Journal of Pure and Applied Algebra 228 (2024) 107543

Contents lists available at ScienceDirect

Journal of Pure and Applied Algebra

journal homepage: www.elsevier.com/locate/jpaa

Jordan 3-graded Lie algebras with polynomial identities

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ARTICLE INFO

Article history:
Received 9 September 2023
Received in revised form 29 September 2023
Available online 5 October 2023
Communicated by C.A. Weibel

MSC:
17B01; 17B05; 17C05

Keywords:
Jordan Lie algebra
Polynomial identity
TKK-construction
Central closure

ABSTRACT

We study Jordan 3-graded Lie algebras satisfying 3-graded polynomial identities. Taking advantage of the Tits-Kantor-Koecher construction, we interpret the PI-condition in terms of their associated Jordan pairs, which allows us to formulate an analogue of Posner-Rowen Theorem for strongly prime PI Jordan 3-graded Lie algebras. Arbitrary PI Jordan 3-graded Lie algebras are also described by introducing the Kostrikin radical of the Lie algebras.

1. Introduction

One of the main structure theorems of associative PI-algebras is Posner theorem for prime associative algebras. The classical, associative, version of this theorem states that any prime ring satisfying a polynomial identity over its centroid is a Goldie ring and it has a primitive PI ring of quotients [5].

A Jordan analogue of Posner, or more accurately of Posner-Rowen, theorem was settled in [20] for strongly Jordan systems having local algebras satisfying polynomial identities. This result was named Posner-Rowen, instead of Posner theorem, since it makes use of the notion of extended central closure that replaces that of the classical central closure construction for Jordan systems. This result was later extended in [21], since as conjectured in [20], extended centroids of strongly prime homotope PI Jordan systems coincide with the field of fractions of their centroids.

In the associative pair setting, the classical results of the associative PI-theory have been recently extended in [23] to associative pairs. These results include, aimed by [19], the treatment of the notion of PI-element

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https://doi.org/10.1016/j.jpaa.2023.107543

for associative pair, and also associative pair analogue of Amitsur (already proved in [22]), Kaplansky and Martindale theorem, and also that of Posner-Rowen theorem for prime associative pairs satisfying homotope polynomial identities, which is again formulated in terms of central closures of the associative pairs and those of their standard embeddings.

In this paper we address the structure theorems of Jordan 3-graded Lie algebras satisfying 3-graded polynomial identities, more precisely, we consider the Jordan 3-graded Lie analogue of Posner-Rowen theorem. Given the available results for strongly prime homotope PI Jordan pairs proved in [20], and the fact that Jordan PI pairs are homotope PI [22], there are two more issues to be tackled. The first one, the relationship between Jordan 3-graded Lie algebras and Jordan pairs is settled by the Tits-Kantor-Koecher construction [8–12,27], that provides us with a suitable channel connecting Jordan and 3-graded Lie constructions, as for instance, those of the extended centroids and (extended) central closures. The second issue is to ensure that the considered 3-graded Lie polynomial identities are smoothly transferred to the associated Jordan pairs. To do this we will consider the relationship, again established through the Tits-Kantor-Koecher construction, between the corresponding Lie and Jordan free objects, to check that there exists a operant version of 3-graded Lie polynomial identity, named essential identity [29], that allows us to transfer the PI condition from Jordan 3-graded Lie algebras to their associated Jordan pairs.

After this introductory section, and a section of preliminaries, in the third section we settle the relationship between the extended centroid of a nondegenerate Jordan pair and that of its TKK-algebra. The natural isomorphism between both constructions obtained in this section, is considered in section four to prove the existence of an isomorphism between the central closure of the TKK-algebra and the TKK-algebra of the extended central closure of a nondegenerate Jordan pair.

We recall here that the notions of extended centroid and central closure were first developed to study prime associative rings satisfying generalized polynomial identities [15], and later extended to the non-associative framework in [2,3]. The Jordan counterpart of these constructions, named extended centroid and extended central closure were introduced in [20], precisely aimed to study Jordan systems having nonzero local algebras satisfying polynomial identities. One of the main results in [20] is precisely the Jordan version of Posner-Rowen theorem for strongly prime Jordan pairs, formulated in terms of the extended central closure of the Jordan pairs.

In section five we settle the operative version of polynomial identity we consider here for Jordan 3-graded Lie algebras. Being our purpose to take advantage of the available results for PI Jordan pairs proved in [20,22], it becomes necessary to ensure that the condition of being a PI-algebra can be smoothly transferred through the TKK-construction, that is, that the 3-graded polynomial identities satisfied by the Jordan 3-graded Lie algebras generate essential identities of their associated Jordan pairs [20, 0,12]. To ensure this, we consider the notion of essential polynomial identity [29], and characterize essential 3-graded polynomial identities as those 3-graded polynomials of the free 3-graded Lie algebra [24] not vanishing on some special Lie algebra $sl(n)$. This condition ensures us that such a 3-graded polynomial identity of a Jordan 3-graded Lie algebra produces an essential Jordan polynomial identity for the associated Jordan pair.

The last two sections of this paper are devoted to study Jordan 3-graded Lie algebras satisfying essential 3-graded polynomial identities. In section six we assume these Lie algebras to be strongly prime, and therefore the TKK-algebras of their associated Jordan pairs, which then result to be also strongly prime [4]. Then the isomorphism obtained in section four relating TKK-constructions and (extended) central closures together to the Jordan version of Posner-Rowen theorem [20], provide us with a Jordan 3-graded Lie version of Posner-Rowen theorem for strongly prime Lie algebras. Additionally, considering all involved objects defined over a base field of characteristic zero or prime at least five, Zelmanov's description of Lie algebras with a finite nontrivial $\mathbb{Z}$-grading [29] applies here providing an accurate description of the central closures of the Jordan 3-graded Lie algebras satisfying essential 3-graded polynomial identities.

Finally, we describe general Jordan 3-graded Lie algebras satisfying essential 3-graded polynomial identities, under no additional regularity requirements. To do this, we divide by suitable radical ideals that provide us with nondegenerate quotient Lie algebras and Jordan pairs. This is the case of the Kostrikin radical of a Lie algebra [28], that we prove here to be related, when the Lie algebra is assumed to be Jordan 3-graded, to the McCrimmon radical of its associated Jordan pair [17]. McCrimmon radical is precisely the Jordan radical that gives rise to nondegenerate quotient Jordan pairs.

Considering the quotient of a Jordan 3-graded Lie algebra by its Kostrikin radical, that turns easily out to be 3-graded, provides us with a nondegenerate Jordan 3-graded Lie algebra that can be written as a subdirect product of strongly prime Jordan 3-graded Lie algebras, all of them still satisfying the same essential polynomial identity, and therefore isomorphic to one of the algebras listed in the Jordan 3-graded version of the Posner-Rowen theorem obtained in the previous section.

2. Preliminaries

We will work with Jordan pairs and Lie algebras over Φ, a unital commutative ring containing $\frac{1}{2}$ that will be fixed throughout. We refer to [6,14] for basic notation, terminology and results on Jordan pairs and Lie algebras.

2.1. Jordan pairs and Lie algebras

A Jordan pair $V = (V^+, V^-)$ has products $Q_x y$ for $x \in V^\sigma$ and $y \in V^{-\sigma}$, $\sigma = \pm$, with linearizations $Q_{x,z}y = D_{x,y}z = \{x, y, z\} = Q_{x+z}y - Q_x y - Q_z y$.

A Lie algebra is a Φ-module L with a bilinear product, denoted $[x, y]$, satisfying $[x, x] = 0$ and the Jacobi identity $[[x, y], z] + [[y, z], x] + [[z, x], y] = 0$ for all x, y, $z \in L$.

2.2. Jordan 3-graded Lie algebras

A Lie algebra L is 3-graded if it admits a decomposition $L = L_1 \oplus L_0 \oplus L_{-1}$, where L_i is a Φ-submodule of L, for $i \in \{0, \pm 1\}$ and $[L_i, L_j] \subseteq L_{i+j}$, with $L_{i+j} = 0$ if $i + j \notin \{0, \pm 1\}$. A 3-graded Lie algebra $L = L_1 \oplus L_0 \oplus L_{-1}$ is Jordan 3-graded if $[L_1, L_{-1}] = L_0$, and the pair (L_1, L_{-1}) admits a Jordan pair structure defined by $\{x, y, z\} = [[x, y], z]$, for any x, $z \in L_\sigma$, $y \in L_{-\sigma}$, $\sigma = \pm$. Then $V = (L_1, L_{-1})$ is called the Jordan pair associated to L [25, 1.5].

Since $\frac{1}{2} \in \Phi$, by [14, Proposition 2.2(a)], any Jordan 3-graded Lie algebra L determines a Jordan product on $V = (L_1, L_{-1})$ given by $Q_x y = \frac{1}{2}\{x, y, x\} = \frac{1}{2}[[x, y], x]$. If, moreover, $\frac{1}{6} \in \Phi$, by [14, Proposition 2.2(b)], any 3-graded Lie algebra $L = L_1 \oplus L_0 \oplus L_{-1}$ defines a Jordan pair structure on (L_1, L_{-1}). Indeed, if $\frac{1}{6} \in \Phi$, any pair of Φ-modules (L_1, L_{-1}) endowed with trilinear mappings

$$\begin{array}{rcl} \{\ ,\ ,\ \} : L_\sigma \times L_{-\sigma} \times L_\sigma & \to & L_\sigma \\ (x, y, z) & \mapsto & \{x, y, z\} = D(x, y)z \end{array}$$

satisfying

(i) $\{x, y, z\} = \{z, y, x\}$,
(ii) $[D(x, y), D(u, v)] = D(\{x, y, u\}, v) - D(u, \{y, x, v\})$,

for all x, z, $u \in L_\sigma$, y, $v \in L_{-\sigma}$, $\sigma = \pm$, has a Jordan pair structure.

Any Jordan pair V defines a Jordan 3-graded Lie algebra given by the Tits-Kantor-Koecher construction, that for any Jordan pair $V = (V^+, V^-)$ gives rise to a 3-graded Lie algebra $L = L_1 \oplus L_0 \oplus L_{-1}$ with $L_1 = V^+$ and $L_{-1} = V^-$. These Lie algebras were (independently) introduced by Kantor [8–10], Koecher [11,12] and Tits [27], and also studied by Meyberg in [18]. We refer to [4, 11.2] for accurate details on this construction and write $TKK(V) = V^+ \oplus \mathrm{IDer}V \oplus V^-$ for the TKK-Lie algebra of a Jordan pair V, where $\mathrm{IDer}V$ is the ideal of the Lie algebra of derivations $\mathrm{Der}V$ of V generated by the inner derivations $\delta(x,y) := (D_{x,y}, -D_{y,x})$ of V, for all $x \in V^+$, $y \in V^-$.

Remark 2.1. The relationship between Jordan 3-graded Lie algebras and TKK-algebras was settled by Neher in [25, 1.5(6)]. For any Jordan 3-graded Lie algebra L with associated Jordan pair V, it holds that $TKK(V) \cong L/C_V$, where $C_V = \{x \in L_0 \mid [x, L_1] = 0 = [x, L_{-1}]\} = Z(L) \cap L_0$ and $Z(L) = \{x \in L \mid [x, L] = 0\}$ denotes the center of L.

