Determination of the symbolic base inertial parameters of planar mechanisms

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Abstract

The inertial information of a planar mechanical system is characterised using 4 inertial parameters per solid. Due to the kinematic constraints, this parametrization turns out be redundant. In order to reduce the computational cost of the model and make it possible to estimate its inertial parameters, the model is usually written in terms of a minimum set of inertial parameters called *base inertial parameters*. These parameters completely determine the dynamics of motion (kinetics) of a mechanism and, since their contributions are independent to each other, their actual values can be estimated experimentally. The base inertial parameters expressions can be written as a linear combination of the inertial parameters and determining their symbolic expressions provides a deeper insight into their physical meaning.

This paper presents a new algorithm to determine the *symbolic* expressions of the base inertial parameters of planar mechanisms. The approach is based on a very well known numerical method to obtain the base inertial parameters and on the fact that these parameters belong to a class of functions that lets us search for symbolic expressions matching with them.

Since the *symbolic* expressions are a function of the geometric constants of the system, the presented algorithm constitutes a very valuable tool in design optimisation and it is also very interesting in dynamic parameter estimation, model reduction and other fields.

Keywords: base parameters, inertial parameters, symbolic, model reduction

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1. INTRODUCTION

Realistic simulation, optimisation and advanced control schemes of mechanical systems are based on accurate Dynamics Models. These models are dependent on the constant values of the inertial and other dynamic parameters of the system such as friction, damping and elastic coefficients. The inertial information of a single planar solid is provided by 4 parameters: the mass, the two components of the position vector of the centre of mass and the second moment of inertia. The Inverse Dynamic Models (IDM) can always be written in a linear form with respect to them (Shome et al., 1998) as:

$$\mathbf{K}\boldsymbol{\phi} = \boldsymbol{\tau} \tag{1}$$

where $\mathbf{K}$ is the observation matrix, $\boldsymbol{\phi}$ are the inertial parameters, and $\boldsymbol{\tau}$ are the external forces and torques. It turns out that the observation matrix of a mechanism generally has not full rank. This is due to the kinematic constraints that couple the movement of the solids. As a consequence, the IDM is better expressed in terms of linear combinations of the inertial parameters. These combinations are called *base inertial parameters* and represent a minimum set of inertial parameters whose values can uniquely determine the kinetics of the mechanism. Accordingly, the actual values of the base inertial parameters can be uniquely estimated from experimental data, making them an essential tool for mechanical model identification. Moreover, since they uniquely determine the kinetics of the mechanism, they are very useful in fields like design optimisation and model reduction.

The IDM models not only can be written linearly with respect to the *inertial parameters*, but also with respect to the *base inertial parameters*. Atkeson et al. (1986) were among the firsts to show this and Maes et al. (1989) demonstrated the linearity of the equations of motion with respect to the so-called *barycentric parameters*. See also Shome et al. (1998) for the construction of a parameter linear model.

The main approaches to obtain the base inertial parameters can be classified in numerical and symbolic. The numerical methods developed by Gautier (1991) give a tool to obtain the relationships between the inertial parameters based on the SVD or QR decompositions (Golub and Loan, 1989) of the observation matrix. These methods provide expressions in the form of a matrix $\boldsymbol{\beta}$ that defines the base parameters, $\boldsymbol{\phi}_b$, as a linear combination of the inertial parameters, $\boldsymbol{\phi}_b = [\mathbf{I}, \boldsymbol{\beta}]\boldsymbol{\phi}$. However, it should be emphasised that these numerical algorithms give the elements of $\boldsymbol{\beta}$ as numbers, not as symbolic expressions dependent on the geometric parameters of the mechanism.

The work of Gautier and Khalil (1988, 1989), Mayeda et al. (1990) and other authors aimed to find symbolic expressions for the base inertial parameters. However, the methods provided in these early papers were only valid for open-loop systems or parallelogram closed-loops. Khalil and Bennis (1995) were the firsts to develop an algorithm to obtain the base inertial parameters symbolically for any mechanism with closed loops. That method was certainly more complex than the methods developed for open-loop systems. Moreover, it did not always find all the base inertial parameters. More recently, Chen et al. (2002) developed an easier method based on the concept of mass and moment of inertia transfer for planar mechanisms and Ros et al. (2012, 2015) generalised and automated the method for spatial mechanisms.

In the last few years, the *base inertial parameters* and other minimum inertial parametrisations have been analysed and used in different fields of mechanical engineering. They have customarily been used for payload (Khalil et al., 2007) and dynamic parameter estimation (Farhat et al., 2008) in the identification of industrial manipulators, as well as for identifiability calculations of the dynamic parameters of parallel robots (Díaz et al., 2008). Recently, Ebrahimi and Kövecses (2010a,b) developed a procedure for the unit-homogenisation of the observation matrix in order to normalise the equations for parameter identification, and characterised the influence of the inertial parameters on the dynamics of a mechanism. Inertial parameters sensitivity analyses have also been performed within the recent past (Eberhard et al., 2007) so as to know how accurate the estimation of the inertia parameters has to be in order to get reliable simulation results.

In the present paper a new method is proposed to obtain the symbolic expressions of the base inertial parameters for open- and closed-loop planar mechanisms. The method can be easily automated and requires very little effort from the analyst. It uses the method provided by Gautier (1991) to obtain the *numerical* base inertial parameters and looks for symbolic expressions that match the numerical values of $\boldsymbol{\beta}$. This paper demonstrates that for *optimally parametrised* planar mechanisms in which the existing loops are closed through the ground, the elements of $\boldsymbol{\beta}$ can be written as products or quotients of geometric parameters. Therefore, the candidate symbolic expressions to match are the ones that belong to that class of functions. The resulting matching expressions are the desired symbolic expressions for the elements of $\boldsymbol{\beta}$.

Comparing the characteristics of different approaches in the literature, it is worth noting that the presented approach, as the approach of Chen et al. (2002), deals with planar mechanisms. A unique characteristic of the presented approach is that despite being *numerical* as the approach of Gautier (1991) it determines the *symbolic* expressions of the base inertial parameters. Other approaches as Khalil et al. (1995) and Ros et al. (2012) can determine the symbolic expressions of the base inertial parameters of spatial mechanisms but the algorithms are very difficult to automate compared to those of the presented approach. Regarding the modelling of the system, the presented approach and also the approach of Khalil et al. (1995) require the use of an optimal parametrisation. However, this requirement does not impose a practical limitation, as any parametrisation of a planar mechanism can be recast to the required one.

The paper is organised as follows. In Section 2, the *numerical* method for the determination of the base inertial parameters proposed by Gautier (1991) is presented. Section 3 shows the conditions in which the proposed algorithm is applicable. Section 4 presents the class of functions for the elements of $\boldsymbol{\beta}$ and describes the algorithm for the calculation of the symbolic expressions for the base inertial parameters. An example application of the algorithm is shown in Section 5. Finally, some conclusions are drawn in Section 6. In the Appendix it is demonstrated that the elements of $\boldsymbol{\beta}$ belong to the referred class of functions.

2. NUMERICAL DETERMINATION OF THE BASE INERTIAL PARAMETERS

2.1. The parameter linear inverse dynamic model

In order to write the IDM equations in a parameter linear form, the inertial parameters of the i^{th} link should be taken as

$$\boldsymbol{\phi}_i = \{m_i, mx_i, my_i, J_i\}^\top \tag{2}$$

where m_i is the mass. Furthermore, mx_i and my_i are the first moments of inertia, J_i is the second moment of inertia, and these must be defined with respect to a certain location in the link frame. Moreover, they cannot be defined with respect to the centre of gravity.

