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Constructing Interval-Valued Fuzzy Material Implication Functions derived from General Interval-Valued Grouping Functions

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***Abstract*—Grouping functions and their dual counterpart, overlap functions, have drawn the attention of many authors, mainly because they constitute a richer class of operators compared to other types of aggregation functions. Grouping functions are a useful theoretical tool to be applied in various problems, like decision making based on fuzzy preference relations. In pairwise comparisons, for instance, those functions allow one to convey the measure of the amount of evidence in favor of either of two given alternatives. Recently, some generalizations of grouping functions were proposed, such as (i) the n-dimensional grouping functions and the more flexible general grouping functions, which allowed their application in n-dimensional problems, and (ii) n-dimensional and general interval-valued grouping functions, in order to handle uncertainty on the definition of the membership functions in real-life problems. Taking into account the importance of interval-valued fuzzy implication functions in several application problems under uncertainty, such as fuzzy inference mechanisms, this paper aims at introducing a new class of interval-valued fuzzy material implication functions. We study their properties, characterizations, construction methods and provide examples.**

***Index Terms*—Grouping functions, general interval-valued grouping functions, fuzzy material implication functions**

I. Introduction

Bustince et al. introduced the concept of overlap functions [1], which are a type of aggregation function [2] not requiring the associativity property, aimed at the application in image processing. The main idea was to measure the degree of overlapping between two classes or objects. The "dual" notion of overlap functions, also introduced by Bustince et al. [3], are the grouping functions. The gist of those functions is to convey the measure of the amount of evidence in favor of either of two alternatives whenever one performs pairwise comparisons [4] in decision making problems based on relations of fuzzy preference [5]. Likewise, grouping functions have been adopted as the disjunction operator in a variety of situations, e.g., in image thresholding technique [6] and for the development of a class of implication functions to construct fuzzy subsethood measures and entropy [7].

Taking as reference the well-known operators like t-norms and t-conorms [8], overlap and grouping functions are indeed much richer classes, respectively. Those functions do present the self-closeness feature with respect to the convex sum and the aggregation by generalized composition of overlap and/or grouping functions [9], [10], whereas t-norms and t-conorms do not respect analogous properties. Moreover, the maximum t-conorm, for instance, is known to be the only idempotent t-conorm and the unique homogeneous t-conorm. On the other hand, one can find numberless idempotent, as well as, homogeneous grouping functions [9], [10].

Now, observe that overlap and grouping functions are bi-variate functions. Since they may be non associative, they can only be applied in bi-dimensional problems (when only a pair of classes/objects are considered). Therefore, some ideas came up to tackle that issue. In Gómez et al. [11], n-dimensional overlap functions were proposed and applied to

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fuzzy rule based classification systems (FRBCSs), which are usually applied in n-ary problems. Additionally, those functions were applied in decomposition strategies [12]. Gómez et al. [13] also presented the idea of n-dimensional grouping functions, applying them as an alternative method to quantify the quality of a fuzzy community detection output based on n-dimensional operators.

Among the works in the n-dimensional context, it is important to point out the one by De Miguel et al. [14], who introduced the concept of general overlap functions, which are n-ary functions with less restrictive boundary conditions, applied to identify the matching degree in the fuzzy reasoning method of FRBCSs. Then, Santos et al. [15] introduced the theoretical basis of general grouping functions, which is a resource that allows more flexibility to n-dimensional grouping functions and can be interpreted, for example, as the quantity of evidence in favor of one alternative among multiple ones when performing n-ary comparisons in multi-criteria decision making based on n-ary fuzzy heterogeneous, incomplete preference relations [16].

A. What is the importance of general interval-valued overlap/grouping functions?

Modelling fuzzy systems involves challenging tasks such as the definition of adequate membership functions. In the literature, it is common to deal with the underlying uncertainty in this process, usually associated with the linguistic terms [17], by applying interval-valued fuzzy sets (IVFSs) [18]. As addressed by several authors [19]–[21], those sets can model not only uncertainty (regarding the membership function) but also vagueness (soft class boundaries), and so, they have had great performance in various applications, such as game theory [22], pest control [23] and classification problems [24].

Independent contributions given by Bedregal et al. [21] and Qiao and Hu [25] presented the notion of interval-valued overlap functions. More recently, general interval-valued overlap functions were proposed by Asmus et al. [19], who also introduced interval-valued overlap indices, with applications to interval-valued FRBCS. Moreover, Asmus et al. [24] presented the concept of n-dimensional admissibly ordered interval-valued overlap functions, which are increasing functions concerning a total order, addressing their relevance in interval-valued FRBCS.

Despite the introduction of interval-valued grouping functions by Qiao and Hu [25] in 2017, their proposal is limited to problems of two classes, which is an obstacle to be overcome in problems with n classes, as previously discussed. In 2020, Asmus et al. [26] introduced the concepts of n-dimensional and general interval-valued grouping functions, in such a way that the latter offers more flexibility than the former.

B. Which would be the role of interval-valued fuzzy material implication functions?

The usage of if-then clause/rule in fuzzy rule-based systems turns the process of representing inferential knowledge very intuitive, and, so, it has been commonly used in several works. It is possible to construct implication-like operators in various and distinct ways. In the fuzzy logic setting, implication functions have been deeply studied throughout years, both in applied and theoretical fields [27].

Classic logical operators $\wedge$ (and), $\vee$ (or) and $\sim$ (not) can be generalized, to the unit interval $[0,1]$, by t-norms T, t-conorms S and fuzzy negations N, respectively [8]. These fuzzy operators can be used to construct fuzzy implication functions, such as the sub-classes of material implications known as (S,N)-implication functions, the residual or intuitionistic-logic implications (R-implication functions), the quantum logic implications (QL-implication functions), and also the Dishkant implications or implications of orthomodular lattices (D-implication functions) [27]. Several definitions in the interval-valued context were also presented, e.g. [28].

Notwithstanding the foregoing implications functions, other studies appeared fetching less restrictive and more flexible operators than t-norms and t-conorms. For instance, we can mention works adopting uninorms under some conditions by Mas et al. [29], or aggregation functions, by Ouyang [30].

Thus, in order to introduce classes of fuzzy implication functions based on richer operators than t-norms and t-conorms (namely, overlap O and grouping G functions, respectively) and which have features of flexibility and less restrictiveness, Dimuro et al. [31] have presented some relevant concepts, such as the material implications known as (G,N)-implication functions, R_O-implication functions (the residual implications), QL-implication functions derived from tuples (O,G,N) (the quantum logic implications), and D-implication functions derived from grouping functions (the Dishkant implications). Several other works on this subject may be found in the literature [32], some of them considering interval-valued approaches [33].

In particular, in this paper we are going to explore a more flexible approach by introducing interval-valued material implication functions derived from general interval-valued grouping functions and interval-valued fuzzy negations. That is, we use general interval-valued grouping functions in order to generalize the Boolean implication $p \rightarrow q \equiv \sim p \vee q$, studying their properties and characterizations.

C. The objective and the organization of this paper

The objective of the present work is to introduce a new and more flexible class of fuzzy material implications in the interval-valued context, namely the $(\mathcal{GG},\mathcal{N})$-implication functions derived from general interval-valued grouping functions $\mathcal{GG}$, studying properties and providing their characterization (Section III). Moreover, Section II presents some preliminary concepts and Section IV is the Conclusion.

II. Preliminaries

Let $L([0,1])$ be the set of closed subintervals of the unit interval $[0,1]$, $L([0,1]) = \{[x_1,x_2] | 0 \leq x_1 \leq x_2 \leq 1\}$. Denote $\mathbf{x} = (x_1,\ldots,x_n) \in [0,1]^n$ and $\mathbf{X} = (X_1,\ldots,X_n) \in L([0,1])^n$. For any $X = [x_1,x_2]$, the left and right end-points of X are represented, respectively, by $\underline{X} = x_1$ and

$\overline{X} = x_2$. For any $\mathbf{X} \in L([0,1])^n$, $\underline{\mathbf{X}} = (\underline{X_1}, \ldots, \underline{X_n})$ and $\overline{\mathbf{X}} = (\overline{X_1}, \ldots, \overline{X_n})$. In this work, we adopt the product order in $L([0,1])^n$, given by $X \leq_{Pr} Y \Leftrightarrow \underline{X} \leq \underline{Y} \wedge \overline{X} \leq \overline{Y}$, for all $X, Y \in L([0,1])$. An interval-valued mapping $F\colon L([0,1])^n \to L([0,1])$ is said to be $\leq_{Pr}$-increasing if it is increasing with respect to the product order $\leq_{Pr}$, i.e., $X_1 \leq_{Pr} Y_1, \ldots, X_n \leq_{Pr} Y_n \Rightarrow F(\mathbf{X}) \leq_{Pr} F(\mathbf{Y})$ is satisfied, for all $\mathbf{X}, \mathbf{Y} \in L([0,1])^n$.