2.3. $\mathbb{Z}$-graded Lie algebras

A Lie algebra L is $\mathbb{Z}$-graded if there exists a decomposition $L = \sum_{n\in\mathbb{Z}} L_n$, where L_n is a Φ-submodule of L for all $n \in \mathbb{Z}$, and $[L_i, L_j] \subseteq L_{i+j}$ for $i, j \in \mathbb{Z}$. Such a grading is finite if the set $\{n \in \mathbb{Z} \mid L_n \neq 0\}$ is finite, and nontrivial if $\sum_{n\neq 0} L_n \neq 0$. TKK-algebras of Jordan pairs are $\mathbb{Z}$-graded Lie algebras $L = L_{-1} + L_0 + L_1$, with $L_{-1} = V^-$, $L_0 = \mathrm{IDerV}$, $L_1 = V^+$ and $L_i = 0$ for $|i| > 1$ [29].

2.4. Ideals

An ideal I of a Jordan pair V is called essential if it has nontrivial intersection with any nonzero ideal of V. Essential ideals of nondegenerate Jordan pairs are those having zero annihilator. Similarly, an ideal I of a Lie algebra L is essential if and only if it has nontrivial intersection with any nonzero ideal of L. Essential ideals of semiprime Lie algebras are those with zero centralizer. Regularity conditions, such as (semi)primeness, nondegeneracy or simplicity can be transferred between Jordan pairs and their TKK-algebras [4, Proposition 11.25].

2.5. Extended centroid and central closure

The notions of extended centroid and central closure were first introduced to study prime rings satisfying generalized polynomials identities [15], and later generalized to nonassociative rings [2,3].

Let L be a semiprime Lie algebra. We denote by $Ad(L)$ the subalgebra of $End_\Phi(L)$ generated by $ad(L) = \{ad\,x \mid x \in L\}$. A pair (f, I) is a permissible map of L if I is an essential ideal of L and $f : I \to L$ is an additive map that commutes with all elements of $Ad(L)$. Following [2,3] we will say that two permissible maps (f_1, I_1) and (f_2, I_2) of a Lie algebra L are equivalent if there exists an essential ideal I of L, contained into $I_1 \cap I_2$, such that $f_1(x) = f_2(x)$ for all $x \in I$. This defines an equivalence relation on the set of permissible maps of L. We will denote by $\overline{(f, I)}$ the equivalence class determined by the permissible map (f, I), and by $\mathcal{C}(L)$ the set of all equivalence classes of permissible maps of L. The set $\mathcal{C}(L)$ is called the extended centroid of L. If L is a semiprime Lie algebra, then $\mathcal{C}(L)$ is a von Neumann regular unital algebra [2, Theorem 2.5].

Extended centroids of quadratic Jordan systems were introduced in [20]. Let V be a Jordan pair and $U = (U^+, U^-)$ an ideal of V. A pair $g = (g^+, g^-)$ of linear mappings $g^\sigma : U^\sigma \to V^\sigma$, $\sigma = \pm$, is a *V-homomorphism* of V if for all $y^\sigma \in U^\sigma$, x^σ, $z^\sigma \in V^\sigma$:

(a) $g^\sigma(Q_{x^\sigma} y^{-\sigma}) = Q_{x^\sigma} g^{-\sigma}(y^{-\sigma})$,
(b) $g^\sigma(Q_{I^\sigma} V^{-\sigma}) \subseteq I^\sigma$ and $(g^\sigma)^2(Q_{y^\sigma} x^{-\sigma}) = Q_{g^\sigma(y^\sigma)} x^{-\sigma}$,
(c) $g^\sigma(\{y^\sigma, z^{-\sigma}, x^\sigma\}) = \{g^\sigma(y^\sigma), z^{-\sigma}, x^\sigma\}$.

We denote by $Hom_V(U, V)$ the set of all V-homomorphisms from U into V [20, 1.1]. A pair (g, U) is a permissible map of V if U is an essential ideal of V.

Two permissible maps (g_1, U_1) and (g_2, U_2) of a Jordan pair V are equivalent, denoted by $(g_1, U_1) \sim (g_2, U_2)$, if there exists an essential ideal U of V, contained into $U_1 \cap U_2$, such that $g_1^\sigma(x) = g_2^\sigma(x)$ for all $x \in U^\sigma$, $\sigma = \pm$. Again, this defines an equivalence relation, with classes denoted by $[g, U]$, on the set of all permissible maps of V, with quotient set $\mathcal{C}(V)$ called the extended centroid of V. The extended centroid $\mathcal{C}(V)$ of a nondegenerate Jordan pair V is a commutative, associative, unital (von Neumann) regular algebra [20, Theorem 1.15, Proposition 2.7].

Let (g, U) be a permissible map of a Jordan pair V, and K a nonzero ideal of V contained in U. We will say that g restricts to K if $g_K \in Hom_V(K, V)$, where g_K denotes the restriction of g to K. A necessary and sufficient condition for g to restrict to K is $g^\sigma(Q_{K^\sigma} V^{-\sigma}) \subseteq K^\sigma$, $\sigma = \pm$ [20, p. 484].

The corresponding escalar extension given by the extended centroid is called central closure. We refer the reader to [2] for the construction of the central closure $\mathcal{C}(L)L$ of a semiprime Lie algebra L, and to [20] for that of the extended central closure $\mathcal{C}(V)V$ of a nondegenerate Jordan pair V.

3. The extended centroid of the TKK-algebra of a nondegenerate Jordan pair

This section is aimed to study the relationship between the extended centroid of a nondegenerate Jordan pair V and that of its TKK-algebra $TKK(V)$.

Remark 3.1. Let L be a Jordan 3-graded Lie algebra with associated Jordan pair V. The following assertions are straightforward:

(i) For any ideal I of L, $I \cap V = (I \cap V^+, I \cap V^-)$ and $\pi(I) = (\pi_+(I), \pi_-(I))$ are ideals of V such that $\pi(I)^3 \subseteq I \cap V \subseteq \pi(I)$, where π_σ denotes the canonical projections of L on L_σ, $\sigma = \pm$. Thus, if V is semiprime and I nonzero, then $I \cap V$ is a nonzero ideal of V.

(ii) If $U = (U^+, U^-)$ is a nonzero ideal of V, then $\mathcal{I}(U) = U^+ \oplus ([U^+, V^-] + [V^+, U^-]) \oplus U^-$ is a nonzero ideal of L.

Lemma 3.2. *Let V be a semiprime Jordan pair.*

(i) If I is an essential ideal of $TKK(V)$, then $I \cap V$ is an essential ideal of V.

(ii) If $U = (U^+, U^-)$ is an essential ideal of V, then $\mathcal{I}(U)$ is an essential ideal of $TKK(V)$.

Proof. (i) Let K be a nonzero ideal of V. Then, since $\mathcal{I}(K)$ is a nonzero ideal of $TTK(V)$, by the essentiality of I, we have that $I \cap \mathcal{I}(K)$ is a nonzero ideal of $TKK(V)$. Therefore, by 3.1(i), it holds that $0 \neq I \cap \mathcal{I}(K) \cap V = (I \cap V) \cap K$. Hence $I \cap V$ is an essential ideal of V.

(ii) Let now K be a nonzero ideal of L. By 3.1(i), $K \cap V$ is a nonzero ideal of V and thus, by the essentiality of U, it holds that $U \cap (K \cap V) = U \cap K \neq 0$. Therefore $0 \neq U^\sigma \cap K \subseteq \mathcal{I}(U) \cap K$, for some $\sigma = \pm$, implying that $\mathcal{I}(U)$ is an essential ideal of $TKK(V)$. □

Remark 3.3.

(i) Lemma 3.2 applies, in particular, to nondegenerate Jordan pairs, since, by [1, p. 212], nondegenerate Jordan pairs are semiprime.

(ii) Taking advantage of the 3-grading of the TKK-algebras, it follows from Lemma 3.2(i), that any essential ideal I of the TKK-algebra of a semiprime Jordan pair V, contains an essential ideal of the form $\mathcal{I}(U)$, for an essential ideal U of V. Indeed, it suffices to consider $U = I \cap V$.

3.1. Product of ideals

Given two ideals $K = (K^+, K^-)$ and $I = (I^+, I^-)$ of a Jordan pair V, the product $K * I = (Q_{K^+}I^- + Q_{V^+}Q_{K^-}I^+, Q_{K^-}I^+ + Q_{V^-}Q_{K^+}I^-)$ is an ideal of V. If $K = V$, then we have $V * I = (Q_{V^+}I^-, Q_{V^-}I^+)$ [16, p. 221].

Lemma 3.4. *Let I be an ideal of the TKK-algebra of a Jordan pair V. Then $\widetilde{I} = \mathcal{I}\big(V*(\pi_+(I), \pi_-(I))\big)$, where π_σ denotes the canonical projections from $TKK(V)$ onto $TKK(V)_\sigma$, $\sigma \in \{0, \pm\}$, is an ideal of $TKK(V)$ such that:*

(i) $\widetilde{I} \subseteq I$,
(ii) $\widetilde{I} \cap V = V * (\pi_+(I), \pi_-(I)) = \pi(\widetilde{I})$.

Moreover, if V a is nondegenerate Jordan pair, then I is essential if and only if $\widetilde{I}$ is essential.

Proof. Let I be an ideal of $TKK(V)$. We first note that, by 3.1(i), $\pi(I) = (\pi_+(I), \pi_-(I))$ is an ideal of V that contains $I \cap V$.

Take now $l \in TKK(V)$ and elements x_σ, $z_\sigma \in TKK(V)_\sigma$, $\sigma = \pm$. Since by the 3-grading of $TKK(V)$ we have $\{x_\sigma, \pi_{-\sigma}(l), z_\sigma\} = [[x_\sigma, \pi_{-\sigma}(l)], z_\sigma] = [[x_\sigma, l], z_\sigma]$, it follows from 3.1 that $V * (\pi_+(I), \pi_-(I))$ is an ideal of V such that:

$$\begin{aligned} V * (\pi_+(I), \pi_-(I)) &= \big(Q_{V^+}\pi_-(I), Q_{V^-}\pi_+(I)\big) = \\ &= \big(\{V^+, \pi_-(I), V^+\}, \{V^-, \pi_+(I), V^-\}\big) = \\ &= \big([V^+, [V^+, I]], [V^-, [V^-, I]]\big). \end{aligned}$$

By 3.1(ii), $\widetilde{I} = \mathcal{I}\big(V * (\pi_+(I), \pi_-(I))\big)$ is an ideal of $TKK(V)$, clearly contained in I, and by the 3-grading of $\widetilde{I}$ it holds that $\widetilde{I} \cap V = V * (\pi_+(I), \pi_-(I))$. Hence $\widetilde{I} \cap V = V * (\pi_+(I), \pi_-(I)) = \pi(\widetilde{I})$.

