma(a, z)$ analyzed in [19], we may not split the exponent a as the sum of its integer and fractional parts as we did in the second part of remark 3.3.6 and simply take $g(t) = t^{a-1}$ and $h(t, z) = e^{-zt}$. As we have an explicit expression for the functions $g(t)$ and $h(t, z)$ we may derive and explicit bound for the remainder [19, eq. 13] that shows $R_n(z) = \mathcal{O}(n^{-\Re a})$ instead of the implicit bound (3.14) that would lead to $R_n(z) = \mathcal{O}\left(n^{-(a-\lceil \Re a \rceil+1)}\right)$, after applying theorem 3.3.1, where we lose information on the speed of convergence.

Remark 3.3.7. *As we have just seen in the example of the incomplete gamma function, a bound of the form $|h(t, z)| \leq Ht^{\alpha}(1-t)^{\beta}$ with $H > 0$, constant, and α and/or $\beta \geq 0$ is common in practice. We could make the theory a bit more general by relaxing hypothesis 2 replacing "$H > 0$ a positive number independent of z and t" by "H an integrable function of the variable t". On the one hand, the price to pay would be a more involved derivation of the uniform bounds of $R_n(z)$. On the other hand, the theses of theorem 3.3.1 would be essentially the same ones, with a slight modification of the form of* (3.14)*. Therefore, we do not consider this possible generalization and we simply assume that $h(t, z)$ is bounded in the form given in hypothesis 2.*

In the next chapter of the thesis, we derive new uniformly convergent expansions of many special functions of the mathematical physics having an integral representation of the form (3.12), by means of a direct application of theorem 3.3.1. We also compare the uniform approximation with the classical power series and asymptotic expansions of those special functions.

Chapter 4

New Uniform Expansions of Some Special Functions

This chapter of the thesis is devoted to the application of the theory of uniform approximation of integral transforms developed in chapter 3 to some particular special functions of the mathematical physics having an integral representation of the form (3.2). That is, we are going to obtain new convergent expansions, given in terms of elementary functions, of many special functions. Those expansions hold uniformly in large sets of the complex plane that may be unbounded and contain both, large and small values of the uniform variable. Several such representations have already been found by the advisors of this thesis and collaborators. For example, in [66] we can find an expansion of the Bessel function of the first kind $J_\nu(z)$ that holds uniformly in z in any fixed horizontal strip of the complex plane $|\Im z| \leq \Lambda$ whereas in [20] we can find two uniform expansions of the confluent hypergeometric $M(a, b, z)$ function uniformly valid for either $\Re z \geq 0$ or $\Re z \leq 0$. On the other hand, in [43] we can find an expansion of the Gauss hypergeometric function ${}_2F_1(a, b, c; z)$ that holds uniformly for z in an unbounded region of the complex plane that depends on a certain angle $\theta \in (0, \pi/2]$ that, in the limit $\theta \to 0$ becomes the cutted plane $\mathbb{C} \setminus [1, \infty)$.

In all the examples that we are going to analyze, we follow the notation of chapter 3. In particular, we identify the functions $g(t)$ and $h(t, z)$ to classify the problem in one of the cases 1-4 analyzed in theorem 3.3.1. Specially important is the value of the parameters α, β, σ and γ in order to estimate the speed of convergence of the expansion. However, a remark on the value of those parameters must be done: In chapter 3 the parameters σ and γ were restricted to the unit interval, $0 < \sigma, \gamma \leq 1$. This is not a restriction on the applicability of theorem 3.3.1 as in the event of a branch point in the integrand, say t^a, we could simply take σ to be the fractional part of a. However, by doing so and using theorem 3.3.1 we may be losing information on the speed of convergence of the approximation. In order not to lose that information, we can consider that the analytic function $t^{a-\sigma}$ is part of the function $h(t, z)$ (see remark 3.3.6). Both choices are possible and produce similar, but slightly different approximations.

On the other hand, in the particular examples of special functions, we have an explicit expression for the functions $g(t)$ and $h(t, z)$ and many times it is possible to find an explicit expression for the remainder of the Taylor expansion of $g(t)$ and, with it, to derive an explicit bound for the remainder $R_n(z)$ of the uniform approximation of the special function. Typically, that bound would show, as $n \to \infty$, that $R_n(z) = \mathcal{O}(n^{-a})$ and not only $R_n(z) = \mathcal{O}(n^{-\sigma})$, being σ the fractional part of a, as a direct application of theorem 3.3.1 would indicate. The same applies for γ if the function $g(t)$ had an integrable singularity at $t = 1$. Then, in many examples of special functions below, such an explicit bound for the remainder has been found in the literature.

In summary, we consider several special functions and we obtain new convergent expansions that are valid in a large region of the complex plane of a selected variable. Some of these expansions have been derived by the advisors of this thesis and by collaborators of my researching group, but I did not contribute to its derivation. Therefore, I have decided to cite the approximations while omitting many details in their derivation. On the other hand, in the derivation of the uniform approximations of the special functions in which I took part (elliptic functions, Struve function, incomplete gamma $\Gamma(a, z)$ function) more details are given.

To illustrate the kind of expansions that we can derive using theorem 3.3.1, we give the first terms of the uniform approximation, graphics and numerical tables for the different special functions that we analyze. In this way, we can see that the derived expansions are given in terms of elementary functions, that they are convergent and also their uniform features. We also compare the uniform approximations with the well-known power series and asymptotic expansions of the corresponding special function.

4.1 The Struve function $H_\nu(z)$

The results of this section are based on a chapter of my Master's Thesis [111, Ch. 3].

The Struve functions [117] are solution to an inhomogeneous Bessel's differential equation. One of them, the Struve function $H_\nu(z)$ admits the following integral representation [117, eq. 11.5.1]

$$H_\nu(z) = \frac{2(z/2)^\nu}{\sqrt{\pi}\Gamma(\nu+1/2)} \int_0^1 (1-t^2)^{\nu-1/2} \sin(zt)dt, \qquad \Re\nu > -1/2, \tag{4.1}$$

valid for all $z \in \mathbb{C}$. We want to derive a convergent expansion of $H_\nu(z)$ uniformly valid for $z \in \mathcal{D}$, with $\mathcal{D}$ a large set of the complex plane to be determined. To this end, we identify $g(t) = (1-t^2)^{\nu-1/2}$ and $h(t, z) = \sin(zt)$. Obviously, $|h(t, z)| \leq \cosh(\Im z) \leq \cosh(\Lambda)$ for all $t \in (0, 1)$ and z in any fixed horizontal strip $|\Im z| \leq \Lambda$, with $\Lambda > 0$. Therefore, we find $\alpha = \beta = 0$. On the other hand, for general values of $\nu \in \mathbb{C}$, the function $g(t)$ has two singular points, located at $t = \pm 1$.

We consider the Taylor series expansion of $g(t)$ at an arbitrary point $\lambda \in [0, 1/2)$. This choice of λ assures that the integration interval $(0, 1)$ is contained in the disk of convergence of the Taylor series of $g(t)$. Therefore, we can apply theorem 3.3.1 (case 3) to the integral in the right hand side of (4.1) to obtain the expansion

$$\frac{\sqrt{\pi}\Gamma(\nu+1/2)}{2}\left(\frac{2}{z}\right)^\nu H_\nu(z) = \sum_{k=0}^{n-1} \frac{(1/2-\nu)_k}{k!}(1-\lambda)^{\nu-k-1/2} F_k(\lambda, z) + R_n(\lambda, z), \tag{4.2}$$

where $R_n(\lambda, z) := \int_0^1 \sin(zt) g_n(t, \lambda) dt$ being $g_n(t, \lambda)$ the Taylor remainder of $g(t)$ at $t = \lambda$ after n terms of the approximation.

On the one hand, the functions $F_k(\lambda,z)$ are elementary functions of z. They are given by

$$
\begin{aligned}
F_k(\lambda,z) &= \sum_{j=0}^{k}\binom{k}{j}(-\lambda)^{k-j}\int_0^1 t^{2j}\sin(zt)dt = (-1)^k\sum_{j=0}^{k}\binom{k}{j}\lambda^{k-j}\frac{d^{2j}}{dz^{2j}}\left(\frac{1-\cos z}{z}\right) \\
&= (-1)^k\sum_{j=0}^{k}\binom{k}{j}\lambda^{k-j}\frac{(2j)!}{z^{2j+1}}\left[1-\cos z\sum_{l=0}^{j}\frac{z^{2l}(-1)^l}{(2l)!}-\sin z\sum_{l=0}^{j-1}\frac{z^{2l+1}(-1)^l}{(2l+1)!}\right],
\end{aligned}
\tag{4.3}
$$

where, for $z=0$, the right hand side of this formula must be understood in the limit sense.

On the other hand, for any $\lambda\in[0,1/2)$ and according to (3.15) in theorem 3.3.1 (case 3), the remainder $R_n(\lambda,z)$ satisfies, as $n\to\infty$, $R_n(\lambda,z)=\mathcal{O}\left(n^{-\gamma}\right)$, being γ the fractional part of $\nu+1/2$. However, as we will see below, we are losing information on the speed of convergence and the remainder actually satisfies $R_n(\lambda,z)=\mathcal{O}\left(n^{-(\nu+1/2)}\right)$, as $n\to\infty$. To show this, we derive an explicit, accurate expression for the remainder $R_n(\lambda,z)$.

We have, for all z with $|\Im z|\le\Lambda$

$$
|R_n(\lambda,z)|\le\cosh(\Lambda)\int_0^1|g_n(t,\lambda)|dt. \tag{4.4}
$$

As $g_n(t,\lambda)$ is the Taylor remainder of $g(t)$ and $t\in(0,1)$ is contained in the disk of convergence of the Taylor series, we have that

$$
\begin{aligned}
g_n(t,\lambda) &= \sum_{k=n}^{\infty}\frac{(1/2-\nu)_k}{k!}(1-\lambda)^{\nu-k-1/2}(t^2-\lambda)^k \\
&= \frac{(1/2-\nu)_n}{n!}(1-\lambda)^{\nu-n-1/2}(t^2-\lambda)^n\,{}_2F_1\left(\left.\begin{matrix}n+1/2-\nu,\ 1\\ n+1\end{matrix}\right|\frac{t^2-\lambda}{1-\lambda}\right).
\end{aligned}
$$

Introducing the last formula into (4.4), splitting that integral at $t=\sqrt{\lambda}$ and after some computations, we may find the bound

$$
|R_n(\lambda,z)|\le\cosh(\Lambda)\frac{|\,(1/2-\nu)_n\,|}{n!}\left[G_1(\nu,\lambda,n)+G_2(\nu,\lambda,n)\right] \tag{4.5}
$$

with

$$
\begin{aligned}
G_1(\nu,\lambda,n) &= \frac{\sqrt{\pi}}{2}\frac{n!}{\Gamma(n+3/2)}\left(\frac{\lambda}{1-\lambda}\right)^n\sqrt{\lambda}(1-\lambda)^{\Re\nu+1/2}\,{}_2F_1\left(\left.\begin{matrix}1,\ 1+\Re\nu\\ n+3/2\end{matrix}\right|\lambda\right), \\
G_2(\nu,\lambda,n) &= \frac{(1-\lambda)^{\Re\nu+1/2}}{1+2\Re\nu}\,{}_3F_2\left(\left.\begin{matrix}1,\ 1/2,\ 1/2+\Re\nu\\ n+1,\ \Re\nu+3/2\end{matrix}\right|1-\lambda\right).
\end{aligned}
$$

For $\lambda\in[0,1/2]$ we have that, as $n\to\infty$, $G_1(\nu,\lambda,n)=\mathcal{O}\left(n^{-1/2}\left(\frac{\lambda}{1-\lambda}\right)^{n+1/2}\right)$ and $G_2(\nu,\lambda,n)=\mathcal{O}(1)$. Therefore, from (4.5) and using the asymptotic behavior of the quotient of two gamma functions, we find $R_n(\lambda,z)=\mathcal{O}\left(n^{-(\nu+1/2)}\right)$, as $n\to\infty$, for any $\lambda\in[0,1/2]$.

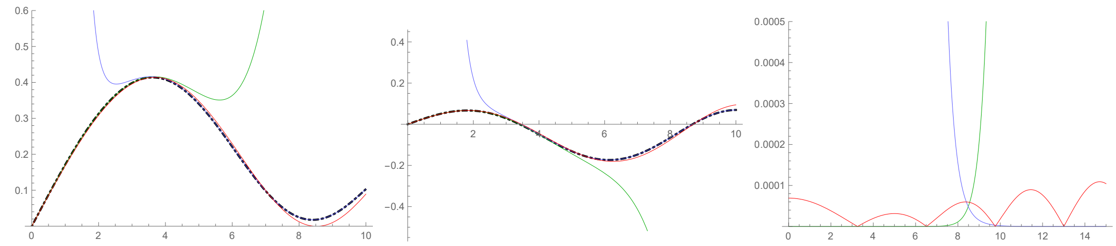


Figure 4.1: Real (left) and imaginary (middle) parts of the approximations of the function on the left hand side of (4.2) for $\nu = 11/5$ (black, dashed) provided by the Taylor expansion [117, eq. 11.2.1] (green), the asymptotic expansion [117, eqs. 11.2.5, 11.6.1 and 10.17.4] (blue) and the uniform expansion (4.2) for $\lambda = 0$ (red), for $z \in [0, 10e^{i\pi/10}]$ after $n = 4$ terms. The figure on the right shows their relative errors for $n = 10$, $\nu = 18/5$ and real $z \in [0, 15]$. The behavior is similar for other values of ν and z.

n	$z = 0.1$	$z = 1$	$z = 4$	$z = 10$	$z = 30$
1	$1.24917 \cdot 10^{-1}$	$1.22911 \cdot 10^{-1}$	$9.00918 \cdot 10^{-2}$	$6.52739 \cdot 10^{-2}$	$1.561 \cdot 10^{-1}$
10	$7.00137 \cdot 10^{-3}$	$6.84592 \cdot 10^{-3}$	$4.3707 \cdot 10^{-3}$	$5.46204 \cdot 10^{-3}$	$4.50228 \cdot 10^{-2}$
20	$2.77883 \cdot 10^{-3}$	$2.71491 \cdot 10^{-3}$	$1.70172 \cdot 10^{-3}$	$2.83448 \cdot 10^{-3}$	$2.21033 \cdot 10^{-2}$
30	$1.5754 \cdot 10^{-3}$	$1.53867 \cdot 10^{-3}$	$9.57482 \cdot 10^{-4}$	$1.74778 \cdot 10^{-3}$	$1.32901 \cdot 10^{-2}$

Table 4.1: Relative error provided by the right hand side of the uniform approximation (4.2) with $\lambda = 0$ to approximate de Struve function, when we truncate the series after n terms, for different values of z and $\nu = 1$.

Moreover, the function $G_1(\nu, \lambda, n) + G_2(\nu, \lambda, n)$ can be shown to be strictly decreasing for $\lambda \in (0, 1/4)$, for all ν with $\Re\nu > -1/2$ and for all $n \in \mathbb{N}$. Hence, $\lambda = 1/4$ is the most convenient election as base point for the Taylor series of $g(t)$, since that value minimizes the value of the error bound. On the other hand, for $\lambda = 0$ the expansion (4.2) becomes simpler, as the sum in the right hand side of (4.3) consists of only one term.

The convergent and uniform features of expansion (4.2) are exhibited in figure 4.1 and table 4.1. Moreover, the uniform approximation (4.2) is compared with the well-known power series expansion [117, eq. 11.2.1] and the asymptotic expansion provided by combining the formulas [117, eqs. 11.2.5, 11.6.1 and 10.17.4] for the Struve $H_\nu(z)$ function.

Remark 4.1.1. *For $\nu = m + 1/2$ with $m = 0, 1, 2, \ldots$ the Struve function is an elementary function and for any $\lambda \in [0, 1/2]$ and n sufficiently large ($n \geq m + 1$), expansion (4.2) is exact, as the bound (4.5) for the remainder vanishes.*

4.2 The Bessel function of the first kind $J_\nu(z)$

Bessel and Struve functions are closely related as the latter are solution to an inhomogeneous Bessel differential equation. Therefore, the results of this section are similar to the ones found in the previous section 4.1. Many details are omitted, but they can be found in [66].

We consider the Poisson's integral representation of the Bessel function of the first

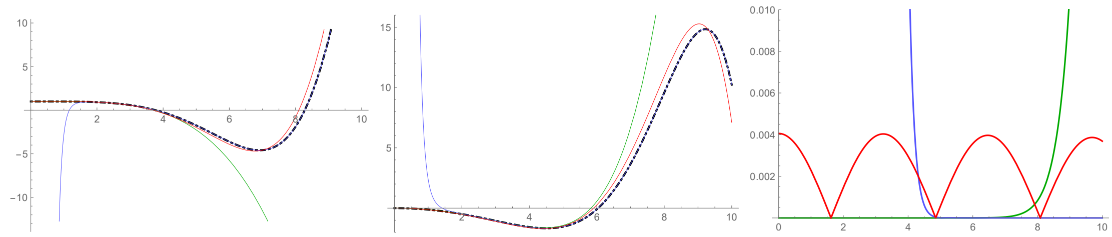


Figure 4.2: Real (left) and imaginary (middle) parts of the approximations of the Bessel function $\frac{2}{z}J_1(z)$ (black, dashed) provided by the Taylor expansion [107, eq. 10.2.2] (green), the asymptotic expansion [107, eq. 10.17.3] (blue) and the uniform expansion [66, Theorem 1] (red), for $z \in [0, 10e^{i\pi/4}]$ after $n = 4$ terms. The figure on the right shows their absolute errors for $n = 10$, $\nu = 1$ and real $z \in [0, 10]$. The behavior is similar for other values of ν and z.

n	$z = 0.1$	$z = 1$	$z = 4$	$z = 10$	$z = 30$
1	$1.50749 \cdot 10^{-1}$	$1.08407 \cdot 10^{-1}$	$6.94103 \cdot 10^{-1}$	3.22872	$3.75696 \cdot 10^{1}$
10	$3.488 \cdot 10^{-4}$	$2.15005 \cdot 10^{-4}$	$1.40729 \cdot 10^{-3}$	$1.57468 \cdot 10^{-2}$	$1.92925 \cdot 10^{-1}$
20	$6.99511 \cdot 10^{-5}$	$4.2304 \cdot 10^{-5}$	$2.69741 \cdot 10^{-4}$	$3.10866 \cdot 10^{-3}$	$2.53792 \cdot 10^{-2}$
30	$2.65193 \cdot 10^{-5}$	$1.59193 \cdot 10^{-5}$	$1.00296 \cdot 10^{-4}$	$1.16031 \cdot 10^{-3}$	$5.83231 \cdot 10^{-3}$

Table 4.2: Relative error provided by the right hand side of the uniform approximation [66, Theorem 1] to approximate the function $(2/z)^2 J_2(z)$, when we truncate the series after n terms, for different values of z.

kind

$$J_\nu(z) = \frac{2(z/2)^\nu}{\sqrt{\pi}\Gamma(\nu+1/2)} \int_0^1 (1-t^2)^{\nu-1/2} \cos(zt)dt, \qquad \Re\nu > -1/2,$$

valid for all $z \in \mathbb{C}$. Following the notation of chapter 3, we identify $g(t) = (1-t^2)^{\nu-1/2}$ and $h(t,z) = \cos(zt)$ and we consider any fixed horizontal strip of the complex plane $\mathcal{D} := \{z \in \mathbb{C} : |\Im z| \le \Lambda\}$, for any $\Lambda > 0$. Then, for all $z \in \mathcal{D}$ and $t \in (0,1)$ we have $|h(t,z)| \le \sinh(\Lambda)$. Therefore $\alpha = \beta = 0$. On the other hand, the function $g(t)$ is the same as in the Struve $H_\nu(z)$ function. Hence, we are in case 3 of the four cases analyzed in chapter 3 and we have $\sigma = 0$ and $\gamma = \nu + 1/2$, with $(A, B) = (0, 1)$. We can apply theorem 3.3.1 to find a convergent expansion of the Bessel function of the first kind that is uniformly convergent in $z \in \mathcal{D}$. This expansion is given in [66, eq. 9]. The remainder of the expansion satisfies $R_n(z,\nu) = \mathcal{O}\left(n^{-\nu-1/2}\right)$, as $n \to \infty$ [66, eq. 12] (see (3.15) in theorem 3.3.1). As an illustration, we obtain the following approximation valid for $x > 0$ [66, eq. 7]:

$$\frac{15\pi}{2x^3} J_3(x) = \left[\frac{3x^4 - 140x^2 + 360}{8x^6} + \theta_1(x)\right] x \sin x + \left[\frac{5(x^2-18)}{2x^4} + \theta_2(x)\right] \cos x,$$

with $|\theta_1(x)| \le 0.0062$ and $|\theta_2(x)| \le 0.051$.

In figure 4.2 and table 4.2 the convergent and uniform features of the approximation are shown. The uniform approximation is also compared with the well-known power series expansion [107, eq. 10.2.2] and asymptotic expansion [107, eq. 10.17.3] of the Bessel function.

Remark 4.2.1. *For $\nu = m + 1/2$ with $m = 0, 1, 2, \ldots$ the Bessel function is an elementary function and the uniform expansion [66, Theorem 1] after n terms is exact for n sufficiently large ($n \ge m + 1$) as the bound [66, eq. 12] for the remainder vanishes.*

n	$z=0.1$	$z=1$	$z=4$	$z=10$	$z=30$
1	$7.1398\cdot10^{-2}$	$1.7953\cdot10^{-1}$	$6.42097\cdot10^{-1}$	1.52345	3.37019
10	$2.80867\cdot10^{-3}$	$4.72791\cdot10^{-3}$	$1.62259\cdot10^{-2}$	$5.4146\cdot10^{-2}$	$2.05248\cdot10^{-1}$
20	$1.08205\cdot10^{-3}$	$1.80653\cdot10^{-3}$	$6.31074\cdot10^{-3}$	$2.21431\cdot10^{-2}$	$9.38413\cdot10^{-2}$
30	$6.06611\cdot10^{-4}$	$1.01017\cdot10^{-3}$	$3.55562\cdot10^{-3}$	$1.27393\cdot10^{-2}$	$5.68929\cdot10^{-2}$

Table 4.3: Relative error provided by the right hand side of the uniform approximation [19, Theorem 1] of the incomplete gamma function $z^{-a}\gamma(a,z)$ for $a=3/2$, when we truncate the series after n terms, for different values of n and z.

4.3 The incomplete gamma function $\gamma(a,z)$

We consider the incomplete gamma function $\gamma(a,z)$ that has already been analyzed in remark 3.3.6. For details on the uniform approximation and explicit bounds for the remainder we refer to [19]. Only some indications are given below.

The starting point is the integral definition of the incomplete gamma function $\gamma(a,z)$ [116, 8.2.1]

$$\gamma(a,z)=z^a\int_0^1 t^{a-1}e^{-zt}dt, \qquad \Re a>0. \tag{4.6}$$

We want to obtain a convergent expansion uniformly valid for z in a large region of the complex plane. Thus, we take $g(t)=t^{a-1}$ and $h(t,z)=e^{-zt}$. Therefore, we are in case 2 of theorem 3.3.1. We consider the standard Taylor expansion of $g(t)$ at a point $t=\lambda\in[1/2,1]$ so that the interval of integration $(0,1)$ is contained in the disk of convergence of $g(t)$ (except for the point $t=0$). On the other hand, the function $h(t,z)$ can be bounded independently of z in the form $|h(t,z)|\leq\max\{1,e^{-\Lambda}\}$ for any $\Lambda\in\mathbb{R}$ whenever $z\in\mathcal{D}:=\{z\in\mathbb{C}:\Re z\geq\Lambda\}$. As a result we find $\alpha=\beta=0$. Moreover, the moments of $h(t,z)$ can be written in terms of the derivatives of $(1-e^{-z})/z$. Thus, when we apply theorem 3.3.1 we find a convergent expansion given in terms of elementary functions that holds uniformly in $z\in\mathcal{D}$ [19, eq. 8]. When we truncate the expansion after n terms we find a remainder $R_n(z)$ that satisfies $R_n(z)=\mathcal{O}\left(n^{-a}\right)$ as $n\to\infty$ [19, eq. 13] (see (3.15) in theorem 3.3.1).

Furthermore, in [19] the authors show that the middle point of the integration interval $\lambda=1/2$ minimizes an explicit bound for the remainder $R_n(z)$ [19, eq. 21]. To illustrate the type of approximation that we obtain, we may find, for any $\Re z\geq0$ [19, eq. 6]

$$z^{-5/2}\gamma\left(\frac{5}{2},z\right)=\frac{24+12z-z^2}{16\sqrt{2}z^3}-e^{-z}\frac{24+36z+23z^2}{16\sqrt{2}z^3}+\epsilon(z),$$

with $|\epsilon(z)|\leq0.0066$. For $z=0$ the right hand side of this formula must be understood in the limit sense.

Finally, some plots comparing the uniform approximation [19, Theorem 1] with the well-known power series expansion [116, eq. 8.7.1] and the asymptotic expansion [116, eqs. 8.2.3 and 8.11.2] of the incomplete gamma function are shown in figure 4.3. On the other hand, in table 4.3 the relative error provided by the uniform approximation is shown for several values of z and n. The convergent and uniform features can be appreciated.

Remark 4.3.1. *For $a=1,2,3,\ldots$ the incomplete gamma function $\gamma(a,z)$ is an elementary function and the uniform expansion [19, eq. 8] after n terms is exact for n sufficiently large ($n\geq a$) as the bound [19, eq. 11] for the remainder vanishes.*

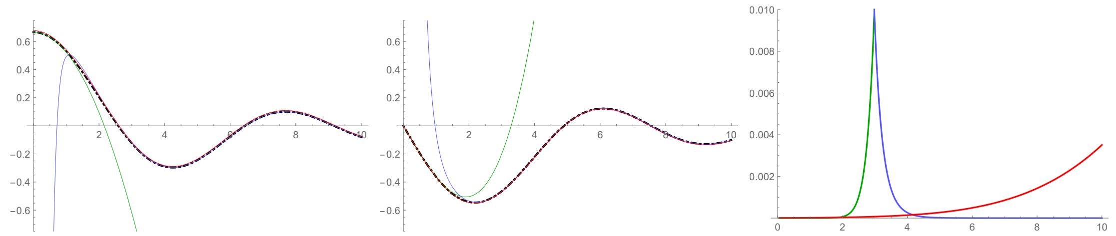


Figure 4.3: Real part (left) and imaginary part (middle) of the approximations of the incomplete gamma function $z^{-a}\gamma(a, z)$ (black, dashed) provided by the power series expansion [116, eq. 8.7.1] (green), the asymptotic expansion [116, eqs. 8.2.3 and 8.11.2] (blue) and the uniform expansion [19, Theorem 1] (red), for $z \in [0, 10i]$ after $n = 4$ terms, for $a = 3/2$. The figure on the right shows their relative errors for $n = 10$, $a = 9/2$ and real $z \in [0, 10]$. The behavior is similar for other values of z and a.

4.4 The incomplete gamma function $\Gamma(a, x)$

A convergent expansion for the incomplete gamma function $\Gamma(a, z)$ in terms of elementary functions and uniformly valid in z bounded from above can be derived from the expansion of $\gamma(a, z)$ in the previous section 4.3 and the well-known relation $\gamma(a, z) + \Gamma(a, z) = \Gamma(a)$ [116, eq. 8.2.3].

Instead, for convenience in the analysis of chapter 5 we are interested in the derivation of a convergent expansion of $\Gamma(a, x)$ with $x > 0$ uniformly valid in a region of the complex $a-$plane that contains the interval $(-\infty, 0)$.

Therefore, we assume that $x > 0$ and we consider the integral definition of the incomplete gamma function [116, eq. 8.2.2]. We perform a change of variables of the form $s \mapsto u$ given by $s = xe^u$ to obtain

$$\Gamma(a, x) := \int_x^\infty s^{a-1} e^{-s} ds = x^a \int_0^\infty e^{au} e^{-xe^u} du.$$

We have found the unbounded integration interval $[0, \infty)$. For this reason, and following the steps of chapter 3, we perform a further change of variables $u \mapsto t$ given by $u = -\log t$. We get

$$\Gamma(a, x) = x^a \int_0^1 t^{-a-1} e^{-x/t} dt. \tag{4.7}$$

As we are considering a as the uniform parameter, we identify $g(t) = \frac{e^{-x/t}}{t}$ and $h(t, a) = t^{-a}$. It is easy to check that the hypotheses of theorem 3.3.1 are satisfied and to derive a convergent expansion of $\Gamma(a, x)$ valid uniformly in $\Re a < 0$ by directly applying theorem 3.3.1. Again, for the convenience of the analysis in chapter 5 we derive the expansion with an bound for the remainder.

The function $g(t)$ admits the following representation, in terms of the Laguerre polynomials $L_n(x) \equiv L_n^{(0)}(x)$ [57],

$$g(t) = \frac{e^{-x/t}}{t} = e^{-x} \sum_{k=0}^{n-1} L_k(x)(1-t)^k + r_n(x, t), \tag{4.8}$$

which directly follows from the generating function of $L_n(x)$ [57, eq. 18.12.13], namely,

$$\frac{e^{x\frac{w}{w-1}}}{1-w} = \sum_{n=0}^{\infty} L_n(x) w^n, \qquad |w| < 1,$$

by setting $w = 1 - t$.

Thus, expansion (4.8) is convergent for $|1-t| < 1$. As a consequence, the remainder $r_n(x,t)$ satisfies

$$r_n(x,t) = e^{-x} \sum_{k=n}^{\infty} L_k(x)(1-t)^k.$$

On the other hand, introducing (4.8) in (4.7) and using the integral representation of the beta function we obtain

$$\Gamma(a,x) = e^{-x} x^a \sum_{k=0}^{n-1} L_k(x) B(1-a, k+1) + R_n(a,x), \tag{4.9}$$

with

$$R_n(a,x) := x^a \int_0^1 t^{-a} r_n(x,t) dt = x^a e^{-x} \int_0^1 t^{-a} \sum_{k=n}^{\infty} L_k(x)(1-t)^k dt.$$

In [57, eq. 18.14.8] we find the inequality $e^{-x/2}|L_n(x)| \le 1$, valid for $0 \le x < +\infty$. Then

$$|R_n(a,x)| \le x^{\Re a} e^{-x/2} \int_0^1 t^{-\Re a} \sum_{k=n}^{\infty} (1-t)^k dt = x^{\Re a} e^{-x/2} \int_0^1 t^{-\Re a - 1}(1-t)^n dt.$$

Using one more time the integral representation of the beta function, we find the bound

$$|R_n(a,x)| \le e^{-x/2} x^{\Re a} B\left(-\Re a, n+1\right) \le e^{-x/2} x^{\Re a} \Gamma\left(-\Re a\right) n^{\Re a}. \tag{4.10}$$

The second inequality above follows from [4, eq. 5.6.8]. Indeed, we can write

$$B\left(-\Re a, n+1\right) = \frac{\Gamma(-\Re a)\Gamma(n+1)}{\Gamma(n+1-\Re a)} = \frac{\Gamma(-\Re a) n \Gamma(n)}{\Gamma(n+1-\Re a)} \le \Gamma(-\Re a) n n^{\Re a - 1} = \Gamma(-\Re a) n^{\Re a}.$$

The results of this section for the incomplete gamma $\Gamma(a,z)$ function will be used in chapter 5. For this reason, we summarize them in the form of a corollary:

Corollary 4.4.1. *Let $a \in \mathbb{C}$ and $x \in \mathbb{R}$ such that $x > 0 > \Re a$. Then, for any $n \in \mathbb{N}$, the incomplete gamma function $\Gamma(a,x)$ admits the representation (4.9) whose remainder $R_n(a,x)$ can be bounded in the form (4.10). In particular, the expansion is convergent for any complex number a in the semi-plane $\Re a \le \Lambda < 0$, for any fixed $\Lambda < 0$, and the convergence rate is of power order. That is, $R_n(a,x) = \mathcal{O}\left(n^a\right)$, as $n \to \infty$.*

Some plots comparing the Taylor series expansion at $a = 0$ of the incomplete gamma funcion $\Gamma(a,x)$, the asymptotic expansion for large a [116, eqs. 8.11.4 and 8.2.3] and the uniform approximation (4.9) are shown in figure 4.4. On the other hand, in table 4.4 the uniform and convergent features of the expansion (4.9) are exhibited.

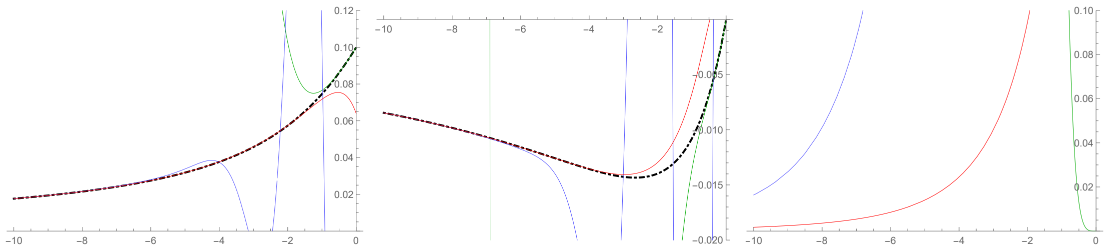


Figure 4.4: Real part (left) and imaginary part (middle) of the approximations of the incomplete gamma function $x^{-a}\Gamma(a,x)$ (black, dashed) provided by the Taylor series expansion at $a=0$ (green), the asymptotic expansion [116, eqs. 8.11.4 and 8.2.3] (blue) and the uniform expansion (4.9) (red), for $a\in[-10e^{i\pi/6},0]$ after $n=4$ terms, for $x=3/2$. The figure on the right shows their relative errors for $n=4$, $x=3$ and $a\in[-10e^{i\pi/4},0]$. The behavior is similar for other values of x and a.

n	$a=-0.1$	$a=-1$	$a=-4$	$a=-10$	$a=-30$
1	1.2643	$6.19307\cdot 10^{-1}$	$1.95536\cdot 10^{-1}$	$7.43882\cdot 10^{-2}$	$2.32522\cdot 10^{-2}$
10	$2.40285\cdot 10^{-1}$	$2.99775\cdot 10^{-2}$	$2.66149\cdot 10^{-4}$	$7.21155\cdot 10^{-7}$	$3.77759\cdot 10^{-11}$
20	$1.49679\cdot 10^{-1}$	$1.16827\cdot 10^{-2}$	$2.57919\cdot 10^{-5}$	$6.62086\cdot 10^{-9}$	$1.31103\cdot 10^{-15}$
30	$8.40945\cdot 10^{-2}$	$4.39893\cdot 10^{-3}$	$2.68919\cdot 10^{-6}$	$4.07639\cdot 10^{-11}$	$2.76772\cdot 10^{-15}$

Table 4.4: Relative error provided by the right hand side of the uniform approximation (4.9) of the incomplete gamma function $x^{-a}\Gamma(a,x)$ for $x=5/3$, when we truncate the series after n terms, for different values of n and a.

4.5 The hypergeometric confluent M function

Details on the uniform approximation that we are going to derive and explicit, accurate bounds for its remainder may be found in [20].

We consider the following integral representation of the hypergeometric confluent $M(a,b,z)$ function [103, eq. 13.4.1]

$$M(a,b,z)=\frac{\Gamma(b)}{\Gamma(a)\Gamma(b-a)}\int_0^1 e^{zt}t^{a-1}(1-t)^{b-a-1}dt,\qquad \Re b>\Re a>0,$$

and we assume that neither a nor $b-a$ is an integer, as in those cases the $M(a,b,z)$ function can be expressed as incomplete gamma functions and the analysis is reduced to section 4.3.

We want to obtain a convergent expansion for $M(a,b,z)$ uniformly valid in the variable z, with z in a large region of $\mathbb{C}$. Following chapter 3 we have two main choices for the functions $g(t)$ and $h(t,z)$.

- We can take $h(t,z)=e^{zt}t^{a-1}$ and $g(t)=(1-t)^{b-a-1}$. Then, we are in case 3 of theorem 3.3.1 and we may find $\alpha=a-1$, $\beta=0$, $\sigma=1$ and $\gamma=b-a$, with $(A,B)=(0,1)$. Considering $t=0$ as the only base point of the Taylor series of $g(t)$ and using theorem 3.3.1 we obtain expansion [20, eq. 10] that was first found by K.E. Muller [90]. The remainder term of the expansion is of the order $R_n(z)=\mathcal{O}(n^{-a})$, as $n\to\infty$. However, the moments of $h(t,z)$ are not given in terms of elementary functions, but rather in terms of incomplete gamma functions.

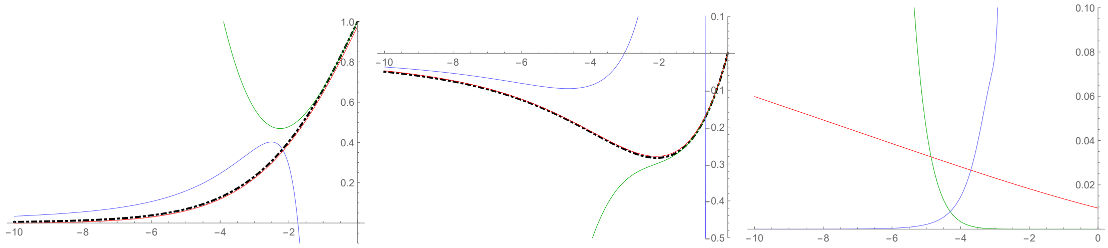


Figure 4.5: Real part (left) and imaginary part (middle) of the approximations of the hypergeometric $M(a,b,z)$ function (black, dashed) provided by the power series expansion [103, eq. 13.2.2] (green), the asymptotic expansion [103, eq. 13.7.2] (blue) and the uniform approximation [20, eq. 21] (red) for $z \in [-10e^{i\pi/4}, 0]$, $a = 2.1$ and $b = 4.2$ when we truncate the series after $n = 4$ terms. The figure in the right shows their relative error after $n = 10$ terms for $a = 1.2$, $b = 3.3$ and real $z \in [-10, 0]$.

- We want an expansion of $M(a,b,z)$ given in terms of elementary functions. Therefore, we choose $h(t,z) = e^{zt}$ and $g(t) = t^{a-1}(1-t)^{b-a-1}$. In this case, the moments of $h(t,z)$ are elementary functions, as they can be written in terms of the derivatives of the function $(e^z - 1)/z$. We are now in case 4 of theorem 3.3.1, with $\alpha = \beta = 0$ and $\sigma = a$, $\gamma = b - a$. If we consider a standard Taylor expansion of $g(t)$, we must take $t_1 = 1/2$ as the base point. Hence, following theorem 3.3.1 we obtain the expansion [20, eq. 21]. According to (3.15) the remainder term when we truncate the expansion after n terms is of the order $R_n(z) = \mathcal{O}\left(n^{-a} + n^{-(b-a)}\right)$ as $n \to \infty$ [20, eq. 28]. The expansion is convergent and uniformly valid in z for $\Re z \le \Lambda$, for any fixed $\Lambda \in \mathbb{R}$. As an illustration of the expansion that we may obtain we have the following approximation, that can be found when we truncate the series [20, eq. 21] after $n = 4$ terms

$$M\left(\frac{4}{3}, \frac{7}{3}, z\right) = \frac{2^{2/3}16\left[30 + 24z + 15z^2 - 5z^3 + e^z\left(13z^3 - 6z^2 + 6z - 30\right)\right]}{243z^4} + \varepsilon(z)$$

  with $|\varepsilon(z)| \le 0.223$ in $\Re z \le 0$. For $z = 0$ the right hand side of this formula must be understood in the limit sense. The bound for the remainder $\varepsilon(z)$ follows from [20, eq. 28].

  In figure 4.5 and table 4.5 the accuracy of the uniform expansion is exhibited. It is compared with the well-known power series expansion [103, eq. 13.2.2] and the asymptotic expansion [103, eq. 13.7.2] of the confluent M function.

  Moreover, the previous expression is uniformly convergent for z with $\Re z \le \Lambda$, for any fixed $\Lambda \in \mathbb{R}$. But, using Kummer's transformation $M(a,b,z) = e^z M(b - a, b, -z)$ [103, eq. 13.2.39] and the previous expansion for the function M, we can obtain an expansion for M that holds uniformly in z for $\Re z \ge \Lambda$ for any $\Lambda \in \mathbb{R}$ [20, eq. 29].

Remark 4.5.1. *For $a, b = 1, 2, 3, \ldots$ the hypergeometric confluent function $M(a,b,z)$ is an elementary function and the uniform expansion [20, eq. 21] after n terms is exact for n sufficiently large ($n \ge b$) as the bound [20, eq. 28] for the remainder vanishes.*

n	$z=0.1$	$z=0.8$	$z=2$	$z=7$	$z=15$
1	$6.21723\cdot10^{-1}$	$6.41421\cdot10^{-1}$	$7.44509\cdot10^{-1}$	1.90089	4.91544
10	$2.40658\cdot10^{-3}$	$2.54205\cdot10^{-3}$	$3.26228\cdot10^{-3}$	$1.26241\cdot10^{-2}$	$4.64366\cdot10^{-2}$
20	$5.08559\cdot10^{-4}$	$5.39019\cdot10^{-4}$	$7.01489\cdot10^{-4}$	$2.87675\cdot10^{-3}$	$1.12976\cdot10^{-2}$
30	$2.02992\cdot10^{-4}$	$2.15442\cdot10^{-4}$	$2.81939\cdot10^{-4}$	$1.18373\cdot10^{-3}$	$4.78494\cdot10^{-3}$

Table 4.5: Relative error provided by the right hand side of the uniform approximation [20, eq. 21] of the hypergeometric confluent $M(a,b,z)$ function for $a=2.3$ and $b=4.6$, when we truncate the series after n terms, for different values of n and z.

4.6 The hypergeometric confluent U function

We consider now another hypergeometric confluent function, namely, the $U(a,b,z)$ hypergeometric function [103]. It admits the following integral representation [103, eq. 13.4.4]

$$U(a,b,z)=\frac{1}{\Gamma(a)}\int_0^\infty e^{-zu}u^{a-1}(1+u)^{b-a-1}du, \qquad \Re a>0, \Re z>0.$$

We want to derive a convergent expansion of $U(a,b,z)$ uniformly valid for z in a large region of the complex plane.

In [20, §3] the following approach is taken: As the integration interval is unbounded, the integral is broken at $u=1$ into two different integrals. In the first integral (the one that runs from $u=0$ to $u=1$) the authors consider the Taylor series expansion of $f(u)=(1+u)^{b-a-1}$ at $u=0$ and apply theorem 3.3.1. On the second integral, they use the Taylor expansion at $u=\infty$ of the function $f(u^{-1})$. In this way, they find expansion [20, eq. 39] whose remainder is shown to satisfy $R_n(a,b,z)=\mathcal{O}\left(n^{-(b-a+1))}\right)$, as $n\to\infty$, whenever $\Re b-\Re a>-1$. The expansion is uniformly convergent in $|z|$ for $\Re z>0$, but given in terms of incomplete gamma functions.

We seek an expansion given in terms of elementary functions. To this end, we follow the ideas of chapter 3 and perform a logarithmic change of variables $u=-\log t$. We have

$$U(a,b,z)=\frac{1}{\Gamma(a)}\int_0^1 t^{z-1}\left(-\log t\right)^{a-1}\left(1-\log t\right)^{b-a-1}dt.$$

We take $g(t)=\left(-\log t\right)^{a-1}\left(1-\log t\right)^{b-a-1}$ and $h(t,z)=t^{z-1}$. For any fixed $\Lambda>0$ we define the region $\mathcal{D}:=\{z\in\mathbb{C}:\Re z\geq\Lambda>0\}$ and then, for $z\in\mathcal{D}$ we find $|h(t,z)|\leq t^{\Lambda-1}$. Therefore, we have $\alpha=\Lambda-1$ and $\beta=0$. On the other hand, for general $a,b\in\mathbb{C}$ both end points $t=0$ and $t=1$ are singular points of $g(t)$. Hence, we are in case 4 of theorem 3.3.1 and, since $t^{1-\sigma}g(t)$ is bounded as $t\to0$ for any $0<\sigma<1$ we can choose any σ, and in particular, a σ as close to 1 as we wish. On the other hand, we take $\gamma=\max\{1,\Re a\}$. Therefore, we can apply theorem 3.3.1 by considering $t_1=1/2$ as the only base point of the standard Taylor expansion of $g(t)$. We have

$$g(t)=\sum_{k=0}^{n-1}A_k(a,b)\left(t-\frac{1}{2}\right)^k+g_n(t).$$

The coefficients $A_k(a,b)$ are elementary functions of a and b. They can be computed as follows, which can be found from Faà di Bruno's formula [35]:

$$\begin{cases} A_0(a,b) = (\log 2)^{a-1}\left(1+\log 2\right)^{b-a-1}, \\ A_n(a,b) = \dfrac{A_0(a,b)}{n!}\displaystyle\sum_{k=1}^{n}\frac{(-1)^k(b-c-k)_k}{(1+\log 2)^k} b(n,k)\,{}_2F_1\left(\left.\begin{matrix}-k,\ 1-a\\ b-a-k\end{matrix}\right| 1+\frac{1}{\log 2}\right), \quad n\geq 1. \end{cases}$$

In this formula, $b(n,k)$ are certain partial ordinary Bell polynomials [131] that can be computed recursively by means of

$$\begin{cases} b(0,0)=1, \qquad b(n,0)=0, \qquad b(0,k)=0, \\ b(n,k) = \displaystyle\sum_{j=1}^{n-k+1}\frac{(n-1)!}{(n-j)!}(-1)^{j+1}2^j b(n-j,k-1). \end{cases}$$

From theorem 3.3.1 we find

$$U(a,b,z) = \frac{1}{\Gamma(a)}\left[\sum_{k=0}^{n-1} A_k(a,b)G_k(z) + R_n(z)\right], \tag{4.11}$$

where $G_k(z)$ are rational functions given by

$$G_k(z) := \int_0^1 t^{z-1}\left(t-\frac{1}{2}\right)^k dt = \left(\frac{1}{2}\right)^k \frac{1}{z}\,{}_2F_1\left(-k,1,z+1;2\right) = \sum_{j=0}^{k}\binom{k}{j}\left(\frac{-1}{2}\right)^{k-j}\frac{1}{z+j}.$$

According to theorem 3.3.1 the remainder terms satisfies $R_n(z) = \mathcal{O}\left(n^{1-\sigma-\Lambda}+n^{-\max\{\Re a,1\}}\right)$, with σ as close to 1 as we wish. Expansion (4.11) is given in terms of rational functions of z. For example, defining $\ell := \log 2$ and truncating the series (4.11) after $n=3$ terms, we find

$$U\left(2,\frac{3}{2},z\right) = \frac{(16\ell^3+20\ell^2+34\ell)+(24\ell^3+54\ell^2+15\ell)z+(16\ell^3+18\ell^2+9\ell-16)z^2}{8z(z+1)(z+2)(1+\ell)^{7/2}} + R_3(z),$$

with $R_3(z)$ of the order specified above. We can not give more information on the remainder as we do not have an explicit bound for it due to the difficult expression for the coefficients $A_n(a,b)$ of the expansion. The expansion holds uniformly in z with $\Re z \geq \Lambda > 0$.

Expansion (4.11) is given in [77, Example 6].

Figure 4.6 and table 4.6 show the accuracy of the uniform expansion (4.11). It is compared with the well-known power series expansion [103, eqs. 13.2.42 and 13.2.2] and the asymptotic expansion [103, eq. 13.7.3] of the confluent U function.

4.7 The exponential integral

We consider now the exponential integral function $E_1(z)$ that is a particular case of the hypergeometric confluent U function as it satisfies $E_1(z) = e^{-z}U(1,1,z)$ [142, 6.11.2]. More precisely, the exponential integral has the following integral represenation [142, eq. 6.2.2]

$$E_1(z) = e^{-z}\int_0^\infty \frac{e^{-zu}}{u+1}du, \qquad \Re z > 0. \tag{4.12}$$

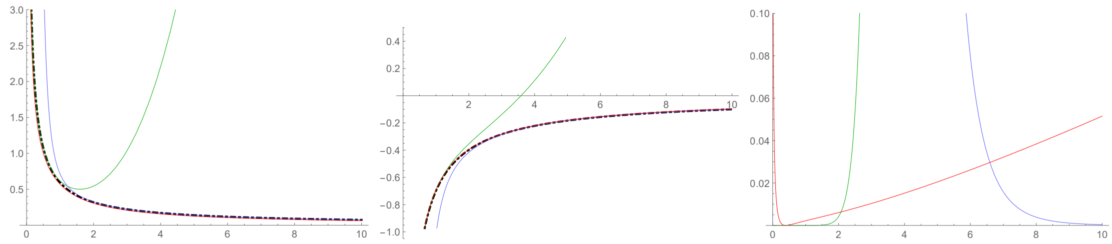


Figure 4.6: Real part (left) and imaginary part (middle) of the approximations of the hypergeometric $U(a,b,z)$ function (black, dashed) provided by the power series expansion [103, eqs. 13.2.42 and 13.2.2] (green), the asymptotic expansion [103, eq. 13.7.3] (blue) and the uniform approximation (4.11) for $z \in [0, 10e^{i\pi/3}]$, $a = 0.9$ and $b = 1.7$ when we truncate the series after $n = 4$ terms. The figure in the right shows their relative error for $a = 1.6$, $b = 1.9$ and real $z \in [0, 10]$ after $n = 10$ terms of the approximations.

n	$z = 0.1$	$z = 0.8$	$z = 2$	$z = 10$	$z = 30$
1	$2.02854 \cdot 10^{-1}$	$9.59021 \cdot 10^{-3}$	$2.56889 \cdot 10^{-2}$	$1.09726 \cdot 10^{-1}$	$2.11021 \cdot 10^{-1}$
10	$1.41691 \cdot 10^{-1}$	$2.33524 \cdot 10^{-3}$	$5.17798 \cdot 10^{-3}$	$2.77002 \cdot 10^{-2}$	$7.20035 \cdot 10^{-2}$
20	$1.26522 \cdot 10^{-1}$	$1.45469 \cdot 10^{-3}$	$2.75221 \cdot 10^{-3}$	$1.54559 \cdot 10^{-2}$	$4.38528 \cdot 10^{-2}$
30	$1.18178 \cdot 10^{-1}$	$1.08179 \cdot 10^{-3}$	$1.84634 \cdot 10^{-3}$	$1.05891 \cdot 10^{-2}$	$3.13643 \cdot 10^{-2}$

Table 4.6: Relative error provided by the right hand side of the uniform approximation (4.11) of the hypergeometric confluent $U(a,b,z)$ function for $a = 1.1$ and $b = 2.2$, when we truncate the series after n terms, for different values of n and z.

A convergent expansion of $E_1(z)$ uniformly valid in $\Re z > 0$ may be derived from (4.11). But in this particular case we may derive a simpler expansion. To obtain it, we perform again a logarithmic change of variables defined by $u = -\log t$. We get

$$E_1(z) = e^{-z} \int_0^1 \frac{t^{z-1}}{1 - \log t} dt.$$

We identify $g(t) = \frac{1}{1-\log t}$ and $h(t,z) = t^{z-1}$. We define the region $\mathcal{D} := \{z \in \mathbb{C} : \Re z \geq \Lambda\}$, for any $\Lambda > 0$. Then, we can take $\alpha = \Lambda - 1$ and $\beta = 0$. On the other hand, the function $f(t) = t^{\sigma-1} g(t)$ is bounded as $t \to 0^+$, for any $\sigma \in (0,1)$. Therefore, we can take any $\sigma \in (0,1)$ and in particular, we may choose σ as close to 1 as we wish. We also have $\gamma = 1$ as the singular point of $g(t)$ are located at $t = 0$ (integrable singularity) and at $t = e$. Hence, we consider the classical Taylor expansion of $g(t)$ at the point $t_1 = 1$:

$$g(t) = \sum_{k=0}^{n-1} A_k (1-t)^k + g_n(t),$$

where $(-1)^k A_k$ are the Taylor coefficients of $g(t)$ at $t = 1$ and $g_n(t)$ is the Taylor remainder.

The coefficients A_k can be computed recursively using the differential equation satisfied by $g(t)$: $t(1 - \log t) g'(t) = g(t)$, and equating the coefficients of equal powers after replacing the Taylor expansion for $g(t)$ and $g'(t)$. In this way, and after some manipulations we can find the recursive formula [142, eq. 6.10.3]. On the other hand, an explicit

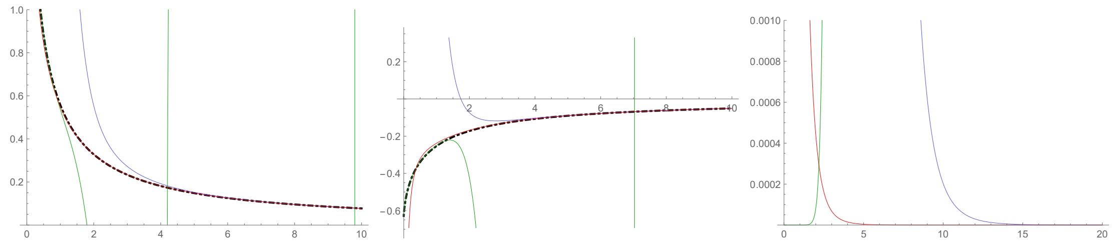


Figure 4.7: Real part (left) and imaginary part (middle) of the approximations of the function $e^z E_1(z)$ function (black, dashed) provided by the power series expansion [142, eq. 6.6.2] (green), the asymptotic expansion [142, eq. 6.12.1] (blue) and the uniform approximation (4.14) for $z \in [0, 10e^{i\pi/5}]$ when we truncate the series after $n = 4$ terms. The figure in the right shows their relative error for real $z \in [0, 20]$ after $n = 10$ terms of the approximations.

formula for A_k is the following

$$A_k = (-1)^k \sum_{j=0}^{k} \frac{j}{k} \frac{B_{k-j}^{(k)}}{(k-j)!}, \tag{4.13}$$

where $B_n^{(\alpha)}$ are the Nørlund polynomials [36, eq. 24.16.9]. Formula (4.13) follows from corollary A.0.2 in appendix A and the derivatives $g^{(n)}(0) = (-1)^n n!$. In particular, the first coefficients are $A_0 = 1$, $A_1 = -1$, $A_2 = 1/2$, $A_3 = -1/3$, $A_4 = 1/6, \ldots$ When we apply theorem 3.3.1 (case 2) to the integral (4.12) we find the expansion[1]

$$E_1(z) = e^{-z} \left[\sum_{k=0}^{n-1} \frac{A_k k!}{(z)_{k+1}} + R_n(z) \right]. \tag{4.14}$$

For any z with $\Re z \geq \Lambda$ expansion (4.14) is uniformly convergent and the remainder term is bounded in the form (3.14)

$$|R_n(z)| \leq M \frac{\Gamma(\Lambda + \sigma - 1)\Gamma(n+1)}{\Gamma(n + \Lambda + \sigma)},$$

for a certain $M > 0$ independent of n and z. Therefore, $R_n(z) = \mathcal{O}\left(n^{1-\sigma-\Lambda}\right)$ with σ as close to 1 as we wish.

Expansion (4.14) may be found in [142, eq. 6.10.1] or [145, Ch. 17, §3].

In figure 4.7 we compare the accuracy of the well-known power series [142, eq. 6.6.2] and asymptotic expansion [142, eq. 6.12.1] of the function $e^z E_1(z)$ with the also well-known factorial series (4.14) that holds uniformly in $\Re z \geq \Lambda > 0$. The convergent features are exhibited in table 4.7.

4.8 The symmetric elliptic integrals $R_D(x, y, z)$ and $R_F(x, y, z)$

The results of this section have been published in the papers [21] and [22].

[1]Expansions of the form (4.14) are called *factorial series*. They are convergent and asymptotic for large z. They are further investigated in chapter 7 of this thesis.

n	$z=0.1$	$z=0.8$	$z=2$	$z=10$	$z=30$
1	$5.48758\cdot10^{-1}$	$1.96298\cdot10^{-1}$	$7.74787\cdot10^{-2}$	$7.14525\cdot10^{-3}$	$9.80938\cdot10^{-4}$
10	$4.3919\cdot10^{-1}$	$8.536236\cdot10^{-3}$	$2.97239\cdot10^{-4}$	$9.48078\cdot10^{-9}$	$2.59773\cdot10^{-13}$
20	$3.59054\cdot10^{-1}$	$4.54237\cdot10^{-3}$	$8.21288\cdot10^{-5}$	$8.21742\cdot10^{-11}$	$3.91617\cdot10^{-16}$
30	$3.16566\cdot10^{-1}$	$3.02423\cdot10^{-3}$	$3.52504\cdot10^{-5}$	$2.90707\cdot10^{-12}$	$1.30539\cdot10^{-16}$

Table 4.7: Relative error provided by the right hand side of the uniform approximation (4.14) of the exponential integral $E_1(z)$, when we truncate the series after n terms, for different values of n and z.

We consider the symmetric elliptic integrals $R_F(x,y,z)$ and $R_D(x,y,z)$. These functions were introduced by Carlson who showed that they are more appropriate for numerical purposes than the standard elliptic integrals [23, 24, 25, 26, 27]. They are defined by the following integral representations [28, eq. 19.16.1], [28, eq. 19.16.5]

$$R_F(x,y,z) := \frac{1}{2}\int_0^\infty \frac{ds}{\sqrt{s+x}\sqrt{s+y}\sqrt{s+z}}, \tag{4.15}$$

$$R_D(x,y,z) := \frac{3}{2}\int_0^\infty \frac{ds}{\sqrt{s+x}\sqrt{s+y}\sqrt{(s+z)^3}}, \tag{4.16}$$

where, in both integrals, $x,y,z\in\mathbb{C}\setminus(-\infty,0]$ except that one or more of x,y,z may be 0 when the corresponding integral converges. We assume that the three variables x, y and z are different, because otherwise the functions R_F and R_D are elementary functions. Moreover, for the sake of simplicity and due to the symmetry of the variables we also assume that one of the variables of R_F, say z, is positive. Similarly, we assume for R_D that either $x,y\in\mathbb{C}\setminus(-\infty,0]$ and $z>0$ or $y,z\in\mathbb{C}\setminus(-\infty,0]$ and $x>0$. The results that we are going to derive can be extended to complex values of the positive variables by using analytic continuation arguments (see Appendix B).

The functions R_F and R_D are homogeneous functions of degree, respectively, $-1/2$ and $-3/2$ in its three variables. Moreover, the function R_F is symmetric in its three variables whereas only the variables x and y play the same role in the integrand of the function R_D. Therefore, instead of the function $R_F(x,y,z)$ we consider the function $F(x,y)$ defined in (4.17) below. On the other hand, instead of considering the function $R_D(x,y,z)$, we consider, if $z>0$ the function $G_1(x,y)$ given in (4.18) or, if $x>0$, the function $G_2(y,z)$ defined in (4.19). These functions are defined by means of

$$F(x,y) := \sqrt{z}R_F\left(z(1+x),z(1+y),z\right) = \frac{1}{2}\int_0^\infty \frac{ds}{\sqrt{s+x+1}\sqrt{s+y+1}\sqrt{s+1}}, \tag{4.17}$$

with $x,y\in\mathbb{C}\setminus(-\infty,-1]$.

$$G_1(x,y) := \sqrt{z^3}R_D\left(z(1+x),z(1+y),z\right) = \frac{3}{2}\int_0^\infty \frac{ds}{\sqrt{s+x+1}\sqrt{s+y+1}\sqrt{(s+1)^3}}, \tag{4.18}$$

with $x,y\in\mathbb{C}\setminus(-\infty,-1]$.

$$G_2(y,z) := \sqrt{x^3}R_D\left(x,x(1+y),x(1+z)\right) = \frac{3}{2}\int_0^\infty \frac{ds}{\sqrt{s+1}\sqrt{s+y+1}\sqrt{(s+z+1)^3}}, \tag{4.19}$$

$y,z\in\mathbb{C}\setminus(-\infty,-1]$.

