f-HybridMem: A consensual analysis via fuzzy consensus measures and penalty functions

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***Abstract*—This paper considers the consensual analysis in decision-making (CDM) processes based on fuzzy logic (FL) and interval-valued fuzzy logic (IVFL), providing a CDM-strategy, by exploring the axiomatic properties of fuzzy consensus measures (FCM) via penalty functions. Thus, two models are formalized, FS-FCM and IVFS-FCM. In the former, the fuzzy-valued lattice enables the analysis of fuzzy information for linguistic variables (LV), which is obtained by the aggregation of penalty functions. And, in the latter, the consensus measures of fuzzy sets are aggregated to build a new consensual analysis modeling. Thus, e.g., the cohesion of several terms related to the same LV can be analyzed, and also the coherence between fuzzy sets referring to the lowest and highest projections. Such models decide based on relevance criteria and qualitative assessments, via the selection of alternatives, supporting the corresponding algorithmic strategies: FS-FCM strategy, applied to fuzzy values, and IVFS-FCM strategy, covering fuzzy sets. The Intf-HybridMem approach explores the access patterns to volatile and non-volatile memories related to decision-making in two steps: (i) the FS-FCM strategy explores consensus measures of fuzzy values from membership functions; and (ii) the IVFS-FCM strategy, modeling inaccuracy inherent in input variables, as read/write frequency and access recency, also including the migration recommendation as output, which is validated by evaluations carried out in both proposed strategies.**

***Index Terms*—Hybrid Memory Management, Decision Making Problem, Fuzzy Consensus Measure, Penalty Functions**

I. Introduction

One of the challenges involving memory architecture still lies in the balance between high-speed, high-capacity, and low-power consumption. Hybrid memories (HM) seem to be a good strategy to tackle that [1]–[3]. On one hand, Non-Volatile Memories (NVMs) furnish lower static power and higher density when compared to volatile memories. On the other hand, NVM applications bring new demands like low endurance and time asymmetry between reading and writing operations. Simply replacing DRAM with NVM architecture would face high durability issues [4], justifying then the use of HM.

This work provides a technique to recommend page migration using the *f-HybridMem* approach [5], considering consensual analysis in decision-making (CDM) processes based on fuzzy logic (FL) and interval-valued fuzzy logic (IVFL). Fuzzy Consensus Measures (FCM) are applied to fuzzy values from membership functions. So, our proposal explores the axiomatic properties of FCM, which for the first time is defined from penalty functions. The study focuses on the fuzzy-valued lattice allowing the analysis of fuzzy information for linguistic variables (LV), which is obtained by aggregating penalty functions. The cohesion of several terms related to the same LV can be analyzed and, therefore, the methodology decides based on relevant criteria and qualitative assessments, via the selection of alternatives, supporting the corresponding algorithmic strategy, named FS-FCM method.

This paper is organized as follows. Section II recalls some relevant concepts used in our proposal. In Section III, we discuss some operators that are constructors of FCM. Section IV describes our case study based on penalty functions. Section V shows our algorithmic approach for CDM-strategy, followed by Section VI with the conclusions.

II. Preliminaries

A. Aggregation functions

An aggregation function merges a set of inputs into a single number that represents the whole set. Based on [6], we report their essential properties in the following definition.

Definition 2.1: A mapping $A\colon [0,1]^n \to [0,1]$ is an n-ary aggregation function (AF) if it verifies the following conditions

A1: If $x_i \leq y_i$ for each $i = 1, \ldots, n$, then $A(x_1, \ldots, x_n) \leq A(y_1, \ldots, y_n)$ (monotonicity);

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A2: $A(0,\ldots,0)=0$ and $A(1,\ldots,1)=1$ (boundary conditions).

Consider $\mathbb{N}_n=\{1,2,\ldots,n\}$, for a natural number n such that $n\geq 1$. Other properties which can be required for an aggregation function A:

A3: $(x_1,\ldots,x_n)=A(x_{(1)},\ldots,x_{(n)})$ for each permutation $(\,):\mathbb{N}_n\to\mathbb{N}_n$ (symmetry);

A4: $A(x_1,\ldots,x_n)=A(x_1,\ldots,x_n,\ldots,x_1,\ldots,x_n)$ (invariance for replications);

A5: $A(x_1,\ldots,x_n)=A(x_1,\ldots,x_i,1,x_{i+1},\ldots,x_n)$ (invariance for 1).

A6: If $A(x,1,\ldots,1)=x$, for each $x\in[0.1]$, then A verifies truth dominance;

A7: If $A(x_1,\ldots,x_n)=1\Leftrightarrow x_1=\ldots=x_n=1$, for each $x_1,\ldots,x_n\in[0,1]$, then A verifies the 1-boundary condition;

A8: If $A(x_1,\ldots,x_n)=0\Leftrightarrow x_1=0$ or $\ldots$ or $x_n=0$ then, for each $x_1,\ldots,x_n\in[0,1]$, A verifies the strong 0-boundary condition;

A8(a): If $A(x,\ldots,x)=x$ for each $x\in[0,1]$, then A is idempotent.

Definition 2.2: A mapping $A\colon\bigcup_{n=1}^{\infty}[0,1]^n\to[0,1]$ is called an extended aggregation function (EAF) if the following properties hold:

(i) $A\restriction[0,1]^n\colon[0,1]^n\to[0,1]$ is an AF, $\forall n\in\mathbb{N}^{+2}=\{2,3,\ldots\}$; and

(ii) $A(x)=x$, $\forall x\in[0,1]$.

Therefore, from Def. 2.2, one can identify an EAF A as a family of functions $(A_n)_{n\in\mathbb{N}^{+2}}$, such that $A_n\colon[0,1]^n\to[0,1]$ is an AF.

