Generalization of QL-operators based on general overlap and general grouping functions

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***Abstract*—Firstly, this work discusses the main conditions guarantying that general overlap (grouping) functions can be obtained from n-dimensional overlap (grouping) functions. Focusing on QL-implications, which are usually generated by strong negations together with t-norms and t-conorms, we consider a non-restrictive construction, by relaxing not only the associativity and the corresponding neutral elements (NE) but also the reverse construction of other properties. Thus, the main properties of the QL-implication class are studied, considering a tuple (G,N,O) generated from grouping and overlap functions together with the greatest fuzzy negation. In addition, in order to provide more flexibility, we define a subclass of QL-implications generated from general overlap and general grouping functions. Some examples are introduced, illustrating the constructive methods to generate such operators.**

***Index Terms*—General Overlap Function, General Grouping Function, QL-implication, Fuzzy Implication, Aggregation function**

I. Introduction

In the scope of fuzzy logic, implication functions are a vital element of its constitution. Defining and using implication functions to represent different scenarios in fuzzy inference systems is still an open challenge, exploring different classes of implications [1]–[6].

Firstly, this paper provides the essential conditions guarantying that general overlap/grouping functions can be constructed from the equivalent n-dimensional operators. This study also includes a study on extensions of fuzzy implication operators obtained via general overlap and grouping functions, addressing mainly the QL-operators and QL-implications [3]. The main proposal of this article is to explore constructive methods to generate implications through the concepts of general overlap and general grouping functions, relaxing some properties. The methods studied provide more flexibility to the underlying fuzzy inference system structure.

This paper is organized as follows. Section 2 brings the main theoretical concepts. In Section 3, we discuss how to obtain general overlap functions from n-dimensional overlap functions, and likewise, in Sect. 4, we present general grouping functions from n-dimensional grouping functions. Sect. 5 addresses the QL-operations and their properties. Moreover, in Sect. 6 we discuss the conditions for generating QL-implications from a tuple $(\mathcal{G}, N, \mathcal{O})$. We conclude in Sect. 7 with the final remarks and future works.

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II. Preliminaries

A. Fuzzy Negation

Definition 2.1: A function $N\colon [0,1] \to [0,1]$ is fuzzy negation if it: (N1) is decreasing; (N2) satisfies $N(0) = 1$ and $N(1) = 0$ (boundary conditions).

A fuzzy negation N is said to be strong if it is involutive (N3) $N(N(x)) = x, \forall x \in [0,1]$, as the strict negation $N_S(x) = 1 - x$. However, a counterexample of the involution is the greatest fuzzy negation:

$$N_\top(x) = \begin{cases} 0, \text{if } x = 1; \\ 1, \text{if } x \in [0,1[. \end{cases} \quad (1)$$

Other properties for a fuzzy negation N, $\forall x \in [0,1]$:
(N4) N is frontier: $N(x) \in \{0,1\}$ iff $x = 0$ or $x = 1$.
(N5) N is non-filling: $N(x) = 1$ iff $x = 0$.
(N6) N is crisp: $N(x) \in \{0,1\}$.

B. Aggregation Functions

An aggregation operator is characterized by merging a tuple of objects in a given set into a single object of the same set. In the context of Fuzzy Logic, the n values in each tuple are real numbers in $[0,1]$. Frequently, conditions must be imposed for an aggregation function, which must be compatible with the applications in the field.

Definition 2.2: [7] An aggregation operator is a function $A : [0,1]^n \to [0,1]$, that:
(A1) satisfies boundary conditions $A(0,\ldots,0) = 0$ and $A(1,\ldots,1) = 1$, and
(A2) $A(x_1,\ldots,x_n) \le A(x'_1,\ldots,x'_n)$ if $x_i \le x'_i$, $\forall x_i, x'_i \in [0,1], i \in \mathbb{N}_n$, is monotone increasing in all arguments.

C. General Overlap Functions

Definition 2.3: [8] A function $\mathcal{O}\colon [0,1]^n \to [0,1]$ is called a general overlap function (GOF, for short) if, $\forall \vec{x} = (x_1,\ldots,x_n) \in [0,1]^n$,
($\mathcal{O}1$) $\mathcal{O}$ is commutative;
($\mathcal{O}2$) If $\prod_{i=1}^n x_i = 0$ then $\mathcal{O}(\vec{x}) = 0$;
($\mathcal{O}3$) If $\prod_{i=1}^n x_i = 1$ then $\mathcal{O}(\vec{x}) = 1$;
($\mathcal{O}4$) $\mathcal{O}$ is increasing;
($\mathcal{O}5$) $\mathcal{O}$ is continuous.

Definition 2.4: [9] A bivariate function $O : [0,1]^2 \to [0,1]$ is an overlap function if it satisfies:
($O1$) $O(x,y) = O(y,x)$ $\forall x,y \in [0,1]$;
($O2$) $O(x,y) = 0$ if and only if $x = 0$ or $y = 0$;
($O3$) $O(x,y) = 1$ if and only if $x = y = 1$;
($O4$) O is increasing;
($O5$) O is continuous.

One can observe that any overlap function is a bivariate GOF, but the converse does not hold. And, an overlap function satisfies 1-section deflation property if:
(O6) $O(x,1) \le x, \forall x \in [0,1]$.

Definition 2.5: [10] An n-dimensional overlap function $\mathcal{O}_n\colon [0,1]^n \to [0,1]$ satisfies, in addition to the properties ($\mathcal{O}1$), ($\mathcal{O}4$) and ($\mathcal{O}5$), the following conditions:
($\mathcal{O}_n2$) $O_n(\vec{x}) = 0$ if and only if $\Pi_{i=1}^n x_i = 0$;
($\mathcal{O}_n3$) $O_n(\vec{x}) = 1$ if and only if $\Pi_{i=1}^n x_i = 1$.

Remark 2.1: In [2], by considering a bivariate overlap function $O\colon [0,1]^2 \to [0,1]$ and $a \in (0,1)$, the mapping $\mathcal{O}_a : [0,1]^2 \to [0,1]$ given, for all $x,y \in [0,1]$, by

$$\mathcal{O}_a(x,y) = \frac{\max(0, O(x,y) - O(\max(x,y),a))}{1 - O(\max(x,y),a)}, \quad (2)$$

is a bivariate GOF which is not an overlap function since it does not verify ($O2$), but verifies ($O3$). In particular, $\mathcal{O}_{\frac{1}{2}}\colon [0,1]^2 \to [0,1]$, which is given as

$$\mathcal{O}_{\frac{1}{2}}(x,y) = \frac{\max(0, O(x,y) - O(\max(x,y),\frac{1}{2}))}{1 - O(\max(x,y),\frac{1}{2})}. \quad (3)$$

Table I reports a listing of GOF $\mathcal{O}\colon [0,1]^n \to [0,1]$ in the first column, and their subclasses within the overlap operators. The second column O indicates the bivariate overlap functions and, in the third, $\mathcal{O}_n$ indicates n-dimensional overlap functions.

Remark 2.2: See in Table I, the general overlap functions:
(i) The function $\mathcal{O}_L\colon [0,1]^n \to [0,1]$ is not an n-dimensional overlap function since it verifies ($\mathcal{O}2$) but does not verify ($\mathcal{O}_n2$). In particular, the function $\mathcal{O}_L\colon [0,1]^2 \to [0,1]$ is given as $\mathcal{O}_L(x,y) = \max(x+y-1, 0)$.
(ii) The function $\mathcal{O}_U\colon [0,1]^n \to [0,1]$ is not an n-dimensional overlap function since it verifies ($\mathcal{O}3$) but does not verify ($\mathcal{O}_n3$). In particular, the function $\mathcal{O}_U\colon [0,1]^2 \to [0,1]$ is given as $\mathcal{O}_U(x,y) = 2xy$ if $xy \le \frac{1}{2}$ and $\mathcal{O}_U(x,y) = 1$, otherwise.
(iii) The function $\mathcal{O}_G\colon [0,1]^n \to [0,1]$ is not an n-dimensional overlap function since it verifies ($\mathcal{O}3$) but does not verify ($\mathcal{O}_n3$). In particular, function $\mathcal{O}_G\colon [0,1]^2 \to [0,1]$ is given as $\mathcal{O}_G(x,y) = 2\max(x+y-1, 0)$ if $x+y \le \frac{3}{2}$ and $\mathcal{O}_G(x,y) = 1$, otherwise.

D. General Grouping Functions

Definition 2.6: [11] A function $\mathcal{G}\colon [0,1]^n \to [0,1]$ is called a general grouping function (GGF, for short) if, $\forall \vec{x} = (x_1,\ldots,x_n) \in [0,1]^n$,
($\mathcal{G}1$) $\mathcal{G}$ is commutative;
($\mathcal{G}2$) If $\sum_{i=1}^n x_i = 0$ then $\mathcal{G}(\vec{x}) = 0$;
($\mathcal{G}3$) If $\exists i \in \mathbb{N}_n$ such that $x_i = 1$ then $\mathcal{G}(\vec{x}) = 1$;
($\mathcal{G}4$) $\mathcal{G}$ is increasing;
($\mathcal{G}5$) $\mathcal{G}$ is continuous.