The functions F, G_1 and G_2 are functions of only two variables and they are more convenient than the functions R_F and R_D to derive new convergent expansions uniformly valid when one of its variables runs in a large set of the complex plane. And all the results that we are going to obtain can be translated to the symmetric elliptic functions by means of the connection formulas

$$R_F(x,y,z) = \frac{1}{\sqrt{z}} F\left(\frac{x-z}{z}, \frac{y-z}{z}\right), \qquad x, y \in \mathbb{C} \setminus (-\infty, 0],\ z > 0. \tag{4.20}$$

$$R_D(x,y,z) = \frac{1}{\sqrt{z^3}} G_1\left(\frac{x-z}{z}, \frac{y-z}{z}\right), \qquad x, y \in \mathbb{C} \setminus (-\infty, 0],\ z > 0. \tag{4.21}$$

$$R_D(x,y,z) = \frac{1}{\sqrt{x^3}} G_2\left(\frac{y-x}{x}, \frac{z-x}{x}\right), \qquad y, z \in \mathbb{C} \setminus (-\infty, 0],\ x > 0. \tag{4.22}$$

Nevertheless, the integrals (4.17), (4.18) and (4.19) are not suitable for the derivation of uniform expansions, as the interval of integration is unbounded. In the general theory of uniform expansion developed in chapter 3 a logarithmic change of variables was indicated to obtain a compact interval of integration. That change of variables was useful from a theoretical point of view and in some examples 4.6, 4.7, but it would be too difficult to work with it if we introduce it in any of the formulas (4.17), (4.18) or (4.19). Instead, we introduce a simpler change of variables $s \mapsto t$ given by $1 + s = 1/t$. In this way, we obtain

$$F(x,y) = \frac{1}{2}\int_0^1 \frac{dt}{\sqrt{t}\sqrt{1+xt}\sqrt{1+yt}}, \qquad x, y \in \mathbb{C} \setminus (-\infty, -1], \tag{4.23}$$

$$G_1(x,y) = \frac{3}{2}\int_0^1 \frac{\sqrt{t}dt}{\sqrt{1+xt}\sqrt{1+yt}}, \qquad x, y \in \mathbb{C} \setminus (-\infty, -1] \tag{4.24}$$

and

$$G_2(y,z) = \frac{3}{2}\int_0^1 \frac{\sqrt{t}dt}{\sqrt{1+yt}\sqrt{(1+zt)^3}}, \qquad y, z \in \mathbb{C} \setminus (-\infty, -1]. \tag{4.25}$$

For the sake of convenience and generality, we consider the more general function

$$\mathcal{F}(a,b,c;x,y) := \int_0^1 \frac{t^c dt}{(1+xt)^a(1+yt)^b}, \qquad x, y \in \mathbb{C} \setminus (-\infty, -1] \tag{4.26}$$

with $a, b, c \in \mathbb{C}$ restricted so that $\Re a, \Re b \geq 0$ and $\Re c > -1$.

We are going to obtain a uniform approximation of the function $\mathcal{F}$ being y the uniform variable. For this reason, we identify $g(t) = (1+xt)^{-a}$ and $h(t,y) = t^c(1+yt)^{-b}$. However, it is not clear if $h(t,y)$ can be bounded independetly of y nor in which region we can do it. Moreover, it is also not clear which point we should consider as base point for the Taylor expansion of $g(t)$, as the function $g(t)$ depends on x. In the following two lemmas we give an answer to this two questions.

Lemma 4.8.1. *For any fixed angle $\theta \in [\pi/2, \pi)$, consider the extended sector*

$$\begin{aligned} S(\theta) := &\{y \in \mathbb{C} : |\arg(y)| \leq \theta\} \bigcup \\ &\left(\{y \in \mathbb{C} : |\arg(y)| > \theta\} \bigcap \{y \in \mathbb{C} : |y+1| > \sin\theta\} \bigcap \{y \in \mathbb{C} : |y+1/2| \leq 1/2\}\right), \end{aligned} \tag{4.27}$$

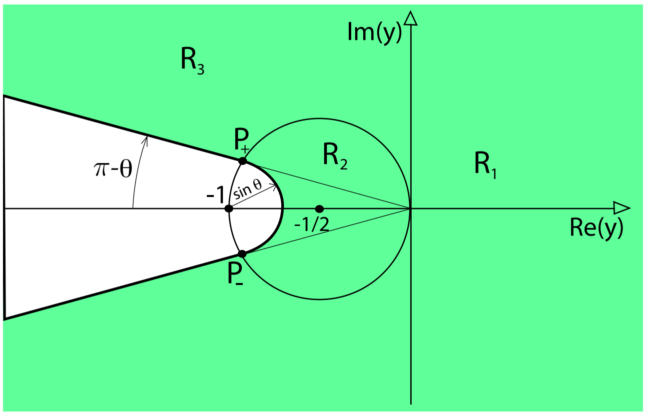


Figure 4.8: The region $S(\theta)$ defined in equation (4.27) is marked in green. Its boundary is the portion of the rays $\arg y = \pm\theta$ exterior to the open disk R_2 (the disk of center $-1/2$ and radius $1/2$) and also the portion of the circle $|y+1| = \sin\theta$ interior to R_2. In the limit case $\theta \to \pi$ the region $S(\theta)$ becomes the cutted complex plane $\mathbb{C}\setminus(-\infty,-1]$.

that is depicted in green in figure 4.8. Then, for any $y \in S(\theta)$, $t \in [0,1]$ and $\Re b \geq 0$ the function $h(t,y) := t^c(1+yt)^{-b}$ can be uniformly bounded in the form

$$|h(t,y)| \leq t^{\Re c} e^{\pi|\Im b|} (\sin\theta)^{-\Re b}. \tag{4.28}$$

Proof. By splitting b into its real and imaginary parts, it is straightforward to check that

$$|h(t,y)| = |t^c|\,|(1+yt)^{-b}| \leq |(1+yt)^{-\Re b}|\, t^{\Re c}\, e^{\pi|\Im b|}. \tag{4.29}$$

To bound the first factor, we divide the region $\mathbb{C}\setminus(-\infty,-1]$ in three different parts named R_1, R_2 and R_3 that are shown in figure 4.8. In each region, we search the absolute maximum of the function $|(1+yt)^{-\Re b}|$ for $t \in [0,1]$ (absolute that depends on the value of $\Re y$). We have:

- For $y \in R_1 := \{y \in \mathbb{C} : \Re y \geq 0\}$, the maximum is attained at $t = 0$ and its value is 1.
- If $y \in R_2 := \{y \in \mathbb{C} : |y + 1/2| < 1/2\} = \{y \in \mathbb{C} : \Re y < -|y|^2\}$, the maximum is attained at $t = 1$ and its value is $|1+y|^{-\Re b}$.
- When $y \in R_3 := \{y \in \mathbb{C}\setminus(-\infty,-1] : -|y|^2 \leq \Re y < 0\}$, the maximum is attained at $t = -\Re y/|y|^2$ and its value is $|\sin(\arg y)|^{-\Re b}$. Note that the region R_3 is the left half complex plane $\Re y < 0$ with both, the straight $(-\infty,-1]$ and the disk R_2, removed.

For any fixed angle $\theta \in [\pi/2,\pi)$, the rays $\arg y = \pm\theta$ cut the boundary of the disk R_2 at the points $P_\pm = (-\cos^2\theta, \pm\sin\theta\,\cos\theta)$. On the one hand, at the portions of the rays $\arg y = \pm\theta$ that are inside the region R_3 (that is a portion of the boundary of $S(\theta)$) we have that

$$|\sin(\arg y)|^{-\Re b} \leq (\sin\theta)^{-\Re b}.$$

On the other hand, at the portion of the circle $|y+1| = \sin\theta$ that is inside the disk R_2, that is, the remaining portion of the boundary of $S(\theta)$, we have that

$$|1+y|^{-\Re b} \leq (\sin\theta)^{-\Re b}.$$

In any case, the last two inequalities together with (4.29) prove the uniform bound (4.28) given in the lemma. □

Remark 4.8.2. *In the limit case $\theta \to \pi$, the region $S(\theta)$ becomes the cutted complex plane $\mathbb{C} \setminus (-\infty, -1]$.*

Lemma 4.8.3. *For any $x \in \mathbb{C} \setminus (-\infty, -1]$, consider a point of the complex plane $w(x)$ given by any of the following three formulas:*

$$w(x) = \frac{1}{2} \times \begin{cases} 1 & \text{if } |\arg(x+1)| < \pi/2, \\ 1 + i\frac{\Re(x+1)-|x+1|}{\Im(x+1)} & \text{if } 0 < |\arg(x+1)| < \pi, \\ 0 & \text{if } |x| < 1. \end{cases} \tag{4.30}$$

Then,

$$|xw(x)| < |1 + xw(x)|, \qquad |x(1 - w(x))| < |1 + xw(x)|. \tag{4.31}$$

Moreover, for $\arg(x+1) = \pi$, the two inequalities $|xw| < |1+xw|$ and $|x(1-w)| < |1+xw|$ can not be simultaneously satisfied for any value of w.

Proof. In order to simplify the notation, we simply write $w(x) = w$. If we divide the second inequality of (4.31) by $|x + 1|$, then (4.31) becomes

$$|xw| < |1 + xw|, \qquad \left|\frac{x}{x+1}(w-1)\right| < \left|1 + \frac{x}{x+1}(w-1)\right|,$$

which mean that the distance from both points, xw and $\frac{x}{x+1}(w-1)$, to the point -1 must be larger than the distance to the point 0. This statement is equivalent to the following two inequalities:

$$\Re(xw) > -\frac{1}{2}, \qquad \Re\left(\frac{x}{x+1}(w-1)\right) > -\frac{1}{2}.$$

We define $\theta := \arg(x+1)$, $r := |x+1|$ and we choose w to be of the form $w = (a+1/2)+ib$ with $a, b \in \mathbb{R}$. Then, the above inequalities read

$$\begin{aligned} \left(\cos\theta - \frac{1}{r}\right) a - b\sin\theta + \frac{1}{2}\cos\theta > 0, \\ (r - \cos\theta)\, a - b\sin\theta + \frac{1}{2}\cos\theta > 0. \end{aligned} \tag{4.32}$$

If we take $\theta = \pi$, the inequalities in (4.32) become

$$\frac{1}{2(r+1)} < a < -\frac{r}{2(r+1)}.$$

Therefore, when $\arg(x + 1) = \pi$, inequalities (4.31) do not hold for any $w \in \mathbb{C}$. On the other side, we have that:

- For $|\theta| < \pi/2$ the two inequalities in (4.32) hold for $a = 0 = b$, that is, $w = 1/2$ (first line in (4.30)).
- For $\theta \neq 0, \pi$ the two inequalities in (4.32) hold for $a = 0$ and $b < \cot(\theta)/2$ if $0 < \theta < \pi$ or $b > \cot(\theta)/2$ if $-\pi < \theta < 0$. In particular, the inequalities hold for the value of w given in the second line of (4.30).

- For $r < 2\cos\theta$ the two inequalities in (4.32) hold for $a = -1/2$ and $b = 0$, that is, for $w = 0$. The condition $r < 2\cos\theta$ is equivalent to $|x| < 1$ (last line in (4.30)).

$\square$

Remark 4.8.4. *Note that, for many values of x, there is an overlaping in the sector that defines $w(x)$ in equation (4.30), that is, there are values of x for which several values of $w(x)$ are possible. This is not important and lemma 4.8.3 states that for any possible choice, inequalities (4.31) hold.*

With the help of the lemmas proved above we are in conditions for obtaining a uniformly convergent expansion of the function $\mathcal{F}$ introduced in (4.26) and defined by the integral representation

$$\mathcal{F}(a,b,c;x,y) := \int_0^1 \frac{t^c dt}{(1+xt)^a(1+yt)^b}, \qquad x,y \in \mathbb{C}\setminus(-\infty,-1] \tag{4.33}$$

with $a,b,c \in \mathbb{C}$ restricted so that $\Re a, \Re b \geq 0$ and $\Re c > -1$.

Following the steps of chapter 3 we identify $g(t) = (1+xt)^{-a}$ and $h(t,y) = t^c(1+yt)^{-b}$. It is clear that $g(t)$ is analytic in $\Omega = \{t \in \mathbb{C} : 1+xt \notin (-\infty,0]\}$. We consider its Taylor series expansion at the point $t = w(x)$, with $w(x)$ given in any of the first two lines of (4.30) in lemma 4.8.3 to find

$$g(t) = \frac{1}{(1+xw)^a}\sum_{k=0}^{n-1}\frac{(a)_k(-x)^k}{k!}\frac{(t-w)^k}{(1+xw)^k} + r_n(t;x,w,a). \tag{4.34}$$

With this election of the base point and with the help of lemma 4.8.3, we have that $\left|\frac{x(w(x)-t)}{1+xw(x)}\right| < 1$ for any $t \in [0,1]$. Thence, expansion (4.34) is convergent and the remainder $r_n(t;x,w,a)$ can be written in the form

$$\begin{aligned} r_n(t;x,w,a) &= \frac{1}{(1+xw)^a}\sum_{k=n}^{\infty}\frac{(a)_k(-x)^k}{k!}\frac{(t-w(x))^k}{(1+xw(x))^k} \\ &= \frac{(a)_n}{(1+xw(x))^a\, n!}\left[\frac{x\,(w(x)-t)}{1+xw(x)}\right]^n {}_2F_1\left(\left.\begin{matrix}1,\ n+a\\ n+1\end{matrix}\,\right|\,\frac{x\,(w(x)-t)}{1+xw(x)}\right). \end{aligned}$$

For $t \in [0,1]$, it holds that $|t-w(x)| \leq |w(x)|$ and then, using the series definition of the Gauss hypergeometric function [104, eq. 15.2.1], we find that

$$|r_n(t;x,w,a)| \leq \frac{|(a)_n|}{|(1+xw(x))^a|\; n!}\left|\frac{xw(x)}{1+xw(x)}\right|^n {}_2F_1\left(\left.\begin{matrix}1,\ n+\Re a\\ n+1\end{matrix}\,\right|\,\left|\frac{xw(x)}{1+xw(x)}\right|\right). \tag{4.35}$$

On the other hand, for any fixed angle $\theta \in [\pi/2,\pi)$ we have that, using lemma 4.8.1, the function $h(t,y)$ can be bounded in the form

$$|h(t,y)| := \left|\frac{t^c}{(1+yt)^b}\right| \leq t^{\Re c} e^{\pi|\Im b|}|\sin\theta|^{-\Re b} := H(t), \tag{4.36}$$

for any $y \in S(\theta)$, being $S(\theta) \subset \mathbb{C}\setminus(-\infty,-1]$ the region defined in lemma 4.8.1 and depicted in figure 4.8. Note that $H(t)$ is an integrable function on $[0,1]$.

We replace $g(t)$ in the integral (4.33) by the right hand side of (4.34) and, interchanging summation and integration, we obtain

$$\mathcal{F}(a,b,c;x,y) = \frac{1}{[1+xw(x)]^a}\sum_{k=0}^{n-1}\frac{(a)_k}{k!}\left(\frac{-x}{1+xw(x)}\right)^k \mathcal{A}_k\left(b,c;y,w(x)\right) + R_n(a,b,c;x,y), \tag{4.37}$$

where the coefficients $\mathcal{A}_k(b,c;y,w)$ are related to the moments of the function $h(t,y)$ in the form

$$\mathcal{A}_k(b,c;y,w) := \int_0^1 (t-w)^k h(t,y)dt = \int_0^1 (t-w)^k t^c(1+yt)^{-b}dt. \tag{4.38}$$

As the moments of h are given by

$$M[h(\cdot,y);n] := \int_0^1 h(t,y)t^n dt = \frac{1}{c+n+1}{}_2F_1\left(\begin{matrix} b, & n+c+1 \\ & n+c+2 \end{matrix}\middle| -y\right),$$

we consider the binomial expansion of the factor $(t-w)^k$ in (4.38) to find

$$\mathcal{A}_k(b,c;y,w) = \sum_{j=0}^{k}\binom{k}{j}\frac{(-w)^{k-j}}{c+j+1}{}_2F_1\left(\begin{matrix} b, & j+c+1 \\ & j+c+2 \end{matrix}\middle| -y\right). \tag{4.39}$$

On the other hand, the remainder term $R_n(a,b,c;x,y)$ in (4.37) is given by

$$R_n(a,b,c;x,y) := \int_0^1 \frac{t^c}{(1+yt)^b} r_n(t;x,w,a)dt.$$

Taking into account the bounds (4.36) and (4.35) and integrating, we find the bound

$$|R_n(a,b,c;x,y)| \leq \frac{e^{\pi|\Im b|}(\sin\theta)^{-\Re b}}{|\Gamma(a)|(\Re c+1)}\frac{\Gamma(\Re a+n)\,|xw(x)|^n}{n!\,|\,(xw(x)+1)^{n+a}\,|}{}_2F_1\left(\begin{matrix} 1, & n+\Re a \\ & n+1 \end{matrix}\middle| \frac{|xw(x)|}{|xw(x)+1|}\right) \tag{4.40}$$

This bound shows that the right hand side of (4.37) is a uniformly convergent expansion of $\mathcal{F}(a,b,c;x,y)$ for $y \in S(\theta)$. The uniform feature follows from the fact that the bound of the remainder is independent of the variable y. To see clearer that the expansion is convergent, we use in (4.40) the definition of the symbol of Pochhammer [4, eq. 5.2.5] and the asymptotic behaviour [4, eq. 5.11.12] for the quotient of two gamma functions, as well as the asymptotic behaviour of the Gauss hypergeometric function [141, eq. 15] to obtain a exponential order of convergence:

$$R_n(a,b,c;x,y) = \mathcal{O}\left(\left[\frac{xw(x)}{xw(x)+1}\right]^n n^{a-1}\right), \qquad \text{as } n\to\infty. \tag{4.41}$$

Remark 4.8.5. *Due to the election of $w(x)$ given by lemma 4.8.3, the branch point $t = -1/x$ of the function $g(t) = (1+xt)^{-a}$ is located outside the disk of convergence of the Taylor expansion of $g(t)$ at $t = w(x)$. Moreover, the integration interval $[0,1]$ is completely contained in that disk of convergence and then, according to case 1 of theorem 3.3.1 the convergence of expansion (4.37) should be of exponential type as shown by (4.41) (recall that, by lemma 4.8.3, $\left|\frac{xw(x)}{xw(x)+1}\right| < 1$.)*

Remark 4.8.6. *We have considered above the point $t = w(x)$ as base point for the Taylor expansion of the function $g(t)$. This election is valid for any value of x. However, if we impose the more demanding restriction $|x| < 1$ (see the third line of (4.30) in lemma 4.8.3) we could consider $t = 0$ as the base point for the Taylor expansion of $g(t)$. Then, following the same steps as above we would obtain a simpler expansion of $\mathcal{F}(a, b, c; x, y)$ with a more accurate error bound (see [21, Theorem 3.2]).*

Recall that the function $\mathcal{F}(a, b, c; x, y)$ generalizes the functions $F(x, y)$, $G_1(x, y)$ and $G_2(y, z)$ given in (4.17), (4.18) and (4.19) respectively and related with the symmetric elliptic integral by means of the connection formulas (4.20), (4.21) and (4.22). We have that $F(x, y) = \frac{1}{2}\mathcal{F}(1/2, 1/2, -1/2; x, y)$, $G_1(x, y) = \frac{3}{2}\mathcal{F}(1/2, 1/2, 1/2; x, y)$ and $G_2(y, z) = \frac{3}{2}\mathcal{F}(1/2, 3/2, 1/2; y, z)$. Hence, a convergent expansion for those functions, that holds uniformly for $y \in S(\theta)$ follows from (4.37).

In general, the coefficients $\mathcal{A}_k(b, c; x, y)$ given in the right hand side of approximation (4.37) are given in (4.39) in terms of Gauss hypergeometric ${}_2F_1$ functions. That is, expansion (4.37) is not given in terms of elementary functions. However, in the particular cases when $\mathcal{F}$ represents any one of the functions F, G_1 or G_2 related with the elliptic symmetric integrals, then the corresponding coefficients $\mathcal{A}_k(b, c; x, y)$ are elementary functions of x and y. More precisely, from (4.39) we have

$$\mathcal{A}_k(b, c; y, w) = \sum_{j=0}^{k} \binom{k}{j} \frac{(-w)^{k-j}}{c+j+1} {}_2F_1\left(\left. \begin{matrix} b, \ j+c+1 \\ j+c+2 \end{matrix} \right| -y \right).$$

- If $b = 1/2$ and $c = -1/2$ (that is, we are considering the coefficients of the function $F(x, y)$) we have

$$\begin{aligned} {}_2F_1\left(\left. \begin{matrix} 1/2, \ k+1/2 \\ k+3/2 \end{matrix} \right| -y \right) &:= \int_0^1 \frac{t^{k-1/2}}{\sqrt{1+yt}} dt \\ &= \frac{2}{\sqrt{\pi}} \frac{(-1)^k}{y^{k+1/2}} \frac{\Gamma(k+1/2)}{k!} \operatorname{arcsinh}(\sqrt{y}) - \sqrt{1+y} \sum_{j=1}^{k} \frac{(-y)^{-j}}{k-j+1} \frac{(1/2-k)_{j-1}}{(-k)_{j-1}}. \end{aligned} \tag{4.42}$$

- If $b = 1/2$ and $c = 1/2$ (we are considering the coefficients of the function $G_1(x, y)$) we have

$$\begin{aligned} {}_2F_1\left(\left. \begin{matrix} 1/2, \ k+3/2 \\ k+5/2 \end{matrix} \right| -y \right) &:= \int_0^1 \frac{t^{k+1/2}}{\sqrt{1+yt}} dt \\ &= \frac{(3/2)_k}{(k+1)!(-y)^{k+1}} \left(\frac{\operatorname{arcsinh}(\sqrt{y})}{\sqrt{y}} - \sqrt{1+y} \sum_{j=0}^{k} \frac{j!(-y^j)}{(3/2)_j} \right). \end{aligned} \tag{4.43}$$

- If $b = 3/2$ and $c = 1/2$ (we are considering the coefficients of the function $G_2(y, z)$) we have

$$\begin{aligned} {}_2F_1\left(\left. \begin{matrix} 3/2, \ k+3/2 \\ k+5/2 \end{matrix} \right| -z \right) &:= \int_0^1 \frac{t^{k+1/2}}{\sqrt{(1+zt)^3}} dt \\ &= \frac{(3/2)_k}{k!(-z)^k} \left[\frac{2\operatorname{arcsinh}(\sqrt{z})}{\sqrt{z^3}} + \frac{1}{z\sqrt{1+z}} \left(-2 + \sum_{j=1}^{k} \frac{(j-1)!(-z)^j}{(3/2)_j} \right) \right]. \end{aligned} \tag{4.44}$$

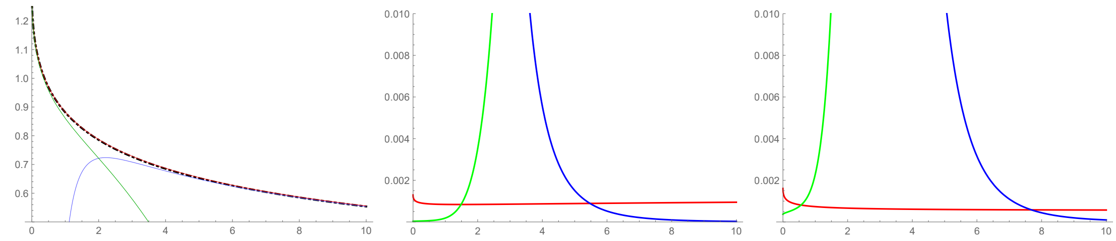


Figure 4.9: The figure on the left shows the function $R_F(1, y, 2)$ (black, dashed) and the approximations provided by the uniform approximation (4.37) using the connection formula (4.20) (red), the series expansion for small values of y [65, Corollary 3.1, eq. 3.1] (green) and the series expansion for large values of y [65, Corollary 3.1, eq. 3.1] (blue) after $n = 2$ terms of the approximations. The other two pictures show the relative error for $x = 1$, $z = 3$ and real $y \in [0, 10]$ after $n = 4$ terms on the approximations of $R_F(x, y, z)$ (middle) and $R_D(x, y, z)$ (right) using the uniform approximation (4.37) and the connection formula (4.20) or (4.21) (red), the series expansion for small values of y [65, Corollary 3.1, eq. 3.1] or [65, Corollary 3.4, eq. 3.14] (green) and the series expansion for large values of y [65, Corollary 3.1, eq. 3.1] or [65, Corollary 3.4, eq. 3.14] (green) (blue). We have taken $w = 1/2$ as base point for the uniform approximations.

n	$y = 0.1$	$y = 0.8$	$y = 2$	$y = 10$	$y = 30$
1	$3.31689 \cdot 10^{-2}$	$3.08361 \cdot 10^{-2}$	$2.95432 \cdot 10^{-2}$	$3.06833 \cdot 10^{-2}$	$3.3182 \cdot 10^{-2}$
10	$1.17583 \cdot 10^{-6}$	$1.01754 \cdot 10^{-6}$	$9.74489 \cdot 10^{-7}$	$1.09487 \cdot 10^{-6}$	$1.29547 \cdot 10^{-6}$
20	$1.25844 \cdot 10^{-10}$	$1.0777 \cdot 10^{-10}$	$1.05111 \cdot 10^{-10}$	$1.21977 \cdot 10^{-10}$	$1.47871 \cdot 10^{-10}$
30	$3.7437 \cdot 10^{-13}$	$4.21052 \cdot 10^{-13}$	$4.18897 \cdot 10^{-11}$	$2.1214 \cdot 10^{-14}$	$2.14431 \cdot 10^{-14}$

Table 4.8: Relative error provided by the right hand side of the uniform approximation (4.37) of the symmetric elliptic integral $R_F(2, y, 5)$ taken $w = 1/2$ as base point, when we truncate the series after n terms, for different values of n and y.

Remark 4.8.7. *To obtain the integral representations (4.17) and (4.18) for the functions $F(x, y)$ and $G_1(x, y)$ we have assumed that $z > 0$. Then, if we apply any of the connection formulas (4.20) or (4.21) to obtain a uniformly convergent expansion of R_F or R_D from (4.37) we are restricted, in principle, to $z > 0$. However, in appendix B it is shown by means of analytical continuation arguments that the integral representations (4.17) and (4.18), and therefore the corresponding approximations, are valid in the bigger sector*

$$\Lambda_1 := \left\{ (x, y, z) \in (\mathbb{C} \setminus (-\infty, 0])^3 : |\arg x - \arg z| < \pi, \ |\arg y - \arg z| < \pi \right\}.$$

The situation is similar for the $G_2(y, z)$ function.

As an illustration of the expansion that we obtain, we have that [22, eq. 4]

$$F(x, y) = \frac{(x + 4y) \arcsin \sqrt{y}}{4y^{3/2}} - \frac{x\sqrt{y+1}}{4y} + \varepsilon(x),$$

with $|\varepsilon(x)| \leq 0.075x^2 \leq 0.075$, valid for $0 \leq x < 1$ and uniformly in y, with $\Re y > 0$.

4.9 The ${}_2F_1$ Gauss hypergeometric function and the incomplete beta function

In this section we find two different expansions, depending on the value of the parameter a, of the ${}_2F_1(a,b,c;z)$ function. Moreover, the incomplete beta function [116, §8.17] is a particular case of the hypergeometric function. In particular, we have [116, eq. 8.17.7] $B_z(a,b) = \frac{z^a}{a}{}_2F_1(1-b,a,a+1;z)$. As a consequence, from the uniformly convergent expansion of the Gauss hypergeometric function that we are going to derive, we can obtain a uniformly convergent expansion for the incomplete beta function.

The expansions for the hypergeometric function are derived in [43] whereas the expansions for the incomplete beta function are given in [44].

4.9.1 A uniformly convergent expansion of the ${}_2F_1(a,b,c;z)$ function for $\Re a \geq 0$.

We consider the Euler integral representation [104, eq. 15.6.1] of the Gauss hypergeometric function

$$ {}_2F_1(a,b,c;z) \equiv {}_2F_1\left(\begin{matrix} a, \ b \\ c \end{matrix}\middle| z\right) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)}\int_0^1 \frac{t^{b-1}(1-t)^{c-b-1}}{(1-zt)^a}dt, \qquad \Re c > \Re b > 0 \tag{4.45} $$

valid for $z \in \mathbb{C} \setminus (1,\infty)$.

We want to obtain a convergent expansion of the Gauss hypergeometric function that holds uniformly for z in a large region of the complex plane. Therefore, we identify $g(t) = t^{b-1}(1-t)^{c-b-1}$ and $h(t,z) = (1-zt)^{-a}$. On the one hand, for general values of $b, c \in \mathbb{C}$ the function $g(t)$ has two branch points located at $t = 0$ and $t = 1$. Hence, we are in case 4 of theorem 3.3.1 and we find $\sigma = b$, $\gamma = c - b$. Then, we consider the standard Taylor expansion of $g(t)$ at the middle point of the integration interval $t = 1/2$. On the other hand, we need to find a bound for $h(t,z)$ uniformly valid in z. For $\Re a \geq 0$ we can bound the function $h(t,y)$ using lemma 4.8.1. We have

$$ |h(t,z)| \leq e^{\pi|\Im a|}\left[\sin\theta\right]^{-\Re a} := H > 0, $$

valid for $t \in [0,1]$ and $-z \in S(\theta)$, with $S(\theta)$ defined in (4.27) and depicted in figure 4.8. Therefore, we find $\alpha = 0$ and $\beta = 0$.

Thus, using theorem 3.3.1 we obtain the convergent expansion [43, eq. 2.2], uniformly for $-z \in S(\theta)$. The remainder of the expansion after n terms satisfies $R_n(z,a,b,c) = \mathcal{O}\left(n^{-b} + n^{-(c-b)}\right)$, as $n \to \infty$ [43, eq. 2.9].

As an illustration of the kind of approximation [43, eq. 2.2] that we find, we have, for example

$$ {}_2F_1\left(\begin{matrix} \frac{1}{2}, \ \frac{3}{2} \\ \frac{5}{2} \end{matrix}\middle| z\right) = \frac{384 - 1120z + 2100z^2 + 525z^3 + \sqrt{1-z}\left(-384 + 928z - 1684z^2 - 1275z^3\right)}{560\sqrt{2}z^4} + \varepsilon(z) $$

valid for $\Re z \leq 0$ with $|\varepsilon(z)| < 0.0784$. For $z = 0$ the right hand side of this formula must be understood in the limit sense. The above formula follows from [43, eq. 2.2] after 3 terms. The bound for the remainder $\varepsilon(z)$ follows from [43, eq. 2.19].

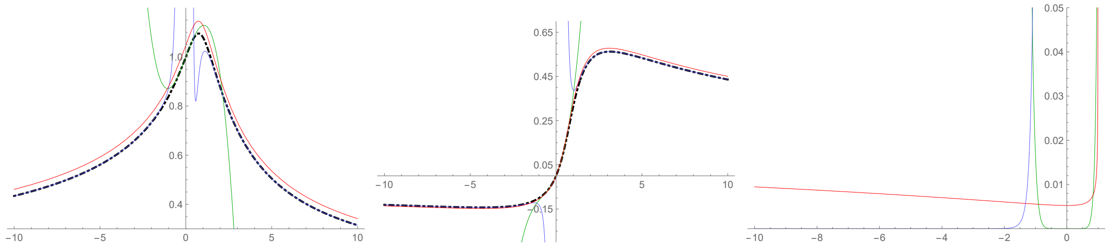


Figure 4.10: Real part (left) and imaginary part (middle) of the approximations of the Gauss hypergeometric ${}_2F_1(a,b,c;z)$ function (black, dashed) provided by the power series definition [104, eq. 15.2.1] (green), the asymptotic expansion [104, eqs. 15.2.1 and 15.8.2] (blue) and the uniform expansion [43, eq. 2.2] (red), for $z\in[-10e^{i\pi/4},10e^{i\pi/4}]$ after $n=4$ terms, for $a=0.5$, $b=1.3$ and $c=2.5$. The figure on the right shows their relative errors for $n=10$, $a=0.8$, $b=1.7$, $c=3.4$ and real $z\in[-10,1]$. The behavior is similar for other values of z and a, b, c.

n	$z=0.75$	$z=-1$	$z=-4$	$z=-15$	$z=-30$
1	$4.02002\cdot10^{-1}$	$3.41553\cdot10^{-1}$	$4.30274\cdot10^{-1}$	$6.50772\cdot10^{-1}$	$8.34703\cdot10^{-1}$
10	$1.06926\cdot10^{-2}$	$8.07005\cdot10^{-3}$	$1.20361\cdot10^{-2}$	$2.49721\cdot10^{-2}$	$3.88179\cdot10^{-2}$
20	$3.90646\cdot10^{-3}$	$2.89177\cdot10^{-3}$	$4.43741\cdot10^{-3}$	$9.86427\cdot10^{-3}$	$1.61565\cdot10^{-2}$
30	$2.11844\cdot10^{-3}$	$1.55559\cdot10^{-3}$	$2.41583\cdot10^{-3}$	$5.54824\cdot10^{-3}$	$9.3415\cdot10^{-3}$

Table 4.9: Relative error provided by the right hand side of the uniform approximation [43, eq. 2.2] of the Gauss hypergeometric function ${}_2F_1(a,b,c;z)$ for $a=1.2$, $b=1.6$ and $c=3.2$, when we truncate the series after n terms, for different values of n and z.

In figure 4.10 and table 4.9 we show the uniform and asymptotic properties of expansion [43, eq. 2.2]. We also compare it with the well-known power series definition [104, eq. 15.2.1] and asymptotic expansion [104, eqs. 15.2.1 and 15.8.2] for large z of the Gauss hypergeometric function.

Moreover, for $\Re b\le 1$ and $-z\in S(\theta)$ we can obtain a uniformly convergent expansion of the incomplete beta function $B_z(a,b)$. In particular, we may derive the expansion [44, eq. 2.2]. The remainder of this expansion after n terms satisfies $R_n(z,a,b)=\mathcal{O}\left(n^{-a}\right)$, as $n\to\infty$ [44, eq. 2.7].

Remark 4.9.1. *For integers values of b and c, the hypergeometric ${}_2F_1$ function is a rational function of a and z. In this case, the uniform expansion [43, eq. 2.2] after n terms is exact if n is large enough ($n\ge c$) as the bound [43, eq. 2.18] for the remainder vanishes.*

4.9.2 A uniformly convergent expansion of the ${}_2F_1(a,b,c;z)$ function for $\Re a\le 0$.

The bound $|(1-zt)^{-a}|\le e^{\pi|\Im a|}\left[\sin\theta\right]^{-\Re a}$ for $t\in[0,1]$ and $-z\in S(\theta)$ that we used in the previous subsection is no longer valid for $\Re a\le 0$. Therefore, we must consider a new approach. We perform the change of variables $t\mapsto 1-t$ in the integral (4.45) to find

$$
{}_2F_1\left(\left.\begin{matrix} a,\ b\\ c\end{matrix}\right| z\right)=\frac{\Gamma(c)(1-z)^{-a}}{\Gamma(b)\Gamma(c-b)}=\int_0^1\frac{t^{c-b-1}(1-t)^{b-1}}{\left[1+\frac{z}{1-z}t\right]^a}dt,\qquad \Re c>\Re b>0,
$$

valid for $z\in\mathbb{C}\setminus(1,\infty)$. Now, we take $g(t)=t^{c-b-1}(1-t)^{b-1}$ and $h(t,z)=\left[1+\frac{z}{1-z}t\right]^{-a}$ and we consider the Taylor expansion of $g(t)$ at $t=1/2$. For general values of $b,c\in\mathbb{C}$

n	$z = 0.75$	$z = -1$	$z = -4$	$z = -15$	$z = -30$
1	$1.06626 \cdot 10^{-1}$	$3.13901 \cdot 10^{-2}$	$2.61334 \cdot 10^{-2}$	$2.55677 \cdot 10^{-2}$	$2.56173 \cdot 10^{-2}$
10	$4.06071 \cdot 10^{-3}$	$4.71742 \cdot 10^{-4}$	$2.14439 \cdot 10^{-4}$	$1.78221 \cdot 10^{-4}$	$1.77492 \cdot 10^{-4}$
20	$1.4392 \cdot 10^{-3}$	$1.4231 \cdot 10^{-4}$	$4.88566 \cdot 10^{-5}$	$3.50847 \cdot 10^{-5}$	$3.45866 \cdot 10^{-5}$
30	$7.59742 \cdot 10^{-4}$	$7.02096 \cdot 10^{-5}$	$2.04139 \cdot 10^{-5}$	$1.29343 \cdot 10^{-5}$	$1.26151 \cdot 10^{-5}$

Table 4.10: Relative error provided by the right hand side of the uniform approximation [44, eq. 3.2] of the incomplete beta function $z^{-a}(1-z)^{1-b}B_z(a,b)$ for $a = 1.7$ and $b = 3.4$, when we truncate the series after n terms, for different values of n and z.

the point $t = 0$ and $t = 1$ are singular points of $g(t)$. Then, we are again in case 4 of theorem 3.3.1 with $\sigma = c - b$ and $\gamma = b$.

We need to bound the function $h(t,z)$ uniformly in a large set of the complex z-plane. To this end, we define, for any $r \in (0,1]$, the region

$$\mathcal{C}_r = \{z \in \mathbb{C} : |z-1| \geq r, \, |\arg(1-z)| < \pi\}.$$

Then, for $z \in \mathcal{C}_r$ and $t \in [0,1]$ we have that [43, §3]

$$|h(t,z)| \leq e^{\pi|\Im a|} r^{\Re a} := H > 0.$$

Hence, we find $\alpha = 0$ and $\beta = 0$. After an application of theorem 3.3.1 we obtain the convergent expansion [43, eq. 3.2] uniformly valid for $z \in \mathcal{C}_r$. If we truncate the expansion after n terms we obtain a remainder $R_n(z,a,b,c)$ that satisfies $R_n(z,a,b,c) = \mathcal{O}\left(n^{-b} + n^{-(c-b)}\right)$, as $n \to \infty$ [43, eq. 3.6].

Besides, from the relation between the Gauss hypergeometric function and the incomplete beta function we may derive a convergent expansion of $B_z(a,b)$ for $\Re b \geq 1$ uniformly valid for $z \in \mathcal{C}_r$ [44, eq. 3.2] whose remainder term of the order n satisfies $R_n(z,a,b) = \mathcal{O}\left(n^{-a}\right)$, as $n \to \infty$ [44, eq. 3.6].

As an illustration of the kind of approximation that we find, we have, for example

$$z^{-5/2}(1-z)^{-1/2}B_z\left(\frac{5}{2},\frac{3}{2}\right) = \\ \frac{397z^3 - 5\left(7\sqrt{\frac{1}{1-z}} + 17\right)z^2 + 24\left(7\sqrt{\frac{1}{1-z}} - 9\right)z + 96\left(\sqrt{\frac{1}{1-z}} - 1\right)}{840\sqrt{2}z^3} + \epsilon(z),$$

valid for $\Re z \leq 0$ with $|\epsilon(z)| < 0.0089$. For $z = 0$ this formula must be understood in the limit sense. This approximation follows from [44, eq. 3.2] by truncating the expansion after 3 terms. The error bound for $\epsilon(z)$ follows from [44, Proposition 3.2]. In figure 4.11 and table 4.10 we show the uniform and convergent features of expansion [44, eq. 3.2]. We also compare it with the well-known power series representation [124] and the asymptotic expansion for large z [116, eq. 8.17.7]+[104, eqs. 15.2.1 and 15.8.2] of the incomplete beta function.

Remark 4.9.2. *For integers values of b and c, the hypergeometric ${}_2F_1$ function is a rational function of a and z. In this case, the uniform expansion [43, eq. 2.2] after n terms is exact if n is large enough ($n \geq c$) as the bound [43, eq. 2.9] for the remainder vanishes.*

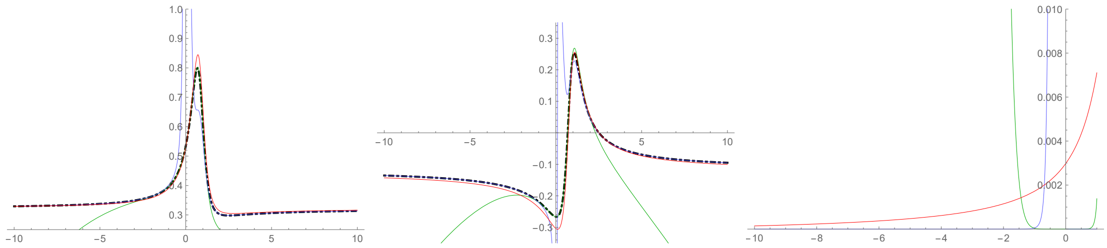


Figure 4.11: Real part (left) and imaginary part (middle) of the approximations of the incomplete beta function $z^{-a}(1-z)^{1-b}B_z(a,b)$ (black, dashed) provided by the power series definition [124] (green), the asymptotic expansion [116, eq. 8.17.7]+[104, eqs. 15.2.1 and 15.8.2] (blue) and the uniform expansion [44, eq. 3.2] (red) for $z \in [-10e^{i\pi/6}, 10e^{i\pi/6}]$ after $n = 4$ terms, for $a = 1.5 + 0.75i$ and $b = 2.25 + 0.25i$. The figure on the right shows their relative errors for $n = 10$, $a = 1.1$, $b = 2.2$ and real $z \in [-10, 1]$. The behavior is similar for other values of z and a, b.

Chapter 5

An Application of the Uniform Expansions: A Series Representation of the Volterra Function

In chapter 3 we have developed a new theory of uniformly convergent expansions of integral transforms of the form $F(z) = \int_0^1 g(t)h(t,z)dt$. In contrast to other expansions that we may find in the literature, like for example power series expansions or asymptotic approximations, the uniform expansions hold in a large (possibly unbounded) region of the complex plane that contains small values of the uniform variable z. That is, the expansion is valid not only in a neighborhood of a selected point, but for large and small values of the selected variable $|z|$. Therefore, we can use these expansions in the numerical evaluation of integral transforms, and in particular, in the evaluation of special functions admitting and integral representation $F(z)$. However, as we have seen in theorem 3.3.1 specially in cases 2-4, the speed of convergence is only of power type if one (or both) of the end points of the integration interval $[0,1]$ are singular points of the function $g(t)$ in the integrand. For this reason, power series and asymptotic expansions are preferable over uniform expansions in the numerical evaluation of integrals when the variable z is, respectively, small or large. In spite of that, as we have seen numerically in chapter 4, there is an intermediate region where, in general, uniform expansions perform better than the power series and the asymptotic expansions.

In chapter 4 we have derived uniform approximation of several special functions and we have seen, with the help of tables and graphics, that they do globally well in the numerical evaluation of integrals. Nevertheless, the main advantage of the uniform expansions is not numerical, but analytic. As uniform expansions hold in a large set of the complex plane, they can replace the function $F(z)$ when it appears in a certain computation, like for example a differential equation or as a factor in the integrand of an integral. In this manner the computation becomes easier to carry out, since the uniform approximation is given in terms of elementary functions.

In this chapter, we illustrate this idea by considering the Volterra function, that is defined by means of the following definite integral [32], [40, Ch. 18, §18.3]

$$\mu(t,\beta,\alpha) := \frac{1}{\Gamma(\beta+1)} \int_0^\infty \frac{t^{u+\alpha} u^\beta}{\Gamma(u+\alpha+1)} du, \tag{5.1}$$

with $\Re\beta > -1$ and $t > 0$. After certain manipulations and using a uniform approximation of a factor in the integrand, we derive a convergent expansion of $\mu(t,\beta,\alpha)$ in terms of incomplete gamma functions. The results of this chapter are based on [70].

5.1 Introduction

The Volterra function $\mu(t,\beta,\alpha)$ is defined by the integral representation (5.1), but there are some particular notations that are usually adopted in some particular cases, namely:

$$\nu(t) := \mu(t,0,0) = \int_0^\infty \frac{t^u du}{\Gamma(u+1)}, \qquad \nu(t,\alpha) := \mu(t,0,\alpha) = \int_0^\infty \frac{t^{u+\alpha} du}{\Gamma(u+\alpha+1)},$$

$$\mu(t,\beta) := \mu(t,\beta,0) = \frac{1}{\Gamma(\beta+1)} \int_0^\infty \frac{t^u u^\beta du}{\Gamma(u+1)}.$$

The function $\mu(t,\beta,\alpha)$ and any of its particular cases, $\nu(t)$, $\nu(t,\alpha)$ or $\mu(t,\beta)$ are analytic functions of the variable t with branch-points at $t=0$ and ∞ and no more singularities [40, Ch. 18, §18.3]; and $\nu(t,\alpha)$ and $\mu(t,\beta,\alpha)$ are entire functions of α. Moreover, the variable β is restricted so that $\Re\beta > -1$ in order for the integral (5.1) to be convergent, but the definition of $\mu(t,\beta,\alpha)$ can be extended to the whole $\beta-$plane by repeated integration by parts [40, Ch. 18, §18.3, eq. 4]

$$\begin{aligned}\mu(t,\beta,\alpha) :&= \frac{1}{\Gamma(\beta+1)} \int_0^\infty \frac{t^{u+\alpha} u^\beta}{\Gamma(u+\alpha+1)} du = \frac{-1}{\Gamma(\beta+2)} \int_0^\infty u^{\beta+1} \frac{d}{dt}\left[\frac{t^{u+\alpha}}{\Gamma(u+\alpha+1)}\right] du \\ &= \ldots = \frac{(-1)^m}{\Gamma(\beta+m+1)} \int_0^\infty u^{\beta+m} \frac{d}{dt}\left[\frac{t^{u+\alpha}}{\Gamma(u+\alpha+m)}\right] du\end{aligned}$$

for $\Re\beta > -m-1$, with $m \in \mathbb{N}\cup\{0\}$. The so extended function $\mu(t,\beta,\alpha)$ is also an entire function of β.

These functions are named after the italian mathematician Vito Volterra who, in 1916, introduced them as solution to certain integral equations with a logarithmic kernel [148]. Since then, these functions have played an important role in differents fields in mathematics. Specially important are their applications in integral equations and operational calculus, as shown by many french mathematicians in the forties of the last century [30, 31, 53, 121, 122, 123]; and due to integral formulas such as [40, Ch. 18, §18.3, eq. 22]

$$\int_0^\infty \exp\left(-\frac{t^2}{4y}\right) \mu(t,\beta,\alpha) dt = 2^{\beta+1} y^{1/2} \pi^{1/2} \mu(y,\beta,\alpha/2) \qquad \Re\alpha > -1,\ \Re y > 0,$$

that relate the Volterra function with a certain integral transform of itself.

For this reason, Volterra functions take a central role in the theory of the Laplace transform as they are the Laplace transform of simple functions. For example [40, Ch. 18, §18.3, eq. 17],

$$\int_0^\infty \frac{t^\beta e^{-st}}{\Gamma(\alpha+t+1)} dt = e^{\alpha s} \mu\left(e^{-s},\beta,\alpha\right), \qquad \Re\beta > -1,$$

and

$$\int_0^\infty e^{-st} \mu\left(t,\beta,\alpha\right) dt = s^{-\alpha-1} \left(\log s\right)^{-\beta-1}, \qquad \Re\alpha > -1,\ \Re s > 1.$$

Besides, the function $\nu(t,\alpha)$ appears in the formula of Paley-Wiener for the inversion of a Laplace transformation [112]. That is, if $f(s)=\int_0^\infty e^{-st}F(t)dt$, then

$$F(t)=\lim_{\lambda\to\infty}\frac{1}{2\pi i}\int_0^\infty f(s)\left[\nu\left(st,-1/2+\lambda i\right)-\nu\left(st,-1/2-\lambda i\right)\right]ds.$$

More recently, Mainardi et al. [49, 85, 86] have studied the Volterra functions in connection with solutions to fractional relaxation/diffusion equations of distributed order; and Apelblat [2] has collected a comprehensive set of information about this function, providing a historical perspective, abundant bibliography, important identities and several integral transforms.

On the other hand, the so called Fransén-Robinson constant, denoted by F, and defined by menas of

$$F:=\int_0^\infty\frac{1}{\Gamma(x)}dx,$$

is a special case of the Volterra function as $F=\mu(1,0,-1)=\mu(1,1,0)$. This constant has interesting applications in statistics [48]: Imagine a certain probability model whose density function decreases faster than e^{-cx}, for any positive constant c. Then, the reciprocal gamma function may be used as density function and the value of the constant F is needed for the sake of normalization.

In spite of the importance that the Volterra function has in many branches of mathematics, most books on special function do not consider it. For this reason, asymptotic expansions have not yet been fully investigated, although asymptotic expansions as $t\to 0$ and $t\to\infty$ may be found, respectively, in [40] and [155]. The results are summarized in [49] and given below:

Define the coefficients

$$D_n^{(\alpha)}:=\frac{(-1)^n}{n!}\mu\left(1,-n-1,\alpha\right)=\frac{1}{n!}\frac{d^n}{dx^n}\left[\frac{1}{\Gamma(\alpha+x+1)}\right]_{x=0}.$$

Then,

$$\mu(t,\beta,\alpha)\sim t^\alpha\sum_{n=0}^\infty(\beta+1)_nD_n^{(\alpha)}\left(\log\frac{1}{t}\right)^{-\beta-1-n},\qquad\text{as } t\to 0,\ \text{with } t\in\mathbb{C}\setminus[1,+\infty). \tag{5.2}$$

On the other hand,

$$\mu(t,\beta,\alpha)\sim E(t,\beta,\alpha)+H(t,\beta,\alpha),\qquad\text{as } |t|\to\infty,\quad |\arg t|<\pi, \tag{5.3}$$

where $H(t,\beta,\alpha)$ is the expansion in the right hand side of (5.2) and

$$E(t,\beta,\alpha):=e^t\sum_{n=0}^\infty\frac{E_n^{(\alpha,\beta)}}{\Gamma(\beta+1-n)}t^{\beta-n},$$

being $E_n^{(\alpha,\beta)}$ the coefficients given by

$$E_n^{(\alpha,\beta)}:=\frac{(-1)^n}{n!}\frac{d^n}{dx^n}\left[\frac{(1-x)^{-\alpha-1}(-x)^{\beta+1}}{\log^{\beta+1}(1-x)}\right]_{x=0}.$$

Nevertheless, convergent expansions are not available in the literature. The aim of this chapter of the thesis is to fill this gap by deriving a family of convergent series of

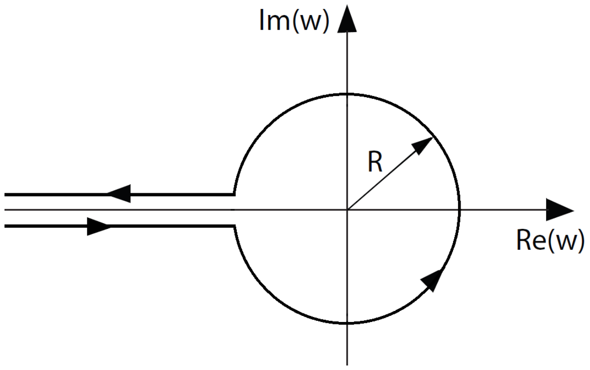


Figure 5.1: A possible path C is obtained by joining a circle of radius $R > |t|$ with the two straight lines $\Im(w) = \pm\varepsilon$, for any $0 < \varepsilon < R$, traversed in the counterclockwise direction.

the Volterra function. To obtain it, in section 5.2 we derive an integral representation of the Volterra function different from the one given as definition in (5.1) and that is more suitable for our analysis. Then, a uniform expansion of a factor of the integrand in the new integral representation is used to obtain a convergent expansion of the Volterra function (section 5.3). The resulting expansion is given in terms of a family of definite integrals that are related to the incomplete gamma function. Finally, some numerical experiments are shown in section 5.4.

5.2 A convenient representation of the Volterra function

The starting point of our analysis is the integral definition of the Volterra function (5.1)

$$\mu(t,\beta,\alpha) := \frac{1}{\Gamma(\beta+1)} \int_0^\infty \frac{t^{u+\alpha} u^\beta}{\Gamma(u+\alpha+1)} du. \tag{5.4}$$

We consider Hankel's loop integral representation of the reciprocal gamma function [4, eq. 5.9.2]

$$\frac{1}{\Gamma(z)} = \frac{1}{2\pi i} \int_C e^w w^{-z} dw, \qquad z \in \mathbb{C},$$

where the integration contour C is the closed loop $(-\infty, 0+)$ that starts at $w = -\infty$ with $\arg w = -\pi$, surrounds the point $w = 0$ counterclockwise and comes back to $w = -\infty$ but with $\arg w = \pi$ (see figure 5.1). Replacing the factor $1/\Gamma(u+\alpha+1)$ in (5.4) by its corresponding Hankel's loop integral and invoking to Fubini's theorem, we find

$$\mu(t,\beta,\alpha) = \frac{t^\alpha}{\Gamma(\beta+1)} \int_C e^w w^{-\alpha-1} \left(\int_0^\infty t^u u^\beta w^{-u} du \right) dw, \qquad \left| \frac{t}{w} \right| < 1,$$

where the restriction $|t| < |w|$ is necessary for convergence reasons. The inner integral above can be computed as follows:

$$\int_0^\infty t^u u^\beta w^{-u} du = \int_0^\infty \exp\left[-u \log(w/t)\right] u^\beta du = \frac{\Gamma(\beta+1)}{\log^{\beta+1}(w/t)},$$

where the last equality follows from formula [4, eq. 5.9.1] for the Γ function. Therefore,

$$\mu(t,\beta,\alpha) = \frac{t^\alpha}{2\pi i} \int_C \frac{e^w w^{-\alpha-1}}{\log^{\beta+1}(w/t)}.$$

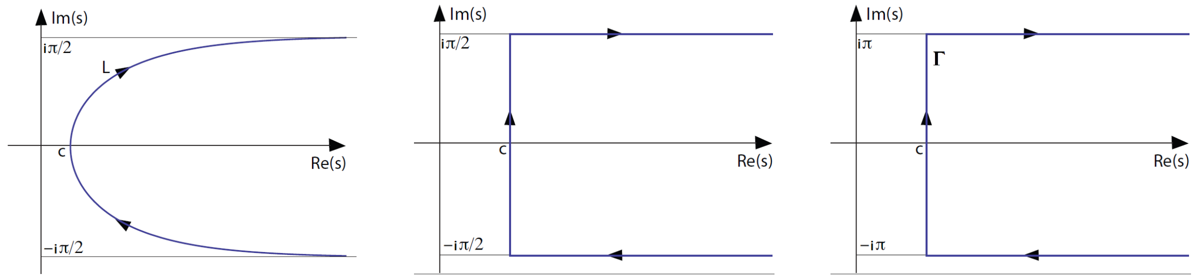


Figure 5.2: The integration contour L (left) in the integral (5.5) crosses the real line at a certain $c > 0$. It can be deformed to the rectangle-like contour (middle) consisting of a vertical line at $\Re s = c$ and two straight lines at height $\Im s = \pm i\pi/2$ that go up to infinity parallel to the real axis. This contour can be further deformed to a similar rectangle Γ, but with height $\Im s = \pm\pi$ (right). All three paths are traversed in the clockwise direction.

The contour C is, for example, the one depicted in figure 5.1 and made up of two straight lines and a circle centered at $w = 0$ with radius $R > |t|$. If $\Re\alpha \geq -1$ we can deform the contour C to the vertical line $C' := \{R + iu : -\infty < u < +\infty\}$, with $R > |t|$. Then

$$\mu(t, \beta, \alpha) = \frac{e^R t^\alpha}{2\pi} \int_{-\infty}^{\infty} \frac{e^{iu}(R+iu)^{-\alpha-1}}{\log^{\beta+1}\left[(R+iu)/t\right]} du, \qquad R > |t|.$$

We perform a further change of variables $u \mapsto s$ defined by $R + iu = te^s$ or, equivalently, $s = \log\left(\frac{R+iu}{t}\right)$ with $|\Im s| < \pi$. Then, the Volterra function is given by

$$\mu(t, \beta, \alpha) = \frac{1}{2\pi i} \int_L e^{-\alpha s} s^{-\beta-1} e^{te^s} ds, \tag{5.5}$$

where the contour $L := \{s = \log\left(\frac{R+iu}{t}\right) : -\infty < u < +\infty,\ |\Im s| \leq \pi/2\}$ is shown in figure 5.2 (left). It cuts the real axis at a certain point $c := \log(R/t) > 0$.

We note that the right hand side of (5.5) is an analytic function of β and therefore it is an explicit expression for the analytic continuation of $\mu(t, \beta, \alpha)$ defined in (5.4) from the half-plane $\Re\beta > -1$ to the entire complex $\beta-$plane.

The curved path L in (5.5) shown in figure 5.2 (left) can be deformed to the rectangle-like contour shown in figure 5.2 (middle) and given by $\{s = u - i\pi/2 : c < u < +\infty\} \cup \{s = c + iu : -\pi/2 < u < \pi/2\} \cup \{s = u + i\pi/2 : c < u < +\infty\}$. We can further deform this path to a similar rectangle Γ whose height is not located at $\Im w = \pm\pi/2$, but at $\Im w = \pm\pi$ (figure 5.2 (right)). This new path is given by $\Gamma := \{s = u - i\pi : c < u < +\infty\} \cup \{s = c + iu : -\pi < u < \pi\} \cup \{s = u + i\pi : c < u < +\infty\}$ where, again, $c := \log(R/t) > 0$ is any postive constant. As a result, the Volterra function can be written as the sum of the integrals

$$\mu(t, \beta, \alpha) = \mu_0(t, \beta, \alpha, c) + \mu_\infty(t, \beta, \alpha, c), \qquad \text{for any } c > 0, \tag{5.6}$$

where

$$\mu_0(t, \beta, \alpha, c) := \frac{e^{-\alpha c}}{2\pi} \int_{-\pi}^{\pi} \frac{e^{-i\alpha u} e^{te^c e^{iu}}}{(c+iu)^{\beta+1}} du, \tag{5.7}$$

and

$$\mu_\infty(t, \beta, \alpha, c) := \frac{e^{-i\alpha\pi}}{2\pi i} \int_c^\infty \frac{e^{-\alpha u} e^{-te^u}}{(u+i\pi)^{\beta+1}} du - \frac{e^{i\alpha\pi}}{2\pi i} \int_c^\infty \frac{e^{-\alpha u} e^{-te^u}}{(u-i\pi)^{\beta+1}} du. \tag{5.8}$$

5.3 Series representations of the Volterra function

In this section we analyze both integrals $\mu_0(t,\beta,\alpha,c)$ and $\mu_\infty(t,\beta,\alpha,c)$ given in (5.6), (5.7), (5.8) and valid for $\Re\alpha \geq -1$, $\Re\beta > -1$, $t>0$ and arbitrary $c>0$. We derive a series representation for each one of them to obtain the main result of this chapter: a family of convergent expansions of the Volterra function in terms of incomplete gamma functions and elementary functions. However, before we obtain it, we need some previous results.

We define the following family of integrals:

$$\phi(a,b,c) := \int_{-\pi}^{\pi} \frac{e^{-a(c+iu)}}{(c+iu)^{b+1}} du, \qquad a,b,c\in\mathbb{C}, \tag{5.9}$$

with the only restriction $\Re b<0$ if c simultaneously satisfies $\Re c=0$ and $\Im c\in[-\pi,\pi]$.

