League competitions and fairness

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Abstract

We formulate two fairness principles and characterize the league competition systems that satisfy them. The first principle requires that all players should have the same chance of being the final winner if all players are equally strong, while the second states that the league competition should not favor weaker players. We apply these requirements to a class of systems which includes round-robin tournaments as a particular case.

1 Introduction

One of the most popular problems when it comes to select an alternative from a set of available options is probably that of selecting a winner in a sports competition. In this paper we present an axiomatic approach to the fairness aspect of such selection problems.

Any precise definition of a competition system entails answering the following basic questions (cf. Haigh 2009): Who the players in a match are, at what stage of the competition this match takes place, and how the final winner is decided. In general,

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competition rules may be set with many different objectives in mind, such as match intensity, attracting the interest of the spectators (cf. Dagaev and Suzdaltsev 2018), minimizing organizational costs, and so on. Fairness is, or should be, certainly one the main goals of any competition designer. Moreover, as outlined in Bartsch et al. (2006), in real world-sport leagues, fairness issues are quite important.

In a league competition, every player participates in a given number of matches against other players. A certain number of points is assigned to the winner of each match, and the final winner is the player with the most points.

The most widely used leagues are "single round-robin tournaments", where each player plays against every other player once, and "double round-robin tournaments", where each player plays against every other player twice (cf. Rasmussen and Trick 2008). Regular seasons in most sports have a league structure. However, there are well-known examples of leagues that are not round-robin tournaments, e.g. the US National Football League (NFL), where each team plays 16 matches in the regular season but does not play the same number of times against every opponent and does not play at all against some teams. In the US National Basketball Association (NBA) each team plays 82 matches in the regular season and plays against every other team, but does not play the same number of matches against every opponent.

Leagues where different players play different number of matches are more difficult to find. However we believe that this possibility should not be disregarded, at least at the theoretical stage. The same principle that supports giving byes in knockout tournaments (favoring more meritorious participants) could be applied to leagues, so that certain players gain an advantage by playing more matches. It should be noted that such an advantage is given if points are assigned only for winning matches, though there are exceptions to this rule. For instance, a bye can be awarded the same number of points as for winning a match in Swiss-system tournaments with an odd number of players.[1]

We nevertheless include the mentioned theoretical possibility of gaining by playing more matches in our formal model, if only to investigate its implications. Recent proposals to change standard round-robin domestic leagues in football also support the pertinence of such a more general analysis (cf. Feehely 2021). We refer the reader to Lasek and Gagolewski (2018) for an overview of tournament formats used in the majority of European top-tier association football competitions as well as to Csató (2020) for the relationship between the existence of ground-robin groups of different strength in tournaments and the quality of matches in the corresponding competitions.

Many of the discussions about fairness in competition systems are based on rather informal arguments. In this paper we offer a formal setup within which such discussions can be framed and precisely define two fairness principles that are commonly pursued in real practice. The first principle requires a competition system not to favor weaker players (we refer to this as "monotonicity in strength"), and the second requires that all players should have the same chance of being the final winner if all players are equally strong ("equal treatment"). Our main purpose is thus to study the extent to which league-type competition systems can be considered fair according to these principles.

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[1] We thank an anonymous referee for drawing our attention to this issue.

Literature overview

There are several papers on modeling and studying fairness in league-type competition systems. One possibility is to focus on the influence of scheduling in sequential round-robin tournaments (cf. Durán et al. 2017, Krumer and Lechner 2017, Sahm 2019), including carry-over effects (cf. Russell 1980, Lambrechts et al. 2018) or on the requirement that a team should not play against extremely weak or extremely strong teams in consecutive periods (cf. Briskorn 2009, Briskorn and Knust 2010, Nemhauser and Trick 1998). Moon and Pullman (1970) concentrate on "equalizing" handicapping methods. Rubinstein (1980) shows axiomatically that the points system used in round-robin competitions is the only one that satisfies three axioms inspired by social choice theory. Levin and Nalebuff (1995) draw an analogy between round-robin points systems and voting systems.

Fairness in sports has also been analyzed with respect to tie-breaking mechanisms within the field of economics (cf. Apesteguía and Palacios-Huerta 2010and Che and Hendershott 2008). Most studies that compare competition systems apply simulation techniques to check that particular properties are satisfied (cf. Appleton 1995, McGarry and Schutz 1997, Scarf et al. 2009, Ryvkin and Ortmann 2008 and Ryvkin 2010). There is also a sizeable number of papers in operations research related to sports that focus on topics other than fairness and thus lie outside the scope of this study. Readers are referred to Csató (2021), Kendall and Lenten (2017), Kendall et al. (2010), Lenten and Kendall (2021),and Wright (2014) for surveys of this literature.

Our contribution

We start our analysis by formulating two basic fairness axioms. The first is an "equal treatment" requirement which states that "if all players are equally strong then all should have the same probability of being the final winner". The second is a "monotonicity in strength" condition requiring that "a weaker player should not have a higher probability of being the final winner than a stronger player".

Each of these axioms is presented in a strong form and in a weak form. The strong versions impose that a league-type competition should fulfill the property for every possible assignment of the players, while the weak forms only require that the property be fulfilled for at least one such assignment. The effect of these weak versions of the axioms was also studied in Arlegi and Dimitrov (2020) and Arlegi (2022). However, the analysis in these works is set within the framework of elimination-type competitions (knockout tournaments) which is completely different from the model of league-type competitions we develop and study in the current paper. In particular, the corresponding characterizations for the case of elimination-type competitions are heavily based on both the tree structure of the knockout tournaments and on the specific way in which the sequential structure of the competition determines players' winning probabilities.

In contrast, we study in the current paper the other main class of competitions (league-type competitions) and focus exclusively on the effect of the *assignment* of the players, i.e. on who is playing against whom and how many times, seeking to isolate this effect from others related to scheduling when the competition is structured sequentially. To that end, we investigate what type of league-type competitions satisfies the said fairness properties. Generally speaking, *equal treatment* leads to competitions where every player plays the same number of matches (Theorems 1 and

2), while *monotonicity in strength* drastically restricts the number of participants to two (Theorem 3). When *weak monotonicity in strength* is considered, the class of competitions that fulfills the property increases (Theorem 4).

The rest of the work is organized as follows. Section 2 presents the basic elements of our framework and introduces the fairness axioms. Section 3 sets out our results. Section 4 concludes and addresses possible extensions of the model. The proofs are relegated to the Appendix.

2 Framework

Let N be a finite set of at least two players who compete with each other by playing a total number s of matches. An assignment $\mathbf{A}$ is a symmetric matrix $\mathbf{A} = (a_{ij})_{i,j \in N}$ with a_{ij} being the number of matches played between players i and j. Thus, $\mathbf{A}$ determines the participants in each of the s matches, with $(\Sigma_{i,j \in N} a_{ij})/2 = s$. For example, for $N = \{1, 2, 3, 4\}$, the assignment $\mathbf{A}$ defined by $a_{ii} = 0$ and $a_{ij} = 1$ for all $i, j \in N$, $i \neq j$, describes a single round-robin tournament between four players with the set of $s = 6$ matches being played $\{(1, 2), (1, 3), (1, 4), (2, 3), (2, 4), (3, 4)\}$. As shown below, our definition of a league allows (and the corresponding results account) for the possibility of players playing different numbers of matches.

The rest of this section describes the main ingredients of our model. These include the ordinal information about each player's relative strengths represented by a binary relation R over the player set N and the set of "winning probability matrices" that are consistent with R (Subsection 2.1), plus the notion of an assignment and the probability of each player being a final winner of a league competition (Subsection 2.2). Subsection 2.3 presents the fairness axioms, making use of all the previous formal elements.