These two properties are also considered for an EAF A:

A9: If $A(x_1,\ldots,x_n)=A(x_1,\ldots,x_n,\ldots,x_1,\ldots,x_n)$ then A is invariant for replications;

A10: If $A(x_1,\ldots,x_n)=A(x_1,\ldots,x_i,1,x_{i+1},\ldots,x_n)$ then A is 1-invariant.

There exist two interesting binary continuous aggregation functions (non necessarily associative), presented by [7], [8], which particularly interest us. Their axiomatic definitions are given as follows.

Definition 2.3: [7] A binary mapping $O\colon[0,1]^2\to[0,1]$ is called an overlap function if it satisfies the following conditions, for all $x,y,z\in[0,1]$:

(O1) $O(x,y)=O(y,x)$;

(O2) $O(x,y)=0$ iff $x=0$ or $y=0$;

(O3) $O(x,y)=1$ iff $x=y=1$;

(O4) if $x\leq y$ then $O(x,z)\leq O(y,z)$;

(O5) O is continuous;

Remark 2.1: Note that whenever an overlap function has a neutral element, then, by (O3), it is necessarily 1.

Definition 2.4: [8] A binary mapping $G\colon[0,1]^2\to[0,1]$ is called a grouping function if it verifies, for all $x,y,z\in[0,1]$:

(G1) $G(x,y)=G(y,x)$;

(G2) $G(x,y)=0$ iff $x=y=0$;

(G3) $G(x,y)=1$ iff $x=1$ or $y=1$;

(G4) if $x\leq y$ then $G(x,z)\leq G(y,z)$;

(G5) G is continuous;

Remark 2.2: Note that whenever a grouping function has a neutral element, then, by (G2), this element is necessarily 0.

B. Restricted Equivalence Functions

The idea of restricted equivalence functions (REF) [9] is proposed as an extension of equivalence functions, in the sense of [10]. In this subsection, we also find restricted dissimilarity functions (RDF) [11].

Next definition generalizes the concept of $\mathcal{L}_{[0,1]}$-REF seen in [9], [12].

Definition 2.5: [13, Def. 2.16] Let $\mathcal{N}$ be a negation operator on the lattice $\mathcal{L}\equiv(\mathcal{L},\leq_{\mathcal{L}})$. An REF over $\mathcal{L}$ ($\mathcal{L}$-REF) $f\colon\mathcal{L}^2\to\mathcal{L}$, regarding the $\mathcal{L}$-negation $\mathcal{N}$, verifies, $\forall a,b\in\mathcal{L}$, the next conditions:

REF1: $f(a,b)=f(b,a)$;

REF2: $f(a,b)=\top_{\mathcal{L}}\Leftrightarrow a=b$;

REF3: $f(a,b)=\bot_{\mathcal{L}}\Leftrightarrow(a=\top_{\mathcal{L}}$ and $b=\bot_{\mathcal{L}})$ or $(a=\bot_{\mathcal{L}}$ and $b=\top_{\mathcal{L}})$;

REF4: $f(a,b)=f(\mathcal{N}(a),\mathcal{N}(b))$;

REF5: $a\leq b\leq c\Rightarrow f(a,b)\geq f(a,c)$ and $f(b,c)\geq f(a,c)$.

Whenever one considers a dual construction $\mathcal{N}$ on $\mathcal{L}\equiv(\mathcal{L},\leq_{\mathcal{L}})$, the concept of an RDF was provided, mainly connected with penalty functions [14]–[16]. The concept of $\mathcal{L}$-RDF, based on previous studies, is presented as follows [13]:

Definition 2.6: [13, Def. 5.1] A mapping $h_L:\mathcal{L}^2\to\mathcal{L}$ is named an RDF over $\mathcal{L}$ ($\mathcal{L}$-RDF) if $\forall a,b,c\in\mathcal{L}$, it satisfies:

RDF1: $h_{\mathcal{L}}(a,b)=h_{\mathcal{L}}(b,a)$;

RDF2: $h_{\mathcal{L}}(a,b)=\top_{\mathcal{L}}$ iff $\{a,b\}=\{\top_{\mathcal{L}},\bot_{\mathcal{L}}\}$;

RDF3: $h_{\mathcal{L}}(a,b)=\bot_{\mathcal{L}}$ iff $a=b$;

RDF4: If $a\leq_{\mathcal{L}}b\leq_{\mathcal{L}}c$, then $h(a,b)\leq_{\mathcal{L}}h(a,c)$ and $h(b,c)\leq_{\mathcal{L}}h(a,c)$.

If $\mathcal{L}$-RDF h verifies the property

RDF5: $h(a,b)=h(\mathcal{N}(a),\mathcal{N}(b))$, $\forall a,b\in\mathcal{L}$.

regarding an $\mathcal{L}$-negation, we say h is an $\mathcal{L}$-RDF with respect to a negation $\mathcal{N}$.

The next proposition shows under which conditions on the lattice $[0,1]$, we obtain from a quasi-concave $\mathcal{L}$-REF a quasi-convex $\mathcal{L}$-RDF, considering its $\mathcal{N}$-dual construction.

Proposition 2.1: Let $\mathcal{N}:\mathcal{L}\to\mathcal{L}$ be a strong negation on the lattice $[0,1]$. A mapping $\mathcal{L}$-REF $f\colon[0,1]^2\to[0,1]$ is quasi-concave, i.e., $\forall a,b,a',b\in[0,1]$, so we have that

QCO**:** $f(\lambda(a,b)+(1-\lambda)(a',b'))\leq\max(f(a,b),f(a',b'))$

iff a corresponding mapping $\mathcal{N}$-dual $\mathcal{L}$-RDF, denoted by $f_{\mathcal{N}}$, is a quasi-convex function, i.e., $\forall a,b,a',b\in[0,1]$, where

QCE**:** $f_{\mathcal{N}}(\lambda(a,b)+(1-\lambda)(a',b'))\geq\min(f_{\mathcal{N}}(a,b),f_N(a',b'))$.