The inertial parameter vector of the system is defined as the set of the inertial parameters of the constituent links,

$$\boldsymbol{\phi} = \{\boldsymbol{\phi}_1^{\top}, \ldots, \boldsymbol{\phi}_i^{\top}, \ldots, \boldsymbol{\phi}_N^{\top}\}^{\top}. \tag{3}$$

Then, it is possible to write the IDM of a mechanism in a linear form with respect to the inertial parameters ($\boldsymbol{\phi}$):

$$\mathbf{K}(\boldsymbol{q}, \dot{\boldsymbol{q}}, \ddot{\boldsymbol{q}}, \boldsymbol{\lambda}, \boldsymbol{\Lambda})\boldsymbol{\phi} = \boldsymbol{\tau} \tag{4}$$

where $\boldsymbol{q}$, $\dot{\boldsymbol{q}}$ and $\ddot{\boldsymbol{q}}$ are the generalised coordinates and their first and second derivatives, and vectors $\boldsymbol{\lambda}$ and $\boldsymbol{\Lambda}$ represent the geometric parameters of the model. There is no restriction on the type of coordinates chosen to model the mechanism.

2.2. Determination of the base inertial parameters

In this section Gautiers numerical method to obtain the base parameters of a mechanism is summarised. The reader is referred to the original paper (Gautier, 1991) for a thorough description of the algorithm.

The algorithm starts from the standard *least squares* identification problem, and calculates the parameters that minimise the error from the measurements of the torques $\boldsymbol{\tau}_i$ and the extended state $(\boldsymbol{q}, \dot{\boldsymbol{q}}, \ddot{\boldsymbol{q}})_i$ for a set of time instants $i = 1, \ldots, n$.

$$\begin{bmatrix} \mathbf{K}(\boldsymbol{q}_1, \dot{\boldsymbol{q}}_1, \ddot{\boldsymbol{q}}_1) \\ \mathbf{K}(\boldsymbol{q}_2, \dot{\boldsymbol{q}}_2, \ddot{\boldsymbol{q}}_2) \\ \cdots \\ \mathbf{K}(\boldsymbol{q}_n, \dot{\boldsymbol{q}}_n, \ddot{\boldsymbol{q}}_n) \end{bmatrix} \boldsymbol{\phi} = \begin{Bmatrix} \boldsymbol{\tau}_1 \\ \boldsymbol{\tau}_2 \\ \cdots \\ \boldsymbol{\tau}_n \end{Bmatrix} \tag{5}$$

The above system is usually written in a compact form as

$$\mathbf{W}\boldsymbol{\phi} = \boldsymbol{\chi}, \tag{6}$$

where $\mathbf{W}$ is called the observation or regression matrix for those time instants.

The algorithm of Gautier is an algebraic procedure devoted to reduce the model of Eq. (6). It takes advantage of the fact that $\mathbf{W}$ has not maximum column rank and selects a set of independent columns of $\mathbf{W}$ to form $\mathbf{W}_R$ so that,

$$\mathbf{W}_R\boldsymbol{\phi}_b = \boldsymbol{\chi}, \tag{7}$$

where $\mathbf{W}_R$ is the reduced observation matrix, $rank(\mathbf{W}) = rank(\mathbf{W}_R)$ and $\boldsymbol{\phi}_b$ (the *base parameters*) are a linear combination of the elements of $\boldsymbol{\phi}$.

As proposed by Gautier (1991) for open-loop mechanisms modelled with the parametrisation of Denavit and Hartenberg (1955), the matrix $\mathbf{W}$ can be evaluated using random values for $\boldsymbol{q}$, $\dot{\boldsymbol{q}}$ and $\ddot{\boldsymbol{q}}$. However, in order to ensure that in a more general model, the geometric, velocity and acceleration constraints are always satisfied, it is customary to evaluate the model with a sufficiently exciting trajectory (Swevers et al., 1996, 1997) in which the constraints are implicitly satisfied.

No experimental estimation of the values of $\boldsymbol{\phi}_b$ will be performed. Therefore, vector $\boldsymbol{\chi}$ will play no role in this procedure.

Using the SVD, $\mathbf{W}$ can be decomposed as follows:

$$\mathbf{W} = \mathbf{U}\boldsymbol{\Sigma}_0\mathbf{V}^\top = \mathbf{U}\begin{bmatrix}\boldsymbol{\Sigma} & \mathbf{0}\\ \mathbf{0} & \mathbf{0}\end{bmatrix}\begin{bmatrix}\mathbf{V}_1 & \mathbf{V}_2\end{bmatrix}^\top, \tag{8}$$

being $n \times m$ the dimensions of $\mathbf{W}$ and b the rank of $\mathbf{W}$. As a consequence, $\mathbf{U}$ is $n \times n$, $\boldsymbol{\Sigma}_0$ is $n \times m$, $\mathbf{V}$ is $m \times m$, $\boldsymbol{\Sigma}$ is $b \times b$, $\mathbf{V}_1$ is $m \times b$ and $\mathbf{V}_2$ is $m \times (m - b)$. Matrix $\mathbf{V}$ is split out into $\mathbf{V}_1$ and $\mathbf{V}_2$ so that $\mathbf{V}_1$ are the first b columns of $\mathbf{V}$. The columns of $\mathbf{V}_2$ happen to be a base for the null space of $\mathbf{W}$. This fact becomes more evident if Eq. (8) is right-multiplied by $\mathbf{V} = \begin{bmatrix}\mathbf{V}_1 & \mathbf{V}_2\end{bmatrix}$. Since $\mathbf{V}$ is orthonormal $\mathbf{V}^{-1} = \mathbf{V}^\top$ and it holds that

$$\mathbf{W}\mathbf{V} = \mathbf{W}\begin{bmatrix}\mathbf{V}_1 & \mathbf{V}_2\end{bmatrix} = \mathbf{U}\begin{bmatrix}\boldsymbol{\Sigma} & \mathbf{0}\\ \mathbf{0} & \mathbf{0}\end{bmatrix} \implies \begin{cases}\mathbf{W}\mathbf{V}_1 = \mathbf{U}\begin{bmatrix}\boldsymbol{\Sigma}\\ \mathbf{0}\end{bmatrix}\\ \mathbf{W}\mathbf{V}_2 = \mathbf{0}.\end{cases} \tag{9}$$

Therefore, for an arbitrary vector $\boldsymbol{\phi}_a$ the following equation is satisfied:

$$\mathbf{W}\boldsymbol{\phi} = \mathbf{W}(\boldsymbol{\phi} + \mathbf{V}_2\boldsymbol{\phi}_a). \tag{10}$$

Then,

$$\boldsymbol{\phi}_R \triangleq \boldsymbol{\phi} + \mathbf{V}_2\boldsymbol{\phi}_a \tag{11}$$

represents an alternative equivalent parametrisation of the system, where $\boldsymbol{\phi}_R$ is $m \times 1$ and $\boldsymbol{\phi}_a$ is $(m - b) \times 1$.