Let F be an interval-valued mapping. F is Moore-continuous if it is continuous with respect to the Moore metric [34] $d_M : L([0,1])^2 \to \mathbb{R}$, given, for all $X, Y \in L([0,1])$, as follows: $d_M(X,Y) = \max(|\underline{X} - \underline{Y}|, |\overline{X} - \overline{Y}|)$. It is possible to generalize the Moore-metric to $L([0,1])^n$, for all $\mathbf{X}, \mathbf{Y} \in L([0,1])^n$: $d_M^n(\mathbf{X}, \mathbf{Y}) = \sqrt{d_M(X_1,Y_1)^2 + \ldots + d_M(X_n,Y_n)^2}$.

Given two functions $f, g : [0,1]^n \to [0,1]$ such that $f \leq g$, the mapping $\widehat{f,g} : L([0,1])^n \to L([0,1])$ is defined, for all $\mathbf{X} \in L([0,1])^n$, by $\widehat{f,g}(\mathbf{X}) = [f(\underline{\mathbf{X}}), g(\overline{\mathbf{X}})]$.

Definition 2.1: [35] A mapping $F\colon L([0,1])^n \to L([0,1])$ is an $\leq_{Pr}$-increasing interval-valued function that is called representable if there are increasing functions $f, g : [0,1]^n \to [0,1]$ such that $f \leq g$ and $F = \widehat{f,g}$. f and g are said to be the *representatives* of the interval-valued function F

When $F = \widehat{f,f}$, we simply represent by $\widehat{f}$.

Next, we provide some interval operations used in this paper [34], for all $X, Y \in L([0,1])$:

Supremum: $\sup(X,Y) = [\max(\underline{X}, \underline{Y}), \max(\overline{X}, \overline{Y})]$;
Sum: $X + Y = [\underline{X} + \underline{Y}, \overline{X} + \overline{Y}]$;
Limited Sum: $X \dot{+} Y = [\min(1, \underline{X} + \underline{Y}), \min(1, \overline{X} + \overline{Y})]$;
Subtraction: $X - Y = [\underline{X} - \overline{Y}, \overline{X} - \underline{Y}]$;
Power: $X^p = [\underline{X}^p, \overline{X}^p], p > 0$.

Definition 2.2: [36] $\mathcal{N}\colon L([0,1]) \to L([0,1])$ is an interval-valued (iv) fuzzy negation if, for all $X, Y \in L([0,1])$:

($\boldsymbol{\mathcal{N}1}$) $\mathcal{N}$ is decreasing, i.e. $\mathcal{N}(X) \leq_{Pr} \mathcal{N}(Y)$ whenever $Y \leq_{Pr} X$;
($\boldsymbol{\mathcal{N}2}$) $\mathcal{N}([0,0]) = [1,1]$ and $\mathcal{N}([1,1]) = [0,0]$.
Moreover, $\mathcal{N}$ is *strict* if
($\boldsymbol{\mathcal{N}3}$) $\mathcal{N}$ is Moore continuous and
($\boldsymbol{\mathcal{N}4}$) $\mathcal{N}(X) <_{Pr} \mathcal{N}(Y)$ whenever $Y <_{Pr} X$.
It is *strong* if
($\boldsymbol{\mathcal{N}5}$) $\mathcal{N}(\mathcal{N}(X)) = X$, for each $X \in L([0,1])$.
It is *crisp* if
($\boldsymbol{\mathcal{N}6}$) $\mathcal{N}(X) \in \{[0,0],[1,1]\}$, for all $X \in L([0,1])$.

Proposition 2.1: [36] Every strong iv-fuzzy negation $\mathcal{N}$ is also strict.

Definition 2.3: [28] A mapping $\mathcal{I}\colon L([0,1])^2 \to L([0,1])$ is said to be an interval-valued (iv) fuzzy implication function on $\langle L([0,1]), \leq_{Pr}\rangle$ if the following properties hold by $\mathcal{I}$, for all $X, Y, Z \in L([0,1])$:

($\boldsymbol{\mathcal{I}1}$) If $X \leq_{Pr} Y$ then $\mathcal{I}(Y,Z) \leq_{Pr} \mathcal{I}(X,Z)$;
($\boldsymbol{\mathcal{I}2}$) If $Y \leq_{Pr} Z$ then $\mathcal{I}(X,Y) \leq_{Pr} \mathcal{I}(X,Z)$;
($\boldsymbol{\mathcal{I}3}$) $\mathcal{I}([0,0],Y) = [1,1]$;
($\boldsymbol{\mathcal{I}4}$) $\mathcal{I}(X,[1,1]) = [1,1]$;
($\boldsymbol{\mathcal{I}5}$) $\mathcal{I}([1,1],[0,0]) = [0,0]$.

Remark 2.1: A similar definition to Def. 2.3 is obtained whenever one respectively permutes ($\boldsymbol{\mathcal{I}3}$) and ($\boldsymbol{\mathcal{I}4}$) by:

($\boldsymbol{\mathcal{I}3}$)* $\mathcal{I}([0,0],[0,0]) = [1,1]$;
($\boldsymbol{\mathcal{I}4}$)* $\mathcal{I}([1,1],[1,1]) = [1,1]$.

Definition 2.4: [9], [27] The following properties may be studied for an iv-fuzzy implication function $I\colon [0,1]^2 \to [0,1]$, for all $X, Y, Z \in L([0,1])$:

(NP) *Left neutrality property:* $\mathcal{I}(1,Y) = Y$;
(EP) *Exchange principle:* $\mathcal{I}(X,\mathcal{I}(Y,Z)) = \mathcal{I}(Y,\mathcal{I}(X,Z))$;
(IB) *Iterative Boolean property:* $\mathcal{I}(X,\mathcal{I}(X,Y)) = \mathcal{I}(X,Y)$;
(CP) *Law of contraposition* (or, the contrapositive symmetry) with respect to an iv-fuzzy negation $\mathcal{N}$: $\mathcal{I}(X,Y) = \mathcal{I}(\mathcal{N}(Y),\mathcal{N}(X))$;
(L-CP) *Left contraposition law* with respect to an iv-fuzzy negation $\mathcal{N}$: $\mathcal{I}(\mathcal{N}(X),Y) = \mathcal{I}(\mathcal{N}(Y),X)$;

Definition 2.5: The natural iv-fuzzy negation of an iv-fuzzy implication function $\mathcal{I}\colon L([0,1])^2 \to L([0,1])$ is defined as the function $\mathcal{N}_{\mathcal{I}}\colon L([0,1]) \to L([0,1])$, such that $\mathcal{N}_{\mathcal{I}}(X) = \mathcal{I}(X,[0,0])$, for all $X \in L([0,1])$.

A function $\varrho\colon L([0,1]) \to L([0,1])$ is an interval-valued (iv) automorphism if it is bijective and increasing with respect to the product order ($\leq_{Pr}$). Thus, for an interval-valued function $F\colon L([0,1])^n \to L([0,1])$, we define the interval-valued function F^ϱ by [36]:

$$F^\varrho(X_1, \ldots, X_n) = \varrho^{-1}(F(\varrho(X_1), \ldots, \varrho(X_n))). \quad (1)$$

Proposition 2.2: [36] Let $\varrho\colon L([0,1]) \to L([0,1])$ be an iv-automorphism. Then ϱ^{-1} is also an iv-automorphism.

Proposition 2.3: [35] Let $\varrho\colon L([0,1]) \to L([0,1])$. ϱ is an iv-automorphism if and only if ϱ is Moore-continuous, strictly increasing, $\varrho([0,0]) = [0,0]$ and $\varrho([1,1]) = [1,1]$.

Theorem 2.1: [36] Let $\mathcal{N}$ be an iv-(strict,strong) fuzzy negation and let ϱ be an iv-automorphism. Then $\mathcal{N}^\varrho$ is also an iv-(strict,strong) fuzzy negation.