Example 2.1: Let $\mathcal{L}_{[0,1]}=([0,1],\leq,\vee,\wedge,1,0)$ be the lattice of all fuzzy values. According to [17], the mappings $F_2,H_2\colon[0,1]^2\to[0,1]$ given by

$$F_2(x,y)=1-(x-y)^2\text{ and }H_2(x,y)=(x-y)^2,\qquad(1)$$

are $\mathcal{L}_{[0,1]}$-REF and $\mathcal{L}_{[0,1]}$-RDF functions, both related to fuzzy negation N_S, meaning that (F_2,H_2) is a pair of dual-N_S functions satisfying QCE and QCO properties, respectively.

Example 2.2: According to [18], the mappings $f_{|\,|}, h_{|\,|}\colon [0,1]^2 \to [0,1]$ defined by

$$f_{|\,|}(x,y) = 1 - |x-y| \text{ and } h_{|\,|}(x,y) = |x-y|, \quad (2)$$

are $\mathcal{L}_{[0,1]}$-REF and $\mathcal{L}_{[0,1]}$-RDF functions, both w.r.t. N_S negation and satisfying QCE and QCO, respectively.

C. Fuzzy consensus measures

Fuzzy consensus measures (FCM) are functions promoting a formal model to achieve a concordance analysis among fuzzy system inputs, often employed in decision-making contexts. The formal study of consensus measures is an indication when an overall assessment reflecting an expert group overview is achieved, establishing minimum/maximum consensus limits and promoting the final decision. Such consensus analysis is mainly concerned with two properties, namely: (i) the unanimity, interpreting the complete consensus when all inputs are the same; and (ii) the minimal consensus, resulting in a null-consensus whenever the inputs lie on both extremes (0 and 1) in the unit interval. The formal definition is given as follows.

Definition 2.7: [19, Definition 7]. A function $C\colon [0,1]^n \to [0,1]$ is called a consensus measure if it holds:

$C1$: $C(a,\dots,a) = 1$, $\forall a \in [0,1]$ (unanimity);

$C2$: $C(0,1) = C(1,0) = 0$ (minimum consensus for $n=2$).

In case one needs to highlight the strict definition of the unanimity property, we have the following concept of FCM:

Definition 2.8: A function $C\colon \bigcup_{n=1}^{\infty}[0,1]^n \to [0,1]$ is a strict FCM if C verifies $C2$ and the strict unanimity property defined by:

$C1(a)$: $C(x_1,...,x_n) = 1$ if, and only if, $x_i = x_j$ for $i,j \in \mathbb{N}_n$ and $\mathbb{N}_n = \{1,2,\dots,n\}$.

Other properties of FCM are also described in [19]:

$C3$: $C(x_1,x_2,\dots,x_n) = C(x_{\sigma(1)},x_{\sigma(2)},\dots,x_{\sigma(n)})$, for all σ-permutation on {1,…,n} and $x \in [0,1]^n$ (symmetry);

$C4$: $C(x_1,x_2,\dots,x_n) = 0$, when $n = 2k$ and $k = \#\{x_i\colon x_i = 0\} = \#\{x_i\colon x_i = 1\}$ (maximum dissension);

$C5$: $C(x_1,x_2,\dots,x_n) = C(N(x_1),N(x_2),\dots,N(x_n))$, when N is a strong fuzzy negation (reciprocity);

$C6$: $C(x) = C(x,x) = C(x,\dots,x)$, $\forall x \in [0,1]$ (replication invariance);

$C7$: For $n = 2k$, let half of the evaluations equals $\mathbf{a} = (a,\dots,a)$, where $\mathbf{a} \in [0,1]^k$. If $|a - x_j| \leq |a - y_j|$ for $j = 1,\dots,k$, then $C(a,x_1,x_2,\dots,x_k) \geq C(a,y_1,y_2,\dots,y_k)$. (monotonicity w.r.t. the majority).

$C8$: For an extended aggregation operator A, C satisfies the A-monotonicity regarding $\mathcal{L}_{[0,1]}$-RDF $h\colon [0,1]^2 \to [0,1]$, if, for any positive integer n, and $x_1,...,x_n,y_1,...,y_n \in [0,1]$, we have that:

$$h(x_i, A(x_1,...,x_n)) \leq h(y_i, A(y_1,....,y_n)), \forall i \in \mathbb{N}_n \Rightarrow C(y_1,...,y_n) \leq C(x_1,...,x_n).$$

Some examples of FCM presented in [19] also verifying Ck properties, $\forall k \in \{3,4,5,6\}$, are reported next, illustrating the definitions given previously.

Example 2.3: Considering the arithmetic mean of the distance between pairs, the mapping $C^d_{SK}\colon \bigcup_{i=1}^{\infty}[0,1]^n \to [0,1]$, defined by

$$C^d_{SK}(x_1,\dots,x_n) = 1 - \frac{2}{n^2}\sum_{\forall i,j | i\neq j}^{n} h(x_i,x_j); \quad (3)$$

is an FCM satisfying $C7$, regarding $\mathcal{L}_{[0,1]}$-RDF given as $h\colon [0,1]^2 \to [0,1]$, where $h(x,y) = (x-y)^2$.

Example 2.4: Take $C_{Tastle}\colon \bigcup_{i=1}^{\infty}[0,1]^n \to [0,1]$, given by

$$C_{Tastle}(x_1,\dots,x_n) = 1 + \frac{1}{n}\sum_{i=1}^{n}\log_2(1 - |x_i - \overline{x}|) \quad (4)$$

C_{Tastle} is an FCM related to the arithmetic mean over the logarithm operator applied to the difference between 1 and $\mathcal{L}_{[0,1]}$-RDF $h : [0,1]^2 \to [0,1]$ defined by $h(x,y) = |x-y|$. Both operators C^d_{SK} and C_{Tastle} verify $C6$ regarding $\mathcal{L}_{[0,1]}$-RDF h and $C8$ considering the arithmetic mean (AM).