Let $\mathbf{P}$ be an arbitrary permutation matrix that reorders the rows of $\mathbf{V}_2$ so that

$$\mathbf{P}^\top\mathbf{V}_2 = \begin{bmatrix}\mathbf{V}_{21}\\ \mathbf{V}_{22}\end{bmatrix} \tag{12}$$

where $\mathbf{V}_{22}$ is required to be a full rank square matrix[2]. Reordering the rows of vectors $\boldsymbol{\phi}_R$ and $\boldsymbol{\phi}$ using the same $\mathbf{P}$,

$$\mathbf{P}^\top\boldsymbol{\phi}_R = \begin{bmatrix}\boldsymbol{\phi}_{R1}\\ \boldsymbol{\phi}_{R2}\end{bmatrix}, \qquad \mathbf{P}^\top\boldsymbol{\phi} = \begin{bmatrix}\boldsymbol{\phi}_1\\ \boldsymbol{\phi}_2\end{bmatrix}, \tag{13}$$

vectors $\boldsymbol{\phi}_{R1}$, $\boldsymbol{\phi}_{R2}$, $\boldsymbol{\phi}_1$ and $\boldsymbol{\phi}_2$ are defined. Now it is possible to write Eq. (11) as follows:

$$\begin{bmatrix}\boldsymbol{\phi}_{R1}\\ \boldsymbol{\phi}_{R2}\end{bmatrix} = \begin{bmatrix}\boldsymbol{\phi}_{1}\\ \boldsymbol{\phi}_{2}\end{bmatrix} + \begin{bmatrix}\mathbf{V}_{21}\\ \mathbf{V}_{22}\end{bmatrix}\boldsymbol{\phi}_a. \tag{14}$$

Eq. (14) is a system of m equations where $\boldsymbol{\phi}_{R1}$ $(b \times 1)$, $\boldsymbol{\phi}_{R2}$ $((m-b) \times 1)$ and $\boldsymbol{\phi}_a$ $((m-b) \times 1)$ are unknowns. Since there are $m-b$ more unknowns than equations, the value of $m-b$ unknowns can be chosen at will. Choosing $\boldsymbol{\phi}_a$ so that the maximum number of elements of $\boldsymbol{\phi}_R$ equal zero, the columns of $\mathbf{W}$ multiplying to those elements of $\boldsymbol{\phi}_R$ will disappear from the model, and it will be possible to write the model equation as in Eq. (7), making $\boldsymbol{\phi}_{R1}$ a minimum parametrisation of the system.

Therefore, making $\boldsymbol{\phi}_{R2} = \mathbf{0}$ in Eq. (14) and eliminating $\boldsymbol{\phi}_a$, it holds:

$$\boldsymbol{\phi}_{R1} = \boldsymbol{\phi}_1 - \mathbf{V}_{21}\mathbf{V}_{22}^{-1}\boldsymbol{\phi}_2. \tag{15}$$

Defining

$$\boldsymbol{\beta} \triangleq -\mathbf{V}_{21}\mathbf{V}_{22}{}^{-1} \tag{16}$$

we obtain the set of base parameters corresponding to the chosen $\mathbf{P}$:

$$\boldsymbol{\phi}_{R1} = \boldsymbol{\phi}_1 + \boldsymbol{\beta}\boldsymbol{\phi}_2. \tag{17}$$

Since the coefficients of matrix $\boldsymbol{\beta}$ are the weights that determine the linear dependence of the base parameters on the inertial parameters, we will refer to them as the *base parameter weights*, or simply $\boldsymbol{\beta}$ *weights*. Note that the above algorithm gives the numerical values for the entries of the matrix $\boldsymbol{\beta}$. Moreover, observe that the set of base parameters is uniquely determined by the choice of the permutation matrix $\mathbf{P}$.

In order to complete the reduced model, the equations of the IDM can be rewritten as:

$$\mathbf{W}\boldsymbol{\phi} = \mathbf{W}\boldsymbol{\phi}_R = (\mathbf{W}\mathbf{P})(\mathbf{P}^\top\boldsymbol{\phi}_R) = [\mathbf{W}_1\mathbf{W}_2]\begin{Bmatrix}\boldsymbol{\phi}_{R1}\\ \boldsymbol{\phi}_{R2}\end{Bmatrix} = \mathbf{W}_1\boldsymbol{\phi}_{R1}. \tag{18}$$

Note that $\mathbf{P}^{-1} = \mathbf{P}^\top$. Eq. (18) holds since Eq. (10) does. As $\mathbf{P}$ reorders the rows of $\boldsymbol{\phi}_R$ into $\boldsymbol{\phi}_{R1}$ and $\boldsymbol{\phi}_{R2}$, it also reorders the columns of $\mathbf{W}$ into $\mathbf{W}_1$

[2] There usually exist many different $\mathbf{P}$ matrices that fulfil this requirement, each of them leading to a different set of *base parameters*.

and $\mathbf{W}_2$ so that $\mathbf{W}_1$ are the first b columns of $\mathbf{WP}$. Since $\boldsymbol{\phi}_a$ was chosen so that $\boldsymbol{\phi}_{R2} = \mathbf{0}$, $(m - b)$ columns of $\mathbf{WP}$ (the whole matrix $\mathbf{W}_2$) will no longer be part of the reduced model. Finally, calling $\mathbf{W}_R = \mathbf{W}_1$ and $\boldsymbol{\phi}_b = \boldsymbol{\phi}_{R1}$, the reduced IDM can be written as in Eq. (7).

3. Conditions for the applicability of the algorithm

Although the algorithm of Gautier can be applied to any mechanism regardless the manner it is parametrised and the number and kind of closed loops that are present on it, the algorithm proposed in this paper will only find the symbolic expressions of the base inertial parameters if the parametrisation of the links is made in an *optimal* way (as defined in the next section) and the existing loops are closed through the ground.

3.1. Conditions on the Parametrisation

Suppose that we have a mechanism for which the existing loops are closed through the ground, and suppose that at least one of the joints in the closed loop is not prismatic[3]. In this situation the closed loop can be opened cutting it through a R joint, as shown in Fig. 1.

Subsequently, the chain has to be *optimally parametrised* based on the open-loop topology obtained after opening the closed loops. The parametrisation of an open-loop chain will be considered *optimal* if the frame of reference of each link is located at the same point as the R joint joining it to the next link towards the ground. If no link of the chain is joined to the ground, any link of the chain connected only to one link can play the role of the link joined to the ground. For links that are joined to the next link towards the ground with a P joint, if all the joints with other links are also prismatic, the origin of the frame of reference can be located at any point. But if any other joint is R, the origin of the frame of reference will have to be located at the location of any of those R joints. Fig. 1 exemplifies the way in which a chain can be opened and parametrised.

The *Optimal parametrisation*, as defined in this section, is customary in mechanism analysis and it does not involve a practical limitation in the applicability of the method. Additionally, the position of the joining points in a link has to be defined using Cartesian Coordinates (λ_x,λ_y) with respect

[3]If all of them are prismatic, there exists a mechanical redundancy that needs to be eliminated. This in turn implies that at least one of the joints will not be prismatic.

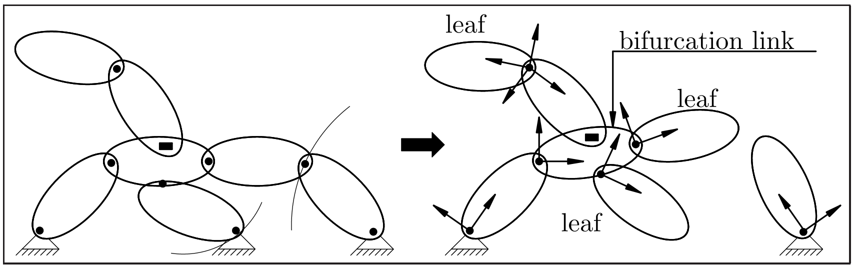

Figure 1: Opening the loop and *optimally parametrising* an articulated chain. ▬ represents prismatic joints while • represents revolution joints.

to the frame of the link. This is not a limitation because any parametrisation can be recast to Cartesian Coordinates.