Assume now that the Jordan pair V is nondegenerate, and let I be an essential ideal of $TKK(V)$. Then, by Lemma 3.2, $I \cap V$ is an essential ideal of V and, since $I \cap V \subseteq \pi(I)$, it follows that $\pi(I)$ is also essential in V. Then, by [20, Lemma 1.2(a)], $V * (\pi_+(I), \pi_-(I))$ is an essential ideal of V, and by Lemma 3.2(ii) we obtain that $\widetilde{I}$ is essential in $TKK(V)$. The converse follows from the fact that $\widetilde{I} \subseteq I$. □

Remark 3.5. Let V be a nondegenerate Jordan pair, and let $\lambda = \overline{(f, I)} \in \mathcal{C}(TKK(V))$. By Lemma 3.4, the pair $(f_{\widetilde{I}}, \widetilde{I})$, where $\widetilde{I} = \mathcal{I}\big(V * (\pi_+(I), \pi_-(I))\big)$, is a permissible map such that $\lambda = \overline{(f, I)} = \overline{(f_{\widetilde{I}}, \widetilde{I})}$ [2, Corollary 2.3]. Thus, since $f\big([V^\sigma, [V^\sigma, I]]\big) = [V^\sigma, [V^\sigma, f(I)]] \subseteq V^\sigma$ and

$$\begin{aligned} &f\big([[V^+, [V^+, I]], V^-] + [V^+, [V^-, [V^-, I]]]\big) = \\ &= [[V^+, [V^+, f(I)]], V^-] + [V^+, [V^-, [V^-, f(I)]]] \subseteq [V^+, V^-], \end{aligned}$$

replacing (f, I) by $(f_{\widetilde{I}}, \widetilde{I})$, if necessary, for any $\lambda = \overline{(f, I)} \in \mathcal{C}(TKK(V))$ we will assume that I is a 3-graded essential ideal of $TKK(V)$ such that $f(I^\sigma) \subseteq TKK(V)_\sigma$, $\sigma \in \{0, \pm1\}$, that is, f is a 3-graded map.

Proposition 3.6. *Let $TKK(V)$ be the TKK-algebra of a nondegenerate Jordan pair V. Then the map:*

$$\begin{array}{rcl} \Psi : \mathcal{C}(TKK(V)) & \to & \mathcal{C}(V) \\ \overline{(f, I)} & \mapsto & [f_{I\cap V}, I \cap V] \end{array}$$

defines a ring homomorphism from the extended centroid $\mathcal{C}(TKK(V))$ of $TKK(V)$ to the extended centroid $\mathcal{C}(V)$ of V.

Proof. Let $\lambda = \overline{(f, I)} \in \mathcal{C}(TKK(V))$, where (f, I) is a permissible map of $TKK(V)$ as in Remark 3.5, so that I is a 3-graded essential ideal of $TKK(V)$ and $f(I^\sigma) \subseteq V^\sigma$ for $\sigma = \pm$.

Let us denote $f^\sigma = f_{I^\sigma}$, and take $x^\sigma, z^\sigma \in V^\sigma$ and $y^\sigma \in I^\sigma$, for $\sigma = \pm$. Clearly $f_{I\cap V}$ is a linear map, and since we are assuming $I \cap V^\sigma = \pi_\sigma(I)$, it holds that:

$$f^\sigma(Q_{x^\sigma} y^{-\sigma}) = f^\sigma\Big(\frac{1}{2}[[x^\sigma, y^{-\sigma}], x^\sigma]\Big) = f\Big(\frac{1}{2}[[x^\sigma, y^{-\sigma}], x^\sigma]\Big) = \\ = \frac{1}{2}[[x^\sigma, f(y^{-\sigma})], x^\sigma] = \frac{1}{2}[[x^\sigma, f^{-\sigma}(y^{-\sigma})], x^\sigma] = Q_{x^\sigma} f^{-\sigma}(y^{-\sigma}).$$

Moreover, since:

$$f^\sigma\big(Q_{I^\sigma} V^{-\sigma}\big) \subseteq f^\sigma\big([[I^\sigma, V^{-\sigma}], I^\sigma]\big) = f\big([[I^\sigma, V^{-\sigma}], I^\sigma]\big) = \\ = \big[[f(I^\sigma), V^{-\sigma}], I^\sigma\big] \subseteq \big[[V^\sigma, V^{-\sigma}], I^\sigma\big] = \big\{V^\sigma, V^{-\sigma}, I^\sigma\big\} \subseteq I^\sigma,$$

it follows that:

$$(f^\sigma)^2(Q_{y^\sigma} x^{-\sigma}) = (f^\sigma)^2(\frac{1}{2}[[y^\sigma, x^{-\sigma}], y^\sigma]) = f^2(\frac{1}{2}[[y^\sigma, x^{-\sigma}], y^\sigma]) = \\ = \frac{1}{2} f([[f(y^\sigma), x^{-\sigma}], y^\sigma]) = \frac{1}{2}[[f(y^\sigma), x^{-\sigma}], f(y^\sigma)] = \\ = \frac{1}{2}[[f^\sigma(y^\sigma), x^{-\sigma}], f^\sigma(y^\sigma)] = Q_{f^\sigma(y^\sigma)} x^{-\sigma},$$

and, finally, we also have:

$$f^\sigma\big(\{y^\sigma, z^{-\sigma}, x^\sigma\}\big) = f^\sigma\big([[y^\sigma, z^{-\sigma}], x^\sigma]\big) = f\big([[y^\sigma, z^{-\sigma}], x^\sigma]\big) = \\ = [[f(y^\sigma), z^{-\sigma}], x^\sigma] = [[f^\sigma(y^\sigma), z^{-\sigma}], x^\sigma] = \{f^\sigma(y^\sigma), z^{-\sigma}, x^\sigma\}.$$

Hence $(f_{I\cap V}, I \cap V)$ is a V-homomorphism, and therefore, by the essentiality of $I \cap V$, it is a permissible map of V.

We next claim that Ψ is well-defined. To prove this claim, let (f_1, I_1) and (f_2, I_2) be permissible maps of $TKK(V)$ such that $\overline{(f_1, I_1)} = \overline{(f_2, I_2)}$ in $\mathcal{C}(TKK(V))$. By Remark 3.5 and [2, Corollary 2.3], we can assume that $I = I_1 \cap I_2$ is an essential 3-graded ideal of $TKK(V)$ such that $(f_1)_I = (f_2)_I$. Therefore, for all $x^\sigma \in I \cap V^\sigma$ it holds that $(f_1)^\sigma(x^\sigma) = f_1(x^\sigma) = f_2(x^\sigma) = (f_2)^\sigma(x^\sigma)$, that results into $\big((f_1)_{I_1\cap V}, I_1 \cap V\big)$ and $\big((f_2)_{I_2\cap V}, I_2 \cap V\big)$ being equivalent permissible maps of V by [20, 1.4]. Thus Ψ is a well-defined map.

Let now $\lambda_1 = \overline{(f_1, I_1)}$ and $\lambda_2 = \overline{(f_2, I_2)} \in \mathcal{C}(TKK(V))$. By [2, p. 1108] and Remark 3.5, $\lambda_1 + \lambda_2 = \overline{\big((f_1)_{I_1\cap I_2} + (f_2)_{I_1\cap I_2}, I_1 \cap I_2\big)}$, and therefore,

$$\Psi(\lambda_1 + \lambda_2) = \big[\big((f_1)_{I_1\cap I_2} + (f_2)_{I_1\cap I_2}\big)_{I_1\cap I_2\cap V}, I_1 \cap I_2 \cap V\big],$$

whereas, on the other hand, by [20, 1.11],

$$\Psi(\lambda_1) + \Psi(\lambda_2) = \big[(f_1)_{I_1\cap V}, I_1 \cap V\big] + \big[(f_2)_{I_2\cap V}, I_2 \cap V\big] = \\ = \big[(f_1)_K + (f_2)_K, K\big],$$

where, by [20, Lemma 1.2(a)], $K = (I_1 \cap I_2 \cap V) * V$ is an essential ideal of V contained in $I_1 \cap I_2 \cap V$. Therefore, since $f_i(I_i^\sigma) \subseteq V^\sigma$, $\sigma = \pm$, $i = 1, 2$, for all $k^\sigma \in K^\sigma$ we have $\big((f_1)_{I_1\cap I_2} + (f_2)_{I_1\cap I_2}\big)_{I_1\cap I_2\cap V}(k^\sigma) = \big((f_1)_K + (f_2)_K\big)(k^\sigma)$ and, consequently, $\Psi(\lambda_1 + \lambda_2) = \Psi(\lambda_1) + \Psi(\lambda_2)$ by [20, Lemma 1.10]. Hence the map Ψ is additive.

To prove that Ψ is a multiplicative map, we note that by [2, p. 1108] $\lambda_1\lambda_2 = \overline{(f_1f_2, f_2^{-1}(I_1))}$ and therefore $\Psi(\lambda_1\lambda_2) = \left[(f_1f_2)_{f_2^{-1}(I_1)\cap V}, f_2^{-1}(I_1)\cap V\right]$, whereas, by [20, 1.13], we have:

$$\Psi(\lambda_1)\Psi(\lambda_2) = \left[(f_1)_{I_1\cap V}, I_1\cap V\right]\left[(f_2)_{I_2\cap V}, I_2\cap V\right] = \left[(f_1f_2)_{K*V}, K*V\right],$$

for the essential ideal $K = (I_1\cap I_2\cap V)*V$ of V [20, Lemma 1.2(a)]. We claim that both $\Psi(\lambda_1\lambda_2)$ and $\Psi(\lambda_1)\Psi(\lambda_2)$ can be restricted to the essential ideal $(f_2^{-1}(I_1)\cap V)\cap(K*V)$ of V.