III. FS-FCM Methodologies: Constructing FCM on the Lattice of Fuzzy Values

In the following subsections, we show that operators like convex sums, conjugate functions, equivalence functions, and penalty functions are constructors of FCM, and we also discuss their main properties.

A. Defining $\mathcal{L}_{[0,1]}$-FCM from covex sums

We propose a method to generate a new FCM $C_1 +_\lambda C_2$ from a convex sum applied to $\mathcal{L}_{[0,1]}$-FCM C_1 and C_2. See that, for $\lambda \in [0,1]$, when it is considered as an operator, it also preserves the main properties of C_1 and C_2.

Proposition 3.1: [20, Prop. 4.1] Consider a fuzzy negation N and let $C_1, C_2\colon \bigcup_{n=1}^{\infty}[0,1]^n \to [0,1]$ be fuzzy consensus measures on $[0,1]$ satisfying Ci for $i \in \{3,4,5,6,7,8\}$, $C6$ (regarding N), $C7$ related to $\mathcal{L}_{[0,1]}$-RDF $h\colon [0,1]^2 \to [0,1]$ and $C8$, w.r.t. an EAF A. For $\lambda \in [0,1]$, the mapping $C_1 +_\lambda C_2\colon \bigcup_{n=1}^{\infty}[0,1]^n \to [0,1]$ given, $\forall x_1,\dots,x_n \in [0,1]$, by

$$C_1 +_\lambda C_2(x_1,\dots,x_n) = \lambda C_1(x_1,\dots,x_n) + (1-\lambda)C_2(x_1,\dots,x_n) \quad (5)$$

is also a fuzzy consensus measure on $[0,1]$ verifying Ci, for the same $i \in \{3,4,5,6,7,8\}$, $C6$ (regarding N), $C7$ related to $\mathcal{L}_{[0,1]}$-RDF h and $C8$ considering AM as an EAF.

B. Defining $\mathcal{L}_{[0,1]}$-FCM from automorphisms

Next, we have a definition illustrating a method to generate a new consensus measure C^ϕ by the action of an automorphism $\phi\colon [0,1] \to [0,1]$ over a fuzzy consensus measure C.

Proposition 3.2: [20, Prop. 4.2] Consider a fuzzy negation N. Take $\phi \in Aut([0,1])$ and let $C\colon \bigcup_{n=1}^{\infty}[0,1]^n \to [0,1]$ be an $\mathcal{L}_{[0,1]}$-FCM verifying Ci, for $i \in \{3,4,5,6,7\}$, $C6$ (regarding N) and $C7$ related to $\mathcal{L}_{[0,1]}$-RDF $h\colon [0,1]^2 \to [0,1]$. The conjugate function $C^\phi\colon \bigcup_{n=1}^{\infty}[0,1]^n \to [0,1]$ given, for all $x_1,\dots,x_n \in [0,1]$, by

$$C^\phi(x_1,\dots,x_n) = \phi^{-1}(C(\phi(x_1),\dots,\phi(x_n))) \quad (6)$$

is an $\mathcal{L}_{[0,1]}$-FCM verifying Ci for $i \in \{3,4,5,6,7\}$, $C6$ (regarding N^{ϕ}, ϕ-conjugate of N) and $C7$ w.r.t. h.

C. Defining $\mathcal{L}_{[0,1]}$-FCM from aggregations and $\mathcal{L}_{[0,1]}$-REF

Next theorem introduces a method $C_{A,f}$ to construct $\mathcal{L}_{[0,1]}$-FCM based on aggregation functions A and restricted equivalence functions f.

Theorem 3.1: Let N be a strong fuzzy negation, $f\colon [0,1]^2 \to [0,1]$ be an $\mathcal{L}_{[0,1]}$-REF regarding N and with an N-dual operator given by $\mathcal{L}_{[0,1]}$-RDF $h\colon [0,1]^2 \to [0,1]$, and let A be an EAF verifying $A3$, $A4$, $A6$ and $A9$ properties. For the permutation $()\colon \mathbb{N}_n \to \mathbb{N}_n$, the mapping $C_{A,f}\colon \bigcup_{n=1}^{\infty}[0,1]^n \to [0,1]$ defined, for all $x_1,\ldots,x_n \in [0,1]$, by

$$\begin{aligned} C_{A,f}(x_1,\ldots,x_n) &= A(f(x_{(1)},x_{(n)}),\ldots,f(x_{(n)},x_{(1)})) \\ &= A_{i=1}^{n} f\left(x_{(i)},x_{(n-i+1)}\right) \end{aligned} \quad (7)$$

is an $\mathcal{L}_{[0,1]}$-FCM verifying $C4$, $C5$, $C6$ (regarding N), $C7$, and $C8$ (considering h).

IV. Fuzzy penalty functions

A penalty function (PF) consists of a powerful tool to determine up to what extent a given output is similar (dissimilar) to a set of inputs. They have been widely studied in the context of consensus analysis in fuzzy systems based on decision making [19], [21]. Penalty functions can dissuade decision-makers from making extreme judgments that make consensual results unfeasible [17]. Aggregation methods based on penalty functions are also capable of proposing different weights, adapting the consensus analysis from the problem modeling to a specific situation [22], [23].

The application and use of PF are not restricted to the aggregation of expert opinions but can be applied to distance measurements aiming at a consensus in benchmarking [24] experiments in current applications via simulations [25], [26].