Definition 2.6: [37] An interval-valued mapping $\mathcal{A}\colon L([0,1])^n \to L([0,1])$ is called an n-dimensional interval-valued (iv)aggregation function if the following conditions hold:

($\boldsymbol{\mathcal{A}1}$) $\mathcal{A}([0,0] \ldots, [0,0]) = [0,0]$, $\mathcal{A}([1,1] \ldots, [1,1]) = [1,1]$;
($\boldsymbol{\mathcal{A}2}$) $\mathcal{A}$ is $\leq_{Pr}$-increasing: for each $i \in \{1, \ldots, n\}$, if $X_i \leq_{Pr} Y$ then

$$\mathcal{A}(X_1, \ldots, X_n) \leq_{Pr} \mathcal{A}(X_1, \ldots, X_{i-1}, Y, X_{i+1}, \ldots, X_n).$$

Definition 2.7: [35] A mapping $\mathcal{S}\colon L([0,1])^2 \to L([0,1])$ is an interval-valued triangular conorm (iv-t-conorm, for short) if, for all $X, Y, Z \in L([0,1])$, the following properties hold:

($\boldsymbol{\mathcal{S}1}$) $\mathcal{S}(X,Y) = \mathcal{S}(Y,X)$;
($\boldsymbol{\mathcal{S}2}$) $\mathcal{S}(X,\mathcal{S}(Y,Z)) = \mathcal{S}(\mathcal{S}(X,Y),Z)$;
($\boldsymbol{\mathcal{S}3}$) If $X \leq_{Pr} Y$ then $\mathcal{S}(X,Z) \leq_{Pr} \mathcal{S}(Y,Z)$;
($\boldsymbol{\mathcal{S}4}$) $\mathcal{S}(X,[0,0]) = X$.

Definition 2.8: [26] A $\mathcal{G}n\colon L([0,1])^n \to L([0,1])$ is an n-dimensional interval-valued (iv) grouping function if it satisfies the following properties, for all $\mathbf{X} \in L([0,1])^n$:

($\mathcal{G}n$1) $\mathcal{G}n$ is commutative;
($\mathcal{G}n$2) $\mathcal{G}n(\mathbf{X}) = [0,0]$ if and only if $X_1 = \ldots = X_n = [0,0]$;
($\mathcal{G}n$3) $\mathcal{G}n(\mathbf{X}) = [1,1]$ if and only if there exists $i \in \{1,\ldots,n\}$ with $X_i = [1,1]$;
($\mathcal{G}n$4) $\mathcal{G}n$ is $\leq_{Pr}$-increasing in the first component: $\mathcal{G}n(X_1,\ldots,X_n) \leq_{Pr} \mathcal{G}n(Y,X_2,\ldots,X_n)$ when $X_1 \leq_{Pr} Y$;
($\mathcal{G}n$5) $\mathcal{G}n$ is Moore continuous.

Observe that $\sup(\mathbf{X}) = [0,0]$ if and only if $X_1 = \ldots = X_n = [0,0]$. Besides, $\sup(\mathbf{X}) = [1,1]$ if and only if there exists $i \in \{1,\ldots,n\}$ with $X_i = [1,1]$.

Example 2.1: The following functions are some examples of n-dimensional iv-grouping functions:

1. $\mathcal{G}n_S(\mathbf{X}) = \sup(\mathbf{X})$;
2. $\mathcal{G}n_p(\mathbf{X}) = [1,1] - \prod_{i=1}^{n}([1,1] - X_i^p)$, for $p > 0$;

Theorem 2.2: [26] Take Gn_1 and Gn_2 as n-dimensional grouping functions such that $Gn_1 \leq Gn_2$. So, the mapping $\widehat{Gn_1, Gn_2}$ is an n-dimensional iv-grouping function.

Definition 2.9: [26] Let a mapping $\mathcal{G}n\colon L([0,1])^n \to L([0,1])$ be an n-dimensional iv-0-grouping function if and only if the property **($\mathcal{G}n$2)** from Definition 2.8 is replaced by: **($\mathcal{G}n$2′)** If $X_1 = \ldots = X_n = [0,0]$, then $\mathcal{G}n(\mathbf{X}) = [0,0]$, for all $\mathbf{X} \in L([0,1])^n$. Similarly, let a mapping $\mathcal{G}n\colon L([0,1])^n \to L([0,1])$ be an n-dimensional iv-1-grouping function if and only if the property **($\mathcal{G}n$3)** from Definition 2.8 is replaced by: **($\mathcal{G}n$3′)** If there exists $i \in \{1,\ldots,n\}$ with $X_i = [1,1]$, then $\mathcal{G}n(\mathbf{X}) = [1,1]$, for all $\mathbf{X} \in L([0,1])^n$.

Example 2.2: Consider the n-dimensional interval-valued limited sum, defined, for all $X \in L([0,1])^n$, by $\mathcal{G}n_S(\mathbf{X}) = X_1 \dot{+} \ldots \dot{+} X_n$. $\mathcal{G}n_S$ is an n-dimensional iv-1-grouping function. However, $\mathcal{G}n_S$ is not an n-dimensional iv-grouping function, since ($\mathcal{G}n$3) does not hold.

Definition 2.10: [26] A function $\mathcal{GG}\colon L([0,1])^n \to L([0,1])$ is said to be a general interval-valued (iv) grouping function if, for all $\mathbf{X} \in L([0,1])^n$:

($\mathcal{GG}$1) $\mathcal{GG}$ is commutative;
($\mathcal{GG}$2) If $X_1 = \ldots = X_n = [0,0]$, then $\mathcal{GG}(\mathbf{X}) = [0,0]$;
($\mathcal{GG}$3) If there exists $i \in \{1,\ldots,n\}$ with $X_i = [1,1]$, then $\mathcal{GG}(\mathbf{X}) = [1,1]$;
($\mathcal{GG}$4) $\mathcal{GG}$ is $\leq_{Pr}$-increasing in the first component: $\mathcal{GG}(X_1,\ldots,X_n) \leq_{Pr} \mathcal{GG}(Y,X_2,\ldots,X_n)$, when $X_1 \leq_{Pr} Y$;
($\mathcal{GG}$5) $\mathcal{GG}$ is Moore continuous.

Example 2.3: The function given, for all $\mathbf{X} \in L([0,1])^n$, by

$$\mathcal{GG}_L(\mathbf{X}) = \begin{cases} [0,0] & \text{if } \overline{m} \leq \frac{1}{n}, \\ [0, \min(1, n \cdot \overline{m})] & \text{if } \min(1, \underline{m}) \leq \frac{1}{n} \\ & \text{ and } \min(1, \overline{m}) > \frac{1}{n}, \\ n \cdot (X_1 \dot{+} \ldots \dot{+} X_n) & \text{otherwise,} \end{cases}$$

with $\underline{m} = \min\left(1, \sum_{i=1}^{n} \underline{X_i}\right)$ and $\overline{m} = \min\left(1, \sum_{i=1}^{n} \overline{X_i}\right)$, is a general iv-grouping function, which is neither an n-dimensional iv-0-grouping function, nor an n-dimensional iv-1-grouping function. Consequently, it is neither an n-dimensional iv-grouping function.

Proposition 2.4: [26] If $F\colon L([0,1])^n \to L([0,1])$ is a mapping that is either an n-dimensional iv-grouping, or an n-dimensional iv-0-grouping and an n-dimensional iv-1-grouping function, then F is also said to be a general iv-grouping function.

Theorem 2.3: [26] Take two general grouping functions GG_1 and GG_2, such that $GG_1 \leq GG_2$. So, the mapping $\widehat{GG_1, GG_2}$ is a general iv-grouping function.

Taking into account the results obtained in Theorem 2.3 and Prop. 2.4, it is possible to get a representable general iv-grouping function. One may construct it by using n-dimensional grouping functions (which, in turn, is called a g-representable general iv-grouping function) or any of their generalizations, such as 0-grouping, 1-grouping or general grouping functions. Nevertheless, if a general iv-grouping function is representable, then its representatives must be general grouping functions.

Example 2.4: Let GG_B be the general grouping function defined by $GG_B(\mathbf{x}) = \min\left(1, n - \sum_{i=1}^{n}(1-x_i)^2\right)$. So, the representable general iv-grouping function $\mathcal{GG}_B$ can be given by taking GG_B as both its representatives, given by $\mathcal{GG}_B(\mathbf{X}) = \widehat{GG_B}(\mathbf{X})$, for all $\mathbf{X} \in L([0,1])^n$.

III. Material iv-fuzzy implications derived from general iv-grouping functions

In this section, we introduce interval-valued implication functions constructed by means of a bivariate general interval-valued grouping function (or simply, general iv-grouping function) and an iv-fuzzy negation, called $(\mathcal{GG},\mathcal{N})$-implication functions.