3.2. Some definitions.

For the description of the following conditions and for the accuracy of the demonstrations given in the Appendix, the next definitions are given:

- A *leaf* will be any link, excluding the ground, that is joined only to one link.
- A *bifurcation link* will be any link, excluding the ground, that is joined to more than two links.
- A *serial chain* will be a chain of links including a *leaf* or *bifurcation link* and all the links towards the ground down to a link joined to a bifurcation link or to the ground.

Whenever the terms *leaf*, *bifurcation link* and *serial chain* are used throughout this paper, they will refer to these definitions. A graph describing three *serial chains* and a *bifurcation link* is described in Fig. 2.

3.3. Conditions on the selection of the dependent parameters.

Once the model has been parametrised, an IDM can be built. In order to apply Gautiers algorithm and determine the *base parameters* as in Eq. (17), vector $\boldsymbol{\phi}$ has to be divided into $\boldsymbol{\phi}_1$ and $\boldsymbol{\phi}_2$. If $rank(\mathbf{W}_{n\times m}) = b$, then $\boldsymbol{\phi}_2$ will be a $(m-b)\times 1$ vector. As each row of $\mathbf{V}_2$ in Gautiers algorithm is associated with a parameter of vector $\boldsymbol{\phi}$, selecting the rows that form $\mathbf{V}_{22}$ is equivalent to selecting the parameters that will form $\boldsymbol{\phi}_2$.

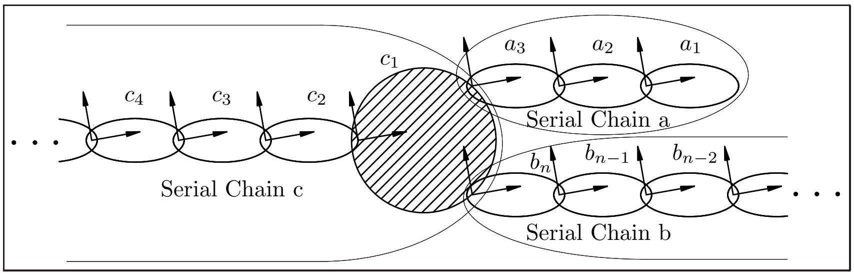

Figure 2: Three *serial chains* (a, b and c) and a *bifurcation link* (c_1).

A necessary condition for the algorithm to success is that the mass parameter of the closest-to-the-ground link of each *serial chain* is included in vector $\boldsymbol{\phi}_2$.

3.4. Summary of conditions

The conditions that the mechanism has to satisfy for the applicability of the algorithm can be summarised as follows:

- The loops of the analysed mechanism (if any) have to be closed through the ground.
- The mechanism has to be *optimally parametrised*, and the location of each fixed point in a link has to be expressed in Cartesian coordinates.
- The mass parameter of the link closest to the ground of each *serial chain* has to be selected as part of vector $\boldsymbol{\phi}_2$ in Gautier's algorithm.

Although the first condition limits the applicability of the algorithm, for an important number of practical mechanisms, the ground is part of the closed loops. Moreover, the second and third conditions only constraint the way in which the procedure has to be applied: any parametrisation (linear in the parameters) can be cast in this form and the determined symbolic base parameters recast back in terms of the original parameters.

4. ALGORITHM DESCRIPTION

The algorithm to obtain the symbolic expressions of the base inertial parameters is divided into three steps: 1) build the linear-in-the-dynamic-parameters IDM using an *optimal parametrisation*, 2) obtain numerical values for the $\boldsymbol{\beta}$ weights, and 3) search for the symbolic expressions that match the numerical $\boldsymbol{\beta}$ weights.

The first and second steps are based on Gautier's numerical algorithm described in Section 2.

The third step is based on the fact that, under the conditions mentioned in Section 3, the base parameter weights ($\boldsymbol{\beta}$) belong to a specific symbolic class of functions. The algorithm compares the numerical value of an element of matrix $\boldsymbol{\beta}$ with a sequence of symbolic candidates within the corresponding class until one of the candidates is found to match the current element of $\boldsymbol{\beta}$ within the required tolerance. The process is repeated for all the elements of $\boldsymbol{\beta}$.

4.1. Symbolic class of the $\boldsymbol{\beta}$ weights

As matrix $\boldsymbol{\beta}$ is constant, the $\boldsymbol{\beta}$ weights can only depend on the parameters of the geometric model, i.e. $\boldsymbol{\beta}_{ij} = \boldsymbol{\beta}_{ij}(\boldsymbol{\lambda}, \boldsymbol{\Lambda})$.

As mentioned before, vectors $\boldsymbol{\lambda}$ and $\boldsymbol{\Lambda}$ represent the set of geometric parameters. Vector $\boldsymbol{\lambda}$ includes the Cartesian coordinates of the joint point between two links, expressed in the reference frame of one of those, as depicted in Fig. 3. Vector $\boldsymbol{\Lambda}$ includes the squares of the distances between the joint points and the origin of the reference for all joint points, as depicted in Fig. 3 as well.

As shown in the Appendix, if the conditions mentioned in Section 3 are met, the $\boldsymbol{\beta}$ weights can be written as in Eq. (19).

$$\boldsymbol{\beta}_{ij} = \prod_{k=1}^{2N} \boldsymbol{\lambda}_k^{a_{ijk}} \prod_{h=1}^{N} \boldsymbol{\Lambda}_h^{b_{ijh}}, \qquad a_{ijk}, b_{ijh} = -1, 0, 1, \tag{19}$$

where N is the number of elements of vector $\boldsymbol{\Lambda}$. $\boldsymbol{\lambda}_k$ and $\boldsymbol{\Lambda}_h$ are the k^{th} and h^{th} elements of $\boldsymbol{\lambda}$ and $\boldsymbol{\Lambda}$ respectively, and coefficients a_{ijk} and b_{ijh} are the exponents of $\boldsymbol{\lambda}_k$ and $\boldsymbol{\Lambda}_h$ for the $\boldsymbol{\beta}_{ij}$ base parameter weight. Eq. (19) represents a Product Function Class (PFC) and if a base parameter weight belongs to this class, it will be said that *it is* PFC. Additionally, if the base parameter weights of a link are PFC, it will also be said that the base parameters are PFC.

At this point it is known that each $\boldsymbol{\beta}_{ij}$ can be written as in Eq. (19), however the actual values of the corresponding exponents a_{ijk} and b_{ijh} for each $\boldsymbol{\beta}_{ij}$ are still unknown. In order to determine the exponents for a single $\boldsymbol{\beta}_{ij}$, the RHS of Eq. (19) is evaluated for different combinations of the exponents

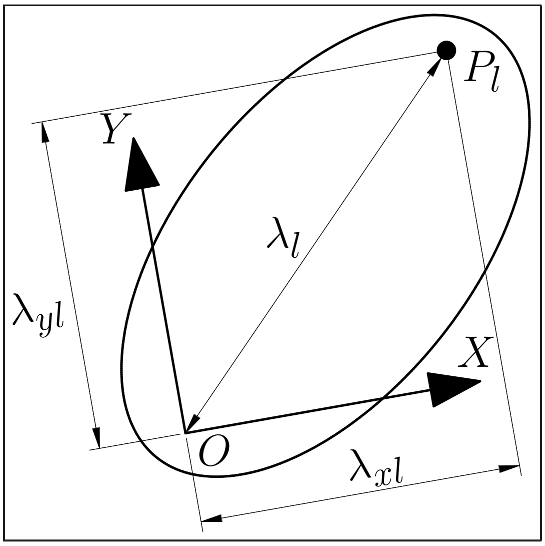

Figure 3: Location of joint point P_l in terms of the Cartesian coordinates λ_{xl} and λ_{yl}.

until it is equal to the given $\boldsymbol{\beta}_{ij}$. For a certain N, the number of different possible combinations for a single $\boldsymbol{\beta}_{ij}$ would be $3^{2N} \cdot 3^N = 3^{3N}$. Checking such number of combinations could be unmanageable even for small values of N. Fortunately, as it will be shown next, some simplifications can be done that dramatically reduce the number of combinations that need to be checked.