The diversity of definitions of fuzzy penalty functions has been formalized via preference relations [15], fuzzy subsethood measures [27], pre-aggregations [28] and many other fuzzy aggregators [18].

Definition 4.1: [29, Def. 10] A mapping $P\colon [0,1]^{n+1} \to \mathbb{R}$ is a PF if and only if the following conditions hold:

P1: $P(x_1,\ldots,x_n,,y) \geq c$, $\forall x_i \in [0,1], i \in \mathbb{N}_n, y \in [0,1]$, and $c \in \mathbb{R}$;

P2: $P(x_1,\ldots,x_n,y) = c$ iff $x_i = y$, $\forall i \in \mathbb{N}_n$, and

P3: P is quasi-convex in y, i.e., for $x_1,\ldots,x_n,y,x'_1,\ldots,x'_n,y' \in [0,1]$, we have that

$$\begin{aligned} P(\lambda(x_1,\ldots,x_n,y) + (1-\lambda)(x'_1,\ldots x'_n,y')) \leq \\ \max(P(x_1,\ldots,x_n,y), P(x'_1,\ldots,x'_n,y')). \end{aligned}$$

We consider a penalty function restricted to the lattice $\mathcal{L}([0,1])$, defined by [18, Def. 10] and [14, Def. 3.7].

Definition 4.2: [18, Def. 10] A penalty-based mapping $p\colon [0,1]^n \to [0,1]$ is defined, $\forall x_1,\ldots,x_n \in [0,1]$, by

$$p(x_1,\ldots,x_n) = arg\min_y P(x_1,\ldots,x_n,y), \quad (8)$$

if y is the only minimizer, and $y = \frac{1}{2}$ if the minimizer set is either the interval $]0,1[$ or $[0,1]$.

According to [14, Ex. 2.2], for each PF P, we seek to define the number $y \in [0,1]$ called fusion value that indicates the merge value of n input data, interpreted by $x_1,\ldots,x_n \in [0,1]$, in order to optimize the penalty-based function p. In this sense, when minimizing y, we have the optimization of p.

Example 4.1: According to example 2.1, discussing results in [18], a mapping $F_2\colon [0,1]^2 \to [0,1]$, defined by $F_2(x,y) = (x-y)^2$ is a quasi-convex $\mathcal{L}([0,1])$-REF function. Let $()\colon \{1,\ldots,n\} \to \{1,\ldots,n\}$ be a permutation. Then, the mapping $P_2\colon [0,1]^{n+1} \to \mathbb{R}$, given by

$$P_2(x_1,\ldots,x_n,y) = \sum_{i=1}^{n} F_2(x_{(i)},y) = \sum_{i=1}^{n}(x_{(i)}-y)^2, \quad (9)$$

is a penalty function, where y is the corresponding fusion value. Note that in [17] it was proved, according to the optimization theory, that when this fusion value is the arithmetic mean, $y = AM(x_1,\ldots,x_n)$, then we obtain the value that minimizes P_2.

Example 4.2: See Example 2.2 with results from [14, Ex. 2.2], showing that $f_{||}(x,y) = |x-y|$ is a quasi-convex $\mathcal{L}([0,1])$-REF function. When $()\colon \{1,\ldots,n\} \to \{1,\ldots,n\}$ defines a permutation, the mapping $P_{||}\colon [0,1]^{n+1} \to \mathbb{R}$, defined by

$$P_{||}(x_1,\ldots,x_n,y) = \sum_{i=1}^{n} f_{||}(x_{(i)}-y) = \sum_{i=1}^{n}|x_{(i)}-y|, \quad (10)$$

is also a penalty function on $\mathcal{L}([0,1])$. In case the corresponding fusion value y refers to the median of the observed values, $y = Med(x_1,\ldots,x_n)$, we have the fusion value that minimizes $P_{||}$ [17].

Further contributions on new studies of classes of penalty functions and their properties, generalizations, and applications can be found in [14], [17], [18].

V. $\mathcal{L}_{[0,1]}$-FCM-PF via Aggregation of Penalty Functions

Based on the results presented previously, we introduce a methodology to obtain $\mathcal{L}_{[0,1]}$-FCM from penalty functions.

Proposition 5.1: Let $M\colon \bigcup_{n=1}^{\infty}[0,1]^n \to [0,1]$ be an EAF and let $C\colon [0,1]^2 \to [0,1]$ be an $\mathcal{L}_{[0,1]}$-FCM. Then, $C_{M,n}\colon \bigcup_{i=2}^{\infty}[0,1]^i \to [0,1]$ defined, $\forall n > 1$, by the expression,

$$C_{M,n}(x_1,....,x_n,y) = M(C(x_1,y),....,C(x_n,y)), \quad (11)$$

is an $\mathcal{L}_{[0,1]}$-FCM. Moreover, if M verifies $A9$ and $C_{M,n}$ also satisfies $C1(a)$, then $C_{M,n}$ verifies $C7$.

Proof: Let M be an EAF and let C be an $\mathcal{L}_{[0,1]}$-FCM.

$C1$: If $x_1 = ... = x_n = y$, $C_{M,n}(x_1,....,x_n,y) = M(C(y,y),...,C(y,y)) = M(1,....,1) = 1$.

$C2$: $C_{M,2}(0,0,1) = M(C(0,1),C(0,1)) = M(0,0) = 0$ and, similarly $C_{M,2}(1,1,0) = M(C(1,0),C(1,0)) = M(0,0) = 0$.

$C7$: If $C_{M,n}(x_1,...,x_n,y) = 1$ then

$M(C(x_1,y),....,C(x_n,y)) = 1$. So, by $A9$, $C(x_i,y) = 1$ for each $i = 1,\dots,n$. And, by $C1(a)$, $x_i = y$ for each $i = 1,\dots,n$. Therefore, $C_{M,n}$ satisfies $C1(a)$. □

The following result provides the conditions to obtain penalty functions from fuzzy consensus measures.