2.1 Players' strength and winning probabilities

We follow Arlegi and Dimitrov (2020) and assume that the members of the player set N are completely ordered according to a binary relation R of *strength* so that, for all $i, j \in N$, iRj is interpreted as *"player i is at least as strong as player j"*. The corresponding asymmetric and symmetric factors of R are denoted by P and I respectively, so iPj reads *"i is strictly stronger than j "* and iIj reads *"i and j are equally strong"*.[2]

Now assume that information is available about the probabilities of one player defeating another in a match. For $i, j \in N$, denote by p_{ij} the corresponding probability and note the following natural connection between R and these probabilities: A player i is at least as strong as another player j if and only if $p_{ij} \geq 0.5$. Since the binary relation of strength is complete, iPj means $p_{ij} > 0.5$ and iIj implies $p_{ij} = 0.5$. For

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[2] In practice, the relation of strength is usually based on criteria such as previous performance, betting odds, etc. Thus, the assumption of the existence of such a relation does not seem to be restrictive. See Prince et al. (2013) for *assuming* that the probability of winning is proportional to the players' powers.

convenience, in what follows we order the players in N according to R; that is, if $i\,P\,j$ then $i < j$ (if $i\,I\,j$ then either $i < j$ or $j < i$).

We denote by $\mathcal{P}_R$ the set of all probability matrices such that, for $\mathbf{P} \in \mathcal{P}_R$, $p_{ij} \geq 0.5$ if and only if $i\,R\,j$. Moreover, we assume that each probability matrix in $\mathcal{P}_R$ satisfies the following two conditions:

$$\forall i, j \in N, \ p_{ij} + p_{ji} = 1. \tag{1}$$

$$\forall i, j \in N, \ p_{ij} \geq 0.5 \text{ implies } p_{ik} \geq p_{jk} \text{ for each } k \in N \setminus \{i, j\}. \tag{2}$$

These conditions are commonly used in the literature (cf. David 1963, Hwang 1982, Horen and Riezman 1985, Schwenk 2000) and taken together they are equivalent to the "strong stochastic transitivity" of the corresponding probability matrix (cf. David 1963). The crucial condition here is Condition (2), which states that any player is more likely to beat a weaker player than a stronger one. It is worth mentioning that, for a fixed binary relation of strength R, our fairness properties are imposed to hold for each $\mathbf{P} \in \mathcal{P}_R$, so that no further details on these probability matrices are needed for our results.

Finally, it is easy to see that the strong stochastic transitivity of the probability matrices in $\mathcal{P}_R$ implies that the binary relation of strength R is transitive. Moreover, for $i, j, k, \ell \in N$ the following useful interval property connecting the strength of these players to their corresponding winning probabilities emerges:

$$i\,R\,j\,R\,k\,R\,\ell \text{ implies } p_{i\ell} \geq p_{jk}. \tag{3}$$

2.2 Assignments and final winners

Given a league competition (s, N), an *assignment of size* $s \in \mathbb{Z}^+$ is a symmetric matrix $\mathbf{A} = \left(a_{ij}\right)_{i,j\in N}$ with $a_{ij} \in \mathbb{Z}^+ \cup \{0\}$ for all $i, j \in N$ being the number of matches played between players i and j such that:

- $a_{ii} = 0$ for each $i \in N$ (no player plays against herself);
- $\Sigma_{j\in N} a_{ij} > 0$ for each $i \in N$ (each player plays at least one match);
- $2 \leq |N| \leq \Sigma_{i,j\in N} a_{ij} = 2s$ (the total number of matches is s).

The set of all assignments for N of size s satisfying the above conditions is denoted by $\mathcal{A}^{(s,N)}$. We assume that player $i \in N$ wins the competition (s, N) at assignment $\mathbf{A} \in \mathcal{A}^{(s,N)}$ if no other player wins more matches than i from the matches induced by $\mathbf{A}$. Given a probability matrix $\mathbf{P}$ containing the probabilities of a player winning a match, we denote by $\varphi_i(\mathbf{A}, \mathbf{P})$ the probability that player $i \in N$ wins the competition. The exact mathematical expression of $\varphi_i(\mathbf{A}, \mathbf{P})$ is provided in the Appendix and it should be noted that it does not use tie-breaking criteria (cf. Berker 2014 and Csató 2021), i.e., multiple final winners are allowed.

2.3 Fairness axioms

We now formally introduce the two fairness principles (cf. Arlegi and Dimitrov 2020). Each of these ideas is presented in a strong form and in a weak form. To state

them, we assume that a binary relation R of strength is defined on the player set N.

Equal Treatment (ET) A league competition (s, N) satisfies ET if for all $\mathbf{A} \in \mathcal{A}^{(s,N)}$ we have $\varphi_i(\mathbf{A}, \mathbf{P}) = \varphi_j(\mathbf{A}, \mathbf{P})$ holding for all $i, j \in N$ whenever $\mathbf{P} \in \mathcal{P}_R$ is such that $p_{ij} = 0.5$ for all $i, j \in N$.

Weak Equal Treatment (WET) A league competition (s, N) satisfies WET if there exists $\mathbf{A} \in \mathcal{A}^{(s,N)}$ with $\varphi_i(\mathbf{A}, \mathbf{P}) = \varphi_j(\mathbf{A}, \mathbf{P})$ holding for all $i, j \in N$ whenever $\mathbf{P} \in \mathcal{P}_R$ is such that $p_{ij} = 0.5$ for all $i, j \in N$.

Monotonicity in Strength (MS) A league competition (s, N) satisfies MS if for all $\mathbf{A} \in \mathcal{A}^{(s,N)}$, for all $i, j \in N$, and for all $\mathbf{P} \in \mathcal{P}_R$ such that $p_{ij} > 0.5$, $\varphi_i(\mathbf{A}, \mathbf{P}) \geq \varphi_j(\mathbf{A}, \mathbf{P})$ holds.

Weak Monotonicity in Strength (WMS) A league competition (s, N) satisfies WMS if there exists $\mathbf{A} \in \mathcal{A}^{(s,N)}$ such that, for all $i, j \in N$ and for all $\mathbf{P} \in \mathcal{P}_R$ such that $p_{ij} > 0.5$, $\varphi_i(\mathbf{A}, \mathbf{P}) \geq \varphi_j(\mathbf{A}, \mathbf{P})$ holds.

ET and WET express the idea that in regard to the final probability of winning, the competition system should not be biased towards any particular player if all of them are equally skilled.

MS and WMS require the competition system not to reward weaker players under any of the possible probability matrices compatible with the strength of the players. In fact, many tournaments are precisely designed to minimize the role of luck in winning the competition: For example, round-robin tournaments and even double round-robin tournaments minimize that effect by staging a large number of matches, and in knockout tournaments the strongest player is often matched with the weakest, the second strongest with the second weakest, and so on.

Apart from the fact that WET is logically weaker than ET and WMS is logically weaker than MS, the normative power of the weaker versions compared to the strong versions may depend on the specific application intended, and in particular on the conjectures as to the benevolence of the competition designer. On the one hand, ET and MS prevent manipulation by a potentially corrupted competition designer because they ensure that there exists no assignment rule benefiting a particular player in relation to another one who is more or equally skilled. On the other hand, WET and WMS rely on confidence in the benevolence of the competition designer, in the sense that the focus is on competition systems where it is always possible to find an assignment rule that is fair, independently of the values in the probability matrices supporting the strength relation.

In formulating the monotonicity properties we were guided by the general view that the information on the exact values of the probabilities of winning might be rather imprecise. A consistent way of incorporating such a view into the corresponding requirements is to require them to hold *for all* probability matrices. Clearly, MS and WMS would then guarantee fairness for all possible probability values which are compatible with the binary relation of strength R. This is also important from a practical perspective as the rules used for assignment in most professional sports playoffs are usually proxies of R obtained on the basis of end-of-season standings. Hence, it is important to know whether the corresponding competition structure is fair independently of the precise numerical scores that describe the strength of the teams at the end of the regular season.

3 Results

3.1 Leagues and equal treatment

Whether the two equal treatment axioms (ET and WET) are fulfilled by league competitions is closely related to the fact that players should play the same number of matches. The first result in this section shows that a league satisfies ET if and only if either each player plays a single match or only two players play all matches. Clearly, both cases imply that there must be an even number of players participating in the competition.