To prove this claim we first note that, by Remark 3.5, $f_2(I_2^\sigma)\subseteq V^\sigma$ which implies that $f_2^{-1}(I_1)\cap V^\sigma\subseteq I_2^\sigma$, and since $(I_1\cap I_2\cap V)^\sigma = I_1^\sigma\cap I_2^\sigma$, we also have $(K*V)^\sigma\subseteq I_1^\sigma\cap I_2^\sigma$. Then:

$$\Big((f_2^{-1}(I_1)\cap V)\cap(K*V)\Big)^\sigma = (f_2^{-1}(I_1)\cap V^\sigma)\cap(K*V)^\sigma = f_2^{-1}(I_1)\cap(K*V)^\sigma,$$

and it follows that:

$$\begin{aligned}
&(f_1f_2)^\sigma\big(Q_{f_2^{-1}(I_1)\cap(K*V)^\sigma}V^{-\sigma}\big) = \\
&= (f_1f_2)^\sigma\big(\big[\big[f_2^{-1}(I_1)\cap(K*V)^\sigma, V^{-\sigma}\big], f_2^{-1}(I_1)\cap(K*V)^\sigma\big]\big) = \\
&= (f_1f_2)\big(\big[\big[f_2^{-1}(I_1)\cap(K*V)^\sigma, V^{-\sigma}\big], f_2^{-1}(I_1)\cap(K*V)^\sigma\big]\big) = \\
&= f_1\big(\big[\big[f_2(f_2^{-1}(I_1)\cap(K*V)^\sigma), V^{-\sigma}\big], f_2^{-1}(I_1)\cap(K*V)^\sigma\big]\big) \subseteq \\
&\subseteq f_1\big(\big[\big[f_2(f_2^{-1}(I_1)\cap V^\sigma), V^{-\sigma}\big], f_2^{-1}(I_1)\cap(K*V)^\sigma\big]\big) \subseteq \\
&\subseteq f_1\big(\big[\big[I_1\cap V^\sigma, V^{-\sigma}\big], f_2^{-1}(I_1)\cap(K*V)^\sigma\big]\big) = \\
&= f_1\big(\big[\big[I_1^\sigma, V^{-\sigma}\big], f_2^{-1}(I_1)\cap(K*V)^\sigma\big]\big) = \\
&= \big[\big[f_1(I_1^\sigma), V^{-\sigma}\big], f_2^{-1}(I_1)\cap(K*V)^\sigma\big] \subseteq \\
&\subseteq \big[\big[V^\sigma, V^{-\sigma}\big], f_2^{-1}(I_1)\cap(K*V)^\sigma\big] = \\
&= \{V^\sigma, V^{-\sigma}, f_2^{-1}(I_1)\cap(K*V)^\sigma\} \subseteq f_2^{-1}(I_1)\cap(K*V)^\sigma,
\end{aligned}$$

implying that $\Psi(\lambda_1\lambda_2)$ can be restricted to the ideal $(f_2^{-1}(I_1)\cap V)\cap(K*V)$ of V. On the other hand, for $\Psi(\lambda_1)\Psi(\lambda_2)$ we have:

$$\begin{aligned}
&((f_1)^\sigma(f_2)^\sigma)\big(Q_{f_2^{-1}(I_1)\cap(K*V)^\sigma}V^{-\sigma}\big) = \\
&= ((f_1)^\sigma(f_2)^\sigma)\big(\big[\big[f_2^{-1}(I_1)\cap(K*V)^\sigma, V^{-\sigma}\big], f_2^{-1}(I_1)\cap(K*V)^\sigma\big]\big) = \\
&= (f_1)^\sigma\big((f_2)^\sigma\big(\big[\big[f_2^{-1}(I_1)\cap(K*V)^\sigma, V^{-\sigma}\big], f_2^{-1}(I_1)\cap(K*V)^\sigma\big]\big)\big) = \\
&= (f_1)^\sigma\big(f_2\big(\big[\big[f_2^{-1}(I_1)\cap(K*V)^\sigma, V^{-\sigma}\big], f_2^{-1}(I_1)\cap(K*V)^\sigma\big]\big)\big) \subseteq \\
&\subseteq (f_1)^\sigma\big(\big[\big[f_2(f_2^{-1}(I_1)\cap V^\sigma), V^{-\sigma}\big], f_2^{-1}(I_1)\cap(K*V)^\sigma\big]\big) \subseteq \\
&\subseteq (f_1)^\sigma\big(\big[\big[I_1\cap V^\sigma, V^{-\sigma}\big], f_2^{-1}(I_1)\cap(K*V)^\sigma\big]\big) = \\
&= f_1\big(\big[\big[I_1\cap V^\sigma, V^{-\sigma}\big], f_2^{-1}(I_1)\cap(K*V)^\sigma\big]\big) = \\
&= \big[\big[f_1(I_1^\sigma), V^{-\sigma}\big], f_2^{-1}(I_1)\cap(K*V)^\sigma\big] \subseteq \\
&\subseteq \big[\big[V^\sigma, V^{-\sigma}\big], f_2^{-1}(I_1)\cap(K*V)^\sigma\big] = \\
&= \big\{V^\sigma, V^{-\sigma}, f_2^{-1}(I_1)\cap(K*V)^\sigma\big\} \subseteq f_2^{-1}(I_1)\cap(K*V)^\sigma.
\end{aligned}$$

Therefore both $\Psi(\lambda_1\lambda_2)$ and $\Psi(\lambda_1)\Psi(\lambda_2)$ are permissible maps of V that can be restricted to the essential ideal $(f_2^{-1}(I_1)\cap V)\cap(K*V)$. Moreover, since $f_i(I_i^\sigma) = f_i^\sigma(I^\sigma)$, both $\Psi(\lambda_1\lambda_2)$ and $\Psi(\lambda_1)\Psi(\lambda_2)$ agree in $(f_2^{-1}(I_1)\cap V)\cap(K*V)$. This implies that Ψ is a multiplicative map and, consequently, a ring homomorphism. $\square$

Theorem 3.7. *The extended centroid $\mathcal{C}(V)$ of a nondegenerate Jordan pair V is isomorphic to the extended centroid $\mathcal{C}(TKK(V))$ of its TKK-algebra $TKK(V)$.*

Proof. Let V be a nondegenerate Jordan pair and let us define the map

$$\begin{array}{rcl} \Upsilon: \mathcal{C}(V) & \to & \mathcal{C}(TKK(V)) \\ [(g^+, g^-), (U^+, U^-)] & \mapsto & \overline{(f, \mathcal{I}(U))} \end{array}$$

where $f: \mathcal{I}(U) \to TKK(V)$ is defined by

$$f\Big(u^+ + \big(\sum_{i=1}^{n}[u_i^+, v_i^-] + \sum_{j=1}^{m}[v_j^+, u_j^-]\big) + u^-\Big) =$$
$$= g^+(u^+) + \Big(\sum_{i=1}^{n}[g^+(u_i^+), v_i^-] + \sum_{j=1}^{m}[v_j^+, g^-(u_j^-)]\Big) + g^-(u^-)$$

for all u^σ, $u_i^\sigma \in U^\sigma$, $v_i^\sigma \in V^\sigma$, $\sigma = \pm$.

To prove that $f : \mathcal{I}(U) \to TKK(V)$ is well-defined, let u_i^σ, $u_j^\sigma \in U^\sigma$, v_i^σ, $v_j^\sigma \in V^\sigma$ such that $\sum_{i=1}^{n}[u_i^\sigma, v_i^{-\sigma}] + \sum_{j=1}^{m}[v_j^\sigma, u_j^{-\sigma}] = 0$, and write $a = \sum_{i=1}^{n}[g^\sigma(u_i^\sigma), v_i^{-\sigma}] + \sum_{j=1}^{m}[v_j^\sigma, g^{-\sigma}(u_j^{-\sigma})]$. Then, for all $w^\sigma \in V^\sigma$, $\sigma = \pm$:

$$[a, w^\sigma] = \sum_{i=1}^{n}\{g^\sigma(u_i^\sigma), v_i^{-\sigma}, w^\sigma\} + \sum_{j=1}^{m}\{v_j^\sigma, g^{-\sigma}(u_j^{-\sigma}), w^\sigma\} =$$
$$= g^\sigma\big(\sum_{i=1}^{n}\{u_i^\sigma, v_i^{-\sigma}, w^\sigma\} + \sum_{j=1}^{m}\{v_j^\sigma, u_j^{-\sigma}, w^\sigma\}\big) =$$
$$= g^\sigma\big([\sum_{i=1}^{n}[u_i^\sigma, v_i^{-\sigma}] + \sum_{j=1}^{m}[v_j^\sigma, u_j^{-\sigma}], w^\sigma]\big) = g^\sigma([0, w^\sigma]) = g^\sigma(0) = 0,$$

which implies $a \in Z(TKK(V))$. But since, by [4, Proposition 11.25], TKK-algebras of nondegenerate Jordan pairs are centerless, it follows that $a = 0$. Hence the map $f : \mathcal{I}(U) \to TKK(V)$ is well-defined.

Now, since, by Lemma 3.4(ii), $\mathcal{I}(U)$ is an essential ideal of V, to prove that $(f, \mathcal{I}(U))$ is a permissible map of $TKK(V)$, it suffices to check that $[f, ad\, y](\mathcal{I}(U)) = 0$ for all $y \in TKK(V)$. Write $y = y^+ + y_0 + y^-$, with $y_0 = \sum_{i=1}^{n}\delta(a_i^\sigma, b_i^{-\sigma}) = \sum_{i=1}^{n}\big(D(a_i^\sigma, b_i^{-\sigma}), -D(b_i^{-\sigma}, a_i^\sigma)\big)$ for some $a_i^\sigma \in V^\sigma$ and $b_i^{-\sigma} \in V^{-\sigma}$. Then, for all $u^\sigma \in U^\sigma$, $\sigma = \pm$, $[f, ad\, y^\sigma](u^\sigma) = f([y^\sigma, u^\sigma]) - [y^\sigma, f(u^\sigma)] = 0 - [y^\sigma, g^\sigma(u^\sigma)] = 0$, and, it holds that $[f, ad\, y^{-\sigma}](u^\sigma) = f([y^{-\sigma}, u^\sigma]) - [y^{-\sigma}, f(u^\sigma)] = [y^{-\sigma}, g^\sigma(u^\sigma)] - [y^{-\sigma}, g^\sigma(u^\sigma)] = 0$. Moreover,

$$\begin{aligned}[f, ad\, y_0](u^\sigma) &= f([y_0, u^\sigma]) - [y_0, f(u^\sigma)] = \\ &= f\big(\sum\{a_i^\sigma, b_i^{-\sigma}, u^\sigma\}\big) - \sum\{a_i^\sigma, b_i^{-\sigma}, g^\sigma(u^\sigma)\} = \\ &= \sum g^\sigma\big(\{a_i^\sigma, b_i^{-\sigma}, u^\sigma\}\big) - \sum\{a_i^\sigma, b_i^{-\sigma}, g^\sigma(u^\sigma)\} = 0,\end{aligned}$$

and, given $v^\sigma \in V^\sigma$,

$$\begin{aligned}[f, ad\, y^\sigma]([u^\sigma, v^{-\sigma}]) &= f([y^\sigma, [u^\sigma, v^{-\sigma}]]) - [y^\sigma, f([u^\sigma, v^{-\sigma}])] = \\ &= -g^\sigma(\{y^\sigma, v^{-\sigma}, u^\sigma\}) + \{y^\sigma, v^{-\sigma}, g^\sigma(u^\sigma)\} = 0,\end{aligned}$$

and, similarly $[f, ad\, y^\sigma]([v^\sigma, u^{-\sigma}]) = 0$. Finally we have:

$$
\begin{aligned}
[f, ad\, y_0]([u^\sigma, v^{-\sigma}]) &= f([y_0, [u^\sigma, v^{-\sigma}]]) - [y_0, f([u^\sigma, v^{-\sigma}])] = \\
&= \sum f\Big(\big[\{a_i^\sigma, b_i^{-\sigma}, u^\sigma\}, v^{-\sigma}\big] - \big[u^\sigma, \{b_i^{-\sigma}, a_i^\sigma, v^{-\sigma}\}\big]\Big) - \\
&\quad - [y_0, [g^\sigma(u^\sigma), v^{-\sigma}]] = \\
&= \sum \Big(\big[g^\sigma(\{a_i^\sigma, b_i^{-\sigma}, u^\sigma\}), v^{-\sigma}\big] - \big[g^\sigma(u^\sigma), \{b_i^{-\sigma}, a_i^\sigma, v^{-\sigma}\}\big]\Big) - \\
&\quad - \sum \Big(\big[\{a_i^\sigma, b_i^{-\sigma}, g^\sigma(u^\sigma)\}, v^{-\sigma}\big] - \big[g^\sigma(u^\sigma), \{b_i^{-\sigma}, a_i^\sigma, v^{-\sigma}\}\big]\Big) = 0,
\end{aligned}
$$

which implies that $[f, ad\, y](\mathcal{I}(U)) = 0$, and therefore that the pair $(f, \mathcal{I}(U))$ is a permissible map of $TKK(V)$.