Theorem 5.1: Let $N\colon [0.1] \to [0,1]$ be a strong fuzzy negation and $C\colon \bigcup_{i=1}^{n}[0,1]^i \to [0,1]$ be an $\mathcal{L}$-FCM satisfying $C1(a)$ and being a quasi-concave function, i.e., for $x_1,\dots,x_n,y,x'_1,\dots,x'_n,y' \in [0,1]$,

$$C(\lambda(x_1,\dots,x_n,y)+(1-\lambda)(x'_1,\dots x'_n,y')) \geq \min(C(x_1,\dots,x_n,y),C(x'_1,\dots,x'_n,y')).$$

So, the mapping $P_{N,C}\colon [0,1]^{n+1} \to [0,1]$ defined by

$$P_{N,C}(x_1,\dots,x_n,y) = N(C(x_1,\dots,x_n,y)) \qquad (12)$$

is a fuzzy penalty function with the fusion value in y, in the sense of Definition 4.1.

Proof: Let $N\colon [0,1] \to [0,1]$ be a strong fuzzy negation and $C\colon [0,1] \to [0,1]$ be an $\mathcal{L}$-FCM verifying $C1(a)$. For $x_1,\dots,x_n,y,x'_1,\dots,x'_n,y' \in [0,1]$, we have the following results:

$P1$: As the range of fuzzy negations N and consensus measures C are in the interval $[0,1]$, when we take $c = 0$, it is immediate that $P_{N,C}(x_1,\dots,x_n,y) \geq 0$, $\forall x_i \in [0,1], i \in \mathbb{N}_n, y \in [0,1]$.

$P2$: If $x_1 = \dots = x_n = y$, $P_{N,C}(x_1,\dots,x_n,y) = N(C(y,\dots,y)) = N(1) = 0$. On the other hand, if $P_{N,C}(x_1,\dots,x_n,y) = 0$ then $N(C(x_1,\dots,x_n,y)) = 0$. As N is strong, $C(x_1,\dots,x_n,y) = 1$ and so, by $C1(a)$, $x_1 = \dots = x_n = y$.

$P3$: Take C as a quasi-concave function in y (satisfying $C9$):

$$\begin{aligned}
&P_{N,C}(\lambda(x_1,\dots,x_n,y)+(1-\lambda)(x'_1,\dots x'_n,y')) = \\
&N(C(\lambda(x_1,\dots,x_n,y)+(1-\lambda)(x'_1,\dots x'_n,y'))) \leq \\
&N(\min(C(x_1,\dots,x_n,y),C(x'_1,\dots,x'_n,y'))) = \\
&\max(N(C(x_1,\dots,x_n,y)),N(C(x'_1,\dots,x'_n,y')) = \\
&\max(P(x_1,\dots,x_n,y),P(x'_1,\dots,x'_n,y')).
\end{aligned}$$

Therefore, Theorem 5.1 holds. □

Corollary 5.1: Let A be an EAF verifying $A9$ and let $C\colon [0,1]^2 \to [0,1]$ be a quasi-concave $\mathcal{L}_{[0,1]}$-FCM verifying $C3$. A mapping $P_{N,C_A}: [0,1]^n \to [0,1]$ defined by

$$P_{N,C_A}(x_1,\dots,x_n,y) = N(C_A(x_1,\dots,x_n,y)) \qquad (13)$$

is a fuzzy penalty function, in the sense of Def. 4.1.

Proof: It follows from Proposition 5.1 and Theorem 5.2. □

The following examples will be applied on *f-HybridMem* approach [5] detailed in Sect. VII in a decision making problem to recommend page migration, consolidating the CDM strategy.

Example 5.1: Consider Example 4.1, so

$$P_{N_S,C_{f_{sq}}}(x_1,\dots,x_n,AM(x_1,\dots,x_n)) = N_S(C_{f_{sq}}(x_1,\dots,x_n,AM(x_1,\dots,x_n))).$$

In addition, the penalty function given in Eq. (9) can be obtained by composition related to Eq. (16):

$$\begin{aligned}
&P_{N_S,C_{f_{sq}}}(x_1,\dots,x_n,AM(x_1,\dots,x_n)) = \\
&= \sum_{i=1}^{n}(x_{(i)} - AM(x_1,\dots,x_n))^2 \\
&= P_2(x_1,\dots,x_n,AM(x_1,\dots,x_n)),
\end{aligned} \qquad (14)$$

In this case, the fusion point y is the arithmetic mean of all $x_1,\dots,x_n \in [0,1]$.

Example 5.2: Consider Example 4.2 of a penalty function,

$$\begin{aligned}
&P_{N_S,C_{f_{||}}}(x_1,\dots,x_n,Med(x_1,\dots,x_n)) = \\
&N_S(C_{f_{||}}(x_1,\dots,x_n,Med(x_1,\dots,x_n))).
\end{aligned} \qquad (15)$$

In Eq. (15), the penalty function has its fusion point y obtained from the median of the input values $x_1,\dots,x_n \in [0,1]$. According to Eq. (10), the penalty function $P_{N_S,C_{f_{||}}}$ can be expressed by:

$$\begin{aligned}
&P_{N_S,C_{f_{||}}}(x_1,\dots,x_n,Med(x_1,\dots,x_n)) = \\
&= \sum_{i=1}^{n}|x_{(i)} - Med(x_1,\dots,x_n)| \\
&= P_{||}(x_1,\dots,x_n,Med(x_1,\dots,x_n))
\end{aligned} \qquad (16)$$

The reverse construction can be seen in the next theorem, which guarantees the conditions to have consensus measures from penalty function on the lattice $\mathcal{L}([0,1])$.