Lemma 5.3.1. *The function $\phi(a,b,c)$ defined in (5.9) admits, for $\Re c>0$ and $\Re b>-1$, the following integral representation:*

$$\phi(a,b,c) = \frac{2e^{-ca}}{\Gamma(b+1)} \int_0^\infty \frac{x^b e^{-cx}\sin\left[\pi(a+x)\right]}{a+x} dx. \tag{5.10}$$

Proof. We consider the well-known integral representation of the Γ function [4, eq. 5.9.1] given by

$$\frac{\Gamma(\nu)}{z^\nu} = \int_0^\infty e^{-zx} x^{\nu-1} dx, \qquad \Re\nu>0,\ \Re z>0. \tag{5.11}$$

Thus, we can write

$$\frac{1}{(c+iu)^{b+1}} = \frac{1}{\Gamma(b+1)} \int_0^\infty x^b e^{-(c+iu)x} dx, \qquad \Re b>-1,\ \Re c>0,$$

which, replaced into (5.9) and using Fubini's theorem to interchange the order of integration yields

$$\begin{aligned} \phi(a,b,c) &= \frac{1}{\Gamma(b+1)} \int_0^\infty \left(\int_{-\pi}^{\pi} e^{-a(c+iu)} x^b e^{-(c+iu)x} du \right) dx \\ &= \frac{e^{-ca}}{\Gamma(b+1)} \int_0^\infty x^b e^{-cx} \left(\int_{-\pi}^{\pi} e^{-i(a+x)u} du \right) dx. \end{aligned}$$

The inner integral can be straightforwardly computed and equals $\frac{2\sin[\pi(a+x)]}{a+x}$, which gives (5.10) and proves the lemma. $\square$

In the following lemma, we compute the functions $\phi(a,b,c)$ in terms of incomplete gamma functions. In particular, we have:

Lemma 5.3.2. *Let $a,b\in\mathbb{C}$ and $c\in\mathbb{R}$. Then, the following formulas for the function $\phi(a,b,c)$ defined in (5.9) hold true:*

1. *If $a\neq 0$ and $b\notin(\mathbb{N}\cup\{0\})$,*

$$\phi(a,b,c) = i\,\Gamma(-b)\left[\frac{\gamma^*\left(-b,a(c-i\pi)\right)}{(c-i\pi)^b} - \frac{\gamma^*\left(-b,a(c+i\pi)\right)}{(c+i\pi)^b}\right], \tag{5.12}$$

where $\gamma^*(\alpha,z)$ is the regularized incomplete gamma function, defined as $\gamma^*(\alpha,z) = \frac{z^{-\alpha}}{\Gamma(\alpha)}\gamma(\alpha,z)$, [116, eq. 8.2.6].

2. *If $a \neq 0$ and $b \in (\mathbb{N} \cup \{0\})$,*

$$\begin{aligned}\phi(a,b,c) = ia^b \Bigg[& \Gamma(-b, a(c+i\pi)) + \frac{(-1)^b \left[\log\left(a(c+i\pi)\right) - \log(c+i\pi)\right]}{b!} - \\ & -\Gamma(-b, a(c-i\pi)) - \frac{(-1)^b \left[\log\left(a(c-i\pi)\right) - \log(c-i\pi)\right]}{b!} \Bigg].\end{aligned} \tag{5.13}$$

3. *If $a = 0$ and $b \neq 0$,*

$$\phi(0,b,c) = \frac{2}{b} \frac{\sin\left[b \arctan(\pi/c)\right]}{\left(\sqrt{c^2+\pi^2}\right)^b}. \tag{5.14}$$

4. *If $a = 0$ and $b = 0$,*

$$\phi(0,0,c) = 2\arctan\left(\frac{\pi}{c}\right). \tag{5.15}$$

Proof. Formulas (5.14) and (5.15) follow by directly computing the integral (5.9) (with $a = 0$). For example, if $b \neq 0$,

$$\phi(0,b,c) = \int_{-\pi}^{\pi} \frac{du}{(c+iu)^{b+1}} = \frac{i}{b}\left[(c+i\pi)^{-b} - (c-i\pi)^{-b}\right] = \frac{i}{b}\left[\frac{(c-i\pi)^b - (c+i\pi)^b}{(c^2+\pi^2)^b}\right].$$

As

$$(c+i\pi)^b = \left(\sqrt{c^2+\pi^2}\right)^b e^{ib\arctan(\pi/c)} \quad \text{and} \quad (c-i\pi)^b = \left(\sqrt{c^2+\pi^2}\right)^b e^{-ib\arctan(\pi/c)},$$

it follows that

$$(c-i\pi)^b - (c+i\pi)^b = -2i\left(\sqrt{c^2+\pi^2}\right)^b \sin\left[b\arctan(\pi/c)\right]$$

and we find (5.14). In a similar way, or taking the limit value $b \to 0$ in (5.14), formula (5.15) can be found.

In order to prove the remaining formulas (5.12) and (5.13), we consider the series expansion of the exponential function in the integral (5.9) to find

$$\phi(a,b,c) = \int_{-\pi}^{\pi} \frac{e^{-a(c+iu)}}{(c+iu)^{b+1}} du = \int_{-\pi}^{\pi} \sum_{n=0}^{\infty} \frac{(-a)^n}{n!}(c+iu)^{n-b-1} du. \tag{5.16}$$

We invoke to the dominated convergence theorem to interchange summation and integration. Then, we have to distinguish two cases depending on whether $b = 0, 1, 2, \ldots$, or not.

- If $b \notin (\mathbb{N} \cup \{0\})$, we get

$$\begin{aligned}\phi(a,b,c) &= \sum_{n=0}^{\infty} \frac{(-a)^n}{n!} \frac{i\left[(c-i\pi)^{n-b} - (c+i\pi)^{n-b}\right]}{n-b} \\ &= \frac{1}{i(c+i\pi)^b}\sum_{n=0}^{\infty} \frac{\left[-a(c+i\pi)\right]^n}{n!(n-b)} - \frac{1}{i(c-i\pi)^b}\sum_{n=0}^{\infty} \frac{\left[-a(c-i\pi)\right]^n}{n!(n-b)}.\end{aligned}$$

  From the series representation [116, eq. 8.7.1] of the regularized incomplete gamma function $\gamma^*(\alpha, z)$ we have that

$$\phi(a,b,c) = \frac{\Gamma(-b)}{i(c+i\pi)^b}\gamma^*(-b, a(c+i\pi)) - \frac{\Gamma(-b)}{i(c-i\pi)^b}\gamma^*(-b, a(c-i\pi))$$

  and formula (5.12) follows.

- If $b \in (\mathbb{N} \cup \{0\})$ we could consider the limit case $b \to n_0$, being n_0 a postive integer, in equation (5.12). Instead, we derive formula (5.13) directly by writing (5.16) in the form

$$\begin{aligned}\phi(a,b,c) &= \sum_{n=0,\, n\neq b}^{\infty} \frac{(-a)^n}{n!} \int_{-\pi}^{\pi} (c+iu)^{n-b-1} du + \frac{(-a)^b}{b!} \int_{-\pi}^{\pi} (c+iu)^{-1} du \\ &= i \sum_{n=0,\, n\neq b}^{\infty} \frac{(-a)^n}{n!} \frac{(c-i\pi)^{n-b}}{n-b} - i \sum_{n=0,\, n\neq b}^{\infty} \frac{(-a)^n}{n!} \frac{(c+i\pi)^{n-b}}{n-b} + \\ &+ i \frac{(-a)^b}{b!} \left[\log(c-i\pi) - \log(c+i\pi)\right].\end{aligned}$$

Using the series representation [116, eq. 8.4.12] of the incomplete gamma function evaluated at a negative integer, that is, $\Gamma(-n,z)$ with $n = 0,1,2,\ldots$, and after some simplifications, we get formula (5.13).

□

In order to find a convergent expansion of the function $\mu_\infty(t,\beta,\alpha,c)$ defined in (5.8) and given by

$$\mu_\infty(t,\beta,\alpha,c) := \frac{e^{-i\alpha\pi}}{2\pi i} \int_c^\infty \frac{e^{-\alpha u} e^{-te^u}}{(u+i\pi)^{\beta+1}} du - \frac{e^{i\alpha\pi}}{2\pi i} \int_c^\infty \frac{e^{-\alpha u} e^{-te^u}}{(u-i\pi)^{\beta+1}} du$$

we perform first some manipulations to obtain a more suitable integral representation. To this end, we use again the integral (5.11) for the gamma function to write

$$\frac{1}{(u \pm i\pi)^{\beta+1}} = \frac{1}{\Gamma(\beta+1)} \int_0^\infty e^{-(u\pm i\pi)v} v^\beta dv.$$

Then,

$$\begin{aligned}\mu_\infty(t,\beta,\alpha,c) &= \frac{e^{-i\alpha\pi}}{2\pi i \Gamma(\beta+1)} \int_c^\infty e^{-\alpha u} e^{-te^u} \left(\int_0^\infty e^{-(u+i\pi)v} v^\beta dv \right) du - \\ &- \frac{e^{i\alpha\pi}}{2\pi i \Gamma(\beta+1)} \int_c^\infty e^{-\alpha u} e^{-te^u} \left(\int_0^\infty e^{-(u-i\pi)v} v^\beta dv \right) du.\end{aligned}$$

We interchange the order of integration by applying Fubini's theorem to find

$$\begin{aligned}\mu_\infty(t,\beta,\alpha,c) &= \frac{1}{2\pi i \Gamma(\beta+1)} \left[\int_0^\infty v^\beta \left(\int_c^\infty e^{-(u+i\pi)(v+\alpha)} e^{-te^u} du \right) dv - \right. \\ &\qquad \left. - \int_0^\infty v^\beta \left(\int_c^\infty e^{-(u-i\pi)(v+\alpha)} e^{-te^u} du \right) dv \right] \\ &= \frac{-1}{\pi \Gamma(\beta+1)} \int_0^\infty v^\beta \sin\left[(v+\alpha)\pi\right] \left(\int_c^\infty e^{-u(v+\alpha)} e^{-te^u} du \right) dv \\ &= \frac{-1}{\pi \Gamma(\beta+1)} \int_0^\infty v^\beta \sin\left[(v+\alpha)\pi\right] t^{v+\alpha} \Gamma(-v-\alpha, te^c)\, dv,\end{aligned} \tag{5.17}$$

where, in the last equality, we have used that the last integral above with respect to u is in fact an incomplete gamma function, as can be easily seen by performing the change of variables $u \mapsto w$ given by $w = te^u$ and using the integral definition of the incomplete gamma function [116, eq. 8.2.1].

It might seem that representation (5.17) of the function $\mu_\infty(t,\beta,\alpha,c)$ is less convenient than formula (5.8) to derive a convergent expansion of the function $\mu_\infty(t,\beta,\alpha,c)$. However, as we will see below, they key point is to use the uniformly convergent expansion of the incomplete gamma function $\Gamma(a,z)$ valid for all $\Re a < 0$ given in section 4.4. That expansion is uniformly valid in the whole interval of integration and then, if we replace the gamma function $\Gamma(-v-\alpha, te^c)$ in the integrand of (5.17) by its uniformly convergent expansion and we interchange summation and integration, we hope to obtain an expansion of the funcion $\mu_\infty(t,\beta,\alpha,c)$ that is convergent.

In the following theorem we show that the above idea is not only formal, but rigorous, by finding accurate bounds for the remainder of the expansion.

Theorem 5.3.3. *Let $\alpha,\beta \in \mathbb{C}$ with $\Re\alpha > 0$ and $\Re\beta > -1$, and $t,c \in \mathbb{R}$ with $t > 0$ and $c > 0$. Then, for any $n \in \mathbb{N}$, the function $\mu_\infty(t,\beta,\alpha,c)$ defined in (5.8) admits the representation*

$$\mu_\infty(t,\beta,\alpha,c) = \frac{e^{-te^c}}{2\pi}\sum_{k=0}^{n-1} L_k(te^c)\sum_{j=0}^{k}\binom{k}{j}e^{c(j+1)}\phi\left(\alpha+j+1,\beta,c\right)+R_n^\infty(t,c,\beta,\alpha), \quad (5.18)$$

where the functions $\phi(a,b,c)$ have been defined in (5.9) and computed in terms of incomplete gamma functions in lemma 5.3.2, and $L_n(x)$ are the Laguerre polynomials.

The remainder $R_n^\infty(t,c,\beta,\alpha)$ is bounded in the form

$$|R_n^\infty(t,c,\beta,\alpha)| \le \frac{e^{-te^c/2-c\Re\alpha+\pi|\Im\alpha|}\Gamma\left(\Re\beta+1\right)\Gamma\left(\Re\alpha\right)}{\pi|\Gamma(\beta+1)|c^{\Re\beta+1}}\frac{1}{n^{\Re\alpha}}. \quad (5.19)$$

Moreover, if $\Re\beta \ge -1/2$ we have the simpler bound

$$|R_n^\infty(t,c,\beta,\alpha)| \le \frac{e^{-te^c/2-c\Re\alpha+\pi|\Im\alpha|}\Gamma\left(\Re\alpha\right)}{\pi\sqrt{sech(\pi\Im\beta)}c^{\Re\beta+1}}\frac{1}{n^{\Re\alpha}}. \quad (5.20)$$

In any case, expansion (5.18) is convergent with a convergence rate of power type. That is

$$R_n^\infty(t,c,\beta,\alpha) = \mathcal{O}\left(n^{-\Re\alpha}\right), \qquad \text{as } n \to \infty.$$

Proof. We consider the integral representation of the function $\mu_\infty(t,c,\beta,\alpha)$ given in the last line of (5.17) and we replace the function $\Gamma(-v-\alpha, te^c)$ by its convergent expansion (4.9) given in corollary 4.4.1 of section 4.4 and uniformly valid for v in the whole integration interval $(0,\infty)$. We get

$$\begin{aligned}
\mu_\infty(t,\beta,\alpha,c) &= \frac{-1}{\pi\Gamma(\beta+1)}\int_0^\infty v^\beta\sin\left[(v+\alpha)\pi\right]t^{v+\alpha}\Gamma(-v-\alpha,te^c)\,dv \\
&= \frac{-1}{\pi\Gamma(\beta+1)}\int_0^\infty v^\beta\sin\left[(v+\alpha)\pi\right]t^{v+\alpha}\Bigg[e^{-te^c}(te^c)^{-v-\alpha}\sum_{k=0}^{n-1}L_k(te^c)B\left(1+v+\alpha,k+1\right)+ \\
&\qquad + R_n(-v-\alpha,te^c)\Bigg]\,dv \\
&= \frac{-e^{-te^c}e^{-\alpha c}}{\pi\Gamma(\beta+1)}\sum_{k=0}^{n-1}L_k(te^c)\int_0^\infty v^\beta e^{-vc}\sin\left[(v+\alpha)\pi\right]B\left(1+v+\alpha,k+1\right)dv + R_n^\infty(t,c,\beta,\alpha),
\end{aligned} \quad (5.21)$$

where the remainder $R_n^\infty(t,c,\beta,\alpha)$ is defined by

$$R_n^\infty(t,c,\beta,\alpha) := \frac{-1}{\pi\Gamma(\beta+1)}\int_0^\infty v^\beta \sin\left[(v+\alpha)\pi\right] t^{v+\alpha} R_n(-v-\alpha, te^c)dv, \tag{5.22}$$

with $R_n(a,x)$ bounded in the form given in corollary 4.4.1. Before bounding the remainder, we compute the approximants. First, we write, using the integral representation of the beta function,

$$\begin{aligned} B(1+v+\alpha,k+1) &= \int_0^1 u^{v+\alpha}(1-u)^k du = \sum_{j=0}^k \binom{k}{j}(-1)^j\int_0^1 u^{v+\alpha+j}du \\ &= \sum_{j=0}^k \binom{k}{j}\frac{(-1)^j}{v+\alpha+j+1}. \end{aligned} \tag{5.23}$$

Next, from (5.21) we have

$$\begin{aligned} &\mu_\infty(t,\beta,\alpha,c) - R_n^\infty(t,c,\beta,\alpha) = \\ &= \frac{e^{-te^c}e^{-\alpha c}}{\pi\Gamma(\beta+1)}\sum_{k=0}^{n-1} L_k(te^c)\sum_{j=0}^k\binom{k}{j}(-1)^{j+1}\int_0^\infty \frac{v^\beta e^{-vc}\sin\left[(v+\alpha)\pi\right]}{v+\alpha+j+1}dv \\ &= \frac{e^{-te^c}}{2\pi}\sum_{k=0}^{n-1} L_k(te^c)\sum_{j=0}^k\binom{k}{j}e^{c(j+1)}\frac{2e^{-c(\alpha+j+1)}}{\Gamma(\beta+1)}\int_0^\infty \frac{v^\beta e^{-vc}\sin\left[(v+\alpha+j+1)\pi\right]}{v+\alpha+j+1}dv \\ &= \frac{e^{-te^c}}{2\pi}\sum_{k=0}^{n-1} L_k(te^c)\sum_{j=0}^k\binom{k}{j}e^{c(j+1)}\,\phi(\alpha+j+1,\beta,c), \end{aligned}$$

where the last equality follows from lemma 5.3.1. In this manner we get the right hand side of expansion (5.18).

It remains to show that the remainder $R_n^\infty(t,\beta,\alpha,c)$ can be bounded as given in the theorem. Using the first inequality of (4.10) in corollary 4.4.1 we have

$$|t^{v+\alpha}R_n(-v-\alpha,te^c)| \le e^{-te^c}e^{-c\Re\alpha}e^{-cv}B\left(v+\Re\alpha,n+1\right).$$

Now, an easy computation shows that the above beta function is a decreasing function of v. Thus, for all $v\in[0,\infty)$, $B\left(v+\Re\alpha,n+1\right)\le B\left(\Re\alpha,n+1\right)$, and using the inequality [4, eq. 5.6.8] as we did in the last line of the proof of lemma 4.4.1 we find

$$|t^{v+\alpha}R_n(-v-\alpha,te^c)| \le e^{-te^c}e^{-c\Re\alpha}e^{-cv}B\left(\Re\alpha,n+1\right) \le e^{-te^c}e^{-c\Re\alpha}e^{-cv}\Gamma(\Re\alpha)n^{-\Re\alpha},$$

Then, from (5.22), the previous bound for $t^{v+\alpha}R_n(-v-\alpha,te^c)$ and the bound $|\sin\left[\pi(v+\alpha)\right]|\le e^{\pi|\Im\alpha|}$, we find

$$|R_n^\infty(t,c,\beta,\alpha)| \le \frac{e^{-te^c}e^{-c\Re\alpha}\Gamma(\Re\alpha)n^{-\Re\alpha}}{\pi|\Gamma(\beta+1)|}\int_0^\infty e^{-cv}v^{\Re\beta}dv.$$

The last integral is a gamma function and we find (5.19). Moreover, taking into account the bound $\Gamma(\Re\beta+1)/|\Gamma(\beta+1)|\le\sqrt{\cosh\left(\pi\Im\beta\right)}$, valid for $\Re\beta\ge -1/2$, that can be found in [116, eq. 5.6.7], we obtain the bound (5.20). This last bound shows that expansion (5.18) is uniformly convergent in the parameter β in the semi-plane $\Re\beta\ge -1/2$.

□

Remark 5.3.4. *In principle, for a fixed $c>0$, the bounds (5.19) or (5.20) in theorem 5.3.3 show that the expansion (5.18) is convergent, but the order of convergence is only of power type: $\mathcal{O}\left(n^{-\Re\alpha}\right)$, as $n\to\infty$. However, if we let c depende on n in the form $te^c=\lambda n$, for any fixed positive paramater λ, that is, if we take $c=\log(\lambda n/t)$, then (5.20) becomes*

$$|R_n^\infty(t,c,\beta,\alpha)|\leq\frac{e^{\pi|\Im\alpha|-\lambda n/2}t^{\Re\alpha}}{\pi\lambda^{\Re\alpha}\left[\log\left(\frac{\lambda n}{t}\right)\right]^{\Re\beta+1}\sqrt{sech(\pi\Im\beta)}}\frac{\Gamma(\Re\alpha)}{n^{2\Re\alpha}},$$

which is valid for $\Re\beta\geq\frac{-1}{2}$ and whenever $n>t/\lambda$, for any fixed $\lambda>0$. Then,

$$R_n^\infty\left(t,\log\left(\frac{\lambda n}{t}\right)\right)=\mathcal{O}\left(\frac{e^{-\lambda n/2}}{(\log n)^{\Re\beta+1}n^{2\Re\alpha}}\right),\qquad as\ n\to\infty.$$

On the other hand, we obtain a convergent series representation of the function $\mu_0(t,\beta,\alpha,c)$ defined in (5.7):

Theorem 5.3.5. *Let $\alpha,\beta\in\mathbb{C}$ with $\Re\alpha>0$ and $\Re\beta>-1$, and $t,c\in\mathbb{R}$ with $t>0$ and $c>0$. Then, for any positive integer n, the function $\mu_0(t,\beta,\alpha,c)$ defined in (5.7) admits the representation*

$$\mu_0(t,\beta,\alpha,c)=\frac{1}{2\pi}\sum_{k=0}^{n-1}\frac{t^k}{k!}\phi(\alpha-k,\beta,c)+R_n^0(t,c,\beta,\alpha),\tag{5.24}$$

where the functions $\phi(a,b,c)$ are defined in (5.9) and computed in lemma 5.3.2 in terms of incomplete gamma functions. The remainder $R_n^0(t,c,\beta,\alpha)$ can be bounded in the form

$$|R_n^0(t,c,\beta,\alpha)|\leq\frac{e^{-c\Re\alpha+\pi(|\Im\alpha|+|\Im\beta|/2)+te^c}}{c^{\Re\beta+1}}\frac{(te^c)^n}{n!}.\tag{5.25}$$

Therefore, $R_n^0(t,c,\beta,\alpha)=\mathcal{O}\left((te^{c+1})^n n^{-1/2-n}\right)$ as $n\to\infty$ and expansion (5.24) is convergent.

Proof. We recall the definition of $\mu_0(t,\beta,\alpha,c)$, given in (5.7) by

$$\mu_0(t,\beta,\alpha,c):=\frac{e^{-\alpha c}}{2\pi}\int_{-\pi}^{\pi}\frac{e^{-i\alpha u}e^{te^ce^{iu}}}{(c+iu)^{\beta+1}}du.$$

The exponential is an entire function whose well-known series expansion is

$$e^{te^{c+iu}}=\sum_{k=0}^{n-1}\frac{t^k}{k!}e^{k(c+iu)}+r_n^0(t,c,u),\qquad\text{for any } n\in\mathbb{N},$$

with

$$|r_n^0(t,c,u)|\leq\sum_{k=n}^{\infty}\frac{(te^c)^k}{k!}=e^{te^c}\frac{\gamma(n,te^c)}{\Gamma(n)}.\tag{5.26}$$

We can replace the exponential function in the integral definition of $\mu_0(t,\beta,\alpha,c)$ to get

$$\begin{aligned}\mu_0(t,\beta,\alpha,c)&=\frac{e^{-\alpha c}}{2\pi}\int_{-\pi}^{\pi}\frac{e^{-i\alpha u}}{(c+iu)^{\beta+1}}\left(\sum_{k=0}^{n-1}\frac{t^ke^{k(c+iu)}}{k!}+r_n^0(t,c,u)\right)du\\&=\frac{1}{2\pi}\sum_{k=0}^{n-1}\frac{t^k}{k!}\int_{-\pi}^{\pi}\frac{e^{-(\alpha-k)(c+iu)}}{(c+iu)^{\beta+1}}du+R_n^0(t,c,\beta,\alpha).\end{aligned}$$

Comparing with the definition of the $\phi(a,b,c)$, (5.9), we find the right hand side of (5.24) with

$$R_n^0(t,c,\beta,\alpha) := \frac{e^{-\alpha c}}{2\pi}\int_{-\pi}^{\pi}\frac{e^{-i\alpha u}}{(c+iu)^{\beta+1}}\, r_n^0(t,c,u)du.$$

In order to find the bound (5.25) we use the bound obtained in (5.26) for the remainder $r_n^0(t,c,u)$. We immediately find

$$|R_n^0(t,c,\beta,\alpha)| \le \frac{e^{-c\Re\alpha}}{2\pi}\frac{e^{te^c}\gamma(n,te^c)}{\Gamma(n)}e^{\pi|\Im\alpha|}\int_{-\pi}^{\pi}\frac{du}{|(c+iu)^{\beta+1}|}.$$

Now, we have $|(c+iu)^{\beta+1}| \ge c^{\Re\beta+1}e^{|\Im\beta|\pi/2}$ and $\gamma(n,te^c)\le (te^c)^n/n$ which follows from the integral definition of the incomplete gamma function [116, eq. 8.2.1]. Accordingly we find the bound (5.25).

Finally, the asymptotic behavior of the remainder $R_n^0(t,c,\beta,\alpha)$ as $n\to\infty$ is found after an application of the Stirling formula for the factorial $n!$. Then, expression (5.24) is a convergent expansion of the function $\mu_0(t,\beta,\alpha,c)$, for any $c>0$. $\square$

Remark 5.3.6. *The asymptotic behavior of the remainder $R_n^0(t,c,\beta,\alpha)$ shows that expansion (5.24) is convergent. However, as we did in remark 5.3.4, we do not take a fixed parameter $c>0$, but we let c depend on n in the form $te^c=\lambda n$, for a fixed parameter $\lambda>0$. Then, using the Stirling approximation of the gamma function and the asymptotic behavior of the incomplete gamma function $\gamma(n,\lambda n)$ given in [116, eq. 8.11.6] (valid for $0<\lambda<1$) in (5.25), we find*

$$\left|R_n^0\left(t,\log(\lambda n/t),\beta,\alpha\right)\right| \le \frac{t^{\Re\alpha}e^{\pi(|\Im\alpha|+|\Im\beta|/2)}\sqrt{2\pi}}{(1-\lambda)\lambda^{\Re\alpha}\left[\log\left(\frac{\lambda n}{t}\right)\right]^{\Re\beta+1/2}}\frac{(e\lambda)^n}{n^{\Re\alpha+1/2}},$$

which shows that

$$R_n^0\left(t,\log(\lambda n/t),\beta,\alpha\right) = \mathcal{O}\left(\frac{(e\lambda)^n}{(\log n)^{\Re\beta+1}n^{\Re\alpha+1/2}}\right),\qquad \text{as } n\to\infty.$$

In particular, for any number $0<\lambda<e^{-1}$, we can take $c=\log(\lambda n/t)$ and the rate of convergence of expansion (5.24) is exponential.

Finally, a convergent series representation of the Volterra function $\mu(t,\beta,\alpha)$ follows from equations (5.6), (5.7), (5.8) and the expansions derived in theorems 5.3.3 and 5.3.5. We have

Corollary 5.3.7. *Let $\alpha,\beta\in\mathbb{C}$ with $\Re\alpha>0$, $\Re\beta>-1$ and $t>0$. Then, for any positive integer n and any arbitrary positive number $c>0$, the Volterra function $\mu(t,\beta,\alpha)$ satisfies*

$$\begin{aligned}\mu(t,\beta,\alpha) = \frac{1}{2\pi}\sum_{k=0}^{n-1}\Bigg[&e^{-te^c}L_k(te^c)\sum_{j=0}^{k}\binom{k}{j}e^{c(j+1)}\phi(\alpha+j+1,\beta,c)\\ &+\frac{t^k}{k!}\phi(\alpha-k,\beta,c)\Bigg]+\mathcal{R}_n(t,c,\beta,\alpha),\end{aligned}\tag{5.27}$$

where $L_k(x)$ are the Laguerre polynomials and $\phi(a,b,c)$ are the functions defined in (5.9) that may be written in terms of incomplete gamma functions (see lemma 5.3.2). The remainder $\mathcal{R}_n(t,c,\beta,\alpha)$ is bounded in the form

$$|\mathcal{R}_n(t,c,\beta,\alpha)| \leq \frac{e^{-c\Re\alpha+\pi|\Im\alpha|}}{c^{\Re\beta+1}}\left[\frac{e^{-te^c/2}M(\beta)\Gamma(\Re\alpha)}{\pi\, n^{\Re\alpha}}+\frac{e^{\pi|\Im\beta|/2+te^c}\gamma(n,te^c)}{\Gamma(n)}\right], \tag{5.28}$$

with $M(\beta) := \frac{\Gamma(\Re\beta+1)}{|\Gamma(\beta+1)|}$. For $\Re\beta \geq -1/2$, $M(\beta)$ may be replaced by $[sech(\pi\Im\beta)]^{-1/2}$ and therefore, expansion (5.27) is uniformly convergent in $\Re\beta$ in the semi-plane $\Re\beta \geq -1/2$. When $n \to \infty$

$$\mathcal{R}_n(t,c,\beta,\alpha) = \mathcal{O}\left(\frac{1}{n^{\Re\alpha}}+\frac{(te^{c+1})^n}{n^{n+1/2}}\right),$$

and then, the expansion (5.27) is convergent, with a convergence order of power type.

Moreover, let $0<\lambda<e^{-1}$ (ideally $\lambda=\lambda_0 := 0.31436990296762807\ldots$, the unique solution of the transcendental equation $\log(e\lambda)+\frac{\lambda}{2}=0$) and take $c=\log\left(\frac{\lambda n}{t}\right)$. Then, the rate of convergence of expansion (5.27) is of exponential type:

$$\mathcal{R}_n\left(t,\log\left(\frac{\lambda n}{t}\right),\beta,\alpha\right) = \mathcal{O}\left(\frac{n^{-\Re\alpha}}{(\log n)^{\Re\beta+1}}\left[\frac{e^{\frac{-\lambda n}{2}}}{n^{\Re\alpha}}+\frac{(e\lambda)^n}{n^{1/2}}\right]\right) \qquad \text{as } n\to\infty. \tag{5.29}$$

Proof. The results follows from the relation (5.6) and theorems 5.3.3 and 5.3.5, as well as the remarks 5.3.4 and 5.3.6. The value λ_0 corresponds to the only value of λ such that $0<\lambda<e^{-1}$ that makes the terms inside the brackets in (5.29) to behave similarly. That is, if we impose the condition $e^{-\frac{\lambda n}{2}} = (e\lambda)^n$, we get the equation $\log(e\lambda)+\frac{\lambda}{2}=0$, whose only solution is $\lambda=\lambda_0 := 0.31436990296762807\ldots$. That is, for $\lambda=\lambda_0$, the expansions (5.18) of $\mu_\infty(t,\beta,\alpha,c)$ and (5.24) of $\mu_0(t,\beta,\alpha,c)$ have a similar convergence rate, which is exponential. □

Remark 5.3.8. *If we set $c=\log\left(\frac{\lambda n}{t}\right)$, with $0<\lambda<e^{-1}$, we may need to take a large number of terms to obtain a valid result when using expansion (5.27), as $c>0$ if and only if $n \geq \frac{t}{\lambda}$. In any case, as the expansion is convergent, by taking enough terms we can find as many precision as required.*

Remark 5.3.9. *The expansion (5.27) derived for the Volterra function in corollary 5.3.7 is not given in terms of elementary functions [132] but rather in terms of incomplete gamma functions. Nevertheless, the expansion is useful as incomplete gamma functions are simpler than the Volterra function. On the one hand, the Volterra function depends on three parameters α, β and t whereas the incomplete gamma function only depends on two. On the other hand, incomplete gamma functions have been investigated in detail and there is an extensive literature about this funcion. Apart from its integral definition, a lot of information is known about incomplete gamma functions [116]: more integral representations, special values, convergent series expansions, asymptotic expansions in several domains of the variables (standard and uniform), recurrence relations and derivatives, relations to other special functions, continued fractions, inequalities, zeros, integrals, sums... On the other side, as it has been pointed out in the introduction, the Volterra function has barely been investigated, probably due to the gamma function in the denominator of its integral definition (5.1).*

Remark 5.3.10. *The Volterra function $\mu(t,\beta,\alpha)$ is real for real variables. On the other hand, in approximation (5.27) we can find the function $\phi(a,b,c)$ for certain parameters a,b and c, which are computed in lemma 5.3.2 in terms of incomplete gamma functions with complex arguments. This should not lead to confusion, the functions $\phi(a,b,c)$ are also real for real values of its parameters, as it is shown from its integral representation given in lemma 5.3.1.*

Remark 5.3.11. *The expansion derived in corollary 5.3.7 is only valid for $\Re\alpha > 0$, whereas the Volterra function is defined for any $\alpha \in \mathbb{C}$. For negative values of $\Re\alpha$ we may use the recurrence relation [2]*

$$t\mu(t,\beta,\alpha) = (\beta+1)\mu(t,\beta+1,\alpha+1) + (\alpha+1)\mu(t,\beta,\alpha+1),$$

which can be derived from the integral definition (5.1) of the Volterra function and the recurrence relation $\Gamma(z+1) = z\Gamma(z)$ for the gamma function.

5.4 Numerical experiments

Tables 5.1-5.6 are some numerical experiments that show the accuracy of the expansion (5.27) given in corollary 5.3.7 for certain values of the parameters α and β, different values of the variable t and different choices of c. There, the relative error [143, eq. 3.1.9] $E_n(t,\alpha,\beta) := \left|\frac{\mu(t,\alpha,\beta)-\mu_n(t,\alpha,\beta)}{\mu(t,\alpha,\beta)}\right|$ provided by formula (5.27) is shown. In this formula, $\mu_n(t,\alpha,\beta)$ denotes the first n terms of the sum on right hand side of (5.27) (without the remainder term $\mathcal{R}_n(t,c,\beta,\alpha)$). In tables 5.1, 5.3 and 5.5 we have taken a fixed value for the parameter c whereas in tables 5.2, 5.4 and 5.6 we have taken $c = \log(\lambda_0 n/t)$, with $\lambda_0 := 0.31436990296762807\ldots$. In particular, in table 5.2 there are some empty entries as the condition $c > 0$ is not satisfied for some values of t and n. The tables shows that expansion (5.27) is convergent, although it may be necessary to take a large number of terms to obtain a numerically good result for the approximation of $\mu(t,\beta,\alpha)$.

Furthermore, in figure 5.3 we compare the approximation supplied by expansion (5.27) with the asymptotic approximations supplied by (5.2) and (5.3) for small and large t, respectively. The comparision is a little bit tricky, and the following observation is to be made: formulas (5.2) and (5.3) are asymptotic and then, the best approximation is achieved with a certain optimal number of terms that depends on t. Besides, the expansions are divergent and then, considering a large number of terms makes the approximation worse. On the other hand, (5.27) is convergent and then, the more terms we take the better. Therefore, in figure 5.3 the number of terms used in (5.2) and (5.3) is such that these divergent expansions are numerically useful in the largest possible range of the numerical experiment shown in figure 5.4. We have concluded that the best number of terms for expansion (5.2) is 1 and for the expansion (5.3) is 4. That is, if we take more than 1 or 4 terms, respectively, the global approximation supplied by formulas (5.2) and (5.3) is worse. On the other hand, for expansion (5.27) we have taken a moderate number of terms ($n = 12$) in figure 5.3.

From a numerical point of view, expansion (5.2) is more appropriate for small values of t and expansion (5.3) is better for large values of t, whereas expansion (5.27) is more suitable for intermediate values of t. In any case, expansion (5.27) produces a globally more satisfactory approximation.

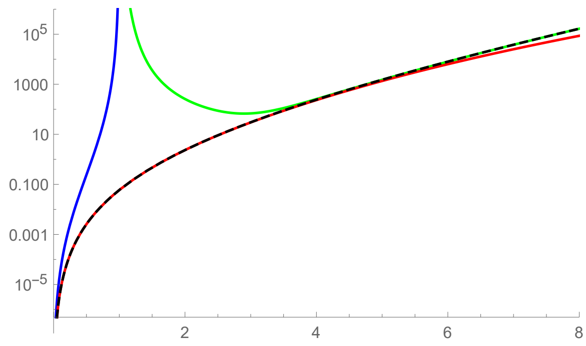


Figure 5.3: Plot, in logarithmic scale, of the Volterra function $\mu(t, 3.1, 2.5)$ (black, dashed), the first term of the asymptotic expansion for small t (5.2) (blue), the 4 first terms of the asymptotic expansion for large t (5.3) (green) and the 12 first terms of the expansion given by (5.27) with $c = 0.1$ (red). The asymptotic expansion (5.2) is to be preferred for small t and the asymptotic expansion (5.3) is to be preferred for large t, whereas the convergent expansion (5.27) given in corollay 5.3.7 performs better for intermediate values and produces an approximation that is in general more satisfactory.

t	$n = 10$	$n = 15$	$n = 20$	$n = 25$
0.3	$8.98 \cdot 10^{-2}$	$3.16 \cdot 10^{-2}$	$2.00 \cdot 10^{-2}$	$6.89 \cdot 10^{-3}$
0.7	$4.80 \cdot 10^{-3}$	$7.43 \cdot 10^{-4}$	$4.96 \cdot 10^{-4}$	$3.51 \cdot 10^{-5}$
1	$4.01 \cdot 10^{-4}$	$1.91 \cdot 10^{-4}$	$5.57 \cdot 10^{-5}$	$3.44 \cdot 10^{-5}$
1.3	$8.82 \cdot 10^{-4}$	$2.86 \cdot 10^{-5}$	$1.50 \cdot 10^{-5}$	$6.46 \cdot 10^{-6}$
1.7	$3.98 \cdot 10^{-3}$	$9.12 \cdot 10^{-6}$	$3.53 \cdot 10^{-6}$	$4.54 \cdot 10^{-7}$
2	$9.53 \cdot 10^{-3}$	$4.33 \cdot 10^{-6}$	$9.41 \cdot 10^{-7}$	$5.50 \cdot 10^{-7}$

Table 5.1: Relative error provided by formula (5.27) in corollary 5.3.7 for $\alpha = 2.4 + 1.1i$, $\beta = 0.8 + 2.9i$, $c = 0.5$ and several values of t.

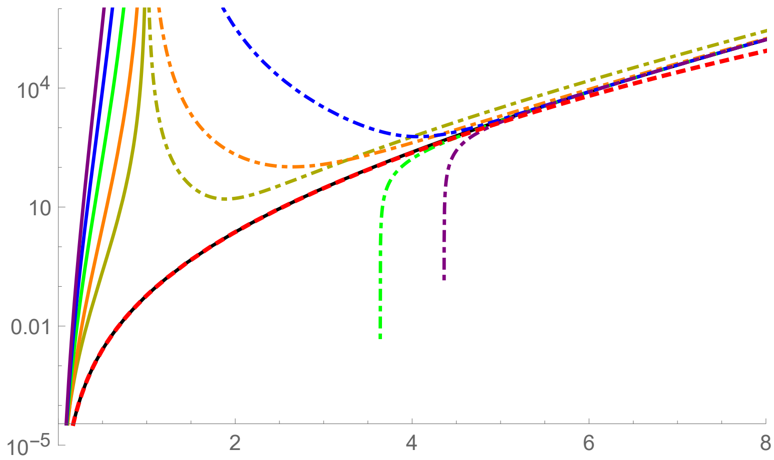


Figure 5.4: Plot, in logarithmic scale, of the Volterra function $\mu(t, 3.1, 2.5)$ (black), the asymptotic approximations for small t (5.2) (continuous lines), the asymptotic expansions for large t (5.3) (dot-dashed lines) and the 12 first terms of the expansion (5.27) given in corollary 5.3.7 with $c = 0.1$ (red, dashed). In the asymptotic formulas (5.2), (5.3) for small and large t we have taken several terms of the approximation: $n = 1$ (dark yellow), $n = 3$ (orange), $n = 6$ (green), $n = 9$ (blue) and $n = 12$ (purple).

t	$n=30$	$n=40$	$n=50$	$n=60$
4	$8.06\cdot10^{-3}$	$8.52\cdot10^{-9}$	$8.00\cdot10^{-11}$	$7.84\cdot10^{-11}$
6	$8.46\cdot10^{-2}$	$4.36\cdot10^{-6}$	$7.96\cdot10^{-12}$	$3.47\cdot10^{-12}$
8	$4.80\cdot10^{-1}$	$2.11\cdot10^{-4}$	$1.49\cdot10^{-9}$	$5.50\cdot10^{-12}$
10	–	$2.88\cdot10^{-3}$	$1.57\cdot10^{-7}$	$4.18\cdot10^{-11}$
12	–	$2.09\cdot10^{+9}$	$5.37\cdot10^{-6}$	$1.18\cdot10^{-9}$
15	–	–	$2.67\cdot10^{+7}$	$2.42\cdot10^{-6}$

Table 5.2: Relative error provided by formula (5.27) in corollary 5.3.7 for $\alpha=14$, $\beta=17$, $c=\log(\lambda_0 n/t)$ and several values of t.

β	$n=10$	$n=15$	$n=20$	$n=25$
$(3+2i)\cdot10^{-8}$	$5.82\cdot10^{-3}$	$2.90\cdot10^{-5}$	$5.15\cdot10^{-8}$	$3.27\cdot10^{-8}$
$0.5+1.2i$	$1.18\cdot10^{-2}$	$5.77\cdot10^{-5}$	$8.00\cdot10^{-8}$	$3.13\cdot10^{-8}$
$4.6+3.7i$	$6.17\cdot10^{-3}$	$2.72\cdot10^{-5}$	$3.07\cdot10^{-8}$	$1.43\cdot10^{-9}$
$8.1+6.9i$	$4.46\cdot10^{-1}$	$9.81\cdot10^{-5}$	$8.33\cdot10^{-8}$	$4.29\cdot10^{-10}$

Table 5.3: Relative error provided by formula (5.27) in corollary 5.3.7 for $\alpha=2.3+0.7i$, $t=1.2$, $c=1.4$ and several values of β.

β	$n=30$	$n=40$	$n=50$	$n=60$
5	$1.34\cdot10^{-4}$	$5.79\cdot10^{-12}$	$9.99\cdot10^{-13}$	$7.22\cdot10^{-13}$
15	$1.43\cdot10^{-1}$	$2.69\cdot10^{-6}$	$3.54\cdot10^{-11}$	$3.19\cdot10^{-11}$
30	1.41	$2.19\cdot10^{-2}$	$3.15\cdot10^{-7}$	$5.84\cdot10^{-10}$
50	$8.00\cdot10^{+9}$	4.27	$7.15\cdot10^{-3}$	$4.20\cdot10^{-6}$

Table 5.4: Relative error provided by formula (5.27) in corollary 5.3.7 for $\alpha=18$, $t=5$, $c=\log(\lambda_0 n/t)$ and several values of β.

α	$n=10$	$n=15$	$n=20$	$n=25$
$(6+7i)\cdot10^{-5}$	$1.31\cdot10^{-4}$	$2.60\cdot10^{-7}$	$1.12\cdot10^{-7}$	$4.50\cdot10^{-9}$
$1.4+0.2i$	$4.70\cdot10^{-4}$	$2.98\cdot10^{-7}$	$9.36\cdot10^{-9}$	$1.00\cdot10^{-9}$
$2+3.2i$	$7.28\cdot10^{-1}$	$1.22\cdot10^{-3}$	$5.33\cdot10^{-6}$	$9.08\cdot10^{-7}$
$3.6+4.1i$	$2.28\cdot10^{+1}$	$3.64\cdot10^{-2}$	$1.93\cdot10^{-5}$	$9.02\cdot10^{-7}$

Table 5.5: Relative error provided by formula (5.27) in corollary 5.3.7 for $\beta=4.9+2.7i$, $t=0.8$, $c=1.6$ and several values of α.

α	$n=30$	$n=40$	$n=50$	$n=60$
10	$1.61\cdot10^{-2}$	$1.43\cdot10^{-8}$	$3.44\cdot10^{-11}$	$1.51\cdot10^{-11}$
20	3.48	$9.93\cdot10^{-4}$	$2.26\cdot10^{-9}$	$6.90\cdot10^{-10}$
30	$1.61\cdot10^{+1}$	3.39	$7.23\cdot10^{-5}$	$8.40\cdot10^{-8}$
40	$7.76\cdot10^{+7}$	$2.03\cdot10^{+2}$	2.69	$3.91\cdot10^{-2}$

Table 5.6: Relative error provided by formula (5.27) in corollary 5.3.7 for $\beta=19.7+11.4i$, $t=4$, $c=\log(\lambda_0 n/t)$ and several values of α.

Chapter 6

A Dual Vision of Uniform Expansions: New Expansions of the Generalized Hypergeometric ${}_{p+1}F_p$ Function

In chapter 3 of this thesis we have developed a theory of uniformly convergent expansions for integral transforms of the form

$$F(z) = \int_0^1 h(t,z)g(t)dt \tag{6.1}$$

that hold in a large, unbounded region $\mathcal{D}$ of the complex plane containing a selected point, say $z = 0$. The basic idea to obtain this expansion was simple: to consider a multi-point Taylor expansion of the function $g(t)$ in m selected points $0 \leq t_1 < t_2 < \ldots < t_m \leq 1$. That expansion is uniformly and absolutely convergent in a certain lemniscate D_r. It avoids all singular points of the function $g(t)$ and, if the base point are cleverly chosen, the integration interval $(0, 1)$ is contained in the lemniscate D_r. Moreover, as the function $g(t)$ is independent of the variable z, so is its multi-point Taylor remainder $r_n(t)$. Therefore, when we replace $g(t)$ by its multi-point Taylor series and interchange summation and integration, we obtain a convergent expansion for the integral $F(z)$. If, in addition, the function $h(t,z)$ can be bounded independently of z in a certain region $\mathcal{D}$, the resulting remainder $R_n(z) = \int_0^1 h(t,z)r_n(t)dt$ can also be bounded independently of z and, as a result, the expansion for the integral transform $F(z)$ is valid in a large region $\mathcal{D}$ of the complex plane.

On the other hand, if we consider an expansion for the factor $h(t,z)$ we usually obtain an expansion for the integral transform $F(z)$ that is valid in a small region. Typically, if $h(t,z) = h(tz)$, the classical power series expansion of $F(z)$ is found by considering the Taylor series expansion of $h(tz)$ at $t = 0$. In general, the function $h(t,z)$ has some finite singularities and then the radius of its Taylor series is small. As a consequence, the power series of many integral transforms (6.1) converge in a small disk. On the other hand, asymptotic expansions are found by considering an expansion of $g(t)$ at the asymptotically relevant point of $h(t,z)$. In general, these expansions are a powerful numerical tool and valid for large values of the variable, but they are divergent. Therefore, it is necessary to study the optimal truncation term of the expansion and also to find accurate bounds for the remainder. In this chapter we investigate a new approach that may be viewed as the complementary technique considered in chapter 3: to consider the multi-point Taylor expansion of $h(t, z)$ at certain appropriately selected base points in order to better avoid the singularities of $h(t, z)$, obtaining larger domains of convergence for the series expansion of $F(z)$.

In [79] the authors have found three different expansions of the confluent hypergeometric function $M(a, b, z)$ by considering the $1-$, $2-$ and $3-$point Taylor expansion of the factor e^{zt} of the integrand in the integral representation [103, eq. 13.4.1] of $M(a, b, z)$. They show that the new expansions are convergent for all values of a, b and $z \in \mathbb{C}$ with $b \neq 0, -1, -2, \ldots$, and that they are given in terms of elementary functions of a, b and z. Furthermore, the resulting expansions are shown to perform numerically better than the series definition [103, eq. 13.2.2] of $M(a, b, z)$ for a wide range of the parameters a, b and the variable z where the use of the series definition was recommended in the literature.

Similarly, in [83] the multi-point Taylor expansion of the factor $(1 - zt)^{-a}$ in the integral representation [104, eq. 15.6.1] has been used to obtain new expansions of the Gauss hypergeometric ${}_2F_1(a, b, c; z)$ function in terms of rational functions of z. The authors show that the region of validity of the expansion is large, may be unbounded, depending on the choice of the base points for the Taylor expansion. Moreover, the expansions converge fast near the points $z = e^{\pm i\pi/3}$. Those points were excluded from the domain of convergence of the Taylor series [104, eq. 15.2.1], even after some transformations of variable [104, §15.8] (see [52, §2.3] or [17] for more details). On the other hand, in [76] new series expansions of the generalized hypergeometric function ${}_3F_2$ have been obtained in a similar manner. The expansions are valid in large (sometimes unbounded) regions of the complex plane. But, moreover, it has been shown that those expansions are, in general, faster and more precise than other series representation of the ${}_3F_2$ function, like for example its power series definition [3, eq. 16.2.1] or Bühring's analytic continuation formula [17].

The approach taken in [76, 79, 83] is the following, that we illustrate with the Gauss hypergeometric ${}_2F_1$ function: Consider its integral representation

$${}_2F_1\left(\begin{matrix} a, & b \\ & c \end{matrix} \middle| \, z \right) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 t^{b-1}(1-t)^{c-b-1}(1-zt)^{-a} dt,$$

valid for $\Re c > \Re b > 0$ and $|\arg(1-z)| < \pi$. The standard Taylor series expansion of the ${}_2F_1$ function can be found by considering the Taylor series expansion at $t = 0$ of $f(t) := (1 - zt)^{-a}$ and interchanging summation and integration. The validity of that expansion (the disk $|z| < 1$) is determined by the following two requirements:

(i) The interval of integration $(0, 1)$ must be completely contained in the domain of convergence of the series expansion of $f(t)$, which is a disk of center $t = 0$ and radius $r \geq 1$, say D_r.

(ii) The branch point $t = 1/z$ of $f(t)$ must be located outside the domain D_r. In other words, z must be located in a region $S_r =$ *the inverse to the exterior of* D_r: $S_r = (D_r^{EXT})^{-1} = \{z \in \mathbb{C} : |z| < r^{-1}\}$.

Therefore, the smaller D_r is, the bigger S_r would be. As D_r must verify (i), the smallest possible value of r is $1+\varepsilon$, with $\varepsilon > 0$ small, and then $S_r = \{z \in \mathbb{C} : |z| < 1\}$. But, using multi-point Taylor expansions, the region D_r becomes thinner and one expects that the corresponding multi-point region S_r becomes bigger. Following this idea, in [76, 79, 83] multi-point Taylor expansions of the function $f(t)$ are considered with requirements (i) and (ii) adapted to the lemniscate of convergence of the multi-point Taylor expansion. This procedure is used to obtain new series expansions of the ${}_2F_1$, ${}_3F_2$ and M hypergeometric functions that hold in large regions of the complex plane. Those regions are unbounded and contain small values of the variable.

We can generalize this idea to any given integral transform $F(z)$ of the form (6.1) by considering the multi-point Taylor series expansion of the function $h(t, z)$ at certain, cleverly selected base points $0 \le t_1 < t_2 < \ldots < t_m \le 1$. That expansion is uniformly and absolutely convergent in a certain lemniscate O_m. Then, we choose the m points satisfying the following two requirements:

(I) The integration interval $(0, 1)$ in (6.1) is contained in O_m.

(II) The variable z runs in a certain region S such that the singularities of $h(t, z)$ as a function of t, say $t_0(z)$ (in general, they depend on z), are not contained in O_m.

Depending on the location of the singularities of $h(t, z)$ the region S may be very large, possibly unbounded and containing small values of the variable z. This method can be viewed as a dual version of the uniform expansions introduced in chapter 3.

We illustrate this idea with the generalized hypergeometric ${}_{p+1}F_p(a, \vec{b}, \vec{c}; z)$ function. In particular, we generalize the results given in [76, 83] for the ${}_2F_1$ and ${}_3F_2$ functions to an arbitrary number of parameters $2p+1$. Moreover, we not only consider the Euler form of the integral representation of the generalized hypergeometric function, but a more general integral (6.4) given below and derived in appendix C.

The results of this chapter have been published in [72].

6.1 The generalized hypergeometric ${}_{p+1}F_p$ function

First, we introduce the generalized hypergeometric ${}_{p+1}F_p(a, \vec{b}, \vec{c}; z)$ function. We need some vectorial notation: for $p, n \in \mathbb{N}$ and $b_k \in \mathbb{C}$, $k = 1, 2, \ldots, p$ we use the notation $\vec{b} := (b_1, b_2, \ldots, b_p)$ and $(\vec{b})_n := (b_1)_n (b_2)_n \cdots (b_p)_n$. The hypergeometric function ${}_{p+1}F_p(a, \vec{b}, \vec{c}; z)$ is defined by means of the power series expansion [3, eq. 16.2.1]

$$ {}_{p+1}F_p(a, \vec{b}, \vec{c}; z) \equiv {}_{p+1}F_p\left(\left. \begin{matrix} a, \ \vec{b} \\ \vec{c} \end{matrix} \right| z \right) := \sum_{n=0}^{\infty} \frac{(a)_n (\vec{b})_n}{(\vec{c})_n n!} z^n, \tag{6.2} $$

that is absolutely convergent inside the unit disk $|z| < 1$ for any complex value of the parameters a, b_k and c_k, $k = 1, 2, \ldots, p$, with $1 - c_k \notin \mathbb{N}$. Then, for numerical computations, the right hand side of (6.2) may be used only in the disk $|z| \le \rho < 1$, with ρ depending on numerical requirements such as precision and efficiency.

At $z = 1$ the hypergeometric functions have a branch point and they are defined outside the unit disk by means of analytic continuation. In particular, we have the following connection formula [3, eqs. 16.8.7 and 16.8.8], valid when no two a_k differ by an integer and $|\arg(-z)| \le \pi$,

$$ \begin{aligned} {}_{p+1}F_p\left(\left. \begin{matrix} a_1, \ a_2, \ \ldots, \ a_{p+1} \\ b_1, \ b_2, \ \ldots, \ b_p \end{matrix} \right| z \right) &= \sum_{k=1}^{p+1} \frac{\prod_{s=1, s\ne k}^{p+1} \Gamma(a_s - a_k)/\Gamma(a_s)}{\prod_{s=1}^{p} \Gamma(b_s - a_k)/\Gamma(b_s)} \frac{1}{(-z)^{a_k}} \times \\ {}_{p+1}F_p\left(\left. \begin{matrix} a_k, \ b_1 + 1 - a_k, \ b_2 + 1 - a_k, \ \ldots, \ b_p + 1 - a_k \\ a_k + 1 - a_1, \ a_k + 1 - a_2, \ \ldots, \ *, \ \ldots, \ a_k + 1 - a_{p+1} \end{matrix} \right| \frac{1}{z} \right), \end{aligned} \tag{6.3} $$

where the symbol $*$ indicates that the entry $a_k + 1 - a_k$ is omitted. Thus, using (6.2) and (6.3) the function ${}_{p+1}F_p(a, \vec{b}, \vec{c}; z)$ can be computed in the exterior of the unit disk $|z| \geq \rho > 1$, with ρ depending on numerical requirements.

Other expansions for the hypergeometric function ${}_{p+1}F_p$, in terms of elementary functions, may be found in the literature. For example, Nørlund [100] found an expansion in powers of $\frac{z}{z-1}$ [3, eq. 16.10.2] that converges in the half plane $\Re z < 1/2$, Bühring [18] obtained an expansion in inverse powers of $z - z_0$ [3, eq. 16.8.9], for arbitrary complex z_0, valid in the region $|z - z_0| > \max\{|z_0|, |z_0 - 1|\}$ and $|\arg(z_0 - z)| < \pi$, and in [55, §4.1] we can find an inverse factorial series representation of the generalized hypergeometric ${}_{p+1}F_p$ function that converges in the half-plane $\Re z < 0$.

All those expansions, the power series defintion (6.2), the Taylor expansion at $z = \infty$ (6.2)-(6.3), Nørlund's expansion and Bühring's expansion are given in terms of elementary functions but, for general values of $(a, \vec{b}, \vec{c})$, none of them is convergent in unbounded regions of the complex plane containing the indented closed unit disk $D^* := \{z \in \mathbb{C} : |z| \leq 1, z \neq 1\}$. Therefore, we seek new convergent expansions of ${}_{p+1}F_p$ given in terms of elementary functions that are convergent in large regions of the complex plane and, in particular, in unbounded regions containing the indented closed unit disk D^*.

The starting point is the following integral representation of the generalized hypergeometric function ${}_{p+1}F_p$ [72, eq. 5] that we derive in appendix C

$$ {}_{p+1}F_p\left(\begin{array}{c} a, \ \vec{b} \\ \vec{c} \end{array} \middle| \, z\right) = \prod_{s=1}^{p} \left(A(b_s, c_s) \int_L t_s^{b_s-1}(1-t_s)^{c_s-b_s-1} dt_s \right) (1 - Tz)^{-a}, \tag{6.4} $$

where $T := \prod_{s=1}^{p} t_s$, every t_s-integration path L is identical and $(1 - Tz)^{-a} = 1$ if $z = 0$. There are four different possibilities for the path L and the constants $A(b_s, c_s)$, depending on the value of the parameters a, $\vec{b}$ and $\vec{c}$ (see appendix C). For example, if $\Re(c_s) > \Re(b_s) > 0$, for all $s = 1, 2, \ldots, p$, we can choose the Euler form of the integral (6.4), that is, $L = [0, 1]$ and $A(b_s, c_s) := \Gamma(c_s)/\left[\Gamma(b_s)\Gamma(c_s - b_s)\right]$. For other values of b_k and c_k, the integration path is a complex contour that can be sticked to the real interval $[0, 1]$ and that surrounds the points $t_s = 0$ and/or $t_s = 1$. The four possibilities are necessary to have at our disposal at least one integral representation valid for every $(a, \vec{b}, \vec{c}) \in \Lambda$, where $\Lambda := \{(a, \vec{b}, \vec{c}) \in \mathbb{C}^{2p+1} : 1 - c_s \notin \mathbb{N}, s = 1, 2, \ldots, p\}$, as each one is valid in different regions of the parameters $\vec{b}$ and $\vec{c}$ but none of them is valid in the whole region $(a, \vec{b}, \vec{c}) \in \Lambda$.

For any possibility in the choice of L and $A(b_s, c_s)$ in the integral (6.4), we consider the function $f(T) := (1 - zT)^{-a}$. When we replace $f(T)$ in (6.4) by its standard Taylor series expansion at $T = 0$, interchange summation and integration and use the corresponding integral representation of the beta function given in [4, §5.12] we obtain the power series definition (6.2). If $|z| < 1$, the Taylor series expansion of $f(T)$ at $T = 0$ converges uniformly in z for any $(t_1, \ldots, t_p) \in L \times \ldots \times L$ in any of the four possible integration domains considered in (6.4). Therefore, expansion (6.2) is convergent in the unit disk $|z| < 1$.

Following [76, 83] and pointed out in the introduction of this chapter, the region of convergence $|z| < 1$ of expansion (6.2) is determined by requiremets (i) and (ii) above. Therefore, the smaller D_r is, the bigger the convergence region of the expansion of ${}_{p+1}F_p$ would be. For this reason, we consider multi-point Taylor expansions as the lemniscate of convergence D becomes thinner and thinner. In particular, D must satisfy requirements (I) and (II), but adapted to any of the four possibilities for the integration path L in (6.4). For this reason, we introduce the following notation: $\mathbf{t} := (t_1, t_2, \ldots, t_p)$, $T = \prod_{s=1}^{p} t_s$ and $X := \{z \in \mathbb{C} : |z - x| < \varepsilon \text{ for some } x \in [0,1] \text{ and small enough } \varepsilon > 0\}$. Then, the requirements for the lemniscate of convergence D are:

(i) Appropriately selecting the base points of the multi-point Taylor expansion of $f(T)$, the domain X (that is squeezed as much as possible around the real interval $[0,1]$) is contained in D, the domain of convergence of the multi-point Taylor expansion of $f(T)$.

(ii) The singular point $T = 1/z$ is located outside D.

As we will see, the use of multi-point Taylor expansions result in expansions that are covergent in large (possibly unbounded) regions of the complex plane that contain the indented closed unit disk D^*.

6.2 New expansions for the ${}_{p+1}F_p(a,\vec{b},\vec{c};z)$ function

6.2.1 A generalization of Nørlund's expansion

First, we consider the standard Taylor expansion of $f(T) = (1 - zT)^{-a}$ at a generic point $T = w$ of the complex plane. We have the following theorem:

Theorem 6.2.1. *For arbitrary $w \in \mathbb{C}$ define $W := \max\{|w|, |1-w|\}$ and the region $S := \{z \in \mathbb{C} : |1 - wz| > |z|W\}$. Then, for any $z \in S$ and $(a,\vec{b},\vec{c}) \in \Lambda$,*

$$ {}_{p+1}F_p\left(\begin{matrix} a, \ \vec{b} \\ \vec{c} \end{matrix} \middle| z \right) = (1 - wz)^{-a} \sum_{k=0}^{n-1} \frac{(a)_k}{k!} \left(\frac{wz}{wz-1} \right)^k {}_{p+1}F_p\left(\begin{matrix} -k, \ \vec{b} \\ \vec{c} \end{matrix} \middle| \frac{1}{w} \right) + R_n\left(\begin{matrix} a,\vec{b} \\ \vec{c} \end{matrix} \middle| z \right). \tag{6.5}$$

On the one hand, the functions ${}_{p+1}F_p(-k,\vec{b},\vec{c};1/w)$ are polynomials in the variable $1/w$ that can be computed using lemma D.0.1 in Appendix D.