Definition 2.7: [12] A bivariate function $G\colon [0,1]^2 \to [0,1]$ is a grouping function if it satisfies the following conditions, for all $x,y,z \in [0,1]$:
($G1$) $G(x,y) = G(y,x)$;
($G2$) $G(x,y) = 0$ if and only if $x = y = 0$;
($G3$) $G(x,y) = 1$ if and only if $x = 1$ or $y = 1$;
($G4$) G is increasing;
($G5$) G is continuous;

Note that any grouping function is a bivariate GGF, but the converse does not hold.

Definition 2.8: [6] Consider a grouping function $G\colon [0,1]^2 \to [0,1]$ and a fuzzy negation $N\colon [0,1] \to [0,1]$. The pair (G,N) satisfies the excluded middle law if:
(LEM) $G(N(x),x) = 1$, $\forall x \in [0,1]$.

The notion of n-dimensional grouping function was also provided by [10], being defined as follows.

Definition 2.9: An n-dimensional grouping function $\mathcal{G}_n\colon [0,1]^n \to [0,1]$ satisfies, in addition to the properties ($\mathcal{G}1$), ($\mathcal{G}4$) and ($\mathcal{G}5$), the following conditions:

($\mathcal{G}_n2$) $\mathcal{G}_n(\vec{x}) = 0$ if and only if $x_i = 0, \forall i \in \mathbb{N}_n$;
($\mathcal{G}_n3$) $\mathcal{G}_n(\vec{x}) = 1$ if and only if $\exists i \in \mathbb{N}_n$ with $x_i = 1$.

Proposition 2.1: [11, Prop. 1] Let $\mathcal{G}\colon [0,1]^n \to [0,1]$ be an n-dimensional grouping function, then $\mathcal{G}$ is also a GGF.

TABLE I
CLASSIFYING GENERAL OVERLAP FUNCTIONS

General Overlap Function $\mathcal{O}$	O	O_n
$\mathcal{O}_{mM}(\vec{x}) = \min_{i=1}^n x_i \cdot \max_{i=1}^n x_i^p,\, p>0$	✓	✓
$\mathcal{O}_{EP}(\vec{x}) = \frac{\prod_{i=1}^n x_i}{1+\prod_{i=1}^n (1-x_i)}$	✓	✓
$\mathcal{O}_S(\vec{x}) = \sin(\frac{\pi}{2}(\prod_{i=1}^n x_i)^p), p>0$	✓	✓
$\mathcal{O}_M(\vec{x}) = \min_{i=1}^n x_i^p,\, p>0$	✓	✓
$\mathcal{O}_L(\vec{x}) = \max((\Sigma_{i=1}^n x_i) - (n-1), 0)$	✗	✗
$\mathcal{O}_U(\vec{x}) = \begin{cases} n\prod_{i=1}^n x_i, & \text{if } \prod_{i=1}^n x_i \leq \frac{1}{n}, \\ 1, & \text{otherwise.}\end{cases}$	✗	✗
$\mathcal{O}_G(\vec{x}) = \begin{cases} nO_L(\vec{x}), & \text{if } O_L(\vec{x}) \leq \frac{1}{n}, \\ 1, & \text{otherwise.}\end{cases}$	✗	✗

See Table II reporting the GGF (first column), also illustrating some grouping functions G (second column) and n-dimensional grouping functions $\mathcal{G}_n$ (third column).

Remark 2.3: Note that:

(i) $\mathcal{G}_B\colon [0,1]^n \to [0,1]$ is not an n-dimensional grouping function, since it verifies ($\mathcal{G}3$) but does not verify ($\mathcal{G}_n3$);

(ii) $\mathcal{G}_k\colon [0,1]^n \to [0,1]$ is not an n-dimensional grouping function, since it verifies ($\mathcal{G}2$) but does not verify ($\mathcal{G}_n2$).

(iii) The function $\mathcal{G}_{LK}\colon [0,1]^n \to [0,1]$ is not an n-dimensional grouping function since it verifies ($\mathcal{G}3$) but does not verify ($\mathcal{G}_n3$). In particular, the Łukasiewicz t-conorm $\mathcal{G}_{LK}\colon [0,1]^2 \to [0,1]$, given as

$$\mathcal{G}_{LK}(x,y) = \min(x+y, 1) \tag{4}$$

is a GGF which is not a grouping function.

Remark 2.4: Note that if GOF and GGF have an NE denoted by $n_{\mathcal{O}}$ and $n_{\mathcal{G}}$, respectively, then $n_{\mathcal{O}}, n_{\mathcal{G}} \in]0,1[$. An illustration of a GOF with an $n_{\mathcal{O}} \in]0,1[$ is found in [2, Remark 5], and a GGF with an $n_{\mathcal{G}} \in]0,1[$ in [11, Remark 1].

TABLE II
CLASSIFYING GENERAL GROUPING FUNCTIONS

General Grouping Function $\mathcal{G}$	G	$\mathcal{G}_n$
$\mathcal{G}_{EP}(\vec{x}) = \frac{\sum_{i=1}^n x_i}{1+\prod_{i=1}^n x_i}$	✓	✓
$\mathcal{G}_O(\vec{x}) = \max_{i=1}^n x_i^p,\, p>0$	✓	✓
$\mathcal{G}_L(\vec{x}) = \left(1 - \prod_{i=1}^n (1-x_i)\right)\min(\sum_{i=1}^n x_i, 1)$	✓	✓
$\mathcal{G}_{ML}(\vec{x}) = \left(1 - \sqrt[n]{\prod_{i=1}^n (1-x_i)}\right)\min(\sum_{i=1}^n x_i, 1)$	✓	✓
$\mathcal{G}_{LK}(\vec{x}) = \min((\Sigma_{i=1}^n x_i), 1)$	✗	✗
$\mathcal{G}_B(\vec{x}) = \min(1, n - \sum_{i=1}^n (1-x_i)^2)$	✗	✗
$\mathcal{G}_k(\vec{x}) = \begin{cases} 0, & \text{if } \max_{i=1}^n x_i \leq k \\ \frac{1}{1-k}(\max_{i=1}^n x_i - k), & 0 \leq k \leq 1 \end{cases}$	✗	✗

E. Fuzzy Implication

Next, we study the properties of fuzzy implication functions, and also the class of QL-implications.

Definition 2.10: [13] A mapping $I\colon [0,1]^2 \to [0,1]$ is called a fuzzy implication function if, $\forall x,y,z \in [0,1]$, it holds:

(FPA) First place antitonicity: if $x \leq y$ then $I(y,z) \leq I(x,z)$;
(SPI) Second place isotonicity: $y \leq z \Rightarrow I(x,y) \leq I(x,z)$;
(BC1) Boundary condition 1: $I(0,0) = 1$;
(BC2) Boundary condition 2: $I(1,1) = 1$;
(BC3) Boundary condition 3: $I(1,0) = 0$.

By Def. 2.10, a fuzzy implication function $I\colon [0,1]^2 \to [0,1]$ may satisfy other properties, $\forall x,y,z \in [0,1]$:

(NP) Left neutrality property: $I(1,y) = y$.
(EP) Exchange principle: $I(x, I(y,z)) = I(y, I(x,z))$;
(IP) Identity principle: $I(x,x) = 1$;
(OP) Ordering property: $I(x,y) = 1 \Leftrightarrow x \leq y$.
(LOP) Left ordering property: $x \leq y \Rightarrow I(x,y) = 1$.
(ROP) Right ordering property: $I(x,y) = 1 \Rightarrow x \leq y$.
(LF) Lowest falsity property: $I(x,y) = 0 \Leftrightarrow x = 1$ and $y = 0$.
(LT) Lowest truth property: $I(x,y) = 1 \Leftrightarrow x = 0$ or $y = 1$.
(CP) Contrapositivity property for a fuzzy negation N: $I(x,y) = I(N(y), N(x))$.
(LCP) Left contrapositivity property for a fuzzy negation N: $I(N(x),y) = I(N(y),x)$.
(RCP) Right contrapositivity property for a fuzzy negation N: $I(x, N(y)) = I(y, N(x))$.
(LBC) Left boundary condition: $I(0,y) = 1$.
(RBC) Right boundary condition: $I(x,1) = 1$.

Remark 2.5: Note that (LBC) and (RBC) are always valid for any fuzzy implication function since they are obtained directly from the boundary conditions seen in Definition 2.10.

Definition 2.11: [1] A function $I_{S,N,T}\colon [0,1]^2 \to [0,1]$ is a QL-implication if there exists a t-conorm $S\colon [0,1]^2 \to [0,1]$, a strong fuzzy negation $N\colon [0,1] \to [0,1]$ and a t-norm $T\colon [0,1]^2 \to [0,1]$, such that

$$I_{S,N,T}(x,y) = S(N(x), T(x,y)), \forall x,y \in [0,1]. \tag{5}$$

III. Generating general overlap operators from n-dimensional overlap operators

Next, as the first contribution, we introduce a methodology to generate GOF from n-dimensional overlap functions.