Theorem 5.2: Let $N\colon [0,1] \to [0,1]$ be a strong fuzzy negation and $P\colon [0,1]^{n+1} \to [0,1]$ be a quasi-convex fuzzy penalty function, with fusion value y, by Def. 4.1. So the mapping $C_{A,P}\colon [0,1]^n \to [0,1]$, for all $x_1,\dots,x_n,y,x'_1,\dots,x'_n,y' \in [0,1]$ given by

$$C_{N,P}(x_1,\dots,x_n,y) = N(P(x_1,\dots,x_n,y)) \qquad (17)$$

is a quasi-concave $\mathcal{L}$-FCM.

VI. Algorithmic approach for CDM-strategy

This Section describes the algorithmic strategy developed for applying fuzzy consensus measures in the decision-making process, which, in Section VI, is used in the management of hybrid memories. We consider two classes of FCM methodologies, according to the nature of the fuzzy modeling:

(i) FS-FCM uses methods for the consensual approach over fuzzy values;

(ii) IVFS-FCM promotes consensual analysis of families of type-1 fuzzy sets, which may include the projections of interval-valued fuzzy sets.

The algorithmic presentation for fuzzy consensus analysis via CDM-strategy is graphically represented in the flowchart of Figure 1, detailing the corresponding sequences of steps and identifying both methodological proposals: FS-FCM and IVFS-FCM, including their selection.

This strategy consolidates the fuzzy consensus approach for decision making in the management of hybrid memories, making it possible to apply it to multiple attributes and experts.

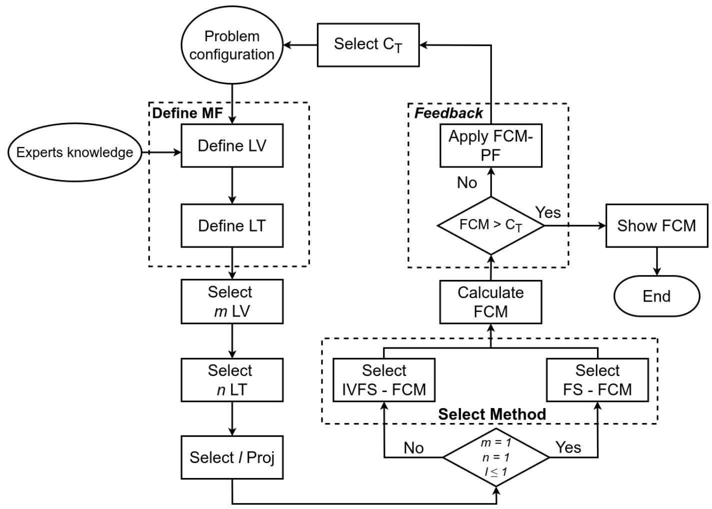

Fig. 1. CDM-strategy flowchart.

With regard to fuzzy consensus, the strategy is comprehensive as it provides a significant set of aggregation functions, penalty functions and restricted equivalence relations, which can be defined by the composition of implication functions and quasi-overlap functions.

Step 1 Selection of membership-functions and their corresponding fuzzy sets.

1.1 Normalization of input/output data.

1.2 Definition of the universe χ_k, where k indicates the number of elements in χ which are associated to a membership degree based on the expert opinions.

1.3 Definition of input/output fuzzy sets:

- Election of linguist variables (AR, RF, WF and R).
- Election of linguist terms (Low, Medium or High).
- Definition of related membership-functions $\mu_X(u)$.

Step 2 Modeling FCM-strategy: introducing the parameter description.

2.1 Election of parameter $m \in \{1, 2, 3\}$ of linguist variables.

2.2 Election of parameter $n \in \{1, 2, 3\}$ of linguist terms.

2.3 Election of parameter $l \in \{0, 1, 2\}$ of projection-functions.
$l = 0$ for a type-1 fuzzy set;
$l = 1$ for the lower bound of an IVFS;
$l = 2$ for the upper bound of an IVFS.

Step 3 Modeling FCM-strategy: providing the method selection

3.1 If $m = 1$ and $n = 1$ and $l \leq 1$ then it will be considered the first methodologies (FS-FCM), and then go to Step 4; otherwise,

3.2 Selected method considers IVFS-FCM methodology and then go to Step 5;

Step 4 Selection of $\mathcal{L}_{[0,1]}$-FCM method: Describing the sub-steps

4.1 Define λ for the convex sum operator.

4.2 Define ϕ-automorphism for the conjugate operator.

4.3 Define the average operator (AM, exp) based on the following conditions:

4.3.1 Define the overlap function and the fuzzy implication function to select the $\mathcal{L}_{[0,1]}$-REF operator;

4.3.2 Define the fusion vector for the selection of the penalty function.

4.4 Go to Step 6.

Step 5 Selection of the $\mathcal{L}_{\mathcal{F}_\chi}$-FCM method, based on the following sub-steps:

5.1 Define λ for the convex sum operator.

5.2 Define ϕ-automorphism for the conjugate operator.

5.3 Define aggregation operator and go to Step 4.

Step 6 Calculation of the Consensus C, based on the selected method.

Step 7 Testing Consensus C **- Applying a threshold** C_T

7.1 When $C \leq C_T$ then go to Step 8, indicating the minimum criteria for the target consensual analysis; otherwise,

7.2 Apply the penalty functions (FCM-PF methods) and go to Step 1.

Step 8 Present the final result by concluding the algorithm for the consensual analysis.

Thus, the above algorithmic description for the strategic CDM was conceived by integrating the theoretical results from the previous sections.

VII. Case study: *f-HybridMem* consensual analysis using FS-FCM method based on penalty functions

Migration policy verifies each page priority to be switched between memory modules. It considers a Rule Base acting on three steps: Fuzzification, Inference, and Defuzzification. A fuzzy-based system to support the uncertainty in data management for hybrid memory architectures, called *f-HybridMem*, was presented in [5] to recommend a correct selection between memory modules by improving the data management, in a page level organization [20], [30]–[32]. *f-HybridMem* outputs each page priority based on the Juzzy module [33].