Next we claim that Υ is a well-defined map. To prove this claim let (g_1, U_1) and (g_2, U_2) be permissible maps of V, such that $\mu = [g_1, U_1] = [g_2, U_2] \in \mathcal{C}(V)$. Using [20, Lemma 1.10], let U be an essential ideal of V, contained into $U_1 \cap U_2$, such that $(g_1)_U = (g_2)_U \in Hom_V(U, V)$. Therefore $[g_1, U] = [g_2, U]$ in $\mathcal{C}(V)$, and consequently $\Upsilon([g_1, U]) = \overline{(f_1, \mathcal{I}(U))}$ and $\Upsilon([g_2, U]) = \overline{(f_2, \mathcal{I}(U))}$ agree on $\mathcal{I}(U)$. Hence Υ is well defined.

To complete the proof it suffices to prove that Υ is a two-sided inverse for the map Ψ defined in Proposition 3.6. Let $\mu = [g, U] \in \mathcal{C}(V)$. Since $\mathcal{I}(U) \cap V = U$, by Lemma 3.4, it holds that $\Psi\Upsilon(\mu) = \Psi\big(\,\overline{(f, \mathcal{I}(U))}\,\big) = [f_{V*U}, V * U]$. Thus, since by [20, Lemma 1.9] any permissible map (g, U) of V restricts to $V * U$, we have $[f_{V*U}, V * U] = [g, U]$, which implies that $\Psi\Upsilon = id_{\mathcal{C}(V)}$. Conversely given $\lambda = \overline{(f, I)} \in \mathcal{C}(TKK(V))$, again using Lemma 3.4, we obtain $I = \mathcal{I}(I \cap V)$ and, therefore that $\Upsilon\Psi(\lambda) = \Upsilon([f_{I\cap V}, I \cap V]) = \overline{(f, I)}$, which implies that $\Upsilon\Psi = id_{\mathcal{C}(TKK(V))}$.

Therefore, Ψ and Υ are mutually inverse ring homomorphisms, defining an isomorphism between the extended centroid $\mathcal{C}(V)$ of a nondegenerate Jordan pair V and the extended centroid $\mathcal{C}(TKK((V))$ of its TKK-algebra. □

4. The central closure of the TKK-algebra of a nondegenerate Jordan pair

We begin this section recalling some facts that can be found, or easily derived, from [2,3,20].

Remark 4.1. Let L be a semiprime Lie algebra.

(i) Any element $x \in \mathcal{C}(L) \otimes L$ admits a (non necessarily unique) representation $x = \sum \lambda_i \otimes a_i$, where $\lambda_i \in \mathcal{C}(L)$, $a_i \in L$. A representation $\sum \lambda_i \otimes a_i$ of x is I-vanishing, for an essential ideal I of L, if there exists $(f_i, I) \in \lambda_i$ such that $\sum [f_i(y), p(a_i)] = 0$ for all $y \in I$, $p \in Ad(L)$ [2].

(ii) The central closure $\mathcal{C}(L)L$ of L is $\mathcal{C}(L)L = \big(\mathcal{C}(L) \otimes L\big)/M$, where M denotes the set of all I-vanishing elements of $\mathcal{C}(L) \otimes L$ for some essential ideal I of L [2, p. 1111]. Moreover, M is the unique ideal of $\mathcal{C}(L) \otimes L$ maximal with respect to $R \subseteq M$ and $M \cap (1 \otimes L) = 0$, where R is the ideal of $\mathcal{C}(L) \otimes L$ generated by all elements of the form $\lambda \otimes u - 1 \otimes f(u)$ with $\lambda = \overline{(f, U)}$ and $u \in U$ [2, Lemma 2.11].

Lemma 4.2. *Let L be the TKK-algebra of nondegenerate Jordan pair V. Then*

$$R = \Big\{ \sum (\rho_i \lambda_i \otimes x_i - \rho_i \otimes f_i(x_i)) \mid \rho_i, \lambda_i \in \mathcal{C}(L), (f_i, I_i) \in \lambda_i, x_i \in I_i \Big\}$$

is a 3-graded ideal of $\mathcal{C}(L) \otimes L$ with respect to the grading induced in $\mathcal{C}(L) \otimes L$.

Proof. The statement follows from Remark 4.1 and [20, Lemma 3.2]. □

Theorem 4.3. *Let V be a nondegenerate Jordan pair. Then, the central closure of the TKK-algebra of V is isomorphic to the TKK-algebra of the extended central closure $\mathcal{C}(V)V$ of V.*

Proof. Let us denote $L = TKK(V)$ and consider the map:

$$\begin{array}{rlll} F: & \mathcal{C}(L) \times L & \to & TKK(\mathcal{C}(V)V) \\ & (\lambda, a_\sigma) & \mapsto & \Psi(\lambda)a_\sigma \in \mathcal{C}(V)V^\sigma \\ & (\lambda, \delta(x,y)) & \mapsto & \Psi(\lambda)\delta(x,y) \in \mathrm{IDer}(\mathcal{C}(V)V) \end{array}$$

where $\lambda \in \mathcal{C}(L)$, $a_\sigma \in V^\sigma$, $\sigma = \pm$, $\delta(x,y) \in L_0 = \mathrm{IDer}\, V$ and Ψ is the ring homomorphism defined in Proposition 3.6.

It is not difficult to prove that F is a well-defined bilinear map and, using that Ψ is a ring homomorphism, it also follows easily that F is a balanced map. That is, for all $\lambda, \lambda_1, \lambda_2 \in \mathcal{C}(L)$, $l^\sigma \in L_\sigma$, $l^\beta \in L_\beta$, $\sigma, \beta \in \{0, \pm\}$ and $\alpha \in \Phi$, it holds that:

(i) $F\big((\lambda_1 + \lambda_2, l^\sigma) - (\lambda_1, l^\sigma) - (\lambda_2, l^\sigma)\big) = \Psi(\lambda_1 + \lambda_2)l^\sigma - \Psi(\lambda_1)l^\sigma - \Psi(\lambda_2)l^\sigma = 0$,
(ii) $F\big((\lambda, l_1^\sigma + l_2^\beta) - (\lambda, l_1^\sigma) - (\lambda, l_2^\beta)\big) = \Psi(\lambda)(l_1^\sigma + l_2^\beta) - \Psi(\lambda)l_1^\sigma - \Psi(\lambda)l_2^\beta = 0$,
(iii) $F\big((\alpha\lambda, l^\sigma) - (\lambda, \alpha l^\sigma)\big) = \Psi(\alpha\lambda)l^\sigma - \Psi(\lambda)\alpha l^\sigma = 0$.

This results into F defining a (3-graded) homomorphism of 3-graded Lie Φ-algebras (also denoted by F) $F : \mathcal{C}(L) \otimes L \to TKK(\mathcal{C}(V)V)$. Moreover, since $\mathcal{C}(V)$ is von Neumann regular [20, Theorem 1.15], $\mathcal{C}(V)[V^+, V^-] = \mathcal{C}(V)^2[V^+, V^-] = [\mathcal{C}(V)V^+, \mathcal{C}(V)V^-]$, and F is an epimorphism.

Next we claim that $KerF$ equals the ideal M described in Remark 4.1(ii). To prove this claim let us first consider an element $1 \otimes a \in KerF \cap (1 \otimes L)$. Then $0 = F(1 \otimes a) = \Psi(1)a = a$, that results into $KerF \cap (1 \otimes L) = 0$. Hence, by Remark 4.1(ii), to prove that $KerF \subseteq M$, it suffices to check that $R \subseteq KerF$.

Let $\lambda \otimes y - 1 \otimes f(y) \in R$, where $\lambda = \overline{(f, I)} \in \mathcal{C}(L)$ and, by Remark 3.5, we can assume that I is 3-graded. By Proposition 3.6, $\Psi(\lambda) = [f_{I \cap V}, I \cap V]$, and $F(\lambda \otimes y - 1 \otimes f(y)) = F(\lambda \otimes y) - F(1 \otimes f(y)) = \Psi(\lambda)y - \Psi(1)f(y) = \Psi(\lambda)y - f(y)$. Thus, for any $y^\sigma \in I \cap V^\sigma$, $\sigma = \pm$, it follows that $\Psi(\lambda)y^\sigma - f(y^\sigma) = f^\sigma(y^\sigma) - f(y^\sigma) = 0$ and, therefore, for all $v^\sigma \in V^\sigma$, $u^{-\sigma}, w^{-\sigma} \in V^{-\sigma}$, $y^\sigma \in I^\sigma$, the element $\Psi(\lambda)[v^\sigma, [u^{-\sigma}, [w^{-\sigma}, y^\sigma]]] - f\big([v^\sigma, [u^{-\sigma}, [w^{-\sigma}, y^\sigma]]]\big) = \Psi(\lambda)[v^\sigma, [u^{-\sigma}, [w^{-\sigma}, y^\sigma]]] - [v^\sigma, [u^{-\sigma}, [w^{-\sigma}, f(y^\sigma)]]]$ defines an inner derivation on $\mathcal{C}(V)V$, such that for any $a^\sigma \in \mathcal{C}(V)V^\sigma$:

$$\begin{aligned} &\Big(\Psi(\lambda)[v^\sigma, [u^{-\sigma}, [w^{-\sigma}, y^\sigma]]] - [v^\sigma, [u^{-\sigma}, [w^{-\sigma}, f(y^\sigma)]]]\Big)a^\sigma = \\ &= -\Psi(\lambda)\{v^\sigma, \{u^{-\sigma}, y^\sigma, w^{-\sigma}\}, a^\sigma\} + \{v^\sigma, \{u^{-\sigma}, f(y^\sigma), w^{-\sigma}\}, a^\sigma\} = \\ &= -\{v^\sigma, \Psi(\lambda)\{u^{-\sigma}, y^\sigma, w^{-\sigma}\}, a^\sigma\} + \{v^\sigma, \{u^{-\sigma}, f(y^\sigma), w^{-\sigma}\}, a^\sigma\} = \\ &= -\{v^\sigma, \{u^{-\sigma}, \Psi(\lambda)y^\sigma, w^{-\sigma}\}, a^\sigma\} + \{v^\sigma, \{u^{-\sigma}, f(y^\sigma), w^{-\sigma}\}, a^\sigma\} = \\ &= -\{v^\sigma, \{u^{-\sigma}, f(y^\sigma), w^{-\sigma}\}, a^\sigma\} + \{v^\sigma, \{u^{-\sigma}, f(y^\sigma), w^{-\sigma}\}, a^\sigma\} = 0. \end{aligned}$$

Similarly $\Big(\Psi(\lambda)[v^\sigma, [u^{-\sigma}, [w^{-\sigma}, y^\sigma]]] - [v^\sigma, [u^{-\sigma}, [w^{-\sigma}, f(y^\sigma)]]]\Big)a^{-\sigma} = 0$ holds for all $a^{-\sigma} \in \mathcal{C}(V)V^{-\sigma}$, which implies, by the maximality of M (see Remark 4.1), that $R \subseteq KerF \subseteq M$.