Definition 3.1: Take $0 \leq a < b \leq 1$ and let $A\colon [0,1]^n \to [0,1]$ be an aggregation function. Then, $A_a^b, A_a, A^b\colon [0,1]^n \to [0,1]$ are defined, $\forall \vec{x} = (x_1, \ldots, x_n) \in [0,1]^n$, as follows:

$$A_a^b(\vec{x}) = \begin{cases} 0, & \text{if } A(\vec{x}) \leq a \\ 1, & \text{if } A(\vec{x}) \geq b \\ \frac{A(\vec{x})-a}{b-a}, & \text{if } a < A(\vec{x}) < b. \end{cases} \tag{6}$$

For $b = 1$ we have $A_a = A_a^1$ and for $a = 0$, $A^b = A_0^b$.

Now, we show the conditions under which GOF can be obtained from n-dimensional overlap functions.

Proposition 3.1: Take $0 \leq a < b \leq 1$ and let $\mathcal{O}_n\colon [0,1]^n \to [0,1]$ be an n-dimensional overlap function. Then,

(a) $\mathcal{O}_a^b$ is a GOF that satisfies neither ($\mathcal{O}_n2$) nor ($\mathcal{O}_n3$).

(b) $\mathcal{O}_a$ is a GOF not satisfying ($\mathcal{O}_n2$) but holding ($\mathcal{O}_n3$).
(c) $\mathcal{O}^b$ is a GOF holding ($\mathcal{O}_n2$) but not satisfying ($\mathcal{O}_n3$).

Proof: (a.1) First, let us prove $\mathcal{O}_a^b$ is a GOF. It is straightforward that $\mathcal{O}_a^b$ satisfies ($\mathcal{O}1$), ($\mathcal{O}4$), and ($\mathcal{O}5$). For ($\mathcal{O}2$), note that if $x_i = 0$ for some $i \in \mathbb{N}_n$, $\mathcal{O}_n(x_1,\ldots,x_{i-1},0,x_{i+1},\ldots,x_n) = 0 < a$. Then, $\mathcal{O}_a^b(x_1,\ldots,0,\ldots,x_n) = 0$. And for ($\mathcal{O}3$), if $x_i = 1, \forall i \in \mathbb{N}_n$, $\mathcal{O}_n(1,\ldots,1) = 1 > b$. So, $\mathcal{O}_a^b(1,\ldots,1) = 1$.
(a.2) Now, we verify that $\mathcal{O}_a^b$ does not satisfy ($\mathcal{O}_n2$). Since $\mathcal{O}_n$ satisfies ($\mathcal{O}5$) and given a, there exists $\vec{x} \in [0,1]^n$ such that $\mathcal{O}_n(\vec{x}) = a > 0$. So, $\mathcal{O}_a^b(\vec{x}) = \frac{\mathcal{O}_n(\vec{x})-a}{b-a} = 0$, but $x_i \neq 0\ \forall i \in \mathbb{N}_n$, since $\mathcal{O}_n$ satisfies (O_n2).
(a.3) At last, $\mathcal{O}_a^b$ does not satisfy ($\mathcal{O}_n3$). Since $\mathcal{O}_n$ satisfies ($\mathcal{O}_5$) and given b, there exists $\vec{x} \in [0,1]^n$ such that $\mathcal{O}_n(\vec{x}) = b < 1$. So, $\mathcal{O}_a^b(\vec{x}) = \frac{\mathcal{O}_n(\vec{x})-a}{b-a} = 1$, but, for some i, there must be $x_i \neq 1$ since $\mathcal{O}_n$ satisfies ($\mathcal{O}_n3$).
(b.1) Analogously, we start proving $\mathcal{O}_a = \mathcal{O}_a^1$, given by

$$\mathcal{O}_a^1(\vec{x}) = \begin{cases} 0, & \text{if } \mathcal{O}_n(\vec{x}) < a \\ \frac{\mathcal{O}_n(\vec{x})-a}{1-a}, & \text{if } \mathcal{O}_n(\vec{x}) \geq a \end{cases}$$

is a GOF. It is straightforward that $\mathcal{O}_a$ satisfies ($\mathcal{O}1$), ($\mathcal{O}4$), and ($\mathcal{O}5$). $\mathcal{O}_a$ satisfies ($\mathcal{O}2$), similarly obtained using item (a.1). $\mathcal{O}_a$ satisfies ($\mathcal{O}3$) because if $x_i = 1\ \forall i \in \mathbb{N}_n$, then $\mathcal{O}_n(1,\ldots,1) = 1$. Therefore, $\mathcal{O}_a(\vec{x}) = \frac{\mathcal{O}_n(\vec{x})-a}{1-a} = 1$.
(b.2) $\mathcal{O}_a$ does not satisfy ($\mathcal{O}_n2$), similar to item (a.2).
(b.3) $\mathcal{O}_a$ satisfies ($\mathcal{O}_n3$). If $\mathcal{O}_a(\vec{x}) = 1$, then $\mathcal{O}_a(\vec{x}) = \frac{\mathcal{O}_n(\vec{x})-a}{1-a} = 1$, so, $\mathcal{O}_n(\vec{x}) = 1$ if and only if $x_1 = \ldots = x_n = 1$.
(c.1) Observe that $\mathcal{O}^b = \mathcal{O}_0^b$, given by

$$\mathcal{O}_0^b(\vec{x}) = \begin{cases} \frac{\mathcal{O}_n(\vec{x})}{b}, & \text{if } 0 \leq \mathcal{O}_n(\vec{x}) \leq b \\ 1, & \text{if } \mathcal{O}_n(\vec{x}) > b \end{cases}$$

is a GOF. It is straightforward that $\mathcal{O}^b$ satisfies ($\mathcal{O}1$), ($\mathcal{O}4$), and ($\mathcal{O}5$). $\mathcal{O}^b$ also satisfies ($\mathcal{O}2$), similar to item (a.1). $\mathcal{O}^b$ satisfies ($\mathcal{O}3$), because if $x_i = 1\ \forall i \in \mathbb{N}_n$, then $\mathcal{O}_n(1,\ldots,1) = 1 > b$. So, $\mathcal{O}^b(\vec{x}) = 1$.
(c.2) $\mathcal{O}^b$ satisfies ($\mathcal{O}_n2$). If $\mathcal{O}^b(\vec{x}) = 0$ then $\mathcal{O}^b(\vec{x}) = \frac{\mathcal{O}_n(\vec{x})}{b} = 0$, so, $\mathcal{O}_n(\vec{x}) = 0$ if and only if $x_i = 0$ for some $i \in \mathbb{N}_n$.
(c.3) $\mathcal{O}^b$ does not satisfy ($\mathcal{O}_n3$). Since $\mathcal{O}_n$ satisfies ($\mathcal{O}_5$), so there exists $\vec{x} \in [0,1]^n$ such that $\mathcal{O}_n(\vec{x}) = b < 1$. So, $\mathcal{O}^b(\vec{x}) = \frac{b}{b} = 1$, but $\vec{x} \neq (1,\ldots,1)$ since $\mathcal{O}_n$ satisfies ($\mathcal{O}_n3$). □

Example 3.1: Consider $a = \frac{1}{4}$, $b = \frac{3}{4}$ and let $O_m\colon [0,1]^2 \to [0,1]$ given as $O_m(x,y) = \min(\sqrt{x},\sqrt{y})$ be an overlap function. Based on Prop. 3.1, see GOF generated from $\mathcal{O}_m$:

$$(\mathcal{O}_m)^{\frac{3}{4}}(x,y) = \begin{cases} \frac{4}{3}\sqrt{x}, \text{ if } x \leq y \text{ and } x \leq \frac{9}{16} \\ \frac{4}{3}\sqrt{y}, \text{ if } y < x \text{ and } y < \frac{9}{16} \\ 1, \text{ if } x > \frac{9}{16} \text{ and } y > \frac{9}{16}. \end{cases}$$

$$(\mathcal{O}_m)^{\frac{3}{4}}_{\frac{1}{4}}(x,y) = \begin{cases} 0, \text{ if } \min(x,y) \leq \frac{1}{16} \\ 1, \text{ if } \min(x,y) \geq \frac{9}{16} \\ 2(\sqrt{x} - \frac{1}{4}), \text{ if } x \leq y \text{ and } \frac{1}{16} < x < \frac{9}{16} \\ 2(\sqrt{y} - \frac{1}{4}), \text{ if } y \leq x \text{ and } \frac{1}{16} < y < \frac{9}{16}. \end{cases}$$

$$(\mathcal{O}_m)_{\frac{1}{4}}(x,y) = \begin{cases} 0, \text{ if } \min(x,y) < \frac{1}{16} \\ \frac{4}{3}(\sqrt{x} - \frac{1}{4}), \text{ if } \frac{1}{16} \leq x \leq y \\ \frac{4}{3}(\sqrt{y} - \frac{1}{4}), \text{ if } \frac{1}{16} \leq y \leq x. \end{cases}$$