4.2. Simplification on the exponents

The dimensions of the $\boldsymbol{\beta}$ weights can be easily determined by looking at the expressions for the numerical base inertial parameters. Thus, denoting the dimensions of $\boldsymbol{\beta}_{ij}$ by $[\boldsymbol{\beta}_{ij}]$, and based on Eq. 17, $[\boldsymbol{\beta}_{ij}]$ can be obtained as a quotient of the dimensions of the inertial parameters of vectors $\boldsymbol{\phi}_1$ and $\boldsymbol{\phi}_2$:

$$[\boldsymbol{\beta}_{ij}] = \frac{[\boldsymbol{\phi}_{1,i}]}{[\boldsymbol{\phi}_{2,j}]} \tag{20}$$

Since the dimensions of the inertial parameters are ML^0, ML^1 or ML^2 (being M and L the dimensions of mass and length respectively) the dimensions of $\boldsymbol{\beta}_{ij}$ will always be $L^{e_{ij}}$ with $e_{ij} = 0, \pm 1, \pm 2$. Moreover, as will be demonstrated in Appendix A.6, all $\boldsymbol{\beta}_{ij}$ will be either equal to 1, equal to an element (or its inverse) of vectors $\boldsymbol{\lambda}$ or $\boldsymbol{\Lambda}$, or a product (or quotient) of two elements of $\boldsymbol{\lambda}$ or $\boldsymbol{\Lambda}$.

Taking into account the dimensional information in the $\boldsymbol{\beta}$ weights, and the reduced number of products between the elements of $\boldsymbol{\lambda}$ and $\boldsymbol{\Lambda}$, the proposed PFC is simplified to:

$$
\begin{aligned}
&\text{if } [\boldsymbol{\beta}_{ij}] = \varnothing \qquad && \boldsymbol{\beta}_{ij} = 1, \text{ or} && \text{(21a)}\\
& && \boldsymbol{\beta}_{ij} = \frac{\boldsymbol{\lambda}_k}{\boldsymbol{\lambda}_h}, \text{ for certain } k \text{ and } h. && \text{(21b)}\\
&\text{if } [\boldsymbol{\beta}_{ij}] = L, && \boldsymbol{\beta}_{ij} = \boldsymbol{\lambda}_k, \text{ for certain } k, \text{ or} && \text{(21c)}\\
& && \boldsymbol{\beta}_{ij} = \frac{\boldsymbol{\Lambda}_k}{\boldsymbol{\lambda}_h}, \text{ for certain } k \text{ and } h. && \text{(21d)}\\
&\text{if } [\boldsymbol{\beta}_{ij}] = L^2, && \boldsymbol{\beta}_{ij} = \boldsymbol{\Lambda}_k, \text{ for certain } k, \text{ or} && \text{(21e)}\\
& && \boldsymbol{\beta}_{ij} = \boldsymbol{\lambda}_k\boldsymbol{\lambda}_h, \text{ for certain } k \text{ and } h. && \text{(21f)}
\end{aligned}
$$

For $[\boldsymbol{\beta}_{ij}] = L^{-1}, L^{-2}$, the inverse of $\boldsymbol{\beta}_{ij}$ will be checked for the PFC of Eqs. (21).

Notice, comparing Eq. (19) with Eq. (21), that the number of possible combinations has been reduced dramatically. Accordingly, exploring the class of functions reduces to check all the possible combinations of the elements of vectors $\boldsymbol{\lambda}$ and $\boldsymbol{\Lambda}$ until the matching expression is found. Therefore the number of possible combinations to check, for Expressions (21a) to (21f), are: 1, $4N^2$, $2N$, $2N^2$, N and $4N^2$, respectively.

4.3. Numerical values for the geometric parameters

Since a one-to-one correspondence is sought between the symbolic geometric parameters and their numerical counterparts, if we select the same numerical value for two (symbolically) different parameters, the symbolic searching algorithm will not be able to distinguish which symbolic parameter corresponds to a given numerical value. Therefore, it is necessary to assign different numerical values to each of the lengths in the model. Unit lengths have to be avoided and we also have to ensure that the product of two lengths is not equal to any other length. This can be achieved, for instance, assigning to each length a different prime number and multiplying them by a common factor. To guarantee the assemblability and mobility of the mechanism, these values should be chosen close enough to the actual geometric values. Obviously, the numerical base inertial parameters have to be calculated using this set of geometric values.

4.4. The algorithm summarised

In order to make the algorithm clearer, it is written here in pseudo-code form:

```
Eliminate mechanical redundancies if any;
Open the loops cutting through a R joint;
Parametrise optimally;
Define joint point position in terms of λ_xl, λ_yl and λ_l;
Choose appropriate numerical values for geometric
parameters;
Define the λ, Λ vectors;
Select the inertial parameters to form φ_2;
Find numerical matrix β with Gautier's SVD algorithm;
forall the β_ij do
    Find the dimensions of β_ij;
    forall the possible candidates A according to
    Eq. (21) do
        if |A − β_ij| < TOL then
            write symbolic equivalent and go to next β_ij;
        end
    end
end
```

Algorithm 1. Pseudo-code for the searching algorithm.

In this algorithm, letter A represents the product combinations of elements of vectors $\boldsymbol{\lambda}$ and $\boldsymbol{\Lambda}$. TOL represents the numerical tolerance allowed for the matching.

5. THE FOUR BAR EXAMPLE

In this section it is shown how the symbolic base inertial parameters of the four bar mechanism represented in Fig. (4) can be calculated using the proposed algorithm.

5.1. Parametrisation

Since the four bar is a closed-loop mechanism, it is necessary to cut the closed loop in a R joint. In this case the chain has been cut in the joint located at point D (see Fig. 4). Once the loop is open, the *optimal parametrisation* has been performed, locating the origins of the reference of each link in the R joint with the next link towards the ground. Reference 2 has been defined with a general orientation with respect to vector $\overline{BC}$, and for the sake of simplicity, references 1 and 3 have been aligned to vectors $\overline{AB}$ and $\overline{CD}$, respectively.

5.2. Generalised coordinates and geometric parameters

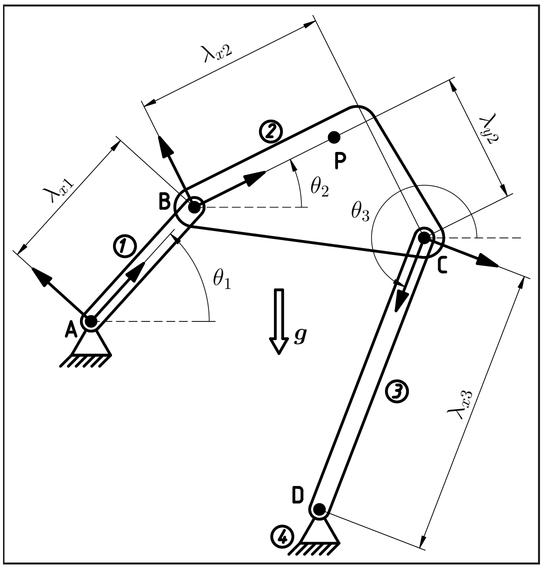

Figure 4: Geometric model of the four bar mechanism

The generalised coordinates used in this case are absolute orientation angles with respect to the horizontal line ($\theta_1, \theta_2, \theta_3$). However, there is no constraint in the selection of coordinates.