On the other hand, the remainder term $R_n(a,\vec{b},\vec{c};z)$ is uniformly bounded in compact sets of S in the form

$$ \left| R_n\left(\begin{matrix} a,\vec{b} \\ \vec{c} \end{matrix} \middle| z \right) \right| \le M(a,\vec{b},\vec{c};z) \left| \frac{zW}{1-wz} \right|^n, \tag{6.6}$$

with $M(a,\vec{b},\vec{c};z) > 0$ independent of n. Furthermore, if $\Re c_k > \Re b_k > 0$ for all $k = 1,2,\ldots,p$, then $M(a,\vec{b},\vec{c};z)$ is also independent of $\vec{b}$ and $\vec{c}$. In this case

$$ \left| R_n\left(\begin{matrix} a,\vec{b} \\ \vec{c} \end{matrix} \middle| z \right) \right| \le \frac{e^{\pi|\Im a|}\Gamma(n+\Re a)|zW|^n}{|\Gamma(a)|n!(|1-wz| - |zW|)^{n+\Re a}}. \tag{6.7}$$

Consequently

$$ R_n\left(\begin{matrix} a,\vec{b} \\ \vec{c} \end{matrix} \middle| z \right) = \mathcal{O}\left(\frac{|zW|^n n^{\Re a - 1}}{(|1-wz| - |zW|)^{n+\Re a}} \right), \qquad \text{as } n \to \infty. \tag{6.8}$$

Proof. Consider the Taylor series expansion of $f(T) = (1-zT)^{-a}$ at the generic point $T = w \in \mathbb{C}$:

$$f(T) = (1-wz)^{-a} \sum_{k=0}^{n-1} \frac{(a)_k}{k!} \left(\frac{z(T-w)}{1-wz} \right)^k + r_n(T). \tag{6.9}$$

Define $D = \{T \in \mathbb{C} : |T-w| < W + \varepsilon \text{ with } \varepsilon > 0 \text{ small}\}$. Then, condition (i) is satisfied, that is, D contains the domain X. For $z \in S$ condition (ii) is satisfied too, that is, if $z \in S$ then $1/z \notin D$.

Moreover, for $T \in [0,1]$ we have $|T-w| \leq W$. Therefore, expansion (6.9) is convergent for all $z \in S$ and the remainder $r_n(T)$ satisfies

$$\begin{aligned} r_n(T) &:= (1-wz)^{-a} \sum_{k=n}^{\infty} \frac{(a)_k}{k!} \left(\frac{z(T-w)}{1-wz} \right)^k \\ &= (1-wz)^{-a} \frac{(a)_n}{n!} \left(\frac{z(T-w)}{1-wz} \right)^n {}_2F_1\left(\left. \begin{matrix} 1, \ n+a \\ n+1 \end{matrix} \right| \frac{z(T-w)}{1-wz} \right). \end{aligned} \tag{6.10}$$

For $z \in S$ we can replace $f(T)$ in (6.4) by the right hand side of (6.9). After interchanging summation and integration we find

$$\begin{aligned} {}_{p+1}F_p\left(\left. \begin{matrix} a, \ \vec{b} \\ \vec{c} \end{matrix} \right| z \right) &= (1-wz)^{-a} \sum_{k=0}^{n-1} \frac{(a)_k}{k!} \left(\frac{z}{wz-1} \right)^k \\ &\times \prod_{s=1}^{p} \left(A(b_s,c_s) \int_L t_s^{b_s-1}(1-t_s)^{c_s-b_s-1} dt_s \right) (w-T)^k + R_n\left(\left. \begin{matrix} a, \vec{b} \\ \vec{c} \end{matrix} \right| z \right). \end{aligned} \tag{6.11}$$

Writing $(w-T)^k = w^k(1-T/w)^k$ and using the integral representation (6.4) we find, in any of the four possibilities for the path L and the constants $A(b_s,c_s)$, the right hand side of expansion (6.5). The remainder $R_n(a,\vec{b},\vec{c};z)$ of the expansion is given by

$$R_n\left(\left. \begin{matrix} a, \vec{b} \\ \vec{c} \end{matrix} \right| z \right) = \prod_{s=1}^{p} \left(A(b_s,c_s) \int_L t_s^{b_s-1}(1-t_s)^{c_s-b_s-1} dt_s \right) r_n(T), \tag{6.12}$$

From Cauchy's integral representation of the remainder $r_n(T)$ we have, for any $T \in [0,1]$

$$r_n(T) = \frac{(T-w)^n}{2\pi i} \oint_C \frac{(1-zv)^{-a}}{(v-T)(v-w)^n} dv, \tag{6.13}$$

where the integration contour C is a circle $|v-w| = r$ of radius $r < |w-1/z|$ that encircles the points $v = w, v = 0$ and $v = 1$ in the positive direction. In particular, we may take $r = |w-1/z| - \varepsilon$ for any $0 < \varepsilon < |w-1/z| - W$. Then, for all $v \in C$ we have $|v-w| = |w-1/z| - \varepsilon$. As we also have $|T-w| \leq W$, for all $T \in [0,1]$, we find

$$|r_n(T)| \leq C(a,z) \frac{W^n}{(|w-1/z| - \varepsilon)^n}, \tag{6.14}$$

where $C(a,z) := \frac{1}{2\pi} \oint_C \left| \frac{(1-zv)^{-a}}{v-T} \right| dv > 0$ can be bounded independently of T and N.

As the bound (6.14) is valid for arbitrarily small positive ε, we take $\varepsilon \to 0$ to find

$$|r_n(T)| \leq C(a,z) \left| \frac{zW}{1-wz} \right|^n, \tag{6.15}$$

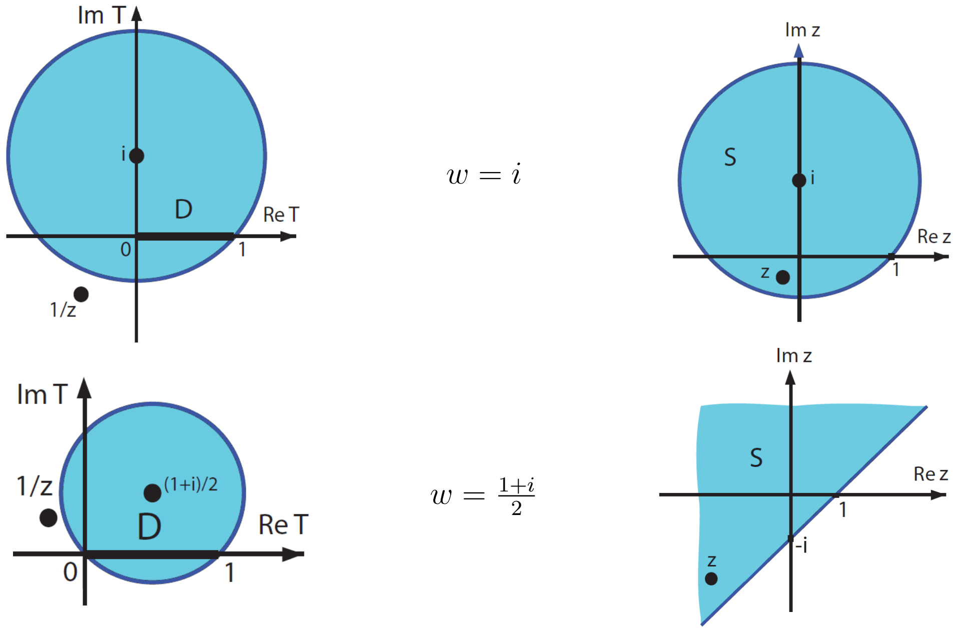


Figure 6.1: The minimal domain of convergence of the standard Taylor expansion of $f(T)$ at $T = w$ containing X is a disk of center w and radius $r = W + \varepsilon$ (figures left). The region S (figures right) is the inverse of the exterior of D: it is a disk for $\Re w < 1/2$ whereas for $\Re w \geq 1/2$ it is a half-plane.

Introducing this bound into (6.12) and using the corresponding integral representation of the beta function [4, §5.12] to integrate we obtain (6.6).

On the other hand, from (6.10) and using the integral representation of the hypergeometric ${}_2F_1$ function [104, eq. 15.6.1] we find the bound

$$|r_n(T)| \leq |(1-wz)^{-a}| \frac{|(a)_n|}{n!} \left|\frac{zW}{1-wz}\right|^n {}_2F_1\left(\begin{matrix} 1, \ n+\Re a \\ n+1 \end{matrix} \middle| \left|\frac{zW}{1-wz}\right|\right) \tag{6.16}$$

In the case that, for all $k = 1, 2, \ldots, p$, $\Re c_k > \Re b_k > 0$ we can consider Euler's form for the path L, that is, $L = [0, 1]$, and introducing bound (6.16) into (6.12) and after some easy computations we find (6.7). The asymptotic behavior as $n \to \infty$ (6.8) follows from (6.7) and Stirling's formula for the gamma function. □

Remark 6.2.2. *The domain of convergence of the expansion* (6.5) *is the region* $S := \{z \in \mathbb{C} : |1 - wz| > |z|W\}$. *If* $\Re w < 1/2$ *the region* S *is a disk of center* $\bar{w}/(2\Re w - 1)$ *and radius* $(|w - 1|)/(1 - 2\Re w)$. *For* $\Re w \geq 1/2$ *the region* S *is the half-plane* $\Re(wz) < 1/2$ *(see figure 6.1).*

Remark 6.2.3. *When* $w = 1$ *expansion* (6.5) *is Nørlund's expansion [3, eq. 16.10.2] given in terms of powers of* $z/(z - 1)$. *It converges in the half-plane* $\Re z < 1/2$.

6.2.2 An expansion using a two-point Taylor expansion

From section 2.1 we know that the use of multi-point Taylor expansions is preferable over the standard Taylor series expansion as we can avoid the singularity at $T = 1/z$ of $f(T) = (1 - zT)^{-a}$ in a more efficient way, while containing the domain X in the interior of the lemniscate of convergence. Therefore, we consider the two-point Taylor expansion of $f(T)$ at the points $T=q$ and $T=1-q$, with $q\in[0,1/2)$. We have the following result:

Theorem 6.2.4. *For arbitrary $q\in[0,q_0]$, with $q_0:=(2-\sqrt{2})/4$, consider the region (see figure 6.2)*

$$S_q^2:=\{z\in\mathbb{C}:|(1-qz)(1+qz-z)|>|(1/2-q)z|^2\}. \tag{6.17}$$

Then, for $z\in S_q^2$ and $(a,\vec{b},\vec{c})\in\Lambda$,

$$\begin{aligned}{}_{p+1}F_p\left(\begin{matrix}a,\ \vec{b}\\ \vec{c}\end{matrix}\middle| z\right)=\sum_{n=0}^{N-1}\sum_{k=0}^{n}\binom{n}{k}[q(1-q)]^{n-k}(-1)^k\prod_{s=1}^{p}\frac{(b_s)_k}{(c_s)_k}\left[A_n^2(a,z)\,{}_{p+1}F_p\left(\begin{matrix}-k,\ \overrightarrow{b+k}\\ \overrightarrow{c+k}\end{matrix}\middle|1\right)\right.\\ \left.+\prod_{s=1}^{p}\frac{b_s+k}{c_s+k}B_n^2(a,z)\,{}_{p+1}F_p\left(\begin{matrix}-k,\ \overrightarrow{b+k+1}\\ \overrightarrow{c+k+1}\end{matrix}\middle|1\right)\right]+R_N\left(\begin{matrix}a,\vec{b}\\ \vec{c}\end{matrix}\middle|z\right),\end{aligned} \tag{6.18}$$

where, for $n=1,2,\ldots$, the coefficients $A_n^2(a,z)$ and $B_n^2(a,z)$ and ${}_{p+1}F_p(-n,\vec{b},\vec{c};1)$ are rational functions of a and z and $\vec{b}$ and $\vec{c}$ respectively. They can be computed recursively using lemmas D.0.2 and D.0.1 in Appendix D.

On the other hand, the remainder $R_N(a,\vec{b},\vec{c};z)$ is bounded in the forn

$$\left|R_N\left(\begin{matrix}a,\vec{b}\\ \vec{c}\end{matrix}\middle|z\right)\right|\leq M(a,\vec{b},\vec{c};z)\left|\frac{(1/2-q)^2z^2}{(1-qz)\left[1+(q-1)z\right]}\right|^N, \tag{6.19}$$

with $M(a,\vec{b},\vec{c};z)>0$ independent of N. Moreover, if $\Re c_k>\Re b_k>0$, for all $k=1,2,\ldots,p$, $M(a,\vec{b},\vec{c};z)$ is also independent of $\vec{b}$ and $\vec{c}$.

Proof. We consider the two-point Taylor expansion of $f(T)=(1-zT)^{-a}$ at $T=q$ and $T=1-q$

$$f(T)=\sum_{n=0}^{N-1}\left[A_n^2(a,z)+B_n^2(a,z)T\right]\left[(T-q)(T+q-1)\right]^n+r_N(T), \tag{6.20}$$

valid in a Cassini oval D_q with foci at $T=q$ and $T=1-q$ and a certain radius $r>0$ to be determined, that is, $D_q=\{T\in\mathbb{C}:|(T-q)(T+q-1)|<r\}$. In formula (6.20) $r_N(T)$ is the two-point Taylor remainder after N terms.

An explicit formula for the coefficients $A_n^2(a,z)$ and $B_n^2(a,z)$ can be derived from formulas (2.2)-(2.3) and also using the recurrent algorithm described after (2.8) in chapter 2. Instead, the recurrence relation (D.3) given in Appendix D is found from the differential equation satisfied by the function $f(T)$, namely, $(1-zT)f'(T)=azf(T)$.

We determine the radius r of the Cassini oval D_q as follows: The interval $[0,1]$ is contained in D_q whenever

$$r\geq\sup_{T\in[0,1]}\{|(T-q)(T+q-1)|\}=\max\{q(1-q),(1/2-q)^2\}.$$

This happens for $r\geq(1/2-q)^2$ when $0\leq q\leq q_0:=(2-\sqrt{2})/4$, where q_0 is the solution of the equation $q(1-q)=(1/2-q)^2$. Then, expansion (6.20) satisfies condition (i) for

$r>(1/2-q)^2$, with $q\in[0,q_0]$. On the other hand, requirement (ii), that is, $1/z\notin D_q$, is satisfied if $r<\left|\left(\frac{1}{z}-q\right)\left(\frac{1}{z}+q-1\right)\right|$. The smallest possible r is $r=(1/2-q)^2+\varepsilon$, with small $\varepsilon>0$ and therefore the largest possible region is S_q^2, given in (6.17).

For $z\in S_q^2$ and $T\in[0,1]$ expansion (6.20) is convergent. It can be introduced into the integral (6.4) and interchange summation and integration to obtain (6.18) with

$$R_N\left(\begin{array}{c}a,\vec{b}\\ \vec{c}\end{array}\middle| z\right):=\prod_{s=1}^{p}\left(A(b_s,c_s)\int_L t_s^{b_s-1}(1-t_s)^{c_s-b_s-1}dt_s\right)r_N(T), \tag{6.21}$$

where the Taylor remainder $r_N(T)$ can be written in terms of a Cauchy integral (2.4). Analogously to the proof in theorem 6.2.1, the remainder $r_N(T)$ can be bounded in the form

$$|r_N(T)|\leq C(a,z)\left|\frac{(T-q)(T+q-1)}{\left(\frac{1}{z}-q\right)\left(\frac{1}{z}+q-1\right)-\varepsilon}\right|^N, \tag{6.22}$$

with $C(a,z)>0$ independent of T and N. Moreover, when $T\in X$ the absolute value of the numerator above is bounded by $(1/2-q)^2$ and, as (6.22) is valid for arbitrarily small $\varepsilon>0$, we find

$$|r_N(T)|\leq C(a,z)\left|\frac{(1/2-q)^2z^2}{(1-qz)\left[1+(q-1)z\right]}\right|^N \tag{6.23}$$

Introducing this bound into (6.21) we obtain (6.19). If in addition $\Re c_k>\Re b_k>0$ for all $k=1,2,\dots,p$ we can use the Euler form of (6.4) (when $L=[0,1]$) as we did in the proof of theorem 6.2.1 to show that $M(a,\vec{b},\vec{c};z)$ is independent of $\vec{b}$ and $\vec{c}$. □

Remark 6.2.5. *Two particular cases are to be highlighted. On the one hand, when we take $T=0$ and $T=1$ as base points for the two-point Taylor expansion of $f(T)=(1-zT)^{-a}$, that is, for $q=0$, expansion (6.18) is the simplest one. In particular, the inner sum that runs from $k=0$ to $k=n$ in (6.18) contains only one non-zero term and formulas (D.2), (D.3) and (D.4) become easier too. On the other hand, for $q=q_0:=(2-\sqrt{2})/4$ the base points for the Taylor expansions are $T=(2-\sqrt{2})/4$ and $T=(2+\sqrt{2})/4$. In this case the region $S_{q_0}^2$ is the largest possible one. In both cases, $q=0$ and $q=q_0$ the indented closed unit disk $D^*=\{z\in\mathbb{C}:|z|\leq 1,\ z\neq 1\}$ is containded in the region S_q^2 of convergence. Moreover, for $q=q_0$ the region $S_{q_0}^2$ is unbounded (see figure 6.2).*

6.2.3 Expansions in larger domains

Now, in order to avoid the branch point $T=1/z$ of $f(T)=(1-zT)^{-a}$ in a more efficient way, we consider the three-point Taylor expansion of $f(T)$ at the base point $T=q$, $T=1/2$ and $T=1-q$ with $q\in[0,1/2)$. We have the following theorem.

Theorem 6.2.6. *For arbitrary $q\in[0,q_1]$ with $q_1:=(2-\sqrt{3})/4$, define the region (see figure 6.3)*

$$S_q^3:=\{z\in\mathbb{C}:6\sqrt{3}|(1-qz)(2-z)(1+qz-z)|>|(1-2q)z|^3\}. \tag{6.24}$$

Then, for $z\in S_q^3$ and $(a,\vec{b},\vec{c})\in\Lambda$,

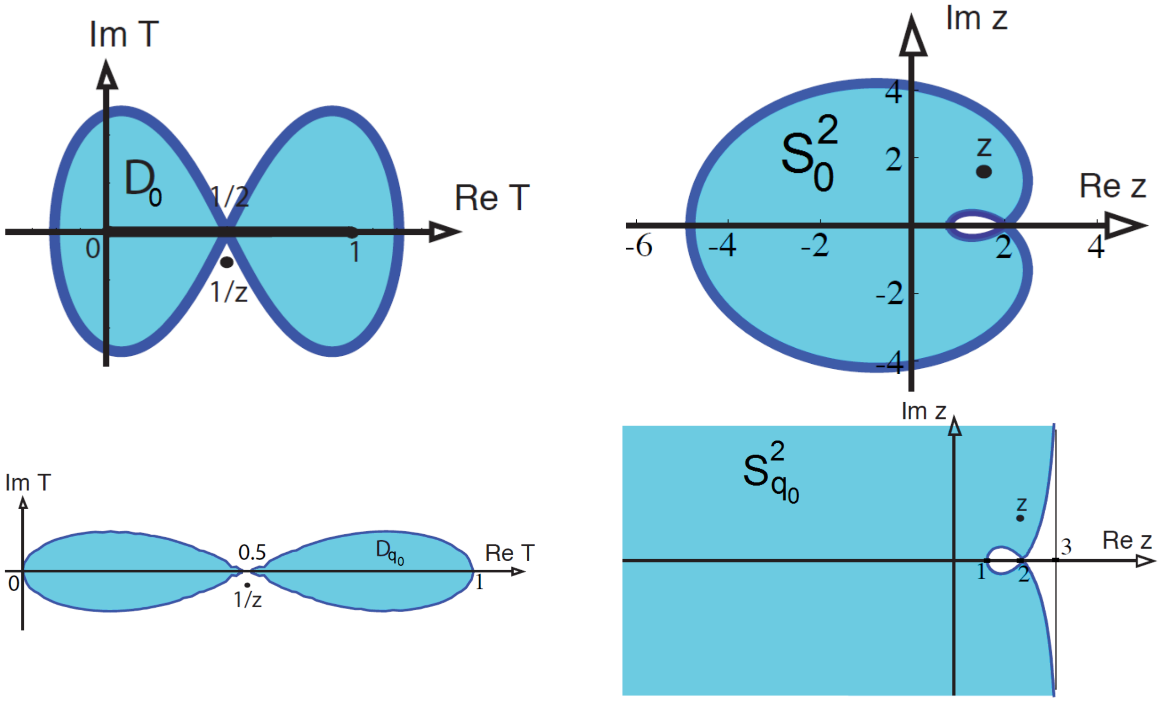


Figure 6.2: The Cassini oval D_q of convergence of the two-point Taylor expansion of $f(T)$ with foci at $T=q$, $T=1-q$ and radius r (left) contains the interval $[0,1]$ and avoids the branch point $T=1/z$ for $r>(\frac{1}{2}-q)^2$ and $q\in[0,q_0]$. The region S_q^2 (right) is the inverse of the exterior of the disk D_q. For $q=0$ (top) the region of convergence S_0^2 is bounded. For $q=q_0$ (bottom), the region $S_{q_0}^2$ is the largest possible one and it is unbounded. We see that the thinner D_q is the larger S_q^2 becomes.

$$
\begin{aligned}
{}_{p+1}F_p\left(\left.\begin{matrix} a,\ \vec{b} \\ \vec{c}\end{matrix}\right| z\right) &= \sum_{n=0}^{N-1}\sum_{k=0}^{n}\binom{n}{k}(-2)^{k-n}\prod_{s=1}^{p}\frac{(b_s)_k}{(c_s)_k}\left[A_n^3(a,z)H_n^q\left(\begin{matrix}\overrightarrow{b+k}\\ \overrightarrow{c+k}\end{matrix}\right)\right.\\
&\left.+\prod_{s=1}^{p}\frac{b_s+k}{c_s+k}B_n^3(a,z)H_n^q\left(\begin{matrix}\overrightarrow{b+k+1}\\ \overrightarrow{c+k+1}\end{matrix}\right)+\prod_{s=1}^{p}\frac{(b_s+k)(b_s+k+1)}{(c_s+k)(c_s+k+1)}C_n^3(a,z)H_n^q\left(\begin{matrix}\overrightarrow{b+k+2}\\ \overrightarrow{c+k+2}\end{matrix}\right)\right]\\
&+R_N\left(\left.\begin{matrix} a,\vec{b} \\ \vec{c}\end{matrix}\right| z\right),
\end{aligned}
\tag{6.25}
$$

with

$$
H_n^q\left(\begin{matrix}\vec{b}\\ \vec{c}\end{matrix}\right) := \sum_{m=0}^{n}\binom{n}{m}\left[q(1-q)\right]^{n-m}(-1)^m\prod_{s=1}^{p}\frac{(b_s)_m}{(c_s)_m}\,{}_{p+1}F_p\left(\left.\begin{matrix}-m,\ \overrightarrow{b+m}\\ \overrightarrow{c+m}\end{matrix}\right|1\right). \tag{6.26}
$$

The coefficients $A_n^3(a,z)$, $B_n^3(a,z)$ and $C_n^3(a,z)$ and the functions $H_n^q(\vec{b},\vec{c})$ are, respectively, rational functions of a and z and $\vec{b}$ and $\vec{c}$. They can be computed recursively using lemmas D.0.3 and D.0.1 in Appendix D.

On the other hand, the rate of convergence of (6.25) is of power type:

$$\left|R_N\left(\begin{array}{c|}a,\vec{b}\\ \vec{c}\end{array}\, z\right)\right| \le M(a,\vec{b},\vec{c};z)\left|\frac{(1-2q)^3z^3}{6\sqrt{3}(1-qz)\left[1+(q-1)z\right](2-z)}\right|^N, \tag{6.27}$$

with $M(a,\vec{b},\vec{c};z)>0$ independent of N. Moreover, if $\Re c_k > \Re b_k > 0$ for all $k = 1,2,\dots,p$, then $M(a,\vec{b},\vec{c};z)$ is also independent of $\vec{b}$ and $\vec{c}$.

Proof. The proof follows the same steps as the proof of theorems 6.2.1 and 6.2.4, but considering the three-point Taylor expansion of $f(T)=(1-zT)^{-a}$ at the base points $T=q$, $T=1/2$ and $T=1-q$, with $q\in[0,1/2)$. We have

$$f(T)=\sum_{n=0}^{N-1}\left[A_n^3(a,z)+B_n^3(a,z)T+C_n^3(a,z)T^2\right]\left[(T-q)(T-1/2)(T+q-1)\right]^n+r_N(T), \tag{6.28}$$

valid in a lemniscate D_q of foci $T=q$, $T=1/2$ and $T=1-q$ and a certain radius $r>0$, that is, $D_q=\{T\in\mathbb{C}:|(T-q)(T-1/2)(T+q-1)|<r\}$. To determine de radius r, we require that the lemniscate D_q satisfy requirements (i) and (ii).

On the one hand, the interval $[0,1]$ is contained in D_q if

$$r\ge\sup_{T\in[0,1]}\{|(T-q)(T-1/2)(T-1+q)|\}=\max\left\{\frac{q(1-q)}{2},\frac{(1-2q)^3}{12\sqrt{3}}\right\}.$$

Let $q_1:=\frac{2-\sqrt{3}}{4}$ be the solution of the equation $\frac{q(1-q)}{2}=\frac{(1-2q)^3}{12\sqrt{3}}$. Then, for $0\le q\le q_1$, expansion (6.28) satisfies condition (i) for $r\ge\frac{(1-2q)^3}{12\sqrt{3}}$. On the other hand, it satisfies condition (ii) if $1/z\notin D_q$, that is, for any

$$r<\left|\left(\frac{1}{z}-q\right)\left(\frac{1}{z}-\frac{1}{2}\right)\left(\frac{1}{z}+q-1\right)\right|.$$

The smallest possible r is $r=\frac{(1-2q)^3}{12\sqrt{3}}+\varepsilon$, for $\varepsilon>0$ small and then, the largest possible region of convergence S is the one given in (6.24): S_q^3.

For $z\in S_q^3$ we introduce the expansion (6.28) into the integral (6.4), we interchange summation and integration and we obtain (6.25), with

$$R_N\left(\begin{array}{c|}a,\vec{b}\\ \vec{c}\end{array}\, z\right):=\prod_{s=1}^{p}\left(A(b_s,c_s)\int_L t_s^{b_s-1}(1-t_s)^{c_s-b_s-1}dt_s\right)r_N(T). \tag{6.29}$$

From Cauchy's formula for the Taylor remainder $r_N(T)$ we find the bound

$$|r_N(T)|\le C(a,z)\left|\frac{(T-q)(T-1/2)(T+q-1)}{\left(\frac{1}{z}-q\right)\left(\frac{1}{z}-\frac{1}{2}\right)\left(\frac{1}{z}+q-1\right)-\varepsilon}\right|^N,$$

with $C(a,z)>0$ independent of T and N. This bound is valid for arbitrarily small $\varepsilon>0$ and, when $T\in X$, we also have $|(T-q)(T-1/2)(T+q-1)|\le(1-2q)^3/(12\sqrt{3})$. Therefore

$$|r_N(T)|\le C(a,z)\left|\frac{(1-2q)^3z^3}{6\sqrt{3}\,(1-qz)\,(2-z)\left[1+(q-1)z\right]}\right|^N.$$

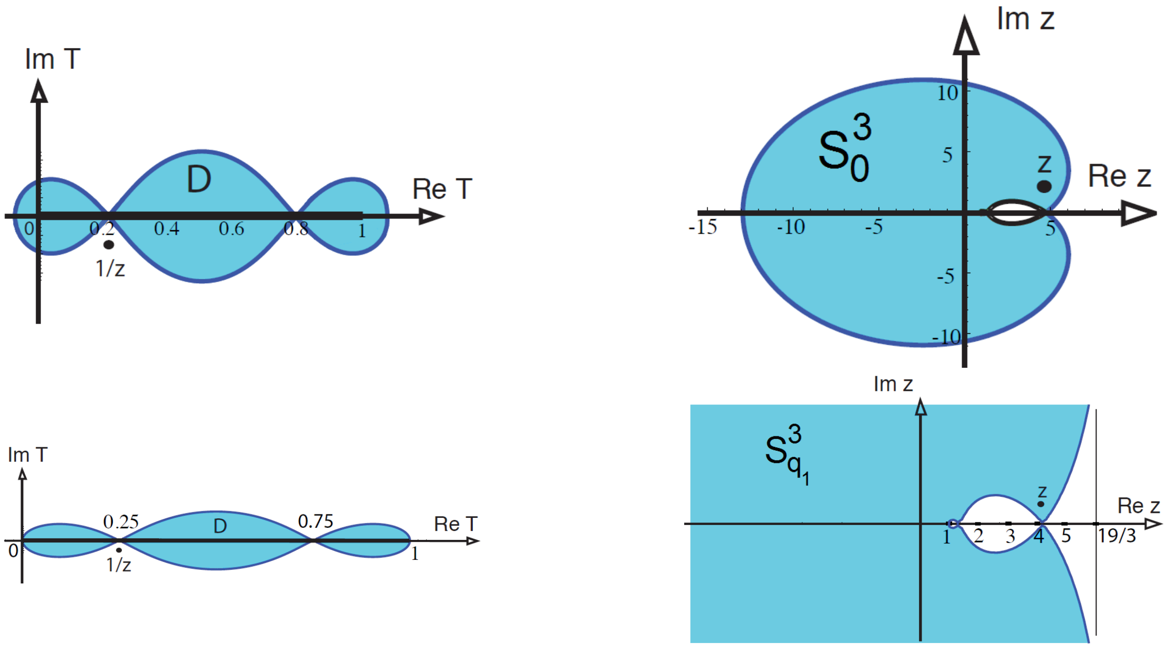


Figure 6.3: The minimal domain of convergence D_q of the three-point Taylor expansion of $f(T)$ at the base point $T = q$, $T = 1/2$ and $T = 1 - q$ is a lemniscate of radius $r > (1-2q)^3/(12\sqrt{3})$ (figures left with $q = 0$ (top) and $q = q_1 := (2-\sqrt{3})/4$ (bottom)). The corresponding region S_q^3, the exterior of the inverse of D_q, is shown on the right for $q = 0$ (top) and $q = q_1$ (bottom).

Introducing this bound into (6.29) and integrating we find (6.27). If in addition $\Re c_k > \Re b_k > 0$ for all $k = 1, 2, \dots, p$, the Euler form of (6.4) can be used to show that $M(a, \vec{b}, \vec{c}; z)$ is independent of $\vec{b}$ and $\vec{c}$, as we did in the proof of theorem 6.2.1.

□

Remark 6.2.7. *We highlight the two extremal cases $q = 0$ and $q = q_1 := (2-\sqrt{3})/4$. In the former case, expansion (6.25) becomes the simplest one whereas in the latter case the region $S_{q_1}^3$ of convergence is the largest possible one. Both regions contain the indented closed unit disk $D^* := \{z \in \mathbb{C} : |z| \leq 1,\ z \neq 1\}$. Moreover, the region $S_{q_1}^3$ is unbounded (see figure 6.3).*

6.3 Final remarks and numerical experiments

In the previous section we have derived new analytical approximations of the generalized hypergeometric function ${}_{p+1}F_p(a, \vec{b}, \vec{c}; z)$ valid in large (sometimes unbounded) regions of the complex plane containing the point $z = 0$ too. The key point of the analysis has been to consider a multi-point Taylor expansion of the factor $(1 - zT)^{-a}$ in the integral representation (6.4). This idea has already been used in [76, 83] to obtain, respectively, new analytic approximations of the ${}_2F_1$ and the ${}_3F_2$ hypergeometric functions. Then, obviously, the results in section 6.2 generalize the results in [76, 83] from 3 or 5 parameters to an arbitrary number $2p + 1$. Moreover, the results in [76, 83] are derived from the Euler-type integral representation of the hypergeometric functions and then the results are only valid in the situation when $\Re c_k > \Re b_k > 0$, for all $k = 1, 2, \dots, p$. Using loop integrals we have seen that the expansions are also valid for any $(a, \vec{b}, \vec{c}) \in \Lambda$, that is, for any complex value of the parameters a, $\vec{b}$ and $\vec{c}$ except $1 - c_k \in \mathbb{N}$.

We have considered the 2− and 3−point Taylor expansion of $f(T)$ at the generic base points $(q,\, 1-q)$ and $(q,\, 1/2,\, 1-q)$ with $q \in [0, 1/2)$ in contrast to [76, 83] where only the case $q = 0$ is studied. In this manner, we have been able to find large regions of convergence for the new expansions of the hypergeometric functions. Most of these regions S_q contain the indented closed unit disk D^* and some of them are unbounded (see figures 6.1, 6.2, 6.3). We could also have considered the possibility of expanding $f(T)$ at four (or even more) points located along the interval $[0, 1]$. In this way, the region of convergence S would be larger than the regions S_q^2 and S_q^3 shown in figures 6.2 and 6.3. However, the integrals defining the coefficients of the expansion as well as the recurrence relations for the coefficients $A_n(a, z)$, $B_n(a, z)$, $C_n(a, z)$, ... (see Appendix D), become more and more complicated.

Furthermore, the derived expansions (6.5), (6.18) and (6.25) are given in terms of rational functions of z and the parameters b_k and c_k. We have obtained bounds for the remainder of the expansions that show the convergence of the expansions in the corresponding regions S_q with an exponential rate of convergence: $R_N(a, \vec{b}, \vec{c}; z) = \mathcal{O}\left(A(z)^{-N}\right)$, as $N \to \infty$, for a certain $A(z) > 1$ that is, in each case, related with the convergence region S_q. We have only given a precise bound for the remainder in subsection 6.2.1 when we used the classical Taylor expansion of the function $f(T)$. In the other cases when we used multi-point Taylor expansion, we have given the bound for the remainder in terms of a certain constant $M(a, \vec{b}, \vec{c}; z)$. The reasons for this are the following: (i) we do not have at our disposal an appropriate expression for the remainder $r_N(T)$ of the two- and three-point Taylor exansions of $f(T)$ that would lead to the derivation of an accurate constant $C(a, z)$; and (ii) the loop integrals are less appropriate than the Euler form when we want to find an explicit and accurate value of the bounding constant $M(a, \vec{b}, \vec{c}; z)$. And Euler's form is only valid in the case when $\Re c_k > \Re b_k > 0$, for all $k \in 1, 2, \ldots, p$.

In tables 6.1-6.6 we show some numerical experiments about the accuracy of the expansions (6.5), (6.18) and (6.25) to evaluate the functions ${}_4F_3(a, \vec{b}, \vec{c}; z)$ and ${}_8F_7(a, \vec{b}, \vec{c}; z)$ for some values of the parameters a, $\vec{b}$ and $\vec{c}$ and different values of the variable z. In expansion (6.5) we have taken $w = 1/2$ as base point for the Taylor expansion. On the other hand, in expansions (6.18) and (6.25) we have taken the simplest case $q = 0$ and the value of q that maximizes the size of the region of convergence, that is, $q = q_0$ for (6.18) and $q = q_1$ for (6.25). We have also compared the accuracy of those expansions with the approximations provided by the power series definition (6.2), the connection formula (6.3) combined with (6.2) and also with Bühring's formula [3, eq. 16.8.9] with $z_0 = 1/2$ as base point. The tables show the relative error provided by the different approximations when we truncate the expansion after N terms.

The numerical results show that the expansions derived from a three-point Taylor expansion (6.25) (with $q = 0$ or $q = q_1$) are more accurate than the expansions derived from a two-point Taylor expansion (6.18) or a standard Taylor expansion (6.5). But moreover, the new expansions are more competititive than Bühring's formula or the standard Taylor series definition when z is close to 0 or when the difference between any couple of the parameters $(a, b_1, b_2, \ldots, b_p)$ is close or equal to an integer. When z is away from 0 the expansions derived from a three-point Taylor expansion (6.25) seems to perform better than Bühring's formula which, on the other hand, seems to perform better than the expansion obtained using a one- or a two-point Taylor expansion. In any case, expansion (6.25) with $q = q_1$ seems to be the most accurate.

Approximation	$N=0$	$N=2$	$N=4$	$N=6$
Power series definition (6.2)	$8.971 \cdot 10^{-2}$	$2.829 \cdot 10^{-3}$	$1.368 \cdot 10^{-4}$	$7.742 \cdot 10^{-6}$
Connection formula: (6.3)+(6.2)	–	–	–	–
Bühring's formula $z_0 = 1/2$	$6.371 \cdot 10^{-1}$	$1.844 \cdot 10^{-1}$	$6.933 \cdot 10^{-2}$	$2.828 \cdot 10^{-2}$
1-point expansion $w = 1/2$ (6.5)	$4.702 \cdot 10^{-2}$	$5.770 \cdot 10^{-4}$	$7.870 \cdot 10^{-4}$	$1.127 \cdot 10^{-7}$
2-point expansion $q = 0$ (6.18)	$8.932 \cdot 10^{-3}$	$1.551 \cdot 10^{-6}$	$3.325 \cdot 10^{-10}$	$7.676 \cdot 10^{-14}$
2-point expansion $q = q_0$ (6.18)	$7.280 \cdot 10^{-4}$	$3.829 \cdot 10^{-8}$	$2.201 \cdot 10^{-12}$	$2.376 \cdot 10^{-16}$
3-point expansion $q = 0$ (6.25)	$1.943 \cdot 10^{-4}$	$8.069 \cdot 10^{-11}$	$4.189 \cdot 10^{-17}$	$2.345 \cdot 10^{-23}$
3-point expansion $q = q_1$ (6.25)	$2.155 \cdot 10^{-6}$	$4.879 \cdot 10^{-13}$	$1.190 \cdot 10^{-19}$	$3.003 \cdot 10^{-26}$

Table 6.1: $p = 3,\ a = 1,\ \vec{b} = \left(\frac{1}{2}, \frac{4}{3}, \frac{5}{6}\right),\ \vec{c} = \left(\frac{5}{3}, \frac{7}{5}, \frac{5}{7}\right),\ z = \frac{-1-i}{5}$.

Approximation	$N=0$	$N=2$	$N=4$	$N=6$
Power series definition (6.2)	–	–	–	–
Connection formula: (6.3)+(6.2)	$8.182 \cdot 10^{-2}$	$2.442 \cdot 10^{-3}$	$1.384 \cdot 10^{-4}$	$9.510 \cdot 10^{-6}$
Bühring's formula $z_0 = 1/2$	$1.101 \cdot 10^{-1}$	$1.361 \cdot 10^{-3}$	$2.099 \cdot 10^{-5}$	$3.474 \cdot 10^{-7}$
1-point expansion $w = 1/2$ (6.5)	$3.474 \cdot 10^{-1}$	$1.021 \cdot 10^{-1}$	$3.301 \cdot 10^{-2}$	$1.115 \cdot 10^{-2}$
2-point expansion $q = 0$ (6.18)	$2.705 \cdot 10^{-1}$	$5.812 \cdot 10^{-2}$	$1.645 \cdot 10^{-2}$	$5.089 \cdot 10^{-3}$
2-point expansion $q = q_0$ (6.18)	$2.799 \cdot 10^{-2}$	$1.134 \cdot 10^{-3}$	$5.157 \cdot 10^{-5}$	$2.476 \cdot 10^{-6}$
3-point expansion $q = 0$ (6.25)	$4.963 \cdot 10^{-2}$	$6.413 \cdot 10^{-4}$	$1.053 \cdot 10^{-5}$	$1.873 \cdot 10^{-7}$
3-point expansion $q = q_1$ (6.25)	$3.102 \cdot 10^{-3}$	$1.518 \cdot 10^{-5}$	$8.371 \cdot 10^{-8}$	$4.880 \cdot 10^{-10}$

Table 6.2: $p = 3,\ a = 1,\ \vec{b} = \left(\frac{1}{2}, \frac{4}{3}, \frac{5}{6}\right),\ \vec{c} = \left(\frac{5}{3}, \frac{7}{5}, \frac{5}{7}\right),\ z = -3 + i$.

Approximation	$N=0$	$N=2$	$N=4$	$N=6$
Power series definition (6.2)	$3.701 \cdot 10^{-1}$	$1.598 \cdot 10^{-1}$	$9.907 \cdot 10^{-2}$	$7.070 \cdot 10^{-2}$
Connection formula: (6.3)+(6.2)	$4.657 \cdot 10^{-1}$	$1.722 \cdot 10^{-1}$	$1.029 \cdot 10^{-1}$	$7.220 \cdot 10^{-2}$
Bühring's formula $z_0 = 1/2$	$6.007 \cdot 10^{-1}$	$1.757 \cdot 10^{-1}$	$6.512 \cdot 10^{-2}$	$2.606 \cdot 10^{-2}$
1-point expansion $w = 1/2$ (6.5)	$2.393 \cdot 10^{-1}$	$7.884 \cdot 10^{-2}$	$2.952 \cdot 10^{-2}$	$1.171 \cdot 10^{-2}$
2-point expansion $q = 0$ (6.18)	$1.959 \cdot 10^{-1}$	$1.416 \cdot 10^{-2}$	$1.222 \cdot 10^{-3}$	$1.125 \cdot 10^{-4}$
2-point expansion $q = q_0$ (6.18)	$4.168 \cdot 10^{-2}$	$1.167 \cdot 10^{-3}$	$3.610 \cdot 10^{-5}$	$1.169 \cdot 10^{-6}$
3-point expansion $q = 0$ (6.25)	$2.256 \cdot 10^{-2}$	$1.085 \cdot 10^{-4}$	$6.332 \cdot 10^{-7}$	$3.959 \cdot 10^{-9}$
3-point expansion $q = q_1$ (6.25)	$4.522 \cdot 10^{-3}$	$1.232 \cdot 10^{-5}$	$3.713 \cdot 10^{-8}$	$1.172 \cdot 10^{-10}$

Table 6.3: $p = 3,\ a = 1,\ \vec{b} = \left(\frac{1}{2}, \frac{4}{3}, \frac{5}{6}\right),\ \vec{c} = \left(\frac{5}{3}, \frac{7}{5}, \frac{5}{7}\right),\ z = e^{\frac{i\pi}{4}}$.

Approximation	$N=0$	$N=2$	$N=4$	$N=6$
Power series definition (6.2)	$1.179 \cdot 10^{-1}$	$5.519 \cdot 10^{-3}$	$4.757 \cdot 10^{-4}$	$5.429 \cdot 10^{-5}$
Connection formula: (6.3)+(6.2)	–	–	–	–
Bühring's formula $z_0 = 1/2$	–	–	–	–
1-point expansion $w = 1/2$ (6.5)	$1.081 \cdot 10^{-1}$	$2.746 \cdot 10^{-3}$	$8.254 \cdot 10^{-5}$	$2.695 \cdot 10^{-6}$
2-point expansion $q = 0$ (6.18)	$2.492 \cdot 10^{-2}$	$2.843 \cdot 10^{-5}$	$4.064 \cdot 10^{-8}$	$6.312 \cdot 10^{-11}$
2-point expansion $q = q_0$ (6.18)	$3.416 \cdot 10^{-3}$	$1.276 \cdot 10^{-6}$	$5.058 \cdot 10^{-10}$	$2.032 \cdot 10^{-13}$
3-point expansion $q = 0$ (6.25)	$1.823 \cdot 10^{-3}$	$1.454 \cdot 10^{-8}$	$1.322 \cdot 10^{-13}$	$3.370 \cdot 10^{-15}$
3-point expansion $q = q_1$ (6.25)	$6.574 \cdot 10^{-4}$	$2.401 \cdot 10^{-9}$	$7.157 \cdot 10^{-15}$	$3.265 \cdot 10^{-15}$

Table 6.4: $p = 3,\ a = 1,\ \vec{b} = (2, 3, 1),\ \vec{c} = (2, 3, 4),\ z = \frac{1}{2} e^{\frac{7i\pi}{6}}$.

Approximation	$N=0$	$N=2$	$N=4$	$N=6$
Power series definition (6.2)	$3.748\cdot 10^{-2}$	$2.369\cdot 10^{-3}$	$3.921\cdot 10^{-4}$	$8.987\cdot 10^{-5}$
Connection formula: (6.3)+(6.2)	–	–	–	–
Bühring's formula $z_0=1/2$	$4.515\cdot 10^{-2}$	$2.352\cdot 10^{-3}$	$2.963\cdot 10^{-4}$	$5.377\cdot 10^{-5}$
1-point expansion $w=1/2$ (6.5)	$2.464\cdot 10^{-1}$	$1.715\cdot 10^{-2}$	$1.236\cdot 10^{-3}$	$9.052\cdot 10^{-5}$
2-point expansion $q=0$ (6.18)	$1.200\cdot 10^{-2}$	$3.011\cdot 10^{-5}$	$1.244\cdot 10^{-7}$	$5.885\cdot 10^{-10}$
2-point expansion $q=q_0$ (6.18)	$2.640\cdot 10^{-2}$	$3.445\cdot 10^{-5}$	$4.698\cdot 10^{-8}$	$6.538\cdot 10^{-11}$
3-point expansion $q=0$ (6.25)	$1.797\cdot 10^{-3}$	$6.212\cdot 10^{-8}$	$3.113\cdot 10^{-12}$	$2.047\cdot 10^{-16}$
3-point expansion $q=q_1$ (6.25)	$2.923\cdot 10^{-3}$	$7.305\cdot 10^{-8}$	$1.908\cdot 10^{-12}$	$1.419\cdot 10^{-16}$

Table 6.5: $p=7,\ a=1,\ \vec{b}=\left(\frac{1}{2},\frac{1}{3},\frac{3}{4},\frac{5}{7},\frac{3}{8},\frac{7}{6},\frac{5}{9}\right),\ \vec{c}=\left(\frac{12}{11},\frac{2}{3},\frac{5}{4},\frac{6}{7},\frac{6}{5},\frac{9}{10},\frac{5}{12}\right),\ z=\frac{2}{3}e^{\frac{2i\pi}{3}}.$

Approximation	$N=0$	$N=2$	$N=4$	$N=6$
Power series definition (6.2)	–	–	–	–
Connection formula: (6.3)+(6.2)	$9.630\cdot 10^{-3}$	$1.115\cdot 10^{-4}$	$4.577\cdot 10^{-6}$	$2.786\cdot 10^{-7}$
Bühring's formula $z_0=1/2$	$2.122\cdot 10^{-2}$	$1.591\cdot 10^{-4}$	$2.206\cdot 10^{-6}$	$3.710\cdot 10^{-8}$
1-point expansion $w=1/2$ (6.5)	$5.809\cdot 10^{-1}$	$2.132\cdot 10^{-1}$	$8.042\cdot 10^{-2}$	$3.075\cdot 10^{-2}$
2-point expansion $q=0$ (6.18)	$6.942\cdot 10^{-2}$	$7.813\cdot 10^{-3}$	$1.608\cdot 10^{-3}$	$3.893\cdot 10^{-4}$
2-point expansion $q=q_0$ (6.18)	$1.779\cdot 10^{-1}$	$8.749\cdot 10^{-3}$	$4.497\cdot 10^{-4}$	$2.359\cdot 10^{-5}$
3-point expansion $q=0$ (6.25)	$2.684\cdot 10^{-2}$	$2.381\cdot 10^{-4}$	$3.198\cdot 10^{-6}$	$4.812\cdot 10^{-8}$
3-point expansion $q=q_1$ (6.25)	$1.057\cdot 10^{-1}$	$1.809\cdot 10^{-3}$	$3.248\cdot 10^{-5}$	$5.975\cdot 10^{-7}$

Table 6.6: $p=7,\ a=1,\ \vec{b}=\left(\frac{1}{2},\frac{1}{3},\frac{3}{4},\frac{5}{7},\frac{3}{8},\frac{7}{6},\frac{5}{9}\right),\ \vec{c}=\left(\frac{12}{11},\frac{2}{3},\frac{5}{4},\frac{6}{7},\frac{6}{5},\frac{9}{10},\frac{5}{12}\right),\ z=-2-2i.$

Chapter 7

A Convergent Laplace's Method for Integrals

We consider the Laplace's transform of a smooth enough function $g(t)$

$$F(z) := \int_0^\infty g(t)e^{-zt}dt, \qquad \Re z > x_0 > 0, \tag{7.1}$$

with large $|z|$ (for fixed $\arg z \in (-\pi/2, \pi/2)$). Imagine that we want to obtain an approximation of $F(z)$. On the one hand, we could apply the lemma of Watson introduced in subsection 2.2.1 of chapter 2. On the other hand, we could obtain a uniformly convergent expansion of $F(z)$ by applying the theory of chapter 3 in the situation where $h(t,z) = e^{-zt}$ and the interval of integration is unbounded:

- Following Watson's lemma we consider an expansion of the function $g(t)$ at $t = 0$. If $g(t)$ is analytic at $t = 0$, it has a Taylor series expansion of the form

$$g(t) = \sum_{k=0}^{n-1} a_k t^k + g_n(t), \qquad a_k := \frac{g^{(k)}(0)}{k!}, \tag{7.2}$$

  with $g_n(t) = \mathcal{O}(t^n)$ as $t \to 0$. This expansion is convergent in a certain open disk $D_r(0)$ centered at $t = 0$ of radius $r > 0$ that is, in general, finite. Then, when we replace this expansion into the integral (7.1) and we interchange summation and integration, we obtain

$$F(z) := \int_0^\infty g(t)e^{-zt}dt = \sum_{k=0}^{n-1} a_k \frac{k!}{z^{k+1}} + R_n(z), \tag{7.3}$$

  with

$$R_n(z) := \int_0^\infty e^{-zt} g_n(t)dt = \mathcal{O}\left(z^{-n-1}\right), \qquad \text{as } z \to \infty.$$

  This last order estimate means that the expansion (7.3) is an asymptotic expansion of $F(z)$ for large $|z|$. However, as we know from chapter 2, the expansion (7.3) is usually divergent. This can be regarded as a penalty for interchanging summation and integration when the integration interval $[0,\infty)$ is not contained in the disk of convergence of the Taylor expansion of $g(t)$.[1]

For example, we may consider the function $F(z)=e^z E_1(z)$, where $E_1(z)$ is the exponential integral and derive the expansion (2.15) which is asymptotic for large $|z|$, but it is not convergent for any value of $z\in\mathbb{C}$.

- In order to apply the theory of uniformly convergent expansion of chapter 3 we must perform a logarithmic change of variables in (7.1) to obtain a bounded integration interval:

$$F(z):=\int_0^\infty g(t)e^{-zt}dt=\int_0^1 g(-\log\tau)\tau^{z-1}d\tau. \tag{7.4}$$

Typically, the function $g(-\log\tau)$ is analytic at $\tau=1$ and has a Taylor expansion of the form

$$g(-\log\tau)=\sum_{k=0}^{n-1}A_k(1-\tau)^k+r_n(\tau),\qquad A_k:=\frac{(-1)^k}{k!}\frac{d^k}{d\tau^k}\left[g(-\log\tau)\right]_{\tau=1},$$

that, due to the singularity at $\tau=0$, is convergent in any open disk $D_r(1)$ centered at $\tau=1$ with radius $r\le 1$. In the best scenario when $r=1$ we can apply theorem 3.3.1, case 3. In this situation, the coefficients A_k can also be computed from the Taylor coefficients of $g(t)$ at $t=0$ using formula (A.5) in corollary A.0.2 of appendix A. Using theorem 3.3.1 we can interchange summation and integration and we find

$$F(z):=\int_0^\infty g(t)e^{-zt}dt=\int_0^1 g(-\log\tau)\tau^{z-1}d\tau=\sum_{k=0}^{n-1}A_k\frac{k!}{(z)_{k+1}}+\mathcal{R}_n(z), \tag{7.5}$$

where $(z)_n$ denotes the Pochhammer's symbol and the remainder $\mathcal{R}_n(z)$ is given by

$$\mathcal{R}_n(z)=\int_0^1\tau^{z-1}r_n(\tau)d\tau.$$

From (3.14), the remainder can be bounded by means of $|\mathcal{R}_n(z)|\le M\frac{\Gamma(n+1)\Gamma(z)}{\Gamma(z+n+1)}$ with $M>0$ independent of n and z. Therefore, we have

$$\mathcal{R}_n(z)=\mathcal{O}\left(n^{-z}\right)\text{ as } n\to\infty,\qquad\text{and}\qquad \mathcal{R}_n(z)=\mathcal{O}\left(z^{-n-1}\right)\text{ as } z\to\infty.$$

In other words, expansion (7.5) is convergent and, as the sequence $\{\frac{1}{(z)_{k+1}}\}_{k=0}^\infty$ is an asymptotic scale according to definition (2.10), (7.5) is also an asymptotic expansion of $F(z)$, as $|z|\to\infty$.

If we consider again the function $F(z)=e^zE_1(z)$ and we use the above procedure we find expansion (4.14) in section 4.7 of chapter 4.

In summary, we have derived two asymptotic expansions of the Laplace transform (7.1). On the one hand, the expansion derived by applying Watson's lemma is easier to compute as the asymptotic scale is given by inverse powers of the asymptotic variable, but the expansion is, in general, divergent. Then, it is necessary to study the optimal truncation term and to have an accurate bound for the remainder. In contrast, the

[1]Even if the function $g(t)$ is entire the convergence of (7.3) is not assured.

expansion obtained by applying the theory of chapter 3 is both convergent (and then, the more terms considered the better) and asymptotic for large $|z|$. The main drawback is that the coefficients of the expansion are more difficult to compute as they are the Taylor coefficients of a composite function.

Expansions of the form (7.3) are named *factorial series* [145, §17.3] and they were first studied by Newton and Stirling. Their relation with a compact Mellin transform $\int_0^1 u^{z-1}f(u)du$ was found by Schlömilch [133] although it was Nörlund [99] who showed the neccesary and sufficient conditions for a factorial series to be expressible as such an integral. Moreover, Nielsen [97] showed the relation between factorial series and asymptotic series by using the Stirling numbers [15, §26.8]. A comprehensive study of factorial series was made in [34, 96]. However, in those manuscripts, the starting point of the analysis is the factorial series itself. Therefore, neither formulas for the coefficients of the approximation (7.5) nor error bounds are given. But moreover, the integrals considered are of any of the forms given in (7.5) where the phase function (in the unbounded integral representation of $F(z)$) is simpy $f(t) = t$. Generalizations of asymptotic and convergent series to integrals of the form

$$F(z) := \int_0^\infty e^{-zf(t)}g(t)dt, \tag{7.6}$$

for a more general function $f(t)$, are not considered in the literature.

The aim of the present chapter of this thesis is to fill this gap, that is, to derive a new convergent and asymptotic expansion to the integral (7.6) with general functions $f(t)$ and $g(t)$. We establish the conditions that the functions $f(t)$ and $g(t)$ must satisfy and we estimate the remainder term. Furthermore, we obtain an explicit and simple formula for the coefficients of the new expansion, in contrast to the classical Laplace method, where the formulas for the coefficients of the expansion involve combinatorial objects whose complexity increases with the numbers of terms considered in the expansion [60, 152, 153].

The chapter is organized as follows: First, we derive a convergent and asymptotic expansion for Mellin transforms over a compact interval. This analysis is used in section 7.2 where we consider the integral (7.6) in the particular case $f(t) = t^m$, with $m \in \mathbb{N}$. The main idea is to consider a logarithmic change of variables that transforms the unbounded integration interval $[0, +\infty)$ into the interval $[0, 1)$ considered in section 7.1. We may check that we require the analyticity of a certain function in a large region. Then, in section 7.3 we perform some tricks to enlarge the applicability of the results derived in section 7.2. Finally, in section 7.4 we use the modified Laplace's method described in section 2.2.3 of chapter 2 to reduce the study of integrals (7.6) to the particular case $f(t) = t^m$ analyzed in section 7.2, obtaining a new convergent Laplace's method for integrals. We illustrate the method by deriving a new convergent and asymptotic expansion for a parabolic cylinder function.

The results of the chapter are based on [71].

7.1 A convergent and asymptotic expansion for compact Mellin transforms of analytic functions

The first step is the derivation of an asymptotic and convergent expansion of the compact Mellin transform

$$F(z) := \int_0^\rho (1-x^m)^{z-1} x^{s-1} f(x) dx, \tag{7.7}$$

where $0 < \rho \le 1$, $m \in \mathbb{N}$, $\Re s > 0$ and $\Re z > x_0 > 0$ is large. We also assume that the function $f(x)$ satisfies one of the following two hypotheses:

- Either the function $f(x)$ is analytic inside an open disk of center 0 and radius $r > \rho$,
- Or the function $f(x)$ has an integrable singularity at $x = \rho$ and has no more singularities inside the open disk of center 0 and radius $r = \rho$. More precisely, there exists a number $0 < \sigma \le 1$ such that $f(x) = \mathcal{O}\left((\rho - x)^{\sigma-1}\right)$ as $x \to \rho$. For convergence reasons we also require $x_0 + \sigma > 1$ if $\rho = 1$.

We have the following theorem.

Theorem 7.1.1. *Consider the integral (7.7) with the above mentioned hypotheses on the function $f(x)$. Then, for $n = 1, 2, 3\ldots$*

$$F(z) = \frac{1}{m}\sum_{k=0}^{n-1} \frac{f^{(k)}(0)}{k!} B_{\rho^m}\left(\frac{k+s}{m}, z\right) + R_n(z), \tag{7.8}$$

where $B_y(w,z)$ is the incomplete beta function [116, §8.17]. The remainder $R_n(z)$ can be bounded in the form

$$|R_n(z)| \le \begin{cases} \frac{M}{(\rho_0)^n} B_{\rho^m}\left(\frac{n+\Re s}{m}, \Re z\right) & \text{if } r > \rho, \\ M(z) B\left(n + \Re s, \sigma\right) & \text{if } r = \rho < 1, \\ M(z) B\left(n + \Re s, \Re z + \sigma - 1\right) & \text{if } r = \rho = 1, \end{cases} \tag{7.9}$$

where $B(w,z)$ is the beta function, M is a certain positive constant independent of n and z, and $M(z) > 0$ is independent of n. On the other hand, in the first line of (7.9) we have $\rho_0 := r - \epsilon > \rho$, with ϵ small. Therefore, expansion (7.8) is convergent, with an exponential rate of convergence if $r > \rho$ and a power rate if $r = \rho$. More precisely, as $n \to \infty$,

$$R_n(z) = \begin{cases} \mathcal{O}\left(n^{-1}\left(\rho/\rho_0\right)^n\right) & \text{if } r > \rho,\ \rho < 1, \\ \mathcal{O}\left(n^{-\Re z}\rho_0^{-n}\right) & \text{if } r > \rho = 1, \\ \mathcal{O}\left(n^{-\sigma}\right) & \text{if } r = \rho < 1, \\ \mathcal{O}\left(n^{-(\Re z + \sigma - 1)}\right) & \text{if } r = \rho = 1. \end{cases} \tag{7.10}$$

Expansion (7.8) is also an asymptotic expansion of $F(z)$ for large $|z|$. The terms of the expansion and the remainder term satisfy, as $|z| \to \infty$,

$$B_{\rho^m}\left(\frac{n+s}{m}, z\right) = \mathcal{O}\left(z^{-\frac{n+s}{m}}\right), \qquad R_n(z) = \mathcal{O}\left(z^{-\frac{n+s}{m}}\right), \qquad n = 1, 2, 3, \ldots \tag{7.11}$$

Proof. The key point of the proof is an accurate analysis of the remainder $r_n(x)$ of the Taylor series expansion of $f(x)$ at $x=0$, similar to the one carried out in section 3.2 of chapter 3.

Consider the Taylor series expansion of $f(x)$ at $x=0$:

$$f(x)=\sum_{k=0}^{n-1}\frac{f^{(k)}(0)}{k!}x^k+r_n(x), \qquad |x|<r. \tag{7.12}$$

We distinguish two cases, depending on whether $r>\rho$ or $r=\rho$.

- We assume first that $r>\rho$. Then, the interval of integration $[0,\rho]$ is completely contained in the disk $D_0(r)$ of convergence of the Taylor series expansion of $f(x)$. If we replace $f(x)$ in (7.7) by expansion (7.12) and we interchange summation and integration we get

$$F(z)=\sum_{k=0}^{n-1}\frac{f^{(k)}(0)}{k!}\int_0^\rho(1-x^m)^{z-1}x^{n+s-1}dx+R_n(x), \tag{7.13}$$

with

$$R_n(x):=\int_0^\rho(1-x^m)^{z-1}x^{s-1}r_n(x)dx. \tag{7.14}$$

Then, (7.8) follows from (7.13) by performing the change of variables $x\mapsto y$ given by $x^m=y$ and using the integral definition [116, eq. 8.17.1] of the incomplete beta function.