Example 3.2: Take $a = \frac{1}{2}$, $b = \frac{\sqrt{3}}{2}$ and consider $O_S\colon [0,1]^2 \to [0,1]$ defined by $O_S(x,y) = \sin(\frac{\pi}{2}xy)$ as an overlap function. By Prop. 4.1, see GOF given as:

$$(\mathcal{O}_S)^{\frac{\sqrt{3}}{2}}(x,y) = \begin{cases} \frac{2\sqrt{3}}{3}\sin(\frac{\pi}{2}xy), \text{ if } 0 \leq xy \leq \frac{2}{3} \\ 1, \text{ otherwise.} \end{cases}$$

$$(\mathcal{O}_S)^{\frac{\sqrt{3}}{2}}_{\frac{1}{2}}(x,y) = \begin{cases} 0, \text{ if } 0 \leq xy \leq \frac{1}{3} \\ 1, \text{ if } \frac{2}{3} \leq xy \leq 1 \\ \frac{2\sin(\frac{\pi}{2}xy)-1}{\sqrt{3}-1}, \text{ otherwise.} \end{cases}$$

$$(\mathcal{O}_S)_{\frac{1}{2}}(x,y) = \begin{cases} 0, \text{ if } 0 \leq xy \leq \frac{1}{3} \\ 2\sin(\frac{\pi}{2}xy) - \frac{1}{2}, \text{ otherwise.} \end{cases}$$

IV. Generating general grouping operators from n-dimensional grouping operators

Similarly, next proposition shows under which conditions one can obtain GGF from n-dimensional grouping functions.

Proposition 4.1: Take $0 \leq a < b \leq 1$ and let $\mathcal{G}_n\colon [0,1]^n \to [0,1]$ be an n-dimensional grouping function. Then,
(a) $\mathcal{G}_a^b$ is a GGF that satisfies neither ($\mathcal{G}_n2$) nor ($\mathcal{G}_n3$).
(b) $\mathcal{G}_a$ is a GGF not satisfying ($\mathcal{G}_n2$) but holding ($\mathcal{G}_n3$).
(c) $\mathcal{G}^b$ is a GGF holding ($\mathcal{G}_n2$) but not satisfying ($\mathcal{G}_n3$).

Proof: The proof is similar to the one done in Prop. 3.1 □

Example 4.1: Let $a = \frac{1}{4}$, $b = \frac{3}{4}$ and $G : [0,1]^2 \to [0,1]$, $G(x,y) = 1 - \min(\sqrt{1-x}, \sqrt{1-y})$ be a grouping function. By Prop. 4.1, see the related GGF:

$$(\mathcal{G})^{\frac{3}{4}}(x,y) = \begin{cases} 0, \text{ if } x = y = 0 \\ 1, \text{ if } x,y \in [\frac{15}{16}, 1] \\ \frac{4}{3}(1 - \min(\sqrt{1-x}, \sqrt{1-y}), \text{otherwise.} \end{cases}$$

$$(\mathcal{G})^{\frac{3}{4}}_{\frac{1}{4}}(x,y) = \begin{cases} 0, \text{ if } x,y \in [0, \frac{7}{16}] \\ 1, \text{ if } x,y \in [\frac{15}{16}, 1] \\ \frac{3}{2} - 2\sqrt{1-x}, \text{ if } x \geq y \text{ and } \frac{7}{16} < x < \frac{15}{16} \\ \frac{3}{2} - 2\sqrt{1-y}, \text{ if } y \geq x \text{ and } \frac{7}{16} < y < \frac{15}{16}. \end{cases}$$

$$(\mathcal{G})_{\frac{1}{4}}(x,y) = \begin{cases} 0, \text{ if } x,y \in [0, \frac{7}{16}] \\ 1, \text{ if } x = 1 \text{ or } y = 1 \\ 1 - \frac{4}{3}\sqrt{1-x}, \text{ if } x \geq y \text{ and } \frac{7}{16} < x < 1 \\ 1 - \frac{4}{3}\sqrt{1-y}, \text{ if } y \geq x \text{ and } \frac{7}{16} < y < 1. \end{cases}$$

Example 4.2: Let $a = \frac{1}{4}$, $b = \frac{3}{4}$ and $G_B\colon [0,1]^2 \to [0,1]$, $G_B(x,y) = \min(1, 2 - (1-x)^2 - (1-y)^2)$ be a grouping function. By Prop. 4.1, G_B generates the following GGF:

$$(\mathcal{G}_B)^{\frac{3}{4}}(x,y) = \begin{cases} 0, \text{ if } x = y = 1 \\ 1, \text{ if } 0 \leq \min(\sqrt{1-x}, \sqrt{1-y}) \leq \frac{1}{4} \\ \frac{4}{3}(1 - \min(\sqrt{1-x}, \sqrt{1-y}), \text{otherwise.} \end{cases}$$

$$(\mathcal{G}_B)^{\frac{3}{4}}_{\frac{1}{4}}(x,y) = \begin{cases} 0, \text{ if } \frac{7}{4} \leq (1-x)^2 + (1-y)^2 \leq 2 \\ 1, \text{ if } 0 \leq (1-x)^2 + (1-y)^2 \leq \frac{5}{4} \\ \frac{7}{2} - 2((1-x)^2 + (1-y)^2), \text{ otherwise.} \end{cases}$$

$$(\mathcal{G}_B)_{\frac{1}{4}}(x,y) = \begin{cases} 0, \text{ if } \frac{7}{4} \leq (1-x)^2 + (1-y)^2 \leq 2 \\ 1, \text{ if } x = 1 \text{ or } y = 1 \\ \frac{4}{3}\min(\frac{2}{3}, \frac{5}{3} - (1-x)^2 - (1-y)^2), \text{otherwise.} \end{cases}$$

V. Generating QL-operators from GOF and GGF

In [14] (Lemma 4.1 and Theorem 4.1), a QL-implication was defined from overlap and grouping functions as follows.

Definition 5.1: A QL-operator $I_{G,N,O}\colon [0,1]^2 \to [0,1]$ derived from a tuple (G,N,O), is given as: $I_{G,N,O}(x,y) = G(N(x), O(x,y)), \forall x,y \in [0,1]$.

Proposition 5.1: Let $N\colon [0,1] \to [0,1]$ be the greatest fuzzy negation $N_\top$, the overlap function $O\colon [0,1]^2 \to [0,1]$ verifies (O6), and the grouping function $G\colon [0,1]^2 \to [0,1]$ verifies (LEM), then $I_{G,N,O}\colon [0,1]^2 \to [0,1]$ is a QL-implication.

In Table III, dual bivariate grouping/overlap functions is used to construct some $I_{G,N,O}$, illustrated in Table IV, considering the fuzzy negation $N_\top$ given in Eq. (1).

TABLE III
DUAL CONSTRUCTIONS OF GROUPING AND OVERLAP FUNCTIONS

Bivariate Grouping Functions	**Bivariate Overlap Functions**
$G_2^V(x,y)=\begin{cases}\frac{1}{2}(1-(1-x)^2(1-y)^2), \\ \text{if } x,y\in[0,\frac{1}{2}]; \\ \max\{x,y\}, \text{ otherwise.}\end{cases}$	$O_2^V(x,y)=\begin{cases}\frac{1}{2}\left(1+(2x-1)^2(2y-1)^2\right), \\ \text{if } x,y\in[0,\frac{1}{2}[\\ \min\{x,y\}, \text{otherwise.}\end{cases}$
$G_D(x,y)=\begin{cases}\frac{x+y-2xy}{2-(x+y)} \text{ if } x+y\neq 2; \\ 0, \text{ if } x+y=2.\end{cases}$	$O_D(x,y) = \begin{cases}\frac{2xy}{x+y} \text{ if } x+y\neq 0; \\ 0, \text{ otherwise.}\end{cases}$
$G_m(x,y)=1-\min\{\sqrt{1-x},\sqrt{1-y}\}$	$O_m(x,y)=\min\{\sqrt{x},\sqrt{y}\}$

TABLE IV
QL-IMPLICATIONS GENERATED BY TUPLES $(G, N_\top, O)$

$I_{G,N_T,O}$ - **QL-implications**
$I_{G_2^V,N_\top,O_2^V}(x,y)=\begin{cases}\frac{1}{2}\left(1+(2y-1)^2\right), \text{if } x=1 \text{ and } y\geq\frac{1}{2}; \\ \frac{1}{2}\left(1-(1-y)^2\right), \text{if } x=1 \text{ and } y<\frac{1}{2}; \\ 1, \text{ otherwise.}\end{cases}$
$I_{G_D,N_\top,O_D}(x,y)=\begin{cases}y, \text{if } x=1; \\ 1, \text{otherwise.}\end{cases}$
$I_{Gm,N_\top,Om}(x,y)=\begin{cases}1-\sqrt{1-\sqrt{y}}, \text{if } x=1; \\ 1, \text{ otherwise.}\end{cases}$

As an extension of the above results, we propose an operator constructed from tuples $(\mathcal{G}, N, \mathcal{O})$, as a generalization of the QL-operator given in [1] and inspired in the classical logical equivalence $p \to q \equiv \neg p \vee (p \wedge q)$, substituting $\vee, \wedge$, and $\neg$ by a GGF $\mathcal{G}$, a GOF $\mathcal{O}$, and a fuzzy negation N, respectively.