As a requirement of the algorithm, the constant lengths λ_{x1}, λ_{x2}, λ_{y2} and λ_{x3}, are used to define the position of the joint points B, C and D with respect to the link frames in Cartesian coordinates.

5.3. Choose appropriate numerical values

The following numerical values are used for the parameters:

$$\lambda_{x1} = \frac{3}{10}, \quad \lambda_{x2} = \frac{7}{10}, \quad \lambda_{y2} = \frac{11}{10}, \quad \lambda_{x3} = \frac{13}{10}. \tag{22}$$

Note that since λ_{y2} has been considered a positive length, the position vector from point B to point C has been given as $\{\overline{BC}\}_{Ref2} = \{\lambda_{x2}, -\lambda_{y2}\}_{Ref2}$ when building the IDM.

5.4. The $\boldsymbol{\lambda}$ and $\boldsymbol{\Lambda}$ vectors

Following the definitions given in Section 4.1, the $\boldsymbol{\lambda}$ and $\boldsymbol{\Lambda}$ vectors are:

$$\boldsymbol{\lambda} = \{\lambda_{x1}, \lambda_{x2}, \lambda_{y2}, \lambda_{x3}\} \tag{23a}$$

$$\boldsymbol{\Lambda} = \{\lambda_1^2, \lambda_2^2, \lambda_3^2\} \tag{23b}$$

where $\lambda_i^2 = \lambda_{xi}^2 + \lambda_{yi}^2$ for $i = 1, \cdots, N$. For the current example $N = 3$ and $\lambda_{y1} = \lambda_{y3} = 0$.

5.5. Find matrix $\boldsymbol{\beta}$ with Gautier's SVD algorithm

The symbolic multibody program 3D_MEC (Ros et al., 2005) has been used to obtain the IDM of the system based on the geometric model data presented in Fig. (4). The first and second moments of inertia of each link are defined with respect to the reference frame of the link.

In order to evaluate matrix $\mathbf{W}$, an inverse kinematic simulation has been performed. The trajectory for the independent coordinate θ_1 has been the one shown in Fig. (5). The trajectory has been built as a finite Fourier Series, and it has been optimised using 11 harmonics in order to sufficiently excite the mechanism (Swevers et al., 1996, 1997).

After evaluating $\mathbf{W}$ in Eq.(6), the algorithm presented in Section 2 is applied to obtain the matrix $\boldsymbol{\beta}$ numerically. Taking the SVD of $\mathbf{W}$, matrices $\mathbf{U}$, $\boldsymbol{\Sigma}_0$ and $\mathbf{V}$ have been obtained. From the diagonal of $\boldsymbol{\Sigma}_0$ it is observed that $b = rank(\mathbf{W}) = 8$, from which it can be deduced that $\boldsymbol{\phi}_1$ is a 8×1 vector and that $\boldsymbol{\phi}_2$ is a 4×1 vector. Taking the last 4 columns of $\mathbf{V}$ matrix $\mathbf{V}_2$ is obtained. Each of the 12 rows of $\mathbf{V}_2$ corresponds to an inertial parameter, with the order given by the original parametrisation $\boldsymbol{\phi}$. Since the rows of $\mathbf{V}_2$ associated to parameters J_1, J_2, J_3 and m_1 are independent[4], matrix $\mathbf{P}$ is determined such that it reorders $\boldsymbol{\phi}$ making these parameters the last 4

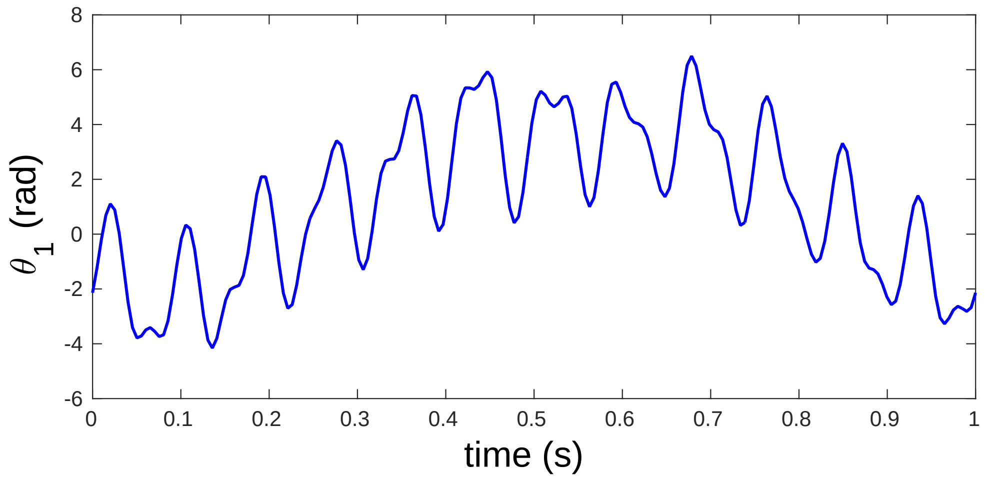

Figure 5: Independent coordinate θ_1 in the inverse dynamic simulation.

elements of $\mathbf{P}^\top \boldsymbol{\phi}$. Note that, as explained in Section 3, selecting m_1 was compulsory since link 1 is the closest-to-the-ground link of a *serial chain*.

Matrix $\mathbf{V}_{22}$ is therefore obtained as the last $m - b = 4$ rows of $\mathbf{P}^\top \mathbf{V}_2$. Accordingly the first $b = 8$ rows of $\mathbf{P}^\top \mathbf{V}_2$ give the 8×4 matrix $\mathbf{V}_{21}$.

Finally, $\boldsymbol{\beta}$ is obtained making use of Eq. (16). The numerical base inertial parameters obtained (omitting the null column of $\boldsymbol{\beta}$ multiplying to m_1) are:

$$\boldsymbol{\phi}_b = \boldsymbol{\phi}_1 + \boldsymbol{\beta}\boldsymbol{\phi}_2 = \begin{Bmatrix} mx_1 \\ my_1 \\ mx_2 \\ my_2 \\ m_2 \\ mx_3 \\ my_3 \\ m_3 \end{Bmatrix} + \begin{bmatrix} -3.333 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & -0.4118 & 0 \\ 0 & 0.6471 & 0 \\ 11.11 & -0.5882 & 0 \\ 0 & 0 & -0.7692 \\ 0 & 0 & 0 \\ 0 & 0.5882 & -0.5917 \end{bmatrix} \begin{Bmatrix} J_1 \\ J_2 \\ J_3 \end{Bmatrix} \tag{24}$$

[4]If no independent group of rows of $\mathbf{V}_2$ can intuitively be selected, an algorithm to calculate the *reduced row echelon form* of a matrix (as `rref` in Matlab) can be used to determine a group of $m - b$ rows of $\mathbf{V}_2$ that are independent.

Matrix $\mathbf{W}_R$ for this example is built with the columns of $\mathbf{W}$ corresponding to the parameters in $\boldsymbol{\phi}_1$.