On the other hand, the remainder $r_n(x)$ can be written in the form

$$r_n(x)=\frac{x^n}{2\pi i}\oint_C\frac{f(w)dw}{w^n(w-x)}, \qquad x\in[0,\rho), \tag{7.15}$$

where C is a simple closed circle around the origin of radius $\rho_0:=r-\epsilon>\rho$ encircling the points $w=0$ and $w=x$ in the positive direction. Then

$$|r_n(x)|\le\frac{x^n}{2\pi}\oint_C\left|\frac{f(w)}{w^n(w-x)}\right|dw=\frac{x^n}{2\pi\rho_0^n}\oint_C\left|\frac{f(w)}{w-x}\right|dw, \qquad x\in[0,\rho). \tag{7.16}$$

As the function $f(x)$ is analytic in the disk $D_0(r)$, its modulus is bounded in C by a positive constant independent of x and n, and so is the factor $1/(w-x)$. Therefore,

$$|r_n(x)|\le\bar{M}\frac{x^n}{\rho_0^n},$$

where $\bar{M}>0$ is a positive constant independent of x and n. Finally, introducing this bound for $r_n(x)$ into (7.14) and integrating, we find the first line of (7.9), with $M:=\bar{M}/m>0$.

- Consider now the case $r=\rho$. The function $f(x)$ may have an integrable singularity at $x=\rho$. We proceed in the same way as we did in the previous point, considering the Taylor series expansion of $f(x)$ at $x=0$ and interchanging summation and integration in (7.7) to find the right hand side of (7.8) with $R_n(z)$ defined in (7.14) and $r_n(x)$ given by the Cauchy's integral (7.15). However, we have to be careful when bounding the remainder $r_n(x)$ as the function $f(x)$ no longer is analytic in the closed disk $D_0(\rho)$ and therefore the integral on the right hand side of (7.16) no longer can be bounded by a positive constant. Nevertheless, we note that the integral (7.15) is a constant function of ϵ that is defined for $\epsilon = 0$ (that is, when we set $\rho_0 = \rho$) and it is a continuous function of ϵ since it is the integral of an integrable function. Thus, formula (7.15) is also valid for $r = \rho$ if we take the limit $\epsilon \to 0$ and we consider that C is a circle of radius $r = \rho$. Moreover, we can perform a translation of the integration variable $w \mapsto v + \rho$ to find

$$r_n(x) = \frac{x^n}{2\pi i} \oint_{\bar{C}} \frac{f(v+\rho)dv}{(v+\rho)^n(v+\rho-x)}, \qquad x \in [0,\rho),$$

where the new integration path $\bar{C}$ is a circle of center $v = -\rho$ and radius ρ: $\bar{C} = \{v \in \mathbb{C} : |v+\rho| = \rho\}$.

By hypothesis, $(\rho - x)^{1-\sigma} f(x)$ is bounded as $x \to \rho$ and then $v^{1-\sigma} f(v+\rho)$ is bounded as $v \to 0$. Consequently, we find

$$\begin{aligned} |r_n(x)| &\leq \frac{x^n}{2\pi} \oint_{\bar{C}} \frac{|v^{1-\sigma} f(v+\rho)|}{|v+\rho|^n} \frac{|v^{\sigma-1}|}{|v+\rho-x|} dv \leq \\ &\leq \bar{M} \frac{x^n}{\rho^n} \oint_{\bar{C}} \frac{|v^{\sigma-1}|}{|v+\rho-x|} dv, \qquad x \in [0,\rho), \end{aligned}$$

with $\bar{M} > 0$ independent of n and x. After the further change of variables $v \mapsto u$ defined by means of $v = (\rho - x)\, u$, with $x \neq \rho$, we find

$$|r_n(x)| \leq \bar{M} \frac{x^n(\rho-x)^{\sigma-1}}{\rho^n} \oint_{\bar{C}/(\rho-x)} \frac{|u|^{\sigma-1}}{|u+1|} du, \tag{7.17}$$

where the integration contour $\bar{C}/(\rho - x)$ is the scaled circle of center $u = -\rho/(\rho - x)$ and radius $\rho/(\rho - x)$ traversed in the positive direction. In other words,

$$\bar{C}/(\rho - x) = \left\{ u \in \mathbb{C} : \left| u + \frac{\rho}{\rho - x} \right| = \frac{\rho}{\rho - x} \right\}.$$

In the limit $x \to \rho$, the scaled circle becomes the imaginary axis traversed upwards and the integral along this path in formula (7.17) is finite. Therefore, the right hand side of (7.17) can be bounded in the form

$$|r_n(x)| \leq \tilde{M} \frac{x^n(\rho-x)^{\sigma-1}}{\rho^n}, \qquad x \in [0,\rho),$$

for a certain $\tilde{M} > 0$ independent of n and x. Introducing this bound for the remainder $r_n(x)$ into (7.14) we get

$$|R_n(z)| \leq \frac{\tilde{M}}{\rho^n} \int_0^\rho (1-x^m)^{\Re z - 1} x^{n+\Re s-1} (\rho - x)^{\sigma-1} dx. \tag{7.18}$$

In order to find (7.9) we distinguish two further subcases, depending on whether $\rho < 1$ or $\rho = 1$.

– If $\rho < 1$ we use the bound $(1-x^m)^{z-1} \leq \bar{M}(z) := \max\{1, (1-\rho^m)^{\Re z-1}\}$, independent of x, and we find

$$|R_n(z)| \leq \frac{\tilde{M}\bar{M}(z)}{\rho^n}\int_0^\rho x^{n+\Re s-1}(\rho-x)^{\sigma-1}dx = \tilde{M}\bar{M}(z)\rho^{\Re s+\sigma-1}B\left(n+\Re s,\sigma\right) = \\ = M(z)B\left(n+\Re s,\sigma\right),$$

with $M(z) := \rho^{\Re s+\sigma-1}\tilde{M}\bar{M}(z)$, and we obtain the second line of (7.9).

– If $\rho = 1$ we find

$$|R_n(z)| \leq \tilde{M}\int_0^1 x^{n+\Re s-1}(1-x)^{\Re z+\sigma-2}(1+x+\ldots+x^{m-1})^{\Re z-1}dx.$$

When $\Re z > 1$ we have the bound $(1+x+\ldots+x^{m-1})^{\Re z-1} \leq m^{\Re z-1}$ whereas when $\Re z < 1$ we find $(1+x+\ldots+x^{m-1})^{\Re z-1} \leq 1$. Therefore, $(1+x+\ldots+x^{m-1})^{\Re z-1} \leq \tilde{m}(z) := \max\{1, m^{\Re z-1}\}$, independent of n, and then

$$|R_n(z)| \leq \tilde{M}\tilde{m}(z)\int_0^1 x^{n+\Re s-1}(1-x)^{\Re z+\sigma-2}dx = M(z)B\left(n+\Re s, \Re z+\sigma-1\right),$$

with $M(z) := \tilde{M}\tilde{m}(z)$ and we deduce the third line of (7.9).

The bounds (7.9) show that expansion (7.8) is convergent. More precisely, using the asymptotic behavior of the complete or incomplete beta functions [4, eqs. 5.12.1 and 5.11.12], [116, eqs. 8.17.2 and 8.18.1] involved in formula (7.9) we obtain the convergence rate (7.10) of expansion (7.8).

It remains to show the asymptotic features of the expansion (7.8). On the one hand, the sequence $F_k(z) := B_{\rho^m}\left(\frac{k+s}{m}, z\right)$ is an asymptotic scale according to definition (2.10). Indeed, using the relations [116, eqs. 8.17.2 and 8.17.4] for the incomplete beta function and its asymptotic behavior [116, eq. 8.18.1] or the asymptotic behavior of the (complete) beta function [4, eqs. 5.12.1 and 5.11.12] we have that

$$\frac{F_{k+1}(z)}{F_k(z)} = \mathcal{O}\left(z^{-1/m}\right) \qquad \text{as } z \to \infty.$$

On the other hand, in the case $r > \rho$, from the first line of (7.9) we find, using again the asymptotic behavior of the incomplete beta function [116, eqs. 8.17.2 and 8.17.4], that

$$R_n(z) = \mathcal{O}\left(z^{-\frac{n+s}{m}}\right) \qquad \text{as } z \to \infty,$$

which gives (7.11).

For $r = \rho$ we have to be more careful. In this case, from (7.18) we have

$$|R_n(x)| \leq \frac{\bar{M}}{\rho^n}\int_0^\rho (1-x^m)^{\Re z-1}x^{n+\Re s-1}(\rho-x)^{\sigma-1}dx.$$

For any $0 < \varepsilon < 1$ we split the above integral in the form

$$\int_0^\rho (1-x^m)^{\Re z-1}x^{n+\Re s-1}(\rho-x)^{\sigma-1}dx = \int_0^{\varepsilon\rho}(1-x^m)^{\Re z-1}x^{n+\Re s-1}(\rho-x)^{\sigma-1}dx + \\ + \int_{\varepsilon\rho}^\rho (1-x^m)^{\Re z-1}x^{n+\Re s-1}(\rho-x)^{\sigma-1}dx. \tag{7.19}$$

In the first integral on the right hand side of (7.19) we use that $(\rho-x)^{\sigma-1}\le[\rho(1-\varepsilon)]^{\sigma-1}$, for all $x\in[0,\varepsilon\rho]$. In the second integral we use that, for $\Re z>1$, $(1-x^m)^{\Re z-1}\le(1-(\varepsilon\rho)^m)^{\Re z-1}$ for all $x\in[\varepsilon\rho,\rho]$. Then

$$\int_0^\rho(1-x^m)^{\Re z-1}x^{n+\Re s-1}(\rho-x)^{\sigma-1}dx\le[\rho(1-\varepsilon)]^{\sigma-1}\int_0^{\varepsilon\rho}(1-x^m)^{\Re z-1}x^{n+\Re s-1}dx+$$
$$+[1-(\varepsilon\rho)^m]^{\Re z-1}\int_{\varepsilon\rho}^{\rho}x^{n+\Re s-1}(\rho-x)^{\sigma-1}dx.$$

The last integral above can be bounded by a positive constant, say K, independent of z, and the first integral is an incomplete beta function. Therefore

$$\int_0^\rho(1-x^m)^{\Re z-1}x^{n+\Re s-1}(\rho-x)^{\sigma-1}dx\le$$
$$\le\frac{[\rho(1-\varepsilon)]^{\sigma-1}}{m}B_{(\varepsilon\rho)^m}\left(\frac{n+\Re s}{m},\Re z\right)+K\left[1-(\varepsilon\rho)^m\right]^{\Re z-1}.$$

For large $|z|$, the second term on the right hand side above is exponentially small compared to the first one, that is of the order $\mathcal{O}\left(z^{-(n+s)/m}\right)$. Therefore, the asymptotic behavior (7.11) for the remainder $R_n(z)$ also follows in the case $r=\rho$ and expansion (7.8) is not only convergent, but asymptotic, as $|z|\to\infty$.

$\square$

Example 2. Consider the integral representation of the Bessel $J_\nu(z)$ function [107, eq. 10.9.4], given by

$$J_\nu(z)=\frac{2(z/2)^\nu}{\pi^{1/2}\Gamma(\nu+1/2)}\int_0^1(1-x^2)^{\nu-1/2}\cos(zx)dx,$$

for large ν. To obtain a convergent and asymptotic expansion for large ν of the function $J_\nu(z)$ we can apply theorem 7.1.1 with $m=2$ and $f(x):=\cos(zx)$. The function $f(x)$ is entire and then $r>\rho:=1$. We also have $f^{(2k+1)}(0)=0$ and $f^{(2k)}(0)=(-1)^kz^{2k}$, for all $k=0,1,2,\ldots$. Therefore, using approximation (7.8) we find

$$J_\nu(z)=\frac{2(z/2)^\nu}{\pi^{1/2}\Gamma(\nu+1/2)}\left[\frac{1}{2}\sum_{k=0}^\infty\frac{(-1)^kz^{2k}}{(2k)!}B\left(\frac{2k+1}{2},\nu+\frac{1}{2}\right)\right]$$
$$=\frac{(z/2)^\nu}{\pi^{1/2}}\sum_{k=0}^\infty\frac{(-1)^kz^{2k}}{\Gamma(2k+1)}\frac{\Gamma(k+\frac{1}{2})}{\Gamma(k+\nu+1)}$$

Using the duplication formula [4, eq. 5.5.5] for the gamma function $\Gamma(2k+1)$ and after some simplifications we find the power series definition of $J_\nu(z)$ [107, eq. 10.2.2], which is a convergent series that is also asymptotic for large $|\nu|$.

7.2 A convergent version of a generalized Watson's lemma

The second step of our analysis is the derivation of an asymptotic and convergent expansion of generalized Laplace's transforms. For the sake of generality we consider the integration interval $(0,b)$ with b finite or infinite and we allow a branch point at the end point $t = 0$ of the integration interval. Moreover, if b is finite we also allow a branch point at the other end point $t = b$ whereas if the integration interval is unbounded we let an exponential growth of the integrand at infinity. More precisely, we consider generalized Laplace's transforms of the form

$$F(z) := \int_0^b e^{-zt^m} t^{s-1} (1 - t/b)^{\sigma-1} h(t) dt, \tag{7.20}$$

with $m \in \mathbb{N}$, $\Re s > 0$, $0 < \Re\sigma \leq 1$, $0 < b \leq \infty$ and large $|z|$ with $|\arg z| < \pi/2$. That is, we assume that $\Re z \geq x_0$ for some fixed $x_0 > 0$. Moreover, in the case $b = \infty$ we assume that the factor $(1 - t/b)^{\sigma-1}$ is replaced by 1 and that $h(t)$ satisfies some growth condition at ∞ to ensure the convergence of the integral (7.20).

The main idea to obtain a convergent and asymptotic approximation of the integral transform $F(z)$ given in (7.20) is to reduce the study to one of the cases analyzed in theorem 7.1.1 by performing a logarithmic change of variables. More precisely, we take

$$t = x \left[-\frac{\log(1 - x^m)}{x^m} \right]^{1/m}, \tag{7.21}$$

where $t > 0$ for $x > 0$. The inverse of the transformation is given by

$$x = t \left[\frac{1 - e^{-t^m}}{t^m} \right]^{1/m}, \tag{7.22}$$

with $x > 0$ for $t > 0$. As we will see below, this change of variables transforms the integral (7.20) to one of the form considered in theorem 7.1.1. Moreover, in lemma A.0.1 of appendix A we can find a formula for the composite function after the change of variables (7.21).

Before deriving an expansion for the integral $F(z)$ on the right hand side of (7.20), we must give some insight in the transformation (7.21)-(7.22). To this end, we define the following region:

Definition. For any $0 < b \leq \infty$ and $m \in \mathbb{N}$, we define the open complex region

$$S_m(b, 0) := \left\{ t \in \mathbb{C} : \left| 1 - e^{-t^m} \right| < \rho \right\}, \qquad \rho := 1 - e^{-b^m} \leq 1, \tag{7.23}$$

where only the branch that contains the real interval $(0, b)$ is considered (see figure 7.1). (The "extra" argument 0 in the notation of $S_m(b, 0)$ will be clear later). Observe that $\rho < 1$ when $0 < b < \infty$ and $\rho = 1$ if and only if $b = \infty$. Moreover, for any $b_1 < b_2$ we have that $S_m(b_1, 0) \subset S_m(b_2, 0) \subset S_m(\infty, 0)$.

The importance of the region $S_m(b, 0)$ in the following analysis is that its image under the transformation (7.21) is a disk of center $x = 0$ and radius $\rho^{1/m}$.

Now, we obtain a convergent and asymptotic expansion for integrals of the form (7.20). To this end, we assume one of the following hypotheses for the function $h(t)$:

- Either $b < +\infty$ and the function $h(t)$ is analytic in $S_m(b_0, 0)$ for some $b_0 > b$,
- Or the function $h(t)$ is analytic in $S_m(b, 0)$ and, if $b = \infty$, $h(t)$ is of exponential order at infinity: $h(t) = \mathcal{O}\left(e^{\alpha t^m}\right)$ when $t \to \infty$, with $0 < \alpha < \min\{1, x_0\}$.

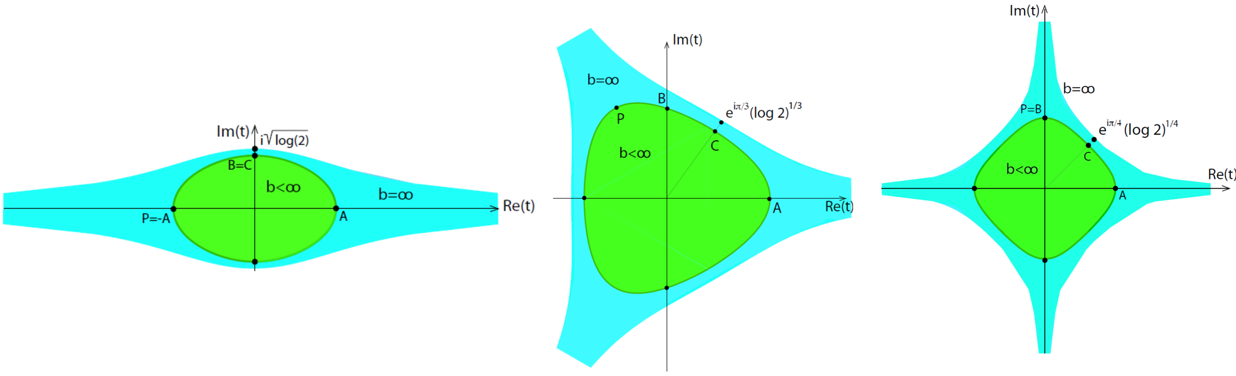


Figure 7.1: Region $S_m(b,0)$ for $m=2$ (left), $m=3$ (middle) and $m=4$ (right) with $b=\infty$ (larger blue unbounded regions, including green areas) and finite b (inner green bounded figures). For any value of b, finite or infinite, the boundary of these figures is comprised by the curve $t(\theta)$ parametrized in the form $t(\theta)=[-\log(1-\rho e^{im\theta})]^{1/m}$, $-\pi<\theta\le\pi$, selecting the continuous branch for which $t>0$ for $\theta=0$; that is, $t(\theta)=|\log(1-\rho e^{im\theta})|^{1/m}e^{i\theta}$, $-\pi<\theta\le\pi$. In these figures we have highlighted four points of the boundary of $S_m(b,0)$: $A:=t(0)=[-\log(1-\rho)]^{1/m}>0$, $B:=t(\pi/2)=[-\log(1-\rho e^{im\pi/2})]^{1/m}=i|B|$, $C:=t(\pi/m)=[-\log(1+\rho)]^{1/m}=e^{i\pi/m}|C|$, $P:=t(2\pi/m)=e^{2i\pi/m}A$. In this figures we have depicted these points for $\rho<1$ (green region with $b<\infty$). When $b\to\infty$ ($\rho\to 1$) we have that $A\to+\infty$, $P\to e^{2i\pi/m}\infty$ and $C\to(\log 2)^{1/m}e^{i\pi/m}$ $\forall\, m$. Roughly speaking, the region $S_m(b,0)$ is a circle around the origin of radius $|C|=|\log(1+\rho)|^{1/m}$ spiked along the m rays determined by the $m-$th roots of the unity $e^{2\pi ik/m}$, $k=0,1,\ldots,m-1$ (the point A is on the first ray, the point P is on the second one, the point C is on the middle angle between A and P). For $b=\infty$ the (blue + green) regions are unbounded, as those spikes go up to the infinity. For $b<\infty$ the (green) regions are similar, but bounded, as those spikes are bounded, and contained in $S_m(\infty,0)$: observe that $S_m(b_1,0)\subset S_m(b_2,0)$ for $b_1<b_2$.

Then, we have the following theorem.

Theorem 7.2.1. *Consider the integral (7.20) with the above analytic hypotheses on the function $h(t)$. Then, for $n=1,2,3,\ldots$*

$$F(z)=\frac{1}{m}\sum_{k=0}^{n-1}A_kB_\rho\left(\frac{k+s}{m},z\right)+R_n(z), \tag{7.24}$$

where $\rho:=1-e^{-b^m}\le 1$ and A_k are the Taylor coefficients of the function

$$\tilde{h}(x):=\left(-\frac{\log(1-x^m)}{x^m}\right)^{\frac{s}{m}-1}\left(1-\frac{x}{b}\left[-\frac{\log(1-x^m)}{x^m}\right]^{1/m}\right)^{\sigma-1}h\left(x\left[-\frac{\log(1-x^m)}{x^m}\right]^{1/m}\right) \tag{7.25}$$

at $x=0$; assuming that, when $b=\infty$, the middle factor is replaced by 1. The Taylor coefficients A_k can be computed either directly using an algebraic manipulator or by means of the following formula (see lemma A.0.1)

$$A_k=\sum_{j=0}^{\lfloor\frac{k}{m}\rfloor}\frac{(-1)^j}{j!}\frac{B_j^{\left(\frac{k+s}{m}\right)}(1)}{(k-jm)!}\frac{d^{k-jm}}{dt^{k-jm}}\left[\left(1-\frac{t}{b}\right)^{\sigma-1}h(t)\right]_{t=0}, \tag{7.26}$$

where $B_n^{(\alpha)}(x)$ are the generalized Bernoulli polynomials of order α [36, §24.16], [88, Ch. VI], and the factor $(1-t/b)^{\sigma-1}$ is replaced by 1 if $b=\infty$.

The remainder term $R_n(z)$ is bounded in the form

$$|R_n(z)| \leq \begin{cases} \frac{M}{\rho_0^n} B_\rho\left(\frac{n+\Re s}{m}, \Re z\right) & \text{if } b < +\infty \text{ and } \sigma = 1, \\ M(z) B\left(n + \Re s, \Re\sigma\right) & \text{if } b < +\infty \text{ and } \sigma \neq 1, \\ M(z) B\left(n + \Re s, \Re z - \alpha\right) & \text{if } b = +\infty, \end{cases} \tag{7.27}$$

for a certain $\rho_0 > \rho > 0$, $M > 0$ independent of z and n and $M(z) > 0$ independent of n. Therefore, expansion (7.24) is convergent, with an exponential rate of convergence for $b < +\infty$ and $\sigma = 1$, and a power rate otherwise. More precisely, as $n \to \infty$,

$$R_n(z) = \begin{cases} \mathcal{O}\left(n^{-1}(\rho/\rho_0)^n\right) & \text{if } b < +\infty \text{ and } \sigma = 1, \\ \mathcal{O}\left(n^{-\sigma}\right) & \text{if } b < +\infty \text{ and } \sigma \neq 1, \\ \mathcal{O}\left(n^{-(\Re z - \alpha)}\right) & \text{if } b = +\infty. \end{cases} \tag{7.28}$$

Expansion (7.24) is also an asymptotic expansion of $F(z)$ for large $|z|$: The terms of the expansion (7.24) and the remainder satisfy

$$B_\rho\left(\frac{k+s}{m}, z\right) = \mathcal{O}\left(z^{-\frac{k+s}{m}}\right) \qquad R_n(z) = \mathcal{O}\left(z^{-\frac{n+s}{m}}\right), \qquad \text{as } z \to \infty \tag{7.29}$$

Proof. We consider the change of variables $t \mapsto x$ given in (7.21). Under this transformation, the region $S_m(b, 0)$ for the variable t is transformed into the open disk of center $x = 0$ and radius $\rho^{1/m}$ for the variable x, where $\rho := 1 - e^{-b^m} \leq 1$. The integral (7.20) is transformed into

$$F(z) = \int_0^{\rho^{1/m}} (1 - x^m)^{z-1} x^{s-1} \tilde{h}(x) dx, \tag{7.30}$$

with $\tilde{h}(x)$ defined in (7.25). The analyticity hypotheses on the function $h(t)$ guarantee that the function $\tilde{h}(x)$ satisfies one of the two analyticity conditions for the function $f(x)$ in theorem 7.1.1. Then, we can apply theorem 7.1.1 to this integral with $f(x)$ replaced by $\tilde{h}(x)$ to find the approximation (7.24). The formula for the Taylor coefficients A_k follows from lemma A.0.1 with $\lambda = \frac{s}{m} - 1$ and $\phi(t) = \left(1 - \frac{t}{b}\right)^{\sigma-1} h(t)$ and the bounds for the remainder are a consequence of (7.9) in theorem 7.1.1. More precisely we find:

- When $b < +\infty$ and $\sigma = 1$, we have $\rho < 1$ and the function $\tilde{h}(x)$ is analytic in a certain open disk $D_0(r)$ of radius $r > \rho_0^{1/m} > \rho^{1/m}$, where $\rho_0 := 1 - e^{-b_0^m}$ and the first line of (7.27) follows from the first line of (7.9).

- If $b < +\infty$ and $\sigma \neq 1$ we have that $\rho < 1$ and the function $\tilde{h}(x)$ has a branch point at $\rho^{1/m} < 1$. Therefore, the function $\tilde{h}(x)$ is analytic in the open disk $D_0(\rho^{1/m})$ and the third line of (7.9) implies the second line of the bound (7.27).

- In the case $b = +\infty$ we have $\rho = 1$ and the middle factor of $\tilde{h}(x)$ in (7.25) is replaced by 1. The function $\tilde{h}(x)$ is analytic in the open unit disk $D_0(1)$ and satisfies the growth condition $\tilde{h}(x) = \mathcal{O}\left((x-1)^{-\alpha}\right)$ as $x \to 1$. Therefore, the fourth line of (7.9) implies the third line of the bound (7.27) with σ replaced by $1 - \alpha$.

□

To illustrate the applicability of theorem 7.2.1 we consider a function that will be useful in the next sections to derive a convergent and asymptotic expansion of the parabolic cylinder function [144].

Example 3. Consider the function

$$\bar{U}_1(a,z) := \int_0^1 (1-t)^{a-1/2} e^{-zt^2} dt, \qquad \Re a > -1/2.$$

It has the form considered in (7.20) with $b = 1$ and $m = 2$. To obtain a convergent and asymptotic expansion of $\bar{U}_1(a,z)$ by applying theorem 7.2.1 we identify $s = 1$ (there is no branch point at $t = 0$), $\sigma = a - 1/2 - \lfloor \Re a - 1/2 \rfloor$ and $h(t) = (1-t)^{a-\sigma+1/2} = (1-t)^{\lfloor \Re a - 1/2 \rfloor + 1}$. This definition of σ guarantees that $0 < \Re\sigma \leq 1$. Besides the function $h(t)$ is analytic in the complex region $S_2(1,0)$ and we can apply theorem 7.2.1 to find the expansion

$$\bar{U}_1(a,z) = \frac{1}{2} \sum_{k=0}^{\infty} A_k B_{1-e^{-1}} \left(\frac{k+1}{2}, z \right), \tag{7.31}$$

where the coefficients A_k are the Taylor coefficients at $x = 0$ of the function

$$\tilde{h}(x) = \left[-\frac{\log(1-x^2)}{x^2} \right]^{-\frac{1}{2}} \left[1 - x \left(-\frac{\log(1-x^2)}{x^2} \right)^{\frac{1}{2}} \right]^{a-\frac{1}{2}}$$

that can be computed, using formula (7.26), in the form

$$A_k = \sum_{j=0}^{\lfloor \frac{k}{2} \rfloor} \frac{(-1)^j}{j!} \frac{B_j^{\left(\frac{k+1}{2}\right)}(1)}{(k-2j)!} \frac{d^{k-2j}}{dt^{k-2j}} \left[(1-t)^{a-1/2} \right]_{t=0} = \sum_{j=0}^{\lfloor \frac{k}{2} \rfloor} \frac{(-1)^j}{j!} \frac{B_j^{\left(\frac{k+1}{2}\right)}(1)}{(k-2j)!} \left(\frac{1}{2} - a \right)_{k-2j}.$$

The first few coefficients are

$$A_0 = 1; \quad A_1 = \frac{1}{2} - a; \quad A_2 = \frac{1}{2}\left(a^2 - 2a + \frac{1}{4} \right); \quad A_3 = \frac{-1}{6}\left(a^3 - \frac{9}{2}a^2 + \frac{23}{4}a - \frac{15}{8} \right).$$

Table 7.1 contains a numerical experiment about approximation (7.31) that shows its convergent and asymptotic character for large $|z|$.

n	$z = 0.5$	$z = 2$	$z = 10$	$z = 30$	$z = 100$
0	1.26	$8.18474 \cdot 10^{-1}$	$3.0011 \cdot 10^{-1}$	$1.54866 \cdot 10^{-1}$	$7.93323 \cdot 10^{-2}$
5	$8.46202 \cdot 10^{-3}$	$3.1662 \cdot 10^{-3}$	$5.92959 \cdot 10^{-5}$	$1.44491 \cdot 10^{-6}$	$2.77723 \cdot 10^{-8}$
10	$3.15303 \cdot 10^{-3}$	$1.06383 \cdot 10^{-3}$	$6.64822 \cdot 10^{-6}$	$1.88215 \cdot 10^{-8}$	$2.5038 \cdot 10^{-11}$
15	$1.34204 \cdot 10^{-3}$	$4.25023 \cdot 10^{-4}$	$1.27042 \cdot 10^{-6}$	$4.3148 \cdot 10^{-10}$	$7.86635 \cdot 10^{-14}$
20	$7.06936 \cdot 10^{-4}$	$2.15652 \cdot 10^{-4}$	$4.16676 \cdot 10^{-7}$	$2.36881 \cdot 10^{-11}$	$4.40583 \cdot 10^{-14}$
25	$4.2018 \cdot 10^{-4}$	$1.24947 \cdot 10^{-4}$	$1.81634 \cdot 10^{-7}$	$2.31511 \cdot 10^{-12}$	$4.38895 \cdot 10^{-14}$

Table 7.1: Relative error [143, eq. 3.1.9] provided by the right hand side of (7.31) when we truncate the series after n terms, for different values of z and $a = 1.8$.

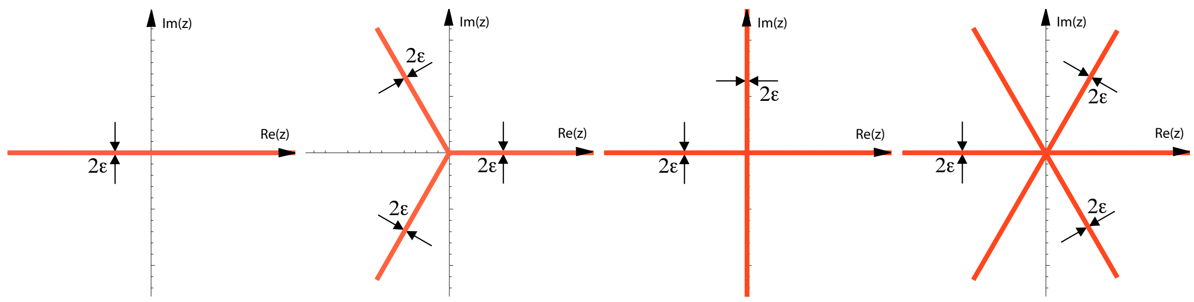


Figure 7.2: Region $s_m(b,0)$ for $m=2$ (left), $m=3$ (middle left), $m=4$ (middle right) and $m=6$ (right). In every picture, the m rays in the directions $e^{2ik\pi/m}$, $k=0,1,2,...,m-1$, have width 2ε and length $b\varepsilon/\left|\sqrt[m]{\log 2}\right|$.

7.3 Some magic tricks

Theorem 7.2.1 requires the analyticity of the function $h(t)$ in the region $S_m(b,0)$ defined in (7.23) and depicted in figure 7.1 for $m=2,3,4$. From a practical point of view, this region may be too large as integral representations of many special functions do not satisfy this requirement. For example, consider the function

$$\bar{U_2}(a,z) := \int_0^\infty (t+1)^{a-1/2} e^{-zt^2} dt, \qquad \Re a > -1/2, \qquad \Re z > 0 \tag{7.32}$$

that we will approximate later and is related with the parabolic cylinder function [144]. The function $h(t)=(t+1)^{a-1/2}$ possesses a branch point at $t=-1$ that belongs to the region $S_2(\infty,0)$ (see figure 7.1 left) and then theorem 7.2.1 cannot be applied. However, as we will see below, after some manipulations, the analyticity requirement in theorem 7.2.1 for the function $h(t)$ can be relaxed, requiring its analyticity in a smaller region and enlarging in this way the range of applicability of theorem 7.2.1.

In order to understand what manipulation we need, we first give some insight into the geometry of the transformation $t\mapsto x$ defined in (7.21) with inverse given in (7.22). The key point is that, after the change of variables $t\mapsto x$, the original integration interval $[0,b)$ is transformed into a new integration interval in (7.30): $x\in[0,\rho^{1/m})\subset[0,1)$ for all $b>0$, where $\rho := 1-e^{-b^m}$. Expansion (7.24) follows after a Taylor expansion of the function $\tilde{h}(x)$ given in (7.25) at $x=0$ and an interchange of summation and integration in (7.30). Then, the resulting expansion (7.24) is convergent whenever $\tilde{h}(x)$ is analytic in the disk $D_0(\rho^{1/m})$, that contains the integration interval. But $\tilde{h}(x)$ is analytic in that disk whenever $h(t)$ is analytic in the image of the disk $D_0(\rho^{1/m})$ under the transformation (7.22), that is the region $S_m(b,0)$.

Then, the trick to relax the analyticity condition of theorem 7.2.1 is a dilatation of the original integration variable $t\in[0,b)$ that squeezes the region $S_m(b,0)$ into a smaller region $s_m(b,0)$. (Again, the "extra" argument 0 in the notation will be clear in a moment.) Such a region $s_m(b,0)$ may be a starlike domain with center at $t=0$ and with m arbitrarily narrow spikes of width 2ε and length $b\varepsilon/\left|\sqrt[m]{\log 2}\right|$ in the directions defined by the m-th roots of the unity, with $\varepsilon>0$ arbitrarily small (see Figure 7.2).

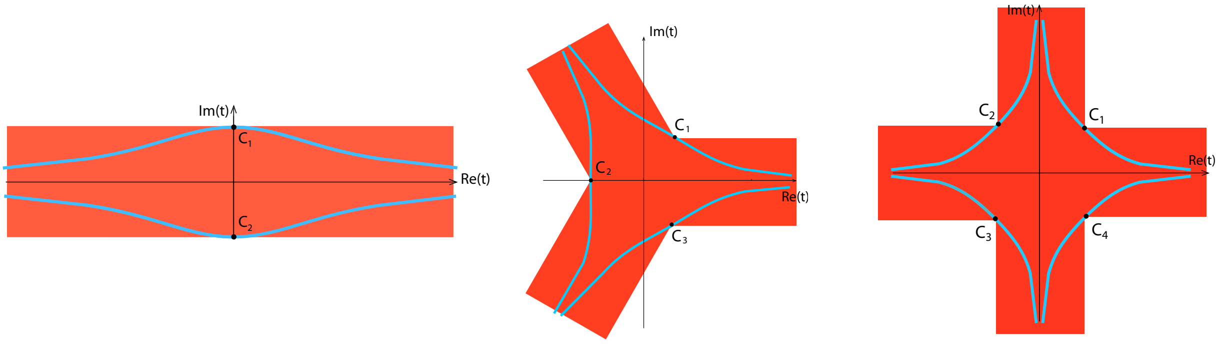


Figure 7.3: The thick red star-like regions $\Lambda s_m(\infty,0)$ contain the corresponding regions $S_m(\infty,0)$ limited by the blue curves; for $m=2$ (left), $m=3$ (middle) and $m=4$ (right). In these figures C_1, C_2,..., C_m are the m different $m-$th roots of $-\log(2)$, that is, $C_k=|C|e^{i(2k+1)\pi/m}$, $k=0,1,2,...,m-1$, $|C|:=\left|\sqrt[m]{\log(2)}\right|$ and $\Lambda=\left|\sqrt[m]{\log(2)}\right|/\varepsilon$ for the given $\varepsilon>0$ used in the definition (7.33) of $s_m(b,0)$.

$$s_m(b,0):=\left\{t\in\mathbb{C}:t=(r+iy)e^{2i\pi k/m};0\le r<\frac{b\varepsilon}{|\sqrt[m]{\log 2}|};k=0,1,\ldots,m-1;|y|<\varepsilon\right\}. \tag{7.33}$$

Now, consider a certain *dilatation parameter* $\Lambda>1$. After the dilatation $u\mapsto t$ given by $t=\Lambda u$, the region $s_m(b,0)$ for the variable u becomes the larger region $\Lambda s_m(b,0)$ for the new variable t. And the region $\Lambda s_m(b,0)$ contains the region $S_m(b,0)$ for any $b>0$ whenever $\varepsilon\Lambda$ is larger than the distance from the origin $t=0$ to the complementary of the region $S_m(\infty,0)$. That distance is attained at the intersecion of the boundary of $S_m(\infty,0)$ with the rays $t=re^{i(2k+1)\pi/m}$, $r>0$, $k=0,1,\ldots,m-1$, which occurs at a distance $C=\left|\sqrt[m]{\log 2}\right|$ (see figure 7.3).

Therefore, the region $\Lambda s_m(b,0)$ contains the region $S_m(b,0)$ for any $b>0$ whenever $\Lambda>\left|\sqrt[m]{\log 2}\right|/\varepsilon$. Then

$$F(z):=\int_0^b e^{-zu^m}u^{s-1}\left(1-\frac{u}{b}\right)^{\sigma-1}h(u)du=\frac{1}{\Lambda^s}\int_0^{\Lambda b}e^{-(z/\Lambda^m)t^m}t^{s-1}\left(1-\frac{t}{\Lambda b}\right)^{\sigma-1}\bar{h}(t)dt,$$

with $\bar{h}(t):=h(t/\Lambda)$. Whenever $h(u)$ is analytic in $s_m(b,0)$, $\bar{h}(t)$ is analytic in $S_m(b,0)\subset\Lambda s_m(b,0)$ and theorem 7.2.1 may be applied to this last integral with z replaced by z/Λ^m and h replaced by $\bar{h}$.

The key point of why the dilatation trick discussed above to enlarge the applicability of theorem 7.2.1 works is hidden in the shape of the region $S_m(\infty,0)$: any ray $\arg t=\theta$ will eventually cut the boundary of the region $S_m(\infty,0)$ unless $\theta=2k\pi/m$, $k=0,1,\ldots,m-1$. Then, for sufficiently large Λ, the dilatation $u\mapsto t=\Lambda u$ will move the singularities of $h(t)$ away from the region $S_m(b,0)$ except for those singularities located in the axes of the starlike region $s_m(b,0)$ as they are invariant under the dilatation no matter how large Λ is. Nevertheless, in the case $b=\infty$ we can still use another trick to avoid those singularities: an appropriate rotation of the original integration path $[0,\infty)$ whenever the function $h(t)$ is analytic in that sector. The effect of this rotation is a rotation of the axes so that we can avoid the singularities located there. Then, choose an angle θ such that $|\arg z+m\theta|<\pi/2$ (ideally $\theta=-\frac{\arg z}{m}$ if $\arg z\neq 0$). If $e^{-\alpha t^m}h(t)$ is bounded as $t \to \infty$ in the sector $\arg t \in [0, \theta]$ (and not only for $t > 0$ as it was required in the hypothesis of theorem 7.2.1) and $h(t)$ is analytic in that sector, we can invoke Cauchy's theorem to rotate the path of integration that angle θ. Then

$$F(z) = e^{is\theta} \int_0^\infty e^{-ze^{im\theta}t^m} t^{s-1} h(e^{i\theta}t)dt.$$

Now, the singularities of $h(e^{i\theta}t)$ are not located on the rays $\arg t = 2k\pi/m$, $k = 0, 1, \ldots, m-1$. Therefore, we can (if necessary) perform a dilatation $\Lambda > 1$ and apply theorem 7.2.1 afterwards.

We can combine both tricks to enlarge the range of applicability of theorem 7.2.1. The result is summarized in the following theorem.

Theorem 7.3.1. *Consider the integral*

$$F(z) := \int_0^b e^{-zt^m} t^{s-1} (1-t/b)^{\sigma-1} h(t)dt, \tag{7.34}$$

with $m \in \mathbb{N}$, $\Re s > 0$, $0 < \Re\sigma \le 1$, $0 < b \le +\infty$ and large $|z|$, that is, $\Re z \ge x_0 > 0$, assuming that the factor $(1-t/b)^{\sigma-1}$ is replaced by 1 if $b = +\infty$. Assume also one of the following hypotheses for the function $h(t)$:

- *Whether the function $h(t)$ is analytic in the starlike region $s_m(b, 0)$ defined in (7.33) and, if $b = +\infty$, the function $e^{-\alpha t^m} h(t)$ is bounded as $t \to +\infty$, for a certain $\alpha < \min\{1, x_0\}$.*

- *Or $b = +\infty$ and there exists an angle θ with $|\arg z + m\theta| < \pi/2$ such that $e^{-\alpha t^m} h(t)$ is bounded as $t \to \infty$ in the sector $\arg t \in [0, \theta]$ for a certain $\alpha < \min\{1, x_0\}$, and the function $h(t)$ is analytic in the sector $\arg t \in [0, \theta]$ and also in the starlike region $s_m(\infty, 0)$ defined in (7.33) rotated an angle θ:*

  $$s_m(\infty, \theta) := \left\{ t \in \mathbb{C} : t = (r+iy)e^{i(\theta + 2\pi k/m)}; r \ge 0; k = 0, 1, \ldots, m-1; |y| < \varepsilon, \right\} \tag{7.35}$$

  for a certain $\varepsilon > 0$.

Then, for $n = 1, 2, 3, \ldots$

$$F(z) = \frac{1}{m\Lambda^s} \sum_{k=0}^{n-1} A_k B_{\rho(\Lambda)} \left(\frac{k+s}{m}, \frac{z}{\Lambda^m} \right) + R_n(z), \tag{7.36}$$

for any $\Lambda \in \mathbb{C}$ with $|\Lambda| > \left| \sqrt[m]{\log 2} \right| / \varepsilon$ and $\arg(\Lambda) = \theta$, with $\theta = 0$ if $h(t)$ is analytic in $s_m(b, 0)$. The parameter $\rho(\Lambda)$ is defined by $\rho(\Lambda) = 1 - e^{-(|\Lambda| b)^m}$ and A_k are the Taylor coefficients of

$$\tilde{h}(x) := \left[-\frac{\log(1-x^m)}{x^m} \right]^{\frac{s}{m}-1} \left(1 - \frac{x}{b\Lambda} \left[-\frac{\log(1-x^m)}{x^m} \right]^{\frac{1}{m}} \right)^{\sigma-1} h\left(\frac{x}{\Lambda} \left[-\frac{\log(1-x^m)}{x^m} \right]^{\frac{1}{m}} \right)$$

at $x = 0$, with the middle factor replaced by 1 when $b = +\infty$. These Taylor coefficients A_k can be computed either, directly using an algebraic manipulator, or by means of the following formula (see lemma A.0.1):

$$A_k = \sum_{j=0}^{\lfloor \frac{k}{m} \rfloor} \frac{(-1)^j}{j!} \frac{B_j^{\left(\frac{k+s}{m}\right)}(1)}{(k-mj)!\,\Lambda^{k-mj}} \frac{d^{k-mj}}{dt^{k-mj}} \left[\left(1 - \frac{t}{b} \right)^{\sigma-1} h(t) \right]_{t=0}, \tag{7.37}$$

where the factor $(1-t/b)^{\sigma-1}$ *is replaced by* 1 *if* $b=+\infty$ *and* $B_k^{(\alpha)}(x)$ *are the generalized Bernoulli polynomials of order* α.

The remainder term $R_n(z)$ *can be bounded in the form*

$$|R_n(z)| \leq \begin{cases} \frac{M}{\rho_0^n} B_{\rho(\Lambda)}\left(\frac{n+\Re s}{m}, \Re\left(\frac{z}{\Lambda^m}\right)\right) & \text{if } b<+\infty \text{ and } \sigma=1, \\ M(z)B\left(n+\Re s, \Re\sigma\right) & \text{if } b<+\infty \text{ and } \sigma\neq 1, \\ M(z)B\left(n+\Re s, \Re\left(\frac{z}{\Lambda^m}\right)-\alpha\right) & \text{if } b=+\infty, \end{cases} \tag{7.38}$$

for a certain $\rho_0>\rho>0$, $M>0$ *independent of* z *and* n *and* $M(z)>0$ *independent of* n. *Therefore, expansion* (7.36) *is convergent, with an exponential rate of convergence for* $b<+\infty$ *and* $\sigma=1$, *and a power rate otherwise. More precisely, as* $n\to\infty$,

$$R_n(z) = \begin{cases} \mathcal{O}\left(n^{-1}(\rho/\rho_0)^n\right) & \text{if } b<+\infty \text{ and } \sigma=1, \\ \mathcal{O}\left(n^{-\sigma}\right) & \text{if } b<+\infty \text{ and } \sigma\neq 1, \\ \mathcal{O}\left(n^{-(\Re\left(\frac{z}{\Lambda^m}\right)-\alpha)}\right) & \text{if } b=+\infty. \end{cases} \tag{7.39}$$

Expansion (7.36) *is also an asymptotic expansion of* $F(z)$ *for large* $|z|$.

Proof. Let $\Lambda\in\mathbb{C}$ such that $|\Lambda| > \left|\sqrt[m]{\log 2}\right|/\varepsilon$ and $\arg\Lambda=\theta$. We perform a change of variables given by $t\mapsto\frac{t}{\Lambda}$. The effect of this change of variables is a dilatation and a rotation in the integral (7.34) in the manner discussed before theorem 7.3.1. Then, we find

$$F(z) = \frac{1}{\Lambda^s}\int_0^{\Lambda b} e^{-\left(\frac{z}{\Lambda^m}\right)t^m} t^{s-1}\left(1-\frac{t}{\Lambda b}\right)^{\sigma-1} h\left(\frac{t}{\Lambda}\right)dt. \tag{7.40}$$

In the case $b<+\infty$ we have $\arg\Lambda=0$ whereas in the case $b=\infty$ and $\arg\Lambda\neq 0$ we invoke Cauchy's theorem to rotate the integration path to the real interval $[0,|\Lambda|b)=[0,\infty)$. As the function $h(t)$ is analytic in the region $s_m(b,\theta)$, after the dilatation and rotation the function $h\left(\frac{t}{\Lambda}\right)$ is analytic in the region $S_m(b,0)$ and we can apply theorem 7.2.1 to the integral on the right hand side of (7.40). As a result we find expansion (7.36) and the bound (7.38) for the remainder $R_n(z)$. □

Example 4. Consider the function $\bar{U_2}(a,z)$ defined in (7.32) and given by

$$\bar{U_2}(a,z) := \int_0^\infty (t+1)^{a-1/2}e^{-zt^2}dt, \qquad \Re a>-1/2, \qquad \Re z>0.$$

Comparing with (7.34) we identify $m=2$, $b=+\infty$, $s=1$ and $h(t)=(t+1)^{a-1/2}$. The function $h(t)$ has a branch point at $t=-1$, which is inside the region $S_2(\infty,0)$. Then, we can not apply theorem 7.2.1 to obtain a convergent an asymptotic expansion of $\bar{U_2}(a,z)$ for large z. Nevertheless, the function $h(t)$ is analytic in the starlike region $s_2(\infty,\theta)$ defined in (7.33) with $\varepsilon=\sin\theta$ and we can apply theorem 7.3.1 if we perform a certain rotation and dilatation. Take $\Lambda\in\mathbb{C}$ with $\arg\Lambda\neq 0$ satisfying $|\arg z+2\arg\Lambda|<\pi/2$ or $|\arg z-2\arg\Lambda|<\pi/2$ and, for the chosen value of $\arg\Lambda$, take $|\Lambda|>|\log 2|/\sin(\arg\Lambda)$. Then, we can apply theorem 7.3.1 to obtain a convergent expansion of $\bar{U_2}(a,z)$ that is also an asymptotic expansion of $F(z)$ for large $|z|$. As $b=\infty$ we find $\rho=1$ and approximation (7.36) in this example reads

$$\bar{U}_2(a,z)=\frac{1}{2\Lambda}\sum_{k=0}^{\infty}A_kB\left(\frac{k+1}{2},\frac{z}{\Lambda^2}\right). \tag{7.41}$$

The coefficients A_k are the Taylor coefficients of the function

$$\tilde{h}(x):=\left(-\frac{\log(1-x^2)}{x^2}\right)^{-\frac{1}{2}}\left[1+\frac{x}{\Lambda}\left(-\frac{\log(1-x^2)}{x^2}\right)^{\frac{1}{2}}\right]^{a-\frac{1}{2}}$$

at $x=0$. They can be computed using formula (7.37) in the form

$$A_k=\sum_{j=0}^{\lfloor\frac{k}{2}\rfloor}\frac{(-1)^j}{j!}\frac{B_j^{\left(\frac{k+1}{2}\right)}(1)}{(k-2j)!}\frac{\left(\frac{1}{2}-a\right)_{k-2j}}{\Lambda^{k-2j}},$$

where we have used that $\frac{d^n}{dt^n}\Big[h(t)\Big]_{t=0}=(-1)^n\left(\frac{1}{2}-a\right)_n$. The first few coefficients are

$$A_0=1,\qquad A_1=\frac{a-1/2}{\Lambda},\qquad A_2=\frac{(1/2-a)_2}{2\Lambda^2}-\frac{1}{4},\qquad A_3=\frac{-(1/2-a)_3}{6\Lambda^3}.$$

We can choose, for example, $\arg\Lambda=\pi/6$ and $|\Lambda|=2$. With this election of Λ we obtain the numerical experiment detailed in table 7.2 , that shows the convergent and asymptotic character of expansion (7.41).

n	$z=0.5$	$z=2$	$z=30$	$z=100$	$z=250$
0	$6.07615\cdot10^{-1}$	$4.77225\cdot10^{-1}$	$1.84414\cdot10^{-1}$	$1.08389\cdot10^{-1}$	$7.08001\cdot10^{-2}$
5	$1.31198\cdot10^{-1}$	$3.68296\cdot10^{-2}$	$1.21413\cdot10^{-4}$	$3.98426\cdot10^{-6}$	$2.72387\cdot10^{-7}$
10	$1.18385\cdot10^{-1}$	$2.50816\cdot10^{-2}$	$4.67633\cdot10^{-6}$	$7.70153\cdot10^{-9}$	$4.13833\cdot10^{-11}$
15	$1.14372\cdot10^{-1}$	$2.18815\cdot10^{-2}$	$1.08835\cdot10^{-6}$	$2.78716\cdot10^{-10}$	$2.68262\cdot10^{-13}$
20	$1.10831\cdot10^{-1}$	$1.92642\cdot10^{-2}$	$2.4277\cdot10^{-7}$	$9.90124\cdot10^{-12}$	$6.95367\cdot10^{-15}$

Table 7.2: Relative error [143, eq. 3.1.9] provided by the right hand side of (7.41) when we truncate the series after n terms, for different values of z and $a=2.6$. We have taken $\Lambda=2e^{i\pi/6}$.

7.4 A convergent and asymptotic Laplace's method

Finally, we consider integrals of the form

$$F(z):=\int_0^{\infty}e^{-zf(t)}g(t)t^{a-1}dt,\qquad \Re a>0,\qquad \Re z>x_0>0 \text{ large}. \tag{7.42}$$

The main idea to obtain a convergent and asymptotic Laplace's method for integrals is to use the idea of the modified Laplace's method introduced in [74] and summarized in section 2.2.3 of chapter 2, that is, to split the phase function $f(t)$ into its asymptotically dominant monomial (as $|z|\to\infty$) and a subdominant remainder. Then, we obtain a monomial in the exponent of the exponential and theorems 7.2.1 or 7.3.1 may be applied. We clearly state the hypotheses H7.4.(i)-H7.4.(vi) that the functions $f(t)$ and $g(t)$ must satisfy in order for the new Laplace's method for integrals to be convergent and asymptotic, as $|z|\to\infty$.

H7.4.(i). *The function $f(t)$ is real for real t.*

H7.4.(ii). *The function $f(t)$ in (7.42) has only one absolute minimum in the positive real axis at a certain point $t_0 \geq 0$.*

Then, there exists a number $m \in \mathbb{N}$ such that $f^{(m)}(t_0) > 0$ and $f^{(k)}(t_0) = 0$ for all $k = 1, 2, \ldots, m-1$, with m even if $t_0 > 0$. Following the ideas of the modified Laplace method introduced in [74] we consider the Taylor polynomial of $f(t)$ at $t = t_0$ of degree m:

$$p(t) := f(t_0) + \eta(t - t_0)^m, \qquad \eta := \frac{f^{(m)}(t_0)}{m!} > 0.$$

Roughly speaking, $p(t)$ is the asymptotically dominant monomial of the phase function $f(t)$, as $|z| \to \infty$ whereas the remainder term $f_m(t) := f(t) - p(t)$ is subdominant:

$$f_m(t) := f(t) - p(t) = \sum_{k=p}^{\infty} \frac{f^{(k)}(t_0)}{k!}(t - t_0)^k, \qquad |t - t_0| < r,$$

where $r > 0$ is the radius of convergence of the Taylor series of $f(t)$ at $t = t_0$ and $p > m$ is the first derivative of $f(t)$ after the $m-$th derivative that does not vanish at $t = t_0$.

H7.4.(iii). *Both function $f(t_0 - t)$ and $g(t_0 - t)$ are analytic in the region $t_0 - S_m(t_0, 0)$, with $S_m(t_0, 0)$ defined in (7.23).*

H7.4.(iv). *For a certain angle θ satisfying $|\arg z + m\theta| < \pi/2$, both functions $f(t_0 + t)$ and $g(t_0 + t)$ are analytic in the sector $\arg(t_0 + t) \in [0, \theta]$ and also in the starlike region $t_0 + s_m(\infty, \theta)$, with $s_m(\infty, \theta)$ defined in (7.33).*

H7.4.(v). *For t in the sector $\arg(t_0 + t) \in [0, \theta]$, there exists a number $0 < \alpha < \min\{1, x_0\}$ such that the function $e^{-\alpha t^m} e^{-z f_m(t_0+t)} g(t_0 + t)$ is bounded as $t \to \infty$.*

H7.4.(vi). *The function $e^{-zf(t)}g(t)t^{a-1}$ is absolutely integrable on $[0, \infty)$, for $\Re z > x_0$ large.*

Remark 7.4.1. *Hypotheses H7.4.(i), H7.4.(ii) and H7.4.(vi) are similar to the hypotheses required for the functions $f(t)$ and $g(t)$ in order to apply the classical method of Laplace for integrals (see subsection 2.2.2) whereas hypotheses H7.4.(iii), H7.4.(iv) and H7.4.(v) will guarantee the convergence of the expansion.*

Then, according to hypothesis H7.4.(ii) the integral (7.42) can be written in the form

$$F(z) = e^{-zf(t_0)} \int_0^{\infty} e^{-z\eta(t-t_0)^m} h(t, z) t^{a-1} dt, \qquad h(t, z) := e^{-zf_m(t)} g(t), \tag{7.43}$$

where the exponent of the exponential function consists only of the asymptotically relevant monomial of the phase function, as $|z| \to \infty$. On the other hand, the subdominant part $f_m(t)$ is included, together with $g(t)$, in the function $h(t, z)$. We split the integral (7.43) at $t = t_0$ to find

$$F(z) = e^{-zf(t_0)} \int_0^{t_0} e^{-z\eta(t-t_0)^m} h(t, z) t^{a-1} dt + e^{-zf(t_0)} \int_{t_0}^{\infty} e^{-z\eta(t-t_0)^m} h(t, z) t^{a-1} dt.$$

If $t_0 = 0$ the first integral vanishes whereas if $t_0 > 0$, m is even and we can perform a shift in the integration variable of the form $t_0 - t \mapsto t$. On the other hand, no matter the value of t_0, in the second integral above we perform a shift of the form $t - t_0 \mapsto t$. We get

$$F(z) = e^{-zf(t_0)} \int_0^{t_0} e^{-z\eta t^m} h(t_0 - t, z)(t_0 - t)^{a-1} dt + e^{-zf(t_0)} \int_0^{\infty} e^{-z\eta t^m} h(t + t_0, z)(t + t_0)^{a-1} dt.$$

With the aim of treating both integrals above at the same time, we define $b^- := t_0$ and $b^+ = +\infty$ and we define the functions

$$F^{\pm}(z) := \int_0^{b^{\pm}} e^{-z\eta t^m} h\left(t_0 \pm t, z\right)\left(t_0 \pm t\right)^{a-1} dt. \tag{7.44}$$

Then

$$F(z) = e^{-zf(t_0)}\left[F^-(z) + F^+(z)\right]. \tag{7.45}$$

Now, hypothesis H7.4.(iii) assures that we can apply theorem 7.2.1 to the function $F^-(z)$ whereas hypotheses H7.4.(iv) and H7.4.(v) guarantee that we can perform a dilatation and a rotation and apply theorem 7.3.1 to $F^+(z)$. But, the function h considered in theorems 7.2.1 and 7.3.1 does not depend on the asymptotic variable z and now the function $h(t,z)$ in the integrand of both integrals $F^{\pm}(z)$ does. Therefore, we are introducing a new ingredient in the analysis: the function h and therefore its Taylor coefficients A_k do depend on the asymptotic variable z.

On the one hand, this dependence on the variable z does not have any influence on the convergence of expansion (7.24) or (7.36) of theorems 7.2.1 and 7.3.1 and the corresponding convergence rate for the remainder $R_n(z)$ of those expansions (7.28) and (7.39), as $n \to \infty$, remain valid. On the other hand, that dependence on z does have an effect on the asymptotic behavior of those expansions and formulas (7.11) are no longer valid. That is, when the function h depends on z the asymptotic features as $|z| \to \infty$ of the new expansion must be proved.

The situation is similar to the one analyzed in [74] as the function $h(t,z)$ also depends on the asymptotic variable z. This dependence is translated to the Taylor coefficients. However, the asymptotic behavior, as $|z| \to \infty$, of the coefficients A_k is known and it does not disrupt the asymptotic character of the expansion.

We have the following result, based on the fact that the $n-$coefficient of the Taylor expansion at $t = t_0$ of the function $h(t,z)$ defined in (7.43) is a polynomial in z of degree $\lfloor n/p \rfloor$, where $p > m$ denotes the first non-vanishing derivative of $f(t)$ at $t = t_0$ after the $m-$derivative (see subsection 2.2.3).

Lemma 7.4.2. *Let $f(t)$ and $g(t)$ be analytic functions at $t = t_0$ and define the function $h(t,z) := e^{-zf_m(t)} g(t)$ given in (7.43) with $f_m(t)$ defined in hypothesis H7.4.(ii). For any $m \in \mathbb{N}$ and $\Lambda, \lambda, \mu \in \mathbb{C}$, consider the function*

$$\begin{aligned}\tilde{h}(x,z) := &\left[-\frac{\log(1-x^m)}{x^m}\right]^{\frac{\lambda}{m}-1}\left(t_0 + \frac{x}{\Lambda}\left[-\frac{\log(1-x^m)}{x^m}\right]^{\frac{1}{m}}\right)^{\mu-1} \times \\ &\times h\left(t_0 + \frac{x}{\Lambda}\left[-\frac{\log(1-x^m)}{x^m}\right]^{\frac{1}{m}}, z\right).\end{aligned} \tag{7.46}$$

Then, the $n-$th Taylor coefficient $A_n(z)$ of $\tilde{h}(x,z)$ at $x=0$ is a polynomial in z of degree $\lfloor \frac{n}{p} \rfloor$, where $p>m$ is the first non-vanishing derivative of $f(t)$ at $t=t_0$ after the $m-$th derivative. This implies that $A_n(z)=\mathcal{O}\left(z^{\lfloor \frac{n}{p} \rfloor}\right)$ as $|z|\to\infty$.