Definition 5.2: A function $I\colon [0,1]^2 \to [0,1]$ is a QL-operation derived from a tuple $(\mathcal{G}, N, \mathcal{O})$ if there exist a bivariate GGF $\mathcal{G}\colon [0,1]^2 \to [0,1]$, a fuzzy negation $N\colon [0,1] \to [0,1]$ and a bivariate GOF $\mathcal{O}\colon [0,1]^2 \to [0,1]$, such that

$$I(x,y) = \mathcal{G}(N(x), \mathcal{O}(x,y)), \tag{7}$$

for all $x,y \in [0,1]$. We denote such QL-operation by $I_{\mathcal{G},N,\mathcal{O}}$.

Theorem 5.1: Let $I_{\mathcal{G},\mathcal{O},N}$ be a QL-operation built from a tuple $(\mathcal{G}, N, \mathcal{O})$, when a GOF $\mathcal{O}$ and a GGF $\mathcal{G}$ have $n_\mathcal{O}$ and $n_\mathcal{G}$ as NE, respectively. One can state that:

(i) $I_{\mathcal{G},N,\mathcal{O}}$ satisfies (SPI), (BC1), (BC2) and (BC3), regarding the properties of Definition 2.10.
(ii) $I_{\mathcal{G},N,\mathcal{O}}$ satisfies (LBC).
(iii) If $n_\mathcal{O} = 1$ then $(\mathcal{G}, N)$ satisfies (LEM) if and only if $I_{\mathcal{G},N,\mathcal{O}}$ satisfies (RBC).
(iv) If $n_\mathcal{O} = 1$ and $n_\mathcal{G} = 0$, then $I_{\mathcal{G},N,\mathcal{O}}$ satisfies (NP).
(v) If $n_\mathcal{G} = 0$ then $N_{I_{\mathcal{G},N,\mathcal{O}}} = N$.
(vi) $N_{I_{\mathcal{G},N_\top,\mathcal{O}}} = N_\top$.
(vii) If $I_{\mathcal{G},N,\mathcal{O}}$ satisfies (LT) then N is non-filling.
(viii) If $\mathcal{G}$ and $\mathcal{O}$ satisfy $(\mathcal{G}_n 2)$ and $(\mathcal{O}_n 2)$, respectively, and N is a frontier negation then $I_{\mathcal{G},N,\mathcal{O}}$ satisfies (LF).
(ix) If $I_{\mathcal{G},N,\mathcal{O}}$ satisfies (ROP) then N is non-filling.
(x) If $N = N_\top$ then $I_{\mathcal{G},N,\mathcal{O}}$ satisfies (LOP).
(xi) If $\mathcal{G}$ and $\mathcal{O}$ satisfy $(\mathcal{G}_n 3)$ and $(\mathcal{O}_n 3)$, respectively, then:
  (a) If N is a frontier negation then $I_{\mathcal{G},N,\mathcal{O}}$ satisfies (LT);
  (b) If N is non-filling then $I_{\mathcal{G},N,\mathcal{O}}$ satisfies (ROP);
  (c) If $I_{\mathcal{G},N,\mathcal{O}}$ satisfies (LOP) then $N = N_\top$
  (d) $I_{\mathcal{G},N,\mathcal{O}}$ only satisfies (RCP) for $N_\top$ negation.
(xii) If $\mathcal{O}$ is idempotent and $(\mathcal{G}, N)$ satisfies (LEM), then $I_{\mathcal{G},N,\mathcal{O}}$ satisfies (IP).
(xiii) $I_{\mathcal{G},N_\top,\mathcal{O}}$ satisfies (EP).

Proof:

(i) It follows that:
(SPI) If $y \leq z$, since $\mathcal{G}$ and $\mathcal{O}$ are increasing: $I_{\mathcal{G},N,\mathcal{O}}(x,y) = \mathcal{G}(N(x), \mathcal{O}(x,y)) \leq \mathcal{G}(N(x), \mathcal{O}(x,z))$. Concluding, $I_{\mathcal{G},N,\mathcal{O}}(x,y) \leq I_{\mathcal{G},N,\mathcal{O}}(x,z)$.
(BC1) $I_{\mathcal{G},N,\mathcal{O}}(0,0) = \mathcal{G}(N(0), \mathcal{O}(0,0)) = \mathcal{G}(1,0) = 1$.
(BC2) $I_{\mathcal{G},N,\mathcal{O}}(1,1) = \mathcal{G}(N(1), \mathcal{O}(1,1)) = \mathcal{G}(0,1) = 1$.
(BC3) $I_{\mathcal{G},N,\mathcal{O}}(1,0) = \mathcal{G}(N(1), \mathcal{O}(1,0)) = \mathcal{G}(0,0) = 0$.

(ii) For all $y \in [0,1]$, it holds that $I_{\mathcal{G},N,\mathcal{O}}(0,y) = \mathcal{G}(N(0), \mathcal{O}(0,y)) = \mathcal{G}(1, \mathcal{O}(0,y)) = 1$.

(iii) Since $n_\mathcal{O} = 1$, we have, for all $x \in [0,1]$, $I_{\mathcal{G},N,\mathcal{O}}(x,1) = 1 \Leftrightarrow \mathcal{G}(N(x), \mathcal{O}(x,1)) = 1 \Leftrightarrow \mathcal{G}(N(x), x) = 1$.

(iv) Since $n_\mathcal{O} = 1$ and $n_\mathcal{G} = 0$, then one has that $I_{\mathcal{G},N,\mathcal{O}}(1,y) = \mathcal{G}(N(1), \mathcal{O}(1,y)) = \mathcal{G}(0,y) = y$.

(v) Since $n_\mathcal{G} = 0$, $N_{I_{\mathcal{G},N,\mathcal{O}}}(x) = I_{\mathcal{G},N,\mathcal{O}}(x,0) = \mathcal{G}(N(x), \mathcal{O}(x,0)) = \mathcal{G}(N(x), 0) = N(x)$.

(vi) $N_{I_{\mathcal{G},N_\top,\mathcal{O}}}(x) = \mathcal{G}(N_\top(x), \mathcal{O}(x,0)) = \mathcal{G}(N_\top(x), 0)$. Then, $N_{I_{\mathcal{G},N_\top,\mathcal{O}}}(x) = 0$ if $x = 1$ and $N_{I_{\mathcal{G},N_\top,\mathcal{O}}}(x) = 1$ if $x < 1$. Therefore, $N_{I_{\mathcal{G},N_\top,\mathcal{O}}} = N_\top$.

(vii) Suppose that N is not non-filling. Then, there exists $x \in]0,1[$ such that $N(x) = 1$. So, $I_{\mathcal{G},N,\mathcal{O}}(x,0) = \mathcal{G}(N(x), \mathcal{O}(x,0)) = \mathcal{G}(1, \mathcal{O}(x,0)) = 1$. But, this is a contradiction due to (LT), $I_{\mathcal{G},N,\mathcal{O}}(x,0) = 1$ iff $x = 0$. So, N is non-filling.

(viii) (LF): $(\Rightarrow)$ Take $I_{\mathcal{G},N,\mathcal{O}}(x,y) = 0$. Then, $I_{\mathcal{G},N,\mathcal{O}}(x,y) = \mathcal{G}(N(x), \mathcal{O}(x,y)) = 0$. So, since $\mathcal{G}$ satisfies (G2), $N(x) = 0$ and $\mathcal{O}(x,y) = 0$. Now, since N is frontier and $\mathcal{O}$ satisfies (O2), then $x = 1$ and $y = 0$. $(\Leftarrow)$ It follows from (BC3).

(ix) Suppose that N is not non-filling. Then, there exists $x \in]0,1[$ such that $N(x) = 1$. Consider $y = \frac{x}{2}$. In this case, $I_{\mathcal{G},N,\mathcal{O}}(x,y) = \mathcal{G}(N(x), \mathcal{O}(x,y)) = \mathcal{G}(1, \mathcal{O}(x,y)) = 1$, but $x > y$. So $I_{\mathcal{G},N,\mathcal{O}}$ does not satisfy (ROP). Therefore, if $I_{\mathcal{G},N,\mathcal{O}}$ satisfies (ROP) then N is non-filling.

(x) Take $N = N_\top$ and consider $x \leq y$. If $x = 1$ then $y = 1$, so $I_{\mathcal{G},N_\top,\mathcal{O}}(1,1) = \mathcal{G}(0, \mathcal{O}(1,1)) = 1$. Now, if $x < 1$, then $I_{\mathcal{G},N_\top,\mathcal{O}}(x,y) = \mathcal{G}(N_\top(x), \mathcal{O}(x,y)) = \mathcal{G}(1, \mathcal{O}(x,y)) = 1$.