Let $\mathcal{GG}\colon L([0,1])^2 \to L([0,1])$ be a general iv-grouping function and let $\mathcal{N}\colon L([0,1]) \to L([0,1])$ be an iv-fuzzy negation. Define the iv-function $\mathcal{I}_{\mathcal{GG},\mathcal{N}}\colon L([0,1])^2 \to L([0,1])$, for all $X, Y \in L([0,1])$, by:

$$\mathcal{I}_{\mathcal{GG},\mathcal{N}}(X,Y) = \mathcal{GG}(\mathcal{N}(X), Y). \tag{2}$$

Theorem 3.1: $\mathcal{I}_{\mathcal{GG},\mathcal{N}}\colon L([0,1])^2 \to L([0,1])$ is an iv-fuzzy implication, entitled $(\mathcal{GG},\mathcal{N})$-implication function.

Proof: Indeed, for all $X, Y, Z \in L([0,1])$, it holds that:

($\mathcal{I}$1) $X \leq_{Pr} Y \overset{(\mathcal{N}1)}{\Rightarrow} \mathcal{N}(Y) \leq_{Pr} \mathcal{N}(X) \overset{(\mathcal{GG}4)}{\Rightarrow} \mathcal{GG}(\mathcal{N}(Y),Z) \leq_{Pr} \mathcal{GG}(\mathcal{N}(X),Z)$. So, $\mathcal{I}_{\mathcal{GG},\mathcal{N}}(Y,Z) \leq_{Pr} \mathcal{I}_{\mathcal{GG},\mathcal{N}}(X,Z)$.
($\mathcal{I}$2) $Y \leq_{Pr} Z \overset{(\mathcal{GG}4)}{\Rightarrow} \mathcal{GG}(\mathcal{N}(X),Y) \leq_{Pr} \mathcal{GG}(\mathcal{N}(X),Z)$. So, $\mathcal{I}_{\mathcal{GG},\mathcal{N}}(X,Y) \leq_{Pr} \mathcal{I}_{\mathcal{GG},\mathcal{N}}(X,Z)$.
($\mathcal{I}$3) $\mathcal{I}_{\mathcal{GG},\mathcal{N}}([0,0],Y) = \mathcal{GG}(\mathcal{N}([0,0]),Y) \overset{(\mathcal{N}2)}{=} \mathcal{GG}([1,1],Y) \overset{(\mathcal{GG}3)}{=} [1,1]$.
($\mathcal{I}$4) $\mathcal{I}_{\mathcal{GG},\mathcal{N}}(X,[1,1]) = \mathcal{GG}(\mathcal{N}(X),[1,1]) \overset{(\mathcal{GG}3)}{=} [1,1]$.
($\mathcal{I}$5) $\mathcal{I}_{\mathcal{GG},\mathcal{N}}([1,1],[0,0]) = \mathcal{GG}(\mathcal{N}([1,1]),[0,0]) \overset{(\mathcal{N}2)}{=}$

$\mathcal{GG}([0,0],[0,0]) \overset{(\mathcal{GG}2)}{=} [0,0]$.

Therefore, $\mathcal{I}_{\mathcal{GG},\mathcal{N}}$ is an iv-fuzzy implication function. ■

Example 3.1: Take the general grouping function GG_B defined by $GG_B(x,y) = \min\{1, 2-(1-x)^2-(1-y)^2\}$. By Ex. 2.4, $\mathcal{GG}_B(X,Y) = \widehat{GG_B}(X,Y)$ is a general iv-grouping function. Moreover, let $\mathcal{N}(X) = \left[1-\overline{X}, 1-\underline{X}\right]$ be the iv-fuzzy negation. Hence, $\mathcal{I}_{\mathcal{GG}_B,\mathcal{N}}\colon L([0,1])^2 \to L([0,1])$ is a $(\mathcal{GG},\mathcal{N})$-implication given, for all $X,Y \in L([0,1])$, by:

$$\mathcal{I}_{\mathcal{GG}_B,\mathcal{N}}(X,Y) = \left[\min\left\{1, 2-\overline{X}^2-(1-\underline{Y})^2\right\}, \min\left\{1, 2-\underline{X}^2-(1-\overline{Y})^2\right\}\right]$$

Next proposition presents some properties of $(\mathcal{GG},\mathcal{N})$-implication functions.

Proposition 3.1: Let $\mathcal{GG}$ be a general iv-grouping function, $\mathcal{N}$ be an iv-fuzzy negation and $\mathcal{I}_{\mathcal{GG},\mathcal{N}}$ be a $(\mathcal{GG},\mathcal{N})$-implication function. Then:

(i) $\mathcal{I}_{\mathcal{GG},\mathcal{N}}$ satisfies (NP) if and only if $[0,0]$ is the neutral element of $\mathcal{GG}$;
(ii) $\mathcal{I}_{\mathcal{GG},\mathcal{N}}$ satisfies R-CP($\mathcal{N}$);
(iii) If $\mathcal{I}_{\mathcal{GG},\mathcal{N}}$ satisfies (NP), then $\mathcal{N}_{\mathcal{I}_{\mathcal{GG},\mathcal{N}}} = \mathcal{N}$;
(iv) If $[0,0]$ is the neutral element of $\mathcal{GG}$, then $\mathcal{N}_{\mathcal{I}_{\mathcal{GG},\mathcal{N}}} = \mathcal{N}$;
(v) If $\mathcal{GG}$ is associative, then $\mathcal{I}_{\mathcal{GG},\mathcal{N}}$ satisfies (EP);
(vi) If $\mathcal{N}$ is strong and $\mathcal{I}_{\mathcal{GG},\mathcal{N}}$ satisfies (EP), then $\mathcal{GG}$ is associative;
(vii) If $\mathcal{N}$ is strong, then $\mathcal{I}_{\mathcal{GG},\mathcal{N}}$ satisfies L-CP($\mathcal{N}$);
(viii) If $[0,0]$ is the neutral element of $\mathcal{GG}$ and $\mathcal{I}_{\mathcal{GG},\mathcal{N}}$ satisfies L-CP($\mathcal{N}$), then $\mathcal{N}$ is strong;
(ix) If $\mathcal{N}$ is strong, then $\mathcal{I}_{\mathcal{GG},\mathcal{N}}$ satisfies CP($\mathcal{N}$);
(x) If $\mathcal{I}_{\mathcal{GG},\mathcal{N}}$ satisfies CP($\mathcal{N}$) and $[0,0]$ is the neutral element of $\mathcal{GG}$, then $\mathcal{N}$ is strong,

where $\mathcal{N}_{\mathcal{I}_{\mathcal{GG},\mathcal{N}}}$ is the natural iv-fuzzy negation of $\mathcal{I}_{\mathcal{GG},\mathcal{N}}$.

Proof: Indeed, one has that:

(i) for all $Y \in L([0,1])$, $\mathcal{I}_{\mathcal{GG},\mathcal{N}}([1,1],Y) = Y \Leftrightarrow \mathcal{GG}(\mathcal{N}([1,1]),Y) = Y \overset{(\mathcal{N}2)}{\Leftrightarrow} \mathcal{GG}([0,0],Y) = Y \Leftrightarrow [0,0]$ is the neutral element of $\mathcal{GG}$.

(ii) For all $X,Y \in L([0,1])$, $\mathcal{I}_{\mathcal{GG},\mathcal{N}}(X,\mathcal{N}(Y)) = \mathcal{GG}(\mathcal{N}(X),\mathcal{N}(Y)) \overset{(\mathcal{GG}1)}{=} \mathcal{GG}(\mathcal{N}(Y),\mathcal{N}(X)) = \mathcal{I}_{\mathcal{GG},\mathcal{N}}(Y,\mathcal{N}(X))$. Therefore, $\mathcal{I}_{\mathcal{GG},\mathcal{N}}$ satisfies R-CP($\mathcal{N}$).

(iii) By item (ii), for all $X \in L([0,1])$, $\mathcal{N}_{\mathcal{I}_{\mathcal{GG},\mathcal{N}}}(X) = \mathcal{I}_{\mathcal{GG},\mathcal{N}}(X,[0,0]) = \mathcal{I}_{\mathcal{GG},\mathcal{N}}(X,\mathcal{N}([1,1])) \overset{\text{(ii)}}{=} \mathcal{I}_{\mathcal{GG},\mathcal{N}}([1,1],\mathcal{N}(X)) \overset{\text{(NP)}}{=} \mathcal{N}(X)$.