To prove that $M \subseteq KerF$, take now $x \in M$ and let $x = \sum \lambda_i \otimes a_i$ be a I-vanishing representation of x, where I is assumed to be a 3-graded ideal by Remark 3.5. Then $F(x) = \sum \Psi(\lambda_i)a_i \in TKK(\mathcal{C}(V)V)$, where $\Psi(\lambda_i) = [(f_i)_{I \cap V}, I \cap V]$, and for all $u^\sigma \in I \cap V^\sigma$, we have

$$\begin{aligned} \big[F(x), u^\sigma\big] &= \big[\sum \Psi(\lambda_i)a_i, u^\sigma\big] = \sum [\Psi(\lambda_i)a_i, u^\sigma] = \\ &= \sum [a_i, \Psi(\lambda_i)u^\sigma] = \big[\sum a_i, f_i(u^\sigma)\big] = 0. \end{aligned}$$

Moreover, for all $v^\sigma \in V^\sigma$, $v^{-\sigma}$, $w^{-\sigma} \in V^{-\sigma}$, $y^\sigma \in I^\sigma$, it holds that:

$$\begin{aligned}
&\big[F(x), [v^\sigma, [u^{-\sigma}, [w^{-\sigma}, y^\sigma]]]\big] = \\
&= [[F(x), v^\sigma], [u^{-\sigma}, [w^{-\sigma}, y^\sigma]]] + [v^\sigma, [[F(x), u^{-\sigma}], [w^{-\sigma}, y^\sigma]]] + \\
&+ [v^\sigma, [u^{-\sigma}, [[F(x), w^{-\sigma}], y^\sigma]]] + [v^\sigma, [u^{-\sigma}, [w^{-\sigma}, [F(x), y^\sigma]]]] = \\
&= [[\sum \Psi(\lambda_i)a_i, v^\sigma], [u^{-\sigma}, [w^{-\sigma}, y^\sigma]]] + [v^\sigma, [[\sum \Psi(\lambda_i)a_i, u^{-\sigma}], [w^{-\sigma}, y^\sigma]]] + \\
&+ [v^\sigma, [u^{-\sigma}, [[\sum \Psi(\lambda_i)a_i, w^{-\sigma}], y^\sigma]]] + [v^\sigma, [u^{-\sigma}, [w^{-\sigma}, [\sum \Psi(\lambda_i)a_i, y^\sigma]]]] = \\
&= \sum [[a_i, v^\sigma], [u^{-\sigma}, [w^{-\sigma}, f_i(y^\sigma)]]] + \sum [v^\sigma, [[a_i, u^{-\sigma}], [w^{-\sigma}, f_i(y^\sigma)]]] + \\
&+ \sum [v^\sigma, [u^{-\sigma}, [[a_i, w^{-\sigma}], f_i(y^\sigma)]]] + \sum [v^\sigma, [u^{-\sigma}, [w^{-\sigma}, [a_i, f_i(y^\sigma)]]]] = \\
&= \sum \big[a_i, [v^\sigma, [u^{-\sigma}, [w^{-\sigma}, f_i(y^\sigma)]]]\big] = \\
&= \sum \big[a_i, f_i\big([v^\sigma, [u^{-\sigma}, [w^{-\sigma}, y^\sigma]]]\big)\big] = 0,
\end{aligned}$$

since $\sum \lambda_i \otimes a_i$ is a I-vanishing representation of x. Indeed, it suffices to consider $p(x) = x$ to be the polynomial appearing in Remark 4.1. This implies that $F(x)$ belongs to the centralizer of the ideal $\mathcal{I}(\mathcal{C}(V)(I \cap V))$ in $TKK(\mathcal{C}(V)V)$, that vanishes, since the essentiality of I in L, implies that of $\mathcal{C}(V)(I \cap V)$ in $\mathcal{C}(V)V$, and therefore the essentiality of $\mathcal{I}(\mathcal{C}(V)(I \cap V))$ in $TKK(\mathcal{C}(V)V)$. Hence $F(x) = 0$ and $Ker\, F = M$ follows.

Finally since $\mathcal{C}(L)L = \big(\mathcal{C}(L) \otimes L\big)/M$, we obtain that $F : \mathcal{C}(L)L \to TKK(\mathcal{C}(V)V)$ is an isomorphism of 3-graded Lie algebras. □

5. Jordan 3-graded Lie algebras with polynomial identities

In this section we begin the study of Jordan 3-graded Lie algebras satisfying essential 3-graded polynomial identities.

5.1. Free 3-graded Lie algebra

Following [24, 2.7], we denote by $\mathcal{L}(X) = \mathcal{L}(X^+ \cup X^-)$ the free Lie algebra on a polarized set $X = X^+ \cup X^-$, which admits a $\mathbb{Z}$-grading $\mathcal{L}(X) = \oplus_{n\in\mathbb{Z}} \mathcal{L}^{(n)}(X)$, defined by the map $\vartheta : X \to \mathbb{Z}$ given by $\vartheta(x^\sigma) = \sigma$, for all $x^\sigma \in X^\sigma$, $\sigma = \pm$. The quotient algebra of $\mathcal{L}(X)$ by the ideal generated by all monomials:

$$[x_1^{\sigma_1}[x_2^{\sigma_2}[x_3^{\sigma_3}[\ldots x_{2n}^{\sigma_{2n}}]\ldots]]], \quad \sigma_1 = \sigma_2,\ \sigma_{2i-1} + \sigma_{2i} = 0 \ \text{ with } \ i \geq 2,$$

where $\sigma = \pm$, $x_i^{\sigma_i} \in X^{\sigma_i}$ and $n \geq 1$, is a 3-graded Lie algebra $\mathcal{L}(X^+, X^-) = \mathcal{L}(X^+, X^-)_1 \oplus \mathcal{L}(X^+, X^-)_0 \oplus \mathcal{L}(X^+, X^-)_{-1}$, where $\mathcal{L}(X^+, X^-)_n$ denotes the canonical projection of $\mathcal{L}^{(n)}(X)$ in $\mathcal{L}(X^+, X^-)$, for $n = 0, \pm 1$. Then $\mathcal{L}(X^+, X^-)$ is the free 3-graded Lie algebra, that is, for every 3-graded Lie algebra $G = G_1 \oplus G_0 \oplus G_{-1}$ and every map $f : X \to G$ such that $f(X^\sigma) \subseteq G_\sigma$, $\sigma = \pm$, there exists a unique Lie algebra homomorphism $F : \mathcal{L}(X^+, X^-) \to G$ such that $F \circ \iota = f$, where $\iota : X \to \mathcal{L}(X^+, X^-)$ denotes the canonical map. Moreover F is 3-graded. It also holds (see [24, 2.7]):

(i) The map $\iota : X \to \mathcal{L}(X^+, X^-)$ is injective,
(ii) $\big(\mathcal{L}(X^+, X^-)_1, \mathcal{L}(X^+, X^-)_{-1}\big)$ is a Jordan pair,
(iii) $\mathcal{L}(X^+, X^-)_0 = \big[\mathcal{L}(X^+, X^-)_1, \mathcal{L}(X^+, X^-)_{-1}\big]$.

Definition 5.1. A 3-graded polynomial $f(x_1^+, \ldots, x_n^+, x_1^-, \ldots, x_m^-) \in \mathcal{L}(X^+, X^-)$ is a 3-graded polynomial identity of a 3-graded Lie algebra $L = L_1 \oplus L_0 \oplus L_{-1}$ if it is mapped to zero under every homomorphism $\varphi : X \to L$ such that $\varphi(X^\sigma) \subseteq L_\sigma$ [29, p. 377].

Lemma 5.2. *Let $FJP(X^+, X^-)$ be the free Jordan pair on the polarized set $X = X^+ \cup X^-$. Then:*

(i) $FJP(X^+, X^-) \cong \big(\mathcal{L}(X^+, X^-)_1, \mathcal{L}(X^+, X^-)_{-1}\big)$.
(ii) $\mathcal{L}(X^+, X^-)$ *is a central extension of* $TKK\big(FJP(X^+, X^-)\big)$, *that is,* $\mathcal{L}(X^+, X^-)/C_V \cong TKK\big(FJP(X^+, X^-)\big)$, *where*

$$C_V = \big\{x \in \mathcal{L}(X^+, X^-)_0 \mid [x, \mathcal{L}(X^+, X^-)_1] = 0 = [x, \mathcal{L}(X^+, X^-)_{-1}]\big\}.$$

Proof. (i) follows from the universal properties of $FJP(X^+, X^-)$ (see [26]) and $\mathcal{L}(X^+, X^-)$, and (ii) follows from 2.1 and 5.1, since $C_V = Z(\mathcal{L}(X^+, X^-) \cap \mathcal{L}(X^+, X^-)_0$. See also [24, Lemma 2.8] □

5.2. Free special 3-graded Lie algebra

Let $Ass[X] = Ass[X^+ \cup X^-]$ be the free associative algebra on the polarized set of generators $X = X^+ \cup X^-$, with $\mathbb{Z}$-grading defined by $\vartheta(x^\sigma) = \sigma 1$ for all $x^\sigma \in X^\sigma$. The quotient algebra of $Ass[X] = \oplus_{n\in\mathbb{Z}} Ass^{(n)}[X]$ by the ideal generated by $\sum_{|n|>1} Ass^{(n)}[X]$ (equivalently by the set $\{x^\sigma y^\sigma \mid x^\sigma, y^\sigma \in X^\sigma, \sigma = \pm\}$) is the free 3-graded associative algebra, denoted $Ass[X^+, X^-]$, on $X = X^+ \cup X^-$ [29, p. 352]. We denote by $S\mathcal{L}(X^+, X^-)$ the subalgebra of $Ass[X^+, X^-]^{(-)}$ generated by the elements of X. Then, $S\mathcal{L}(X^+, X^-)$ is the free special 3-graded Lie algebra, and, by the universal property of $\mathcal{L}(X^+, X^-)$, there exists a unique homomorphism of 3-graded Lie algebras $\pi : \mathcal{L}(X^+, X^-) \to S\mathcal{L}(X^+, X^-)$ extending the inclusion $X \subseteq S\mathcal{L}(X^+, X^-)$.

Definition 5.3. We will say that a 3-graded polynomial $f \in \mathcal{L}(X^+, X^-)$ is essential if its image $\pi(f) \in S\mathcal{L}(X^+, X^-) \subseteq Ass[X^+, X^-]$ is nonzero and has a monic leading term (of highest *deg* degree) as an associative polynomial [29, p. 377].

As usual, polynomial identities will be assumed to be multilinear polynomial identities.

Lemma 5.4. *Let $L = L_1 + L_0 + L_{-1}$ be a 3-graded Lie algebra satisfying an essential 3-graded polynomial identity $f \in \mathcal{L}(X^+, X^-)$ of degree d. Then L satisfies a multilinear essential 3-graded polynomial identity of degree less than or equal to d.*

Proof. See [5, 6.2.4]. □

Next we characterize essential 3-graded polynomial identities in terms of the special Lie algebras $sl(n)$, endowed with 3-gradings defined by decompositions of the form $n = p + q$, for some $p, q \in \mathbb{N}$.