(xi) Indeed,
(a) Consider $I_{\mathcal{G},N,\mathcal{O}}(x,y) = 1$. So, $I_{\mathcal{G},N,\mathcal{O}}(x,y) = \mathcal{G}(N(x), \mathcal{O}(x,y)) = 1$ and it follows that, since $\mathcal{G}$ satisfies $(\mathcal{G}_n 3)$, $N(x) = 1$ or $\mathcal{O}(x,y) = 1$. Now, take

$N(x)=1$. Since N is frontier, then $x=0$. On the other hand, if $\mathcal{O}(x,y)=1$, then, since $\mathcal{O}$ satisfies ($\mathcal{O}_n 3$), $x=y=1$. Clearly, if $I_{\mathcal{G},N,\mathcal{O}}(x,y)=1$, then $x=0$ or $y=1$.

(b) Consider $I_{\mathcal{G},N,\mathcal{O}}(x,y)=\mathcal{G}(N(x),\mathcal{O}(x,y))=1$. Then, since $\mathcal{G}$ satisfies ($\mathcal{G}_n 3$), $N(x)=1$ or $\mathcal{O}(x,y)=1$. Take $N(x)=1$, so, since N is non-filling, then $x=0$. Now, if $\mathcal{O}(x,y)=1$, then, by $\mathcal{O}$ satisfies ($\mathcal{O}_n 3$), $x=y=1$. Therefore, $x\leq y$.

(c) Suppose that $N\neq N_\top$. Then, there exists $x\in]0,1[$ such that $N(x)<1$. Take $y=\frac{1+x}{2}>x$. As $\mathcal{O}$ satisfies ($\mathcal{O}_n 3$), and $x,y<1$ then $\mathcal{O}(x,y)<1$, so $I_{\mathcal{G},N,\mathcal{O}}(x,y)=\mathcal{G}(N(x),\mathcal{O}(x,y))\neq 1$, since $\mathcal{G}$ satisfies ($\mathcal{G}_n 3$). Thus, $I_{\mathcal{G},N,\mathcal{O}}$ does not satisfy (LOP). Therefore, if $I_{\mathcal{G},N,\mathcal{O}}$ satisfies (LOP) then $N=N_\top$.

(d) For any $x,y\in[0,1]$,

$$I_{\mathcal{G},N_\top,\mathcal{O}}(x,N(y))=\mathcal{G}(N_\top(x),\mathcal{O}(x,N(y))) = \begin{cases} 1 & \text{if } x<1 \\ \mathcal{G}(0,\mathcal{O}(1,N(y))) & \text{if } x=1 \end{cases} \quad (8)$$

$$I_{\mathcal{G},N_\top,\mathcal{O}}(y,N(x))=\mathcal{G}(N_\top(y),\mathcal{O}(y,N(x))) = \begin{cases} 1 & \text{if } y<1 \\ \mathcal{G}(0,\mathcal{O}(1,N(x))) & \text{if } y=1 \end{cases}. \quad (9)$$

Imagine that $N\neq N_\top$. Then, there is an $x\in]0,1[$, such that $N(x)<1$. Since $n_\mathcal{O}=1$ and $n_\mathcal{G}=0$ by Eq. (8), $I_{\mathcal{G},N_\top,\mathcal{O}}(x,N(1))=1$, and, by Eq. (9), $I_{\mathcal{G},N_\top,\mathcal{O}}(1,N(x))=\mathcal{G}(0,\mathcal{O}(1,N(x)))<1$, since $\mathcal{G}$ and $\mathcal{O}$ satisfy ($\mathcal{G}_n 3$) and ($\mathcal{O}_n 3$), respectively. So, $I_{\mathcal{G},N_\top,\mathcal{O}}$ does not satisfy (RCP) for $N\neq N_\top$. Now, take $N=N_\top$. So, by Eq. (8) and Eq. (9),

$$I_{\mathcal{G},N_\top,\mathcal{O}}(x,N_\top(y))=\begin{cases} 1, & \text{if } x<1 \text{ or } y<1 \\ 0, & \text{if } x=1 \text{ and } y=1, \end{cases}$$

$$I_{\mathcal{G},N_\top,\mathcal{O}}(y,N_\top(x))=\begin{cases} 1, & \text{if } y<1 \text{ or } x<1 \\ 0, & \text{if } y=1 \text{ and } x=1. \end{cases}$$

So, $I_{\mathcal{G},N_\top,\mathcal{O}}$ satisfies (RCP) for $N=N_\top$.

(xii) Indeed, for all $x\in[0,1]$, $\mathcal{O}(x,x)=1$, since $\mathcal{O}$ is idempotent. Now, by $(\mathcal{G},N)$ satisfying (LEM), $I_{\mathcal{G},N,\mathcal{O}}(x,x)=\mathcal{G}(N(x),\mathcal{O}(x,x))=\mathcal{G}(N(x),x)=1$.

(xiii) Take $\mathcal{I}=I_{\mathcal{G},N_\top,\mathcal{O}}$, for $x,y,z\in[0,1]$, we have that

$$\mathcal{I}(x,\mathcal{I}(y,z))=\mathcal{G}(N_\top(x),\mathcal{O}(x,\mathcal{I}(y,z)))= \\ =\begin{cases}\mathcal{G}(0,\mathcal{O}(1,\mathcal{I}(y,z)))=\mathcal{G}(0,\mathcal{O}(1,\mathcal{G}(0,\mathcal{O}(1,z)))), \text{if } x=y=1\\ 1, \text{ if } x<1 \text{ or } y<1;\end{cases}$$

$$\mathcal{I}(y,\mathcal{I}(x,z))=\mathcal{G}(N_\top(y),\mathcal{O}(y,\mathcal{I}(x,z))) \\ =\begin{cases}\mathcal{G}(0,\mathcal{O}(1,\mathcal{I}(x,z))), \text{ if } y=1\\ 1, \text{ if } y<1\end{cases} \\ =\begin{cases}\mathcal{G}(0,\mathcal{O}(1,\mathcal{G}(0,\mathcal{O}(1,z)))), \text{ if } \ x=y=1\\ 1, \text{ if } \ x<1 \text{ or } y<1\end{cases}$$

So, $\mathcal{I}(x,\mathcal{I}(y,z))=\mathcal{I}(y,\mathcal{I}(x,z))$ and, $\mathcal{I}=I_{\mathcal{G},N_\top,\mathcal{O}}$ satisfies (EP). Therefore, Theorem 5.1 holds. □

The reciprocal idea of item **(vii)** in Theorem 5.1 is not true.

Proposition 5.2: Under the conditions of Theorem 5.1, when N is non-filling then $I_{\mathcal{G},N,\mathcal{O}}$ does not satisfy (LT).

Proof: Take $\mathcal{G}$ and $\mathcal{O}$ satisfying (G3) and (O3), respectively. Since N is non-filling, $N(x)\neq 1$, for all $x\in]0,1[$, i.e., $N(x)<1$, therefore $\mathcal{G}(N(x),\mathcal{O}(x,1))<1$. Otherwise, because $\mathcal{G}$ satisfies (G2), we would have $N(x)=1$ or $\mathcal{O}(x,1)=1$, which is a contradiction, since $N(x)<1$ and $\mathcal{O}$ satisfies (O3). So $I_{\mathcal{G},N,\mathcal{O}}(x,1)\neq 1$ and $I_{\mathcal{G},N,\mathcal{O}}$ does not satisfy (LT). □

Remark 5.1: One can also observe that the reciprocal ideas of items **(ix)** and **(x)** in Theorem. 5.1 hold when $\mathcal{G}$ verifies ($\mathcal{G}_n 3$) and $\mathcal{O}$ verifies ($\mathcal{O}_n 3$).

Proposition 5.3: If $\mathcal{G}$ and $\mathcal{O}$ satisfy ($\mathcal{G}_n 3$) and ($\mathcal{O}_n 3$), respectively, then:
(i) If $n_\mathcal{O}=1$ and $n_\mathcal{G}=0$ are NE of $\mathcal{O}$ and $\mathcal{G}$, respectively, then $I_{\mathcal{G},N,\mathcal{O}}$ does not satisfy (LCP) for any negation N';
(ii) $I_{\mathcal{G},N,\mathcal{O}}$ does not satisfy (OP).