In the *f-HybridMem* approach, it was defined four linguistic variables: where RF (reading frequency), WF (writing frequency), and AR (access recency) are the input values and R (recommendation) is the output.

For the current case study, we use penalty functions to obtain consensual analysis applied on fuzzy sets considering RF, WF, AR and R linguistic variables (LV). The linguistic terms (LT) "high" (H), "medium" (M) and "low" (L) were associated to each of the four linguistic variables and, therefore, the consensus measures were calculated based on the *FS-FCM* method considering Eqs. (9) and (10).

Figure 2 depicts the results for the consensus measure using the AM aggregation function and the cases where the penalty functions were considered.

Now we discuss how penalty functions influence the consensus measure for the input and output of the LV. First, we see that for the AR input the consensus is greater for L and M linguistic terms using AM aggregation function without a penalty function. For H LT, however, the consensus is greater when penalty functions are taken into account.

For the input RF and WF, the consensus is greater for M using the AM aggregation function. For the others LT, H and L, the consensus is greater with the $|x-y|$ penalty function, where x is constituted by the input values and y is obtained by the median of the non-null input values.

The output R presents the same behavior as the input RF and WF, where the consensus is greater for M. However, for H and L linguistic terms, the consensus is smaller with $(x-y)^2$ penalty function, where x denotes the input values and y corresponds to the arithmetic mean of the input values.

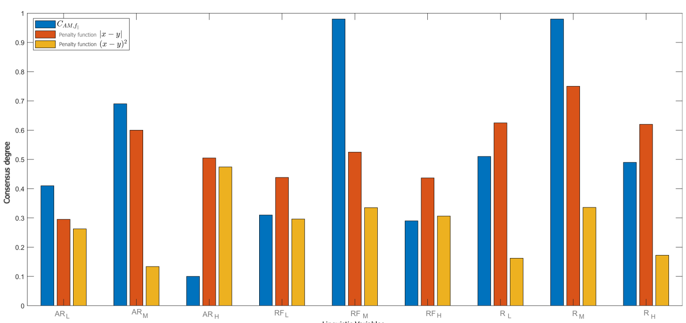

Fig. 2. Penalty function influence on Consensus Measures for AM aggregation function.

Table I summarizes the results depicted in Figure 2. Table II shows the consensus measures over input-output variables to interval-valued membership functions, applying AM and exp aggregation functions. In addition, the consensus measures for C_{SK} and C_{Tastle} operators are presented.

TABLE I
CONSENSUS MEASURES OVER INPUT-OUTPUT VARIABLES TO MODEL FUZZY ($REF_{f_{||}}(x,y)$) AND PENALTY FUNCTIONS.

X	C_{AM}	$C_{Penalty}$	
		$\|x-y\|$	$(x-y)^2$
AR_L	0.4100	0.2950	0.2626
AR_M	0.6900	0.6000	0.1339
AR_H	0.1000	0.5050	0.4743
RF_L	0.3100	0.4383	0.2962
RF_M	0.9800	0.5250	0.3351
RF_H	0.2900	0.4367	0.3063
R_L	0.5100	0.6250	0.1623
R_M	0.9800	0.7500	0.3359
R_H	0.4900	0.6200	0.1724

TABLE II
CONSENSUS MEASURES OVER INPUT-OUTPUT VARIABLES TO INTERVAL-VALUED MEMBERSHIP FUNCTIONS.

X	$[\underline{X}, \overline{X}]$	C_{SK}	C_{Tastle}	$C_{AM,f_{\|\|}}$	$C_{\exp,f_{\|\|}}$
AR_L	$\underline{L}_{In}$	0.6047	0.4734	0.5500	0.6832
	$\overline{L}_{In}$	0.2419	0.1918	0.3300	0.4127
AR_M	$\underline{M}_{In}$	0.8268	0.7135	0.7900	0.8185
	$\overline{M}_{In}$	0.4596	0.3710	0.6220	0.6817
AR_H	$\underline{H}_{In}$	0.6686	0.5181	0.4360	0.4434
	$\overline{H}_{In}$	0.1411	0.1171	0.1200	0.1543
RF_L	$\underline{L}_{In}$	0.7165	0.5770	0.5860	0.6153
	$\overline{L}_{In}$	0.2358	0.1933	0.2100	0.2549
RF_M	$\underline{M}_{In}$	0.7116	0.5683	0.9880	0.9882
	$\overline{M}_{In}$	0.2680	0.2238	0.9800	0.9803
RF_H	$\underline{H}_{In}$	0.7133	0.5727	0.5740	0.6019
	$\overline{H}_{In}$	0.2286	0.1880	0.1980	0.2421
R_L	$\underline{L}_{Out}$	0.8056	0.6815	0.7060	0.7327
	$\overline{L}_{Out}$	0.4293	0.3502	0.3460	0.3910
R_M	$\underline{M}_{Out}$	0.7838	0.6489	0.9880	0.9880
	$\overline{M}_{Out}$	0.7267	0.6034	0.9840	0.9840
R_H	$\underline{H}_{Out}$	0.7984	0.6726	0.6940	0.7209
	$\overline{H}_{Out}$	0.4173	0.3410	0.3340	0.3798

CONCLUSIONS

In this paper, the *f-HybridMem* method is used to recommend page migration based on a consensual analysis using penalty functions and fuzzy consensus measures. The fuzzy consensus analysis was performed over data regarding the linguist variables related to access recency, reading and writing frequencies, and recommendation attributes.

Future works will apply our results to analyze the correlation and entropy in FL and IVFL for the different linguistic variables of a system for decision making problems.

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