Theorem 1 *A league competition* (s, N) *satisfies ET if and only if either* $2s = |N|$ *or* $2s > |N| = 2$

It is sometimes suggested that leagues are fair competition systems. However, this is not true in our framework because our definition of league is much broader than the sense in which the term is popularly used, which is usually identified with round-robin tournaments. In fact, the theorem given above excludes leagues as fair structures unless every player plays exactly one match or there are only two players. We now show that weakening ET to WET extends the class of fair league competitions by including those competitions where each player plays the same number of matches against every other player.

Theorem 2 *Any league competition* (s, N) *with either* $2s = |N|$ *or* $2s = k(|N| - 1)|N|$ *for some integer* $k \geq 1$ *satisfies WET.*

It should be noted that Theorem 2 is not superfluous in the sense that not every league competition satisfies WET. For example, it can easily be checked that there is no way to assign three players to a two-match competition so that WET is fulfilled.

3.2 Leagues and monotonicity in strength

Our next result shows that MS restricts the number of players in a competition to only two.

Theorem 3 *A league competition* (s, N) *satisfies MS if and only if* $|N| = 2$

When MS is weakened to WMS, the class of fair competition systems is considerably enlarged because any league competition turns out to satisfy this axiom provided that either each player participates in exactly one match or the total number of matches is at least $(|N| - 1)$. In particular, this implies that any league competition covered by Theorem 2 satisfies the weak versions of both the equal treatment and monotonicity in strength properties.

Theorem 4 *Any league competition* (s, N) *with* $2s = |N|$ *or* $s \geq |N| - 1$ *satisfies WMS.*

It is not difficult to prove that the assignment used in the proof of Theorem 4 (see the Appendix) when $2s = |N|$ is the only one for which (s, N) satisfies WMS. However,

there are assignments other than those used when $s \geq |N| - 1$ holds for which (s, N) satisfies WMS.

Finally, as in Theorem 2, it should be noted that Theorem 4 is not superfluous in the sense that there exist league competitions that do not satisfy WMS. For example, it can be proved that a league competition consisting of three matches and five players does not satisfy WMS.

3.3 Round-robin tournaments

Let $k \geq 1$ be an integer. A *k-round-robin tournament* is a league competition (s, N) *together* with an assignment $\mathbf{A} \in \mathcal{A}^{(s,N)}$ satisfying the additional condition $a_{ij} = k$ for all $i, j \in N, i \neq j$. That is, according to $\mathbf{A}$, each player in N plays exactly k matches against every other player in N. Clearly then, a single round-robin tournament requires $k = 1$, while a double round-robin competition implies $k = 2$. Notice finally that, by $\mathbf{A} \in \mathcal{A}^{(s,N)}$, $2\,s = k\,(|N| - 1)\,|N|$ follows.

Given the way we introduced our axioms and the fact that a round-robin tournament is defined with respect to a specific assignment, it makes sense to check whether such a competition system satisfies WET and WMS only i.e. the relevant results are to be derived from Theorem 2 and Theorem 4.

It follows directly from Theorem 2 that the first condition in the above definition of a k-round-robin tournament is satisfied. In the proof of Theorem 2, we show further that a league competition (s, N) with $2\,s = k\,(|N| - 1)$ satisfies WET with respect to $\mathbf{A} \in \mathcal{A}^{(s,N)}$ letting each player participate in exactly k number of matches against every other player. Thus, we conclude that every k-round-robin tournament satisfies WET.

As for WMS, Theorem 4 first tells us that any single round-robin tournament with two players playing exactly one match fulfills WMS. This follows from $2s = |N|$ which, in order to be satisfied for a k-round-robin tournament (with $2\,s = k\,(|N| - 1)\,|N|$), requires $k = 1$ and $|N| = 2$. Further recall that, according to Theorem 4, any league competition where the total number of matches is at least $(|N| - 1)$ also satisfies WMS. Since the latter condition is necessary for a league competition to be a round-robin tournament, one might be tempted to conclude that round-robin tournaments also satisfy WMS. It should be noted though, that the assignment used here to prove the fulfillment of WMS in this case *does not match* the one in the definition of a round-robin tournament. It is highly likely that league competitions (s, N) with $2\,s = k\,(|N| - 1)\,|N|$ for some integer $k \geq 1$ do generally satisfy WMS with respect to the round-robin tournament assignment as defined above. However, providing a formal proof is much more complex than what it may appear at first sight.[3]

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[3] See Saarinen et al. (2015) for a brief impression of the computational complexity of calculating the probability of a given player finally winning a round-robin tournament, even if all values in the probability matrix belong to $\{0, 1/2, 1\}$. See also Kulhanek and Ponomarenko (2020) about the complexity of computing the probability of winning in knockout tournaments and the possibility of counterintuitive outcomes.

4 Conclusion

The results reported in this paper enable different league-type competition systems to be compared in terms of their fairness. In particular, we have shown that there are not many league-type competitions that are *fair* in the sense of *simultaneously* satisfying both types of fairness in their corresponding strong or weak forms. MS combined with either ET or WET produces a degenerate competition consisting of a two-player league (Theorem 3). Weakening MS to WMS and imposing it together with ET slightly enlarges the class of admissible leagues to those where each participant plays a single match. In contrast, the combination of WMS and WET results in an expansion of the class of admissible leagues to include leagues that allow players to play the same number of times in at least $|N| - 1$ matches (Theorems 2 and 4). An interesting direction for future research in that respect could be the quantification of the extent to which these properties are violated in certain competition formats. A first attempt in that general research direction was made in Csató (2022), where simulation techniques are used for an investigation of monotonicity and incentive compatibility properties of sports tournaments.

Theorem 2 proves that k-round-robin tournaments (which include single and double round-robin tournaments as special cases) satisfy WET. It seems quite likely that also other league-type competitions do satisfy WET; for instance, one could consider the assignment recently adopted by the UEFA Champions League from the 2024/25 season (cf. UEFA 2022), where each player plays the same number of matches against the same number of other players (not necessarily against every other player).[4] Most likely the mentioned assignments also satisfy WMS, although the formal proof remains an open question even if one restricts probabilities to take two possible values; that is, if $p_{ij} = \alpha \in [0.5, 1]$ for all $i, j \in N$ with $i < j$.

Our analysis in this paper does not cover Swiss-system tournaments as the assignment in the later class of tournaments, in contrast to our definition of league-type competitions, is not fixed a priori but it rather depends on the results of the matches already played. The possibility for such assignment path-dependence leads to very interesting operational research problems as how to pair the players (cf. Biró et al. 2017, Führlich et al. 2021, Kujansuu et al. 1999, Ólafsson 1990) or how to generate a players' ranking at the end of the tournament (cf. Csató (2013, 2017)).

Finally, it is worth mentioning that the strong stochastic transitivity of the probability matrices plays a crucial role in the results obtained. Any potential weakening of this condition would clearly enlarge the set of matrices that satisfy it and thus leads to a narrower class of fair competition systems.

Data Availability Not applicable.

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[4] We thank an anonymous referee for suggesting this assignment type as well as for sharing the corresponding reference.

Declarations

Conflict of interest The authors have no relevant financial or non-financial interests to disclose.