(iv) It is straightforward from items (i) and (iii).

(v) Since $\mathcal{GG}$ is associative, for all $X,Y,Z \in L([0,1])$:

$$\begin{aligned}
&\mathcal{I}_{\mathcal{GG},\mathcal{N}}(X,\mathcal{I}_{\mathcal{GG},\mathcal{N}}(Y,Z))\\
&= \mathcal{GG}(\mathcal{N}(X),\mathcal{GG}(\mathcal{N}(Y),Z))\\
&\overset{\text{Assoc.}}{=} \mathcal{GG}(\mathcal{GG}(\mathcal{N}(X),\mathcal{N}(Y)),Z)\\
&\overset{(\mathcal{GG}1)}{=} \mathcal{GG}(\mathcal{GG}(\mathcal{N}(Y),\mathcal{N}(X)),Z)\\
&\overset{\text{Assoc.}}{=} \mathcal{GG}(\mathcal{N}(Y),\mathcal{GG}(\mathcal{N}(X),Z))\\
&= \mathcal{I}_{\mathcal{GG},\mathcal{N}}(Y,\mathcal{I}_{\mathcal{GG},\mathcal{N}}(X,Z)).
\end{aligned}$$

Therefore, $\mathcal{I}_{\mathcal{GG},\mathcal{N}}$ satisfies (EP).

(vi) For all $X,Y,Z \in L([0,1])$:

$$\begin{aligned}
&\mathcal{GG}(X,\mathcal{GG}(Y,Z))\\
&\overset{(\mathcal{GG}1)}{=} \mathcal{GG}(X,\mathcal{GG}(Z,Y))\\
&\overset{\mathcal{N}\text{strong}}{=} \mathcal{GG}(\mathcal{N}(\mathcal{N}(X)),\mathcal{GG}(\mathcal{N}(\mathcal{N}(Z)),Y))\\
&= \mathcal{I}_{\mathcal{GG},\mathcal{N}}(\mathcal{N}(X),\mathcal{I}_{\mathcal{GG},\mathcal{N}}(\mathcal{N}(Z),Y))\\
&\overset{\text{(EP)}}{=} \mathcal{I}_{\mathcal{GG},\mathcal{N}}(\mathcal{N}(Z),\mathcal{I}_{\mathcal{GG},\mathcal{N}}(\mathcal{N}(X),Y))\\
&\overset{\mathcal{N}\text{strong}}{=} \mathcal{GG}(Z,\mathcal{GG}(X,Y))\\
&\overset{(\mathcal{GG}1)}{=} \mathcal{GG}(\mathcal{GG}(X,Y),Z).
\end{aligned}$$

Therefore, $\mathcal{GG}$ is associative.

(vii) Since $\mathcal{N}$ is a strong iv-fuzzy negation, for all $X,Y \in L([0,1])$:

$$\begin{aligned}
&\mathcal{I}_{\mathcal{GG},\mathcal{N}}(\mathcal{N}(X),Y)\\
&= \mathcal{GG}(\mathcal{N}(\mathcal{N}(X)),Y) = \mathcal{GG}(X,Y) \overset{(\mathcal{GG}1)}{=} \mathcal{GG}(Y,X)\\
&= \mathcal{GG}(\mathcal{N}(\mathcal{N}(Y)),X) = \mathcal{I}_{\mathcal{GG},\mathcal{N}}(\mathcal{N}(Y),X).
\end{aligned}$$

Therefore, $\mathcal{I}_{\mathcal{GG},\mathcal{N}}$ satisfies L-CP($\mathcal{N}$).

(viii) Since $[0,0]$ is the neutral element of $\mathcal{GG}$, we have that, for all $X \in L([0,1])$,

$$\begin{aligned}
X &= \mathcal{GG}([0,0],X) \overset{(\mathcal{N}2)}{=} \mathcal{GG}(\mathcal{N}(\mathcal{N}([0,0])),X)\\
&= \mathcal{I}_{\mathcal{GG},\mathcal{N}}(\mathcal{N}([0,0]),X) \overset{\text{(L-CP)}}{=} \mathcal{I}_{\mathcal{GG},\mathcal{N}}(\mathcal{N}(X),[0,0])\\
&= \mathcal{GG}(\mathcal{N}(\mathcal{N}(X)),[0,0]) = \mathcal{N}(\mathcal{N}(X)).
\end{aligned}$$

Therefore, $\mathcal{N}$ is strong.

(ix) Since $\mathcal{N}$ is strong, then:

$$\begin{aligned}
\mathcal{I}_{\mathcal{GG},\mathcal{N}}(\mathcal{N}(Y),\mathcal{N}(X)) &= \mathcal{GG}(\mathcal{N}(\mathcal{N}(Y)),\mathcal{N}(X))\\
&= \mathcal{GG}(Y,\mathcal{N}(X))\\
&\overset{(\mathcal{GG}1)}{=} \mathcal{GG}(\mathcal{N}(X),Y)\\
&= \mathcal{I}_{\mathcal{GG},\mathcal{N}}(X,Y).
\end{aligned}$$

Therefore, $\mathcal{I}_{\mathcal{GG},\mathcal{N}}$ satisfies CP($\mathcal{N}$).

(x) Since $\mathcal{I}_{\mathcal{GG},\mathcal{N}}$ satisfies CP($\mathcal{N}$), we have for all $Y \in L([0,1])$ and $X = [1,1]$, $\mathcal{I}_{\mathcal{GG},\mathcal{N}}(\mathcal{N}(Y),\mathcal{N}([1,1])) = \mathcal{I}_{\mathcal{GG},\mathcal{N}}([1,1],Y)$, i.e., $\mathcal{GG}(\mathcal{N}(\mathcal{N}(Y)),\mathcal{N}([1,1])) = \mathcal{GG}(\mathcal{N}([1,1]),Y)$, thus, by **($\mathcal{N}$2)**, $\mathcal{GG}(\mathcal{N}(\mathcal{N}(Y)),[0,0]) = \mathcal{GG}([0,0],Y)$. Since $[0,0]$ is the neutral element of $\mathcal{GG}$, $\mathcal{N}(\mathcal{N}(Y)) = Y$, for all $Y \in L([0,1])$. Hence, $\mathcal{N}$ is strong. ■

Proposition 3.2: Let $\mathcal{GG}$ be an associative general iv-grouping function and let $\mathcal{GG}_{[0,0]}\colon L([0,1]) \to L([0,1])$ be given by $\mathcal{GG}_{[0,0]}(Y) = \mathcal{GG}([0,0],Y)$, for all $Y \in L([0,1])$. Then, $\mathcal{GG}_{[0,0]}$ is a surjective function if and only if $\mathcal{GG}$ is an iv-t-conorm.

Proof: Assume that $\mathcal{GG}_{[0,0]}$ is surjective. By **($\mathcal{GG}$1)** and **($\mathcal{GG}$4)**, **($\mathcal{S}$1)** and **($\mathcal{S}$3)** are satisfied, respectively. Since $\mathcal{GG}$ is associative, we have that **($\mathcal{S}$2)** is verified. Again, by the associativity of $\mathcal{GG}$, for all $Z \in L([0,1])$, $\mathcal{GG}([0,0],\mathcal{GG}([0,0],Z)) = \mathcal{GG}(\mathcal{GG}([0,0],[0,0]),Z)$. So, by **($\mathcal{GG}$2)**:

$$\mathcal{GG}([0,0],\mathcal{GG}([0,0],Z)) = \mathcal{GG}([0,0],Z), \tag{3}$$

for all $Z \in L([0,1])$. Now, given any $Y \in L([0,1])$, as $\mathcal{GG}_{[0,0]}$ is surjective, there exists $Z \in L([0,1])$ such that $\mathcal{GG}([0,0],Z) = Y$. So, $\mathcal{GG}([0,0],Y) = \mathcal{GG}([0,0],\mathcal{GG}([0,0],Z)) \overset{\text{Eq.(3)}}{=} \mathcal{GG}([0,0],Z) = Y$. Thus, **($\mathcal{S}$4)** is also satisfied. Therefore, $\mathcal{GG}$ is an iv-t-conorm. The fact that if $\mathcal{GG}$ is an iv-t-conorm, then $\mathcal{GG}_{[0,0]}$ is a surjective function follows straight from **($\mathcal{S}$4)**. ∎

Next, we present some results considering that $\mathcal{N}$ is a crisp iv-fuzzy negation. We ensure, e.g., that one of the most important properties of fuzzy implication, the Exchange principle (EP), is satisfied by $(\mathcal{GG},\mathcal{N})$-implication functions.