Proof. The Taylor coefficients $A_n(z)$ of $\tilde{h}(x,z)$ can be computed using formula (A.2) in lemma A.0.1. Then, in the computation of the $n-$th coefficient $A_n(z)$, the variable z only appears in the derivatives of the function $h(t,z)$. But it has been shown in [74] that the $k-$th Taylor coefficient of $h(t,z)$ at $t=t_0$ is in fact a polynomial in z of degree $\lfloor \frac{k}{p} \rfloor$. As the range of the index of summation in (A.2) goes from $k=0$ to $k=\lfloor \frac{n}{p} \rfloor$, we conclude that the Taylor coefficients $A_n(z)$ of $\tilde{h}(x,z)$ at $x=0$ are also polynomials in z of degree $\lfloor \frac{n}{p} \rfloor$. □

Finally, we give the main result of this chapter of the thesis: a convergent and asymptotic expansion of the integral (7.42) with an explicit formula for the coefficients of the expansion and with estimates for the remainder. The main idea to derive it is to write the integral (7.42) in the form (7.44)-(7.45) and apply theorem 7.2.1 to $F^-(z)$ and theorem 7.3.1 to $F^+(z)$. However, as the function $h(t,z)$ depends on the asymptotic variable z we must prove that the new expansion is still asymptotic, as $|z|\to\infty$. The proof of this property is technical and somewhat long and tedious. We have:

Theorem 7.4.3. *Consider the integral (7.42) and assume hypotheses H7.4.(i)-H7.4.(vi) given above with the function $h(t,z)$ defined in (7.43) and the parameters m,t_0 and η given in hypothesis H7.4.(ii). Then, for $n=1,2,3,\dots$*

$$F(z)=e^{-zf(t_0)}\times \left\{\frac{1}{m}\sum_{k=0}^{n-1}\left[A_k^-(z)B_\rho\left(\frac{k+1}{m},\eta z\right)+\frac{A_k^+(z)}{(\Lambda^+)^{\lambda^+}}B\left(\frac{k+\lambda^+}{m},\frac{\eta z}{(\Lambda^+)^m}\right)\right]+R_n(z)\right\}, \tag{7.47}$$

where $\rho:=1-e^{-t_0^m}$, $\lambda^+=1$ if $t_0>0$ or $\lambda^+=a$ if $t_0=0$, $|\Lambda^+|>\left|\sqrt[m]{\log 2}\right|/\varepsilon$ and $\arg(\Lambda^+)=\theta$. *The coefficients $A_k^\pm(z)$ are, respectively, the Taylor coefficients of the function*

$$\tilde{h}^\pm(x,z):=\left[-\frac{\log(1-x^m)}{x^m}\right]^{\frac{\lambda^\pm}{m}-1}\left(t_0\pm\frac{x}{\Lambda^\pm}\left[-\frac{\log(1-x^m)}{x^m}\right]^{\frac{1}{m}}\right)^{\mu^\pm-1}\times h\left(t_0\pm\frac{x}{\Lambda^\pm}\left[-\frac{\log(1-x^m)}{x^m}\right]^{\frac{1}{m}},z\right) \tag{7.48}$$

at $x=0$, with $\Lambda^-=\lambda^-=1$, $\mu^-=a$ and either $\mu^+=a$ if $t_0>0$ or $\mu^+=1$ if $t_0=0$. These coefficients $A_k^\pm(z)$ can be computed using the formula

$$A_n^\pm(z)=\sum_{k=0}^{\lfloor \frac{n}{m} \rfloor}\frac{(-1)^k B_k^{(\frac{n+\lambda^\pm}{m})}(1)}{(\Lambda^\pm)^{n-km}\,k!}\frac{(\pm1)^{n-km}}{(n-km)!}\frac{d^{n-km}}{dt^{n-km}}\left[t^{\mu^\pm-1}h(t,z)\right]_{t=t_0}, \tag{7.49}$$

where $B_n^{(\alpha)}(x)$ are the generalized Bernoulli polynomials [36, §24.16], [88, Ch. VI].

The remainder $R_n(z)$ is bounded by the sum of the second line of (7.27) with $s = 1$ and the third line of (7.27) with $s = \lambda^+$. Therefore, expansion (7.47) is convergent. It is also an asymptotic expansion of $F(z)$ for large $|z|$: On the one hand, the terms of the expansion between brackets inside the sum (7.47) are of order $\mathcal{O}\left(z^{\frac{k}{p}-\frac{k+1}{m}} + z^{\frac{k}{p}-\frac{k+\lambda^+}{m}}\right)$, as $|z| \to \infty$. On the other hand, the remainder $R_n(z) = \mathcal{O}\left(z^{\frac{n}{p}-\frac{n+1}{m}} + z^{\frac{n}{p}-\frac{n+\lambda^+}{m}}\right)$, as $|z| \to \infty$, with p given in H7.4.(ii).

Proof. We split the phase function $f(t)$ in the form given in hypothesis H7.4.(ii), and we write the integral (7.42) in the form (7.44)-(7.45). Using hypothesis H7.4.(iii) we can apply theorem 7.2.1 to $F^-(z)$ with $b = t_0$, $s = 1$, $\sigma = a - \lceil \Re a \rceil + 1$ and $h(t)$ replaced by $t_0^{a-1} h(t_0 - t, z)(1 - t/t_0)^{a-\sigma} = t_0^{a-1} h(t_0 - t, z)(1 - t/t_0)^{\lceil \Re a \rceil - 1}$ and z replaced by ηz. On the other hand, using hypotheses H7.4.(iv) and H7.4.(v) we can apply theorem 7.3.1 to $F^+(z)$ with $b = +\infty$, $\sigma = 1$, z replaced by ηz and either $s = 1$ with $h(t)$ replaced by $h(t_0 + t, z)(t_0 + t)^{a-1}$ if $t_0 > 0$, or $s = a$ and $h(t)$ replaced by $h(t_0 + t, z)$ if $t_0 = 0$. In this way, we find the expansion (7.47) being $A_k^{\pm}(z)$ the Taylor coefficients of $\tilde{h}^{\pm}(x, z)$ at $x = 0$. Formula (7.49) for the Taylor coefficients $A_n^{\pm}(z)$ follows from lemma A.0.1 with $\lambda = \frac{\lambda^{\pm}}{m} - 1$ and $\phi(t) = \left(t_0 \pm \frac{t}{\Lambda^{\pm}}\right)^{\mu^{\pm}-1} h(t_0 \pm \frac{t}{\Lambda^{\pm}})$.

The remainder $R_n(z)$ is given by the sum of the remainders $R_n^-(z)$ and $R_n^+(z)$ given by theorems 7.2.1 and 7.3.1 for the functions $F^-(z)$ and $F^+(z)$ respectively, that is,

$$F^{\pm}(z) = \frac{1}{m} \sum_{k=0}^{n-1} A_k^{\pm}(z) \Phi_k^{\pm}(\rho^{\pm}; z) + R_n^{\pm}(z), \tag{7.50}$$

with $\rho^- = \rho = 1 - e^{-t_0^m}$ and $\rho^+ = 1$, and

$$\Phi_k^{\pm}(c; z) := \frac{1}{(\Lambda^{\pm})^{\lambda^{\pm}}} B_c\left(\frac{k + \lambda^{\pm}}{m}, \frac{\eta z}{(\Lambda^{\pm})^m}\right), \qquad k = 0, 1, 2, \dots \qquad 0 \le c \le 1. \tag{7.51}$$

Therefore, the convergence character of expansion (7.47) comes from (7.27) and (7.38). In other words, the dependence on z of the function $h(t, z)$ does not spoil the convergence character of the expansion. However, we must prove that it does not spoil its asymptotic feature either.

First, we show that the terms between brackets inside the sum of (7.47) form an asymptotic sequence, as $z \to \infty$: From lemma 7.4.2 the coefficients $A_k^{\pm}(z) = \mathcal{O}\left(z^{\frac{k}{p}}\right)$, as $|z| \to \infty$. On the other side, from the asymptotic behavior of the beta and incomplete beta functions [4, eqs. 5.12.1 and 5.11.12] and [116, eqs. 8.17.2, 8.17.4 and 8.18.3] we find that the terms of the expansion between brackets inside the sum in (7.47) are of the order $\mathcal{O}\left(z^{\frac{k}{p}-\frac{k+\lambda^+}{m}} + z^{\frac{k}{p}-\frac{k+1}{m}}\right)$, as $|z| \to \infty$. In other words, they form an asymptotic sequence that is not of Poincaré type, but that decreases in the form of a sawtooth.

Secondly, we have to prove that the remainder $R_n(z)$ satisfies the estimate given in the theorem, as $|z| \to \infty$. The key point to prove this is to split both integrals $F^{\pm}(z)$ at the point $t = |z^{-1/p}|$, for $|z| > x_0$ large enough. That is, we write $F^{\pm}(z) = F_0^{\pm}(z) + F_1^{\pm}(z)$, with

$$F_0^{\pm}(z) = \int_0^{|z^{-1/p}|} e^{-z\eta t^m} h(t_0 \pm t, z)(t_0 \pm t)^{a-1} dt, \qquad F_1^{\pm}(z) = \int_{|z^{-1/p}|}^{b^{\pm}} e^{-z\eta t^m} h(t_0 \pm t, z)(t_0 \pm t)^{a-1} dt. \tag{7.52}$$

We remark that, for $|z|$ large enough, we have that $|z^{-1/p}| \leq t_0$ for any positive t_0.

Now, the integral $F_0^-(z)$ is of the form of $F^-(z)$ with t_0 replaced by $|z^{-1/p}|$, and the same applies to the integral $F_0^+(z)$ if we also replace $h(t_0 - t, z)(t_0 - t)^{a-1}$ by $h(t_0 + t, z)(t_0 + t)^{a-1}$. Therefore, we can apply theorems 7.2.1 or 7.3.1 to these two integrals and we get

$$F_0^{\pm}(z) = \frac{1}{m} \sum_{k=0}^{n-1} A_k^{\pm}(z) \Phi_k^{\pm}(\rho_z; z) + R_{n,0}^{\pm}(z), \qquad \rho_z := 1 - e^{-|z^{-m/p}|}, \tag{7.53}$$

with $\Phi_k^{\pm}(c; z)$ given in (7.51) and

$$R_{n,0}^{\pm}(z) := \int_0^{(\rho_z)^{1/m}} (1 - x^m)^{\frac{\eta z}{(\Lambda^{\pm})^m} - 1} x^{\lambda^{\pm} - 1} r_n^{\pm}(x, z) dx, \tag{7.54}$$

where $r_n^{\pm}(x, z)$ is the $n-$th order Taylor remainder of $\tilde{h}^{\pm}(x, z)$ at $x = 0$.

Then, on the one hand, according to the splitting described above, we have

$$F^{\pm}(z) = F_0^{\pm}(z) + F_1^{\pm}(z) = \frac{1}{m} \sum_{k=0}^{n-1} A_k^{\pm}(z) \Phi_k^{\pm}(\rho_z; z) + R_{n,0}^{\pm}(z) + F_1^{\pm}(z). \tag{7.55}$$

On the other hand, we have formula (7.50). Then, from (7.50) and (7.55) we find that

$$R_n^{\pm}(z) = \Psi_n^{\pm}(z) + R_{n,0}^{\pm}(z) + F_1^{\pm}(z), \tag{7.56}$$

with

$$\Psi_n^{\pm}(z) := \frac{1}{m} \sum_{k=0}^{n-1} A_k^{\pm}(z) [\Phi_k^{\pm}(\rho^{\pm}; z) - \Phi_k^{\pm}(\rho_z; z)]. \tag{7.57}$$

In the remaining of the proof we study the asymptotic behavior of every one of the three terms $\Psi_n^{\pm}(z)$, $R_{n,0}^{\pm}(z)$ and $F_1^{\pm}(z)$ on the right hand side of formula (7.56), in order to find out the asymptotic behavior of $R_n^{\pm}(z)$, and then of $R_n(z) = R_n^+(z) + R_n^-(z)$.

- $\Psi_n^{\pm}(z)$. Note that the arguments of the two incomplete beta functions on the right hand side of (7.57) (see (7.51)) are the same, the incomplete beta functions only differ in their index. Then, taking into account the integral representation of the incomplete beta function [4, eq. 8.17.1] we find that

$$\Phi_k^{\pm}(\rho^{\pm}; z) - \Phi_k^{\pm}(\rho_z; z) = \frac{1}{(\Lambda^{\pm})^{\lambda^{\pm}}} \int_{\rho_z}^{\rho^{\pm}} t^{\frac{k+\lambda^{\pm}}{m} - 1} (1 - t)^{\frac{\eta z}{(\Lambda^{\pm})^m} - 1} dt.$$

Hence

$$\left|\Phi_k^{\pm}(\rho^{\pm}; z) - \Phi_k^{\pm}(\rho_z; z)\right| \leq \frac{1}{|(\Lambda^{\pm})^{\lambda^{\pm}}|} \int_{1-e^{-|z^{-m/p}|}}^{1} t^{\frac{k+\Re\lambda^{\pm}}{m} - 1} (1 - t)^{\Re\left(\frac{\eta z}{(\Lambda^{\pm})^m}\right) - 1} dt.$$

Performing the change of variables $t \mapsto u$ defined by $t = 1 - e^{-|z^{-m/p}|}u$ we find

$$\left|\Phi_k^{\pm}(\rho^{\pm}; z) - \Phi_k^{\pm}(\rho_z; z)\right| \le \frac{1}{|(\Lambda^{\pm})^{\lambda^{\pm}}|}\left(e^{-|z^{-m/p}|}\right)^{\Re\left(\frac{\eta z}{(\Lambda^{\pm})^m}\right)} \times$$
$$\int_0^1 (1 - e^{-|z^{-m/p}|}u)^{\frac{k+\Re\lambda^{\pm}}{m}-1} u^{\Re\left(\frac{\eta z}{(\Lambda^{\pm})^m}\right)-1} du \le \frac{1}{|(\Lambda^{\pm})^{\lambda^{\pm}}|}\frac{1}{\Re\left(\frac{\eta z}{(\Lambda^{\pm})^m}\right)} e^{-\Re\left(\frac{\eta z}{(\Lambda^{\pm})^m}\right)|z^{-\frac{m}{p}}|}. \tag{7.58}$$

Thus, from the asymptotic behavior of the coefficients $A_k^{\pm}(z)$ (see lemma 7.4.2) and formulas (7.57) and (7.58) we have that, for all $k = 0, 1, \ldots, n-1$,

$$\Psi_n^{\pm}(z) = \mathcal{O}\left(z^{\lfloor n/p \rfloor - 1} e^{-\Re\left(\frac{\eta z}{(\Lambda^{\pm})^m}\right)|z^{-\frac{m}{p}}|}\right), \qquad \text{as } |z| \to \infty. \tag{7.59}$$

- $F_1^{\pm}(z)$. We have that $f(t_0 \pm t) - f(t_0) = \eta t^m + \mathcal{O}(t^p)$ as $t \to 0^+$ with $\eta > 0$, and therefore $\exists\ \delta^{\pm} > 0$ independent of t (and of course of z) such that

$$f(t_0 \pm t) - f(t_0) - \frac{\eta}{2}t^m > 0 \quad \text{for } 0 < t < \delta^{\pm}. \tag{7.60}$$

On the other hand, as t_0 is an absolute minimum of $f(t)$ in $[0, \infty)$, there exist $\epsilon^{\pm} > 0$ such that

$$f(t_0 \pm t) - f(t_0) - \epsilon^{\pm} > 0 \quad \text{for } t \ge \delta^{\pm}. \tag{7.61}$$

For large enough $|z|$ we have that $\delta^{\pm} > |z^{-1/p}|$ and we can split the integral $F_1^{\pm}(z)$ at $t = \delta^{\pm}$. We write

$$F_1^{\pm}(z) = G_1^{\pm}(z) + G_2^{\pm}(z),$$

with

$$G_1^{\pm}(z) := \int_{|z^{-1/p}|}^{\delta^{\pm}} e^{-z\eta t^m} e^{-z f_m(t_0 \pm t)} g(t_0 \pm t)(t_0 \pm t)^{a-1} dt,$$

$$G_2^{\pm}(z) := \int_{\delta^{\pm}}^{b^{\pm}} e^{-z\eta t^m} e^{-z f_m(t_0 \pm t)} g(t_0 \pm t)(t_0 \pm t)^{a-1} dt.$$

Using $f(t_0 \pm t) = f(t_0) + \eta t^m + f_m(t_0 \pm t)$ and (7.60) we find

$$\begin{aligned}
|G_1^{\pm}(z)| &\le \int_{|z^{-1/p}|}^{\delta^{\pm}} |e^{-z[f(t_0\pm t) - f(t_0) - \eta t^m/2]}|\,|e^{-z\eta t^m/2}|\,|g(t_0 \pm t)(t_0 \pm t)^{a-1}| dt \\
&\le \int_{|z^{-1/p}|}^{\delta^{\pm}} e^{-\eta t^m \Re z/2} |g(t_0 \pm t)(t_0 \pm t)^{a-1}| dt \le \bar{K}\int_{|z^{-1/p}|}^{\delta^{\pm}} e^{-\eta t^m \Re z/2} \\
&\le \bar{K}\int_{|z^{-1/p}|}^{\infty} e^{-\eta t^m \Re z/2} = \frac{K}{(\Re z)^{1/m}}\Gamma\left(\frac{1}{m}, \frac{\eta \Re z}{2|z^{\frac{m}{p}}|}\right),
\end{aligned}$$

with K and $\bar{K}$ positive constants independent of $|z|$. From the asymptotic behavior of the incomplete gamma function [4, eq. 8.11.2] we deduce that

$$G_1^{\pm}(z) = \mathcal{O}\left(z^{\frac{m-1}{p}-1} e^{-\frac{\eta}{2}z^{1-\frac{m}{p}}}\right), \qquad \text{as } |z| \to \infty. \tag{7.62}$$

On the other hand

$$G_2^\pm(z) := \int_{\delta^\pm}^{b^\pm} e^{-z\eta t^m} e^{-zf_m(t_0\pm t)} g(t_0 \pm t)(t_0 \pm t)^{a-1} dt$$
$$= \int_{\delta^\pm}^{b^\pm} e^{-z[f(t_0\pm t)-f(t_0)]} g(t_0 \pm t)(t_0 \pm t)^{a-1} dt.$$

And then,

$$|G_2^\pm(z)| \le e^{-\epsilon^\pm \Re z} \int_{\delta^\pm}^{b^\pm} e^{-\Re z\left[f(t_0\pm t)-f(t_0)-\epsilon^\pm\right]} |g(t_0 \pm t)(t_0 \pm t)^{a-1}| dt.$$

Using (7.61), we find that, for $\Re z > x_0$, $e^{-\Re z\left[f(t_0\pm t)-f(t_0)-\epsilon^\pm\right]} \le e^{-x_0\left[f(t_0\pm t)-f(t_0)-\epsilon^\pm\right]}$. Taking also into account that the last integral above, with $\Re z$ replaced by x_0, is convergent by hypothesis H7.4.(vi), we conclude that

$$|G_2^\pm(z)| \le K e^{-\epsilon^\pm \Re z},$$

with $K > 0$ independent of $|z|$. From this formula and (7.62) we find

$$F_1(z) = \mathcal{O}\left(e^{-\frac{\eta}{2} z^{1-\frac{m}{p}}} + e^{-z\epsilon^\pm}\right), \qquad \text{as } |z| \to \infty. \tag{7.63}$$

- $R_{n,0}^\pm(z)$. Recall the integral representation of $R_{n,0}^\pm(z)$ given in (7.54). The factor $r_n^\pm(x,z)$ is the Taylor remainder of $\tilde{h}^\pm(x,z)$ at $x = 0$. Then, $r_n^\pm(x,z)$ admits the following Cauchy's integral representation:

$$r_n^\pm(x,z) = \frac{x^n}{2\pi i} \oint_{\mathcal{C}} \frac{\tilde{h}^\pm(w,z)}{w^n(w-x)} dw, \qquad x \in D_0(r),$$

for a certain $r > 0$ independent of $|z|$. We choose the integration path $\mathcal{C}$ to be the circle of center 0 and radius $2|z^{-1/p}|$ ($< r$ for large enough $|z|$), oriented in the positive sense. Since (see (7.53)) $\rho_z := 1 - e^{-|z^{-m/p}|} \simeq |z^{-m/p}|$ when $|z| \to \infty$, for sufficiently large $|z|$, both points 0 and x are contained inside the circle $\mathcal{C}$ for any $x \in [0, \rho_z^{1/m}]$.

Recall at this point that $\tilde{h}^\pm(w,z)$ is given in (7.48) and (7.43), and we can write

$$\tilde{h}^\pm(w,z) = e^{-zw^p\psi(w)} \varphi(w),$$

with

$$\psi(w) := w^{-p} f_m(t_0 \pm t(w)), \qquad t(w) := \frac{w}{\Lambda^\pm}\left[\frac{-\log(1-w^m)}{w^m}\right]^{\frac{1}{m}},$$

$$\varphi(w) := g(t_0 \pm t(w)) \left(-\frac{\log(1-w^m)}{w^m}\right)^{\lambda^\pm/m-1} (t_0 \pm t(w))^{\mu^\pm - 1}.$$

The functions $\psi(w)$ and $\varphi(w)$ are analytic in the disk $D_0(r)$ that contains the circle $\mathcal{C}$. Then, due to the choice of the radius of the circle $\mathcal{C}$ to cancel out the variable z in the exponent of the exponential of the function $\tilde{h}^\pm(w,z)$, we have that

$$|\tilde{h}^\pm(w,z)| \le e^{|z(2z^{-1/p})^p|\,|\psi(w)|} |\varphi(w)| = e^{2^p|\psi(w)|}|\varphi(w)| \le \bar{K},$$

for some constant $\bar{K} > 0$ independent of $|z|$. For $w \in \mathcal{C}$ we also have $|w-x| \geq |z^{-1/p}|$ and $|w|^n = 2^n|z^{-n/p}|$. Then

$$|r_n^{\pm}(x,z)| \leq Kx^n|z^{n/p}|,$$

with $K > 0$ independent of x and $|z|$. Then, from (7.54) we have

$$\begin{aligned}
|R_{n,0}^{\pm}(z)| &\leq \int_0^{\rho_z^{1/m}} (1-x^m)^{\Re\left(\frac{\eta z}{(\Lambda^{\pm})^m}\right)-1} x^{\Re\lambda^{\pm}-1}|r_n^{\pm}(x,z)|dx \\
&\leq K|z^{n/p}| \int_0^{\rho_z^{1/m}} (1-x^m)^{\Re\left(\frac{\eta z}{(\Lambda^{\pm})^m}\right)-1} x^{n+\Re\lambda^{\pm}-1}dx \\
&= K|z^{n/p}|B_{\rho_z}\left(\frac{n+\Re\lambda^{\pm}}{m}, \Re\left(\frac{\eta z}{(\Lambda^{\pm})^m}\right)\right).
\end{aligned}$$

Using the asymptotic behavior of the incomplete beta function [4, eqs. 8.17.2, 8.17.4, 8.18.3] we find

$$R_{n,0}^{\pm}(z) = \mathcal{O}\left(z^{\frac{n}{p}-\frac{n+\lambda^{\pm}}{m}}\right), \qquad \text{as } |z| \to \infty. \tag{7.64}$$

Finally, from (7.56), (7.59), (7.63) and (7.64) it follows that

$$R_n^{\pm}(z) = \mathcal{O}\left(z^{\frac{n}{p}-\frac{n+\lambda^{\pm}}{m}}\right) + \text{exp. small terms}, \qquad \text{as } |z| \to \infty,$$

and then, from the relation $R_n(z) = R_n^-(z) + R_n^+(z)$, we get

$$R_n(z) = \mathcal{O}\left(z^{\frac{n}{p}-\frac{n+\lambda^{+}}{m}} + z^{\frac{n}{p}-\frac{n+\lambda^{-}}{m}}\right) + \text{exp. small terms}, \qquad \text{as } |z| \to \infty,$$

which concludes the proof. □

Remark 7.4.4. *If $t_0 = 0$ the integral $F^-(z)$ defined in (7.44)-(7.45) vanishes. Therefore, the first term inside the brackets in expansion (7.47) and the remainder $R_n^-(z)$ defined in the proof of theorem 7.2.1 also vanish. In other words, if $t_0 = 0$ the "$-$" part of expansion (7.47) vanishes and only the "$+$" terms remain.*

Remark 7.4.5. *The extra dependence on the asymptotic variable z on the function $h(t,z)$ does not spoil the convergent character of expansion (7.47) and, as we have proved in theorem 7.4.3 it does not spoil its asymptotic character, as $|z| \to \infty$, either. The effect of the dependence of $h(t,z)$ on the variable z is that now the coefficients $A_k^{\pm}(z)$ depend on the asymptotic variable z. However, they are polynomials in z of degree $\lfloor \frac{k}{p} \rfloor$ (see lemma 7.4.2) and then the only effect is that the asymptotic sequences $A_k^-(z)B_\rho\left(\frac{k+1}{m}, \eta z\right)$ and $A_k^+(z)B\left(\frac{k+1}{\lambda^+}, \frac{\eta z}{(\Lambda^+)^m}\right)$ are no longer Poincaré sequences that decrease monotonically in the form $z^{-k/m}$ (as in the classical Laplace's method), but sequences that decrease in the form of a sawtooth (see [74] for more details), that is,*

$$A_k^-(z)B_\rho\left(\frac{k+1}{m}, \eta z\right) = \mathcal{O}\left(z^{\lfloor \frac{k}{p} \rfloor - \frac{k+1}{m}}\right) \qquad \text{as } |z| \to \infty,$$

and

$$A_k^+(z)B\left(\frac{k+\lambda^+}{m}, \frac{\eta z}{(\Lambda^+)^m}\right) = \mathcal{O}\left(z^{\lfloor \frac{k}{p} \rfloor - \frac{k+\lambda^+}{m}}\right), \quad \text{as } |z| \to \infty.$$

Example 5. We consider the following integral representation of the parabolic cylinder function [144, eq. 12.5.1]

$$U(a,z)=\frac{e^{-\frac{z^2}{4}}}{\Gamma\left(a+1/2\right)}\int_0^\infty u^{a-1/2}e^{-\frac{u^2}{2}-zu}\,dt,\qquad \Re a>-\frac{1}{2}.$$

Assuming that $z<0$, we perform the change of variable $u\mapsto t$ given by $u=-zt$. We get the integral representation

$$U(a,z)=\frac{(-z)^{a+1/2}e^{-\frac{z^2}{4}}}{\Gamma\left(a+1/2\right)}\int_0^\infty t^{a-1/2}e^{-z^2f(t)}\,dt,\qquad \Re a>-\frac{1}{2},$$

with $f(t)=\frac{t^2}{2}-t=\frac{1}{2}\left(t-1\right)^2-\frac{1}{2}$. The function $f(t)$ has a unique absolute minimum that occurs at $t_0=1$. We split the function $f(t)$ into its asymptotically dominant part $p(t)$ and a subdominant remainder $f_m(t)$. We find $p(t)=f(t)=\frac{1}{2}\left(t-1\right)^2-\frac{1}{2}$ and $f_m(t)=0$. Therefore, splitting the integral at the point $t_0=1$ we find

$$U(a,z)=\frac{(-z)^{a+1/2}e^{-\frac{z^2}{4}}}{\Gamma\left(a+1/2\right)}\left[U_1(a,z)+U_2(a,z)\right]$$

with

$$U_1(a,z):=e^{\frac{z^2}{2}}\int_0^1 t^{a-1/2}e^{-z^2\frac{(t-1)^2}{2}}dt=e^{\frac{z^2}{2}}\int_0^1(1-t)^{a-1/2}e^{-\frac{z^2}{2}t^2}dt=e^{\frac{z^2}{2}}\bar{U}_1\left(a,\frac{z^2}{2}\right),$$

and

$$U_2(a,z):=e^{\frac{z^2}{2}}\int_1^\infty t^{a-1/2}e^{-z^2\frac{(t-1)^2}{2}}dt=e^{\frac{z^2}{2}}\int_0^\infty(t+1)^{a-1/2}e^{-\frac{z^2}{2}t^2}dt=e^{\frac{z^2}{2}}\bar{U}_2\left(a,\frac{z^2}{2}\right),$$

with $\bar{U}_1(a,z)$ and $\bar{U}_2(a,z)$ defined in examples 3 and 4 respectively. Therefore, a convergent and asymptotic expansion of the parabolic cylinder function $U(a,z)$ for large negative z follows from expansions (7.31) and (7.41).

Chapter 8

A Systematic "Saddle Point Near An End Point" Asymptotic Method

In this chapter of the thesis we no longer derive a convergent expansion of a certain integral transform. Instead, we continue a line of research initiated by the advisors of this thesis: the systematization of classical asymptotic methods of integrals [68, 74, 75]. In particular, we revisit the "saddle point near an end point" asymptotic method and we derive a new expansion satisfying the following two properties: (i) the asymptotic sequence is the same as in the classical method; (ii) the coefficients of the expansions can be computed systematic and straightforwardly by means of an explicit, closed formula. The results of the chapter are based on [73].

The "saddle point near an end point" problem consists on finding an asymptotic expansion of integrals of the form

$$F(x;\alpha) = \int_a^b e^{-xf(t;\alpha)} g(t)(t-a)^{s-1} dt, \qquad \alpha \in (\alpha_1, \alpha_2), \tag{8.1}$$

for large x uniformly valid in α. In this formula (a, b) and (α_1, α_2) are finite or infinite real intervals, α is a real parameter and $f(t)$ and $g(t)$ are smooth enough functions in (a, b). We assume that $a > -\infty$ and for the sake of generality, we let an integrable singularity of the integrand at $t = a$ given by the factor $(t-a)^{s-1}$, with $s > 0$.

For fixed α, the problem of finding an asymptotic expansion of $F(x)$ for large x has already been discussed in section 2.2.2 of chapter 2: According to Laplace's method, for large x, the main contribution of the integrand to the integral (8.1) should come from the neighborhood of the points of the integration interval $[a, b]$ where the phase function $f(t)$ attains its absolute minima. Without loss of generality we may assume that $f(t)$ has only one absolute minimum that occurs at a point $t = t_0 \in [a, b)$. Thus, in section 2.2.2 we have derived three different expansions (2.18), (2.19) or (2.21) depending on the location of t_0 with respect to the integration interval $[a, b)$: (i), the absolute minimum of $f(t)$ occurs at the interior point $t_0 \in (a, b)$, with $f'(t_0) = 0$ and $f''(t_0) > 0$; (ii), the absolute minimum of the phase function $f(t)$ occurs at one of the end integration points, say $t = a$, and it is a simple, critical point of $f(t)$; or (iii), the function $f(t)$ is strictly incresing in $[a, b)$ and therefore attains its absolute minimum at the end point $t = a$, with $f'(a) > 0$.

The three situations are entirely different and lead to three formally different expansions (2.18), (2.19) or (2.21). This does not generate any problem as far as the absolute minimum t_0 of $f(t)$ is fixed. However, now the phase function $f(t;\alpha)$ is also a function of a certain parameter α. Imagine that as α crosses a certain critical value $\alpha^* \in (\alpha_1, \alpha_2)$, the absolute minimum of $f(t)$ changes of nature from being an interior point to being the end point $t = a$ (or vice versa). We say that there is “a saddle point near an end point”. In this case, which approximation (2.18), (2.19) or (2.21) should be used to estimate the asymptotic behavior of $F(x;\alpha)$, for large x? Comparing the three expansions, we can see that there is a formal discontinuous transition between the approximants of $F(x;\alpha)$ given by (2.18), (2.19) and (2.21) even when the function $F(x;\alpha)$ is a continuous function of the parameter α. But moreover, expansions (2.18) and (2.21) are useless from a numerical point of view when α is close to α^* because the coefficients h_n of those expansions blow up as $\alpha \to (\alpha^*)^+$ and $\alpha \to (\alpha^*)^-$ respectively. We face the problem of finding an asymptotic expansion valid for any value of the parameter α, that is, of finding an expansion uniformly valid for $\alpha \in (\alpha_1, \alpha_2)$. This type of approximations have already been studied in the literature [13, 145, 154] and belong to the class of the so called *uniform asymptotic expansions.*[1].

A similar situation occurs in the context of the approximation of contour integrals $F(z)$ when the contributions of the different saddle points of the phase function to the asymptotic behavior of $F(z)$ for large $x := |z|$ depend on the argument $\alpha := \arg z$ of the asymptotic variable z. As a consequence the asymptotic expansions for large $|z|$ of $F(z)$ are different in different sectors for $\arg z$ [154, Ch. 2, §4]. In general, the transition from one sector to a contiguous sector is discontinuous and originates the famous *Stokes phenomenon* [135, 136, 137]. This abrupt transition can be smoothed by considering the error function as basic approximant [7, 8], obtaining in this manner a more sophisticated asymptotic expansion, but with a smooth behavior of the asymptotic approximation when the asymptotic variable z crosses the Stokes lines.

In the case of the integral (8.1) the origin of the discontinuous behavior of the asymptotic approximations (2.18), (2.19) and (2.21) is different: as the parameter α crosses the critical value α^*, the absolute minimum $t = t_0(\alpha)$ of the phase function crosses the end integration point $t = a$ and is no longer a critical point of $f(t)$. Therefore, when $t_0(\alpha)$ is inside the integration interval we use the change of variables $f(t;\alpha) - f(t_0(\alpha);\alpha) = u^2$ to derive the asymptotic expansion (2.18) of $F(x;\alpha)$. On the other hand, when $t_0(\alpha) = a$ we use another change of variables defined by $f(t;\alpha) - f(t_0(\alpha);\alpha) = u$ to deduce the expansion (2.21) of $F(x;\alpha)$. In order to obtain an expansion valid for either $t_0(\alpha) \in (a, b)$ or $t_0(\alpha) = a$ we should use a change of variables valid for both situations. Then, we consider a change of variables $t \mapsto u$ that depends on α and that encodes the transition from one case to the other one. That is, we consider the change of variables $f(t;\alpha) - f(t_0(\alpha);\alpha) = \frac{u^2}{2} - cu$, with $c := \pm\sqrt{2f(a;\alpha) - 2f(t_0(\alpha);\alpha)}$, where the $\pm$ sign is taken accordingly to $\alpha > \alpha^*$ or $\alpha < \alpha^*$ [154, Ch. 7, §3]. After some computations, the

[1]The concept of *uniform expansion* here is different from the one introduced in chapter 3 of this thesis. There, *uniform* means that the error is bounded by a constant independent of the uniform variable in a large domain. Here, *uniform* means that the expansion is valid when another variable (or parameter) runs in a certain domain.

following compound asymptotic expansion can be derived [145, Ch. 22], [154, Ch. 7, §3]

$$F(x;\alpha) \sim e^{-xf(a;\alpha)} \left[\frac{U_s(c\sqrt{x})}{x^{s/2}} \sum_{n=0}^{\infty} \frac{a_n}{x^n} + \frac{U_s'(c\sqrt{x})}{x^{(s+1)/2}} \sum_{n=0}^{\infty} \frac{b_n}{x^n} \right], \qquad x \to \infty, \tag{8.2}$$

with $U_a(z) := \Gamma(a) \exp{(z^2/4)}\, U(a-1/2,-z)$, where $U(a,z)$ is a parabolic cylinder function [144], and a_n, b_n are certain coefficients that may be computed by inverting the two-point Taylor series at the points $t=a$ and $t=t_0(\alpha)$ of the function $u(t)$ implicitly defined by the change of variables $f(t;\alpha)-f(a;\alpha)=\frac{u^2}{2}-cu$. Expansion (8.2) was first found by Bleistein who introduced an integration by parts method [13] that was later applied in other scenarios, like for example in the derivation of a uniform asymptotic expansion for integrals with two coalescing saddle points.

In the standard Laplace's method, in either case $t_0(\alpha) \in (a,b)$ or $t_0(\alpha)=a$, the asymptotic sequence is given by elementary functions: they are nothing but inverse powers of the asymptotic variable x. In contrast, in the uniform method "saddle point near an end point" the asymptotic sequence is given in terms of a special function: the parabolic cylinder function. This function encodes the abrupt transition between the two standard Laplace's expansions (2.18) and (2.21) as the parameter α crosses the critical value α^*, ensuring that expansion (8.2) is valid for any value of $\alpha \in (\alpha_1,\alpha_2)$.

On the other hand, in either of these expansions, the standard Laplace's expansions or the uniform asymptotic expansion, the computation of the coefficients h_n or $(a_n,\, b_n)$ is not straightforward, as it is neccesary to invert the Taylor series of a function defined implicitly by a certain change of variables. We can not find general analytical formulas for these coefficients in traditional textbooks about asymptotics [105, 145, 154], but rather an indication on how to compute the few first. Nevertheless, some more or less explicit representations of these coefficients can be found in the literature: Perron's method gives them in terms of derivatives of an explicit function whereas Wojdylo [152, 153] found a recurrence formula in terms of partial ordinary Bell polynomials whose complexity increases with the number of terms considered. On the other hand, based on Cauchy-type integral representations, in [56] the authors have given a stable and efficient manner to approximate the coefficients that appear in uniform asymptotic expansions of integrals and in particular, the coefficients that appear in the "saddle point near an end point" uniform asymptotic method.

A simplification of the standard method of Laplace (in the non-uniform case) has been introduced in [74] and summarized in subsection 2.2.3 of this thesis. On the one hand, the computation of the coefficients in this modified method is simpler and systematic, given by formula (2.24). On the other hand, the computation of the asymptotic sequence is as simple as in the standard Laplace's method. The main idea is to split the phase function $f(t)$ into its asymptotically dominant monomial $f^{(m)}(t_0)/m!(t-t_0)^m$ and a subdominant remainder $f_m(t)$. Moreover, in [68] the same simplifying idea has been applied to integrals with two asymptotically relevant points: a saddle point and a pole. By avoiding the change of variables inherent to the classical uniform method "saddle point near a pole" [145, Ch. 21], [154, Ch. 7, §2], this uniform method has been simplified providing a simpler uniform asymptotic expansion. On the one hand, the asymptotic sequence given in [68] is, as in the classical method, written in terms of error functions. On the other hand, as in the modification of the method of Laplace [subsection 2.2.3], the coefficients can also be computed by means of an explicit, closed formula.

The aim of this chapter of the thesis is to continue this line of research and derive a new uniform asymptotic method "saddle point near an end point" whose coefficients can be computed by means of a closed, explicit formula. Again, the main idea is to avoid the change of variable $t \mapsto u$ defined by the relation $f(t;\alpha) - f(a;\alpha) = \frac{u^2}{2} - cu$ that characterizes the classical uniform method. Instead, we split the phase function into its asymptotically dominant part and a subdominant remainder. In the cases analyzed so far, the standard Laplace's method [74], the uniform method "saddle point near a pole" [68] and also in the convergent Laplace's method [Chapter 7], that asymptotically dominant part was a monomial of the form $f^{(m)}(t_0)/m!(t-t_0)^m$. However, as we will see below, the situation in the uniform asymptotic method "saddle point near an end point" is different, as the asymptotically dominant part is not a monomial, but a polynomial of the second degree with two non-constant terms. Despite this extra difficulty, we will see below that the coefficients can be computed straightforwardly by means of a systematic, explicit formula and, as in the classical method, the asymptotic sequence is again given in terms of parabolic cylinder functions.

8.1 Preparatory results

We investigate the integral

$$F(x;\alpha) := \int_a^b e^{-xf(t;\alpha)} g(t)(t-a)^{s-1} dt, \qquad s > 0, \tag{8.3}$$

where we assume that (a,b) is a real interval that may be finite or infinite, but with a finite end point, say $a \in \mathbb{R}$. We also consider that $\alpha \in (\alpha_1, \alpha_2)$, for a real interval (α_1, α_2). We assume that the functions $f(t)$ and $g(t)$ are real valued and, for the sake of generality, we allow a branch point at the point $t = a$, the factor $(t-a)^{s-1}$ with $s > 0$. We also consider that the function $f(t)$ possesses a unique absolute minimum t_0 in $[a,b)$. That absolute minimum may be an interior point $t_0 \in (a,b)$ or an end point $t_0 = a$. But the location of that absolute minimum may change with the parameter α. Imagine that t_0 is an interior point that, eventually, as α crosses the critical value $\alpha^* \in (\alpha_1, \alpha_2)$, the point t_0 crosses the end point of the integration domain $t = a$ and therefore the absolute minimum of $f(t)$ in $[a,b)$ does no longer occur in an interior point but at the end point $t_0 = a$. We are interested in deriving an asymptotic expansion of $F(x;\alpha)$ for large $|x|$ (with $\arg x \in (-\pi/2, \pi/2)$) uniformly valid for $\alpha \in (\alpha_1, \alpha_2)$.

We consider the following hypotheses for the functions $f(t)$ and $g(t)$:

H8.1.(i) The functions $f(t;\alpha)$, $f'(t;\alpha)$, $f''(t;\alpha)$ and $g(t)$ are continuous real functions of their variables in their respective domains, where primes denote derivatives with respect to the variable t.

H8.1.(ii) By subdividing the range of integration in (8.3) and/or reversing the order of integration if necessary, we may assume, without loss of generality, that the function $f(t;\alpha)$ has a unique minimum at the point $t = t_0(\alpha)$ in $[a,b)$. We consider the step function

$$\theta(\alpha) := \begin{cases} 0 & \text{if } \alpha \le \alpha^*, \\ 1 & \text{if } \alpha > \alpha^*. \end{cases} \tag{8.4}$$

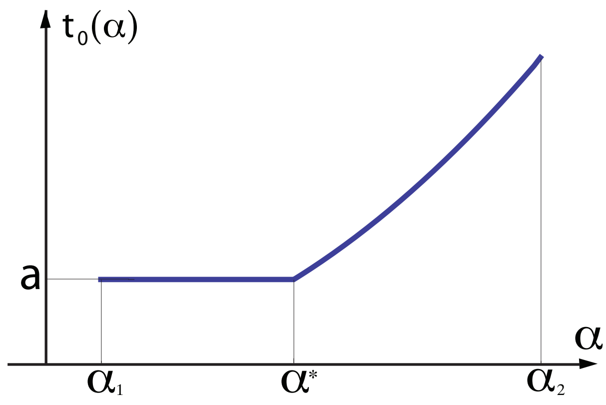


Figure 8.1: Typical portrait of the minimum $t_0(\alpha)$ of the phase function $f(t;\alpha)$ as a function of the parameter α. For $\alpha < \alpha^*$ the absolute minimum of $f(t,\alpha)$ in the integration interval is located at the end point $t = a$. As α crosses the critical value α^*, the absolute minimum of $f(t;\alpha)$ moves to the interior of the integration interval (a,b) and coincides with its only relative minimum $\bar{t_0}(\alpha)$.

Then, we assume that $t_0(\alpha)$ is a function of α of the form

$$t_0(\alpha) = a\,(1-\theta(\alpha)) + \bar{t_0}(\alpha)\theta(\alpha) = \begin{cases} a & \text{if } \alpha_1 \le \alpha \le \alpha^*, \\ \bar{t_0}(\alpha) & \text{if } \alpha^* \le \alpha \le \alpha_2. \end{cases} \tag{8.5}$$

In this formula, $\bar{t_0}(\alpha)$ denotes the only relative minimum point of $f(t;\alpha)$ in $[a,b)$ (when it exists) that we assume to be a simple critical point of $f(t)$. That is, $f'\left(\bar{t_0}(\alpha);\alpha\right) = 0$ and $f''\left(\bar{t_0}(\alpha);\alpha\right) > 0$. We have that $\bar{t_0}(\alpha^*) = a$ and we assume that $\bar{t_0}(\alpha) > a$ for $\alpha > \alpha^*$ and that the function $\bar{t_0}(\alpha)$ is a continuous function of α. Therefore, $t_0(\alpha)$ is also a continuous function of the parameter α (see figure 8.1).

H8.1.(iii) For $\alpha < \alpha^*$ the function $f(t;\alpha)$ attains its absolute minimum in $[a,b)$ at the point $t_0(\alpha) = a$ and $f'(a;\alpha) > 0$. We assume that $f''(t_0(\alpha);\alpha) = f''(a,\alpha) > 0$ also for $\alpha < \alpha^*$ and not only for $\alpha \ge \alpha^*$ as specified in hypothesis H8.1.(ii).

H8.1.(iv) Both functions $f(t;\alpha)$ and $g(t)$ have a Taylor expansion at the point $t = t_0(\alpha)$ with common radius of convergence $\rho > 0$, and the coefficients $f^{(n)}\left(t_0(\alpha);\alpha\right)$ are continuous functions of α for $n = 0, 1, 2, \ldots$.

H8.1.(v) The integral (8.3) is absolutely convergent for sufficiently large $|x|$, say $\Re x > x_0$, for some fixed $x_0 > 0$.

In the classical uniform expansion (8.2) the coefficients (a_n, b_n) are, somehow, the coefficients of the two-point Taylor expansion at the points $u = 0$ (the end point) and $u = c$ (the saddle point) of the function $h(u,\alpha) := g\big(t(u)\big)\big(t(u)-a\big)^{s-1}(u-c)/f'\big(t(u)\big)$ in the integrand of (8.3) after the convenient change of variables $f(t;\alpha) - f(t_0(\alpha);\alpha) = \frac{u^2}{2} - cu$. They can be computed by reverting the Taylor series of the function $u(t)$ defined by the change of variables at the points $t = a$ and $t = \bar{t_0}(\alpha)$. Thus, closed, explicit formulae for those coefficients are not given in traditional text book on asymptotics [105, 145, 154]. But, in [145, Remark 22.2] it is suggested that, may be, a standard

Taylor expansion at only one point, say $u = 0$ or $u = c$, will provide easier formulas for the coefficients (a_n, b_n). Motivated by this remark, we re-consider the spliting of the phase function $f(t;\alpha)$ into an asymptotically dominant Taylor polynomial and a subdominant remainder (see subsection 2.2.3). But now, there are two asymptotically relevant points: either $t = a$ if $\alpha \leq \alpha^*$ or $t = \bar{t_0}(\alpha)$ if $\alpha \geq \alpha^*$. Then, we consider the Taylor expansion of the phase function $f(t;\alpha)$ at the moving point $t_0(\alpha)$ defined in (8.5), that is asymptotically relevant for any value of α. Before we derive our main result we need some definitions.

Definition. (Asymptotically dominant Taylor polynomial). We consider the second-order Taylor polynomial of the phase function $f(t;\alpha)$ at the moving point $t = t_0(\alpha)$ defined in (8.5):

$$p(t;\alpha) := f\left(t_0(\alpha);\alpha\right) + a_1(\alpha)\left(t - t_0(\alpha)\right) + a_2(\alpha)\left(t - t_0(\alpha)\right)^2. \tag{8.6}$$

From hypotheses H8.1.(ii) and H8.1.(iii) we have that $a_2(\alpha) := \frac{1}{2}f''\left(t_0(\alpha);\alpha\right) > 0$, for all $\alpha \in (\alpha_1, \alpha_2)$. We also have that $a_1(\alpha) := f'\left(t_0(\alpha);\alpha\right) \geq 0$, with $a_1(\alpha) = 0$, for $\alpha \geq \alpha^*$.

For the sake of simplicity, in the remaining of the chapter we will omit the dependence of $a_1(\alpha)$, $a_2(\alpha)$, $t_0(\alpha)$, $\bar{t_0}(\alpha)$ and $\theta(\alpha)$ on the parameter α and simply write a_1, a_2, t_0, $\bar{t_0}$ and θ.

Definition. (Asymptotically subdominant remainder). We also consider the remainder of the asymptotically dominant Taylor polynomial of the phase function

$$f_p\left(t;\alpha\right) := f\left(t;\alpha\right) - p\left(t;\alpha\right), \tag{8.7}$$

and we denote by q ($q \geq 3$) the order of the next non-vanishing derivative of the phase function $f\left(t;\alpha\right)$ at $t = t_0$ after the second derivative.

Thus, we split the phase function to re-write integral (8.3) in the form

$$F(x;\alpha) := \int_a^b e^{-xp(t;\alpha)} h(t,x;\alpha)(t-a)^{s-1} dt, \qquad h(t,x;\alpha) := e^{-xf_p(t;\alpha)} g(t). \tag{8.8}$$

On the one hand, the phase function is now simpler as it is just a polynomial $p(t;\alpha)$ of the second degree. On the other hand, the factor $h(t,x;\alpha)$ also contains the asymptotic variable x. As we may guess from similar situations [68, 74], [Ch.7 of this thesis] and we will see below, the dependence on x of the function $h(t,x;\alpha)$ does not spoil the asymptotic character of the expansion that we are going to derive.

From the definition of the function $h(t,x;\alpha)$ (8.8) and hypothesis H8.1.(iv) we have that the function $h(t,x;\alpha)$ has a Taylor expansion at $t = t_0$:

$$h(t,x;\alpha) = \sum_{k=0}^{n-1} h_k(x;\alpha)(t-t_0)^k + r_n(t,x;\alpha), \tag{8.9}$$

valid in a certain disk of center t_0 and radius $\rho > 0$, $D_\rho(t_0)$. In this formula, $r_n(t,x;\alpha)$ is the Taylor remainder of $h(t,x;\alpha)$ at $t = t_0$ and $h_n(x;\alpha)$ are the Taylor coefficients. These coefficients can be computed in the form

$$h_n(x;\alpha) = \sum_{k=0}^{n} C_k\left(x;\alpha\right) B_{n-k}(\alpha), \tag{8.10}$$

where $C_n(x;\alpha)$ and $B_n(\alpha)$ are, respectively, the Taylor coefficients at $t = t_0$ of the functions $e^{-xf_p(t;\alpha)}$ and $g(t)$:

$$e^{-xf_p(t;\alpha)} = \sum_{n=0}^{\infty} C_n(x;\alpha)(t-t_0)^n, \qquad g(t) = \sum_{n=0}^{\infty} B_n(\alpha)(t-t_0)^n, \qquad t \in D_\rho(t_0). \quad (8.11)$$

Moreover, the coefficients $h_n(x;\alpha)$ can be computed in terms of the Taylor coefficients at $t = t_0$ of the functions $e^{-xf(t)}$ and $g(t)$. Denote by $A_n(x;\alpha)$ the Taylor coefficients of the function $e^{-xf(t;\alpha)}$ at $t = t_0$:

$$e^{-xf(t;\alpha)} = \sum_{n=0}^{\infty} A_n(x;\alpha)(t-t_0)^n, \qquad t \in D_\rho(t_0) \quad (8.12)$$

valid in a certain disk centered at t_0 of radius ρ, $D_\rho(t_0)$.

Then, we have the following result for the coefficients $h_n(x;\alpha)$ (8.9) of the function $h(t,x;\alpha)$ defined in (8.8).

Lemma 8.1.1. *Consider the coefficients a_1 and a_2 defined in (8.6) and the Taylor coefficients $A_n(x;\alpha)$ and $B_n(\alpha)$ of the functions $e^{-xf(t;\alpha)}$ and $g(t)$ defined, respectively, in (8.12) and (8.11).*

(i) For $n = 0,1,2,\ldots$, the coefficients $h_n(x;\alpha)$ of the function $h(t,x;\alpha)$ defined in (8.8) can be computed by means of the formula

$$\begin{aligned} h_n(x;\alpha) &= e^{xf(t_0;\alpha)} \sum_{k+2j+i+l=n} \frac{(a_1x)^k}{k!}\frac{(a_2x)^j}{j!} A_i(x;\alpha)B_l(\alpha) \\ &= e^{xf(t_0;\alpha)} \sum_{j=0}^{n} \left(\sum_{k=0}^{j} \left[\sum_{i=0}^{\lfloor k/2 \rfloor} x^{k-i}\frac{a_2^i}{i!}\frac{a_1^{k-2i}}{(k-2i)!} \right] A_{j-k}(x;\alpha) \right) B_{n-j}(\alpha) \end{aligned} \quad (8.13)$$

(ii) The coefficients $h_n(x;\alpha)$ can be computed using formula (8.10) and the following recursive relation for the coefficients $C_n(x;\alpha)$:

$$\begin{cases} C_0(x;\alpha) = 1, \\ C_1(x;\alpha) = C_2(x;\alpha) = \ldots = C_{q-1}(x;\alpha) = 0, \\ C_n(x;\alpha) = -\dfrac{x}{n}\displaystyle\sum_{k=0}^{n-q} \frac{C_k(x;\alpha)\, f^{(n-k)}(t_0)}{(n-k-1)!}, \qquad n = q, q+1, q+2, \ldots, \end{cases} \quad (8.14)$$

where $q \geq 3$ is given in the definition of asymptotically subdominant remainder after equation (8.7).

(iii) The coefficients $h_n(x;\alpha)$ are polynomials in x of degree $\lfloor n/q \rfloor$. Therefore, $h_n(x;\alpha) = \mathcal{O}\left(x^{\lfloor n/q \rfloor}\right)$ as $x \to \infty$.

Proof. (i) Formula (8.13) follows from the very definition of $h(t,x;\alpha)$ in (8.8), the definition of the coefficients $A_n(x;\alpha)$ (8.12) and $B_n(\alpha)$ (8.11), the equality $f_p(t;\alpha) = f(t;\alpha) - a_1(t-t_0) - a_2(t-t_0)^2$ and the Taylor series of the exponential function:

$$e^{a_jx(t-t_0)^j} = \sum_{n=0}^{\infty} \frac{(a_jx)^n}{n!}(t-t_0)^{jn}, \qquad j = 1,2, \qquad t \in \mathbb{R}.$$

(ii) To derive the recurrence relation (8.14) we observe that the function $e^{-xf_p(t;\alpha)}$ satisfies the differential equation

$$\frac{d}{dt}\left(e^{-xf_p(t;\alpha)}\right) = xe^{-xf_p(t;\alpha)}\left[a_1 + 2a_2(t-t_0) - f'(t)\right].$$

Introducing in this equation the expansion of $e^{-xf_p(t;\alpha)}$ given in (8.11), the Taylor expansion of $f'(t)$ at $t = t_0$ and equating coefficients of equal powers we find the relation (8.14) by taking into account the definition of a_1, a_2 and q.

(iii) The recurrence relation (8.14) shows that the coefficients $C_n(x;\alpha)$ are polynomials in x of degree $\lfloor x/q \rfloor$. Therefore, $C_n(x;\alpha) = \mathcal{O}\left(x^{\lfloor n/q \rfloor}\right)$ as $x \to \infty$. Then, (iii) follows from (8.10).

□

Remark 8.1.2. *The index q plays the same role as the index p did in the modified Laplace's method of subsection 2.2.3 with a slight difference: the index q may depend on the parameter α. But, for any $\alpha \in (\alpha_1, \alpha_2)$ we have that $q \geq 3$ and, as we will see below, this possible dependence does not have any effect on the asymptotic analysis of $F(x;\alpha)$.*

8.2 The systematic method "saddle point near an end point"

In this section we give the main result of this chapter of the thesis: a uniform asymptotic method "saddle point near an end point" whose coefficients can be computed by means of an explicit, closed and systematic formula. We have the following theorem.

Theorem 8.2.1. *Consider the integral*

$$F(x;\alpha) = \int_a^b e^{-xf(t;\alpha)} g(t)(t-a)^{s-1}, \qquad s > 0 \tag{8.15}$$

introduced in (8.3) with the notation introduced in the previous section. Assume also that hypotheses H8.1.(i)-H8.1.(v) hold. Then, for $n = 1, 2, 3, \ldots$,

$$F(x;\alpha) = e^{-xf(t_0;\alpha)}\left(\sum_{k=0}^{n-1} h_k(x;\alpha)\Phi_k(x;\alpha,s) + R_n(x;\alpha)\right), \tag{8.16}$$

where the coefficients $h_k(x;\alpha)$ are given in lemma 8.1.1 and can be explicitly computed in terms of the Taylor coefficients of $e^{-xf(t;\alpha)}$ and $g(t)$ at $t = t_0$. The asymptotic sequence $\Phi_k(x;\alpha,s)$ can be obtained from the following recurrence relation

$$\begin{cases} \Phi_0(x;\alpha,s) := \frac{\Gamma(s)}{(2a_2x)^{\frac{s}{2}}} \exp\left\{\frac{x}{2}\left[a_1(t_0-a) - a_2(t_0-a)^2 + \frac{a_1^2}{4a_2}\right]\right\} U\left(s - 1/2, \frac{a_1+2a_2(a-t_0)}{\sqrt{2a_2}}\sqrt{x}\right), \\ \Phi_{k+1}(x;\alpha,s) = \Phi_k(x;\alpha,s+1) + (a-t_0)\Phi_k(x;\alpha,s), \qquad k = 0,1,2,\ldots \end{cases} \tag{8.17}$$

where $U(s,x)$ is a parabolic cylinder function [144, eq. 12.5.1]. An explicit formula for $\Phi_k(x;\alpha,s)$ is the following:

$$\Phi_k(x;\alpha,s) = \sum_{j=0}^{k}\binom{k}{j}(a-t_0)^{k-j}\Phi_0\left(x;\alpha,s+j\right). \tag{8.18}$$

Furthermore, we have, for fixed α, the following asymptotic results as $x \to \infty$,

$$h_k(x;\alpha)\Phi_k(x;\alpha,s) = \begin{cases} \mathcal{O}\left(x^{\lfloor k/q \rfloor-(k+s)}\right) & \text{for } \alpha < \alpha^*, \\ \mathcal{O}\left(x^{\lfloor k/q \rfloor-(k+s)/2}\right) & \text{for } \alpha = \alpha^*, \\ \mathcal{O}\left(x^{\lfloor k/q \rfloor-(k+1)/2}\right) & \text{for } \alpha > \alpha^*. \end{cases} \tag{8.19}$$

The remainder $R_n(x;\alpha)$ in (8.16) satisfies, for any fixed $\alpha \in (\alpha_1, \alpha_2)$, as $|x| \to \infty$,

$$R_n(x;\alpha) = \mathcal{O}\left(h_n(x;\alpha)\Phi_n(x;\alpha,s)\right). \tag{8.20}$$

Therefore, expansion (8.16) is an asymptotic expansion of $F(x;\alpha)$ as $|x| \to \infty$ uniformly valid for $\alpha \in (\alpha_1, \alpha_2)$:

$$F(x;\alpha) \sim e^{-xf(t_0;\alpha)} \left(\sum_{k=0}^{\infty} h_k(x;\alpha)\Phi_k(x;\alpha,s) \right), \qquad \text{as } |x| \to \infty. \tag{8.21}$$

Proof. We split the phase function $f(t;\alpha)$ in the integral $F(x;\alpha)$ (8.15) in the form (8.8). We find

$$F(x,\alpha) = e^{-xf(t_0;\alpha)} \int_a^b e^{-a_1x(t-t_0)-a_2x(t-t_0)^2} h(t,x;\alpha)(t-a)^{s-1}dt. \tag{8.22}$$

We replace expansion (8.9) for the function $h(t,x;\alpha)$ at $t = t_0$ into (8.22) and interchange summation and integration to find

$$F(x;\alpha) = e^{-xf(t_0;\alpha)} \left[\sum_{k=0}^{n-1} h_k(x;\alpha)\tilde{\Phi}_k(x;\alpha,s) + R_n(x;\alpha) \right], \tag{8.23}$$

where the coefficients $h_k(x,\alpha)$ are the Taylor coefficients of the function $h(t,x;\alpha)$ at $t = t_0$ that are given in lemma 8.1.1, and

$$\tilde{\Phi}_k(x;\alpha,s) := \int_a^b e^{-a_1x(t-t_0)-a_2x(t-t_0)^2}(t-t_0)^k(t-a)^{s-1}dt, \qquad k = 0,1,2,\ldots, \tag{8.24}$$

$$R_n(x;\alpha) := \int_a^b e^{-a_1x(t-t_0)-a_2x(t-t_0)^2}(t-a)^{s-1}r_n(t,x;\alpha)dt, \tag{8.25}$$

being $r_n(t,x;\alpha)$ the remainder of the Taylor expansion (8.9) of $h(t,x;\alpha)$ at $t = t_0$.

On the one hand, we have

$$\begin{aligned} \tilde{\Phi}_k(x;\alpha,s) &= \int_0^{b-a} e^{-a_1x(t+a-t_0)-a_2x(t+a-t_0)^2}(t+a-t_0)^k t^{s-1}dt \\ &= \int_0^{\infty} e^{-a_1x(t+a-t_0)-a_2x(t+a-t_0)^2}(t+a-t_0)^k t^{s-1}dt + \mathcal{O}\left(e^{-a_1x(b-t_0)-a_2x(b-t_0)^2}\right), \end{aligned} \tag{8.26}$$

Thus, we define

$$\Phi_k(x;\alpha,s) := \int_0^{\infty} e^{-a_1x(t+a-t_0)-a_2x(t+a-t_0)^2}(t+a-t_0)^k t^{s-1}dt, \qquad k = 0,1,2,\ldots, \tag{8.27}$$

and (8.16) follows from (8.23) except for exponentially small terms.

Now, the first line of formula (8.17) follows directly from (8.27) by using the integral representation of the parabolic cylinder function [144, eq. 12.5.1]. The second formula of the recurrence relation (8.17) comes from (8.27) by replacing k by $k+1$ and splitting the factor $(t+a-t_0)^{k+1} = (t+a-t_0)^k(t+(a-t_0))$. Finally, the explicit formula (8.18) follows by considering the binomial expansion of the term $(t+a-t_0)^k$ in (8.27).