Proof:

(i) Suppose that there exists $x\in]0,1[$ such that $N(N'(x))\in]0,1[$, so $N'(x)\in]0,1[$. Since $\mathcal{G}$ and $\mathcal{O}$ satisfy ($\mathcal{G}_n 3$) and ($\mathcal{O}_n 3$), respectively, the following holds:

$$I_{\mathcal{G},N,\mathcal{O}}(N'(x),1)=\mathcal{G}(N(N'(x)),\mathcal{O}(N'(x),1))\neq 1. \quad (10)$$

On the other hand, we have that:

$$I_{\mathcal{G},N,\mathcal{O}}(N'(1),x)=\mathcal{G}(1,\mathcal{O}(0,x))=1. \quad (11)$$

Then, from Eqs. (10) and (11), one concludes that $I_{\mathcal{G},N,\mathcal{O}}$ does not satisfy (LCP) for such N'. Now suppose that, for all $x\in]0,1[$ it holds that $N(N'(x))\in\{0,1\}$, that is, $N\circ N'$ is crisp. Then one has to consider two cases:
(1) Consider $x\in]0,1[$ such that $(N\circ N')(x)=0$. Then,

$$I_{\mathcal{G},N,\mathcal{O}}(N'(x),0)=\mathcal{G}(N(N'(x)),\mathcal{O}(N'(x),0)) \\ =\mathcal{G}(0,\mathcal{O}(N'(x),0))=G(0,0)=0, \quad (12)$$

$$I_{\mathcal{G},N,\mathcal{O}}(N'(0),x)=\mathcal{G}(N(1),\mathcal{O}(1,x)) \\ =\mathcal{G}(0,\mathcal{O}(1,x))=x\neq 0, \quad (13)$$

since $n_\mathcal{O}=1$ and $n_\mathcal{G}=0$. By Eqs. (12) and (13), $I_{\mathcal{G},N,\mathcal{O}}$ does not satisfy (LCP).
(2) Consider $x\in]0,1[$ such that $(N\circ N')(x)=1$. Then, again by $n_\mathcal{O}=1$ and $n_\mathcal{G}=0$,

$$I_{\mathcal{G},N,\mathcal{O}}(N'(x),0)=\mathcal{G}(N(N'(x)),\mathcal{O}(N'(x),0)) \\ =\mathcal{G}(1,\mathcal{O}(N'(x),0))=1, \quad (14)$$

$$I_{\mathcal{G},N,\mathcal{O}}(N'(0),x)=\mathcal{G}(N(1),\mathcal{O}(1,x)) \\ =\mathcal{G}(0,\mathcal{O}(1,x))=x\neq 1. \quad (15)$$

By Eqs. (14) and (15), $I_{\mathcal{G},N,\mathcal{O}}$ does not satisfy (LCP). So, $I_{\mathcal{G},N,\mathcal{O}}$ does not satisfy (LCP) for any negation N'.

(ii) Since $\mathcal{G}$ and $\mathcal{O}$ satisfy ($\mathcal{G}_n 3$) and ($\mathcal{O}_n 3$), respectively, the following holds: $I_{\mathcal{G},N,\mathcal{O}}(x,y)=\mathcal{G}(N(x),\mathcal{O}(x,y))=1 \Leftrightarrow N(x)=1$ or $x=y=1$. Imagine that $x\neq 0$ such that $N(x)=1$ and consider $y=\frac{x}{2}$. In this case, $I_{\mathcal{G},N,\mathcal{O}}(x,y)=1$, but $x>y$ and $I_{\mathcal{G},N,\mathcal{O}}$ does not satisfy (OP). Now, suppose that there is not an $x\neq 0$ such that $N(x)=1$, i.e, N is non-filling. Then, take $x=0.5$ and

$y = 0.7$. So, $N(x) \neq 1$ and $I_{\mathcal{G},N,\mathcal{O}}(x,y) \neq 1$ does not satisfy (OP) either.

Therefore, Proposition 5.3 holds. □

VI. Generating QL-Implications from $(\mathcal{G}, N, \mathcal{O})$

QL-implication functions can be obtained from tuples $(\mathcal{G}, N, \mathcal{O})$. In fact they are QL-operations constructed from tuples $(\mathcal{G}, N, \mathcal{O})$ regarding the greatest fuzzy negation $N_\top$.

The main properties satisfied by QL-implications are derived from the properties of QL-operations given from tuples $(\mathcal{G}, N, \mathcal{O})$. Observe that, in some cases, those QL-implication functions may satisfy weaker versions of properties satisfied by the standard QL-implications. However, they are richer than the ones derived from t-norms and positive t-conorms.

Proposition 6.1: If $I_{\mathcal{G},N,\mathcal{O}}$ satisfies (RBC) and the GOF $\mathcal{O}$ satisfies (O6), then the pair $(\mathcal{G}, N)$ satisfies (LEM).

Proof: Since $I_{\mathcal{G},N,\mathcal{O}}$ satisfies (RBC), i.e., $\forall x \in [0,1]$, it holds that $I_{\mathcal{G},N,\mathcal{O}}(x,1) = 1$, then $\mathcal{G}(N(x), \mathcal{O}(x,1)) = 1$. Conversely, since $\mathcal{G}$ is increasing, then, by (O6), $\mathcal{G}(N(x), \mathcal{O}(x,1)) \leq \mathcal{G}(N(x), x)$. Thus, one concludes that $\mathcal{G}(N(x), x) = 1$, for all $x \in [0,1]$ (LEM). □

Corollary 6.1: If $I_{\mathcal{G},N,\mathcal{O}}$ is a fuzzy implication function and the GOF $\mathcal{O}$ satisfies (O6), then the pair $(\mathcal{G}, N)$ satisfies (LEM).

Proof: If $I_{\mathcal{G},N,\mathcal{O}}$ is a fuzzy implication then $I_{\mathcal{G},N,\mathcal{O}}$ satisfies (RBC). Therefore, the result follows from Prop. 6.1. □

Theorem 6.1: Let $\mathcal{O}$ be a GOF, and $\mathcal{G}$ be a GGF, so:

(i) If a QL-operation constructed from the tuple $(\mathcal{G}, N, \mathcal{O})$, where $\mathcal{G}$ satisfies $(\mathcal{G}_n3)$ and $\mathcal{O}$ satisfies $(\mathcal{O}_n3)$, is a fuzzy implication function then $N = N_\top$.

(ii) If $N = N_\top$ then a QL-operation constructed from the tuple $(\mathcal{G}, N, \mathcal{O})$ is a fuzzy implication function.

Proof: (i) Suppose that $N \neq N_\top$. Then, there exists $x \in]0,1[$, such that $N(x) < 1$. Then, since $\mathcal{G}$ and $\mathcal{O}$ satisfy $(\mathcal{G}_n3)$ and $(\mathcal{O}3)$, respectively, we have that: $I_{\mathcal{G},N,\mathcal{O}}(x,1) = \mathcal{G}(N(x), \mathcal{O}(x,1)) < 1$, which is a contradiction, since any fuzzy implication satisfies (RBC). Therefore, $N = N_\top$.
(ii) If $N = N_\top$, $I_{\mathcal{G},N_\top,\mathcal{O}}(x,y) = \mathcal{G}(N_\top(x), \mathcal{O}(x,y))$. So,

$$I_{\mathcal{G},N_\top,\mathcal{O}}(x,y) = \begin{cases} \mathcal{G}(0, \mathcal{O}(1,y)), \text{if } x = 1; \\ \mathcal{G}(1, \mathcal{O}(x,y)), \text{if } x < 1. \end{cases} \quad (16)$$

Since $\mathcal{G}(0, \mathcal{O}(1,y)) \leq 1$, then it is immediate that $I_{\mathcal{G},N,\mathcal{O}}$ satisfies (FPA), and the result follows from Thorem 5.1 **(i)**. □

Observe that any QL-implication function constructed from a tuple $(\mathcal{G}, N_\top, \mathcal{O})$ has the form of Eq. (16).

Proposition 6.2: Take a GOF $\mathcal{O}\colon [0,1]^2 \to [0,1]$, a GGF $\mathcal{G}\colon [0,1]^2 \to [0,1]$, with $n_\mathcal{O}$ and $n_\mathcal{G}$ as the NE of $\mathcal{O}$ and $\mathcal{G}$, respectively, and let $N_\top\colon [0,1] \to [0,1]$ be the greatest fuzzy negation. So, the following statements hold:
(i) If $n_\mathcal{O} = 1$, then the QL-implication function constructed from the tuple $(\mathcal{G}, N_\top, \mathcal{O})$ is defined by:

$$I_{\mathcal{G},N_\top,\mathcal{O}}(x,y) = \begin{cases} 1, \text{ if } x < 1 \text{ or } y = 1 \\ \mathcal{G}(0,y), \text{ if } x = 1 \text{ and } y < 1. \end{cases}$$

(ii) If $n_\mathcal{G} = 0$, then the QL-implication function constructed from the tuple $(\mathcal{G}, N_\top, \mathcal{O})$ is defined by:

$$I_{\mathcal{G},N_\top,\mathcal{O}}(x,y) = \begin{cases} \mathcal{O}(1,y), \text{ if } x = 1 \text{ and } y < 1 \\ 1, \text{ if } x < 1 \text{ or } y = 1. \end{cases}$$

Proof: Considering Eq. (16), the following holds:
(i) Since $\mathcal{O}$ has 1 as NE, there are two cases: (1) if $x < 1$, then $I_{\mathcal{G},N_\top,\mathcal{O}}(x,y) = 1$ and (2) if $x = 1$, then $I_{\mathcal{G},N_\top,\mathcal{O}}(1,y) = \mathcal{G}(0, \mathcal{O}(1,y)) = \mathcal{G}(0,y)$. Now, if $y = 1$ then $I_{\mathcal{G},N_\top,\mathcal{O}}(1,y) = 1$ and, if $y < 1$ then $I_{\mathcal{G},N_\top,\mathcal{O}}(1,y) = \mathcal{G}(0,y)$.
(ii) Since $\mathcal{G}$ has 0 as NE, then there are also two cases: (1) if $x < 1$ then $I_{\mathcal{G},N_\top,\mathcal{O}}(x,y) = 1$ and (2) if $x = 1$, then $I_{\mathcal{G},N_\top,\mathcal{O}}(1,y) = \mathcal{G}(0, \mathcal{O}(1,y)) = \mathcal{O}(1,y)$. So, $I_{\mathcal{G},N_\top,\mathcal{O}}(1,y) = 1$ whenever $y = 1$ and $I_{\mathcal{G},N_\top,\mathcal{O}}(1,y) = \mathcal{O}(1,y)$ whenever $y < 1$. □

See the QL-implications from a tuple $(\mathcal{G}, N, \mathcal{O})$.