Proposition 5.5. *Let $f \in \mathcal{L}(X^+, X^-)$ be a 3-graded polynomial. Then, $\pi(f) \in S\mathcal{L}(X^+, X^-)$ is nonzero if and only if there exist $p, q \in \mathbb{N}$ such that f is not an identity of $sl(p + q)$. Thus, if Φ is a field, f is an essential 3-graded polynomial if and only if there exist $p, q \in \mathbb{N}$ such that f is not an identity of $sl(p + q)$.*

Proof. Let $f = f_1 + f_0 + f_{-1} \in \mathcal{L}(X^+, X^-)$ be a 3-graded polynomial such that $0 \neq \pi(f) = g(x_1^+, \dots, x_1^-, \dots) \in S\mathcal{L}(X^+, X^-)$. We can assume $X^\sigma = \{x_1^\sigma, \dots, x_d^\sigma\}$ for some $d \in \mathbb{N}$ and $\sigma = \pm$.

Let $k = deg(g)$ be the degree of g as an element of $S\mathcal{L}(X^+, X^-) \subseteq Ass[X^+, X^-]$, defined in the obvious way, and let N be the ideal of $Ass[X^+, X^-]$ generated by all monomials of degree *deg* strictly greater that k. Then, N is a 3-graded ideal $N = N_1 + N_0 + N_{-1}$ of $Ass[X^+, X^-]$, such that $N_i = N \cap Ass[X^+, X^-]_i$, for $i = 0, \pm 1$, and the quotient algebra $A = Ass[X^+, X^-]/N$ is a 3-graded associative algebra.

Write now $A = A_1 + A_0 + A_{-1}$, where $A_i = Ass[X^+, X^-]_i/N_i$ for $i \in \{0, \pm\}$. Then (A_1, A_{-1}) is an associative pair, and its standard embedding $\mathcal{E}$ is a finite-dimensional free Φ-module [23, 2.2].

Following the proof of [5, Lemma 6.2.1], consider now the regular representation $\rho : \mathcal{E} \to End_\Phi(\mathcal{E})$ of $\mathcal{E}$ as a right $\mathcal{E}$-module. The Φ-module decomposition of $\mathcal{E}$, in matricial notation:

$$\mathcal{E} = \begin{pmatrix} \mathcal{E}_{11} & \mathcal{E}_{12} \\ \mathcal{E}_{21} & \mathcal{E}_{22} \end{pmatrix} = \begin{pmatrix} \mathcal{E}_{11} & 0 \\ \mathcal{E}_{21} & 0 \end{pmatrix} + \begin{pmatrix} 0 & \mathcal{E}_{12} \\ 0 & \mathcal{E}_{22} \end{pmatrix} = M_1 + M_2,$$

results into a 3-grading of $End_\Phi(\mathcal{E}) = End_\Phi(\mathcal{E})_1 \oplus End_\Phi(\mathcal{E})_0 \oplus End_\Phi(\mathcal{E})_{-1}$, given by:

$$\begin{aligned} End_\Phi(\mathcal{E})_1 &= Hom_\Phi(M_1, M_2), \\ End_\Phi(\mathcal{E})_0 &= End_\Phi(M_1) \oplus End_\Phi(M_2), \\ End_\Phi(\mathcal{E})_{-1} &= Hom_\Phi(M_2, M_1), \end{aligned}$$

that makes ρ a 3-graded homomorphism. Moreover, since $\mathcal{E}$ is an unital algebra, ρ is indeed a 3-graded monomorphism.

Fixing now bases of the free Φ-modules M_1 and M_{-1}, we can obtain a 3-graded isomorphism $End_\Phi(\mathcal{E}) \cong M_n(\Phi)$, where $n = \dim_\Phi(\mathcal{E})$ and $M_n(\Phi) = M_n(\Phi)_1 \oplus M_n(\Phi)_0 \oplus M_n(\Phi)_{-1}$ is the 3-grading given by

$$M_n(\Phi) = \begin{pmatrix} 0 & M_{p,q}(\Phi) \\ 0 & 0 \end{pmatrix} \oplus \begin{pmatrix} M_p(\Phi) & 0 \\ 0 & M_q(\Phi) \end{pmatrix} \oplus \begin{pmatrix} 0 & 0 \\ M_{q,p}(\Phi) & 0 \end{pmatrix},$$

where $p = \dim_\Phi(M_1)$ and $q = \dim_\Phi(M_2)$. (Note that here $p = q$.) Consequently the homomorphism $\varphi : A \to M_n(\Phi)$, resulting from the composition of the inclusion $A \subseteq \mathcal{E}$, the regular representation $\rho : \mathcal{E} \to End_\Phi(\mathcal{E})$ and the above isomorphism $End_\Phi(\mathcal{E}) \cong M_n(\Phi)$, is a 3-graded monomorphism. Moreover, if we denote by $\overline{x_i^\sigma} = x_i^\sigma + N \in A_\sigma$ the homomorphic images of the elements $x_i^\sigma \in X^\sigma$ in the quotient algebra $A = Ass[X^+, X^-]/N$, then, $\varphi(\overline{x_i^\sigma}) = a_i^\sigma \in M_n(\Phi)$ is traceless matrix, and, therefore $a_i^\sigma \in sl(p+q)$, for all $i = 1, \dots, d$ and $\sigma = \pm$.

Assume now that f is a (multilinear, see Lemma 5.4) polynomial identity of $sl(p+q)$. Denoting by $\varphi^{(-)} : A^{(-)} \to M_n(\Phi)^{(-)}$ the Lie algebra monomorphism induced by φ, it follows that:

$$\begin{aligned} 0 &= f(a_1^+, \dots, a_d^+, a_1^-, \dots, a_d^-) = f(\varphi(\overline{x_1^+}), \dots, \varphi(\overline{x_d^+}), \varphi(\overline{x_1^-}), \dots, \varphi(\overline{x_d^-})) \\ &= \varphi(f(\overline{x_1^+}, \dots, \overline{x_d^+}, \overline{x_1^-}, \dots, \overline{x_d^-})) = \varphi(\overline{f(x_1^+, \dots, x_d^+, x_1^-, \dots, x_d^-)}) = \varphi(\overline{g}). \end{aligned}$$

But, since $\varphi^{(-)}$ is a monomorphism, this implies $\overline{g} = 0$, contradicting the choice of N. Hence there exist $p, q \in \mathbb{N}$ such that f is not an identity of $sl(p+q)$.

Conversely, let us suppose that there exist p, $q \in \mathbb{N}$ such that $f \in \mathcal{L}(X^+, X^-)$ is not a polynomial identity for $sl(p+q)$. Writing $n = p+q$, this induces a 3-grading in $sl(n)$ such that $\big(sl(n)_1, sl(n)_{-1}\big) \subseteq \big(M_{p,q}(\Phi), M_{q,p}(\Phi)\big)$ is a special Jordan pair.

Consider now the case $f = f_+ \in \mathcal{L}(X^+, X^-)_1$. Then, since $f = f_+$ is not a polynomial identity of $sl(n)$, by Lemma 5.2, it follows that $f = f_+ \in FJP[X^+, X^-]^+$ is not a polynomial identity for its associated Jordan pair $\big(sl(n)_1, sl(n)_{-1}\big)$, and therefore $f = f_+$ has a nonzero image in $FSJP[X^+, X^-]$, the free special Jordan pair. Hence $\pi(f) = \pi(f_+) \neq 0$. The case when $f = f_- \in \mathcal{L}(X^+, X^-)_{-1}$ follows analogously.

Suppose next that $f = f_0 \in \mathcal{L}(X^+, X^-)_0$. We claim that there exists $y^\sigma \in X^+ \cup X^-$ such that $[f_0, y^\sigma] \neq 0$. Indeed, relabeling if necessary, if $x_1^{\sigma_1} \dots x_{2d}^{\sigma_{2d}}$ is a monic monomial of highest degree in f_0 and y^σ a variable not appearing in f_0 with $\sigma \neq \sigma_1$ (or $\sigma \neq \sigma_{2d}$), it suffices to note that the polynomial $[f_0, y^\sigma]$ contains the monomial $y^\sigma x_1^{\sigma_1} \dots x_{2d}^{\sigma_{2d}}$ with coefficient 1.

We claim that there exists a large enough nonnegative integer m_0 such that neither f nor $[f_0, y^\sigma]$ vanish in $sl(m_0)$. Indeed, the statement for f is clear, since it does not vanish in any $sl(m)$ $m \geq n = p+q$, and for $[f_0, y^\sigma]$ it follows from the case $f = f_\sigma$.

Finally, replacing f_0 by $g_\sigma = [f_0, y^\sigma]$, if necessary, we obtain an essential polynomial such that $\pi(f) = \pi(f_0) \neq 0$. $\square$

We refer to [20, 0.12], and references therein, for the notion of essential polynomial in Jordan systems.

Theorem 5.6. *Let L be a Jordan 3-graded Lie algebra with associated Jordan pair V. If L satisfies an essential 3-graded polynomial identity, then the Jordan pair V satisfies an essential polynomial identity.*

Proof. By Lemma 5.2 and Proposition 5.5, it suffices to note that if L satisfies an essential 3-graded polynomial identity $f = f_1 + f_0 + f_{-1}$, then V satisfies the essential polynomial identity $g = (g_+, g_-) = \big(f_1 + [f_0, y^+], f_{-1} + [f_0, y^-]\big)$. $\square$

6. Posner Rowen's theorem for Jordan 3-graded Lie algebras

In this section we attempt to describe strongly prime Jordan 3-graded Lie algebras satisfying essential 3-graded polynomial identities. To do this we relate their polynomial identities to those of their associated Jordan pairs, to then taking advantage of the results on PI Jordan pairs proved in [22].

In this section we will make use of Zelmanov's classification of simple (infinite-dimensional) Lie algebras with a finite nontrivial $\mathbb{Z}$-grading [29]. Therefore from now Φ is assumed to be a field of characteristic zero or characteristic at least 5.

Remark 6.1. Let L be a Jordan 3-graded Lie algebra. If L satisfies an essential 3-graded polynomial identity $f = f_1 + f_0 + f_{-1}$, then, by Theorem 5.6, its associated Jordan pair V satisfies the essential polynomial identity $g = (g_+, g_-) = \big(f_1 + [f_0, y^+], f_{-1} + [f_0, y^-]\big)$. Since $deg(f) = \max\,\{deg(f_1)\,, deg(f_0)\,, deg(f_{-1})\}$, it holds that:

- (i) If $deg(f) = 2d-1$ is odd, then, for some $\sigma = \pm$, $deg(f) = deg(f_\sigma) > deg(f_0)$. Thus $deg(f) \geq deg(f_0)+1$, and we obtain that $deg(g) = \max\{deg(g_+)\,, deg(g_-)\} = deg(f) = 2d-1$.
- (ii) If $deg(f) = 2(d-1)$ is even, then $deg(f) = deg(f_0) > deg(f_\sigma)$, for $\sigma = \pm$. This results into $deg(f) \geq deg(f_\sigma) + 1$ and therefore $deg(g) = deg(f) + 1 = 2d-1$.