Appendix: Proofs of the theorems

We start by introducing some additional notation and remarks. Given a competition (s, N) and an assignment $\mathbf{A} \in \mathcal{A}^{(s,N)}$, $M(\mathbf{A})$ stands for the set of matches induced by $\mathbf{A}$. No ties are allowed and the player who wins the match $m \in M(\mathbf{A})$ is denoted by $w_m(\mathbf{A})$. Player $i \in N$ is a winner of the competition (s, N) at $\mathbf{A} \in \mathcal{A}^{(s,N)}$ if $|\{m \in M(\mathbf{A}) : w_m(\mathbf{A}) = i\}| \geq |\{m \in M(\mathbf{A}) : w_m(\mathbf{A}) = j\}|$ holds for all $j \in N$. Thus, given $\mathbf{A} \in \mathcal{A}^{(s,N)}$, $w(\mathbf{A}) = (w_m(\mathbf{A}))_{m \in M(\mathbf{A})}$ stands for the vector of match winners, whose dimension is s. Given $\mathbf{A} \in \mathcal{A}^{(s,N)}$ and a probability matrix $\mathbf{P}$, the probability $pr(w(\mathbf{A}))$ of $w(\mathbf{A})$ occurring is given by $pr(w(\mathbf{A})) = \Pi_{m \in M(\mathbf{A})}\varphi_{w_m(\mathbf{A})}(\mathbf{A}, \mathbf{P})$. Thus, for each $i \in N$, $\varphi_i(\mathbf{A}, \mathbf{P})$ can be expressed by $\varphi_i(\mathbf{A}, \mathbf{P}) = \sum_{w(\mathbf{A}) \in W_i(\mathbf{A})} pr(w(\mathbf{A})) = \sum_{w(\mathbf{A}) \in W_i(\mathbf{A})} \prod_{m \in M(\mathbf{A})} \varphi_{w_m(\mathbf{A})}(\mathbf{A}, \mathbf{P})$, where $W_i(\mathbf{A})$ stands for the set of all vectors of winners in the matches induced by $\mathbf{A}$ for which i is a final winner of the competition. Additionally, when the probability matrix $\mathbf{P}$ is such that $p_{ij} = 0.5$ holds for all $i, j \in N$, we have $pr(w(\mathbf{A})) = (0.5)^s$ for each vector of match winners and thus, $\varphi_i(\mathbf{A}, \mathbf{P}) = (0.5)^s \cdot |W_i(\mathbf{A})|$ holding for each $i \in N$.

Proof of Theorem 1 Let (s, N) be a league competition with either $2s = |N|$ or $2s > |N| = 2$. We show first that (s, N) satisfies ET. Let $\mathbf{A} \in \mathcal{A}^{(s,N)}$ be an assignment and $\mathbf{P}$ the probability matrix with $p_{ij} = 0.5$ for all $i, j \in N$. Notice then that either each player in N plays exactly one match (when $2s = |N|$) or there are only two players who play all matches (when $2s > |N| = 2$). Consider the following cases separately. *Case 1* ($2s > |N| = 2$ and s is odd). Let $N = \{1, 2\}$ and notice that in this case we have $\varphi_1(\mathbf{A}, \mathbf{P}) = (0.5)^s \cdot |W_1(\mathbf{A})|$ and $\varphi_2(\mathbf{A}, \mathbf{P}) = (0.5)^s \cdot |W_2(\mathbf{A})|$. Because s is odd, $w(\mathbf{A}) \in W_1(\mathbf{A})$ implies $w(\mathbf{A}) \notin W_2(\mathbf{A})$, and $w(\mathbf{A}) \in W_2(\mathbf{A})$ implies $w(\mathbf{A}) \notin W_1(\mathbf{A})$. Thus, $W_1(\mathbf{A}) \cap W_2(\mathbf{A}) = \emptyset$. Finally, given that $N = \{1, 2\}$ and $W_1(\mathbf{A}) \cap W_2(\mathbf{A}) = \emptyset$, we can define a bijection $f : W_1(\mathbf{A}) \to W_2(\mathbf{A})$ by just replacing 1 by 2 and 2 by 1 as match winners at each $w(\mathbf{A}) \in W_1(\mathbf{A})$ as to get $f(w(\mathbf{A})) \in W_2(\mathbf{A})$. Thus $|W_1(\mathbf{A})| = |W_2(\mathbf{A})|$ and $\varphi_1(\mathbf{A}, \mathbf{P}) = \varphi_2(\mathbf{A}, \mathbf{P})$ holds. We conclude then that (s, N) satisfies ET.
Case 2 ($2s > |N| = 2$ and s is even). Notice that in this case $W_1(\mathbf{A}) \cap W_2(\mathbf{A}) \neq \emptyset$. We have then that $w(\mathbf{A}) \in W_1(\mathbf{A}) \setminus W_2(\mathbf{A})$ implies $w(\mathbf{A}) \notin W_2(\mathbf{A}) \setminus W_1(\mathbf{A})$, and $w(\mathbf{A}) \in$

$W_2(\mathbf{A}) \backslash W_1(\mathbf{A})$ implies $w(\mathbf{A}) \notin W_1(\mathbf{A}) \backslash W_2(\mathbf{A})$. Finally, we can define a bijection $f : W_1(\mathbf{A}) \backslash W_2(\mathbf{A}) \rightarrow W_2(\mathbf{A}) \backslash W_1(\mathbf{A})$ in the same way as in Case 1 and conclude that $|W_1(\mathbf{A}) \backslash W_2(\mathbf{A})| = |W_2(\mathbf{A}) \backslash W_1(\mathbf{A})|$ should follow and thus, $|W_1(\mathbf{A})| = |W_2(\mathbf{A})|$ also holds. We conclude that $\varphi_1(\mathbf{A}, \mathbf{P}) = \varphi_2(\mathbf{A}, \mathbf{P})$ and thus (s, N) satisfies ET also in this case.

Case 3 $(2s = |N|)$. Because each player plays exactly one match in this case, for any vector $w(\mathbf{A})$ of match winners, a player is the final winner of the competition only if she is the winner of the match which she is assigned by $\mathbf{A}$. Thus, $|W_i(\mathbf{A})| = |W_j(\mathbf{A})|$ holds for all $i, j \in N$. By $\varphi_i(\mathbf{A}, \mathbf{P}) = (0.5)^s \cdot |W_i(\mathbf{A})|$ holding for each $i \in N$, (s, N) satisfies ET.

Assume now that (s, N) is a league competition satisfying ET. To show that either $2s = |N|$ or $2\,s > |N| = 2$ holds, let $\mathbf{P}$ be the probability matrix with $p_{ij} = 0.5$ for all $i, j \in N$. It suffices to show that, for some assignment in $\mathcal{A}^{(s,N)}$, $2\,s > |N| > 2$ leads to a contradiction (as $|N| \leq 2\,s$ follows from (s, N) satisfying ET and the definition of an assignment). Consider then the following possible cases.

Case 1 ($|N|$ is even and $2s = |N| + 2$). Let the assignment $\mathbf{A}$ be such that $a_{12} = 2$ and $\Sigma_{j \in N \backslash \{1,2\}} a_{ij} = 1$ for all $i \in N \backslash \{1, 2\}$, and notice that $\mathbf{A} \in \mathcal{A}^{(s,N)}$.

Consider a competition system $(s - 1, N)$ and the assignment $\mathbf{A}' \in \mathcal{A}^{(s-1,N)}$ such that $a'_{12} = 1$ and $a'_{ij} = a_{ij}$ for all $i, j \in N \backslash \{1, 2\}$. Observe that, by $2(s - 1) = |N|$ and as argued in Case 3 above, $|W_1(\mathbf{A}')| = |W_3(\mathbf{A}')|$. Take $w(\mathbf{A}') \in W_1(\mathbf{A}')$ and note that this implies that player 1 wins his single match in $M(\mathbf{A}')$ against player 2. Thus, for each $w^*(\mathbf{A}') \in W_1(\mathbf{A}')$ there are exactly two vectors $w'(\mathbf{A}), w''(\mathbf{A}) \in W_1(\mathbf{A})$ of match winners for (s, N) where, all else being equal, either player 1 wins his other match against player 2 or player 2 wins it. On the other hand, for $w^{**}(\mathbf{A}')$ to belong to $W_3(\mathbf{A}')$ player 3 must necessarily win her single match. Thus, for each $w^{**}(\mathbf{A}') \in W_3(\mathbf{A}')$ there is a unique vector $w'''(\mathbf{A}) \in W_3(\mathbf{A})$ of match winners for (s, N) where, all else being equal, the winners of the two matches between players 1 and 2 are different. We conclude then that $|W_1(\mathbf{A})| > |W_3(\mathbf{A})|$ holds and thus, by $\varphi_1(\mathbf{A}, \mathbf{P}) = (0.5)^s \cdot |W_1(\mathbf{A})| > (0.5)^s \cdot |W_3(\mathbf{A})| = \varphi_3(\mathbf{A}, \mathbf{P})$, we reach a contradiction to (s, N) satisfying ET.