The asymptotic behavior of $\Phi_k(x;\alpha,s)$ follows either by an application of the classical Laplace's method to the integral (8.27) or by using the asymptotic behavior of the parabolic cylinder function for large argument [145, Ch. 11, §§11.2 and 11.3] in the formula (8.18) and some computations, in the different three cases: (i) $\alpha < \alpha^*$ and then $t_0 = a$ and $a_1 > 0$, (ii) $\alpha = \alpha^*$ and then $t_0 = a$ and $a_1 = 0$, and (iii) $\alpha > \alpha^*$ and then $t_0 > a$ and $a_1 = 0$. Using either method, we get $\Phi_k(x;\alpha,s) = \mathcal{O}\left(x^{-\sigma(k)}\right)$ for $x \to \infty$, with $\sigma(k) = (k+s)$ in case (i), $\sigma(k) = (k+s)/2$ in case (ii), and $\sigma(k) = (k+1)/2$ in case (iii). In addition, we know from lemma 8.1.1(iii) that the coefficients $h_k(x;\alpha)$ are polynomials in x of degree $\lfloor k/q \rfloor$. Therefore, the sequence $h_k(x;\alpha)\Phi_k(x;\alpha,s)$ constitutes an asymptotic sequence, as $x \to \infty$, and its behavior is the specified in (8.19).

It remains to prove that the remainder $R_n(x;\alpha)$ satisfies the asymptotic property (8.20). The proof of this fact is a bit long and tedious, and resembles the proof of the asymptotic character of expansion (7.47) in the asymptotic and convergent Laplace's method developed in chapter 7.

We split the integral $F(x;\alpha)$ in the form

$$F(x;\alpha) = e^{-xf(t_0;\alpha)}\left[F_0(x;\alpha) + F_1(x;\alpha) + F_2(x;\alpha)\right], \tag{8.28}$$

with

$$F_1(x;\alpha) := \int_a^{t_0-\theta|x^{-1/q}|} e^{x[f(t_0;\alpha)-f(t;\alpha)]} g(t)(t-a)^{s-1}dt, \tag{8.29}$$

$$F_2(x;\alpha) := \int_{t_0+|x^{-1/q}|}^{b} e^{x[f(t_0;\alpha)-f(t;\alpha)]} g(t)(t-a)^{s-1}dt, \tag{8.30}$$

and

$$\begin{aligned} F_0(x;\alpha) &:= \int_{t_0-\theta|x^{-1/q}|}^{t_0+|x^{-1/q}|} e^{x[f(t_0;\alpha)-f(t;\alpha)]} g(t)(t-a)^{s-1}dt \\ &= \int_{t_0-\theta|x^{-1/q}|}^{t_0+|x^{-1/q}|} e^{-a_1x(t-t_0)-a_2x(t-t_0)^2} h(t,x;\alpha)(t-a)^{s-1}dt, \end{aligned} \tag{8.31}$$

where $\theta = \theta(\alpha)$ is the step function defined in (8.4). We note that if $\alpha \le \alpha^*$ we have $\theta = 0$ and $t_0 = a$. Therefore, if $\alpha \le \alpha^*$, $F_1(x;\alpha) = 0$.

We replace the expansion (8.9) of $h(t,x;\alpha)$ at $t = t_0$ in the integral $F_0(x;\alpha)$ and interchange summation and integration. We find

$$F_0(x;\alpha) = \sum_{k=0}^{n-1} h_k(x;\alpha)\Phi_k^0(x;\alpha) + R_n^0(x;\alpha), \tag{8.32}$$

where we have defined

$$\Phi_k^0(x;\alpha) := \int_{t_0-\theta|x^{-1/q}|}^{t_0+|x^{-1/q}|} e^{-a_1x(t-t_0)-a_2x(t-t_0)^2} (t-t_0)^k (t-a)^{s-1}dt, \tag{8.33}$$

and

$$R_n^0(x;\alpha) := \int_{t_0-\theta|x^{-1/q}|}^{t_0+|x^{-1/q}|} e^{-a_1x(t-t_0)-a_2x(t-t_0)^2}(t-a)^{s-1}r_n(t,x;\alpha)dt. \tag{8.34}$$

Thus, from (8.28), (8.16) and (8.32) we find

$$\begin{aligned} R_n(x;\alpha) &= F_1(x;\alpha) + F_2(x;\alpha) + F_0(x;\alpha) - \sum_{k=0}^{n-1} h_k(x;\alpha)\Phi_k(x;\alpha,s) \\ &= F_1(x;\alpha) + F_2(x;\alpha) + \Psi_n(x;\alpha,s) + R_n^0(x;\alpha), \end{aligned} \tag{8.35}$$

with the obvious definition

$$\Psi_n(x;\alpha,s) := \sum_{k=0}^{n-1} h_k(x;\alpha)\left[\Phi_k^0(x;\alpha,s) - \Phi_k(x;\alpha,s)\right]. \tag{8.36}$$

In the remaining of the proof we show that $R_n^0(x;\alpha)$ is of the order indicated in the right hand side of (8.20) and that $F_1(x;\alpha)$, $F_2(x;\alpha)$ and $\Psi_n(x;\alpha,s)$ are exponentially small compared with $R_n^0(x;\alpha)$, as $|x|\to\infty$. Therefore, from (8.35) we will conclude the asymptotic behavior (8.20) for $R_n(x;\alpha)$, as $x\to\infty$.

- $F_1(x;\alpha)$: We only study the case $\alpha\geq\alpha^*$ as $F_1(x;\alpha)=0$ otherwise. In this case $a_1=0$ and

  $$f(t;\alpha)-f(t_0;\alpha) = a_2(t-t_0)^2 + \mathcal{O}\left((t-t_0)^q\right), \qquad \text{as } t\to t_0.$$

  Then, there exists $\Lambda_1>0$ independent of t and x such that

  $$f(t;\alpha)-f(t_0;\alpha) \geq \frac{a_2}{2}(t-t_0)^2, \qquad \forall t\in(t_0-\theta\Lambda_1, t_0-\theta|x^{-1/q}|).$$

  Moreover, since t_0 is the unique absolute minimum of $f(t,\alpha)$ in $[a,b)$, there exists $\varepsilon_1>0$ independent of x such that

  $$f(t;\alpha)\geq f(t_0;\alpha)+\varepsilon_1, \qquad \forall t\in(a,t_0-\theta\Lambda_1).$$

  Then,

  $$\begin{aligned} &|F_1(x;\alpha)| \leq \\ &\int_a^{t_0-\theta\Lambda_1} e^{\Re x[f(t_0;\alpha)-f(t;\alpha)]}\left|g(t)(t-a)^{s-1}\right|dt + \int_{t_0-\theta\Lambda_1}^{t_0-\theta|x^{-1/q}|} e^{\Re x[f(t_0;\alpha)-f(t;\alpha)]}\left|g(t)(t-a)^{s-1}\right|dt \\ &\leq \int_a^{t_0-\theta\Lambda_1} e^{-\varepsilon_1\Re x}\left|g(t)\right|(t-a)^{s-1}dt + \int_{t_0-\theta\Lambda_1}^{t_0-\theta|x^{-1/q}|} e^{-\Re x(t-t_0)^2a_2/2}\left|g(t)\right|(t-a)^{s-1}dt \\ &\leq C_1e^{-\varepsilon_1\Re x} + C_2e^{-\frac{a_2}{2}|x^{-2/q}|\Re x}, \end{aligned}$$

  with C_1 and C_2 positive constants independent of $|x|$. In the last inequality above we have used that $g(t)$ is continuous in (a,t_0), that its absolute value is bounded and also that $(t-a)^{s-1}$ is integrable. Therefore,

  $$F_1(x;\alpha) = \mathcal{O}\left(e^{-\varepsilon_1x}+e^{-\frac{a_2}{2}x^{1-\frac{2}{q}}}\right), \qquad \text{as } |x|\to\infty. \tag{8.37}$$

- $F_2(x;\alpha)$. A similar analysis (with $a_1 \geq 0$) shows that we can find two numbers $\Lambda_2 > 0$ and $\varepsilon_2 > 0$ independent of x such that

$$F_2(x;\alpha) = \mathcal{O}\left(e^{-\varepsilon_2 x} + e^{-\frac{a_1}{2}x^{1-\frac{1}{q}}-\frac{a_2}{2}x^{1-\frac{2}{q}}}\right), \qquad \text{as } |x| \to \infty. \tag{8.38}$$

- $\Psi_n(x;\alpha)$. We have that

$$\begin{aligned}
&\Phi_k(x;\alpha) - \Phi_k^0(x;\alpha) \\
&= \int_0^{t_0-a-\theta|x^{-1/q}|} e^{-a_1x(t+a-t_0)-a_2x(t+a-t_0)^2}(t+a-t_0)^k t^{s-1}dt \\
&+ \int_{t_0-a+|x^{-1/q}|}^{\infty} e^{-a_1x(t+a-t_0)-a_2x(t+a-t_0)^2}(t+a-t_0)^k t^{s-1}dt \\
&= \mathcal{O}\left(e^{-a_2x^{1-\frac{2}{q}}}\right) + \mathcal{O}\left(e^{-a_1x^{1-\frac{1}{q}}-a_2x^{1-\frac{2}{q}}}\right), \qquad \text{as } |x| \to \infty,
\end{aligned}$$

with $a_2 > 0$ and $a_1 > 0$ for $\alpha > \alpha^*$ and $a_1 = 0$ for $\alpha \leq \alpha^*$. Therefore, for each $k \in \mathbb{N}$ we have that

$$\Phi_k(x;\alpha) - \Phi_k^0(x;\alpha) = \mathcal{O}\left(e^{-a_1x^{1-\frac{1}{q}}-a_2x^{1-\frac{2}{q}}}\right), \qquad \text{as } |x| \to \infty,$$

and, as $h_k(x;\alpha)$ are polynomials in x of degree $\lfloor k/q \rfloor$ (see lemma 8.1.1) we conclude that

$$\Psi_n(x;\alpha) = \mathcal{O}\left(e^{-a_2x^{1-\frac{2}{q}}}\right), \qquad \text{as } |x| \to \infty, \tag{8.39}$$

- $R_n^0(x;\alpha)$. If $|x|$ is large enough, the integration interval $I := (t_0 - \theta|x^{-1/q}|, t_0 + |x^{-1/q}|)$ in the definition (8.34) of $R_n^0(x;\alpha)$ is contained in the disk of convergence of the Taylor series of $h(t,x;\alpha)$ at $t = t_0$. In that case, the remainder $r_n(t,x;\alpha)$ admits the Cauchy's integral representation

$$r_n(t,x;\alpha) = \frac{(t-t_0)^n}{2\pi i}\oint_C \frac{h(w,x;\alpha)dw}{(w-t_0)^n(w-t)},$$

where the integration path C is any simple closed contour around the point $w = t_0$ that contains the point $w = t_0$ and $w = t$, for all $t \in I$, oriented in the positive direction. We choose C to be the circle of center t_0 and radius $2|x^{-1/q}|$. Then, for any $w \in C$, $|w - t_0| = 2|x^{-1/q}|$ and $|w - t| \geq |x^{-1/q}|$, for all $t \in I$. Moreover, we have that

$$h(w,x;\alpha) = e^{[f(w;\alpha)-p(w;\alpha)]}g(w) = e^{-x(w-t_0)^q\tilde{f}(w;\alpha)}g(w),$$

for a certain function $\tilde{f}(w;\alpha)$ analytic in a disk of center t_0 that contains the path C. Thus, for all $w \in C$

$$|h(w,x;\alpha)| \leq e^{-|x||w-t_0|^q|\tilde{f}(w;\alpha)|}|g(w)| \leq e^{-2^qM_1}M_2 \leq M_3,$$

for certain positive constants M_1, M_2 and M_3 independent of w and x. We highlight the choice of the path C to cancel out the parameter x of the exponent above. It follows that

$$|r_n(t,x;\alpha)| \leq K_n|t-t_0|^n|x^{n/q}|, \qquad \forall t \in I, \tag{8.40}$$

with $K_n := M_3/(2^{n-1}) > 0$ independent of x and t.

Then

$$|R_n^0(x;\alpha)| \le K_n |x^{n/q}| \int_{t_0-\theta|x^{-1/q}|}^{t_0+|x^{-1/q}|} e^{-\Re x\left[a_1(t-t_0)+a_2(t-t_0)^2\right]} |t-t_0|^n (t-a)^{s-1} dt. \quad (8.41)$$

With the aim to further analyze this remainder, we distinguish the three different cases $\alpha < \alpha^*$, $\alpha = \alpha^*$ and $\alpha > \alpha^*$.

- Case 1: $\alpha < \alpha^*$. We have $\theta = 0$ and $t_0 = a$. Then, from (8.41)

$$\begin{aligned} |R_n^0(x;\alpha)| &\le K_n |x^{n/q}| \int_a^{a+|x^{-1/q}|} e^{-\Re x\left[a_1(t-a)+a_2(t-a)^2\right]} (t-a)^{n+s-1} dt \\ &= K_n |x^{n/q}| \int_0^{|x^{-1/q}|} e^{-\Re x\left[a_1 t + a_2 t^2\right]} t^{n+s-1} dt \\ &\le K_n |x^{n/q}| \int_0^{\infty} e^{-\Re x\left[a_1 t + a_2 t^2\right]} t^{n+s-1} dt = K_n |x^{n/q}| \Phi_n(\Re x, s; \alpha), \end{aligned}$$

  which follows from the definition of $\Phi_n(x,s;\alpha)$ in (8.27) in the particular case $t_0 = a$. Taking into account the asymptotic behavior of $\Phi_n(x,s;\alpha)$ shown above, we find

$$R_n^0(x;\alpha) = \mathcal{O}\left(x^{n/q-(n+s)}\right), \qquad \text{as } |x| \to \infty. \quad (8.42)$$

- Case 2: $\alpha = \alpha^*$. In this case $\theta = 0$, $t_0 = a$ and $a_1 = 0$. Then, from (8.41)

$$\begin{aligned} |R_n^0(x;\alpha)| &\le K_n |x^{n/q}| \int_a^{a+|x^{-1/q}|} e^{-a_2 \Re x (t-a)^2} (t-a)^{n+s-1} dt \\ &= K_n |x^{n/q}| \int_0^{|x^{-1/q}|} e^{-a_2 \Re x t^2} t^{n+s-1} dt \\ &\le K_n |x^{n/q}| \int_0^{\infty} e^{-a_2 \Re x t^2} t^{n+s-1} dt = K_n |x^{n/q}| \Phi_n(\Re x, s; \alpha), \end{aligned}$$

  From the asymptotic behavior of the function $\Phi_n(x,s;\alpha)$, we obtain

$$R_n^0(x;\alpha) = \mathcal{O}\left(x^{n/q-(n+s)/2}\right), \qquad \text{as } |x| \to \infty. \quad (8.43)$$

- Case 3: $\alpha > \alpha^*$. Then, $\theta = 1$, $t_0 = \bar{t_0} \in (a,b)$ and $a_1 = 0$. From (8.41)

$$|R_n^0(x;\alpha)| \le K_n |x^{n/q}| \int_{t_0-|x^{-1/q}|}^{t_0+|x^{-1/q}|} e^{-\Re x\, a_2 (t-t_0)^2} |t-t_0|^n (t-a)^{s-1} dt.$$

  We further split the integral at $t = t_0$ and write

$$|R_n^0(x;\alpha)| \le K_n |x^{n/q}| \left[H_1(x;\alpha) + H_2(x;\alpha)\right], \quad (8.44)$$

  where

$$\begin{aligned} H_1(x;\alpha) &:= \int_{t_0-|x^{-1/q}|}^{t_0} e^{-a_2 \Re x (t-t_0)^2} (t_0 - t)^n (t-a)^{s-1} dt \\ &= \int_0^{|x^{-1/q}|} e^{-a_2 \Re x t^2} t^n (t_0 - a + t)^{s-1} dt, \end{aligned}$$

and

$$H_2(x;\alpha) := \int_{t_0}^{t_0+|x^{-1/q}|} e^{-a_2 \Re x(t-t_0)^2} (t-t_0)^n (t-a)^{s-1} dt$$
$$= \int_0^{|x^{-1/q}|} e^{-a_2 \Re x t^2} t^n (t+t_0-a)^{s-1} dt,$$

Now, we have

$$(t_0 - a \pm t)^{s-1} \sim (t_0 - a)^{s-1}, \qquad \text{as } t \to 0^+,$$

in other words, as $t_0 > a$, the factor $(t_0 - a \pm t)^{s-1}$ does not have any influence on the asymptotic analysis of the integrals $H_1(x;\alpha)$ and $H_2(x;\alpha)$. Then, a direct application of Laplace method gives

$$H_k(x;\alpha) \sim (t_0-a)^{s-1} \int_0^{|x^{-1/q}|} e^{-a_2 \Re x t^2} t^n dt \leq (t_0-a)^{s-1} \int_0^{\infty} e^{-a_2 \Re x t^2} t^n dt$$
$$= \frac{(t_0-a)^{s-1} \Gamma\left(\frac{n+1}{2}\right)}{2 a_2^{\frac{n+1}{2}} (\Re x)^{\frac{n+1}{2}}},$$

valid for $k = 1, 2$ as $|x| \to \infty$. Consequently, (8.44) yields

$$R_n^0(x;\alpha) = \mathcal{O}\left(x^{n/q-(n+1)/2}\right), \qquad \text{as } |x| \to \infty. \tag{8.45}$$

If we compare the asymptotic behavior of $F_1(x;\alpha)$ (8.37), $F_2(x;\alpha)$ (8.38), and $\Psi_n(x;\alpha)$ (8.39) with the asymptotic behavior of $R_n^0(x;\alpha)$ (8.42), (8.43) and (8.45), it turns out that the former are exponentially small compared to the latter, as $|x| \to \infty$. Then, (8.20) follows from equality (8.28).

□

Remark 8.2.2. *From hypothesis H8.1.(ii), $t_0(\alpha)$ is a continuous function of α and, as the parabolic cylinder function $U(a,x)$ is a continuous function of its arguments, from (8.18) and the first line of (8.17) the functions $\Phi_n(x;\alpha,s)$ are also continuous functions of α, for $\alpha \in (\alpha_1, \alpha_2)$. Moreover, the coefficients $B_n(\alpha)$ defined in (8.11) are continuous functions of α because $t_0(\alpha)$ is a continuous function of α. From hypothesis H8.1.(iv), the coefficients $A_n(x;\alpha)$ and $C_n(x;\alpha)$ are also continuous functions of α and so are the coefficients $h_n(x;\alpha)$. Consequently, the partial sums in the right hand side of (8.16) are continuous functions of α, for all $\alpha \in (\alpha_1, \alpha_2)$. Although the asymptotic behavior of $\Phi_n(x;\alpha,s)$ changes as α crosses the critical value α^*, the transition from one to other asymptotic behavior is not abrupt, but continuous and encoded in the parabolic cylinder functions that define the asymptotic sequence $\Phi_n(x;\alpha,s)$.*

Remark 8.2.3. *The right hand side of (8.16) is not a genuine Poincaré type asymptotic expansion of the function $F(x;\alpha)$, as the asymptotic sequence is not of the form x^{-n}. The asymptotic behavior of the terms in the expansion do not decrease monotonically with the order n of the approximation, but in the form of a sawtooth.*

8.3 Examples

In this section we consider the hypergeometric confluent functions $M(c,d,x)$ and $U(c,d,x)$ [103] for large d and x, and of the same order, that is, we consider the functions $M(c,x/\alpha+c+1,x)$ and $U(c,\alpha x+c+1,x)$ for large x and fixed α and c. In [67] the authors have found three different non-uniform expansions for those functions and large x, according to whether $\alpha>1$, $\alpha=1$ or $\alpha<1$, with explicit formulas for the coefficients. On the other hand, in [139] the author has obtained uniform asymptotic expansions for $M(c,d,x)$ and $U(c,d,x)$ for large d and x, in terms of parabolic cylinder functions, by applying the classical method due to Bleistein, but this method does not offer an explicit general formula for the coefficients of the expansion. Then, we are going to apply theorem 8.2.1 to derive new uniform asymptotic expansions of the functions $M(c,x/\alpha+c+1,x)$ and $U(c,\alpha x+c+1,x)$ for large x and fixed c, uniformly valid for $\alpha\in(0,\infty)$, with explicit formulas for the coefficients.

8.3.1 The confluent hypergeometric function $U(c,d,x)$ for large d and x

We consider the confluent hypergeometric function $U(c,d,x)$ [103, eq. 13.4.4]

$$U(c,d,x)=\frac{1}{\Gamma(c)}\int_0^\infty e^{-xt}t^{c-1}(1+t)^{d-c-1}dt,\qquad \Re x,\Re c>0,$$

for large $|d|$ and $|x|$ and of the same order. More precisely, we take $d=\alpha x+c+1$ with fixed $\alpha\in(0,\infty)$, independent of x. Then,

$$U(c,\alpha x+c+1,x)=\frac{1}{\Gamma(c)}\int_0^\infty e^{-xt}t^{c-1}(1+t)^{\alpha x}dt=\frac{1}{\Gamma(c)}\int_0^\infty e^{-x[t-\alpha\log(t+1)]}t^{c-1}dt,$$

is of the form (8.15) with $f(t;\alpha)=t-\alpha\log(t+1)$, $g(t)=1$, $s=c$, $a=0$ and $b=+\infty$. Therefore, the phase function $f(t,\alpha)$ has a unique critical point at the point $t=\bar{t_0}(\alpha)=\alpha-1$. This point is inside the integration interval $(0,\infty)$ for $\alpha>\alpha^*=1$ and it is outside the interval if $\alpha<1$. Then, for $\alpha<1$ the absolute minimum of $f(t;\alpha)$ is attained at the left end point of the integration interval $t=0$. Following the notation of the previous section, we have

$$t_0(\alpha)=\begin{cases}0, & \text{if } \alpha\le 1,\\ \bar{t_0}(\alpha)=\alpha-1, & \text{if } \alpha\ge 1,\end{cases}$$

$$f\left(t_0;\alpha\right)=\begin{cases}0, & \text{if } \alpha\le 1,\\ \alpha-1-\alpha\log\alpha, & \text{if } \alpha\ge 1,\end{cases}\qquad a_1=\begin{cases}1-\alpha, & \text{if } \alpha\le 1,\\ 0, & \text{if } \alpha\ge 1,\end{cases}$$

$$a_2=\begin{cases}\frac{\alpha}{2}, & \text{if } \alpha\le 1,\\ \frac{1}{2\alpha}, & \text{if } \alpha\ge 1,\end{cases}\qquad f^{(3)}(t_0;\alpha)=\begin{cases}-2\alpha, & \text{if } \alpha\le 1,\\ \frac{-2}{\alpha^2}, & \text{if } \alpha\ge 1.\end{cases}$$

In particular, the third derivative of $f(t,\alpha)$ at $t=t_0$ does not vanish for any value of $\alpha\in(0,\infty)$ and therefore $q=3$. Moreover, hypotheses H8.1.(i)-H8.1.(v) are satisfied and

$\|x\|$	$\alpha = 0.5$	$\alpha = 0.999$	$\alpha = 1$	$\alpha = 1.001$	$\alpha = 2$
10	$9.91163 \cdot 10^{-3}$	$1.50196 \cdot 10^{-2}$	$1.51878 \cdot 10^{-2}$	$1.51406 \cdot 10^{-2}$	$7.92813 \cdot 10^{-2}$
50	$3.78331 \cdot 10^{-4}$	$3.35825 \cdot 10^{-3}$	$3.44214 \cdot 10^{-3}$	$3.41987 \cdot 10^{-3}$	$9.57528 \cdot 10^{-4}$
125	$3.4385 \cdot 10^{-5}$	$1.37483 \cdot 10^{-3}$	$1.4295 \cdot 10^{-3}$	$1.41525 \cdot 10^{-3}$	$3.27337 \cdot 10^{-4}$
500	$6.42147 \cdot 10^{-7}$	$3.41359 \cdot 10^{-4}$	$3.69141 \cdot 10^{-4}$	$3.61993 \cdot 10^{-4}$	$8.29282 \cdot 10^{-5}$

Table 8.1: Absolute value of the relative errors in the approximation of $U(c, \alpha\, x + c + 1, x)$ for $c = 1.3$, $\arg x = \pi/3$ with increasing $|x|$ and several values of α obtained by truncating the right hand side of (8.46) after 4 terms.

we can apply theorem 8.2.1 to find the expansion

$$U(c, \alpha x + c + 1, x) \sim \exp\left[x\left(\frac{a_1 t_0}{2} - \frac{a_2 t_0^2}{2} + \frac{a_1^2}{8a_2} - f(t_0;\alpha)\right)\right]\sum_{n=0}^{\infty} h_n(x;\alpha)\times \\ \times \sum_{k=0}^{n}\binom{n}{k}\frac{(c)_k(-t_0)^{n-k}}{(2a_2 x)^{\frac{k+c}{2}}}U\left(k + c - \frac{1}{2}, \frac{a_1 - 2a_2 t_0}{\sqrt{2a_2}}\sqrt{x}\right). \tag{8.46}$$

Furthermore, the coefficients $h_n(x;\alpha)$ may be computed by means of

$$h_n(x;\alpha) = \sum_{k=0}^{n}\left(\sum_{j=0}^{\lfloor k/2\rfloor} x^{k-j}\frac{a_2^j}{j!}\frac{a_1^{k-2j}}{(k-2j)!}\right)A_{n-k}(x;\alpha),$$

with

$$A_n(x;\alpha) = \frac{(-1)^n}{n!}x^n\sum_{k=0}^{n}\binom{n}{k}\frac{(-\alpha x)_k}{[(1+t_0)x]^k},$$

which follows from lemma 8.1.1(i). On the other hand, they can also be computed recursively in the form

$$\begin{cases} h_0(x;\alpha) = 1, \qquad h_1(x;\alpha) = 0, \qquad h_2(x;\alpha) = 0, \\ h_n(x;\alpha) = \dfrac{\alpha x}{n}\displaystyle\sum_{k=0}^{n-3} h_k(x;\alpha)\frac{(-1)^{n-k+1}}{(1+t_0)^{n-k}}, \qquad \text{for } n \geq 3. \end{cases}$$

which follows from lemma 8.1.1(ii).

In Tables 8.1, 8.2 and 8.3 the relative error of the first n terms of the right hand side of (8.46) are shown, for several orders of approximation n and different values of x and α. The asymptotic features of the expansion as well as its uniform validity as α crosses the critical value $\alpha^* = 1$ are exhibited.

8.3.2 The confluent hypergeometric function $M(c, d, x)$ for large d and x

We consider the confluent hypergeometric function $M(c, d, x)$ [103, eq. 13.4.1]

$$M(c,d,x) = \frac{\Gamma(d)}{\Gamma(c)\Gamma(d-c)}\int_0^1 e^{xt}t^{c-1}(1-t)^{d-c-1}dt, \qquad \Re d > \Re c > 0,$$

$\lvert x\rvert$	$\alpha=0.5$	$\alpha=0.999$	$\alpha=1$	$\alpha=1.001$	$\alpha=2$
10	$7.37175\cdot 10^{-3}$	$3.13925\cdot 10^{-2}$	$3.13836\cdot 10^{-2}$	$3.13035\cdot 10^{-2}$	$7.20888\cdot 10^{-3}$
50	$4.83724\cdot 10^{-5}$	$5.18706\cdot 10^{-3}$	$5.20778\cdot 10^{-3}$	$5.1777\cdot 10^{-3}$	$1.7074\cdot 10^{-4}$
125	$1.09292\cdot 10^{-6}$	$1.54702\cdot 10^{-3}$	$1.56002\cdot 10^{-3}$	$1.54608\cdot 10^{-3}$	$2.57828\cdot 10^{-5}$
500	$1.70064\cdot 10^{-9}$	$2.19347\cdot 10^{-4}$	$2.23829\cdot 10^{-4}$	$2.1996\cdot 10^{-4}$	$1.58766\cdot 10^{-6}$

Table 8.2: Absolute value of the relative errors in the approximation of $U(c,\alpha\,x+c+1,x)$ for $c=1.7$, $\arg x=-\pi/4$ with increasing $|x|$ and several values of α obtained by truncating the right hand side of (8.46) after 7 terms.

x	$\alpha=0.5$	$\alpha=0.999$	$\alpha=1$	$\alpha=1.001$	$\alpha=2$
50	$4.47618\cdot 10^{-6}$	$1.14525\cdot 10^{-2}$	$1.16025\cdot 10^{-2}$	$1.15118\cdot 10^{-2}$	$6.27377\cdot 10^{-4}$
125	$1.51468\cdot 10^{-8}$	$1.88652\cdot 10^{-3}$	$1.92964\cdot 10^{-3}$	$1.90678\cdot 10^{-3}$	$9.16883\cdot 10^{-5}$
300	$3.22221\cdot 10^{-11}$	$3.19066\cdot 10^{-4}$	$3.31144\cdot 10^{-4}$	$3.25267\cdot 10^{-4}$	$1.53032\cdot 10^{-5}$
2000	$2.11941\cdot 10^{-16}$	$6.39149\cdot 10^{-6}$	$7.08214\cdot 10^{-6}$	$6.77489\cdot 10^{-6}$	$3.34849\cdot 10^{-7}$

Table 8.3: Absolute value of the relative errors in the approximation of $U(c,\alpha\,x+c+1,x)$ for $c=2.1$, x real and positive and several values of α obtained by truncating the right hand side of (8.46) after 11 terms.

for large $|d|$ and $|x|$ and of the same order. More precisely, we take $d=\frac{x}{\alpha}+c+1$, with $\alpha\in(0,\infty)$ fixed, independent of x. Then

$$M\left(c,\frac{x}{\alpha}+c+1,x\right)=\frac{\Gamma(\frac{x}{\alpha}+c+1)}{\Gamma(c)\Gamma(\frac{x}{\alpha}+1)}\int_0^1 t^{c-1}e^{-x\left[-t-\frac{1}{\alpha}\log(1-t)\right]}dt,$$

is of the form (8.15) with $f(t;\alpha)=-t-\frac{1}{\alpha}\log(1-t)$, $g(t)=1$, $s=c$, $a=0$ and $b=1$. The phase function $f(t;\alpha)$ has a unique critical point that occurs at $t=\bar{t_0}(\alpha)=\alpha-1$, which is a relative minimum. For $\alpha>1=\alpha^*$ the critical point is inside the integration interval, that is, $\bar{t_0}(\alpha)\in(0,1)$ for $\alpha>1$. On the other hand, for $\alpha<1$, the function $f(t;\alpha)$ is monotonically increasing in the whole integration interval $(0,1)$ and the absolute minimum of $f(t;\alpha)$ occurs at the left end point of the integration interval $t=0$. Following the notation of the previous section, we have

$$t_0(\alpha)=\begin{cases}0, & \text{if } \alpha\le 1,\\ \bar{t_0}(\alpha)=1-\frac{1}{\alpha}, & \text{if } \alpha\ge 1,\end{cases}$$

$$f\left(t_0;\alpha\right)=\begin{cases}0, & \text{if } \alpha\le 1,\\ \frac{1-\alpha+\log\alpha}{\alpha}, & \text{if } \alpha\ge 1,\end{cases}\qquad a_1=\begin{cases}\frac{1}{\alpha}-1, & \text{if } \alpha\le 1,\\ 0, & \text{if } \alpha\ge 1,\end{cases}$$

$$a_2=\begin{cases}\frac{1}{2\alpha}, & \text{if } \alpha\le 1,\\ 2\alpha, & \text{if } \alpha\ge 1,\end{cases}\qquad f^{(3)}(t_0;\alpha)=\begin{cases}\frac{2}{\alpha}, & \text{if } \alpha\le 1,\\ 2\alpha^2, & \text{if } \alpha\ge 1.\end{cases}$$

In particular, the third derivative of $f(t,\alpha)$ at $t=t_0$ does not vanish for any value of $\alpha\in(0,\infty)$ and therefore $q=3$. Moreover, hypotheses H8.1.(i)-H8.1.(v) are satisfied and

$\lvert x\rvert$	$\alpha = 0.5$	$\alpha = 0.999$	$\alpha = 1$	$\alpha = 1.001$	$\alpha = 2$
10	$4.72393 \cdot 10^{-3}$	$6.67767 \cdot 10^{-2}$	$6.74219 \cdot 10^{-2}$	$6.72191 \cdot 10^{-2}$	$2.32909 \cdot 10^{-2}$
50	$1.8438 \cdot 10^{-4}$	$1.11182 \cdot 10^{-2}$	$1.13492 \cdot 10^{-2}$	$1.12687 \cdot 10^{-2}$	$3.09317 \cdot 10^{-3}$
125	$1.73515 \cdot 10^{-5}$	$4.17696 \cdot 10^{-3}$	$4.31349 \cdot 10^{-3}$	$4.26452 \cdot 10^{-3}$	$1.29576 \cdot 10^{-3}$
500	$3.39302 \cdot 10^{-7}$	$9.68183 \cdot 10^{-4}$	$1.03212 \cdot 10^{-3}$	$1.00861 \cdot 10^{-3}$	$3.31046 \cdot 10^{-4}$

Table 8.4: Absolute value of the relative errors in the approximation of $M(c, x/\alpha + c + 1, x)$ for $c = 1.8$, $\arg x = \pi/6$ with increasing $|x|$ and several values of α obtained by truncating the right hand side of (8.47) after 4 terms.

we can apply theorem 8.2.1 to find the expansion

$$M\left(c, \frac{x}{\alpha} + c + 1, x\right) \sim \exp\left[x\left(\frac{a_1 t_0}{2} - \frac{a_2 t_0^2}{2} + \frac{a_1^2}{8a_2} - f(t_0;\alpha)\right)\right] \frac{\Gamma(\frac{x}{\alpha} + c + 1)}{\Gamma(\frac{x}{\alpha} + 1)} \times \\ \times \sum_{n=0}^{\infty} h_n(x;\alpha) \sum_{k=0}^{n} \binom{n}{k} \frac{(c)_k (-t_0)^{n-k}}{(2a_2 x)^{\frac{k+c}{2}}} U\left(k + c - \frac{1}{2}, \frac{a_1 - 2a_2 t_0}{\sqrt{2a_2}} \sqrt{x}\right), \tag{8.47}$$

where $(c)_k$ denotes the Pochhammer's symbol [4, §5.2(iii)]. Furthermore, the coefficients $h_n(x;\alpha)$ may be computed by means of

$$h_n(x;\alpha) = \sum_{k=0}^{n} \left(\sum_{j=0}^{\lfloor k/2 \rfloor} x^{k-j} \frac{a_2^j}{j!} \frac{a_1^{k-2j}}{(k-2j)!} \right) A_{n-k}(x;\alpha),$$

with

$$A_n(x;\alpha) = \frac{x^n}{n!} \sum_{k=0}^{n} \binom{n}{k} \frac{(\frac{-x}{\alpha})_k}{[(t_0 - 1)x]^k},$$

which follows from lemma 8.1.1(i). On the other hand, they can also be computed recursively in the form

$$\begin{cases} h_0(x;\alpha) = 1, \qquad h_1(x;\alpha) = 0, \qquad h_2(x;\alpha) = 0, \\ h_n(x;\alpha) = \dfrac{-x}{\alpha n} \displaystyle\sum_{k=0}^{n-3} \frac{h_k(x;\alpha)}{(1 - t_0)^{n-k}}, \qquad \text{for } n \geq 3. \end{cases}$$

which follows from lemma 8.1.1(ii). Finally, tables 8.4, 8.5 and 8.6 contain some numerical experiments about the accuracy of the expansion on the right hand side of (8.47) for several values of the asymptotic variable x and the uniform parameter α.

$\|x\|$	$\alpha = 0.5$	$\alpha = 0.999$	$\alpha = 1$	$\alpha = 1.001$	$\alpha = 2$
10	$2.30956 \cdot 10^{-3}$	$1.78422 \cdot 10^{-1}$	$1.79622 \cdot 10^{-1}$	$1.79362 \cdot 10^{-1}$	$1.90181 \cdot 10^{-1}$
50	$6.952 \cdot 10^{-6}$	$1.05984 \cdot 10^{-2}$	$1.07168 \cdot 10^{-2}$	$1.06675 \cdot 10^{-2}$	$6.89637 \cdot 10^{-3}$
125	$1.29727 \cdot 10^{-7}$	$2.33898 \cdot 10^{-3}$	$2.37607 \cdot 10^{-3}$	$2.3575 \cdot 10^{-3}$	$1.09354 \cdot 10^{-3}$
500	$1.81357 \cdot 10^{-10}$	$2.56346 \cdot 10^{-4}$	$2.63712 \cdot 10^{-4}$	$2.5939 \cdot 10^{-4}$	$6.81279 \cdot 10^{-5}$

Table 8.5: Absolute value of the relative errors in the approximation of $M(c, x/\alpha + c + 1, x)$ for $c = 1.6$, $\arg x = -\pi/5$ with increasing $|x|$ and several values of α obtained by truncating the right hand side of (8.47) after 7 terms.

x	$\alpha = 0.5$	$\alpha = 0.999$	$\alpha = 1$	$\alpha = 1.001$	$\alpha = 2$
50	$1.0563 \cdot 10^{-7}$	$1.04366 \cdot 10^{-3}$	$1.08498 \cdot 10^{-3}$	$1.06838 \cdot 10^{-3}$	$2.5643 \cdot 10^{-3}$
125	$2.16124 \cdot 10^{-10}$	$9.03597 \cdot 10^{-4}$	$9.34321 \cdot 10^{-4}$	$9.24176 \cdot 10^{-4}$	$6.552 \cdot 10^{-4}$
500	$1.04613 \cdot 10^{-12}$	$8.25788 \cdot 10^{-5}$	$8.75967 \cdot 10^{-5}$	$8.57636 \cdot 10^{-5}$	$7.43843 \cdot 10^{-5}$
1800	$2.07736 \cdot 10^{-16}$	$6.79056 \cdot 10^{-6}$	$7.5558 \cdot 10^{-6}$	$7.25678 \cdot 10^{-6}$	$6.89563 \cdot 10^{-6}$

Table 8.6: Absolute value of the relative errors in the approximation of $M(c, x/\alpha + c + 1, x)$ for $c = 2.1$, x real and positive and several values of α obtained by truncating the right hand side of (8.47) after 11 terms.

Chapter 9

Conclusions and Future Work

This thesis presents new ingredients in the analytical approximation of integral transforms and, in particular, of special functions having an integral representation. In particular, we seek new methods to derive approximations in the form of a series of integral transforms satisfying the following three properties:

(a) The expansions are convergent.

(b) The expansions are given in terms of elementary functions whose coefficients can be straightforwardly computed by means of a simple, closed and systematic formula.

(c) The expansions are valid in a large (possibly unbounded) region $\mathcal{D}$ of the complex plane that contains small values of the selected variable or a selected point, say $z = 0$.

The main conclusions of the work are given in the next section 9.1. The chapter ends with section 9.2 that exhibits some possible future work related with the research of this thesis.

9.1 Conclusions

In **Chapter 3**, under some mild conditions on the functions $h(t, z)$ and $g(t)$, we have designed a general method of constructing new expansions of the parametric integral $F(z) = \int_0^1 h(t, z)g(t)dt$ satisfying properties (a), (b) and (c) above. In contrast, the classical Taylor or asymptotic expansions are only valid for, respectively, small or large values of $|z|$. Therefore, the *uniform approximation* proposed in this chapter provides smaller errors in any $L_p(\mathcal{D})$ norm than the Taylor or asymptotic approximations when the domain $\mathcal{D}$ is large enough[1].

The convergence of the uniform approximations can be either of exponential or power type, depending on whether the end points $t = 0$ and $t = 1$ of the integration interval are regular points of $g(t)$ or not. More precisely, let $R_n(z)$ denote the remainder of the uniform approximation when it is truncated after n terms. We have that $R_n(z) = \mathcal{O}(a^{-n})$, for some $a > 1$ if the integration interval $[0, 1]$ is completely contained in the region of convergence D_r of the (multi-point) Taylor expansion of $g(t)$. On the other hand, if $(0, 1) \subset D_r$ but $[0, 1] \subsetneq D_r$, that is, $t = 0$ and/or $t = 1$ are singular points of $g(t)$, the

[1]We only consider those values of p for which the $L_p(\mathcal{D})$ norm is finite.

remainder satisfies $R_n(z) = \mathcal{O}(n^{-\delta})$, for some $\delta > 0$ that is related with the behavior of the function $g(t)h(t,z)$ at the end points $t = 0$ and/or $t = 1$.

In the worst scenario, when one or both end points of the integration interval are singular points of $g(t)$ the speed of convergence of the uniform expansions is not impressive. Still, they are a nice numerical tool for the evaluation of integral transforms $F(z)$ or special functions admitting such an integral representation. Indeed, in **Chapter 4** we have applied the theory introduced in the previous chapter to many special functions of the mathematical physics. These examples illustrate the applicability of the theory of *uniform approximations*. In particular, we have derived new convergent expansions of the following special functions: the Struve function $H_\nu(z)$; the Bessel function of the first kind $J_\nu(z)$; the incomplete gamma functions $\gamma(a,z)$ and $\Gamma(a,z)$; the confluent hypergeometric $M(a,b,z)$ and $U(a,b,z)$ functions; the exponential integral $E_1(z)$; the symmetric elliptic integrals $R_F(x,y,z)$ and $R_D(x,y,z)$; the Gauss hypergeometric function ${}_2F_1(a,b,c;z)$ and the incomplete beta function $B_z(a,b)$.

In the general theory of chapter 3 we have considered the possibility of expanding the function $g(t)$ at multiple points. This may be necessary from a theoretical point of view to assure that the domain of convergence of the Taylor series contains the integration interval. However, in all the derived uniform expansions of chapter 4 the use of one-point Taylor series was enough to satisfy this requirement. Therefore, for simplicity, we have taken only one base point. In this manner we have derived expansions given in terms of elementary functions and that hold in unbounded regions of the complex plane that contain small values of $|z|$. In the examples analyzed in chapter 4 such regions are horizontal strips, half-planes, or a region $S(\theta)$ that depends on a certain angle $\theta \in [\pi/2, \pi)$ that, in the limit case $\theta \to \pi$, becomes the cutted complex plane $\mathbb{C} \setminus (-\infty, -1]$.

We have compared the uniform expansions with the classical expansions derived from Taylor series or asymptotic expansions by means of numerical tables 4.1-4.10 and figures 4.1-4.7, 4.9-4.11. In general, the numerical results show that the use of Taylor power series is recommended over the uniform approximation when we want to obtain an approximation valid only for small values of the variable. Similarly, asymptotic expansions should be used if an approximation only for large values of the variables is needed. However, there exists a certain intermediate region where the uniform approximation performs better than both, Taylor series and asymptotic expansions. But, moreover, uniform expansions provide approximations that are globally more satisfactory. Therefore, uniform approximations are recommended when we are working with the function $F(z)$ in a large region of the complex plane that contains both, small and large values of the variable $|z|$.

For example, in spectral problems of atomic models in quantum mechanics, the computation of the spectrum requires the knowledge of the analytic behavior of the eigenfunctions for both, large and small values of the radius. In this problem, the use of Taylor or asymptotic expansions is useless as their validity is local. In contrast, uniform approximations are valid in a sufficiently large set and therefore they may be useful to compute the spectrum easily.

More generally, consider a certain computation, may be an integral or a differential equation, that involves the function $F(z)$, and in which the variable z runs in a large domain $\mathcal{D}$. In general, the complicated analytical expression of $F(z)$ makes that calculation impossible, but we may just replace $F(z)$ by its uniform approximation given in terms of elementary functions. This replacement is valid in a large region of the complex plane and, in most cases, it will be valid for the whole range of values of the variable

z required in that calculation (range of integration, domain of the differential equation, etc...). Then, replacing $F(z)$ by its uniform approximation makes that calculation simpler as it is given in terms of elementary functions.

This approach has been taken in **Chapter 5** to obtain a convergent expansion of the Volterra function. The Volterra function appears in many branches of mathematics and plays an important role in the theory of Laplace transforms. Despite its importance, it has barely been investigated, probably due to the presence of an inverse gamma function in the integrand of its integral definition. After some manipulations, we have obtained an incomplete gamma function in an integral representation of the Volterra function. Then, the replacement of that incomplete gamma function by its uniform approximation easily provides an expansion of this function, that has been shown to be convergent.

In **Chapter 6** we have developed a somehow dual expansion of the uniform approximation of integral transforms $F(z)=\int_0^1 h(t,z)g(t)dt$ considered in chapter 3. Roughly speaking, the uniform approximations were found by replacing $g(t)$ in the integrand of $F(z)$ by its multi-point Taylor expansion and interchanging summation and integration. In chapter 6 we consider the multi-point Taylor expansion of the factor $h(t,z)$ at certain selected m points in a manner that the integration interval $[0,1]$ is completely contained in the lemniscate of convergence of the multi-point Taylor series of $h(t,z)$. We let z run in a large set S of the complex plane such that the singularities of $h(t,z)$ (as a function of t) are located outside the lemniscate of convergence of the multi-point Taylor series of $h(t,z)$. In this way, the domain of validity of the expansion for $F(z)$ (the region S) is enlarged. We illustrate this idea in the particular case when $F(z)$ is an integral representation of the generalized hypergeometric function ${}_{p+1}F_p(a,\vec{b},\vec{c};z)$ and $h(t,z)=(1-zT)^{-a}$. This function has a singularity at $T=1/z$ that can be efficiently avoided by using the multi-point Taylor expansion of $h(t,z)$. In particular, we use one-, two- and three-point Taylor expansions at certain, cleverly chosen base points, to obtain unbounded regions of convergence that contain the indented closed unit disk $D^*=\{z\in\mathbb{C}:|z|\le 1,\ z\ne 1\}$. Furthermore, the derived expansions are given in terms of rational functions. Therefore, this approach also produces expansions satisfying the three properties (a), (b) and (c) detailed at the beginning of this chapter.

Moreover, accurate bounds for the remainder of the new derived approximations show that the speed of convergence is exponential. That is, if $R_n(z)$ denotes the remainder when we truncate the expansion after n terms, then $R_n(z)=\mathcal{O}(A(z)^n)$, for some $A(z)<1$ that is related with the convergence region S. This fact is supported by some numerical experiments. They suggest that the expansion derived from a three-point Taylor expansion performs numerically better than other expansions: The three-point Taylor expansion is more accurate and valid in a bigger domain than the one- or two-point Taylor expanions derived in the same chapter. But moreover, it also performs better than other expansions that we may find in the literature such as the power series definition, formulas (6.3)+(6.2) or Büring's connection formula. On the other hand, the use of multi-point Taylor expansions at four o more points would enlarge the region of convergence, but the coefficients of the expansion would become more and more complicated.

In **Chapter 7** we have considered a special integral transform: when the integration interval is unbounded and the kernel $h(t,z)$ is an exponential of the form $e^{-zf(t)}$. In other words, we have considered integrals of the form

$$F(z)=\int_0^\infty e^{-zf(t)}g(t)dt, \tag{9.1}$$

where the Laplace's method is usually used to obtain an asymptotic expansion for large z. This method is very powerful but has three main drawbacks[2]:

- The expansion has a local validity as it is only valid for large values of $|z|$.
- The asymptotic expansion is divergent and then it is required to find accurate error bounds and to study the optimal truncation term of the series.
- The coefficients of the expansion are defined by reverting a certain series implicitly given by a change of variable.

We have solved these problems at once by designing a *convergent and asymptotic Laplace's method* for integrals whose coefficients can be computed by means of an explicit, simple formula, in terms of generalized Bernoulli polynomials and the Taylor coefficients of $e^{-zf(t)}$ and $g(t)$ at the asymptotically dominant point of $f(t)$, as $z \to \infty$. The price to pay is that the asymptotic sequence is not straightforward to compute as in the classical Laplace's method, but given in terms of (incomplete) beta functions.

The conditions on the function $f(t)$ and $g(t)$ to apply the convergent and asymptotic Laplace's method are similar to the ones required for the application of the classical Laplace method, except that we require for the functions to be analytic in a certain region $S_m(\infty, 0)$. However, performing some tricks, this restriction can be relaxed and require, basically, the analyticity of both functions in a neighborhood of the point t_0 where the phase function $f(t)$ attains its absolute minimum.

Then, the designed convergent and asymptotic Laplace's method for integrals provides asymptotic expansions satisfying properties (a) and (c) at the beginning of the chapter, but we have lost property (b). More precisely, although the coefficients are given by an explicit formula, the asymptotic sequence is given in terms of incomplete beta functions.

Finally, in **Chapter 8** we have no longer derived convergent expansions of integrals. Instead, we have again considered integrals of the form

$$F(z;\alpha) = \int_a^b e^{-zf(t;\alpha)} g(t)dt,$$

and we have sought an asymptotic expansion of $F(z;\alpha)$ for large z uniformly valid as α varies. For fixed value of the parameter α, the method of Laplace provides three different asymptotic expansions for this integral, depending on the location of the absolute minimum of the phase function with respect to the integration interval $[a, b]$. However, the three expressions are formally different. Imagine now that the parameter α varies in a way that changes the nature of the absolute minimum of $f(t)$, from being an interior point $t_0 \in (a, b)$ to being one of the integration end points, say $t_0 = a$. Then, neither of the three expansions are useful when α is near that transition point, because the coefficients blow up. Moreover, that transition is not continuous, but abrupt. Then, the classical uniform "saddle point near an end point" asymptotic method has been designed to solve this problem. At the cost of considering the parabolic cylinder function as asymptotic sequence, an expansion valid uniformly in α can be found. However, the coefficients of the expansion are computed by reverting a certain series implictly given by a change of variables. For this reason, simple and explicit formulas for those coefficients

[2]By no means these drawbacks discredites the method of Laplace for the approximation of integrals when z is large.

are not given in traditional text books on asymptotics. In chapter 8 a modification of this method is considered, obtaining an explicit and systematic formula for the coefficients of the expansion. On the other hand, as in the classical method, the asymptotic sequence is given by parabolic cylinder functions.

9.2 Future work

Chapter 3 provides a new ingredient in the approximation of special functions: the uniform approximation of integral transforms. As they are valid in a large set of the complex plane, these expansions could be of interest to many researches around the world that work with special functions. Therefore, it could be interesting to update the famous *NIST Handbook of Mathematical Functions* [106, 108] by adding the uniform expansions of the different special functions in a new edition of this celebrated book.

Moreover, apart from the functions analyzed in **Chapter 4**, uniform expansions of more special functions can be found in the literature. For example, in [66] a uniform expansion of the Bessel function of the second kind $Y_\nu(z)$ given in terms of incomplete gamma functions can be found; and in [69] we can find uniform expansions of the generalized hypergeometric functions ${}_{p-1}F_p$ and ${}_pF_p$ that hold, respectively, in any horizontal strip and any right halfplane of the complex plane. It could be interesting to derive uniform expansion to other special functions not analyzed yet, such as the error function, the Airy function, the modified Bessel functions or the Riemann zeta function. Note that some of this functions are particular cases of other functions analyzed in chapter 4, but it would be better to study them on their own in order to obtain simpler expressions and/or more accurate error bounds.

On the other hand, we have seen in the figures of chapter 4 that there is usually an intermediate region where the uniform expansions perform numerically better than the power series and asymptotic expansions. It could be interesting to somehow determine the precise form of the regions in which each expansion is more accurate.

In **Chapter 6** we have used multi-point Taylor expansions to derive new analytical representations of the generalyzed hypergeometric ${}_{p+1}F_p$ function. The numerical results on the accuracy of the derived expansions (6.5), (6.18) and (6.25) show that they are more accurate than other expansions in the literature. But moreover, the use of multi-point Taylor expansions of a factor in an integral representation has been used in [79] to derive new expansions of the confluent hypergeometric $M(a,b,z)$ function that hold in the entire complex plane. The numerical results of that paper show that the three-point Taylor expansions are more accurate than the expansions obtained from two- or one-point Taylor series. They also show that the three-point Taylor expansion is more accurate than the power series definition of the M function in the cases where the latter was recommeneded for use in the literature.

Then, it would be interesting to consider other special functions having an integral representation of the form $F(z) := \int_0^1 h(t,z)g(t)dt$, for certain functions $g(t)$ and $h(t,z)$. In the particular cases when $h(t,z) = (1-zt)^{-a}$ or $h(t,z) = e^{zt}$ we respectively find the generalyzed hypergeometric ${}_{p+1}F_p$ function and the hypergeometric confluent M function. But, for other kernels $h(t,z)$ we would find other special functions. For example, if $h(t,z) = \cos(zt)$ we find an integral representation of the Bessel function of the first kind; for $h(t,z) = t^z$ we obtain a representation of the parabolic cylinder function or the confluent U function; for $h(t,z)=(-\log t)^z$ we get the gamma function or the Riemann's zeta function, etc.

Typically, using the classical Taylor series of $h(t,z)$ at $t=0$ (in the case when $h(t,z)=h(tz)$) we find a power series of $F(z)$ whereas asymptotic expansions of $F(z)$ are found from the asymptotic expansion of $g(t)$ at the asymptotically relevant point of $h(t,z)$. We can instead consider the multi-point Taylor expansions of $h(t,z)$ to derive new expansions because the lemniscates of convergence of the multi-point Taylor expansion avoid the singular points of $h(t,z)$ (that, in general, depend on z) in a more efficient manner. Then, it could be that the new expansions for $F(z)$ hold in large regions of the complex plane and at the same time are faster than the classical Taylor or asymptotic expansions.

In **Chapter 7** we have applied the developed convergent and asymptotic Laplace's method for integrals to the confluent hypergeometric $U(a,b,z)$ function with the aim to illustrate the applicability of that method. It could be interesting to apply this result to other special functions, in order to have expansions that are simultaneously convergent and asymptotic for large z.

On the other hand, in the case when the phase function $f(t)$ in the integral (9.1) is simply $f(t)=t$, the convergent and asymptotic Laplace method for integrals of chapter 7 produces the well-known *factorial series*. That is, series of the form $\sum_{n=0}^{\infty} c_n n!/(z)_n$ that are both, convergent and asymptotic for large z. This type of series were heavily studied at the beginning of the twentieth century [96, 97, 99, 133]. Back then, the integral representation (9.1) (with $f(t)=t$) was not important. The object of study were the factorial series theirselves. Therefore, it may be interesting to study the resulting series of the convergent and asymptotic Laplace method in more detail, as they are a natural generalization of factorial series.

In **Chapter 8** we have derived a systematic "saddle point near an end point method". This result is a continuation of the line of research initiated by my research group in [68, 74, 75] to systemise classical asymptotic methods. In [68] the uniform asymptotic "saddle point near a pole" method has been simplified, obtaining a systematic and explicit formula for the coefficients of the expansion, obtaining the error function as asymptotic sequence (as the classical method does). In chapter 8 we have systemised the uniform asymptotic method "saddle point near an end point". Continuing in this line, it remains to find a systematic "two coalescing saddle points" uniform asymptotic method.

Another related possible future work is to consider the multi-dimensional version of the methods developed in this thesis. In particular, it would be interesting to uniform approximations of multi-dimensional integrals and to modify the multi-dimensional Laplace's method [154, Ch. 9, §5] to make it convergent. As an application, we could derive, for example, new expansions for the Appell functions [3, §16.13].

Appendices

Appendix A

The Generalized Bernoulli Polynomials

The *generalized Bernoulli polynomials* $B_n^{(\alpha)}(x)$ can be defined by means of the generating function [36, §24.16], [88, Ch.6], [140, Ch.1 §1.1]

$$\left(\frac{t}{e^t-1}\right)^{\alpha} e^{xt} = \sum_{n=0}^{\infty} \frac{B_n^{(\alpha)}(x)}{n!} t^n, \qquad |t| < 2\pi, \tag{A.1}$$

where α and x are arbitrary complex numbers. For $x = 0$ we obtain the generalized Bernoulli numbers $B_n^{(\alpha)}(0) \equiv B_n^{(\alpha)}$, also called Nørlund polynomials [36, eq. 24.16.9], that are polynomials of degree n of the complex variable α.

In this thesis, we use them to compute the Taylor coefficients at an appropriate point of the composite function that results after a logarithmic change of variables of the form (7.21).