Proposition 3.3: Let $\mathcal{I}_{\mathcal{GG},\mathcal{N}}\colon L([0,1])^2 \to L([0,1])$ be a $(\mathcal{GG},\mathcal{N})$-implication function where $\mathcal{N}\colon L([0,1]) \to L([0,1])$ is a crisp iv-fuzzy negation. Then: (i) $\mathcal{N}_{\mathcal{I}_{\mathcal{GG},\mathcal{N}}} = \mathcal{N}$; (ii) $\mathcal{I}_{\mathcal{GG},\mathcal{N}}$ satisfies (EP); (iii) If $[0,0]$ is the neutral element of $\mathcal{GG}$, then $\mathcal{I}_{\mathcal{GG},\mathcal{N}}$ satisfies (IB).

Proof: Take $\mathcal{I}_{\mathcal{GG},\mathcal{N}}(X,Y) = \mathcal{GG}(\mathcal{N}(X),Y)$. (i) Since $\mathcal{N}$ is crisp, then, $\mathcal{N}(X) = [0,0]$ or $\mathcal{N}(X) = [1,1]$. If $\mathcal{N}(X) = [0,0]$, then $\mathcal{N}_{\mathcal{I}_{\mathcal{GG},\mathcal{N}}}(X) = \mathcal{I}_{\mathcal{GG},\mathcal{N}}(X,[0,0]) = \mathcal{GG}(\mathcal{N}(X),[0,0]) = \mathcal{GG}([0,0],[0,0]) \overset{(\mathcal{GG}2)}{=} [0,0] = \mathcal{N}(X)$. Now, if $\mathcal{N}(X) = [1,1]$, therefore $\mathcal{N}_{\mathcal{I}_{\mathcal{GG},\mathcal{N}}}(X) = \mathcal{I}_{\mathcal{GG},\mathcal{N}}(X,[0,0]) = \mathcal{GG}(\mathcal{N}(X),[0,0]) = \mathcal{GG}([1,1],[0,0]) \overset{(\mathcal{GG}3)}{=} [1,1] = \mathcal{N}(X)$.

(ii) For all $X,Y \in L([0,1])$, since $\mathcal{N}$ is crisp, we have that $\mathcal{N}(X),\mathcal{N}(Y) \in \{[0,0],[1,1]\}$. So, one can consider the following two cases:

(1) Let $\mathcal{N}(X) = \mathcal{N}(Y)$. So, for all $Z \in L([0,1])$:

$$\begin{aligned}
&\mathcal{I}_{\mathcal{GG},\mathcal{N}}(X,\mathcal{I}_{\mathcal{GG},\mathcal{N}}(Y,Z)) = \\
&\quad = \mathcal{GG}(\mathcal{N}(X),\mathcal{GG}(\mathcal{N}(Y),Z)) \\
&\quad = \mathcal{GG}(\mathcal{N}(Y),\mathcal{GG}(\mathcal{N}(X),Z)) \\
&\quad = \mathcal{I}_{\mathcal{GG},\mathcal{N}}(Y,\mathcal{I}_{\mathcal{GG},\mathcal{N}}(X,Z)).
\end{aligned}$$

(2) Let $\mathcal{N}(X) \neq \mathcal{N}(Y)$. In this case, $\mathcal{N}(X) = [1,1]$ or $\mathcal{N}(Y) = [1,1]$. So, by **($\mathcal{GG}$3)**, for $\mathcal{N}(X) = [1,1]$,

$$\begin{aligned}
&\mathcal{I}_{\mathcal{GG},\mathcal{N}}(X,\mathcal{I}_{\mathcal{GG},\mathcal{N}}(Y,Z)) = \\
&\quad = \mathcal{GG}(\mathcal{N}(X),\mathcal{GG}(\mathcal{N}(Y),Z)) \\
&\quad = \mathcal{GG}([1,1],\mathcal{GG}(\mathcal{N}(Y),Z)) \\
&\quad = [1,1] \\
&\quad = \mathcal{GG}(\mathcal{N}(Y),[1,1]) \\
&\quad = \mathcal{GG}(\mathcal{N}(Y),\mathcal{GG}([1,1],Z)) \\
&\quad = \mathcal{GG}(\mathcal{N}(Y),\mathcal{GG}(\mathcal{N}(X),Z)) \\
&\quad = \mathcal{I}_{\mathcal{GG},\mathcal{N}}(Y,\mathcal{I}_{\mathcal{GG},\mathcal{N}}(X,Z)).
\end{aligned}$$

For $\mathcal{N}(Y) = [1,1]$, it can be similarly demonstrated.

Therefore, for any case, $\mathcal{I}_{\mathcal{GG},\mathcal{N}}$ satisfies (EP).

(iii) Again, since $\mathcal{N}$ is a crisp iv-fuzzy negation, then $\mathcal{N}(X) = [0,0]$ or $\mathcal{N}(X) = [1,1]$. If $\mathcal{N}(X) = [0,0]$, then, since $[0,0]$ is the neutral element of $\mathcal{GG}$,

$$\begin{aligned}
\mathcal{I}_{\mathcal{GG},\mathcal{N}}(X,\mathcal{I}_{\mathcal{GG},\mathcal{N}}(X,Y)) &= \mathcal{GG}(\mathcal{N}(X),\mathcal{GG}(\mathcal{N}(X),Y)) \\
&= \mathcal{GG}([0,0],\mathcal{GG}([0,0],Y)) \\
&= \mathcal{GG}([0,0],Y) \\
&= \mathcal{GG}(\mathcal{N}(X),Y) \\
&= \mathcal{I}_{\mathcal{GG},\mathcal{N}}(X,Y).
\end{aligned}$$

On the other hand, if $\mathcal{N}(X) = [1,1]$, then

$$\begin{aligned}
&\mathcal{I}_{\mathcal{GG},\mathcal{N}}(X,\mathcal{I}_{\mathcal{GG},\mathcal{N}}(X,Y)) = \\
&\quad = \mathcal{GG}(\mathcal{N}(X),\mathcal{GG}(\mathcal{N}(X),Y)) \\
&\quad = \mathcal{GG}([1,1],\mathcal{GG}([1,1],Y)) \\
&\quad \overset{(\mathcal{GG}3)}{=} [1,1] \\
&\quad = \mathcal{GG}([1,1],Y) \\
&\quad = \mathcal{GG}(\mathcal{N}(X),Y) \\
&\quad = \mathcal{I}_{\mathcal{GG},\mathcal{N}}(X,Y).
\end{aligned}$$

Therefore, in any case, $\mathcal{I}_{\mathcal{GG},\mathcal{N}}$ satisfies (IB). ∎

Lemma 3.1: Let $\mathcal{I}\colon L([0,1])^2 \to L([0,1])$ be a Moore continuous iv-fuzzy implication function. If $\mathcal{N}_{\mathcal{I}}\colon L([0,1]) \to L([0,1])$ is a strong iv-fuzzy negation and $\mathcal{I}$ satisfies L-CP($\mathcal{N}_{\mathcal{I}}$), then, for all $X,Y \in L([0,1])$:

$$\mathcal{GG}_{\mathcal{I}}(X,Y) = \mathcal{I}(\mathcal{N}_{\mathcal{I}}(X),Y) \tag{4}$$

is a general iv-grouping function.

Proof: For all $X,Y,Z \in L([0,1])$,

($\mathcal{GG}$1) $\mathcal{GG}_{\mathcal{I}}(X,Y) = \mathcal{I}(\mathcal{N}_{\mathcal{I}}(X),Y) \overset{\text{(L-CP)}}{=} \mathcal{I}(\mathcal{N}_{\mathcal{I}}(Y),X) = \mathcal{GG}_{\mathcal{I}}(Y,X)$.

($\mathcal{GG}$2) $\mathcal{GG}_{\mathcal{I}}([0,0],[0,0]) = \mathcal{I}(\mathcal{N}_{\mathcal{I}}([0,0]),[0,0]) \overset{(\mathcal{N}2)}{=} \mathcal{I}([1,1],[0,0]) \overset{(\mathcal{I}5)}{=} [0,0]$.

($\mathcal{GG}$3) $\mathcal{GG}_{\mathcal{I}}([1,1],Y) = \mathcal{I}(\mathcal{N}_{\mathcal{I}}([1,1]),Y) \overset{(\mathcal{N}2)}{=} \mathcal{I}([0,0],Y) \overset{(\mathcal{I}3)}{=} [1,1]$ and
$\mathcal{GG}_{\mathcal{I}}(X,[1,1]) = \mathcal{I}(\mathcal{N}_{\mathcal{I}}(X),[1,1]) \overset{(\mathcal{I}4)}{=} [1,1]$.