Case 2 ($|N|$ is even and $2\,s > |N| + 2$). Let $\mathbf{A}$ be such that $a_{12} = \frac{2s - |N|}{2} + 1$ and $\Sigma_{j \in N \backslash \{1,2\}} a_{ij} = 1$ for all $i \in N \backslash \{1, 2\}$, and notice that $\mathbf{A} \in \mathcal{A}^{(s,N)}$. Fix a vector $w(\mathbf{A})$ of match winners and note that, given the definition of $\mathbf{A}$, player 3 is a final winner in $w(\mathbf{A})$ only if, for each $i \in N$, player i wins at most one match. However, by $\frac{2s - |N|}{2} + 1 > 2$, either player 1, or player 2 or both must win more than one match. We conclude then that $W_3(\mathbf{A}) = \emptyset$ should hold. Meanwhile, $W_1(\mathbf{A}) \neq \emptyset$ because player 1 is a final winner of the competition if she wins all her matches. We have then $\varphi_1(\mathbf{A}, \mathbf{P}) = (0.5)^s \cdot |W_1(\mathbf{A})| > 0 = (0.5)^s \cdot |\emptyset| = (0.5)^s \cdot |W_3(\mathbf{A})| = \varphi_3(\mathbf{A}, \mathbf{P})$ in contradiction to (s, N) satisfying ET.

Case 3 ($|N|$ is odd). Let $N = \{1, \ldots, n\}$ and take $\mathbf{A}$ defined as follows: $a_{12} = \frac{2s - |N| + 1}{2}$, $a_{1n} = 1$, and $\Sigma_{j \in N \backslash \{1,2,n\}} a_{ij} = 1$ for all $i \in N \backslash \{1, 2, n\}$. Notice that $\mathbf{A} \in \mathcal{A}^{(s,N)}$. We proceed by considering the following three possible sub-cases.

Case 3.1 ($\frac{2s - |N| + 1}{2} = 1$). In this case player 1 plays exactly one match against player 2 and exactly one match against player n. Denote these matches by m_{12} and m_{1n}, correspondingly. There are four possible combinations of winners in m_{12} and m_{1n}: (i)

1 wins in both matches; (ii) 1 wins against 2 and loses against n; (iii) 1 loses against 2 and wins against n; and (iv) 1 loses in both matches. Notice that each player from $N \setminus \{1, 2, n\}$ can win at most one match. Therefore, for each of the vectors of match winners in $M(\mathbf{A}) \setminus \{m_{12}, m_{1n}\}$, three out of the four possible combinations of winners in $\{m_{12}, m_{1n}\}$ give as a result a vector of match winners in $W_1(\mathbf{A})$ and only two give as a result a vector of match winners in $W_n(\mathbf{A})$. We conclude then that $|W_1(\mathbf{A})| > |W_n(\mathbf{A})|$ holds and thus, by $\varphi_1(\mathbf{A}, \mathbf{P}) = (0.5)^s \cdot |W_1(\mathbf{A})| > (0.5)^s \cdot |W_n(\mathbf{A})| = \varphi_n(\mathbf{A}, \mathbf{P})$, we reach a contradiction to (s, N) satisfying ET.

Case 3.2 ($\frac{2s - |N| + 1}{2} = 2$). Denote by m'_{12} and m''_{12} the two matches between players 1 and 2, and let m_{1n} be the one match played by player n (against player 1). Analogously to Case 3.1, it is easy to compute that, for each of the vectors of match winners in $M(\mathbf{A}) \setminus \{m'_{12}, m''_{12}, m_{1n}\}$, three out of the eight possible combinations of winners in $\{m'_{12}, m''_{12}, m_{1n}\}$ give as a result a vector of match winners in $W_1(\mathbf{A})$ and only two give as a result a vector of match winners in $W_n(\mathbf{A})$. Therefore $|W_1(\mathbf{A})| > |W_n(\mathbf{A})|$ and thus, by $\varphi_1(\mathbf{A}, \mathbf{P}) = (0.5)^s \cdot |W_1(\mathbf{A})| > (0.5)^s \cdot |W_n(\mathbf{A})| = \varphi_n(\mathbf{A}, \mathbf{P})$, we again reach a contradiction to (s, N) satisfying ET.

Case 3.3 ($\frac{2s - |N| + 1}{2} > 2$). Fix $w(\mathbf{A})$ and note that, given the definition of $\mathbf{A}$, $w(\mathbf{A}) \in W_n(\mathbf{A})$ only if, for any $i \in N$, $w_m(\mathbf{A}) = i$ holds for at most one match $m \in M(\mathbf{A})$. However, by $\frac{2s - |N| + 1}{2} > 2$, the latter condition is violated for either player 1, or player 2, or for both players. We conclude then that $W_n(\mathbf{A}) = \emptyset$ should hold. Meanwhile, $W_1(\mathbf{A}) \neq \emptyset$ because player 1 is the winner of the competition if she wins all the matches that she plays. We have then $\varphi_1(\mathbf{A}, \mathbf{P}) = (0.5)^s \cdot |W_1(\mathbf{A})| > 0 = (0.5)^s \cdot |\emptyset| = (0.5)^s \cdot |W_n(\mathbf{A})| = \varphi_n(\mathbf{A}, \mathbf{P})$ in contradiction to (s, N) satisfying ET. $\qquad\square$

Proof of Theorem 2 Let (s, N) be a league competition with either $2s = |N|$ or $2s = k(|N| - 1)|N|$ for some integer $k \geq 1$. Because ET is stronger than WET, it follows from Theorem 1 that (s, N) satisfies WET when $2s = |N|$.

Consider now the case of $2s = k(|N| - 1)|N|$ for some integer $k \geq 1$ and let $\mathbf{A}$ be the assignment where each player plays k matches against every other player from N. By $2s = k(|N| - 1)|N|$, $\mathbf{A} \in \mathcal{A}^{(s,N)}$ follows. We show now that (s, N) satisfies WET with respect to $\mathbf{A}$. To that end, let $\mathbf{P}$ be the probability matrix with $p_{ij} = 0.5$ for all $i, j \in N$, and recall that $\varphi_q(\mathbf{A}, \mathbf{P}) = (0.5)^s \cdot |W_q(\mathbf{A})|$ holds for each $q \in N$. It suffices then to prove that the cardinality of $W_q(\mathbf{A})$ is the same for all $q \in N$. We proceed as follows.

Take $i, j, \ell \in N$ and for $x \in \{1, \ldots, k\}$, denote by $m_{i\ell}^x$ ($m_{j\ell}^x$) the x-th match of player i (j) against player ℓ. Denoting by $W(\mathbf{A})$ the set of all vectors of winners in the matches induced by $\mathbf{A}$, define $f : W_j(\mathbf{A}) \to W(\mathbf{A})$ as follows:

(1) For $\ell = i$ and each $x \in \{1, \ldots, k\}$, set $f(w_{m_{j\ell}^x}(\mathbf{A})) \neq w_{m_{j\ell}^x}(\mathbf{A})$;

(2) For each $\ell \in N \setminus \{i, j\}$ and each $x \in \{1, \ldots, k\}$:

  – if $w_{m_{j\ell}^x}(\mathbf{A}) = j$ and $w_{m_{i\ell}^x}(\mathbf{A}) = \ell$, set $f(w_{m_{j\ell}^x}(\mathbf{A})) = \ell$ and $f(w_{m_{i\ell}^x}(\mathbf{A})) = i$;

  – if $w_{m_{j\ell}^x}(\mathbf{A}) = \ell$ and $w_{m_{i\ell}^x}(\mathbf{A}) = i$, set $f(w_{m_{j\ell}^x}(\mathbf{A})) = j$ and $f(w_{m_{i\ell}^x}(\mathbf{A})) = \ell$;

  – if $w_{m_{j\ell}^x}(\mathbf{A}) = \ell$ and $w_{m_{i\ell}^x}(\mathbf{A}) = \ell$, set $f(w_{m_{j\ell}^x}(\mathbf{A})) = \ell$ and $f(w_{m_{i\ell}^x}(\mathbf{A})) = \ell$;

  – if $w_{m_{j\ell}^x}(\mathbf{A}) = j$ and $w_{m_{i\ell}^x}(\mathbf{A}) = i$, set $f(w_{m_{j\ell}^x}(\mathbf{A})) = j$ and $f(w_{m_{i\ell}^x}(\mathbf{A})) = i$;

(3) For each $q \in N \setminus \{i, j\}$, each $r \in N \setminus \{q\}$, and each $x \in \{1, \ldots, k\}$: set $f(w_{m_{qr}^x}(\mathbf{A})) = w_{m_{qr}^x}(\mathbf{A})$.