Lemma A.0.1. *For any $m \in \mathbb{N}$, let $\phi(t)$ be an analytic function in a disk $D_r(0)$ centered at $t = 0$ with radius $r > 0$. For any $\lambda \in \mathbb{C}$, consider the function*

$$\tilde{\phi}(x) := \left[-\frac{\log(1-x^m)}{x^m}\right]^{\lambda} \phi\left(x\left[-\frac{\log(1-x^m)}{x^m}\right]^{1/m}\right).$$

Then, the $n-$th Taylor coefficient of $\tilde{\phi}(x)$ at $x = 0$, that we denote $\tilde{\phi}_n$, is given in terms of the derivatives of $\phi(t)$ at $t = 0$ by means of the formula

$$\tilde{\phi}_n = \sum_{k=0}^{\lfloor \frac{n}{m} \rfloor} \frac{(-1)^k}{k!} \frac{B_k^{(\lambda+1+\frac{n}{m})}(1)}{(n-km)!} \frac{d^{n-km}}{dt^{n-km}}\left[\phi(t)\right]_{t=0}. \tag{A.2}$$

Proof. It is clear that the function $\tilde{\phi}(x)$ is analytic in the open disk $D_0(\rho)$, for some $0 < \rho < 1$. From Cauchy's formula for the $n-$th derivative of an analytic function, we have

$$\tilde{\phi}^{(n)}(0) = \frac{n!}{2\pi i}\oint_C \frac{\tilde{\phi}(x)}{x^{n+1}}dx = \frac{n!}{2\pi i}\oint_C \left[-\frac{\log(1-x^m)}{x^m}\right]^{\lambda} \phi\left(x\left[-\frac{\log(1-x^m)}{x^m}\right]^{1/m}\right)\frac{dx}{x^{n+1}},$$

where C is the boundary of the disk of center 0 and radius $\rho < 1$, oriented in the positive sense and does not contain any singularity of the function $\phi\left(x\left[-\frac{\log(1-x^m)}{x^m}\right]^{1/m}\right)$. We consider the change of variables $x \mapsto t$ given by (7.22), that is, $x = t\left[\frac{1-e^{-t^m}}{t^m}\right]^{1/m}$, with $t > 0$ for $x > 0$. This transformation maps the disk $D_0(\rho)$ to a certain region $S_m((-\log(1-\rho^m))^{1/m}, 0)$, with $S_m(q,0)$ defined in (7.23). We find

$$\tilde{\phi}^{(n)}(0) = \frac{n!}{2\pi i}\oint_\gamma \phi(t)e^{-t^m}\left[\frac{-t^m}{e^{-t^m}-1}\right]^{\lambda+1+\frac{n}{m}}\frac{dt}{t^{n+1}},$$

where γ is the boundary of the region $S_m((-\log(1-\rho^m))^{1/m}, 0)$, that is a closed curve encircling $t = 0$ in the positive direction. We may choose ρ small enough in order to assure that $S_m((-\log(1-\rho^m))^{1/m}, 0) \subset D_r(0)$. Then, we apply Cauchy's theorem again to find

$$\tilde{\phi}^{(n)}(0) = \frac{d^n}{dt^n}\left[\phi(t)e^{-t^m}\left(\frac{-t^m}{e^{-t^m}-1}\right)^{\lambda+1+\frac{n}{m}}\right]_{t=0}. \tag{A.3}$$

Taking into account the generating function of the generalized Bernoulli polynomials (A.1) and the $k-$th derivative at $x = 0$ of a composite function of the form $\Psi(x) = \psi(x^m)$, we have that

$$\frac{d^k}{dt^k}\left[e^{-t^m}\left(\frac{-t^m}{e^{-t^m}-1}\right)^{\lambda+1+\frac{n}{m}}\right]_{t=0} = \begin{cases} 0 & \text{if } k \neq 0 \,(\text{mod } m), \\ (-1)^{k/m}\frac{k!}{(k/m)!}B_{\frac{k}{m}}^{(\lambda+1+\frac{n}{m})}(1) & \text{if } k = 0 \,(\text{mod } m). \end{cases} \tag{A.4}$$

Then, if we apply Leibniz's formula for the $n-$th derivative of a product in (A.3) and we use (A.4) we find the desired result (A.2). $\square$

The case $\lambda = 0$ and $m = 1$ is specially interesting. We have the following corollary:

Corollary A.0.2. *The $n-$Taylor coefficient of $\phi(-\log t)$ at $t = 1$, A_n, is given in terms of the derivatives of $\phi(t)$ at $t = 0$ and the Nørlund polynomials by means of the formula*

$$A_n = \sum_{k=0}^{n}(-1)^k\frac{\phi^{(k)}(0)}{k!}\frac{k}{n}\frac{B_{n-k}^{(n)}}{(n-k)!}. \tag{A.5}$$

Proof. We have

$$A_n = \frac{d^n}{dt^n}\left[\phi(-\log t)\right]_{t=1} = (-1)^n\frac{d^n}{dt^n}\left[\phi(-\log(1-x))\right]_{x=0}$$

From (A.2) with $\lambda = 0$ and $m = 1$, we have

$$A_n = (-1)^n\sum_{k=0}^{n}\frac{(-1)^k}{k!}\frac{B_k^{(n+1)}(1)}{(n-k)!}\frac{d^{n-k}}{dt^{n-k}}\left[\phi(t)\right]_{t=0}.$$

In [88, pp. 129] we find the relation $B_k^{(n+1)}(1) = \left(1-\frac{k}{n}\right)B_k^{(n)}$. Therefore

$$A_n = \sum_{k=0}^{n}\frac{(-1)^{n-k}}{k!}\frac{1}{(n-k)!}\left(1-\frac{k}{n}\right)B_k^{(n)}\frac{d^{n-k}}{dt^{n-k}}\left[\phi(t)\right]_{t=0},$$

and formula (A.5) follows from rearranging the terms of the summation by means of the change of index $j = n - k$. $\square$

Appendix B

A Larger Domain for a Uniformly Convergent Expansion of the Symmetric Elliptic Functions

In section 4.8 of chapter 4 we have obtained convergent expansions of the functions $F(x,y)$, $G_1(x,y)$ and $G_2(y,z)$ defined, respectively, in (4.17), (4.18) and (4.19) and related with the symmetric elliptic integrals $R_F(x,y,z)$ and $R_D(x,y,z)$ by means of the connection formulas (4.20), (4.21) and (4.22). The expansions were given in terms of elementary functions and uniformly valid for $y \in S(\theta)$, region defined in (4.27) and depicted in figure 4.8. However, to obtain the integral representations (4.17), (4.18) and (4.19) of $F(x,y)$, $G_1(x,y)$ and $G_2(y,z)$ we have required that one of the variables to be real and positive, as we have performed a change of variables $s \mapsto ws$, with $w = z$ to derive $F(x,y)$ from $R_F(x,y,z)$ and $G_1(x,y)$ from $R_D(x,y,z)$; or $w = x$ to obtain $G_2(y,z)$ from $R_D(x,y,z)$. If, instead of restricting that variable to the positive real numbers, we let it run in a the bigger domain $\mathbb{C} \setminus (-\infty, 0]$, then, for the function R_F (and similarly for R_D) we would have obtained

$$R_F(x,y,z) = \frac{z}{2}\int_0^{\infty e^{-i\arg z}} \frac{ds}{\sqrt{z(s+x/z)}\sqrt{z(s+y/z)}\sqrt{z(s+1)}}.$$

When $\arg(-u/z) \notin [0, -\arg z]$, $u = x, y$, we can invoke Cauchy's theorem to rotate the integration contour $[0, \infty e^{-i\arg z})$ to the contour $[0,\infty)$. Besides, $\arg(-u/z) = \arg(u/z) - \pi \mathrm{sign}\,(\arg(u/z))$, and therefore, the condition $\arg(-u/z) \notin [0, -\arg z]$ is equivalent to the condition $|\arg(u/z) + \arg z| < \pi$. Taking into account that

$$|\arg(u/z) + \arg z| = \begin{cases} |\arg u| \ (< \pi) & \text{if } |\arg u - \arg z| < \pi, \\ |\arg u \pm 2\pi| \ (\geq \pi) & \text{if } |\arg u - \arg z| \geq \pi, \end{cases}$$

we deduce that $|\arg(u/z) + \arg z| < \pi$ if and only if $|\arg u - \arg z| < \pi$.

But moreover, when $|\arg(u/z) + \arg z| < \pi$ we have that $\sqrt{z(u/z)} = \sqrt{z}\sqrt{u/z}$ and using that, for all $s \geq 0$, $|\arg(s+u/z)| \leq |\arg(u/z)|$, we also have that $\sqrt{z(s+u/z)} = \sqrt{z}\sqrt{s+u/z}$, for all $s \geq 0$. In conclusion, when $|\arg x - \arg z| < \pi$ and $|\arg y - \arg z| < \pi$,

$$R_F(x,y,z) = \frac{1}{2\sqrt{z}}\int_0^\infty \frac{ds}{\sqrt{s+x/z}\sqrt{s+y/z}\sqrt{s+1}},$$

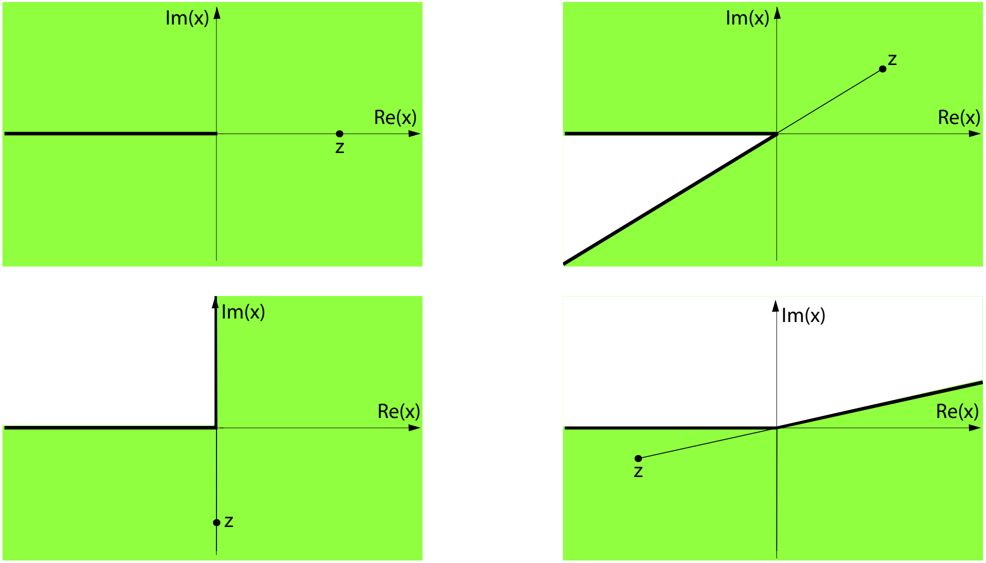


Figure B.1: The argument of the variable x is restricted to the sector $\arg x \in (\arg z - \pi, \arg z + \pi) \cap (-\pi, \pi]$. The green region in all the pictures shows the different shapes of the $x-$section of the region Λ_1 for different arguments of z.

that is, the connection formula (4.20) is valid not only for $z > 0$, but in the bigger domain

$$\Lambda_1 := \{(x, y, z) \in (\mathbb{C} \setminus (-\infty, 0])^3 : |\arg x - \arg z| < \pi, \; |\arg y - \arg z| < \pi\}. \tag{B.1}$$

In figure B.1 we illustrate the shape of the $x-$domain Λ_1 for fixed z (the $y-$domain for fixed z is analogous).

Similar arguments can be applied to the symmetric elliptic integral $R_D(x, y, z)$ to find that the connection formulas (4.21) and (4.22) are respectively valid in the bigger domains Λ_1 and Λ_2, with Λ_1 defined in (B.1) and

$$\Lambda_2 := \{(x, y, z) \in (\mathbb{C} \setminus (-\infty, 0])^3 : |\arg y - \arg x| < \pi, \; |\arg z^3 - \arg x^3| < \pi\}.$$

Therefore, uniform expansions for the symmetric elliptic integrals $R_F(x, y, z)$ and $R_D(x, y, z)$ can be found from the connection formulas (4.20), (4.21) and (4.22) and the expansion (4.37) not only when one of their variables is positive, but in a bigger domain. In particular, the expansions derived for $F(x, y)$ and $G_1(x, y)$ hold for $(x, y, z) \in \Lambda_1$ whereas the expansion derived for $G_2(y, z)$ holds for $(x, y, z) \in \Lambda_2$.

Appendix C

The Integral Representation (6.4) Of the ${}_{p+1}F_p(a, \vec{b}, \vec{c}; z)$ Function

In this appendix we describe the four different possibilities for the integration paths and the constants $A(b_s, c_s)$ for all $s = 1, 2, \ldots, p$ in the integral representation (6.4) of the generalized hypergeometric function ${}_{p+1}F_p$.

(a) The path is $L = [0, 1]$ and

$$A(b, c) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)}.$$

Formula (6.4) is valid for $\Re(c_k) > \Re(b_k) > 0$ for all $k = 1, 2, \ldots, p$.

(b) The integration contour L is a complex path that starts and terminates at $t = 0$. It encircles the point $t = 1$ once in the positive direction. At the point where L cuts the positive real axis we have $\arg(t) = 0$ and $\arg(a - t) = \pi$. This path is depicted in figure C.1 (left). On the other hand,

$$A(b, c) = \frac{e^{i\pi(b-c)}\Gamma(1+b-c)\Gamma(c)}{2\pi i\Gamma(b)}.$$

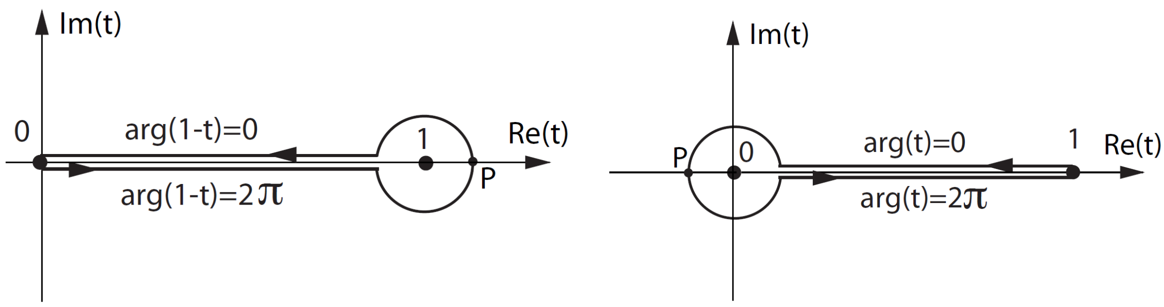
In this case, formula (6.4) is valid for $c_k - b_k \notin \mathbb{N}$ and $\Re(b_k) > 0$ for all $k = 1, 2, \ldots, p$.

(c) This case is, somehow, a mirror case of the previous one. In particular, the path L starts and terminates at $t = 1$ and it encircles the point $t = 0$ once in the positive direction. At the point where L cuts the negative real axis, we have that $\arg(t) = \pi$ and $\arg(1 - t) = 0$. In figure C.1 (right) this path is depicted. We also have

$$A(b, c) = \frac{e^{-i\pi b}\Gamma(1-b)\Gamma(c)}{2\pi i\Gamma(c-b)}.$$

In this case, formula (6.4) is valid for $b_k \notin \mathbb{N}$ and $\Re(c_k - b_k) > 0$ for all $k = 1, 2, \ldots, p$.

(d) The integration path starts and terminates at an arbitrary point P located on the real axis between $t = 0$ and $t = 1$. It encircles the points 0 and 1 once in the positive



Case (b) Case (c)

Figure C.1: The p identical integration contours L in (6.4), cases (b) and (c), may be deformed to the respective contours (b) (left) and (c) (right) depicted in this figure. The horizontal segments are assumed to be sticked to the real interval $[0,1]$ and the radius ϵ of the circles is infinitesimally small. In case (b), at the point P we have that $\arg(t)=0$ and $\arg(1-t)=\pi$. At the point P of figure (c) we have that $\arg(t)=\pi$ and $\arg(1-t)=0$.

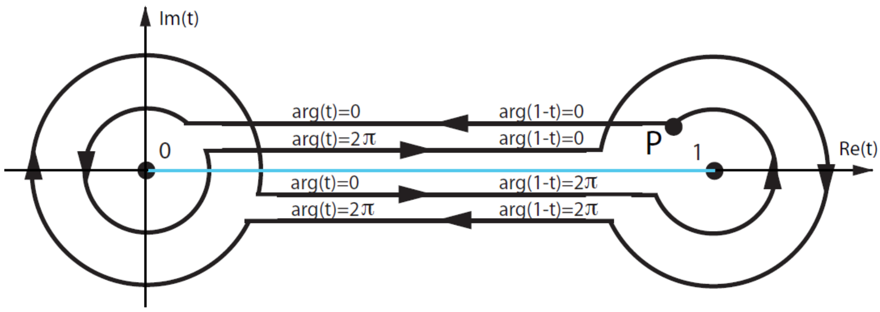


Figure C.2: The p identical integration contours L in (6.4), case (d), may be deformed to the contour depicted in this figure. The horizontal segments are assumed to be sticked to the real interval $[0,1]$ and the radius ϵ of the circles is infinitesimally small. At the point P we have that $\arg(t)=\arg(1-t)=0$.

direction and then once in the negative direction. At the point P of L, we have $\arg(t)=\arg(1-t)=0$. This path is depicted in figure C.2. On the other hand

$$A(b,c)=\frac{e^{i\pi(1-c)}\Gamma(c)}{4\Gamma(b)\Gamma(c-b)\sin(\pi b)\sin[\pi(c-b)]}.$$

In this case, formula (6.4) is valid for b_k and $c_k-b_k\notin\mathbb{N}$, for all $k=1,2,\ldots,p$.

We assume that the contours L described in cases (b), (c) and (d) are squeezed (analytically deformed) as much as possible around the real interval $[0,1]$, as it is shown in figures C.1 and C.2.

The multiple integral representation (6.4) is a generalization of the respective integral representation of the Gauss hypergeometric function $_2F_1(a,b,c;z)$ given in [104, eqs. 15.6.1, 15.6.2, 15.6.4 or 15.6.5]. The derivation of these multiple integral representation is a straightforward generalization of the derivation of the corresponding simple integral representation of the $_2F_1$ function. In particular, for any case (a)-(d) we replace the expansion

$$(1 - Tz)^{-a} = \sum_{n=0}^{\infty} \frac{(a)_n}{n!}(zT)^n, \qquad |zT| < 1,$$

into the right hand side of (6.4), interchange summation and integration and use the reflection formula of the gamma function. We also use the different integral representations of the beta function [4, eqs. 5.12.1, 5.12.10 or 5.12.12] depending on the case (a), (b), (c) or (d) under consideration:

$$\int_L t^{a-1}(1-t)^{b-1}dt = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)} \times \begin{cases} 1, & \text{case (a)}, \\ 2ie^{i\pi b}\sin(\pi b), & \text{case (b)}, \\ 2ie^{i\pi a}\sin(\pi a), & \text{case (c)}, \\ -4e^{i\pi(a+b)}\sin(\pi a)\sin(\pi b), & \text{case (d)}. \end{cases}$$

Then, we obtain the series definition (6.2) of the generalized hypergeometric function ${}_{p+1}F_p(a, \vec{b}, \vec{c}; z)$. Therefore, the right hand side of (6.4) represents the analytic continuation in the variable z of ${}_{p+1}F_p(a, \vec{b}, \vec{c}; z)$, defined by the right hand side of the series (6.2), from the disk $|z| < 1$ to the cutted complex plane $|\arg(1 - z)| < \pi$.

Appendix D

The Coefficients of the New Expansions For the ${}_{p+1}F_p$ Function

In this appendix we give a recursive formula to compute several coefficients that appear in the expansions (6.5), (6.18) and (6.25) of chapter 6 for the generalized hypergeometric ${}_{p+1}F_p$ function.

Lemma D.0.1. *The functions ${}_{p+1}F_p(-k,\vec{b},\vec{c};z)$ are polynomials in the variable z of degree k. They can be computed using the following recurrence relation, valid for $k=1,2,\dots$*

$$\begin{cases} {}_{p+1}F_p\left(\left.\begin{matrix} 0, \; \vec{b} \\ \vec{c} \end{matrix}\right| z\right) = 1, \\ {}_{p+1}F_p\left(\left.\begin{matrix} -k, \; \vec{b} \\ \vec{c} \end{matrix}\right| z\right) = {}_{p+1}F_p\left(\left.\begin{matrix} 1-k, \; \vec{b} \\ \vec{c} \end{matrix}\right| z\right) - z\left(\prod_{s=1}^{p}\frac{b_s}{c_s}\right){}_{p+1}F_p\left(\left.\begin{matrix} 1-k, \; \overrightarrow{b+1} \\ \overrightarrow{c+1} \end{matrix}\right| z\right), \end{cases} \tag{D.1}$$

where $\overrightarrow{b+1} := (b_1+1, b_2+2, \dots, b_p+1)$.

Proof. Using the series definition [3, 16.2.1] it is straightforward to check that the function ${}_{p+1}F_p(-k,\vec{b},\vec{c};z)$ is a polynomial in the variable z of degree k as the defining series terminates after $k+1$ terms. On the other hand, the three term recurrence relation follows from the integral representation (6.4) in the particular case when $a=-k$, with $k\in\mathbb{N}$, by splitting the factor $(1-zT)^k = (1-zT)^{k-1}(1-zT)$ and performing some easy computations. $\square$

Lemma D.0.2. *A recursive formula for the coefficients $A_n^2(a,z)$ and $B_n^2(a,z)$ in the two-point Taylor expansion (6.20) of $f(T)=(1-zT)^{-a}$ at the points $T=q$ and $T=1-q$ with $q\in[0,1/2)$ is the following:*

$$\begin{aligned} A_0^2(a,z) &= \frac{(1-q)(1-zq)^{-a} - q\,[1-z(1-q)]^{-a}}{1-2q}, \\ B_0^2(a,z) &= \frac{[1-z(1-q)]^{-a} - (1-zq)^{-a}}{1-2q}, \end{aligned} \tag{D.2}$$

and

$$\begin{cases} A_{n+1}^2(a,z) &= \dfrac{M_{11}^2(z,n,a,q)A_n^2(a,z)+M_{12}^2(z,n,a,q)B_n^2(a,z)}{(n+1)(4q^2-4q+1)\left[z-1q(1-q)z^2\right]}, \\ B_{n+1}^2(a,z) &= \dfrac{M_{21}^2(z,n,a,q)A_n^2(a,z)+M_{22}^2(z,n,a,q)B_n^2(a,z)}{(n+1)(4q^2-4q+1)\left[z-1q(1-q)z^2\right]}, \end{cases} \tag{D.3}$$

where

$$\begin{aligned} M_{11}^2(z,n,a,q) &:= z(a+2n)\left[1-(1+2q^2-2q)z\right], \\ M_{12}^2(z,n,a,q) &:= z^2\left[-n+q(q-1)(a+1)\right]+z\left[2aq(1-q)+1+3n\right]-2n-1, \\ M_{21}^2(z,n,a,q) &:= -z(2-z)(a+2n), \\ M_{22}^2(z,n,a,q) &:= z^2\left[n(1+4q-4q^2)+2(1+a)q(1-q)\right]-(a+2+6n)z+4n+2. \end{aligned} \tag{D.4}$$

Proof. Consider the diffential equation satisfied by the function $f(T)=(1-zT)^{-a}$, namely $(1-zT)f'(T)=azf(T)$. Consider also the two-point Taylor expansion of $f(T)$ at $T=q$ and $T=1-q$

$$f(T)=\sum_{n=0}^{\infty}\left[A_n^2(a,z)+B_n^2(a,z)T\right]\left[(T-q)(T+q-1)\right]^n.$$

Derivate this expression with respect to T to obtain the two-point Taylor expansion of $f'(T)$ at the points $T=q$ and $T=1-q$. Then, introducing both expressions into the differential equation and equating the coefficients of $[(T-q)(T+q-1)]^n$ and $T\,[(T-q)(T+q-1)]^n$ we obtain the recurrence scheme (D.2)-(D.3)-(D.4). □

Lemma D.0.3. *A recursive formula for the coefficients $A_n^3(a,z)$, $B_n^3(a,z)$ and $C_n^3(a,z)$ in the three-point Taylor expansion (6.28) of $f(T)=(1-zT)^{-a}$ at the points $T=q$, $T=1/2$ and $T=1-q$ with $q\in[0,1/2)$ is the following:*

$$\begin{aligned} A_0^3(a,z) &:= \frac{4q^2(1-z/2)^{-a}+(1-qz)^{-a}+q\left[-4(1-z/2)^{-a}+[1+(q-1)z]^{-a}-(1-qz)^{-a}\right]}{(1-2q)^2}, \\ B_0^3(a,z) &:= \frac{4(1-z/2)^{-a}-[1+(q-1)z]^{-a}-2q\,[1+(q-1)z]^{-a}-3(1-qz)^{-a}+2q(1-qz)^{-a}}{(1-2q)^2}, \\ C_0^3(a,z) &:= \frac{-2^{2+a}(2-z)^{-a}+2\,[1+(q-1)z]^{-a}+2(1-qz)^{-a}}{(1-2q)^2}. \end{aligned} \tag{D.5}$$

For $n=1,2,\dots$ we have the following recurrence relation to compute the coefficients $A_n^3(a,z)$, $B_n^3(a,z)$ and $C_n^3(a,z)$:

$$\begin{aligned} A_{n+1}^3(a,z) &= \frac{M_{11}^3(z,n,a,q)A_n^3(a,z)+M_{12}^3(z,n,a,q)B_n^3(a,z)+M_{13}^3(z,n,a,q)C_n^3(a,z)}{2(n+1)(1-2q)^4(z-2)\left[(q-1)qz^2+z-1\right]}, \\ B_{n+1}^3(a,z) &= \frac{M_{21}^3(z,n,a,q)A_n^3(a,z)+M_{22}^3(z,n,a,q)B_n^3(a,z)+M_{23}^3(z,n,a,q)C_n^3(a,z)}{2(n+1)(1-2q)^4(z-2)\left[(q-1)qz^2+z-1\right]}, \\ C_{n+1}^3(a,z) &= \frac{M_{31}^3(z,n,a,q)A_n^3(a,z)+M_{32}^3(z,n,a,q)B_n^3(a,z)+M_{33}^3(z,n,a,q)C_n^3(a,z)}{(n+1)(1-2q)^4(z-2)\left[(q-1)qz^2+z-1\right]}. \end{aligned} \tag{D.6}$$

where the quantities $M^3_{ij}(z,n,a,q)$, $i,j=1,2,3$ *are given by*

$$\begin{aligned}M^3_{11}(z,n,a,q):=&4z(a+3n)\left[16q^4z^2-32q^3z^2+2q^2\left(9z^2+6z-8\right)-2q\left(z^2+6z-8\right)\right.\\&\left.+z^2-3z+2\right],\end{aligned}$$

$$\begin{aligned}M^3_{12}(z,n,a,q):=&4\left\{8q^4z^2(az+z-2)+q^2\left[7(a+1)z^3+6(2a-1)z^2-12(a+2)z+16\right]\right.\\&+q\left[(a+1)z^3-2(6a+5)z^2+12(a+2)z-16\right]-16q^3z^2(az+z-2)\\&\left.-z^2+3z-2\right\}+6n\left[32q^4(z-1)z^2-64q^3(z-1)z^2+32q^2\left(z^3-2z+1\right)\right.\\&\left.-32q(z-1)^2+(z-2)^2(z-1)\right],\end{aligned}$$

$$\begin{aligned}M^3_{13}(z,n,a,q):=&n\left[-12+28z-21z^2+5z^3-4q\left(3z^3+42z^2-106z+60\right)+12q^2\left(19z^3\right.\right.\\&\left.+2z^2-46z+20\right)+16q^3z\left(16+18z-35z^2\right)+8q^4z\left(75z^2-18z-16\right)\\&\left.-384q^5z^3+128q^6z^3\right]+4(q-1)q\left\{24-6(a+6)z+6(a+2)(q-1)qz^3\right.\\&\left.+z^2\left[a\left(-4q^2+4q+5\right)-24q^2+24q+12\right]\right\},\end{aligned}$$

$$M^3_{21}(z,n,a,q):=4z(a+3n)\left[3\left(4q^2-4q-1\right)z^2+\left(8q^2-8q+26\right)z-24\right],$$

$$\begin{aligned}M^3_{22}(z,n,a,q):=&6n\left[\left(36q^2-36q-3\right)z^3+\left(50+40q-40q^2\right)z^2-96z+48\right]\\&+8z\left[a\left(4q^2-4q-5\right)-18\right]-12z^2\left[a\left(4q^2-4q-3\right)+8q^2-8q-4\right]\\&+48(a+1)(q-1)qz^3+96,\end{aligned}$$

$$\begin{aligned}M^3_{23}(z,n,a,q):=&n\left\{16q^4z^2(3z+10)-32q^3z^2(3z+10)+24q^2\left(9z^3+4z^2-2z-4\right)\right.\\&\left.+8q\left(12+6z+8z^2-21z^3\right)-15z^3+262z^2-516z+264\right\}\\&-4\left\{4(a+2)q^4(z-2)z^2-8(a+2)q^3(z-2)z^2-q^2\left[-16+24z\right.\right.\\&\left.(2+a)z^3+6(a-2)z^2\right]+q\left[5(a+2)z^3-2(a+14)z^2+24z-16\right]\\&\left.-5(a+2)z^2+6(a+5)z-20\right\},\end{aligned}$$

$$M^3_{31}(z,n,a,q):=-4z(a+3n)\left[\left(8q^2-8q-1\right)z^2+12z-12\right],$$

$$\begin{aligned}M^3_{32}(z,n,a,q):=&-6n\left[\left(20q^2-20q-1\right)z^3-24\left(q^2-q-1\right)z^2-48z+24\right]\\&-4\left\{z^2\left[a\left(-4q^2+4q+5\right)-12q^2+12q+6\right]+6(a+1)(q-1)qz^3\right.\\&\left.-6(a+3)z+12\right\},\end{aligned}$$

$$\begin{aligned}M^3_{33}(z,n,a,q):=&n\left[\left(-64q^4+128q^3-168q^2+104q+5\right)z^3+12\left(4q^2-4q-11\right)z^2\right.\\&\left.+12\left(8q^2-8q+23\right)z-144\right]-4\left\{3z^2\left(a-4q^2+4q+2\right)\right.\\&\left.-2z\left[2a\left(q^2-q+1\right)+9\right]+3(a+2)(q-1)qz^3+12\right\}.\end{aligned}$$

Proof. The proof of this lemma is identical to the proof of the previous one, but considering the three-point Taylor expansion of $f(T)$ at $T=q$, $T=1/2$ and $T=1-q$:

$$f(T)=\sum_{n=0}^{\infty}\left[A^3_n(a,z)+B^3_n(a,z)T+C^3_n(a,z)T^2\right]\left[(T-q)(T-1/2)(T+q-1)\right]^n.$$

$\square$

Bibliography

[1] M. Abramowitz and I. Stegun, *Handbook of Mathematical Functions with Formulas, Graphs and Mathematical Tables*, Dover, New York City, 1964.

[2] A. Apelblat, *Integral Transforms and Volterra Functions*, Nova Science Publishers, New York, 2010.

[3] R. A. Askey and A. B. Olde Daalhuis, *Generalized Hypergeometric Functions and Meijer G-Function, in: NIST Handbook of Mathematical Functions*, Cambridge University Press, Cambridge. 2010, pp. 403 – 418 (Chapter 16).

[4] R. A. Askey and R. Roy, *Gamma Function, in: NIST Handbook of Mathematical Functions*, Cambridge University Press, Cambridge. 2010, pp. 135 – 147 (Chapter 5).

[5] A. Aziz and T. Y. Na, *Perturbation Methods in Heat Transfer*, Hemisphere Publishing Corporation, 1984.

[6] A. Bejan, *Convection Heat Transfer*, Wiley, New York, 1984, doi: 10.1002/9781118671627.

[7] M. V. Berry, *Stokes'phenomenon; smoothing a Victorian discontinuity*, Inst. Hautes Études Sci. Publ. Math., 68 (1988), pp. 211–221, doi: 10.1007/BF02698550.

[8] M. V. Berry, *Uniform asymptotic smoothing of Stokes's discontinuities*, Proc. Roy. Soc. London Ser. A, 422 (1989), pp. 7 – 21, doi: 10.1098/rspa.1989.0018.

[9] M. V. Berry, *Asymptotics, Superasymptotics, Hyperasymptotics..., in: Asymptotics beyond all orders*, Springer, Boston. 1991, pp. 1 – 14 (Chapter 1).

[10] M. V. Berry and C. J. Howls, *Hyperasymptotics*, Proc. Roy. Soc. London Ser. A, 430 (1990), pp. 653–668, doi: 10.1098/rspa.1990.0111.

[11] M. V. Berry and C. J. Howls, *Axial and focal-plane diffraction catastrophe integrals*, J. Phys. A: Math. Theor., 43 (2010), doi: 10.1088/1751-8113/43/37/375206.

[12] M. V. Berry and C. J. Howls, *Integrals with Coalescing Saddles, in: NIST Handbook of Mathematical Functions*, Cambridge University Press, Cambridge. 2010, pp. 775 – 793 (Chapter 36).

[13] N. Bleistein, *Uniform asymptotic expansions of integrals with stationary point near algebraic singularity*, Comm. Pure Appl. Math., 19 (1966), pp. 353–370, doi: 10.1002/cpa.3160190403.

[14] N. Bleistein and R. A. Handelsman, *Asymptotic Expansions of Integrals*, Dover Pub., New York, 1986.

[15] D. M. Bressoud, *Combinatorial Analysis, in: NIST Handbook of Mathematical Functions*, Cambridge University Press, Cambridge. 2010, pp. 617 – 636 (Chapter 26).

[16] Y. A. Brychkov, *Handbook of Special Functions, Derivatives, Integrals, Series and Other Formulas*, CRC Press, New Cork, 2008, doi: 10.1201/9781584889571.

[17] W. Bühring, *An Analytic Continuation of the Hypergeometric Series*, SIAM J. Math. Anal., 18 (1987), pp. 884 – 889, doi: 10.1137/0518066.

[18] W. Bühring, *Generalized hypergeometric functions at unit argument*, Proc. Amer. Math. Soc., 114 (1992), pp. 145 – 153, doi: 10.2307/2159793.

[19] B. Bujanda, J. L. López and P. J. Pagola, *Convergent expansions of the incomplete gamma function in terms of elementary functions*, Anal. Appl. (Singap), 16 (2018), pp. 435–448, doi: 10.1142/S0219530517500099.

[20] B. Bujanda, J. L. López and P. J. Pagola, *Convergent expansions of the confluent hypergeometric functions in terms of elementary functions*, Math. Comput., 88 (2019), pp. 1773–1789, doi: 10.1090/mcom/3389.

[21] B. Bujanda, J. L. López, P. J. Pagola and P. Palacios, *An analytic representation of the second symmetric standard elliptic integral in terms of elementary functions*, Results Math., 77 (2022), doi: 10.1007/s00025-022-01707-3.

[22] B. Bujanda, J. L. López, P. J. Pagola and P. Palacios, *Uniform approximations of the first symmetric elliptic integral in terms of elementary functions*, Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Mat. RACSAM, 116 (2022), doi: 10.1007/s13398-021-01151-y.

[23] B. C. Carlson, *A table of elliptic integrals of the second kind*, Math. Comp., 49 (1987), pp. 595–606, doi: 10.1090/S0025-5718-1987-0906192-1.

[24] B. C. Carlson, *A table of elliptic integrals of the thrid kind*, Math. Comp., 51 (1988), pp. 267–280, doi: 10.1090/S0025-5718-1988-0942154-7.

[25] B. C. Carlson, *A table of elliptic integrals: cubic cases*, Math. Comp., 53 (1989), pp. 327–333, doi: 10.1090/S0025-5718-1989-0969482-4.

[26] B. C. Carlson, *A table of elliptic integrals: one quadratic factor*, Math. Comp., 56 (1991), pp. 267–280, doi: 10.1090/S0025-5718-1991-1052087-6.

[27] B. C. Carlson, *A table of elliptic integrals: two quadratic factors*, Math. Comp., 59 (1992), pp. 165–180, doi: 10.1090/S0025-5718-1992-1134720-4.

[28] B. C. Carlson, *Elliptic Integrals, in: NIST Handbook of Mathematical Functions*, Cambridge University Press, Cambridge. 2010, pp. 485 – 522 (Chapter 19).

[29] C. Chester, B. Friedman and F. Ursell, *An extension of the method of steepest descents*, Proc. Cambridge Philos. Soc., 53 (1957), pp. 599–611, doi: 10.1017/S0305004100032655.

[30] S. Colombo, *Sur quelques correspondances symboliques (in french)*, C. R. Acad. Sci. Paris, 216 (1943), pp. 368–369.

[31] S. Colombo, *Sur la fonction $\nu(t,n)$ (in french)*, C. R. Acad. Sci. Paris, 226 (1948), pp. 1235–1236.

[32] S. Colombo, *Sur les fontions $\nu(x,n)$ et $\mu(x,m,n)$, (in french)*, Bull. Sci. Math., 2 (1955), pp. 72–78.

[33] P. J. Davis, *Leonhard Euler's Integral: A Historical Profile of the Gamma Function*, Amer. Math. Monthly, 66 (1959), pp. 849–869, doi: 10.2307/2309786.

[34] E. Delabaere and J.-M. Rasoamanana, *Sommation effective d'une somme de Borel par séries de factorielles (in french)*, Ann. Inst. Fourier, 57 (2007), pp. 421–456, doi: 10.5802/aif.2263.

[35] F. F. Di Bruno, *Théorie des Formes Binaires (in french)*, Librerie Breno, Torino, 1876.

[36] K. Dilcher, *Bernoulli and Euler Polynomials, in: NIST Handbook of Mathematical Functions*, Cambridge University Press, Cambridge. 2010, pp. 587–599 (Chapter 24).

[37] W. Eckhaus, *Matched Asymptotic Expansions and Singular Perturbations*, North-Holland Publishing Company, Amsterdam, 1973, doi: 10.1016/s0304-0208(08)x7073-2.

[38] W. Eckhaus, *Asymptotic Analysis of Singular Perturbations*, North Holland Publishing Company, Amsterdam, 1979, doi: 10.1016/s0168-2024(08)x7001-8.

[39] A. Erdélyi, *Asymptotic Expansions*, Dover, New York, 1956.

[40] A. Erdélyi, W. Magnus, F. Oberhettinger and F. G. Tricomi, *Higher Transcendental Functions, Vol. III*, Robert E. Krieger Publishing Co. Inc., Melbourne, 1981. Based on notes left by Harry Bateman, Reprint of the 1955 original.

[41] L. Euler, *De progressionibus transcendentibus seu quarum termini generales algebraice dari nequeunt (in latin)*, Novi Comment. Acad. Sci. Imp. Petrop., 5 (1738), pp. 36–57.

[42] C. Ferreira, J. L. López and E. Pérez Sinusía, *The use of two-point Taylor expansions in singular one-dimensional boundary value problems I*, J. Math. Anal. Appl., 463 (2018), pp. 708–725, doi: 10.1016/j.jmaa.2018.03.041.

[43] C. Ferreira, J. L. López and E. Pérez Sinusía, *Uniform convergent expansions of the Gauss hypergeometric function in terms of elementary functions*, Integral Transforms Spec. Funct., 29 (2018), pp. 942–954, doi: 10.1080/10652469.2018.1525369.

[44] C. Ferreira, J. L. López and E. Pérez Sinusía, *Uniform representations of the incomplete beta function in terms of elementary functions*, Electron. Trans. Number. Anal., 48 (2018), pp. 450–461, doi: 10.1553/etna_vol48s450.

[45] C. Ferreira, J. L. López and E. Pérez Sinusía, *Analysis of singular one-dimensional linear boundary value problems using two-point Taylor expansions*, Electron. J. Qual. Theory Differ. Equ., 1 (2020), pp. 1–21, doi: 10.14232/ejqtde.2020.1.22.

[46] P. Flajolet, E. Fusy, X. Gourdon, D. Panario and N. Pouyanne, *A hybrid of Darboux's method and singularity analysis in combinatorial asymptotics*, Electron. J. Combin., 13 (2006), doi: 10.37236/1129.

[47] P. Flajolet and A. Odlyzko, *Singularity analysis of generating functions*, SIAM J. Discrete Math., 3 (1990), pp. 216–240, doi: 10.1137/0403019.

[48] A. Fransén and S. Wrigge, *Calculation of the Moments and the Moment Generating Function for the Reciprocal Gamma Distribution*, Math. Comp., 42 (1984), pp. 601–616, doi: 10.2307/2007605.

[49] R. Garrappa and F. Mainardi, *On Volterra function and Ramanujan integrals*, Analysis (Berlin), 36 (2016), pp. 89–105, doi: 10.1515/anly-2015-5009.

[50] C. F. Gauss, *Disquisitiones generales circa seriem infinitam* $1+\frac{\alpha\beta}{1\cdot\gamma}x+\frac{\alpha(\alpha+1)\beta(\beta+1)}{1\cdot2\cdot\gamma(\gamma+1)}x\,x+\frac{\alpha(\alpha+1)(\alpha+2)\beta(\beta+1)(\beta+2)}{1\cdot2\cdot3\cdot\gamma(\gamma+1)(\gamma+2)}x^3+$ *etc. (in latin), (1813)*, Cambridge University Press. Reprinted on pp. 123 – 124 in *Werke*, 2011, doi: 10.1017/CBO9781139058247.

[51] M. Ghandehari and D. Logothetti, *How elliptic integrals K and E arise from circles and points in the Minkowski plane*, J. Geom., 50 (1994), pp. 63–72, doi: 10.1007/bf01222663.

[52] A. Gil, J. Segura and N. M. Temme, *Numerical Methods for Special Functions*, SIAM, Philadelphia, 2007, doi: 10.1137/1.9780898717822.

[53] P. Humbert, *Une nouvelle correspondance symbolique (in french)*, C. R. Acad. Sci. Paris, 218 (1944), pp. 99–100.

[54] D. Kaminski and R. B. Paris, *On the use of Hadamard expansions in hyperasymptotic evaluation: differential equations of hypergeometric type*, Proc. Roy. Soc. Edinburgh Sect. A, 134 (2004), pp. 159–178, doi: 10.1017/S0308210500003139.

[55] D. B. Karp and J. L. López, *Representations of hypergeometric functions for arbitrary parameter values and their use*, J. Approx. Theory, 218 (2017), pp. 42–70, doi: 10.1016/j.jat.2017.03.004.

[56] S. F. Khwaja and A. B. Olde Daalhuis, *Computations of the coefficients appearing in the uniform asymptotic expansions of integrals*, Stud. Appl. Math., 139 (2017), pp. 551–567, doi: 10.1111/sapm.12172.

[57] T. H. Koornwinder, R. Wong, R. Koekoek and R. F. Swarttouw, *Orthogonal Polynomials, in: NIST Handbook of Mathematical Functions*, Cambridge University Press, Cambridge. 2010, pp. 435 – 484 (Chapter 18).

[58] E. Landau, *Vorlesungen über Zahlentheorie [3 volumes] (in german)*, Verlag von S. Hirzel, Leipzig., 1927. English translation in 1996 by AMS Chelsea Publishing: Elementary number theory.

[59] P. S. Laplace, *Théorie Analytique des Probabilités. Secondie Partie: Théorie des Approximations des Formules qui sont Fonctions de Très-Grands Nombres (in french)*, Courcier, Paris, 1812, doi: 10.14711/spcol/b718972.

[60] H. A. Lauwerier, *The calculation of the coefficients of certain asymptotic series by means of linear recurrent relations*, Appl. Sci. Res. B, 2 (1952), pp. 77–84, doi: 10.1007/bf02919761.

[61] D. F. Lawden, *Elliptic Functions and Applications*, Springer-Verlag, New York, 1989, doi: 10.1007/978-1-4757-3980-0.

[62] P. Lecocq, A. Dufay, G. Sourimant and J.-E. Marvie, *Analytic Approximations for Real-Time Area Light Shading*, IEEE Trans. Vis. Comput. Graph., 23 (2017), pp. 1428–1441, doi: .

[63] A. M. Legendre, *Traité des Fonctions Elliptiques et des Intégrales Eulériennes: Avec des Tables pour en faciliter la calcul numérique, Vol. I (in french)*, Imprimerie de Huzart-Courcuer, Paris, 1825.

[64] E. M. Lifshitz and L. P. Pitaevskii, *Statistical Physics, Part 2: Theory of the Condensed State*, Pergamon Press, Oxford, 1980.

[65] J. L. López, *Asymptotic expansions of symmetric standard elliptic integrlas*, SIAM J. Math. Anal., 31 (2000), pp. 754–775, doi: 10.1137/S0036141099351176.

[66] J. L. López, *Convergent expansions of the Bessel functions in terms of elementary function*, Adv. Comput. Math., 44 (2018), pp. 277–294, doi: 10.1007/s10444-017-9543-y.

[67] J. L. López and P. J. Pagola, *The confluent hypergeometric functions $M(a,b,z)$ and $U(a,b,z)$ for large b and z*, J. Comput. Appl. Math., 233 (2010), pp. 1570 – 1576, doi: 10.1016/j.cam.2009.02.072.

[68] J. L. López and P. J. Pagola, *A Systematic "Saddle Point Near a Pole" Asymptotic Method with Application to the Gauss Hypergeometric Funtion*, Stud. Appl. Math., 127 (2011), pp. 24 – 37, doi: 10.1111/j.1467-9590.2010.00510.x.

[69] J. L. López, P. J. Pagola and D. B. Karp, *Uniformly convergent expansions for the generalized hypergeometric functions ${}_{p-1}F_p$ and ${}_pF_p$*, Integral Transforms Spec. Funct., 31 (2020), pp. 820–837, doi: 10.1080/10652469.2020.1752687.

[70] J. L. López, P. J. Pagola and P. Palacios, *Series representations of the Volterra function and the Fransén-Robinson constant*, J. Approx. Theory, 272 (2021), doi: 10.1016/j.jat.2021.105641.

[71] J. L. López, P. J. Pagola and P. Palacios, *A convergent and asymptotic Laplace method for integrals*, 2022, doi: 10.1016/j.cam.2022.114897. Accepted for publication in *J. Comput. Appl. Math.*, 2022.

[72] J. L. López, P. J. Pagola and P. Palacios, *New Analytic Representations of the Hypergeometric Functions* $_{p+1}F_p$, Constr. Approx., 55 (2022), pp. 891 – 917, doi: 10.1007/s00365-021-09537-2.

[73] J. L. López, P. J. Pagola and P. Palacios, *The uniform asymptotic method "saddle point near an end point" revisited (preprint)*, 2022.

[74] J. L. López, P. J. Pagola and E. Pérez Sinusía, *A simplification of Laplace's method: Applications to the Gamma function and Gauss hypergeometric function*, J. Approx. Theory, 161 (2009), pp. 280–291, doi: 10.1016/j.jat.2008.09.004.

[75] J. L. López, P. J. Pagola and E. Pérez Sinusía, *A systematization of the saddle point method. Application to the Airy and Hankel functions*, J. Math. Anal. Appl., 354 (2009), pp. 347–359, doi: 10.1016/j.jmaa.2008.12.032.

[76] J. L. López, P. J. Pagola and E. Pérez Sinusía, *New series expansions of the* $_3F_2$ *function*, J. Math. Anal. Appl., 421 (2015), pp. 982 – 995, doi: 10.1016/j.jmaa.2014.07.065.

[77] J. L. López, P. Palacios and P. J. Pagola, *Uniform convergent expansions of integral transforms*, Math. Comp., 90 (2021), pp. 1357–1380, doi: 10.1090/mcom/3601.

[78] J. L. López and E. Pérez Sinusía, *Two-point Taylor expansions and one-dimensional boundary value problems*, Math. Comp., 79 (2010), pp. 2103–2115, doi: 10.1090/S0025-5718-10-02370-7.

[79] J. L. López and E. Pérez Sinusía, *New series expansions for the confluent hypergeometric function* $M(a, b, z)$, Appl. Math. Comput., 235 (2014), pp. 26 – 31, doi: 10.1016/j.amc.2014.02.099.

[80] J. L. López, E. Pérez Sinusía and N. M. Temme, *Multi-point Taylor approximations in one-dimensional linear boundary value problems*, Appl. Math. Comput., 207 (2009), pp. 519–527, doi: 10.1016/j.amc.2008.11.015.

[81] J. L. López and N. M. Temme, *Two-point Taylor Expansions of Analytic Functions*, Stud. Appl. Math., 109 (2002), pp. 297–311, doi: 10.1111/1467-9590.00225.

[82] J. L. López and N. M. Temme, *Multi-point Taylor expansions of analytic functions*, Trans. Amer. Math. Soc., 356 (2004), pp. 4323 – 4342, doi: 10.1090/S0002-9947-04-03619-0.

[83] J. L. López and N. M. Temme, *New series expansions of the Gauss hypergeometric function*, Adv. Comput. Math., 39 (2013), pp. 349–365, doi: 10.1007/s10444-012-9283-y.

[84] P.-C. Lu, *Introduction to the Mechanics of Viscous Fluids*, Holt, Rinehart and Winston, New York, 1973.

[85] F. Mainardi, A. Mura, R. Gorenflo and M. Stojanovic, *The Two Forms of Fractional Relaxation of Distributed Order*, J. Vib. Control, 13 (2007), pp. 1249–1268, doi: 10.1177/1077546307077468.

[86] F. Mainardi, A. Mura, G. Pagnini and R. Gorenflo, *Time-fractional Diffusion of Distribuited Order*, J. Vib. Control, 14 (2008), pp. 1267–1290, doi: 10.1177/1077546307087452.

[87] G. P. Michon, *Trigonometry and Functions*, 2010, numericana.com.

[88] L. M. Milne-Thomson, *The Calculus of Finite Differences*, Chelsea Publishing Company, New York, 1981.

[89] K. W. Morton, *Numerical Solution of Convection-Diffusion Problems*, Springer Netherlands, 1995.

[90] K. E. Muller, *Computing the confluent hypergeometric function $M(a,b,x)$*, Number. Math., 90 (2001), pp. 179 – 196, doi: 10.1007/s002110100285.

[91] G. Nemes, *On the Coefficients of the Asymptotic Expansion of $n!$*, J. Integer Seq., 13 (2010), doi: 10.48550/arXiv.1003.2907.

[92] G. Nemes, *An Explicit Formula for the Coefficients in Laplace's Method*, Constr. Approx., 38 (2013), pp. 471–487, doi: 10.1007/s00365-013-9202-6.

[93] G. Nemes, *Error bounds and exponential improvement for Hermite's asymptotic expansion for the gamma function*, Appl. Anal. Discrete Math., 7 (2013), pp. 161–179, doi: 10.2298/AADM130124002N.

[94] G. Nemes, *Error bounds and exponential improvement for the asymptotic expansion of the Barnes $G-$function*, Proc. R. Soc. Lond. Ser. A. Math. Phys. Eng. Sci., 470 (2014), doi: 10.1098/rspa.2014.0534.

[95] G. Nemes, *The resurgence properties of the large-order asymptotics of the Hankel and Bessel functions*, Anal. Appl. (Singap.), 12 (2014), pp. 403–462, doi: 10.1142/S021953051450033X.

[96] N. Nielsen, *Recherches sur les séries de factorielles (in french)*, Ann. Sci. École Norm. Sup., 3 (1902), pp. 409–453, doi: 10.24033/asens.515.

[97] N. Nielsen, *Sur la répresentation asymptotique d'une série de factorielles (in french)*, Ann. Sci. École Norm. Sup., 3 (1904), pp. 449–458, doi: 10.24033/asens.544.

[98] A. F. Nikiforov and V. B. Uvarov, *Special Functions of Mathematical Physics: A Unified Introduction with Applications*, Birkhäuser Verlag, Basel, 1988, doi: 10.1007/978-1-4757-1595-8.

[99] N. E. Nørlund, *Sur les séries de facultés (in french)*, Acta Math., 37 (1914), pp. 327–387, doi: 10.1007/BF02401838.

[100] N. E. Nørlund, *Hypergeometric functions*, Acta Math., 94 (1955), pp. 289 – 349, doi: 10.1007/BF02392494.

[101] A. B. Olde Daalhuis, *Hyperasymptotic expansions of confluent hypergeometric functions*, IMA J. Appl. Math., 49 (1992), pp. 203–216, doi: 10.1093/imamat/49.3.203.

[102] A. B. Olde Daalhuis, *On the asymptotics for late coefficients in uniform asymptotic expansions of integrals with coalescing saddles*, Methods Appl. Anal., 7 (2000), pp. 727–745, doi: 10.4310/MAA.2000.v7.n4.a7.

[103] A. B. Olde Daalhuis, *Confluent Hypergeometric Functions, in: NIST Handbook of Mathematical Functions*, Cambridge University Press, Cambridge. 2010, pp. 321 – 349 (Chapter 13).

[104] A. B. Olde Daalhuis, *Hypergeometric Function, in: NIST Handbook of Mathematical Functions*, Cambridge University Press, Cambridge. 2010, pp. 383 – 401 (Chapter 15).

[105] F. W. J. Olver, *Asymptotics and Special Functions*, A K Peters, Ltd., Natick, Massachusetts, 1997, doi: 10.1201/9781439864548.

[106] F. W. J. Olver, D. W. Lozier, R. F. Boisvert and C. W. Clark, *NIST Handbook of Mathematical Functions*, Cambridge University Press, Cambridge, 2010.

[107] F. W. J. Olver and L. C. Maximon, *Bessel Functions, in: NIST Handbook of Mathematical Functions*, Cambridge University Press, Cambridge. 2010, pp. 215 – 286 (Chapter 10).

[108] F. W. J. Olver, A. B. Olde Daalhuis, D. W. Lozier, B. I. Schneider, R. F. Boisvert, C. W. Clark, B. R. Miller, B. V. Saunders, H. S. Cohl and e. M. A. McClain, *NIST Digital Library of Mathematical Functions*, Release 1.1.6 of 2022-06-30. Visit `http://dlmf.nist.gov/`.

[109] F. W. J. Olver and R. Wong, *Asymptotic approximations, in: NIST Handbook of Mathematical Functions*, Cambridge University Press, Cambridge. 2010, pp. 41–70 (Chapter 2).

[110] R. E. O'Malley, *Introduction to Singular Perturbations*, Academic Press, New York, 1974, doi: 10.1016/b978-0-125-25950-7.x5001-3.

[111] P. Palacios, *Desarrollos uniformemente convergentes de funciones especiales (in spanish)*, 2018. Master's Final Project directed by J. L. López and P. J. Pagola. Available online at `https://zaguan.unizar.es/`.

[112] R. E. A. C. Paley and N. Wiener, *Fourier Transforms in the Complex Domain*, American Mathematical Society, New York, 1934, doi: 10.1090/coll/019.

[113] R. B. Paris, *On the use of Hadamard expansions in hyperasymptotic evaluation. I. Real variables*, R. Soc. Lond. Proc. Ser. A. Math. Phys. Eng. Sci., 457 (2001), pp. 2835–2853, doi: 10.1098/rspa.2001.0823.

[114] R. B. Paris, *On the use of Hadamard expansions in hyperasymptotic evaluation. II. Complex variables*, R. Soc. Lond. Proc. Ser. A. Math. Phys. Eng. Sci., 457 (2001), pp. 2855–2869, doi: 10.1098/rspa.2001.0824.

[115] R. B. Paris, *On the use of Hadamard expansions in hyperasymptotic evaluation of Laplace-type integrals. I. Real variable*, J. Comput. Appl. Math., 167 (2004), pp. 293–319, doi: 10.1016/j.cam.2003.10.005.

[116] R. B. Paris, *Incomplete Gamma and Related Functions, in: NIST Handbook of Mathematical Functions*, Cambridge University Press, Cambridge. 2010, pp. 173 – 192 (Chapter 8).

[117] R. B. Paris, *Struve and Related Functions, in: NIST Handbook of Mathematical Functions*, Cambridge University Press, Cambridge. 2010, pp. 287 – 301 (Chapter 11).

[118] R. B. Paris, *Hadamard Expansions and Hyperasymptotic Evaluation. An extension of the Method of Steepest Descents*, Cambridge University Press, Cambridge, 2011, doi: 10.1017/CBO9780511753626.

[119] R. B. Paris and D. Kaminski, *Asymptotics and Mellin-Barnes Integrals*, Cambridge University Press, Cambridge, 2001, doi: 10.1017/cbo9780511546662.

[120] R. B. Paris and D. Kaminski, *Hyperasymptotic evaluation of the Pearcey integral via Hadamard expansions*, J. Comput. Appl. Math., 190 (2006), pp. 437–452, doi: 10.1016/j.cam.2005.01.038.

[121] M. Parodi, *Application du calcul symbolique à la résolution de certaines équations de Fredholm (in french)*, Bull. Sci. Math., 2 (1945), pp. 174–184.

[122] M. Parodi, *Sur les solutions d'équations intégrales dont les noyaux renferment des polynomes d'Hermite (in french)*, C. R. Acad. Sci. Paris, 224 (1947), pp. 780–782.

[123] M. Parodi, *Sur une propriété de l'équation intégrale* $\int_0^\infty (t^x f(x)/\Gamma(1+x))dx = g(t)$ *(in french)*, Ann. Soc. Sci. Bruxelles I, 62 (1948), pp. 24–26.

[124] K. Pearson, *Tables of the Incomplete Beta Functions*, Cambridge University Press, Cambridge, 1968.

[125] O. Perron, *Über die näherungweise Berechnung von Funktionen großer Zhalen (in german)*, Sitzungsber. Bayr. Akad. Wissensch. (Münch. Ber., 11 (1917), pp. 191–219.

[126] M. E. Peskin and D. V. Schroeder, *An Introduction To Quantum Field Theory*, CRC Press, Boca Raton, 1995, doi: 10.1201/9780429503559.

[127] H. Poincaré, *Sur les intégrales irrégulières: Des équations linéaires*, Acta Math., 8 (1886), pp. 295–344, doi: 10.1007/BF02417092.

[128] M. Razpet, *An application of elliptic integrals*, J. Math. Anal. Appl., 168 (1992), pp. 425–429, doi: 10.1016/0022-247X(92)90170-I.

[129] B. Riemann, *Theorie der durch die Gauss'sche Reihe* $F(\alpha, \beta, \gamma, x)$ *darstellbaren Funktionen (in german)*, Abh. Königl. Ges. Wiss. Göttingen, 7 (1857), pp. 3–22.

[130] B. Riemann, *Über die Anzahl der Primzahlen unter einer gegebenen Grösse (in german)*, Ber. Akad. Wiss. Berlin, (1859).

[131] J. Riordan, *Combinatorial identities*, John Wiley & Sons, Inc., New York-London-Sydney, 1968.

[132] R. Roy and F. W. J. Olver, *Elementary functions, in: NIST Handbook of Mathematical Functions*, Cambridge University Press, Cambridge. 2010, pp. 103–134 (Chapter 4).

[133] O. Schlömilch, *Über Facultätenreihen (in german)*, Ber. Verh. K. Sächs. Ges. Wiss. Leipz., Math.-Phys. Kl., 11 (1859), pp. 109–137.

[134] J. Stirling, *Methodus differentialis: Sive tractatus de summation et interpolation serierum infinitarium (in latin), (1730)*, Springer, London. English translation by I. Tweddle, *James Stirling's Methodus Differentialis*, 2003, doi: 10.1007/978-1-4471-0021-8.

[135] G. G. Stokes, *On the discontinuity of arbitrary constants which appear in divergent developments, in: Mathematical and Physical Papers*, vol. 4, Cambridge University Press, Cambridge. (1857), 2009 reprint of the 1904 original, pp. 77 – 109 (Chapter 12), doi: 10.1017/CBO9780511702273.013.

[136] G. G. Stokes, *Supplement to a paper on the Discontinuity of Arbitrary Constants which appear in Divergent Developments, in: Mathematical and Physical Papers*, vol. 4, Cambridge University Press, Cambridge. (1868), 2009 reprint of the 1904 original, pp. 283 – 298 (Chapter 30), doi: 10.1017/CBO9780511702273.031.

[137] G. G. Stokes, *Note on the Determination of Arbitrary Constants which appear as Multipliers of Semi-convergent Series, in: Mathematical and Physical Papers*, vol. 5, Cambridge University Press, Cambridge. (1889), 2009 reprint of the 1905 original, pp. 221 – 225 (Chapter 23), doi: 10.1017/CBO9780511702297.024.

[138] R. H. Swendsen, *An Introduction to Statistical Mechanics and Thermodynamics*, Oxford University Press, New York, 2012, doi: 10.1093/oso/9780198853237.001.0001.

[139] N. M. Temme, *Uniform Asymptotic Expansions of Confluent Hypergeometric Functions*, J. Inst. Math. Appl., 22 (1978), pp. 215 – 223, doi: 10.1093/imamat/22.2.215.

[140] N. M. Temme, *Special Functions: An Introduction to the Classical Functions of Mathematical Physics*, John Wiley & Sons, Inc., New York, 1996, doi: 10.1002/9781118032572.

[141] N. M. Temme, *Large parameter cases of the Gauss hypergeometric function*, J. Comput. Appl. Math., 153 (2003), pp. 441–462, doi: 10.1016/S0377-0427(02)00627-1.

[142] N. M. Temme, *Exponential, Logarithmic, Sine and Cosine Integrals, in: NIST Handbook of Mathematical Functions*, Cambridge University Press, Cambridge. 2010, pp. 149 – 157 (Chapter 6).

[143] N. M. Temme, *Numerical methods, in: NIST Handbook of Mathematical Functions*, Cambridge University Press, Cambridge. 2010, pp. 71–101 (Chapter 3).

[144] N. M. Temme, *Parabolic Cylinder Functions, in: NIST Handbook of Mathematical Functions*, Cambridge University Press, Cambridge. 2010, pp. 303 – 319 (Chapter 12).

[145] N. M. Temme, *Asymptotic Methods for Integrals*, World Scientific, New Jersey, 2015, doi: 10.1142/9195.

[146] A. Urbina, M. León De La Barra, G. León De La Barra and M. Cañas, *Elliptic integrals and limit cycles*, Bull. Austral. Math. Soc., 48 (1993), pp. 195 – 200, doi: 10.1017/S0004972700015641.

[147] M. D. Van Dyke, *Perturbations Methods in Fluid Mechanics*, Academic Press, New York, 1968.

[148] V. Volterra, *Teoria delle potenze, dei logaritmi e delle funzioni di decomposizione (in italian)*, Acc. Lincei., Memoire 5 (1916), pp. 167–249.

[149] W. Wasow, *Asymptotic Expansions for Ordinary Differential Equations*, Dover Publications, Inc. Mineola, New York, 1965.

[150] G. N. Watson, *The Harmonic Functions Associated with the Parabolic Cylinder*, Proc. London. Math. Soc., 2 (1918), pp. 241–246, doi: 10.1112/plms/s2-17.1.116.

[151] W. S. Weiglhofer, *Electromagnetic depolarization dyadics and elliptic integrals*, J. Phys. A: Math. Gen., 31 (1998), pp. 7191–7196, doi: 10.1088/0305-4470/31/34/019.

[152] J. Wojdylo, *Computing the Coefficients in Laplace's Method*, SIAM Rev., 48 (2006), pp. 76–96, doi: 10.1137/S0036144504446175.

[153] J. Wojdylo, *On the coefficients that arise from Laplace's method*, J. Comput. Appl. Math., 196 (2006), pp. 241–266, doi: 10.1016/j.cam.2005.09.004.

[154] R. Wong, *Asymptotic Approximation of Integrals*, Society for industrial and Applied Mathematics, Philadelphia, 2001, doi: 10.1137/1.9780898719260.

[155] M. Wyman and R. Wong, *The Asymptotic behavior Of* $\mu(z, \beta, \alpha)$, Canadian J. Math., 21 (1969), pp. 1013–1023, doi: 10.4153/CJM-1969-112-4.

[156] D. S. Zézé, M. Potier-Ferry and Y. Tampango, *Multi-point Taylor series to solve differential equations*, Discrete Contin. Dyn. Syst. Ser. S, 12 (2019), pp. 1791–1806, doi: 10.3934/dcdss.2019118.

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