($\mathcal{GG}$4) If $X \leq_{Pr} Y$, then, by **($\mathcal{N}$1)**, $\mathcal{N}_{\mathcal{I}}(Y) \leq_{Pr} \mathcal{N}_{\mathcal{I}}(X) \overset{(\mathcal{I}1)}{\Rightarrow} \mathcal{I}(\mathcal{N}_{\mathcal{I}}(X),Z) \leq_{Pr} \mathcal{I}(\mathcal{N}_{\mathcal{I}}(Y),Z) \Rightarrow \mathcal{GG}_{\mathcal{I}}(X,Z) \leq_{Pr} \mathcal{GG}_{\mathcal{I}}(Y,Z)$.

($\mathcal{GG}$5) It follows from the Moore continuity of $\mathcal{I}$ and $\mathcal{N}_{\mathcal{I}}$.

Therefore, $\mathcal{GG}_{\mathcal{I}}$ is a general iv-grouping function. ∎

Example 3.2: Take the iv-fuzzy implication $\mathcal{I}_L\colon L([0,1])^2 \to L([0,1])$ defined, for all $X,Y \in L([0,1])$, by $\mathcal{I}_L(X,Y) = \left[\min\left(1, 1-\overline{X}+\underline{Y}\right), \min\left(1, 1-\underline{X}+\overline{Y}\right)\right]$. By the continuity of the real functions (minimum, sum and subtraction), $\mathcal{I}_L$ is clearly Moore continuous. Besides,

$$\begin{aligned}
\mathcal{N}_{\mathcal{I}_L}(\mathcal{N}_{\mathcal{I}_L}(X)) &= \left[1-(1-\underline{X}), 1-(1-\overline{X})\right] = X \text{ and} \\
\mathcal{I}_L(\mathcal{N}_{\mathcal{I}_L}(X),Y) &= \left[\min\left(1,\underline{X}+\underline{Y}\right), \min\left(1,\overline{X}+\overline{Y}\right)\right] \\
&= \left[\min\left(1,\underline{Y}+\underline{X}\right), \min\left(1,\overline{Y}+\overline{X}\right)\right] \\
&= \mathcal{I}_L(\mathcal{N}_{\mathcal{I}_L}(Y),X),
\end{aligned}$$

i.e., $\mathcal{N}_{\mathcal{I}_L}$ is a strong iv-fuzzy negation and $\mathcal{I}_L$ satisfies L-CP($\mathcal{N}_{\mathcal{I}_L}$). Thus, $\mathcal{GG}_{\mathcal{I}_L}$ is a general iv-grouping function given by $\mathcal{GG}_{\mathcal{I}_L}(X,Y) = \left[\min\left(1,\underline{X}+\underline{Y}\right), \min\left(1,\overline{X}+\overline{Y}\right)\right]$.

The next proposition provides a characterization for the class of $(\mathcal{GG},\mathcal{N})$-implication functions.

Theorem 3.2: Take a function $\mathcal{I}\colon L([0,1])^2 \to L([0,1])$. So, the following statements are equivalent:

(i) $\mathcal{I} = \mathcal{I}_{\mathcal{GG},\mathcal{N}}$ is a $(\mathcal{GG},\mathcal{N})$-implication function, where $\mathcal{N}$ is strong iv-fuzzy negation and $[0,0]$ is a neutral element of the general iv-grouping function $\mathcal{GG}$;

(ii) $\mathcal{I}$ is Moore continuous and satisfies **($\mathcal{I}$1)**, (NP) and L-CP($\mathcal{N}_{\mathcal{I}}$), where $\mathcal{N}_{\mathcal{I}}$ is a strong iv-fuzzy negation.

Proof: $(i) \Rightarrow (ii)$ By Prop. 3.1, $\mathcal{I}$ satisfies **($\mathcal{I}$1)**. Since $\mathcal{N}$ is strong, we have, by Prop. 2.1, that $\mathcal{N}$ is strict, thus $\mathcal{N}$ is Moore continuous. So, it follows from the Moore continuity of $\mathcal{GG}$ and $\mathcal{N}$ that $\mathcal{I}$ is also Moore continuous. Since $[0,0]$ is a neutral element of $\mathcal{GG}$, by Prop. 3.1(i), $\mathcal{I}_{\mathcal{GG},\mathcal{N}}$ satisfies (NP), and for all $X \in L([0,1])$,

$$\mathcal{N}_{\mathcal{I}}(X) = \mathcal{I}(X,[0,0]) = \mathcal{GG}(\mathcal{N}(X),[0,0]) = \mathcal{N}(X).$$

Therefore, $\mathcal{N}_{\mathcal{I}}$ is a strong iv-fuzzy negation, since $\mathcal{N}$ is a strong iv-fuzzy negation. So, by item (vii) of Prop. 3.1, L-CP($\mathcal{N}_{\mathcal{I}}$) holds.

$(ii) \Rightarrow (i)$ By **($\mathcal{N}$2)**, we have that, if $Y \leq_{Pr} Z$ then $\mathcal{N}_{\mathcal{I}}(Z) \leq_{Pr} \mathcal{N}_{\mathcal{I}}(Y)$. So,

$$\begin{aligned}
&\mathcal{I}(X,Y)\\
&\overset{\mathcal{N}_{\mathcal{I}}\text{strong}}{=} \mathcal{I}(\mathcal{N}_{\mathcal{I}}(\mathcal{N}_{\mathcal{I}}(X)),Y) \overset{\text{(L-CP)}}{=} \mathcal{I}(\mathcal{N}_{\mathcal{I}}(Y),\mathcal{N}_{\mathcal{I}}(X))\\
&\overset{(\mathcal{I}1)}{\leq_{Pr}} \mathcal{I}(\mathcal{N}_{\mathcal{I}}(Z),\mathcal{N}_{\mathcal{I}}(X)) \overset{\text{(L-CP)}}{=} \mathcal{I}(\mathcal{N}_{\mathcal{I}}(\mathcal{N}_{\mathcal{I}}(X)),Z)\\
&\overset{\mathcal{N}_{\mathcal{I}}\text{strong}}{=} \mathcal{I}(X,Z).
\end{aligned}$$

So, $\mathcal{I}$ satisfies **($\mathcal{I}$2)**. Now, since $\mathcal{I}$ satisfies (NP), $\mathcal{I}([1,1],[1,1]) = [1,1]$ and $\mathcal{I}([1,1],[0,0]) = [0,0]$. Therefore $\mathcal{I}$ verifies **($\mathcal{I}$4)*** and **($\mathcal{I}$5)**, respectively. Moreover, since $\mathcal{N}_{\mathcal{I}}$ is strong and $\mathcal{I}$ satisfies L-CP($\mathcal{N}_{\mathcal{I}}$), $\mathcal{I}([0,0],[0,0]) = \mathcal{I}(\mathcal{N}_{\mathcal{I}}(\mathcal{N}_{\mathcal{I}}([0,\ 0])),[0,0]) \overset{\text{(L-CP)}}{=} \mathcal{I}(\mathcal{N}_{\mathcal{I}}([0,0]),\mathcal{N}_{\mathcal{I}}([0,0])) = \mathcal{I}([1,1],[1,1]) = [1,1]$. Thus, $\mathcal{I}$ satisfies **($\mathcal{I}$3)***. Therefore, $\mathcal{I}$ is an iv-fuzzy implication function and since, by Prop. 2.1, $\mathcal{N}_{\mathcal{I}}$ is Moore continuous, then $\mathcal{I}$ is also Moore continuous. In addition, by Lemma 3.1, $\mathcal{GG}_{\mathcal{I}}(X,Y) = \mathcal{I}(\mathcal{N}_{\mathcal{I}}(X),Y)$ is a general iv-grouping function. So, for all $X,Y \in L([0,1])$

$$\begin{aligned}
&\mathcal{I}_{\mathcal{GG}_{\mathcal{I}},\mathcal{N}_{\mathcal{I}}}(X,Y)\\
&= \mathcal{GG}_{\mathcal{I}}(\mathcal{N}_{\mathcal{I}}(X),Y) = \mathcal{I}(\mathcal{N}_{\mathcal{I}}(\mathcal{N}_{\mathcal{I}}(X)),Y) \overset{\mathcal{N}_{\mathcal{I}}\text{strong}}{=} \mathcal{I}(X,Y).
\end{aligned}$$

Hence, $\mathcal{I}$ is a $(\mathcal{GG},\mathcal{N})$-implication. By Prop. 3.1(i), $[0,0]$ is a neutral element of $\mathcal{GG}_{\mathcal{I}}$, since $\mathcal{I}$ satisfies (NP), and so, by Prop. 3.1(iv), $\mathcal{N} = \mathcal{N}_{\mathcal{I}}$ is a strong iv-fuzzy negation. ■

Considering an iv-automorphism ϱ, in order to obtain the result proposed in Prop. 3.4 ahead, we need $\mathcal{GG}^{\varrho}$ and $\mathcal{N}^{\varrho}$ to be a general iv-grouping function and an iv-fuzzy negation, respectively. By Theorem 2.1, $\mathcal{N}^{\varrho}$ is an iv-fuzzy negation. So, in the next lemma, we show an analogous result for $\mathcal{GG}^{\varrho}$.