5.6. Searching

The searching phase of the algorithm is performed to find matching expressions for each $\boldsymbol{\beta}_{ij}$ of $\boldsymbol{\beta}$.

According to the dimensions of the inertial parameters of vectors $\boldsymbol{\phi}_1$ and $\boldsymbol{\phi}_2$, it is deduced from Eq. (20) that the $\boldsymbol{\beta}$ weights have dimensions of L^{-1} and L^{-2} as shown in Table 1.

Dimensions	$\boldsymbol{\beta}_{ij}$ or ${\boldsymbol{\beta}_{ij}}^{-1}$
$[L^0]$	$\varnothing$
$[L^1]$	$(\frac{1}{3.333}, \frac{1}{0.4118}, \frac{1}{0.6471}, \frac{1}{0.7692})$
$[L^2]$	$(\frac{1}{11.11}, \frac{1}{0.5882}, \frac{1}{0.5917})$

Table 1: Dimensions of the $\boldsymbol{\beta}_{ij}$ weights.

As described in Eq. (21), depending on the dimensions of each $\boldsymbol{\beta}$ weight, two possible expression types can match it. Checking all the possible combinations of the elements of $\boldsymbol{\lambda}$ and $\boldsymbol{\Lambda}$, the symbolic equivalent expression of the given numerical value for the $\boldsymbol{\beta}$ weight is found. After the search is completed, the symbolic base inertial parameters obtained for the four bar model are:

$$\boldsymbol{\phi}_b = \begin{Bmatrix} mx_1 \\ my_1 \\ mx_2 \\ my_2 \\ m_2 \\ mx_3 \\ my_3 \\ m_3 \end{Bmatrix} + \begin{bmatrix} -\lambda_{x1}^{-1} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & -\lambda_{x2}/\lambda_2^2 & 0 \\ 0 & \lambda_{y2}/\lambda_2^2 & 0 \\ \lambda_1^{-2} & -\lambda_2^{-2} & 0 \\ 0 & 0 & -\lambda_{x3}^{-1} \\ 0 & 0 & 0 \\ 0 & \lambda_2^{-2} & -\lambda_3^{-2} \end{bmatrix} \begin{Bmatrix} J_1 \\ J_2 \\ J_3 \end{Bmatrix} \tag{25}$$

The tolerance used has been $TOL = 10^{-10} \cdot min(|\boldsymbol{\beta}_{ij}|)$, $\forall \boldsymbol{\beta}_{ij} \neq 0$. Note that $\lambda_1 = \lambda_{x1}$ and $\lambda_3 = \lambda_{x3}$ in this example.

The expressions of the base parameters obtained in this example have also been determined using the *inertia transfer concept* described in Chen et al. (2002) and Ros et al. (2012), and the same results have been obtained.

6. CONCLUSIONS

In this paper a new method to obtain the symbolic expressions of the base inertial parameters of open and closed-through-the-ground planar mechanisms has been proposed and illustrated with an example. The method is based on the well known numerical method of Gautier but provides the exact *symbolic* expressions of the base inertial parameters as a function of the constant geometric parameters of the system. Therefore, it constitutes a very valuable tool in design optimisation and it is also very interesting in dynamic parameter estimation, model reduction and other fields.

The proposed method searches for matches of the values of symbolic candidate expressions, with the numerically calculated versions of the $\boldsymbol{\beta}$ weights. Therefore, it requires the same level of user intervention as the most common numerical algorithms, with the advantage of providing symbolic expressions for the base inertial parameters.

For the defined *optimal parametrisation*, using the inertia transfer concept, it has been demonstrated that for planar mechanisms where the existing loops are closed through the ground, the elements of $\boldsymbol{\beta}$ can always be written as products of lengths (or their inverses). The conditions on the parametrisation do not impose a practical limitation, as any parametrisation of a planar mechanism can be recast to the proposed one.

Appendix A.

Appendix A.1. Introduction

In this Appendix it is shown that for a planar mechanism that is *optimally parametrised* (as defined in Section 3) and which can have loops closed through the ground, if the mass property of the last body of each *serial chain* (as defined in Section 3) is selected as a parameter to eliminate, then the resulting base parameter weights belong to the Product Class of Functions (PCF) described by Eq. (19). It is also shown that the PFC described by Eq. (19) can be simplified to that of Eq. (21).

The demonstrations are based on the mass and inertia transfer concepts introduced by Chen et al. (2002). As stated in Propositions 1 and 2 of that paper, when two links are joined by a revolute joint (see Fig. A.1) an arbitrary point mass (at R) can be transferred from one link to the other without changing the kinetics of the kinematic chain. Similarly, when two links are joined with a prismatic joint, an arbitrary amount of second moment of inertia can be transferred without changing the kinetics of the chain. The transferred mass or second moment of inertia are chosen appropriately to remove the desired parameters. In order to obtain the base parameter set, the above process is repeated until the maximum number of parameters is removed.

The algorithm of Gautier (1991), presented in Section 2, is based on the same idea: a set of inertial parameters is removed from the original parameter set. It has been shown that this numerical algorithm produces a unique set of base parameters provided that the permutation matrix ($\mathbf{P}$) is given. This is equivalent to specify which inertial parameters we want to eliminate from the original parameter set. Such eliminations can be traced back to the corresponding mass or second moment of inertia transfers between links. It follows that the $\boldsymbol{\beta}$ matrix obtained from the mass transfer concept and that of Gautier match each other if the same set of original inertial parameters is removed.

So, it is enough to demonstrate that the elimination process of Chen et al. (2002) implies that the expressions of the elements of $\boldsymbol{\beta}$ are PFC.

The demonstration will proceed as follows. In Appendix A.2 the mass and inertia transfers will be explained. In Appendix A.3 it will be shown that the base parameters of *serial chains* are PFC. In Appendix A.4 and Appendix A.5 it will be shown, respectively, that the base parameters of tree-like mechanisms and those of mechanisms with closed loops through the ground are PFC. It will be concluded that the parameters of a general mechanism will also be PFC. Finally, in Appendix A.6 it will be shown that the Product Function Class of Eq. (19) can be simplified to the class described in Eq. (21).

Appendix A.2. Mass and inertia transfers at revolute and prismatic joints in a serial chain

In this section it is explained in detail how an inertial parameter is eliminated from the equations of motion for a pair of links joined by a R or P joint. The concept of inertia transfer described in Chen et al. (2002) is used for that purpose. From here on, the base inertial parameter expressions that

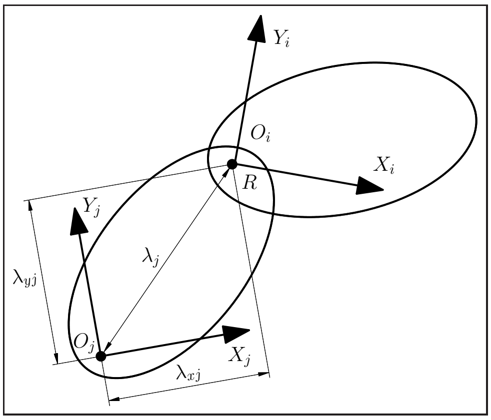

Figure A.1: Links i and j joined by a revolute joint. The position of point O_i in reference (X_j, Y_j) is defined in terms of λ_{xj}, λ_{yj} and λ_j.

result from a single inertia transfer will be called *intermediate base parameters* since they are an intermediate step in the process of calculating the base inertial parameters.