Notice that, by construction, the number of matches won by i at $f(w(\mathbf{A}))$ is the same as the number of matches won by j at $w(\mathbf{A})$ and vice versa. Moreover, also by construction, for each $q \neq i, j$, the number of matches won by q is the same in both $w(\mathbf{A})$ and $f(w(\mathbf{A}))$. Hence, $w(\mathbf{A}) \in W_j(\mathbf{A}) \setminus W_i(\mathbf{A})$ implies $f(w(\mathbf{A})) \in W_i(\mathbf{A}) \setminus W_j(\mathbf{A})$, while $w(\mathbf{A}) \in W_j(\mathbf{A}) \cap W_i(\mathbf{A})$ implies $f(w(\mathbf{A})) \in W_i(\mathbf{A}) \cap W_j(\mathbf{A})$ and, thus, $f(w(\mathbf{A})) \in W_i(\mathbf{A})$ holds.

We now show that $w(\mathbf{A}) \neq w'(\mathbf{A})$ for $w(\mathbf{A}), w'(\mathbf{A}) \in W_j(\mathbf{A})$ implies $f(w(\mathbf{A})) \neq f(w'(\mathbf{A}))$.

If $w_{m_{ij}^x}(\mathbf{A}) \neq w'_{m_{ij}^x}(\mathbf{A})$ or $w_{m_{qr}^x}(\mathbf{A}) \neq w'_{m_{qr}^x}(\mathbf{A})$ holds for some $q \in N \setminus \{i, j\}$, $r \in N \setminus \{q\}$, and $x \in \{1, \dots, k\}$, then $f(w_{m_{ij}^x}(\mathbf{A})) \neq f(w'_{m_{ij}^x}(\mathbf{A}))$ and $f(w_{m_{qr}^x}(\mathbf{A})) \neq f(w'_{m_{qr}^x}(\mathbf{A}))$ clearly follows due to parts (1) and (3), respectively, of the above construction.

If $j = w_{m_{j\ell}^x}(\mathbf{A}) \neq w'_{m_{j\ell}^x}(\mathbf{A}) = \ell$ for some $x \in \{1, \dots, k\}$ and $\ell \in N \setminus \{i, j\}$, then the following four cases are possible:

(a) $w_{m_{i\ell}^x}(\mathbf{A}) = w'_{m_{i\ell}^x}(\mathbf{A}) = i$. We have then $f(w_{m_{j\ell}^x}(\mathbf{A})) = j$, $f(w_{m_{i\ell}^x}(\mathbf{A})) = i$, $f(w'_{m_{j\ell}^x}(\mathbf{A})) = j$, $f(w'_{m_{i\ell}^x}(\mathbf{A})) = \ell$;

(b) $w_{m_{i\ell}^x}(\mathbf{A}) = w'_{m_{i\ell}^x}(\mathbf{A}) = \ell$. We have then $f(w_{m_{j\ell}^x}(\mathbf{A})) = \ell$, $f(w_{m_{i\ell}^x}(\mathbf{A})) = i$, $f(w'_{m_{j\ell}^x}(\mathbf{A})) = \ell$, $f(w'_{m_{i\ell}^x}(\mathbf{A})) = \ell$;

(c) $w_{m_{i\ell}^x}(\mathbf{A}) = i$ and $w'_{m_{i\ell}^x}(\mathbf{A}) = \ell$. We have then $f(w_{m_{j\ell}^x}(\mathbf{A})) = j$, $f(w_{m_{i\ell}^x}(\mathbf{A})) = i$, $f(w'_{m_{j\ell}^x}(\mathbf{A})) = \ell$, $f(w'_{m_{i\ell}^x}(\mathbf{A})) = \ell$;

(d) $w_{m_{i\ell}^x}(\mathbf{A}) = \ell$ and $w'_{m_{i\ell}^x}(\mathbf{A}) = i$. We have then $f(w_{m_{j\ell}^x}(\mathbf{A})) = \ell$, $f(w_{m_{i\ell}^x}(\mathbf{A})) = i$, $f(w'_{m_{j\ell}^x}(\mathbf{A})) = j$, $f(w'_{m_{i\ell}^x}(\mathbf{A})) = \ell$.

Thus, $f(w(\mathbf{A})) \neq f(w'(\mathbf{A}))$ holds for each of the possible cases.

Finally, if $i = w_{m_{i\ell}^x}(\mathbf{A}) \neq w'_{m_{i\ell}^x}(\mathbf{A}) = \ell$ for some $x \in \{1, \dots, k\}$ and $\ell \in N \setminus \{i, j\}$ the situation is completely analogous to the previous one, so it can also be derived in this last case that $f(w(\mathbf{A})) \neq f(w'(\mathbf{A}))$. We conclude then that f is a bijection, so $|W_i(\mathbf{A})| = |W_j(\mathbf{A})|$ holds. Hence, the league competition (s, N) satisfies WET with respect to $\mathbf{A}$. $\qquad \square$

Proof of Theorem 3 We first show that if $N = \{1, 2\}$, then (s, N) satisfies MS. Let $\mathbf{P} \in \mathcal{P}_R$ be an arbitrary but fixed probability matrix. Let $\mathbf{A} \in \mathcal{A}^{(s,N)}$ and notice that, by $N = \{1, 2\}$, $a_{12} = s$. Now assume that $p_{12} > 0.5$. It needs to be shown that $\varphi_1(\mathbf{A}, \mathbf{P}) - \varphi_2(\mathbf{A}, \mathbf{P}) \geq 0$ holds. If $s = 1$ then the inequality follows immediately. Assume then that $s \geq 2$ and consider the following two cases.

Case 1 ($2s > |N| = 2$ and s is odd). Note that in this case we have

$$\varphi_1(\mathbf{A}, \mathbf{P}) = p_{12}^s + p_{12}^{s-1} \cdot p_{21} + p_{12}^{s-2} \cdot p_{21}^2 + \dots + p_{12}^{(s+1)/2} \cdot p_{21}^{(s+1)/2-1}$$

and

$$\varphi_2(\mathbf{A}, \mathbf{P}) = p_{21}^s + p_{21}^{s-1} \cdot p_{12} + p_{21}^{s-2} \cdot p_{12}^2 + \dots + p_{21}^{(s+1)/2} \cdot p_{12}^{(s+1)/2-1}.$$

Thus, $\varphi_1(\mathbf{A}, \mathbf{P}) - \varphi_2(\mathbf{A}, \mathbf{P}) = \left(p_{12}^s - p_{21}^s\right) + p_{12} \cdot p_{21} \cdot \left(p_{12}^{s-2} - p_{21}^{s-2}\right) + \ldots + p_{12}^{(s+1)/2-1} \cdot p_{21}^{(s+1)/2-1} \cdot (p_{12} - p_{21}) > 0$, where the inequality follows from $p_{12} > 0.5$. We conclude that (s, N) satisfies MS.

Case 2 ($2s > |N| = 2$ and s is even). When expressing $\varphi_1(\mathbf{A}, \mathbf{P})$ and $\varphi_2(\mathbf{A}, \mathbf{P})$ for this case there are two differences compared to the corresponding expressions in Case 1. First, $(s+1)/2$ must be replaced by $s/2+1$, and $(s+1)/2 - -1$ by $s/2 - 1$. Second, in both expressions the probabilities of the vectors of match winners where both players, 1 and 2, are winners must be considered; that is, the term $p_{12}^{s/2} + p_{21}^{s/2}$ must be added in both expressions. Since these vectors for player 1 and player 2 do coincide (due to s being even), the added terms are the same for both players. Thus, they cancel out when the difference between $\varphi_1(\mathbf{A}, \mathbf{P})$ and $\varphi_2(\mathbf{A}, \mathbf{P})$ is taken. By reproducing the same reasoning as in Case 1, we can conclude that (s, N) satisfies MS also in this case.