Example 6.1: Consider the GOF $\mathcal{O}(x,y) = \max(x+y-1, 0)$, the $N_\top$ negation in Eq. (1), and the GGF given as:

$$\mathcal{G}(x,y) = \begin{cases} 0, \text{ if } \max(x,y) \leq \frac{1}{2} \\ 2(\max(x,y) - \frac{1}{2}), \text{ otherwise.} \end{cases} \quad (17)$$

The $I_{\mathcal{G},N_\top,\mathcal{O}}$ QL-implication is given as follows:

$$I_{\mathcal{G},N_\top,\mathcal{O}}(x,y) = \begin{cases} 0, \text{ if } x = 1 \text{ and } y \leq \frac{1}{2} \\ 2y - 1, \text{ if } x = 1 \text{ and } y > \frac{1}{2} \\ 1, \text{ if } x \neq 1. \end{cases}$$

Example 6.2: Take the GOF as Eq. (3) and $O(x,y) = xy$:

$$\mathcal{O}_{\frac{1}{2}}(x,y) = \frac{\max(xy - \frac{1}{2}\max(x,y), 0)}{1 - \frac{1}{2}\max(x,y)},$$

the $N_\top$ negation in Eq. (1), and the GGF $\mathcal{G}$ in Eq. (17). So, the $I_{\mathcal{G},N_\top,\mathcal{O}_{\frac{1}{2}}}$ QL-implication is given as:

$$I_{\mathcal{G},N_\top,\mathcal{O}_{\frac{1}{2}}}(x,y) = \begin{cases} 0, \text{ if } x = 1 \text{ and } y \leq \frac{3}{4} \\ 4y - 3, \text{ if } x = 1 \text{ and } y > \frac{3}{4} \\ 1, \text{ if } x \neq 1. \end{cases}$$

Theorem 6.2: Let $I_{\mathcal{G},N_\top,\mathcal{O}}\colon [0,1]^n \to [0,1]$ be a QL-implication function constructed from a tuple $(\mathcal{G}, N_\top, \mathcal{O})$. Then it holds that:
(i) $I_{\mathcal{G},N_\top,\mathcal{O}}$ satisfies (LBC), (LOP), (RCP) and (EP);
(ii) If $n_\mathcal{O} = 1$ and $n_\mathcal{G} = 0$ are the NE of $\mathcal{O}$ and $\mathcal{G}$, respectively, then $I_{\mathcal{G},N_\top,\mathcal{O}}$ satisfies (NP);
(iii) $N_{I_{\mathcal{G},N_\top,\mathcal{O}}} = N_\top$;
(iv) If $\mathcal{G}$ and $\mathcal{O}$ satisfy $(\mathcal{G}_n2)$ and $(\mathcal{O}_n2)$, respectively, then $I_{\mathcal{G},N_\top,\mathcal{O}}$ satisfies (LF);
(v) $I_{\mathcal{G},N_\top,\mathcal{O}}$ satisfies (FP);
(vi) If $n_\mathcal{O} = 1$ and $n_\mathcal{G} = 0$ are NE of $\mathcal{O}$ and $\mathcal{G}$, respectively, then $I_{\mathcal{G},N_\top,\mathcal{O}}$ does not satisfy (LCP) for $N_\top$;
(vii) $I_{\mathcal{G},N_\top,\mathcal{O}}$ does not satisfy (ROP), (LT) and (OP).

Proof:

(i)-(iii) It follows from Theorem 5.1, itens (ii), (x), (xi)(d), (xiii), (iv) and (vi), respectively.

(iv) Indeed, consider $I_{\mathcal{G},N_\top,\mathcal{O}}(x,y) = 0$, then $x = 1$ and $\mathcal{G}(0, \mathcal{O}(1,y)) = 0$. Since $\mathcal{G}$ and $\mathcal{O}$ satisfy $(\mathcal{G}_n2)$ and

$(\mathcal{O}_n2)$, respectively, $\mathcal{G}(0,\mathcal{O}(1,y)) = 0 \Leftrightarrow \mathcal{O}(1,y) = 0 \Leftrightarrow y = 0$. Now, if $x = 1$ and $y = 0$, then $I_{\mathcal{G},N_\top,\mathcal{O}}(1,0) = \mathcal{G}(0,\mathcal{O}(1,0)) = \mathcal{G}(0,0) = 0$.

(v) $I_{\mathcal{G},N_\top,\mathcal{O}}(x,x) = \mathcal{G}(N_\top(x),\mathcal{O}(x,x)), \forall x \in [0,1]$. So,

$$I_{\mathcal{G},N_\top,\mathcal{O}}(x,x) = \begin{cases} \mathcal{G}(0,\mathcal{O}(1,1) & \text{if } x = 1; \\ \mathcal{G}(1,\mathcal{O}(x,x) & \text{if } x < 1. \end{cases}$$

And $I_{\mathcal{G},N_\top,\mathcal{O}}(x,x) = 1$.

(vi) Take $x < y < 1$, then $I_{\mathcal{G},N_\top,\mathcal{O}}(N_\top(x),y) = \mathcal{G}(N_\top(N_\top(x)),\mathcal{O}(N_\top(x),y)) = \mathcal{G}(0,\mathcal{O}(1,y)) = \mathcal{G}(0,y) = y$ and $I_{\mathcal{G},N_\top,\mathcal{O}}(N_\top(y),x) = \mathcal{G}(0,\mathcal{O}(1,x)) = \mathcal{G}(0,x) = x$. So, $I_{\mathcal{G},N_\top,\mathcal{O}}(N_\top(x),y) \neq I_{\mathcal{G},N_\top,\mathcal{O}}(N_\top(y),x)$, since $x < y$.

(vii) Indeed,the next properties hold:

(ROP): It follows from Theorem 5.1(ix).

(LT): It follows from Theorem 5.1(vii).

(OP): For $0 < x < 1$, take $y = \frac{x}{2}$, then $I_{\mathcal{G},N_\top,\mathcal{O}}(x,y) = \mathcal{G}(N_\top(x),\mathcal{O}(x,y)) = \mathcal{G}(1,\mathcal{O}(x,y)) = 1$, but $x > y$.

□

Remark 6.1: As a counterpoint to Theorem 6.2, let $\mathcal{G}_{LK}$ be the GGF given in Eq.(4) which does not verify $(\mathcal{G}_n3)$, let N_S be the standard negation and $\mathcal{O}_S : [0,1]^2 \to [0,1]$ be an overlap function verifying $(\mathcal{O}_n3)$ and defined as

$$\mathcal{O}_S(x,y) = \begin{cases} xy, \text{ if } xy \leq 0.6; \\ 0.6, \text{ if } 0.6 \leq xy \leq 0.8; \\ 2xy - 1, \text{ if } xy \geq 0.8. \end{cases}$$

Then, $I_{\mathcal{G}_{LK},N_S,O_S}$ is a QL-implication function given as:

$$I_{\mathcal{G}_{LK},N_S,\mathcal{O}_S}(x,y) = \begin{cases} \min(1 - x(1-y), 1), \text{ if } xy \leq 0.6; \\ \min(1.6 - x, 1), \text{ if } 0.6 \leq xy \leq 0.8; \\ x(2y-1), \text{ if } xy \geq 0.8. \end{cases}$$

Moreover, taking the overlap function $\mathcal{O}_M$ (for $p = 1$) in Table I, see also the Reichenbach QL-implication $I_{\mathcal{G}_{LK},N_S,\mathcal{O}_M}$.

VII. Conclusion

In this article we study the application of the general overlap and grouping functions to construct QL-implications which are generated not only by the greatest fuzzy negation but also considering the standard negation N_S. The proposal recovers the characteristics of these functions, also proposing a constructive model for the generation of general overlap (grouping) functions by overlap (grouping) functions. Examples of classes are presented in order to validate the methods.

Further works include the generalization of other properties such as the O-conditionality law [15] and distributivity laws [16] of fuzzy implications. Moreover the main results can be extended for the interval-valued approach by considering admissible orders [17].

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