Appendix A.2.1. Mass transfer at a revolute joint

Let links i and j be joined with a revolute joint at point R, as shown in Fig. (A.1). If a point mass m_{ij} is transferred from link i to link j at point R, the equations that define the expressions of the *intermediate base parameters*, denoted by a *prime* ($'$), are:

$$\begin{aligned}
m'_j &= m_j + m_{ij} & m'_i &= m_i - m_{ij} \\
mx'_j &= mx_j + \lambda_{xj} \cdot m_{ij} & mx'_i &= mx_i \\
my'_j &= my_j + \lambda_{yj} \cdot m_{ij} & my'_i &= my_i \\
J'_j &= J_j + \lambda_j^2 \cdot m_{ij} & J'_i &= J_i.
\end{aligned} \tag{A.1}$$

This system of equations consists of eight equations and nine unknowns, so it is possible to assign a value to m_{ij} such that one of the new parameters is equal to zero. This leads to its removal from the base parameter set. The whole set of possible removals by the mass transfer is:

$$\begin{array}{rcl} \underline{\text{Removed Parameters}} & & \underline{\text{Transferred mass}} \\ m'_i = 0 & \Longrightarrow & m_{ij} = m_i \quad \text{(A.2a)} \\ mx'_j = 0 & \Longrightarrow & m_{ij} = -\dfrac{1}{\lambda_{xj}} \cdot mx_j \quad \text{(A.2b)} \\ my'_j = 0 & \Longrightarrow & m_{ij} = -\dfrac{1}{\lambda_{yj}} \cdot my_j \quad \text{(A.2c)} \\ J'_j = 0 & \Longrightarrow & m_{ij} = -\dfrac{1}{\lambda_j^2} \cdot J_j \quad \text{(A.2d)} \\ m'_j = 0 & \Longrightarrow & m_{ij} = -m_j \quad \text{(A.2e)} \end{array}$$

It is clear that making any of the primed parameters equal to zero and substituting back the corresponding m_{ij} expression from Eqs. (A.2) into Eqs. (A.1), the weights of the *intermediate base parameters* will be PFC.

Appendix A.2.2. Second moment of inertia transfer through a prismatic joint

Now, let links i and j be joined by a prismatic joint. If a given part of the second moment of inertia, J_{ij}, is transferred from link i to link j, the equations that define the *intermediate base parameters* for each link are the following:

$$\begin{aligned} m'_j &= m_j & \qquad m'_i &= m_i \\ mx'_j &= mx_j & mx'_i &= mx_i \\ my'_j &= my_j & my'_i &= my_i \\ J'_j &= J_j + J_{ij} & J'_i &= J_i - J_{ij}. \end{aligned} \tag{A.3}$$

In this case, only one of the two *intermediate* moments of inertia can be made equal to zero. To that end, the transferred inertia, J_{ij}, should be one of the following:

$$\begin{array}{rcl} \underline{\text{Removed Parameters}} & & \underline{\text{Transferred moment}} \\ J'_i = 0 & \Longrightarrow & J_{ij} = J_i \quad \text{(A.4a)} \\ J'_j = 0 & \Longrightarrow & J_{ij} = -J_j \quad \text{(A.4b)} \end{array}$$

As in the case of mass transfer, it is clear here too that for any of the two possible cancellations, the weights of the *intermediate base parameters* will be PFC.

Appendix A.3. The base parameters of a serial chain are PFC

Let us first suppose that links i and j might have done a single transfer but in any case their *intermediate base parameters* are PFC. Applying Eqs. (A.1) and (A.2) if joint ij is R (or Eqs. (A.3) and (A.4) if joint ij is P) it is easy to see that the resulting new *intermediate base parameters* of both links are PFC. Note that a joint participates only once in the inertia transfer process.

Appendix A.4. Extension to tree structure mechanisms

A tree structure mechanism is composed of *serial chains* that are joined together in *bifurcation links*, as it has been described in Section 3. For arbitrary inertia transfers between a *bifurcation link* and the link of each *serial chain* joined to it, it is straight forward to see, by application of Eqs. (A.1) to (A.4), that in general the resulting base inertial parameters are not PFC. However, if the mass of the links (a) joined to the leaf side of a *bifurcation link* (b) are removed through a mass transfer , the *intermediate base parameters* become:

$$\begin{aligned}
m'_b &= m_b + \sum_a m'_a & m''_a &= 0, & \forall a. \\
mx'_b &= mx_b + \sum_a \lambda_{xa} \cdot m'_a & mx''_a &= mx'_a, & \forall a. \\
my'_b &= my_b + \sum_a \lambda_{ya} \cdot m'_a & my''_a &= my'_a, & \forall a. \\
J'_b &= J_b + \sum_a \lambda_a^2 \cdot m'_a & J''_a &= J'_a, & \forall a.
\end{aligned} \tag{A.5}$$

where m'_a, mx'_a, my'_a and J'_a are the inertia parameters of link a before the transfer. As m'_a is PFC, the *intermediate base parameters* of the *bifurcation link* b (m'_b, mx'_b, my'_b and J'_b) and those of link a (m''_a, mx''_a, my''_a and J''_a) will be PFC.

Appendix A.5. Extension to mechanisms with closed loops through the ground

For the purposes of the demonstration, mechanisms with closed loops are transformed to tree-like mechanisms cutting the different loops by one of their joints.

It is easy to see that for mechanisms with general closed loops, the resulting base inertial parameters will not generally be PFC. When a chain is cut through a joint to open the closed loop, the transfer that is done makes the *intermediate base parameter* expressions of the chains at both sides of the cut share a common inertial parameter. As a consequence, when subsequent transfers are done towards the ground and reach a *bifurcation link*, a sum of two expressions that contain a common parameter may occur, and the *intermediate base parameters* will not be PFC. Obviously, when the existing loops are closed through the ground, this recombination of parameters does not occur and the *intermediate base parameters* will be PFC.

Appendix A.6. The base parameter weights involve at most two geometric parameters

From Appendix A.2 to Appendix A.5, it has been demonstrated that the base parameters are PFC. In this section, it will be demonstrated that the structure of Eq. (19) can be simplified to that of Eq. (21), where the $\boldsymbol{\beta}$ weights can be written as products of, at most, two elements of vectors $\boldsymbol{\lambda}$ or $\boldsymbol{\Lambda}$.

Let us first suppose that links i and j of Fig. (A.1) could have done one of their two transfers with another link. Second, let us suppose that as a consequence of that possible transfer, the $\boldsymbol{\beta}$ weights of the *intermediate base parameters* satisfy the next two conditions:

a) The $\boldsymbol{\beta}$ weights of the *intermediate base parameters* m'_i and m'_j are functions of, at most, a single geometric parameter.

b) The $\boldsymbol{\beta}$ weights of the *intermediate base parameters* mx'_i, my'_i, J'_i, mx'_j, my'_j and J'_j are functions of, at most, two geometric parameters.

Analysing all the possible parameter eliminations for the transfer between links i and j described by Eqs. (A.1) to (A.4), it is easy to see that the *intermediate base parameters* also satisfy conditions $a)$ and $b)$ after the transfer. As a consequence, all the $\boldsymbol{\beta}$ weights will depend on, at most, two elements of vectors $\boldsymbol{\lambda}$ and/or $\boldsymbol{\Lambda}$. In this step, it is necessary to note that, the first and second moments of inertia of link j before the transfer with link i has taken place, will always coincide with those of j before its first transfer, i.e. they are equal to the inertial parameters of the original link.

Note also that the demonstration holds as well for the case in which j is a *bifurcation link*.

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