Now assume that (s, N) satisfies MS. We show that $|N| > 2$ leads to a contradiction. Consider first the case where $N = \{1, 2, 3\}$ and the probability matrix $\mathbf{P} \in \mathcal{P}_R$ is as defined below:

$$\mathbf{P} = \begin{pmatrix} 0.5 & 0.5 + \varepsilon & 1 - \varepsilon \\ & 0.5 & 1 - 2\varepsilon \\ & & 0.5 \end{pmatrix}$$

Take the assignment $\mathbf{A} \in \mathcal{A}^{(s,N)}$ defined by $a_{12} = 1$ and $a_{23} = s - 1$. If $s = 2$, then $\varphi_1(\mathbf{A}, \mathbf{P}) \approx 0.5$ and $\varphi_2(\mathbf{A}, \mathbf{P}) \approx 1$ in contradiction to $p_{12} > 0.5$ and (s, N) satisfying MS.

If $s > 2$, given $\mathbf{A}$ and $\mathbf{P}$, we have that player 2 has a probability arbitrarily close to 1 of winning all matches except one and therefore a probability close to one of winning most of her matches. Thus, $\varphi_2(\mathbf{A}, \mathbf{P}) \approx 1$ and $\varphi_1(\mathbf{A}, \mathbf{P}) \approx 0$, again reaching a contradiction to $p_{12} > 0.5$ and (s, N) satisfying MS.

Next assume that $|N| > 3$ holds and consider a probability matrix $\mathbf{P} \in \mathcal{P}_R$ such that $p_{23} > 0.5$ and $p_{3k} \approx 1$ for all $k \in N \setminus \{1, 2\}$. We distinguish the following two cases.

Case 1 ($2s = |N|$). In this case each player participates in exactly one match according to every assignment. Take $\mathbf{A} \in \mathcal{A}^{(s,N)}$ to be such that $a_{12} = a_{34} = 1$. Note that, for any vector of match winners, a player is a final winner of the entire competition only if she wins her single match according to $\mathbf{A}$. Thus, $\varphi_3(\mathbf{A}, \mathbf{P}) \approx 1$ and $\varphi_2(\mathbf{A}, \mathbf{P}) \leq 0.5$ in contradiction to $p_{23} > 0.5$ and (s, N) satisfying MS.

Case 2 ($2s > |N|$). Consider the assignment $\mathbf{A} \in \mathcal{A}^{(s,N)}$ defined as follows: $a_{12} = 1$, $a_{1j} = a_{2j} = 0$ for all $j \in N \setminus \{1, 2\}$, and $a_{3j} = 1$ for each $j \in N \setminus \{1, 2\}$. Note that, according to $\mathbf{A}$ and $\mathbf{P}$, player 3 has a probability arbitrarily close to one of winning all her matches, so she has a probability close to one of winning every match in the competition except one (the one played between players 1 and 2). Hence, $\varphi_3(\mathbf{A}, \mathbf{P}) \approx 1$ and $\varphi_2(\mathbf{A}, \mathbf{P}) \approx 0$ in contradiction to $p_{23} > 0.5$ and (s, N) satisfying MS. □

Proof of Theorem 4 We start with the case $2s = |N|$. Let $\mathbf{P} \in \mathcal{P}_R$ be an arbitrary but fixed probability matrix and, recalling that $|N|$ is even, consider the assignment $\mathbf{A} \in \mathcal{A}^{(s,N)}$ defined for all $i, j \in N$ as follows: $a_{ij} = 1$ if and only if $i + j = |N| + 1$.

That is, $\mathbf{A}$ matches the best player with the worst one, the second best with the second worst, and so on. Clearly, given $\mathbf{A}$, a player is a final winner in a vector of match winners if and only if she is the winner of the single match that she plays. Thus, for $k \in N$, we have $\varphi_k(\mathbf{A}, \mathbf{P}) = p_{kk'}$ with $k' \in N$ being the player such that $a_{kk'} = 1$.

Now assume that $p_{ij} > 0.5$ holds for some $i, j \in N$. It needs to be shown that $\varphi_i(\mathbf{A}, \mathbf{P}) \geq \varphi_j(\mathbf{A}, \mathbf{P})$ follows in such a case. To that end, let $i', j' \in N$ be such that $a_{ii'} = a_{jj'} = 1$. If $i' = j$, then $\varphi_i(\mathbf{A}, \mathbf{P}) = p_{ij} > p_{ji} = \varphi_j(\mathbf{A}, \mathbf{P})$ follows. If $i' \neq j$, we have from $p_{ij} > 0.5$ that $i R j$ holds, while $j' R i'$ holds due to the construction of $\mathbf{A}$. Thus, $p_{ii'} \geq p_{jj'}$ follows by (3). We then have $\varphi_i(\mathbf{A}, \mathbf{P}) = p_{ii'} \geq p_{jj'} = \varphi_j(\mathbf{A}, \mathbf{P})$ as required for showing that (s, N) satisfies WMS with respect to $\mathbf{A}$.

Now consider a league (s, N) such that $s \geq |N| - 1$ holds. Fix a probability matrix $\mathbf{P} \in \mathcal{P}_R$ and consider the assignment $\mathbf{A}$ defined as follows:

- $a_{12} = s - |N| + 2$,
- $a_{ij} = 1$, if $i = 1$ and $j \neq 2$,
- $a_{ij} = 0$, if $i \neq 1$.
  According to the above assignment,
- player 1 plays all s matches (due to $a_{12} + (|N| - 2) \cdot 1 = s$),
- each of the remaining players (except player 2) participates in exactly one match (which is, of course, against player 1), and
- player 2 meets player 1 in each of the remaining $(s - |N| + 2)$ matches of the competition.

Clearly, $\mathbf{A} \in \mathcal{A}^{(s,N)}$ follows. We show that (s, N) satisfies WMS with respect to $\mathbf{A}$ in three steps. Below, $M^{12}(\mathbf{A})$ stands for the set of $(s - |N| + 2)$ matches played between 1 and 2 according to $\mathbf{A}$.

Step 1 If $p_{12} > 0.5$, then $\varphi_1(\mathbf{A}, \mathbf{P}) \geq \varphi_2(\mathbf{A}, \mathbf{P})$.

Proof. Recall that $\varphi_1(\mathbf{A}, \mathbf{P}) = \sum\limits_{w(\mathbf{A}) \in W_1(\mathbf{A})} pr(w(\mathbf{A}))$ and $\varphi_2(\mathbf{A}, \mathbf{P}) = \sum\limits_{w(\mathbf{A}) \in W_2(\mathbf{A})} pr(w(\mathbf{A}))$. Given a vector $w(\mathbf{A})$ of match winners, let $M^{12}_{w_m(\mathbf{A})=1}(w(\mathbf{A}))$ be the set of matches in $M^{12}(\mathbf{A})$ which player 1 wins, and let $M^{12}_{w_m(\mathbf{A})=2}(w(\mathbf{A}))$ be the set of matches in $M^{12}(\mathbf{A})$ which player 2 wins.

Define the mapping $f : W_2(\mathbf{A}) \to W_1(\mathbf{A})$ by just interchanging the match winners for each match in $M^{12}(w(\mathbf{A}))$. Notice that for $w(\mathbf{A}), w'(\mathbf{A}) \in W_2(\mathbf{A})$ with $w(\mathbf{A}) \neq w'(\mathbf{A})$, $f(w(\mathbf{A})) \neq f(w'(\mathbf{A}))$ follows and thus, f is a bijection between $W_2(\mathbf{A})$ and a subset of $W_1(\mathbf{A})$.