Hence, if L satisfies an essential 3-graded polynomial identity f of degree either even $2(d-1)$ or odd $2d-1$, we can assume that its associated Jordan pair V satisfies an essential polynomial identity $g = (g_+, g_-) = \big(f_1 + [f_0, y^+], f_{-1} + [f_0, y^-]\big)$ of degree $2d-1$.

Theorem 6.2. (Posner-Rowen's Theorem for Jordan 3-graded Lie algebras.) *Let L be a Jordan 3-graded Lie algebra with associated Jordan pair V over a field Φ of characteristic zero or prime $p \geq 5$. If L is strongly prime and satisfies an essential 3-graded polynomial identity, then its central closure is simple and, therefore, isomorphic to one of following Lie algebras:*

- I. *$[R^{(-)}, R^{(-)}]/Z$, where $R = R_{-1} + R_0 + R_1$ is a simple associative 3-graded algebra, finite dimensional over its center, and Z is the center of the derived algebra $[R^{(-)}, R^{(-)}]$.*
- II. *$[K(R,*), K(R,*)]/Z$, where $R = R_{-1} + R_0 + R_1$ is a simple associative 3-graded algebra, finite dimensional over its center, with involution $* : R \to R$, such that $R_i^* = R_i$ for all $i \in \{0, \pm 1\}$, and $K(R,*) = \{a \in R \mid a^* = -a\}$.*
- III. *The Tits-Kantor-Koecher algebra of the Jordan algebra of a symmetric bilinear form.*
- IV. *An exceptional Lie algebra of type E_6 or E_7.*

Moreover, in cases I and II the isomorphism preserves the grading, that is, it is an isomorphism of 3-graded algebras.

Proof. We first note that being strongly prime, then L is isomorphic to the TKK-algebra of is associated Jordan pair V, and V is strongly prime (see 2.1 and [4, Proposition 11.25]).

By Theorem 5.6, V satisfies an essential multilinear polynomial identity and, then, by [22, Proposition 4.6], V is homotope-PI. Therefore, by the Jordan pair analogue of Posner Rowen theorem [20, Theorem 6.1(ii)], the extended central closure $\mathcal{C}(V)V$ of V is a simple Jordan pair with finite capacity. Moreover, assuming the degree of the essential polynomial identity of L to be as in Remark 6.1, and considering that V and $\mathcal{C}(V)V$ satisfy the same multilinear polynomial identities, we can assume that $\mathcal{C}(V)V$ has capacity at most $2d$.

Now, since by Theorem 4.3, the central closure $\mathcal{C}(L)L$ of L is isomorphic to the TKK-algebra $TKK(\mathcal{C}(V)V)$ of the extended central closure $\mathcal{C}(V)V$ of V, by [4, Proposition 11.25], $\mathcal{C}(L)L$ is a simple Jordan 3-graded Lie algebra, and therefore $\mathcal{C}(L)L$ is isomorphic to one of the Lie algebras listed in [29, Theorem 1].

To finish the proof it suffices to note that the associative algebras appearing in cases I and II are finite dimensional over their centers by [7, p. 57], and that, by [25, 7.2, 7.3], in the exceptional case IV the only possibilities allowed for $\mathcal{C}(L)L$ are types E_6 or E_7. $\square$

7. PI Jordan 3-graded Lie algebras

We devote this last section to arbitrary Jordan 3-graded Lie algebras satisfying essential 3-graded polynomial identities. To cope with the absence of regularity conditions we will consider the Kostrikin radical of the Lie algebras.

7.1. Kostrikin radical

An element z of a Lie algebra L is a *crust of a thin sandwich* if $(ad\, z)^2 = 0$. Lie algebras with no nonzero crusts of thin sandwiches are nondegenerate Lie algebras (also called strongly nondegenerate in the sense of Kostrikin). The smallest ideal of a Lie algebra L that provides a nondegenerate quotient algebra is the Kostrikin radical of L, denoted by $K(L)$ [13]. We also recall here that the Jordan counterpart, that is, the smallest ideal of a Jordan pair V that provides a nondegenerate Jordan system is the McCrimmon radical $Mc(V)$ of V [28, p. 538-539].

Remark 7.1. Given an ideal I of a Lie algebra L we will denote $\widetilde{I} = \{x \in L \mid [x, L] \subseteq I\}$ the pre-image of the center $Z(L/I)$ of the quotient Lie algebra L/I by the canonical projection $L \to L/I$.

Proposition 7.2. *Let L be a Jordan 3-graded Lie algebra with associated Jordan pair V. Then $K(L) = \widetilde{\mathcal{I}(Mc(V))}$.*

Proof. Let $\mathcal{I}(Mc(V))$, defined as in 3.4, be the ideal of L generated by the McCrimmon radical $Mc(V)$ of the Jordan pair V. Then $\overline{L} = L/\mathcal{I}(Mc(V))$ is a Jordan 3-graded Lie algebra, whose associated Jordan pair $(\overline{L}_1, \overline{L}_{-1}) \cong V/Mc(V)$ is nondegenerate.

Next we claim that $L/\widetilde{\mathcal{I}(M(V))} = \overline{L}/C_{(\overline{L}_1,\overline{L}_{-1})} = TKK(\overline{L}_1, \overline{L}_{-1})$, where by 2.1, it suffices to check that

$$\widetilde{\mathcal{I}(Mc(V))}/\mathcal{I}(M(V)) = Z(L/\mathcal{I}(Mc(V))) = C_{(\overline{L}_1,\overline{L}_{-1})}.$$

To prove this claim let $\overline{z}_i \in Z(L/\mathcal{I}(Mc(V))) \cap \overline{L}_i$, $i \neq 0$. Then $\overline{z}_i$ is an absolute zero divisor in $(\overline{L}_1, \overline{L}_{-1})$, but since $Mc(\overline{L}_1, \overline{L}_{-1}) = 0$, it follows that $\overline{z}_i = 0$. Hence $Z(L/\mathcal{I}(Mc(V))) \subseteq \overline{L}_0$ which implies that

$$\mathcal{I}(\widetilde{Mc(V)})/\mathcal{I}(Mc(V)) = Z(L/\mathcal{I}(Mc(V))) = C_{(\overline{L}_1,\overline{L}_{-1})}.$$

Hence $L/\mathcal{I}(\widetilde{Mc(V)}) = TKK(\overline{L}_1, \overline{L}_{-1})$, which is a nondegenerate Lie algebra by [4, Proposition 11.25], and, therefore, by the minimality of the Kostrikin radical, $K(L) \subseteq \mathcal{I}(\widetilde{Mc(V)})$ holds.

Conversely, it is not difficult to prove that $K(L)$ is a 3-graded ideal of L, and therefore $L/K(L)$ is a nondegenerate Jordan 3-graded Lie algebra, with nondegenerate associated Jordan pair $W = \big(L_1/K_1(L), L_{-1}/K_{-1}(L)\big)$. Moreover $C_W = Z(L/K(L)) = 0$, which implies, by 2.1, that $L/K(L) = TKK(W)$. Thus $\big(L_1/K_1(L), L_{-1}/K_{-1}(L)\big)$ is a nondegenerate Jordan pair and $\big(Mc(L_1), Mc(L_{-1})\big) \subseteq \big(K_1(L), K_{-1}(L)\big)$ holds. Hence it follows that $\mathcal{I}(Mc(V)) \subseteq K(L)$ and, therefore, $\mathcal{I}(\widetilde{Mc(V)}) \subseteq K(L)$ [28,29]. □

Lemma 7.3. *Let L be a Jordan 3-graded Lie algebra with associated Jordan pair V. Then:*

$$\mathcal{I}(\widetilde{Mc(V)}) = \bigcap \big\{\widetilde{\mathcal{I}(P)} \mid P \textit{ is a strongly prime ideal of } V\ \big\}.$$

Proof. Since, by [17], $Mc(V) = \bigcap \big\{P \mid P \text{ is a strongly prime ideal of } V\ \big\}$, it is straightforward that $\mathcal{I}(\widetilde{Mc(V)}) \subseteq \widetilde{\mathcal{I}(P)}$ for all strongly prime ideals P of V. To prove the reverse containment, let

$$x_i \in \bigcap \big\{\widetilde{\mathcal{I}(P)} \mid P \text{ is a strongly prime ideal of } V\ \big\} \cap L_i, \quad i \in \{\pm 1, 0\}.$$

If $i = \pm 1$, by 3.1, $x_i \in \mathcal{I}(\widetilde{Mc(V)})_i$. Otherwise $i = 0$ and then, for any strongly prime ideal P of V it holds that $[x_0, L] \subseteq \mathcal{I}(P)$, which implies $[x_0, L_1 + L_{-1}] \subseteq Mc(V)$, and therefore that $x_0 \in C_{V/Mc(V)}$. Hence $x_0 \in \mathcal{I}(\widetilde{Mc(V)})$. □

Proposition 7.4. *Let L be Jordan 3-graded Lie algebra. Then $L/K(L)$ is a subdirect product of strongly prime Jordan 3-graded Lie algebras.*

Proof. By Proposition 7.2 and Lemma 7.3, it now suffices to note that for any strongly prime ideal P of the Jordan pair $V = (L_1, L_{-1})$, the quotient algebra $L/\widetilde{\mathcal{I}(P)}$ is a strongly prime Jordan 3-graded Lie algebra. □

Theorem 7.5. *Let L be a Jordan 3-graded Lie algebra. If L satisfies an essential 3-graded polynomial identity, then the nondegenerate Jordan 3-graded Lie algebra $L/K(L)$ is a subdirect product of strongly prime Jordan 3-graded Lie algebras satisfying the same essential 3-graded polynomial identity. Therefore, the central closure of each subdirect factor is isomorphic to one of the algebras listed in Theorem 6.2.*

Proof. Let $K(L)$ be the Kostrikin radical of the Jordan 3-graded Lie algebra L. Then, by 7.1, the quotient Lie algebra $L/K(L)$ is nondegenerate, and it is Jordan 3-graded by Proposition 7.2, since as noted before $K(L)$ a 3-graded ideal of L. Moreover, by Proposition 7.4, $L/K(L)$ is a subdirect product of strongly prime Jordan 3-graded Lie algebras $\{L_\lambda\}_{\lambda\in\Lambda}$, all them satisfying the same essential 3-graded polynomial identities as the Lie algebra L. Hence, for each λ, the central closure $\mathcal{C}(L_\lambda)L_\lambda$ of L_λ is isomorphic to one of the algebras listed in Theorem 6.2. □

Declaration of competing interest

The authors declare the following financial interests/personal relationships which may be considered as potential competing interests:

Authors are partially supported by grant PID2021-123461NB-C21, funded by MCIN/AEI/10.13039/501100011033 and by “ERDF A way of making Europe” and grant E22-23 Álgebra y Geometría, Gobierno de Aragón.

Acknowledgements

Authors are partially supported by grant PID2021-123461NB-C21, funded by MCIN/AEI/10.13039/501100011033 and by “ERDF A way of making Europe” and grant E22-23 Álgebra y Geometría, Gobierno de Aragón.

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