Lemma 3.2: Let $\mathcal{GG}\colon L([0,1])^2 \to L([0,1])$ be a general iv-grouping function and let $\varrho\colon L([0,1]) \to L([0,1])$ be an iv-automorphism. Then $\mathcal{GG}^{\varrho}\colon L([0,1])^2 \to L([0,1])$ is also a general iv-grouping function.

Proof: Let us verify if $\mathcal{GG}^{\varrho}$ satisfies all conditions of Def. 2.10. Since $\mathcal{GG}$ is a general iv-grouping function and, by Prop. 2.2, ϱ^{-1} is an iv-automorphism, for all $X,Y,Z \in L([0,1])$: **($\mathcal{GG}$1)** $\mathcal{GG}^{\varrho}(X,Y) = \varrho^{-1}(\mathcal{GG}(\varrho(X),\varrho(Y))) \overset{(\mathcal{GG}1)}{=} \varrho^{-1}(\mathcal{GG}(\varrho(Y),\varrho(X))) = \mathcal{GG}^{\varrho}(Y,X)$.

($\mathcal{GG}$2) If $X = Y = [0,0]$, therefore $\mathcal{GG}^{\varrho}(X,Y) = \varrho^{-1}(\mathcal{GG}(\varrho(X),\varrho(Y))) \overset{\text{Prop.2.3}}{=} \varrho^{-1}(\mathcal{GG}([0,0],[0,0])) \overset{(\mathcal{GG}1)}{=} \varrho^{-1}([0,0]) = [0,0]$.

($\mathcal{GG}$3) Let $X = [1,1]$ or $Y = [1,1]$. If $X = [1,1]$, then $\mathcal{GG}^{\varrho}(X,Y) = \varrho^{-1}(\mathcal{GG}(\varrho(X),\varrho(Y))) \overset{\text{Prop.2.3}}{=} \varrho^{-1}(\mathcal{GG}([1,1],\varrho(Y))) \overset{(\mathcal{GG}3)}{=} \varrho^{-1}([1,1]) = [1,1]$. For $Y = [1,1]$, it is analogous.

Since $\mathcal{GG}$, ϱ and ϱ^{-1} are $\leq_{Pr}$-increasing and Moore continuous (Prop. 2.2 and 2.3), then $\mathcal{GG}^{\varrho}$ satisfies **($\mathcal{GG}$4)** and **($\mathcal{GG}$5)**, respectively. Thus, $\mathcal{GG}^{\varrho}$ is a general iv-grouping function. ■

Now, we are able to generate a new interval-valued $(\mathcal{GG},\mathcal{N})$-implication function, from an iv-automorphism ϱ and an interval-valued $(\mathcal{GG},\mathcal{N})$-implication function.

Proposition 3.4: Let $\mathcal{I}\colon L([0,1])^2 \to L([0,1])$ be an iv-fuzzy implication function and $\varrho\colon L([0,1]) \to L([0,1])$ an iv-automorphism. Then, the following statements are equivalent.

(i) $\mathcal{I}$ is an interval-valued $(\mathcal{GG},\mathcal{N})$-implication function.

(ii) $\mathcal{I}^{\varrho}$ is an interval-valued $(\mathcal{GG},\mathcal{N})$-implication function.

Proof: $(i) \Rightarrow (ii)$ Consider $\mathcal{I} = \mathcal{I}_{\mathcal{GG},\mathcal{N}}$, where $\mathcal{GG}$ is a general iv-grouping function and $\mathcal{N}$ an iv-fuzzy negation. Then, for all $X,Y \in L([0,1])$,

$$\begin{aligned}
&\mathcal{I}^{\varrho}(X,Y)\\
&= \mathcal{I}^{\varrho}_{\mathcal{GG},\mathcal{N}}(X,Y) = \varrho^{-1}(\mathcal{I}_{\mathcal{GG},\mathcal{N}}(\varrho(X),\varrho(Y)))\\
&\overset{\text{(i)}}{=} \varrho^{-1}(\mathcal{GG}(\mathcal{N}(\varrho(X)),\varrho(Y)))\\
&= \varrho^{-1}(\mathcal{GG}(\varrho(\varrho^{-1}(\mathcal{N}(\varrho(X)))),\varrho(Y)))\\
&\overset{\text{Eq.(1)}}{=} \mathcal{GG}^{\varrho}(\varrho^{-1}(\mathcal{N}(\varrho(X))),Y) = \mathcal{GG}^{\varrho}(\mathcal{N}^{\varrho}(X),Y)\\
&= \mathcal{I}_{\mathcal{GG}^{\varrho},\mathcal{N}^{\varrho}}(X,Y).
\end{aligned}$$

Therefore, since $\mathcal{N}^{\varrho}$ is an iv-fuzzy negation and, by Lemma 3.2, $\mathcal{GG}^{\varrho}$ is a general iv-grouping function, then $\mathcal{I}^{\varrho}$ is a $(\mathcal{GG},\mathcal{N})$-implication function.

$(ii) \Rightarrow (i)$ Consider $\mathcal{I}^{\varrho} = \mathcal{I}_{\mathcal{GG},\mathcal{N}}$, where $\mathcal{GG}$ is a general iv-grouping function and $\mathcal{N}$ is an iv-fuzzy negation. Then, for all $X,Y \in L([0,1])$,

$$\begin{aligned}
&\mathcal{I}(X,Y)\\
&= \varrho(\varrho^{-1}(\mathcal{I}(\varrho(\varrho^{-1}(X),\varrho(\varrho^{-1}(Y))))))\\
&\overset{\text{Eq.(1)}}{=} \varrho(\mathcal{I}^{\varrho}(\varrho^{-1}(X),\varrho^{-1}(Y)))\\
&\overset{\text{(ii)}}{=} \varrho(\mathcal{I}_{\mathcal{GG},\mathcal{N}}(\varrho^{-1}(X),\varrho^{-1}(Y)))\\
&= \varrho(\mathcal{GG}(\mathcal{N}(\varrho^{-1}(X)),\varrho^{-1}(Y)))\\
&= \varrho(\mathcal{GG}(\varrho^{-1}(\varrho(\mathcal{N}(\varrho^{-1}(X)))),\varrho^{-1}(Y))) \overset{\text{Eq.(1)}}{=}\\
&= \mathcal{GG}^{\varrho^{-1}}(\mathcal{N}^{\varrho^{-1}}(X),Y) = \mathcal{I}_{\mathcal{GG}^{\varrho^{-1}},\mathcal{N}^{\varrho^{-1}}}(X,Y).
\end{aligned}$$

So, $\mathcal{I} = \mathcal{I}_{\mathcal{GG}^{\varrho^{-1}},\mathcal{N}^{\varrho^{-1}}}$, once $\mathcal{N}^{\varrho^{-1}}$ is an iv-fuzzy negation and, by Lemma 3.2, $\mathcal{GG}^{\varrho^{-1}}$ is a general iv-grouping function. Hence, $\mathcal{I}$ is a $(\mathcal{GG},\mathcal{N})$-implication function. ■

IV. Conclusions

In this work, considering the contributions of grouping functions to several application areas, we focused on the development of a more flexible class of fuzzy implication functions, which also consider the modeling of uncertainty, namely, the interval-valued fuzzy material implications, called $(\mathcal{GG},\mathcal{N})$-implication functions. We also studied several properties and provided the characterization.

Ongoing work consider the study of the other implication functions derived from general iv-overlap and grouping functions, e.g., the residual, the quantum logic and the Dishkant implication functions, and their intersections.

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