Note also that, for any vector $w(\mathbf{A})$ of match winners, we have: $pr(w(\mathbf{A})) = \prod\limits_{m \in M(\mathbf{A}) \setminus M^{12}(\mathbf{A})} \varphi_{w_m(\mathbf{A})}(\mathbf{A}, \mathbf{P}) \cdot (p_{12})^{\left| M^{12}_{w_m(\mathbf{A})=1}(w(\mathbf{A})) \right|} \cdot (p_{21})^{\left| M^{12}_{w_m(\mathbf{A})=2}(w(\mathbf{A})) \right|}$.

Finally, note that, by $w(\mathbf{A}) \in W_2(\mathbf{A})$, in $w(\mathbf{A})$ player 2 wins at least as many matches against player 1 as player 1 against player 2; that is, $\left| M^{12}_{w_m(\mathbf{A})=2}(w(\mathbf{A})) \right| \geq \left| M^{12}_{w_m(\mathbf{A})=1}(w(\mathbf{A})) \right|$. Thus, $\left| M^{12}_{f(w(\mathbf{A}))_m=1}(w(\mathbf{A})) \right| \geq \left| M^{12}_{f(w(\mathbf{A}))_m=2}(w(\mathbf{A})) \right|$. We then have from $p_{12} > p_{21}$ that $(p_{12})^{\left| M^{12}_{w_m(\mathbf{A})=1}(w(\mathbf{A})) \right|} \cdot (p_{21})^{\left| M^{12}_{w_m(\mathbf{A})=2}(w(\mathbf{A})) \right|} \leq$

$(p_{12})^{\left|M^{12}_{f(w(\mathbf{A}))m=1}(w(\mathbf{A}))\right|} \cdot (p_{21})^{\left|M^{12}_{f(w(\mathbf{A}))m=2}(w(\mathbf{A}))\right|}$ holds. Therefore, $pr(w(\mathbf{A})) \leq pr(f(w(\mathbf{A})))$ holds for each $w(\mathbf{A}) \in W_2(\mathbf{A})$, which enables us to conclude that $\varphi_1(\mathbf{A},\mathbf{P}) \geq \varphi_2(\mathbf{A},\mathbf{P})$.

Step 2 If $p_{ij} > 0.5$ for $i \in \{1,2\}$ and $j \in N \setminus \{1,2\}$, then $\varphi_i(\mathbf{A},\mathbf{P}) \geq \varphi_j(\mathbf{A},\mathbf{P})$.

Proof Note first that if $\left|M^{12}(\mathbf{A})\right| > 2$ then $W_j(\mathbf{A}) = \emptyset$ and $\varphi_i(\mathbf{A},\mathbf{P}) \geq 0 = \varphi_j(\mathbf{A},\mathbf{P})$ follow immediately.

If $\left|M^{12}(\mathbf{A})\right| = 2$, then $w(\mathbf{A}) \in W_j(\mathbf{A})$ implies that every player wins exactly one match, i.e., all players are winners of the entire competition. Thus, $w(\mathbf{A}) \in W_j(\mathbf{A})$ implies $w(\mathbf{A}) \in W_i(\mathbf{A})$; that is, $W_j(\mathbf{A}) \subseteq W_i(\mathbf{A})$ holds and hence, $\varphi_i(\mathbf{A},\mathbf{P}) \geq \varphi_j(\mathbf{A},\mathbf{P})$ follows.

Now assume that $\left|M^{12}(\mathbf{A})\right| = 1$. Recall that only vectors of match winners in $W_i(\mathbf{A}) \setminus W_j(\mathbf{A})$ and in $W_j(\mathbf{A}) \setminus W_i(\mathbf{A})$ do matter for the comparison of $\varphi_i(\mathbf{A},\mathbf{P})$ and $\varphi_j(\mathbf{A},\mathbf{P})$. If $i = 1$, then $W_j(\mathbf{A}) \setminus W_1(\mathbf{A}) = \{w(\mathbf{A})\}$ with $w(\mathbf{A})$ consisting of player 1 losing all her matches and thus $pr(w(\mathbf{A})) = p_{21} \cdot p_{31} \cdot \ldots \cdot p_{n1}$. Meanwhile, the vector $w'(\mathbf{A})$ of match winners where player 1 wins all her matches definitely belongs to $W_1(\mathbf{A}) \setminus W_j(\mathbf{A})$, with $pr(w'(\mathbf{A})) = p_{12} \cdot p_{13} \cdot \ldots \cdot p_{1n}$. Thus, given that $p_{1k} \geq p_{k1}$ for each $k \in N \setminus \{1\}$ (with strict inequality holding for $k = j$), we have that $\varphi_1(\mathbf{A},\mathbf{P}) > \varphi_j(\mathbf{A},\mathbf{P})$.

In contrast, $i = 2$ implies $W_j(\mathbf{A}) \setminus W_2(\mathbf{A}) = \{w(\mathbf{A})\}$ with $w(\mathbf{A})$ consisting of 1 only winning her match against 2 and $pr(w(\mathbf{A})) = p_{12} \cdot p_{31} \cdot p_{41} \cdot \ldots \cdot p_{n1}$. Consider now the vector $w'(\mathbf{A})$ of match winners which differs from $w(\mathbf{A})$ only in that at $w'(\mathbf{A})$ player 2 wins her single match against player 1 and player j loses her single match against player 1. This involves that $w'(\mathbf{A}) \in W_2(\mathbf{A}) \setminus W_j(\mathbf{A})$ with $pr(w'(\mathbf{A})) \geq pr(w(\mathbf{A}))$ due to $p_{21} \geq p_{j1}$ following from $p_{2j} > 0.5$ and condition (2). We conclude then that $\varphi_2(\mathbf{A},\mathbf{P}) \geq \varphi_j(\mathbf{A},\mathbf{P})$.

Step 3 If $p_{ij} > 0.5$ for some $i, j \in N \setminus \{1,2\}$, then $\varphi_i(\mathbf{A},\mathbf{P}) \geq \varphi_j(\mathbf{A},\mathbf{P})$.

Proof Recall that in this situation each of the players i and j participates in exactly one match against player 1. Consider the following three possible cases.

Case 1 ($\left|M^{12}(\mathbf{A})\right| = 1$). In this case, there is a unique vector of match winners $w(\mathbf{A}) \in W_j(\mathbf{A}) \setminus W_i(\mathbf{A})$ consisting of player j winning her match against player 1 and player 1 winning only her match against player i. We then have $pr(w(\mathbf{A})) = p_{j1} \cdot p_{1i} \cdot \prod_{k \neq i,j} p_{k1}$.

Similarly, there is a unique vector of match winners $w'(\mathbf{A}) \in W_i(\mathbf{A}) \setminus W_j(\mathbf{A})$ with $pr(w'(\mathbf{A})) = p_{i1} \cdot p_{1j} \cdot \prod_{k \neq i,j} p_{k1}$. By $p_{ij} > 0.5$ and condition (2), $p_{1j} \cdot p_{i1} \geq p_{j1} \cdot p_{1i}$

follows. We then have $pr(w'(\mathbf{A})) \geq pr(w(\mathbf{A}))$ which implies $\varphi_i(\mathbf{A},\mathbf{P}) \geq \varphi_j(\mathbf{A},\mathbf{P})$.

Case 2 ($\left|M^{12}(\mathbf{A})\right| = 2$). In this case, each of the two vectors of match winners in $W_j(\mathbf{A})$ has the following structure: each of the players in $N \setminus \{1,2\}$ (including i and j) wins her match against player 1; player 1 wins only one of her matches against player 2, and player 2 wins the other match between players 1 and 2. Clearly, $W_j(\mathbf{A}) = W_i(\mathbf{A})$ and thus, $\varphi_i(\mathbf{A},\mathbf{P}) = \varphi_j(\mathbf{A},\mathbf{P})$ holds.

Case 3 ($\left|M^{12}(\mathbf{A})\right| > 2$). In this case, given that there are more than two matches between players 1 and 2, $W_i(\mathbf{A}) = W_j(\mathbf{A}) = \emptyset$. Thus $\varphi_i(\mathbf{A},\mathbf{P}) = \varphi_j(\mathbf{A},\mathbf{P}) = 0$. □

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